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Abstract: Research indicates that calculus students often imagine objects losing a dimension entirely when a limit is taken, and that this image serves as an obstacle to their understanding of the fundamental theorem of calculus. Similar imagery, in the form of "indivisibles", was similarly unsupportive of the development of the fundamental theorem in the mid-1600s, unlike the more powerful subsequent imagery of infinitesimals. This parallel between student issues and historical issues suggests several implications for how to provide students with imagery that is more productive for understanding the fundamental theorem, such as the imagery of infinitesimals or the more modern quantitative limits approach, which relies heavily on quantitative reasoning.

Keywords: Calculus; infinitesimals; history of Calculus; Fundamental Theorem of Calculus

Introduction

Recently several colleagues told me they think that the fundamental theorem of calculus is not something that undergraduate students can really make sense of when they take a course in calculus, although it may snap into focus later in their education[i]. The implication was that we should not bother spending very much time teaching it. This is a bit too cynical for me. I don't

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think it's too much to ask that students come out of a calculus class understanding the *fundamental* idea of calculus, why there is a deep connection between integrals and derivatives. Nonetheless, research has shown that there are some common mental images that students use to make sense of derivatives and integrals, images that can serve as obstacles to their learning of the fundamental theorem of calculus.

The purpose of this article is to explore one of the most common of these mental images: the loss of dimension. First I describe this mental image and some of its manifestations in student reasoning, and discuss its limitations for promoting understanding of the fundamental theorem of calculus. Then I discuss some historical occurrences of pre-calculus ideas that also involve the loss of dimension, in the form of "indivisibles," and how these indivisibles were likewise limited in promoting the understanding of the fundamental theorem of calculus. I describe how the approach of infinitesimals, which replaced indivisibles, led to the successful conceptualization of calculus. This suggests that such imagery of infinitesimals might be useful for calculus students as well. Finally, I discuss the more modern approach of focusing just on limits of quantities, and why this is also more successful than approaches that involve the loss of dimension.

It is important to clarify what I mean by "mental image" in this article. One way to interpret a mental image is loosely as Vinner does, as a set of mental pictures that a person uses when considering a mathematical term like "limit"—pictures that may not necessarily, if somehow externalized, satisfy the mathematical definition of the term (e.g. Tall & Vinner 1981, Vinner 1989, 1991, Vinner & Dreyfus, 1989). While this notion of mental image captures the spirit of the idea I will be using, a Piagetian interpretation of mental image clarifies a few aspects of this. Piaget generally speaks of a mental image as expressing an anticipated outcome of an action taken on an object (Piaget 1967). These actions and objects can certainly be products of reflective abstraction, and so may be many steps removed from actual physical actions and perceived objects, but nonetheless the idea of mental image is ultimately grounded in physical action and attention (Thompson 1994). This means that a mental image (a) need not be entirely visual, (b) need not directly correspond with "real-world" objects or operations, and (c) need not directly correspond with things that have mathematical definitions. The Piagetian interpretation of mental image helps us keep in mind how such an image functions in the learning, rather than simply noting how it may or may not differ from standard mathematical treatments or definitions. Here this is particularly important, because it is not always clear if the mental images I am discussing can be seen to directly conflict with mathematical definitions. Furthermore, these mental images are results of imagined limit processes or infinite decompositions, so they are not going to be changed by any amount of checking against any real-world objects they might be seen to portray.

Dimension loss, or "collapse," in student reasoning about calculus

Recent research shows that one of the most common mental images that students employ when conceptualizing limits, and calculus concepts like derivatives and integrals that rely on limits, is one of dimension loss (Oehrtman 2009; Thompson 1994). Oehrtman calls this the "collapse metaphor," in which an object is seen as becoming smaller and smaller in one of its dimensions, and when the limit is reached it *becomes* an object whose dimension is one less than it was before (2009). When the limit is taken, one of the object's dimensions collapses and is lost. To illustrate this mental image, I draw quite a bit upon on Oehrtman's study (2009), which details it in a variety of contexts with examples from undergraduate students. Students use this metaphor when making sense of Riemann sums and definite integrals, derivatives, and the fundamental theorem.

A classic example of the collapse metaphor can be seen when students discuss a definite (Riemann) integral. Ten of Oehrtman's 25 undergraduate participants used the collapse metaphor in this context, seeing the integral as a sum of "infinitely many one-dimensional vertical lines over the interval [*a,b*] extending from the *x*-axis to a height *f(x)* and produced by a collapse of two-dimensional rectangles from the Riemann sum as their widths became zero" (p. 413).

Oehrtman found that many of these students also bring this collapse metaphor to bear in the context of the fundamental theorem of calculus. For instance, students were asked why the derivative of the formula for the volume of a sphere, $V = \frac{4}{3}\pi R^3$, with respect to the radius *r*, is equal to the surface area of the sphere, $SA = 4\pi R^2$. The students "described thin concentric spherical shells with 'the last shell of the sphere' getting thinner and thinner and eventually becoming the 'last sphere's surface area'" (ibid, p. 413). Then later the students described the definite integral in the equation

$$\frac{4}{3}\pi R^3 = \int_0^R 4\pi R^2 dr$$

as representing the limit of the sum of all these shells as they became thinner and eventually became two-dimensional surfaces. Thus, "the volume is just adding up the surface area of small spheres" (p. 413). Here we see the loss of dimension that occurs when students use the mental image of dimension loss when they imagine the result of a limit.

