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Visualizations and intuitive reasoning in mathematics

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Abstract. On the basis of historical and didactical examples we consider the role of visualizations and intuitive thinking in mathematics. Examples from the 17th and 19th century have been used as well as smaller empirical studies at upper secondary school level and university level. We emphasize that a mathematical visualization does not reveal its intended meaning. With experience we can learn to interpret the visualization in different ways, depending on what is asked for.

Keywords. Visualization, mathematical concepts, definition.

1. Introduction

The status of visualizations in mathematics has varied from time to time. Mancosu (2005, p. 13) points out that during the 19th century visual thinking fell into disrepute. The reason may have been that mathematical claims that seemed obvious on account of an intuitive and immediate visualization, turned out to be incorrect when new mathematical methods were applied. He exemplifies this with K. Weierstrass' (1815–1897) construction of a continuous but nowhere differentiable function from 1872.[2] Before this discovery, it was not an uncommon belief among mathematicians that a continuous function must be differentiable, except at isolated points. The reason for this was perhaps that mathematicians relied too much on visual thinking. Nevertheless, as the development of visualization techniques in computer science improved in the middle of the 20th century, visual thinking rehabilitated the epistemology of mathematics (Mancosu, 2005, pp. 13–21).

In this paper the role of visualizations and intuitive thinking is discussed on the basis of historical and didactical examples. In a historical study the 17th century debate between the philosopher Thomas Hobbes and the mathematician John Wallis is considered. It seems that one problem was that Hobbes and Wallis were relying a bit too much on visualizations and intuitive thinking instead of formal definitions. Another problem was that at least Hobbes made no clear distinction between mathematical objects and "other objects". We consider

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[2] In 1872 Weierstrass constructed the function

$$f(x) = \sum b^n \cos(a^n x)\pi,$$

where x is a real number, a is odd, $0 < b < 1$ and $ab < 1 + \frac{3\pi}{2}$, which was published by du Bois Reymond (1875, p. 29).

these problems on the basis of two examples that both deal with the connection between the finite and the infinite.

Moreover, we consider some examples from the 19th century, which was a period when mathematical analysis underwent a considerable change. Jahnke (1993) discusses mathematics during the mid 19th century and argues that:

> […] a new attitude in the sciences arose, which led to a sometimes "anti-Kantian" view of mathematics that sought to break the link between mathematics and the intuition of time and space (Jahnke, 1993, p. 265).

In this paper we demonstrate how some fundamental concepts during this time period were defined (or perhaps described) on the basis of a survey written by the Swedish mathematician E.G Björling in 1852. Apparently, one problem was that the definitions of some fundamental concepts were too vague. This caused some problems, for instance the new "rigorous" mathematics contained new types of functions that could be used as counterexamples to some fundamental theorems in mathematical analysis. Another problem was that the definitions were not always generally accepted within the mathematical society.

Furthermore, we discuss some different interpretations of the famous "Cauchy's sum theorem", which was first formulated in 1821. Cauchy claimed that the sum function of a convergent series of real-valued continuous functions was continuous. The validity of Cauchy's sum theorem were frequently discussed by contemporary mathematicians to Cauchy, but there has also been discussions among modern mathematicians regarding what Cauchy really meant with his 1821 theorem. For instance, what did Cauchy mean with his convergence condition, and what did he mean with a function and a variable? In this paper Schmieden and Laugwitz' (1958) non-standard analysis interpretation of Cauchy's sum theorem is considered.

Finally, we consider a didactical issue that deals with Giaquinto's (1994) claim that visual thinking in mathematics can be used to personally "discover" truths in geometry but only in restricted cases in mathematical analysis. In this paper we criticize some of Giaquinto's statements. We stress that it is not proper to distinguish between a "visible mathematics" and a "non-visible" mathematics. Furthermore, we claim that Giaquinto does not take into consideration *what* one wants to visualize and to *whom*.

2. The debate between Hobbes and Wallis

An early example of a discussion on the role of intuition and visual thinking in mathematics is the 17th century debate between the philosopher Thomas Hobbes (1588-1679) and the mathematician John Wallis (1616-1703). Hobbes and Wallis often discussed the relation between the finite and the infinite, or rather, if there is a relation between the finite and the infinite. In this paper two issues of this debate will be considered; *Torricelli's infinitely long solid* and *The angle of contact,* respectively.[3] In connection to the former issue an example

[3] These two examples have also been discussed in (Bråting and Pejlare, 2008).

from a textbook at university level in mathematics will be considered and in connection to the latter issue a didactical study on upper secondary school pupils will be presented.

2.1 Torricelli's infinitely long solid

In 1642 the Italian mathematician Evangelista Torricelli (1608-1647) claimed that it was possible for a solid of infinitely long length to have a finite volume (Mancosu, 1996, p. 130). In modern terminology, if one revolves for instance the function $y = 1/x$ around the x-axis and cuts the obtained solid with a plane parallel to the y-axis, one obtains a solid of infinite length but with finite volume. Sometimes this solid is referred to as *Torricelli's infinitely long solid* (see Figure 1). The techniques behind the determination of Torricelli's infinitely long solid were provided by the Italian mathematician Evangelista Cavalieri's (1598-1647) theory of indivisibles from the 1630:s (Mancosu, 1996, p. 131). However, the difference is that Torricelli was using *curved* indivisibles on solids of *infinitely long* lengths. In (Mancosu, 1996, p. 131) "Torricelli's infinitely long solid" is discussed on the basis of the debate between Hobbes and Wallis.

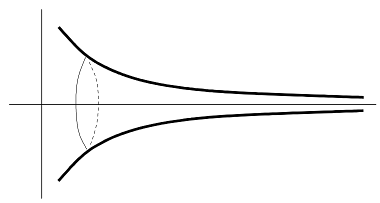

Figure 1: Torricelli's infinitely long solid.

Hobbes rejected the existence of infinite objects, such as "Torricelli's infinitely long solid", since *"[...] we can only have ideas of what we sense or of what we can construct out of ideas so sensed"* (Mancosu, 1996, pp. 145-146). He insisted that every object must exist in the universe and be perceived by "the natural light". Mancosu points out that many 17th century philosophers held that geometry provides us with indisputable knowledge and that all knowledge involves a set of self-evident truths known by "the natural light" (Mancosu, 1996, pp. 137-138). Hobbes stressed that when mathematicians spoke of, for instance, an "infinitely long line" this would be interpreted as a line which could be extended as much as one prefered to. He argued that infinite objects had no material base and therefore could not be perceived by "the natural light". According to Hobbes, it was not possible to speak of an "infinitely long line" as something given. The same thing was valid for solids of infinite length but with finite volume.

Meanwhile, for Wallis, "Torricelli's infinitely long solid" was not a problem as long as it was considered as a *mathematical object*. According to Mancosu, Wallis shared Leibniz' opinion that it was nothing more spectacular about "Torricelli's infinitely long solid" than for instance that the infinite series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$$

equals 1. If the new method led to the result that infinite solids could have finite volumes, then these solids existed within a mathematical context. Unlike Hobbes, it seems that Wallis (and Leibniz) made a distinction between mathematical objects and "other objects". Perhaps one can also say that Wallis and Leibniz were *generalizing* the volume concept to not only be a measure on finite solids, but also a measure on solids of infinitely long lengths.[4]

A similar (but more modern) problem is to show that the number of elements in the set of all natural numbers is equal to (in terms of cardinality) the number of elements in the set of all positive even numbers. This is done by showing that there exists a "one-to-one" correspondance between the elements of the two sets:

$$\begin{array}{c}\{0, 1, 2, 3, \ldots\}\\ \updownarrow \ \updownarrow \ \updownarrow \ \updownarrow\\ \{2, 4, 6, 8, \ldots\}\end{array}$$

In this case the concept of "number" is generalized. It is relatively easy to determine if the number of elements in two finite sets are equal. One simply has to count the elements in the two respective sets. It is also relatively easy to establish a "one-to-one" correspondance between the elements in that case. However, to determine if the elements in two infinite sets are equal is not that easy. In such a case one has to use a certain *method* to establish a "one-to-one" correspondance between the elements in the two sets. It is important to observe that "Torricelli's infinitely long solid" as well as the example with the comparison between the numbers of elements in two infinite sets are contradictory to "everyday situations" since we obtain "paradoxes". In the latter example the set of positive even numbers is included in the set of natural numbers (although the sets have the same cardinality) and in the former example we obtain a solid with finite volume but infinitely long length.

2.1.1 An example of a concept generalization in a textbook

In the textbook *Calculus – a complete course* (Adams, 2002), which covers the first year of studies in one-dimensional calculus at university level, the volume concept is considered in connection to integration. The textbook introduces that volumes of certain regions can be expressed as definite integrals and thereby determined. Then solids of revolution are considered. For instance, a solid ball can be generated by rotating a half-disk about the diameter of that half-disk. Finally, volumes of solids of infinitely long length are considered in connection to improper integrals. In this case one could perhaps say that the integral concept has been generalized. However, there is no explanation in the textbook that the volume concept actually has been generalized. One of the examples (which is equal to "Torricelli's infinitely long solid") is written in the following way:

The volume of the horn is

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx = \pi \lim_{R\to\infty} \frac{1}{x^2} dx = \cdots = \pi \ cubic \ units$$

(Adams, 2002, p. 410).

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[4]In (Bråting and Öberg, 2005) generalizations of mathematical concepts are discussed in more detail.

One problem of not mentioning that the volume concept has been generalized could be that students believe that we calculate a volume of an "everyday object". Another problem could perhaps be that students think that it is possible to intuitively based on "everyday objects" understand that the volume of the above horn is finite.