Constraints and affordances of the image of dimension loss in student reasoning

This metaphor seems to be a very natural and visually appealing way of thinking about what happens to objects in a limit. It does not necessarily serve as a significant obstacle for students who use it to conceptualize derivatives (Oehrtman 2009). Where it hinders the understanding is with understanding the relationship between the derivative and the definite integral, without which there is no calculus. This is reflected in the first part of the fundamental theorem of calculus (which is what I will be referring to when I say "fundamental theorem").

In a teaching experiment focusing on the fundamental theorem, Thompson (1994) asked students to explain why the instantaneous rate of change (with respect to height) of the volume of water filling a container is the same as the water's top surface area at that height (Figure 1). One student responded that the volume of the thin top slice would *become* the area of the top surface when the thickness went to zero in the limit. This is an example of dimensional collapse. According to Thompson, the basic problem with this reasoning is that the student is not seeing the connection between an increment of volume and an increment of height. The increment of volume is being viewed as only an area, having no height. The student's mental image "could be described formally as

$$\lim_{\Delta h \to 0} V(h + \Delta h) - V(h) = A(h)$$

which would have meant that

$$\lim_{\Delta h \to 0} \frac{V(h + \Delta h) - V(h)}{\Delta h} = \lim_{\Delta h \to 0} \frac{A(h)}{\Delta h}$$

an equality I cannot interpret" (Thompson 1994, pp. 263-4).

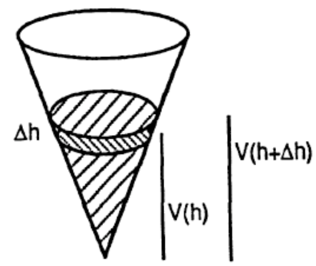

Figure 1—Student's diagram for identifying amount of change in volume as water rises in height within a conical storage tank, from Thompson (1994, p. 259). As $\Delta h \to 0$, some students treated the volume of the top region as becoming the area of the top surface.

The student using this metaphor of collapse is thus not attending to the *ratio* of an increment of volume to an increment of height. The height dimension is lost when the limit occurs, and thus so is the information about relationship between an increment of volume and an increment of height. For the student, when the limit occurs, the resulting quantity *A(h)* is no longer a *rate* at all.

When the students who use the collapse metaphor for Riemann sums conceptualize the fundamental theorem of calculus, the same kind of problem can occur. If a thin rectangular area simply *becomes* a line segment when the limit is taken, then the relationship between the rectangle's area and its width is lost (see Figure 2). Therefore the height of the function *f(x)* is not seen as the *rate* at which the area under the curve changes as *x* changes. This metaphor is thus an obstacle to understanding the fundamental theorem, because it obscures the fact that the value of *f(x)* is a derivative, a rate of change, of the area under the curve. In these ways, the collapse metaphor can preempt or hinder investigation into the ratio structures that are crucial to the fundamental theorem of calculus (Oehrtman 2009).

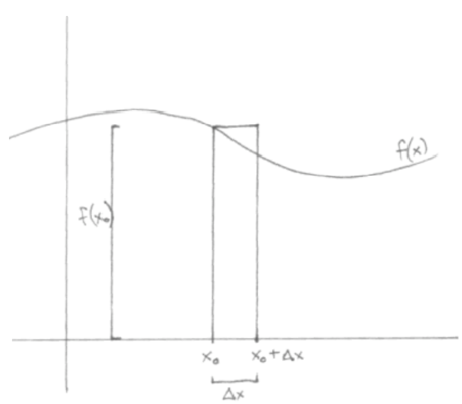

Figure 2—One rectangle in a Riemann sum. As $\Delta x \to 0$ the rectangle might be seen to collapse to the line segment with height $f(x_0)$, obscuring the idea that this height is the *rate* of change of the rectangle's area [$f(x_0)\times\Delta x$] with respect to its width Δx.

Historical dimension loss: indivisible techniques

In the historical development of geometry, techniques that involve the loss of a dimension in a figure have been very prominent, in the form of the technique of indivisibles. Indivisibilist techniques were used rather widely in ancient Greek mathematics, not as a method of proof but as a heuristic tool for solving problems (see Knorr 1996 for a good summary).[2] The technique was rediscovered in Europe in the early 17th century, and gained prominence as a method of solving area and volume problems particularly through Bonaventura Cavalieri's 1635 work *Geometria indivisibilibus continuorum nova quadam ratione promota* (*Geometry developed by a new method through the indivisibles of the continua*). The technique of indivisibles uses lower-dimensional pieces to reason about a figure, and it invokes the same kind of mental imagery that the collapse metaphor invokes. The technique is similarly limited in allowing for the conceptualization of the fundamental theorem of calculus. In fact, I argue later that the abandonment of indivisibles in favor of infinitesimals was crucial to the development of the fundamental theorem.

To use Cavalieri's method of indivisibles, imagine two plane figures ABC and A'B'C' bounded by two parallel lines EO and BC (see Figure 3). Now move the top line down until it meets BC, keeping it always parallel to BC. "The intersections of this moving plane, or fluent, and the figure ABC, which are produced in the overall motion, taken all together, I call: *all the lines of the figure ABC*," taken with "transit" perpendicular to BC (Cavalieri 1635, book 2, pp. 8-9). If each of these lines of figure ABC (say, LM) is as long as its corresponding line of figure A'B'C' (L'M'), then the two figures have the same area. More generally, if the lines of one

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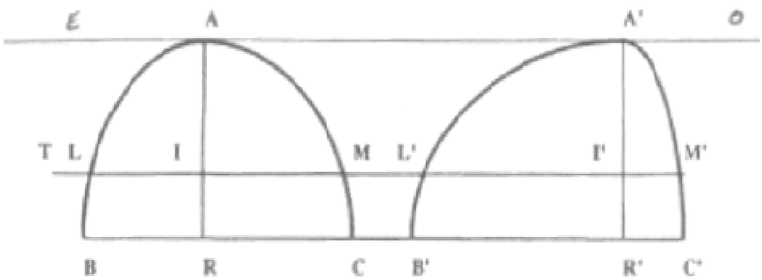
[2] Archimedes, for instance, used this technique extensively and brilliantly in his *Method*, a work found in a written-over palimpsest in 1906. The rediscovery and very recent reconstruction of the text is an amazing story, and can be read about in The Archimedes Codex (Netz & Noel 2007), among other places.

figure are all in a given ratio to the lines of the other, then the two figures' areas are in the same ratio.

Figure 3—Cavalieri's principle, adapted from Mancosu (1996, p. 41)



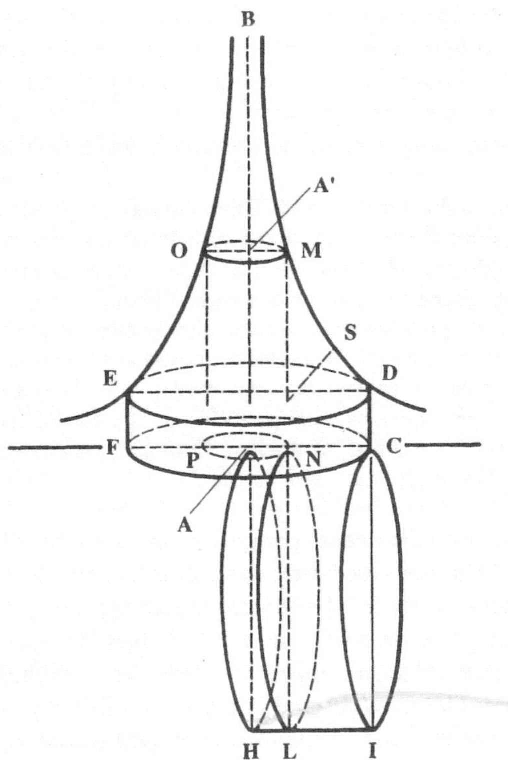
Cavalieri states this method for solids also and applies it to a wide variety of figures. For instance, through a series of comparisons of indivisibles of several figures, one can quickly conclude that a cone takes up a third of the volume of the cylinder that contains it. The principle marked a significant step toward the development of integral calculus, and it is still commonly used today, although it is no longer stated in such a way that requires us to consider a figure as being comprised of "all its lines."

Cavalieri's technique of indivisibles uses the idea of dimensional collapse by treating the two-dimensional figure as being entirely comprised of its one-dimensional line segments. These segments are "indivisible" slices of the figure because each has no width and cannot be further divided in the dimension of thickness. The figure is thus precisely the same as what you get when you conceptualize a definite integral using the collapse metaphor: an area comprised of infinitely many one-dimensional line segments. Although this similarity is striking, there is a key difference between the collapse metaphor and the imagery associated with indivisibles. In the collapse metaphor a lower-dimensional object comes about as a result of an envisioned limit process in which a dimension is *lost*, whereas for Cavalieri the indivisible slices are not the result of a limit process and never really were two-dimensional.

Historical issues with dimension loss

In 1643, E. Torricelli famously adapted the technique of indivisibles to find the volume of his "Trumpet," a figure often now called Gabriel's Horn. In Figure 5 this trumpet is the infinitely long solid created by rotating a hyperbola around its asymptote AB, from some point E upward, together with cylinder FEDC. Torricelli decomposes the solid into all of its indivisible nested shells, two-dimensional cylindrical surfaces (e.g. POMN). Then he shows that each such surface has the same area as a corresponding disc (e.g. NL) whose radius is the same as the hyperbola's diameter AS. All of these discs comprise the cylinder PCIH, so the volume of the trumpet is the same as the volume of this cylinder (e.g. $\pi(AS)^2$).