2.2 The angle of contact

"The angle of contact", which already occurred in Euclid's *Elements*, appeared to be an angle contained by a curved line (for instance a circle) and the tangent to the same curved line. A dispute between Hobbes and Wallis concerning "the angle of contact" was based on the following two questions:

1. Does there *exist* an angle between a circle and its tangent (see Figure 2)?
2. If such an angle exists, what is the size of it?

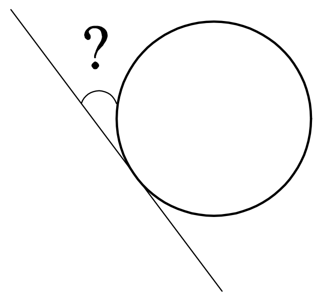

Figure 2: The angle of contact.

Wallis claimed that "the angle of contact" was *nothing*, whereas Hobbes argued that it was not possible that something which could be perceived from a picture could be nothing (Wallis, 1685, p. 71; Hobbes, 1656, pp. 143-144). In fact, this dispute originated from an earlier discussion between Jacques Peletier (1517-1582) and Christopher Clavius (1537-1612) (Peletier, 1563; Clavius, 1607).

According to Hobbes, it was not possible that something that actually could be perceived from a picture drawn on a paper could be nothing. Another reason why "the angle of contact" could not be nothing was the possibility of making proportions in a certain way between different "angles of contact". Hobbes claimed:

> […] an angle of Contingence[5] is a Quantity[6] because wheresoever there is Greater or Less, there is also Quantity (Hobbes, 1656, pp. 143-144).

This statement was perhaps based on Eudoxos' theory of ratios, which is embodied in books V and XII of Euclid's *Elements*. Definitions 3 and 4 of book V states:

> DEFINITION 3. A ratio is a sort of relation in respect of size between two magnitudes of the same kind.
>
> DEFINITION 4. Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another (Heath, 1956, p. 114).

[5] Hobbes used the term "the angle of Contingence", instead of "the angle of contact".
[6] Hobbes' term "quantity" can be interpreted as "magnitude", which is used in for instance Euclid's *Elements.*

Hobbes, as well as Wallis, discussed the possibility of making proportions between different angles of contact on the basis of a picture similar to Figure 3 below. Hobbes' approach was to compare the "openings" (Hobbes' expression) between two different angles of contact. On the basis of Figure 3, Hobbes claimed that it was obvious that the "opening" between the small circle and the tangent line was greater than the "opening" between the large circle and the tangent line. That is, since one "opening" was greater than the other, the angle of contact must be a quantity since *"wherever there is Greater and Less, there is also quantity"* (see Hobbes' quotation above). From this Hobbes concluded that the angle of contact was a quantity (magnitude), and hence it could not be nothing.

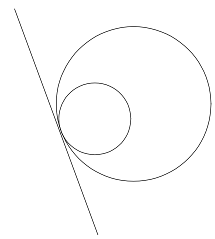

Figure 3: "The angle of contact" in proportion to another angle of contact.

Meanwhile, Wallis (1685) stressed that the "angle of contact" is of "no magnitude". He claimed that *"[...] the angle of contact is to a real angle as 0 is to a number"* (Wallis, 1685, p. 71). That is, according to Wallis it was not possible, by multiplying, to get the "angle of contact" to exceed any real angle (remember Definition 4 above). He pointed out that an "angle of contact" will always be contained in every real angle. However, at the same time he stressed that *"[...] the smaller circle is more crooked than the greater circle"* (Wallis, 1685, p. 91).

Today this is not a problem since we have determined that the answer to the question if the "angle of contact" exists is not dependent on pictures such as Figures 2 and 3 above. Instead the answer depends on which definition of an angle that we are using. In school mathematics an angle is defined as an object that can only be measured between two intersecting segments (Wallin et al., 2000, p. 93). According to such a definition the "angle of contact" is not an angle. However, an angle can be defined differently. For instance, in differential geometry an angle between two intersecting curved lines can be defined as the angle between the two tangents in the intersection point. According to such a definition "the angle of contact" exists and is equal to 0.

2.2.1 Pupils are considering "the angle of contact"

On the basis of a picture similar to Figure 2 above we let 39 pupils of upper secondary school level answer the question:

What is the size of the angle between the circle and its tangent?

According to their teacher these pupils were both motivated and their results in mathematics were above-average. The answers of the pupils were distributed over the following five categories:

1. 9 pupils answered that the angle was 0°.
2. 15 pupils answered that the angle was a fixed value greater than 0°, for instance 45°.
3. 8 pupils answered that the answer depends on where in the picture one measures.
4. 4 pupils answered that the angle does not exist.
5. 3 pupils did not answer.

Several of the 15 pupils in the second category claimed that the angle is 90°. One of the 8 pupils in the third category was formulated in the following way:

> It depends on where one measures. Since the circle is curved the angle gets greater and greater for every point. At the point where the circle and the tangent meets the angle is 0°.

Another pupil in the third category answered:

> Exactly in the tangent point the angle is 0°.

It seems that the pupils' approach was to carefully study the picture to find the answer. Roughly speaking, they were trying to find the answer "in the picture". Apparently, most of the pupils did not base their answers on a formal definition.

One of the 4 pupils in the fourth category gave the following answer:

> It is not an angle since the circle is round.

Of course one cannot be certain that this answer was based on the definition of angle. But perhaps the pupil understood that something was wrong but could not explain why. Perhaps the task was different than the pupil was used to, for instance, normally curved lines have nothing to do with angles. Another pupil in the same category answered:

> Does a circle really have an angle? If it has, it has infinitely many angles.

This pupil does not refer to the definition of angle either, but similar to the answers in this category just mentioned, it seems that this pupil also understood that something was not as it used to be. The two remaining pupils in this category did however refer to the definition of angle.

One possible reason why most of the pupils did not use the definition of an angle could be that they are not used to apply definitions. Perhaps the problems in their textbooks are not based on using formal definitions. Nevertheless, the historical debate concerning the existence of "the angle of contact" could be one way to demonstrate the need of formal definitions in mathematics.

3. Some examples of mathematical analysis from the mid-19th century

In the history of mathematics the 19th century is often considered as a period when mathematical analysis underwent a major change. There was an increasing concern for the lack of "rigor" in analysis concerning basic concepts, such as functions, derivatives, and real numbers (Katz, 1998, 704-705). For instance, the mathematicians wanted to loose the connection between mathematical analysis and geometry. The definitions of several fundamental concepts in analysis were vague and gave rise to different views of not only the definitions, but also of the theorems involving these concepts. Furthermore, mathematicians did not always use the same definition of fundamental concepts. Disputes regarding the meaning of fundamental concepts and the validity of some theorems started to arise. Two reasons to this may have been vague definitions and the lack of generally accepted definitions of fundamental concepts in mathematics.

Jahnke (1993) discusses a new emerging attitude among mathematicians during the mid 19th century whose aim was to erase the link between mathematics and the intuition of time and space. He argues that in the natural sciences in general the ambition of requiring new empirical knowledge became less important, instead, science should focus on the "understanding of nature and culture" (Jahnke, 1993, p. 267). Furthermore, Jahnke states:

> Rather than considering pure mathematics in terms of algorithmic procedures for calculating certain magnitudes, the emphasis fell on *understanding* certain relations from their own presuppositions in a purely conceptual way. To understand given relations in and of themselves one must generalize them and see them abstractly (Jahnke, 1993, p. 267).

Laugwitz (1999) considers the 19th century as a ”turning point” in the ontology as well as the method of mathematics. He argues that instead of using mathematics as a tool for computations, the emphasis fell on conceptual thinking. Laugwitz continues:

> The supreme mastery of computational transformations by Gauss, Jacobi and Kummer was beyond doubt but had reached its practical limits (Laugwitz, 1999, p. 303).

In Sections 3.1 and 3.2 of this paper we consider some examples of how fundamental concepts in analysis were defined (or perhaps described) during the mid 19th century. We also discuss which effect vague definitions of fundamental mathematical concepts can have on theorems in analysis. The examples are based on the Swedish mathematician E.G. Björling (1808-1872), who was an associated professor in Sweden during this time period. In Section 3.1 Björling's view of some fundamental concepts in analysis will be considered. In Section 3.2 we will consider the famous "Cauchy's sum theorem", which was first formulated in 1821. In particular, we will focus on some different interpretations of what Cauchy really meant with his theorem. We will consider modern interpretations of Cauchy's sum theorem as well as interpretations of some contemporary mathematicians to Cauchy.

3.1 E.G. Björling's view of fundamental concepts in mathematical analysis

Björling lived during the above mentioned time period when mathematics underwent a considerable change. One can discern the "old" mathematical approach as well as the "new" mathematical approach in Björling's work. For instance, Björling had an "old-fashioned" way of considering functions when he sometimes considered them as something that already existed and the definition worked as a description. At the same time, Björling tried to develop new concepts in mathematics. A closer look of some of Björling's work will be considered.

In a paper from 1852 Björling included a survey where he defined (or rather described) fundamental concepts in mathematical analysis; for instance functions, derivatives and continuity. Although the purpose with the paper was to consider Cauchy's sufficient condition for expanding a complex valued function in a power series[7], Björling claimed that it was necessary to clarify some fundamental concepts in both real and complex analysis. According to Björling there seemed to exist different views of some of the most fundamental concepts in analysis. He stated:

> It goes without saying, that it has been necessary to return to some of the fundamental concepts in higher analysis, whose conception one has not yet generally agreed on […] It was, from my point of view, necessary, that I in advance – and before the main issue was considered – gave *my own* conception of these fundamental concepts and of these propositions' general applicability, then not only the base, which I have built, would be properly known, but also every misunderstanding of the formulation of the definite result would be prevented (Björling, 1852, p. 171).