Figure 5—Torricelli's trumpet, from Mancosu (1996, p. 133)



Torricelli's result was impressive not only because it shows an infinitely extended solid to have finite volume, but also because it uses *curved* indivisibles to show it. One of the most bizarre aspects of this demonstration is the last indivisible pairing, in which the infinite axis AB "is equal to" the disc AH (Torricelli, 1644, v. 1, p. 194). Not only is there already a collapse of dimension by viewing the original figure as comprised of its indivisibles, but the last indivisible sustains yet another dimensional collapse and as a one-dimensional line is seen as “equal” (in area?) to a two-dimensional disc.[3] This paradoxical conclusion is not the only one of its sort—for instance, Torricelli’s teacher Galileo had already shown that, in using two-dimensional indivisible slices to find the volume of a particular bowl-like figure, the last indivisible degenerates to a one-dimensional circle which is equal to a zero-dimensional point (Galileo 1634).

Note also the strange fact that although each indivisible slice of Torricelli’s trumpet has finite area according to Torricelli’s slicing method, if you sliced the trumpet in vertical planar slices instead you would get one slice with infinite area, namely the slice containing the axis AB. This last point indicates a danger with indivisibles—one slicing method produces correct results while another does not, and it is not clear that there is an a priori way to distinguish between the two. Torricelli was aware of this kind of problem with indivisibles, and used a diagram like the one in Figure 6 to demonstrate it. Suppose we have two right triangles with different area, Triangle 1 (BDA) and Triangle 2 (BDC), as shown. For each point P on side BD there corresponds a vertical indivisible slice JI of Triangle 1 and a vertical indivisible slice KL of Triangle 2. Thus all the slices of the Triangle 1 are in one-to-one correspondence with those of Triangle 2, yet the two triangles have different areas.

[3] It is noteworthy that some of Oehrtman’s students also used a collapse metaphor to make sense of Torricelli’s trumpet, albeit in a different manner; see Oehrtman 2009, pp. 412-13.

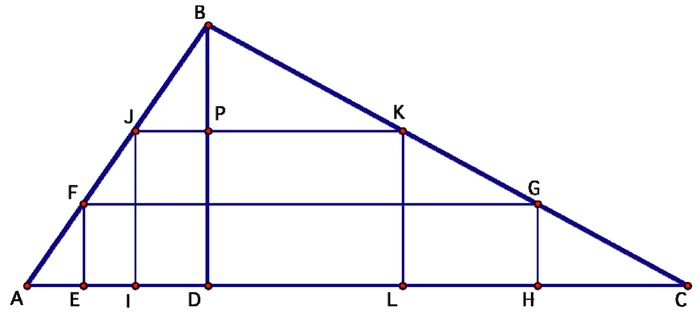


Figure 6—Torricelli's demonstration of problems with correspondences of indivisibles when determining areas

Cavalieri solved this problem by pointing out that his principle only works if the slicings of the two figures are done with the same "transit." The idea is that the indivisibles must have the same *density* in both figures. Indeed, Cavalieri likens the notion to the threads in a cloth: if Triangle 1 and 2 are cut from the same cloth, then the first might have 100 threads and the second 200 (Cavalieri 1966). So any one-to-one correspondence between threads must leave 100 threads of Triangle 2 unaccounted for.

Some of the issues with indivisibles are certainly solved by restricting oneself to cases in which the density of indivisibles is the same. But this also indicates that we are doing something a bit unnatural. It is an odd thing to conveniently lose a dimension by using indivisibles, only to have to pay attention to what is happening again in this dimension by reference to density or transit. Indeed, this issue led Torricelli to develop a theory in which indivisibles have a certain thickness (spissitudo) to them (Mancosu 1996), thus in a nebulous way returning some quality of the lost dimension back to the indivisibles. Thus, almost immediately after indivisibles made a splash in the 1630s, there was an effort to restore information about the collapsed dimension in order to avoid some of these pitfalls. Only when a method came along that retained the powerful technique of dividing something into infinitely many pieces, without losing the thickness dimension of these pieces, that the full power of calculus became available. This was the method of infinitesimals.

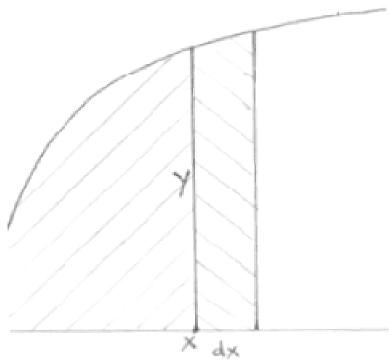
The infinitesimal calculus

Although many 17th century mathematicians found tangent lines to and areas under a variety of curves, most scholars officially credit the development of "Calculus" to Isaac Newton and Gottfried Wilhelm Leibniz (independently). The reason is that these two mathematicians created systems that (a) centrally recognize the relationship between derivatives and integrals, that is the fundamental theorem of calculus, and (b) provide a systematic method for determining derivatives of many types of curves, and hence integrals (e.g. Whiteside 1960). Newton's method of fluxions and fluents is the earlier method (1665-6), but was not widely circulated until it was in a later re-imagined form in *Principia Mathematica* (1687). Leibniz' infinitesimal calculus was developed in the 1670s and circulated in the early 1680s, and it is its notation we generally use today.

There are differences in mental imagery between the two systems, but the image of the infinitesimal is central to both. A more thorough account of Newton and Leibniz' treatments of the fundamental theorem of calculus can be found in Bressoud (2011); here I focus only on the aspects that are important to my point.

The crucial idea for Leibniz is that any curve is a "polygon with infinitely many [infinitesimal] sides," so on infinitely small intervals the curve is straight (Kleiner 2001, p. 146). Thus, for example, an instantaneous rate of change of two quantities is the ratio dy/dx of an infinitesimal difference in one quantity to an infinitesimal difference in the other. This means that an area under a curve can be seen as an infinite collection of infinitesimal rectangles (well, trapezoids, but which quickly can be seen as having equivalent areas as rectangles). Thus in "$\int ydx$", the $\int$ is a big S for "summa" (sum) of all of the rectangles' area, each being the product of a height y and width dx (the difference between two infinitely close x values). Now the question can be asked: for an infinitesimal change in *x*, i.e. *dx*, how much does the area under the curve change? An infinitesimal rectangle is accrued, with area *ydx*, where *y* is the height of the curve (Figure 7). Thus the rate of change of the area accumulating under the curve as *x* increases an infinitesimal amount, is *ydx/dx*, or simply *y*, the curve's height at that point.

Figure 7—The Fundamental Theorem of Calculus in terms of infinitesimals: for an infinitesimal increment of area accrued, the rate of change of this area with respect to the infinitesimal increment *dx* is just *y*, the curve's height



In this version of the fundamental theorem, notice how important it is to retain the dimension of thickness of the rectangle, even though it is infinitesimal. The *ratio* of the (rectangle's area)/(rectangle's width) is only meaningful if the infinitely small rectangle still *has* a width, and hence an area. If the rectangle were instead seen as collapsing to an area-less line segment, this quotient would become meaningless, so the fundamental theorem would remain unavailable.

Newton's understanding of the fundamental theorem of calculus is more dynamic than Leibniz', and is often treated in terms of velocities, yet the key idea is similar: "to recognize the ordinate as the rate of change of the area" (Bressoud 2011, p. 104). As *x* increases, the area under the curve accumulates at a rate given by the height *y* of the curve. Thus the area changes incrementally as *x* changes incrementally, and the ratio of the two is the curve's height. Newton often treats such increments as infinitesimal (Whiteside 1960), although they can also be seen as finite quantities that would vanish simultaneously if shrunk; the key idea is that the ratio between them is maintained and neither of them collapses.

In this way the fundamental theorem of calculus was unavailable using indivisibles, but became accessible using the image of infinitesimals. And it was because of this that the infinitesimal calculus became The Calculus in the subsequent century, transforming mathematics and science in its widespread application.

What is an infinitesimal?

The imagery of infinitesimals made the fundamental theorem of calculus available in ways that indivisibles and the imagery of dimension loss could not. This suggests that the imagery of infinitesimals may also offer students a way to surpass the collapse metaphor they often use, in order to make sense of the fundamental theorem. Although I contend that it is very promising to employ the imagery of infinitesimals with students, infinitesimals bring with them their own issues that were problematic historically and could serve as obstacles for students. Other works have addressed these philosophical issues in more detail, which I do not attempt here (e.g. Mancosu 1996, Knobloch 1994, Bell 2008, Russell 1914). The purpose of this section is to summarize these issues and how they may function for students.

The most important issue is that it is not quite clear what an infinitesimal quantity is; what does it mean for a quantity to be infinitely small, smaller than any finite magnitude, but not be zero? To address this, note that an "infinitesimal" width is precisely one that is not subject to what we now call the Archimedean Axiom. This axiom states that it is always possible to find a finite multiple of one magnitude which will exceed a given other magnitude. An infinitesimal length cannot satisfy the axiom, because it can be multiplied infinitely many times to still comprise a finite, not infinite, length. The rigorous work to construct infinitesimals by denying this axiom is technical and relatively recent (e.g. Robinson 1966). A calculus student who uses the image of infinitesimals might be uneasy about this property of infinitesimals, and is probably not going to study up on the formal construction of infinitesimals in order to allay his concerns. There are two related kinds of conception that are plausibly important for such a student to have in order to overcome this potential obstacle to working with the image of infinitesimals.

First, such a student must be able to imagine an infinitesimal as a convenient formal object, a "useful fiction," as Leibniz says, not as a thing that exists in a tangible way, while yet not having the machinery for its formal creation. This kind of thing may not be a significant obstacle for students; by the time they are in calculus they have often worked with mathematical objects, such as negative and imaginary numbers, in a formal way without worrying about their real-world referents and without the machinery to understand the formal construction of these objects.