Björling begins his survey by considering the function concept. He describes a function as

> […] an analytical expression which contains a real variable *x* (Björling, 1852, p. 171).

Björling certainly defined functions, but sometimes he seemed to consider the definition of a function as a description of something that already exists. As a consequence of his definition of a function, Björling considered every variable expression as a function . Of course this differs form the modern function concept in several ways. For instance, for Björling a function did not need to be single-valued (which will be exemplified below). One possible way to interpret Björling's function concept is as a rule which "tells you what to do with the variable x".

Let us consider three functions that Björling discussed in his survey from 1852. In modern terminology these three functions would be expressed as;

$$f(x) = \frac{x}{|x|}\ ^{8}, \qquad g(x) = \frac{\sqrt{x} - \sqrt{a}}{x - a} \quad and \quad h(x) = |x|.$$

These functions are graphically represented in Figure 4 below.

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[7] Björling's view of Cauchy's theorem on power series expansions of complex valued functions are discussed in (Bråting, 2010+).

[8] Björling used the notation $\frac{x}{\sqrt{x^2}}$.

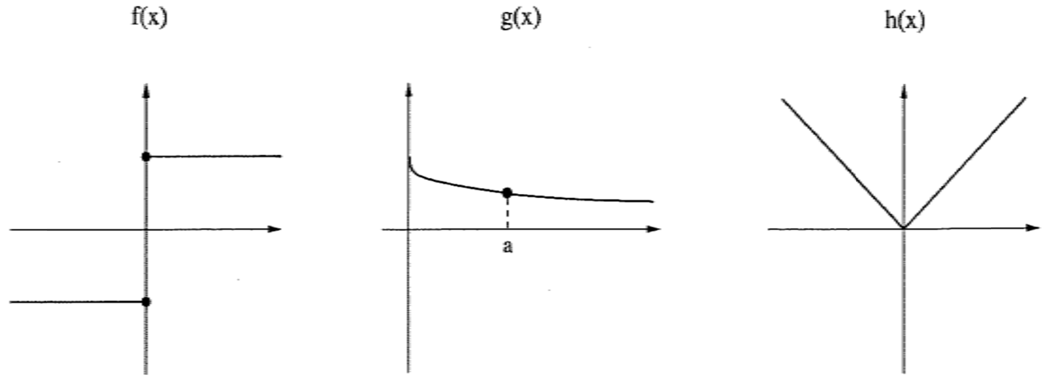

Figure 4: A graphical representation of how Björling may have considered the functions *f, g,* and *h*.

Björling considered $f(x)$ as a function which attains the *two* values ± 1 at $x = 0$. However, it had no derivative at $x = 0$ since the function representing the derivative jumps at $x = 0$. One problem of allowing functions to attain more than one value is that it becomes difficult to conclude what limit function a certain sequence of functions converges to (which is discussed further in Section 3.2 in this paper). Perhaps one can say that we have made it easier for us today since a function is always defined on a specific domain. In the above example this would imply that one need not take into account which value(s) the function $f(x)$ obtains at $x = 0$ since it is not defined at this point.

According to Björling, the function $g(x)$ attains the one value $\frac{1}{2\sqrt{a}}$ at $x = a$, since

$$\lim_{\Delta \to 0} g(a + \Delta) = \frac{1}{2\sqrt{a}}.$$

In modern terminology we would say that Björling considers $g(a)$ as a removable discontinuity. Björling did not consider whether the derivative of $g(x)$ at $x = a$ existed or not, but probably (on the basis of similar examples in Björling's 1852 survey) Björling would have said that the derivative at $x = a$ is equal to $-\frac{1}{8a\sqrt{a}}$ since

$$\lim_{\Delta \to 0} \frac{g(a + \Delta) - g(a)}{\Delta} = -\frac{1}{8a\sqrt{a}}.$$

According to Björling, the function $h(x)$ attains 0 at $x = 0$ and the derivative at $x = 0$ exists and is equal to the two values ± 1. In fact, Björling (1852) claimed that *"[...] generally, the derivative of a function $f(x)$, at a specific point x_0, can only obtain a finite and determined quantity" if $f(x_0)$ is single-valued*" (Björling, 1852, p. 177).

On the basis of these three examples, it seems that Björling's approach was to investigate the behavior of mathematical objects on the account of their "natural properties". At least one gets the impression that, for Björling, it was already presupposed that the expression written on the paper was a function and the task for Björling was to discover its exact properties. Perhaps one can say that Björling tried to "find answers in the graphs of the functions".

3.2 Cauchy's sum theorem

In 1821 the French mathematician A.L Cauchy (1789-1857) claimed that the sum function of a convergent series of real-valued continuous functions was continuous. Cauchy's proof of the theorem was relatively concise and imprecise, which led to different interpretations of Cauchy's formulation of the theorem. Some contemporary mathematicians to Cauchy criticized the validity of the theorem and came up with exceptions as well as corrections of the theorem. Furthermore, the formulation of Cauchy's 1821 theorem has been frequently discussed among mathematicians and historians of mathematics in modern time. For instance, if one interprets Cauchy's convergence condition with, in modern terminology, *pointwise convergence* Cauchy's 1821 theorem is wrong. Meanwhile, if one interprets Cauchy's convergence condition with the modern concept *uniform convergence*, Cauchy's 1821 theorem is correct.

In 1826 the Norwegian mathematician N.H Abel (1802-1829) came up with exceptions to Cauchy's sum theorem. Abel had constructed new types of functions that were sums of trigonometric series, which Cauchy probably had not anticipated when he formulated the theorem in 1821. It turned out that some of these functions could be used as counterexamples to Cauchy's sum theorem. For instance, in (Abel, 1826, p. 316) Abel emphasized that the trigonometric series

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots$$

was an exception to Cauchy's theorem. Apparently, although this series is a convergent series of real valued continuous functions, the sum is discontinuous at $x = 2k\pi$, for each integer *k* (see Figure 5).

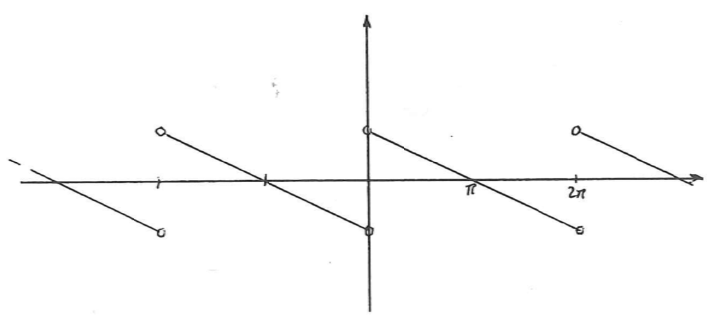

Figure 5: A (modern) graphical representation of the sum of the series $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots$

However, it was not only the new functions that led to problems regarding the validity of Cauchy's sum theorem. It seems that the mathematical theory had reached a point where the convergence condition was not precise enough to exclude counterexamples such as Abel's. It turned out that a series of functions can converge in different ways and it was therefore necessary to specify the convergence condition in Cauchy's theorem. In fact, during this time period there were already attempts to distinguish between different convergence concepts. For instance, Björling (1846) tried to explain and prove Cauchy's sum theorem on the basis of his own distinction between "convergence for every value of *x*" and "convergence for every *given* value of *x*", where the former notion was the stronger convergence condition.[9] Perhaps "convergence for every value of *x*" in connection with Cauchy's sum theorem was an attempt to express what in modern terminology could be described as

$$\sup_x \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{\infty} f_k(x) \right| \to 0$$

when $n \to \infty$, that is uniform convergence of the partial sums. However, one problem for Björling was that he did not have a proper way of connecting the variables *n* and *x*. As Grattan-Guinness (2000) points out

> […] during the 19th century there was a problem to distinguish between the expressions "for all x there is a y such that…" and "there is a y such that for all x…" (Grattan-Guinness, 2000, p. 70).

Later both Stokes (1847) and Seidel (1848) came up with corrections to the theorem, but it was not until 1853 that Cauchy modified his 1821 theorem by adding the stronger convergence condition "always convergent" to his 1821 version. In an example Cauchy clarifies that if an equality holds "always" it must hold for $x = 1/n$, that is, he allows *x* to depend on *n*.

3.2.1 A modern view of Cauchy's sum theorem

A standard theorem in modern analysis is the following: If a sequence of real valued continuous functions, f_n, converges uniformly to a function *f*, then *f* is a continuous function. In this case uniform convergence means that the maximum value of $|f_n(x) - f(x)| \to 0$ when $n \to \infty$. That is, for each *n* we choose the "worst" *x*, which makes $|f_n(x) - f(x)|$ as large as possible. If this absolute value still "tends to 0" while "*n* tends to infinity" then f_n converges uniformly to *f*. One obtains Cauchy's sum theorem for

$$f_n(x) = \sum_{1}^{n} g_k(x),$$

where g_k are continuous functions.

The non-standard analysis interpreters' Schmieden and Laugwitz (1958) claim that Cauchy's 1821 theorem was correct, at least if one uses their own theory based on infinitesimals. However, Schmieden and Laugwitz' interpretation has been discussed among historians of mathematics. The issue has been to interpret what Cauchy meant with his expression (mentioned above)

$$x = \frac{1}{n}.$$

Cauchy defined an infinitesimal as a variable which becomes zero in the limit (Cauchy, 1821, p. 19). However, this definition has been interpreted in different ways. For instance, Giusti (1984) argues that $x = \frac{1}{n}$ should *"[…] be seen as an ordinary sequence having 0 as a limit"*

[9] Björling's convergence concepts are considered in detail in (Bråting, 2007).

(Giusti, 1984, pp. 49-53). In modern terminology this is often written as $x_n = \frac{1}{n}$. Meanwhile, Laugwitz (1980) claims that *"[...] for Cauchy a real number can be decomposed into two parts $x + \alpha$, where x is the standard part and α an infinitesimal quantity"* (Laugwitz, 1980, p. 26). The infinitesimal α can be generated by a sequence having zero as a limit, for instance $\alpha = 1/n$. If one uses Laugwitz' interpretation it appears that Cauchy's 1821 theorem was correct. In fact, by using Laugwitz' infinitesimal theory one can even show that Cauchy's convergence condition from 1821 was weaker than uniform convergence, yet sufficient for the theorem to be true (Palmgren, 2007, pp. 171-172). However, in (Bråting, 2007, p. 534) it is argued that Laugwitz' theory presupposes the modern function concept, which was not available for Cauchy. Remember from Section 3.1 above that Björling, who was a contemporary mathematician to Cauchy, considered a function as *"[...] an analytical expression which contains a variable x"*. Bråting (2009) claims that

> [...] with such an imprecise way of defining functions it seems unlikely that a weaker convergence concept than uniform convergence can guarantee continuity in the limit (Bråting, 2009, p. 18).

Let us consider an example of a sequence of functions which shows the difficulty of deciding what will happen in the limit with an imprecise function concept and without the strong condition uniform convergence. Consider the sequence of functions

$$h_n(x) = \begin{cases} \sin 2\pi nx, & x \in [0, 1/n] \\ 0, & elsewhere \end{cases}$$

The successive graphs of the functions h_1, h_2, h_4 and h_8 are illustrated in Figure 6 below.

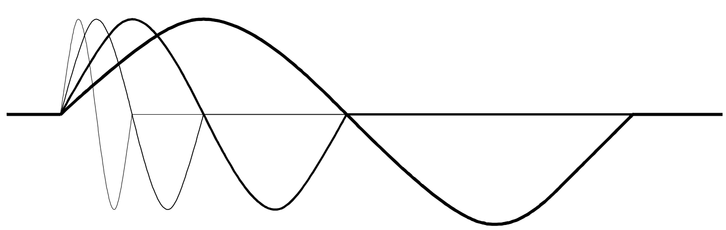

Figure 6: The successive graphs of the functions h_1, h_2, h_4 and h_8.

In modern terminology the sequence of functions h_n converges pointwise to 0 and the 0-function is continuous. However, h_n does not converge uniformly since

$$h_n\left(\frac{1}{4n}\right) = 1$$

for each *n*. Hence, the sequence of functions h_n does not converge uniformly, but yet the limit function is continuous. It is nothing wrong with this since uniform convergence only is a sufficient condition in the modern version of Cauchy's sum theorem.

Without the modern function concept it would be difficult to conclude what happens near 0. One should have in mind that the concept "limit function" is based on pointwise convergence and in this case a clear distinction between $x = 0$ and $x > 0$. In modern terminology we argue that h_n converges pointwise to 0 (and therefore 0 is the limit function) since for each fix $x > 0$ we get $h_n(x) = 0$ if we choose *n* sufficiently large and $h_n(0) = 0$ for each *n*.

However, it is necessary to explain exactly what is meant with ”what happens when *n* grows?” Otherwise, one can perhaps obtain the following answer to what happens with the graphs of h_n:

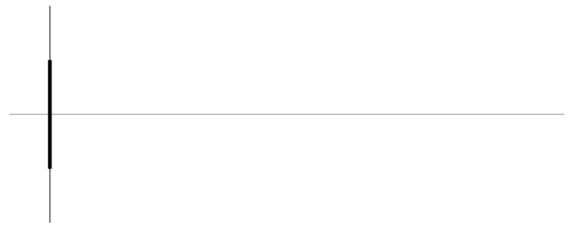

Figure 7: One possible answer to “what happens with the graphs of h_n when *n* grows?”

There is nothing wrong with this answer if we choose another convergence concept, for instance “pointwise convergence in two dimensions” (that is of a graph). Remember from Section 3.1 above that Björling considered ”multi-valued” functions, for instance Björling claimed that $y = \frac{x}{|x|}$ obtained the two values ± 1 at $x = 0$.

In fact, we let twenty university students at the course ”One-dimensional analysis” answer the question ”What happens at 0 when *n* turns to infinity?” on the basis of Figure 6 above. Eighteen of the twenty students gave the answer illustrated in Figure 7. One reason for this may be that one wants to believe that there must be something left when “the sine wave” get compressed? Or perhaps the students’ answers were based on some physical argumentation.

In the above example we have interpreted the expression $\frac{1}{4n}$ as an ordinary sequence and not as an infinitesimal quantity. However, if the expression $\frac{1}{4n}$ is interpreted as an infinitesimal quantity it can be shown, by using non-standard analysis, that the sequence of functions h_n satiesfies Cauchy’s sum theorem.[10]

This example shows that there are limits to what a visualization in mathematics can achieve. In the first place, one has to decide what convergence condition should be used, which requires a formal definition. Then the answer can be uniquely determined. Apparently, although one knows the formal definition of pointwise convergence it can be difficult to conclude that the limit function in the above example becomes 0 (remember the students’ answers that were mentioned above).

4. What can visualizations achieve?

A closely related issue to Björling’s view of mathematical concepts comes up in Giaquinto’s (1994) claim that visual thinking can be a means of discovery in geometry but only in severely restricted cases in analysis. By discoveries he does not mean scientific discoveries, but how one personally realizes that something is true. He suggests that visualizing becomes unreliable whenever it is used to discover the existence or nature of the limit of some infinite processes. One gets the impression that Giaquinto tries to distinguish between a “visible

[10] This is demonstrated in detail in (Bråting, 2007, pp. 531-532).

mathematics" and a "less visible mathematics". However, to divide mathematics in such a way can be interpreted as if one part of mathematics is more directly connected to empirical reality and the other part is abstract. Giaquinto stresses;

> […] geometric concepts are idealizations of concepts, with physical instances, meanwhile the basic concepts of analysis are non-visual since they are not equivalent to any (known) concepts which can be conveyed visually (Giaquinto, 1994, p. 804).

It seems that Giaquinto does not take into account *what* one wants to visualize and to *whom.* The main problem is that Giaquinto does not seem to consider *who* is supposed to make the discovery.

In order to investigate Giaquinto's claims, we will consider one of his own examples.[11] With this example Giaquinto wants to show that visualizing the limit of an infinite process sometimes can be deceptive. He considers the following sequence of curves: the first curve is a semicircle on a segment of length *d*; dividing the segment into equal halves the second curve is formed from the semicircle over the left half and the semicircle under the right half; if a curve consists of 2^n semicircles, the next curve results from dividing the original segment into 2^{n+1} equal parts and forming the semicircles on each of these parts, alternatively over and under the line segment; see Figure 8. We can notice that at those points where two of the semicircles touch each other, we will have singularities. The smaller the semicircles are the more singularities we get.

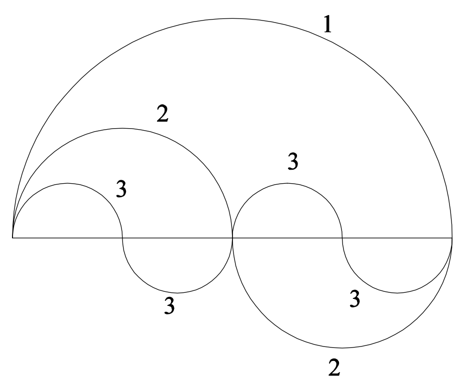

Figure 8: The sequence with semicircle curves.

Giaquinto claims that since the curves converges to the line segment, the limit of the lengths of the curves appears to be the length of this segment. He points out that this belief is wrong, since the sequence of the lengths of the curves will be

$$\frac{\pi d}{2}, \frac{\pi d}{2}, \frac{\pi d}{2}, \frac{\pi d}{2}, \dots$$

and therefore converge to

$$\frac{\pi d}{2}.$$

[11] The investigation of this example is also considered in (Bråting and Pejlare, 2008).

Furthermore, Giaquinto argues that this example lends credence to the idea that visualizing is not reliable when used to discover the nature of the limit of an infinite process.

However, it seems that Giaquinto does not take into consideration that the interpretation of a visualization does not necessarily have to be unique. We could for example in this case consider the limit of the lengths of the curves, or we could consider the length of the limit function. Depending on how we interpret the visualization we get different results. If we look at the lengths of the curves and take the limit we get the result $\frac{\pi d}{2}$. But, if we instead consider the length of the limit function, then the result is the length of the diameter, that is the result is d. Thus, depending on what question we want to answer we have to interpret the visualization in different ways.

In the visualization of the semicircles much is left unsaid. The visualization does for instance not tell us to look for the limit of the lengths of the curves or for the length of the limit function. We believe that our mathematical experience, as well as the context, is important while interpreting the visualization and "seeing" the relation. Giaquinto does not seem to take into account that people are on different levels of mathematical knowledge, and that visualizations can certainly be sufficient for convincing oneself of the truth of a statement in mathematics, if one has sufficient knowledge of what they represent. A person with little mathematical experience may not realize that the visualization can be interpreted in more than one way, giving different results. With experience we can learn to interpret the visualization in different ways, depending on what is asked for. The more familiar we become with mathematics the more we may be able to "read into" the visualization.

Apparently Giaquinto believes that there exists a definite visualization which reveals its meaning to the individual. However, we do not believe that it is quite that simple. We argue that the individual interacts with a mathematical visualization in a way which is better or worse depending on previous knowledge and on the context. This interaction is important and may even be necessary; the meaning of the visualization is not independent of the observer. A concrete example is when a teacher illustrates a circle by drawing it on the blackboard, as in Figure 9 below.