Second, if some of the expected properties of finite numbers, such as the Archimedean Axiom, do not work with infinitesimals, then there must be some compensating idea of what the rules *are* when working with infinitesimals. This is one of the issues that Berkeley famously criticizes about the infinitesimal calculus, in which one is seemingly allowed to treat a rectangle width as finite at one moment and zero at another, depending on which happens to be more convenient (Berkeley 1734). The way to address this issue is to do precisely what Leibniz did by developing a plausible list of rules for working with infinitesimal (and infinite) numbers or magnitudes: e.g., (a) a finite length times an infinitesimal length is always infinitesimal, and (b) when comparing two finite magnitudes, any infinitesimal parts are negligible, and many more (Leibniz 1684). Such rules are sensible enough, and there is evidence that students can even develop them themselves by pursuing the implications of their own conceptions about infinitesimals (Ely 2010).

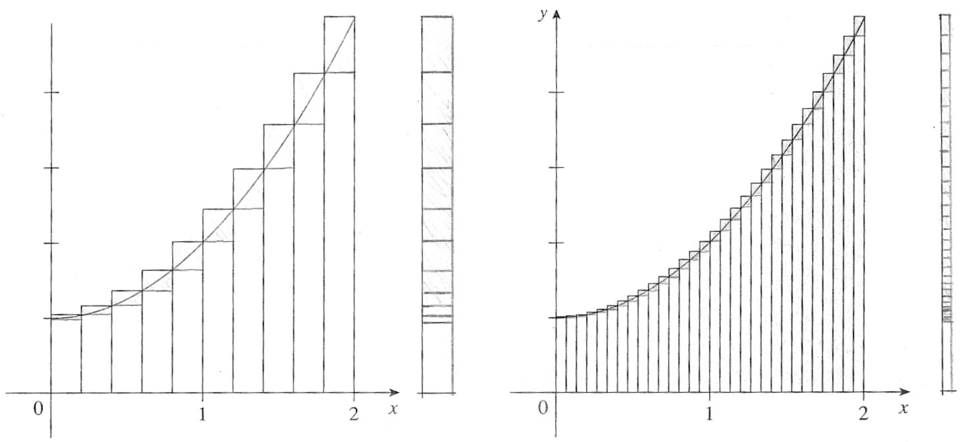
A more modern calculus image: quantitative limits

I have already made the case that the image of infinitesimals promotes the understanding of the fundamental theorem while the image of dimensional collapse is an obstacle to this understanding. Although this supports the case that calculus might be taught using infinitesimals, most instructors are probably not planning to do such a thing.[4] Luckily the image of the infinitesimal is not the only one that is available that helps with the understanding of the fundamental theorem, even though it was the image that, historically, led to this understanding. Another alternative is to have an image of the rectangles vanishing entirely when the limit is taken in a Riemann sum. In other words, in the limit when the number of rectangles goes to infinity, the rectangles disappear and there only remains the envisioned area under the curve as one cohesive region. For the sake of this paper, I will refer to this as the quantitative limits approach, for reasons that will hopefully become clear.

To create the notion of a definite integral using this image, one must rely heavily on quantitative reasoning. Take for the moment a function that is increasing over some interval, and partition the interval into some finite number of equal subintervals. On such a partition, the right-hand sum overestimates the area and the left-hand sum underestimates it. But we note that the area between the right- and left-hand sums, shown by the lightly shaded regions of Figure 8, approaches 0 in size as our partition gets finer (as $\Delta x \to 0$). Thus we can say that the right-hand sums approach the same *value* that the left-hand sums approach as $\Delta x \to 0$. So since the area under the curve is trapped between these, its value must be this same numerical value too.

Figure 8—The area between the right- and left-hand Riemann sums for a given function for two different partitions of the domain. This area is seen telescoped into a single strip to the right of each graph, to illustrate that this region diminishes in area as the partition becomes finer.

[4] In (Ely & Boester 2010) we discuss the merit of such an "infinitesimals" approach.



All of this we can say without a particular image for what happens to each rectangle in the limit. In other words, we can remain agnostic about the limit *object* that is attained as our rectangle widths Δx approach 0, and we only need to understand what limit value, or *quantity* of area, is attained by $\sum_{i=1}^{n} f(x_i^*)\Delta x$ * when $\Delta x \to 0$ (and $n \to \infty$ accordingly). At each step, $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is a quantity of area, and so is its limit. We need not worry about the qualities of the object that this limit value corresponds to. What we do need is a satisfactory way of determining the numerical limit of a sequence or set of quantities.

To conceptualize the first part of the fundamental theorem with this quantitative limits image, one may ask about the ratio of the change in accumulated area to the change in x, and what the limit of this ratio is as the change in x goes to 0 (Figure 2). As this Δx goes to 0, the object becomes more and more like a rectangle (well, trapezoid) divided by its width, so in the limit this ratio can be imagined to approach the height of the function at the given point x. This bit of reasoning can be done without the collapse metaphor and without infinitesimals. As in the case of the definite integral, it just requires imagery for understanding the numerical limit of a sequence of quantities.

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* Supposing the interval over which one wants to find the area is partitioned into equal subintervals of size Δx, with each x_i being, say, the right-hand endpoint of the i^{th} subinterval.

This imagery might be best provided by the metaphor of *approximation*. This metaphor involves (a) an unknown value, (b) a set of known values that approximate this unknown value, and (c) the amount of error between each approximation and the unknown value, with the idea that this error value can be made as small as one likes by taking a good enough approximation (Oehrtman 2009). In the case of the fundamental theorem rectangle there is the unknown value (the limit ratio of $A(x)/\Delta x$), the approximating value (here, $(f(x)\Delta x)/\Delta x$ for x on a small interval $[x, \Delta x]$), and the idea that the error quantity between the two can be made as small as one needs by taking a good enough approximation (i.e., making the width Δx small enough.

This metaphor is particularly compatible with the quantitative limits approach, in which no rectangle is imagined as the final resulting graphical *object* of such a limit process, because the approximation metaphor focuses the attention instead on the *quantities*: the approximating and error quantities, and thus, the limit quantity. And indeed, students readily and resiliently appeal to the approximation metaphor, which can be productively and deliberately built upon for understanding the formal definition of limit (Oehrtman 2009; Oehrtman, Swinyard, Martin, Hart-Weber, & Roh 2011).

While it makes sense that the quantitative limits approach can promote productive conceptualization of the fundamental theorem of calculus, there may also be a cognitive tension that arises with this approach. This tension is based on the fact that the approach specifies images for the objects before the limit is taken, but does not necessarily specify an image for the *resulting* object of a limit process. For example, with this approach, when the limit is reached for a definite integral, the rectangles disappear. Another example is the fundamental theorem: when the limit is reached, there is no accumulation rectangle that remains, nor does the rectangle degenerate to a line segment, which would be the collapse metaphor. And similarly, when the limit is taken in a derivative, there is no differential triangle that remains, nor does the triangle degenerate to a point, which would be the collapse metaphor. In all of these cases the quantitative limits approach offers no clear image for the final result of the object that shrinks in the limit process; there is only the quantity that is reached in the limit, and a broad image of what the quantity measures, such as the total area under the curve.

According to a few theoretical perspectives this may conflict with our tendency to imagine a resulting state for a limit process. For instance, Lakoff & Nuñez claim that learners apply the Basic Metaphor for Infinity (BMI) to limit processes, which entails the envisioning of a final resultant state of the limit (2000). Indeed, they claim that all mathematical objects involving the infinite employ this metaphor in their cognitive construction. If this is the case, then students will imagine some sort of final result of what happens to each rectangle in a Riemann sum, by the fact that the BMI is used when the limit in envisioned. If the quantitative limits approach presents them with no such final result, they may fill in such a result themselves. The most obvious choice might be to imagine that the rectangles degenerate into line segments—that is, the collapse metaphor. And I have already outlined concerns with this metaphor.

Whether this issue actually arises or not with the quantitative limits approach is a testable hypothesis, not just one for speculation: Do students who use the quantitative limits approach often find themselves resorting to the collapse metaphor as an image? Indeed, although there is evidence that the collapse metaphor can cause some problems with the fundamental theorem, might a student's successful understanding of the quantitative limits approach supersede these problems, even if they do at times use the collapse metaphor?

Conclusion

I have argued that both the quantitative limits approach and the infinitesimals approach allow for successful conceptualization of the fundamental theorem of calculus (and of derivatives and integrals) in ways that approaches involving dimensional collapse do not. It is not our purpose here to thoroughly explore the affordances and limitations of using either of these types of images when learning, or teaching, calculus concepts (a brief point/counterpoint related to this can be found in Ely & Boester 2010). Either way, the understanding of the fundamental theorem relies on the coordination of the change in area accumulated under a curve with the change in the underlying variable, an idea that is obscured by images of dimension loss or collapse. Patrick Thompson, Marilyn Carlson, and their colleagues have shown how this understanding is grounded in students' prior experience with covariation of quantities, experience that ideally should begin long before these students step into a calculus classroom (Carlson, Persson, & Smith, 2003; Oehrtman, Carlson, & Thompson, 2008; Thompson & Silverman, 2008).

It is important to note that the images of dimension loss, such as the collapse metaphor, can have some successful affordances too and should not be viewed as failed reasoning on the part of the student. These images may well arise as a result of prior instruction, and certainly any particular student may not use such an image consistently or exclusively across all relevant calculus contexts (Oehrtman 2009). The collapse metaphor will likely be common in any calculus class that focuses on conceptual, and not just procedural, knowledge of the important ideas of calculus. As a result, the instructor should take its affordances and limitations seriously and know how to furnish other imagery that will be more helpful, particularly for understanding the fundamental theorem of calculus.

References

Bell, J. (2005). *The Continuous and the Infinitesimal in Mathematics and Philosophy*. MIlan: Polimetrica.

Berkeley, G. (1734). The analyst; or a discourse addressed to an infidel mathematician. In *The Works of George Berkeley Bishop of Cloyne*, Luce, A. A., & Jessop, T. E. (Eds), vol. 4, pp. 53-102. London/Edinburgh: Nelson, 1948-57.