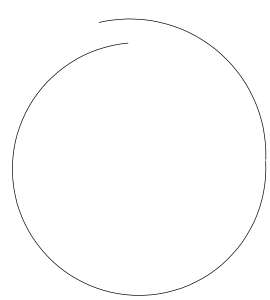

Figure 9: "The circle on the blackboard".

The picture on the blackboard is not a circle, since it is impossible to draw a perfect circle. But for a person who knows that a circle is a set of points in the plane that are equidistant from the midpoint, the picture on the blackboard is sufficient to understand that the teacher is talking about a "mathematical" circle. However, for a child who has never heard of a circle before, the figure of the blackboard probably means something else. By looking at the circle in Figure 9 the child may even think that a circle is a ring which is not connected at the top. The point is that visualizations can certainly be sufficient for convincing oneself of the truth of a statement in mathematics, provided that one has sufficient knowledge of what they represent.

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Was Pythagoras Chinese- Revisiting an Old Debate

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Introduction

Two of the great ancient civilizations were those of the Greeks and the Chinese. Many great works of art, architecture, philosophy and literature have been produced by both of these civilizations. When it comes to mathematics in the Western World, the Greeks have also been credited with many contributions to the field, especially geometry. Anyone who has completed a standard high school mathematical curriculum has been introduced to the names of Pythagoras, Euclid and Archimedes and their methods. Not as widely credited in the Western world are the mathematical contributions of the ancient Chinese civilization. An examination into Chinese mathematics reveals their deep understanding of mathematics, in some areas at a level greater than that of the Greeks. The Chinese civilization had their own mathematical greats, Liu Hui and Zu Chongzhi who were every bit as genius as their Greek counterparts. The purpose for mathematics and the techniques utilized by the two civilizations may differ, but the knowledge of the two civilizations is remarkably similar. This paper provides the reader a summary of the two civilizations works and mathematical philosophies and a comparison of the techniques used to determine π, proof of the right triangle theory, and the famous works of each civilization and the application of the civilizations mathematical knowledge in the science of land surveying.

Chinese Mathematical History

The origins of Chinese mathematics are steeped in legend. It is said that the Yellow Emperor, who ruled sometime around 2698-2598 BC charged one of his subjects named Li Shou to create arithmetic. (Li, 1987) The creation of a mathematical system cannot be credited to one individual, but the legend does give evidence of a mathematical system present in the 26th century BC in China. Evidence from the Shang Dynasty gives the first physical evidence of mathematical application in Chinese culture. The Shang Dynasty was a well developed agricultural society from the 16th to 11th century BC. Remains of large cylindrical grain bins have been discovered, along with bronze coins, evidence of a monetary system. Additional bone artifacts, known as plastrons, have the first known writing system of China. Including in the writing are numerals of a sexagesimal system (Li, 1987).

The *Zhoubi suanjing* is the oldest known Chinese mathematical work. The *Zhoubi suanjing* is also known as *Chou Pei Suan Ching* or *Chou Pi Suan Ching* in some historical references (Dauben 2007). The author of the compilation is unknown, though it is believed to be composed sometime between 100 BC and 100 AD. However, the contents were likely from a much earlier period (Li 1987). The *Zhoubi* is a work focused on astronomy, but the mathematics and measuring methods discussed were also applicable to land surveying and construction. The *Zhoubi* was divided into two sections, the first dealing with mathematics and the second part with astronomy. The mathematics in section one is necessary for the explanations of astronomy in the second portion.

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Particularly of interest is its discussion of the right triangle theory in section one. While commonly referred to as the Pythagoras theorem in the Western World, it is known as the *gou-gu* theorem in ancient Chinese literature. The text concerning the *gou-gu* theorem is written as a dialogue between a teacher Chen Zi and a student Rong Fang, both of whom nothing is known historically(Dauben, 2007). After being asked be Rong Fang, Chen Zi explains how using the shadow cast by the sun at midday it is possible to determine the distance between the sun and the Earth. In this explanation first appears a statement of the *gou-gu* theorem. Commentary added to the work by subsequent authors has been argued to be proofs of the theorem, though again it is difficult to determine the exact time period the commentary was added (Dauben, 2007).

The *Zhoubi* references the Emperor Yu being able to rule the country because of the Gougu theorem. Emperor Yu, or Yu the Great, ruled China in 2070 BC (Yu the Great, 2011). He was responsible for controlling an epic flood by engineering dredges and new flood channels to control the water. In depictions of his likeness, he is often seen holding a set-square, a device similar to a carpenter's square. This is evidence the Chinese understood and were applying the principles of the right angle theory as early as 2070 BC. The *Zhoubi suanjing* not only describes problems for calculating the distances of heavenly bodies and distances on Earth, but also a description of the tools of the trade. The primary tool used is the gnomon, an L-shaped metal or wood measuring device comparable to today's set square. In the *Zhoubi suanjing*, instructions for the use of the gnomon are again given by ways of a conversation between two scholars (Li, 1987).

The *Jiu Zhang suanshu*, translated to The Nine Chapters on Mathematical Art (Dauben, 2007), is the next and perhaps most significant mathematical work in Chinese history. The *Jiu Zhang suanshu*, also known as the *Chua Chang Suan Shu*, has been referred to as the Chinese equivalent to Euclid's Elements (Joseph, 2000). It is one of the oldest mathematical texts in the world, with problems more varied and richer than in any Egyptian text (Joseph, 2000). It is of interest that no English translation of the works has been made to date, despite the fact it is perhaps the most important compilation of ancient Chinese mathematics. Regardless, the *Jiu Zhang* is a comprehensive collection of the accomplishments of the Zhou, Qin and Han dynasties, approximately 11th century BC to 220 AD (Swetz, 1992). It contains the mathematical knowledge formulated over several centuries and through circumstantial evidence has been dated to sometime around the first century AD (Li, 1987). The *Jiu Zhang* is composed of 246 mathematical problems in a problem and solution format, divided into nine chapters and covering a range of math topics (Li, 1987). The text is written as a dialogue between an anonymous teacher and student, similar to the *Zhoubi suanjing*. The problems can generally be generally assigned to one of four categories: problems with applications to everyday real world situations, pseudo-real problems, which are of the same form as the real problems but focus on impractical or implausible situations, recreational problems and finally purely mathematical problems. The real world application problems include problems dealing with topics such as field measurements, bartering, exchanging goods and taxation. Pseudo-problems are utilized to show the principles of the mathematical knowledge that is to demonstrate methods for reframing difficult problems in a simpler manner. Pseudo-problems demonstrate that with mathematics it was possible to measure the immeasurable. Recreational problems include the riddles and counting rhymes, likely included for pedagogical purposes. Purely mathematical problems included methods for calculating π. A breakdown of the *Jiu Zhang* follows: Chapter 1 lays out methods for determining area of cultivated farm land. Chapter 2 and 3 focuses on proportions. Chapter 4 covers the methods for finding square and cube roots. Chapter 5 shows techniques for determining volumes of various solid shapes related to construction. Chapter 6 details calculations for distribution of goods such as grain and labor. Chapter 7 introduces the use of the method of false position. Chapter 8 discusses problems on simultaneous linear equations and the concept of positive and negative numbers. Chapter 9 discusses the Gougu theorem and general methods for solving quadratic equations (Li, 1987). The individual responsible for gathering together the works that compose the *Jiu Zhang* is unknown.

While the material of the *Jiu Zhang* is in and of itself important, the commentary and extension of the works provided by scholars studying the text over the subsequent centuries is what truly makes the *Jiu Zhang* the most important document in ancient Chinese mathematics. One of those scholars was Liu Hui, who in 263 AD wrote an addendum where he supplied theoretical verification of the procedures and extended some theories. The collection of nine additional problems Lui wrote to extend theories on the right triangle became known as the *Haidao suanjing*, translated as The Sea Island Mathematical Manual. Lui Hui is perhaps the most important mathematician in Chinese mathematics, and his contributions will be discussed more in depth later. Zu Chongzhi was another important scholar who studied the *Jiu Zhang* and extended its works. Zu lived from 429 to 500 AD and spent his early years studying the text of the *Jiu Zhang* and the works of other mathematicians such as Lui, correcting several errors of the mathematicians (Li, 1987). He is best known for his calculation of π, which will be discussed later. Chongzhi also produced a method for determining the volume of a sphere by comparing the areas of a cross-section (Li, 1987).

Chinese mathematics continued to advance into the modern era with works such as The Ten Books of Mathematical Classics, but the importance of the *Jiu Zhang* and *Zhoubi suanjing* to the Chinese civilization would continue for the next millennium.

Around the time of the completion of the two works, the imperial civil service in China began to grow in size and stature under the Eastern Han dynasty around 25-220 BC (Lloyd, 2003). Civil servants were the government workers of their day. It has been said that the Chinese creation of civil servant examination system has had far greater impact on the world than their invention of gunpowder, paper or the compass (Crozier, 2002). The civil servant examination system allowed any man to secure a respected position in the civil service; whereas most civilizations at this time held those positions for the ruler's relatives, friends and supporters. This gave great stability to the Chinese civilization because the civil service was not dependent upon whoever was in charge (Crozier, 2002). This civil service system spread throughout the world and can be seen presently in the United States, where a change in presidency or governorship does not trigger a turnover of all civil servants.