Bressoud, D. (2011). Historical reflections of teaching the fundamental theorem of integral calculus. *The American Mathematical Monthly, 118*(2), pp. 99-115.

Carlson, M. P., Persson, J., & Smith, N. (2003). Developing and connecting calculus students' notions of rate-of-change and accumulation: The fundamental theorem of calculus. *Proceedings of the 2003 Meeting of the International Group for the Psychology of Mathematics Education—North America*, vol. 2, University of Hawaii, Honolulu, HI, 165-172.

Cavalieri, B. (1966). *Geometria Degli Indivisibili*. Translated with introduction and notes by L. Lombardo Radice. Turin: Utet.

Cavalieri, B. (1635). *Geometria indivisibilibus continuorum nova quadam ratione promota*. Bononiae.

Ely, R. (2010). Nonstandard student conceptions about infinitesimal and infinite numbers. *Journal for Research in Mathematics Education. 41*(2): 117–146.

Ely, R. & Boester, T. (2010). Point/Counterpoint: Should we teach calculus using infinitesimals? *Online Proceedings for the 13th SIGMAA on Research in Undergraduate Mathematics Education Conference.* Raleigh, NC: North Carolina State University.

Galilei, G. (1638). *Dialogues Concerning Two New Sciences*. Trans. H. Crew & A. de Salvio. 2nd ed., Dover, New York, 1954; reprint of Macmillan, New York, 1914.

Holme, A. (2002). *Geometry: Our Cultural Heritage*. NY: Springer.

Kaput, J. (1994). Democratizing access to calculus: New routes to old roots. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77–156).

Kleiner, I. (2001). History of the infinitely small and the infinitely large in calculus. *Educational Studies in Mathematics*, 48, 137-174.

Knobloch, E. (1994). The infinite in Leibniz's mathematics: The historiographical method of comprehension in context. In K. Gavroglu et al. (Eds.), *Trends in the Historiography of Science*, Dordrecht, pp. 265-278.

Knorr, W. (1996). The Method of Indivisibles in Ancient Geometry. In *Vita Mathematica: Historical Research and Integration with Teaching*, R. Calinger (Ed.). Washington, D.C.: Mathematical Association of America, pp. 67-86.

Leibniz, G. W. (1684). Nova methodus pro maximis et minimis. Translation in Struik, D. J., 1969. *A Source Book in Mathematics, 1200-1800*. Harvard Univ. Press: 271-81.

Mancosu, P. (1996). *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*. New York: Oxford University Press.

Oehrtman, M., Swinyard, C., Martin, J., Hart-Weber, C., & Roh, K. H. (2011). From intuition to rigor: Calculus students' reinvention of the definition of sequence convergence. In the online *Proceedings of the Fourteenth Annual Conference on Research in Undergraduate Mathematics Education*, Portland, OR.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*, M. P. Carlson & C. Rasmussen, eds. Mathematical Association of America, Washington, DC, pp. 27-41.

Piaget, J. (1967). *The Child's Concept of Space*. W. W. Norton, New York.

Robinson, A. (1966). *Non-standard Analysis*. Amsterdam: North-Holland.

Russell, B. (1914). *Our Knowledge of the External World as a Field for Scientific Method in Philosophy*. Open Court Publishing Co.: Chicago.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics 26*, 229-274.

Thompson, P. W. & Silverman, J. (2008). The concept of accumulation in calculus. In *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*, M. P. Carlson & C. Rasmussen, eds. Mathematical Association of America, Washington, DC, pp. 43-52.

Torricelli, E. (1644). De solido hyperbolico acuto. In *Opera Geometrica*, Florentiae. Reprinted in *Opere di Evangelista Torricelli*, G. Loria & G. Vassura (Eds.). Faenza: Stabilimento Tipografico Montanari, 1919-44.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20, 356-366.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall, (Ed.), *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic.

Vinner, S. (1989). Avoidance of visual considerations in calculus students. *Journal of Mathematical Behavior 11*, 149-156.

Whiteside, D. T. (1960). Patterns of mathematical thought in the later seventeenth century. *Archive for the History of Exact Sciences, 1*, pp. 179-388.

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[i] Editorial Comment: The Fundamental Theorem of Calculus (FTC) can be viewed as an instance of "institutionalized" mathematical knowledge. In other words, even though everyone (i.e., mathematicians) know what it is, there is no uniformity in its portrayal in textbooks. It has been argued that its portrayal in textbooks is the "victim" of didactical transposition. Readers interested in this line of thought or "What is really the Fundamental Theorem of Calculus?" are referred to the work of Jablonka and Kliniska (2012) in the Montana Mathematics Enthusiast Monograph Series.

Jablonka, E., & Klisinska, A. (2012). A note on the institutionalization of mathematical knowledge or “What was and is the Fundamental Theorem of Calculus, really”? In B. Sriraman (Ed), *Crossroads in the History of Mathematics and Mathematics Education* (pp.3-40). Information Age Publishing, Charlotte, NC.