To ensure that those entering the prestigious civil service were competent and held the values of the culture, a rigorous examination was required of all applicants. The examination included a grueling essay portion and also an oral session, where students had to exactly complete a sentence chosen at random from the text once they were given the first portion of the sentence. A canon of texts, including works by Confucius, medicinal books and other philosophical works required complete memorization. Included in the canon were mathematical texts, including the *Jiu Zhang* and *Zhoubi suanjing*. Texts of a total of over 400,000 characters had to be completely memorized if a candidate was to have any hope of obtaining a civil service position, even at the entry level, and the pass rate was only 1 or 2% (Crozier, 2002). Mathematical texts were included to ensure those collecting taxes and performing land surveys had the mathematical knowledge to complete the tasks. The civil service examination continued to exist in one form or another up to the late 19th century (Li 1987).

Greek Mathematical Understanding

The Greeks saw mathematics as a philosophical pursuit, using mathematics to prove with certainty the truths of the world around them. It was Plato who stated: "Now logistic and arithmetic treat of the whole of number. Yes. And apparently, they lead us towards truth. They do, indeed." (Thomas, 1939). Considering that mindset, it is understandable why the Greeks considered geometry above all the type of mathematics. Ancient Greek mathematics focused on logic and is known for the great detail and thoroughness used to prove propositions. One only has to examine the depth of a proof in Euclid's Elements, such as the proof of Pythagoras's Theorem (Thomas, 1939) to see this.

The history of Greek Mathematical is marked with a who's who of famous names in all of mathematics. The facts of individual's works before Euclid's time are difficult to verify because of the lack of existing documents. Scholars are in agreement that Greek Mathematics began with Thale's Ionian school, established around 600 BC (Fauvel, 1987). Thales of Miletus was a philosopher, businessman and traveler, who had spent time in Egypt and Babylon, studying their mathematics before returning to Iona in Asia Minor and establishing a school (Struik, 1987). Thales understood the relationship of similar right triangles and used this knowledge to calculate the height of pyramids and the distance of ships from shore. Thales is also credited with four propositions, though it is difficult to verify: 1. A circle is bisected by its diameter. 2. The angles at the base of an isosceles triangle are equal 3. Two intersecting straight lines form two pairs of equal angles. 4. An angle inscribed in a semicircle is a right angle (Dilke, 1987).

Pythagoras was the next in the line of important Greek mathematicians. In 531 BC, Pythagoras moved to Croton in present day Southern Italy and opened a school to study among other things mathematics. The students of Pythagoras, known as the Pythagoreans, considered the universe to be ordered by means of the counting numbers and the person who fully understood the harmony of numerical ratios would become divine and immortal (Groza 1968). Pythagoras's most famous contributions were the proof that $\sqrt{2}$ is an irrational number and a proof of the relationship of the sides of a right triangle. The Babylonians and Egyptians had understood this relationship, but Pythagoras was the first to provide a proof. Pythagoras's proof that $\sqrt{2}$ is irrational was of great concern to the Greeks, because of their belief in the harmony of numbers. Pythagoras's greatest contribution to the world may be his discovery of the mathematical representation of the musical scale.

The schools of Socrates (400 BC), Plato (380 BC) and Aristotle (340 BC) produced students who continued to develop the discipline of mathematics and methods of logic and proof. Eudoxes, a student of Plato, made several significant contributions, including the development of the method of exhaustion and theory of proportions. The method of exhaustion was built upon the work of Antiphon (Groza, 1968). The Greeks knew how to calculate the area of a polygon and by inscribing a progression of n-sided polygons within a shape of unknown area; they could calculate a close approximation of the shape's area. As the number of sides of the polygon increased, the difference between the area of the polygon and the shape decreased. As the difference became arbitrarily small, the possible values for the size of the shape were "exhausted". The method of exhaustion helped Archimedes to determine the value of π, as will be discussed later in the paper. Eudoxes work with proportions advanced the methods of the

Pythagoreans, which dealt only with rational, or measurable, numbers. Eudoxes's work presented proportions in a geometrical sense, so as to show that being able to exactly measure the elements of the proportion was not required. This theory quelled the concern Greek mathematician's held over the irrationality of $\sqrt{2}$ and helped spur other advancements (Struik, 1987).

In 300 BC Euclid produced *The Elements*, a comprehensive collection of the works of all Greek mathematical work to that time. Little is known about Euclid the man, other than he lived in Alexandria around 300 BC. *The Elements* became *the* book on mathematics in the western world and its works continue to be studied and applied even today. Outside of the Christian Bible, *The Elements* has been printed and studied more than any other book in history (Groza, 1968). *The Elements* are divided into 13 books following a logical order. Euclid used a standardized form when presenting the material. A declaration would be made, he would then state a description of the problem (ex: let ABC and DEF be triangles with right angles…), the proof would follow, typically proof by contradiction, and a conclusion would be shown, restating the declaration followed by "which was to be proved". Books 1 through 4 contain definitions, postulates, common notions, and algebraic representation of planar geometry. Books 5 through 10 cover proportions and ratios, number theory, geometric sequences, and the method of exhaustion. Books 11 through 13 deal with special geometry, covering the ideas of books 1 through 4 and extending them to three dimensional shapes. Book 13 describes the five platonic solids and defines their relationships to a sphere. The description of the design and material of Euclid's *The Elements* is drawn from the 1939 translation by Ivor Thomas (Thomas, 1939) and Chapter 3 of Fuavel and Gray's *The History of Mathematics* (Fauevel, 1987).

Archimedes was the next prominent figure to arrive on the scene of Greek mathematics. He lived from 287 to 212 BC in Syracuse and is regarded as one of the greatest mathematicians of all times (Groza, 1968). There are more of the writings of Archimedes than of any other mathematician of antiquity surviving to today (Fauvel, 1987). Though Archimedes made many great contributions to mechanics, his true passion lied with the theoretical proof of mathematics. Archimedes most famous work was the calculation of the area of a circle and the determination of π. His methods involved inscribed and circumscribed polygons about the circle. The method will be explained in more detail later. Archimedes also invented a method for discussing large numbers. He did this by attempting to define the number of sand particles that could be contained in the universe, known as the sand-reckoner problem. The system he laid out to define large numbers is similar to the use of powers today ($10^8 = 10{,}000{,}000$), which was remarkable because the Greeks had a very simplistic way of writing numbers, where powers could not be incorporated (Fauvel, 1987).

One great contribution of Archimedes, known as Archimedes's Palimpsest, was thought to be lost forever until it was discovered in 1906. The Palimpsest included copies of his treatises "Equilibrium of Planes", "Spiral Lines", "Measurement of a Circle", "On the Sphere and Cylinder", "On Floating Bodies", "Stomachion" and "The Method of Mechanical Theorems". The Method of Mechanical Theorems (The Method) discovered is the only known copy, and is an important document in understanding Greek mathematics (Heath, 2002). The Method was a letter written by Archimedes to Eratosthenes. In the letter, Archimedes discusses the manner in which he makes his discoveries of the theorems that he then went on to provide proofs. Archimedes is careful to tell Eratosthenes that discovering a postulate through experimentation and proving the discovery through proof were two separate tasks. Archimedes lays out for Eratosthenes his genius method for determining the volume of solids, such as the cone and pyramid or areas, such as the parabola. Archimedes reasoned that figures could be broken down into an infinitesimal number of elements and these individual elements could be weighed and balanced on a lever opposite a figure of known weight and center of gravity. In physics and engineering terminology, he was calculating the moments of the figures around the fulcrum of the lever. This is the first sign of the idea of an integral in mathematics (Heath 2002). In addition to the actual content of the letter, it is the fact that Archimedes reveals his methods of discovery that is so important. The Greeks only published the proofs of their theorems; leaving historians to wonder how they discovered some of the properties they were so skilled at proving. The Method is a look inside Archimedes's office.

Archimedes other important works include his theorems on areas of plane figures and on volumes of solid bodies. These can be considered the precursors to integral calculus. In his book *On the Sphere and Cylinder*, he developed expressions for the area and volume of a sphere and cylinder. In his book *Quadrature of the Parabola*, Archimedes provides an expression for the area of a parabolic segment and proves the expression true (Struik, 1987). Archimedes had many more contributions in the fields of mathematics, mechanics and hydraulics and was a celebrity in his time.

Archimedes death marks the end of golden age for Greek Mathematics. Archimedes died at the hands of conquering Romans, who imposed a slave economy, more intent on making profits that pursuing mathematical knowledge. However, despite the conquering Romans, new contributions were made to mathematics, albeit less frequently. Around 230 BC, mathematician Eratosthenes proposed a method for calculating the circumference of the Earth (Thomas, 1939). Diophantos, around 250 AD, developed a form of algebra used to solve indeterminate equations. He also made contributions to the field of number theory, such as the theorem that states if each of two integers is the sum of two squares, then their product can be resolved in two ways into two squares (Struik, 1987). Heron (or Hero) of Alexandria was one of the last great mathematicians of ancient Greece, although it is disputed during which time period he lived, most likely it was in the first or third century AD (Thomas, 1939). While his Greek predecessors were more concerned with math theory, Heron was interested in practical uses of mathematics, demonstrated by his many inventions. Heron is credited with a large volume of work covering mechanics of machines. One of his works on mathematics is *Metrica*, a book on how to calculate the area and volume of different objects. Heron is known today for Hero's formula, where the area of a triangle can be calculated if the side lengths of the triangle are known.

Even during the time of Eratosthenes, Diophantus and Heron, there was the influence of the Roman Empire. While the Greeks were more concerned with more noble pursuits, Roman saw mathematics as a tool to be used for economic gains.

Comparisons

Now that a brief history of both cultures mathematical works and understandings have been laid out, a comparison of their works and techniques for determining π and the right angle theorem will be examined, along with a discussion on land surveying techniques

Euclid's Elements vs. *Jiu Zhang suahshu*

Now would be a good time for a more in depth comparison of the form and content of the most famous works of the cultures, Euclid's *Elements* and the *Jiu Zhang suahshu*. Both works are compilations of each culture's mathematical knowledge up to that given point in time. The books, in effect, provided a means for social advancement, for those who could master them were elevated to privileged elite. However, there are significant differences in the structure and content of each book.

The Elements follows a logical progression of material. The early chapters are filled with definitions, postulates and less complex theorems. Progressively, new material is built upon earlier material. The *Jiu Zhang suahshu*'s content does not follow a sequential order. Material presented in later chapters of the book is independent of earlier material.

The books have a different structure in the way they present ideas. The *Jiu Zhang suahshu* announces the problem and then gives a solution, with little or no explanation of the methods used to get the answer. There was also no attempt to provide a proof of the assertions of mathematical properties and theorems. Most of the description of the methods to solving the problems and proofs of the theorems was provided in the commentary, by mathematicians such as Lui Hui. The purpose of the texts was not to prove beyond argument the material but to guide the reader (student) on the principles so that they could master the methods for themselves. The authors wanted to the student to work through the problem themselves. The *Jiu Zhang suahshu* gave the reader the tools to solve problems and did not concern itself with proving the authenticity of the tools.

The Elements was nearly the opposite, as its chief concern was giving rigorous proof of ideas presented. *The Elements* presents a conjecture or idea and then painstakingly goes through reasoning step by step to come to an incontrovertible conclusion that the conjecture is true. *The Elements* was not concerned with having the student think for themselves, but wanted to show the students irrefutable proof of the conclusion. It has been theorized that the Greek's infatuation with proof comes from the system of law and politics practiced by the civilization. The law system provided for jury trials, where each side presented their arguments on the merits of their case. It was the goal of each side to give a sound and uncontroversial argument (Lloyd, 2003). While this may be difficult to near impossible in the courtroom, it was an achievable pursuit in the field of mathematics.

Though the structure of the two civilizations texts would lead one to conclude the Chinese were more open than the Greeks to discovering and critically analyzing mathematics, this was not the case**.** As detailed before, the Chinese were interested in the rote memorization of their mathematical works. Before any analysis or interpretation of the material could be conducted, one first needed to memorize the entire work. The Chinese also held the creators of the works in high regard and were hesitant to question any of the assertion or methods within the work. Chinese mathematical work during this time served the utilitarian purpose of supporting the monarchy (Swetz 1977). The study of mathematics was also considered a mostly solo venture in Chinese culture. Though the Greeks had a rigid structure in their mathematical text, they were more open to oral group discussions of mathematical theory. Because of the nature of proofs being uncontroversial, it was not considered a faux paus to question an aspect of a theorem or proof.

π

It has been said: "In history, the accuracy of the ratio of the circumference to the diameter (pi or π) calculated in a country can serve as a measure of the level of scientific development of that country at that time. " (Li, 1987). Both Greek and Chinese mathematicians had innovative ways of determining the accuracy of π.

In the beginning of Chinese mathematics the ratio of the circumference of a circle to the diameter was 3:1. This was an accurate enough approximation for the tasks at hand. Other ancient Chinese mathematicians began using closer approximations of π, such as 3.1547 and 3.16 ($\sim\sqrt{10}$), but none had developed a method for accurately calculating the true value of π. In 263 AD, in his commentary on Jiu Zhang suanshu, Liu Hui proposed the "method of circle division" to calculate the true value of π. In his method, Liu inscribed regular polygons inside a circle. Liu Hui began by showing that using π = 3 gives the area of a regular dodecagon (12 sided), not that of a circle. By continually doubling the number of sides of the regular polygon, the polygon approaches the shape of the circle. Therefore Area of polygon = Area of circle, and the estimation of π becomes more accurate. Liu understood that if one knows the side length of a hexagon, one can calculate the side length of a dodecagon. Given a polygon with 2n sides and l_{2n} = side length of *2n*- polygon, then the side length of polygon with *4n* sides can be determined, applying the Gougu theorem, such that:

$$l_{4n} = \sqrt{\left\{r - \sqrt{r^2 - \left(\frac{l_{2n}}{2}\right)^2}\right\}^2 + \left(\frac{l_{2n}}{2}\right)^2}$$

Liu understood that this process could be repeated to obtain a more and more accurate value of π. Liu calculated the area of a 192-sided polygon to get $\pi = 3.14 + \frac{64}{625}$. Around 480 AD, Chinese mathematician Zu Chongzhi calculated the value of π to be in the range of 3.1415926 < π < 3.1415927, that is to say he calculated the accuracy of π to seven decimal places (Li, 1987). Chongzhi did not leave evidence of his methods of calculations, but the only known method was Liu's method of circle division. If Chongzhi went this route, he would have run the recursive algorithm for a 24,576-sided polygon. Chongzhi's calculation of π was the most accurate in the known world until the 15th century, meaning his calculations were not bested for over 900 years (Liu, 1987).

The history of the value of π in Greek mathematics before Archimedes is not known. Some scholars have proposed that the Greeks used the Egyptian approximation, 3.16 or the Babylonian estimate of 3.125. However, the methods by which theses civilizations calculated these values had little to do with the ratio of the diameter and circumference of a circle (Rossi, 2004). Around 400 BC, Antiphon, in his attempt to "square the circle" developed the method of inscribing triangles inside a circle (Thomas, 1939). Although he failed to determine a technique to square the circle, his methods were picked up and extended by Archimedes.

Archimedes used a similar iterative method for determining the area of a circle, and thus the value of π. Archimedes laid out his method in his book *Measurement of a Circle.* Proposition 3 of the book states: The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$ (Heath, 2002). Archimedes goes about proving this by calculating the perimeter of a circumscribed polygon, where the perimeter is greater than the circumference of the circle, and an inscribed polygon, where the perimeter is less than the circumference of the circle. Rather than actually calculate the perimeters, Archimedes used Euclid's theorem on similar triangles to produce ratios of segments. His proof demonstrated the ratio of a 96-sided polygon circumscribed. He then had an inequality where $\pi < 3\frac{1}{7}$.

Archimedes then repeated the process with a 96-sided polygon inscribed to get the inequality of $\pi > 3\frac{10}{71}$ (Heath, 2002).

Archimedes work was four hundred years prior to the work of Liu, although there is no evidence of Liu having knowledge of Archimedes work. Both the Greek and Chinese methods utilized an iterative process increasing the number of sides of a polygon. It is of interest that Archimedes method is an arithmetical process and not a geometrical process, as was the norm of Greek mathematics at that time. The Chinese were well versed in arithmetical processes.

Right Triangle Theory

Both cultures had an understanding of the right triangle. That is to say, for a right triangle with side leg lengths *a* and *b* and hypotenuse of length *c* has the relationship $a^2 + b^2 = c^2$. They understood the power of this relationship for surveying. The way each civilization approached a proof of the relationship differs.

The Zhoubi suanjing contained a specific example of the right triangle theory, specifically the 3-4-5 right triangle, known as the Gougu theorem. A proof of how to demonstrate the $a^2 + b^2 = c^2$ relationship was known as the "piling up of rectangles" (Joseph, 2000). The original diagram accompanying the explanation is shown in figure 1 below.

The explanation starts with a triangle of sides 3, 4 and diagonal of length 5. A square is to be drawn on the diagonal side then three additional half rectangles of sides 3 and 4 are to be drawn to circumscribe the square and make a plate of 7x7. The four circumscribed half rectangles form two complete rectangles of area 24. When this area is subtracted from the plate area of 49, the remainder is 25, the area of the diagonal square (Joseph, 2000).

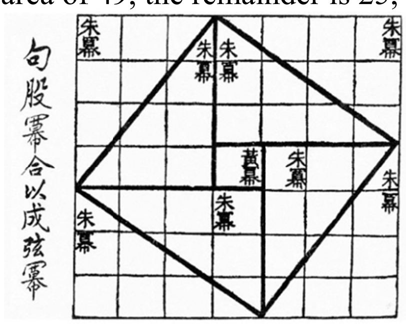

Figure 1

Lui Hui provided an extension of this proof in his commentary on the Zhoubi suanjing, known as the out-in complimentary principle (Joseph, 2000). The proof relied on the assumptions that the area of a figure remained the same under translation to another location and if a figure is divided into several smaller pieces, the summation of the area of those pieces is equal to the area of the original figure. Though a visual figure of Hui's explanation has been lost, there have been several replacements proposed by scholars. The jest of the proof is that taking the squares of the sides of a triangle and transforming the squares by cutting and rotation, a square of the diagonal can be produced (Joseph, 2000).

The Greeks understood the property of $a^2 + b^2 = c^2$ from the Babylonians, although they did not have a proof of the property (Groza, 1968). Pythagoras is credited with providing the first proof of the property, although historians have been unable to verify it was actually Pythagoras who performed the proof. Pythagoras's proof was documented in Euclid's *The*

Elements in book 1, proposition 47 (Thomas, 1939). The proof is quite rigorous and involves geometric properties of similar triangles, parallel lines and rectangles.

As was common for Greek proofs, Pythagoras's proof requires a sound knowledge of the geometric properties of triangles, whereas the understanding of the simplistic Chinese proof requires little background knowledge due to its visual nature. The written proof of the Chinese proof came some three hundred years after Euclid's *Elements* were published. The significant differences in the methods of proof demonstrate the unique origination of each civilization's proof.

Surveying

Both civilizations were expert land surveyors. The science of surveying was based primarily on the right triangle theory properties and geometric properties of similar triangles.

One of the great Greek examples of advanced surveying capabilities is the tunnel of Eupalinus at Samos, completed around 530 BC. The 1,036 meter aqueduct tunnel was excavated from both sides of a mountain and took 10 years to complete. The tunnel was built with great precision; the vertical difference at the middle meeting point is only 60 centimeters or less than one-eighth of one percent of the excavated distance (Apostol, 2004). However no evidence remains on how exactly the surveyors were able to lay out the correct orientation. One theory is that they used a succession of right triangles to traverse around the mountain. This method was later explained by Heron, not specifically for the Samos tunnel, but as a general method for tunnel construction (Dilke, 1987).

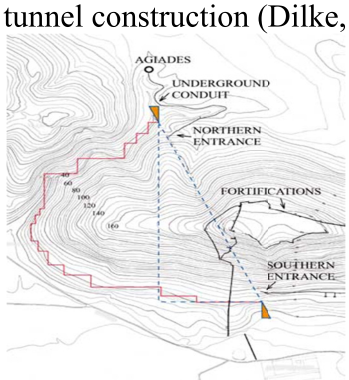

Figure 2 (Reference: Apostol 2004)

As shown in figure 2 above, the surveyor would mark one end of the proposed tunnel and then measure out a succession of right angles around the obstruction at the same elevation to locate the second tunnel entrance. Additional right triangles could be laid out at the tunnel entrances to provide a reference for a line of site to check work during construction (Apostol, 2004)

The understanding of ancient Chinese surveying comes from the *Haidao suanjing* or the Sea Island Mathematical Manual, written by Liu Hui in 263 AD. The Sea Island Manual contains nine problems dealing with the double difference method (also known as *chong cha*) of determining heights and distances. While the use of one simple ratio of a right triangle to determine unknown heights had been used for centuries, Liu developed the double difference method, which enabled surveyors to determine the both the height and distance to an inaccessible point.

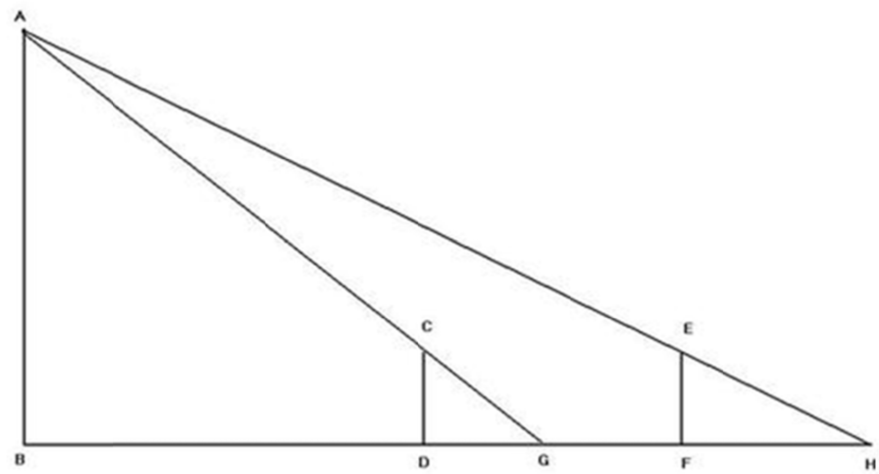

Figure 3

The following is an explanation of Lui's double difference method, referencing Figure 3 above. If point A and B were inaccessible, it would still be possible to measure the distance and height from point D. CD and EF would represent observation poles with known lengths. Stepping back from CD to the point G where A is visible and likewise to point H from EF gives the first required data. The distances DF, DG and FH are all know at this time. Using the properties of similar triangles, $\frac{DF}{FH-DG} = \frac{EA}{EH} = \frac{AB-EF}{EF}$. From these ratios, AB = $\frac{DF}{FH-DG}$ * EF + EF and using the same process of similar triangles, it can be calculated that BD = $\frac{DF}{FH-DG}$ * DG. The unknown distances of AB and BD are in terms of known distances (Li, 1987). By moving from survey observations utilizing a simple ratio and one observation point to problems with multiple observation points and ratio computations, Lui was taking a radical mathematical step (Swetz, 1992).

In a reversal of the overarching mindsets of each respective civilization, the Greeks were more concerned with the practical application of surveying techniques, while the Chinese focused on theoretical explanations (Swetz, 1992). Greeks were more proficient in the technical art of surveying while the Chinese excelled in the application of mathematics to surveying situations. It has been said that in a comparison of the knowledge of mathematical surveying techniques, China was one thousand years in advance of the Western World (Swetz, 1992).

Was there communication between the two civilizations?

It has been well documented that European Jesuit missionaries began entering China by the early 17th century AD (Li 1987). Because China was under the strong central rule of the Ming Dynasty at this time, it was impractical for the missionaries to impart their faith on the Chinese through force. Instead, the missionaries needed to entice the Chinese with science and technology (Li 1987). The Jesuits brought with them the mathematical knowledge and advancement of their respective culture and over time, the distinction between European and Chinese mathematics was gone. There is little debate on the history at this time period, but stepping back further in time, the connection between European (Greek) and Chinese cultures becomes much more unclear and speculative.

When historians examine the historical documents and study specific themes, it is difficult not to speculate there was some link between the two cultures. One such theme examined is Pythagoreans theorem. At the surface, the Pythagorean theorem and the *gou-gu* theorem seem like independent discoveries of both nations. Pythagoras lived in 6th century BC and is credited with the proof and the *gou-gu* theorem was first presented in the *Zhoubi suanjing* which has been dated to sometime around 100 BC to 100 AD. However, a closer examination might lead to a different conclusion. Accounts and facts of Greek work during this time tilt more on the side of legend than fact. No proof has been uncovered that Pythagoras was the one who actually proved the theorem; it has only been attributed to him by others. Historians have speculated that all work done at the Pythagorean School was credited to Pythagorean. Because no proof exists it is difficult to determine the methods Pythagoras used if he did indeed prove the theorem. Although the *Zhoubi suanjing* is dated around 100 BC, it is generally accepted that the work contained within the compilation comes from earlier times. Using contextual evidence, the portion of the work containing the *gou-gu* theorem geometrical proof can be likely dated to the time of Confucius, around 600 BC.

A paper by Liu Dun explores the history of the method of double-false-position, which provides more speculative data to the theory that a link existed between the two civilizations (Dun, 2002). The method of the double-false-was first described in *Jiu Zhang suanshu*, indicating the method was known in China around 50 AD. Subsequently, the method was transferred to Muslim mathematicians at some point and then on to Europeans sometime during the Middle Ages. It then was brought back to China by the Jesuit missionaries, who claimed the method as their own. The paper leaves the question of how the method got to the Muslim mathematicians and at what period in time did this occur?

A similar paper was presented by Kurt Vogel, tracking the survey problem found in the *Haidao suanjing* about the sea island survey across a waterway. The method (explained in the survey section above) was discovered by Liu Hui in China around 260 AD. Two hundred years later, the method and an altered example problem appeared (Vogel, 2002) in Indian mathematics as part of the Aryahbhata text. The Indian knowledge was then transferred to the Arabs through the translation into Sanskrit. The problem, with references to Indian mathematicians, appears in Islamic mathematician al-Biruni in his work *The Exhaustive Treaties on Shadows* around 1000 AD. (Vogel, 2002). The path from the Arab world to Europe is unknown, but the method appears in the *geometria incerti auctoris* in Italy around 1000 AD.

These research papers point to the fact that information was flowing between the two civilizations, however it appears to be moving at such a slow pace as to be practically isolated. While speculation about the origins of Pythagorean's theorem provides an engaging topic, the lack of concrete evidence either way bounds the theory to stay in the conspiracy arena.

Conclusion

Due to the geographical isolation of China in ancient times, the accepted history of mathematics in the western world largely ignores their contributions. By examining the history of Chinese mathematics, one can see they had an understanding of mathematics similar to that of the western world, and Greeks in particular. In cases such as the calculation of π and the proof of the right angle theory, both Chinese and Greeks made astounding discoveries. The most prominent Greek mathematicians, such as Pythagoras and Archimedes are well known, though Chinese mathematicians with equal genius and comparable contributions to mathematics such as Lui Hui and Zu Chongzhi are pushed to the margins of fame.

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Some reflections on mathematics and mathematicians. Simple questions, complex answers

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… *"The Mathematic does not have own existence. It is only an arbitrary code, designed to describe physical observations or philosophical concepts. Each can adapt it to its own needs."*
Dr. John Keyser, Ph.D. in Physics[2]

SUMMARY
In this work we present some reflections on mathematics and mathematicians. Special emphasis is placed on the questions (1) what is mathematics? And (2) what is a mathematician? Some reflections and open questions are posed at the end of the work.

0. Introduction.
Professions have played a key role in the development of disciplinarily, and vice versa. Within some disciplines the direct binding to a profession or a field have over time been loosened and (re)searching knowledge for its own sake has become a main driving force of a new, advanced kind of disciplinarily. For mathematics these historical shifts are symptomatic in the debates over the discipline's *true nature*. While the relationship between science, technology and mathematics historically the last 200 years has been rather symbiotic, mathematics today serve so many different professions and fields, that a unified, valid definition of its *nature* is hard to find.

Mathematical discoveries have come both from the attempt to describe the natural world and from the desire to arrive at a form of inescapable truth from careful reasoning. These remain fruitful and important motivations for mathematical thinking, but in the last century mathematics has been successfully applied to many other aspects of the human world: voting trends in politics, the dating of ancient artifacts, the analysis of automobile traffic patterns, and long-term strategies for the sustainable harvest of deciduous forests, to mention a few. Today, mathematics as a mode of thought and expression is more valuable than ever before. Learning to think in mathematical terms is an essential part of becoming a liberally educated person.

Much of mathematics is itself about mathematical objects. This is part of why mathematics can seem like an arcane and up-approachable field to an outsider. Fortunately in asking *What is Mathematics?* We are asking about the meaning and consequences of Mathematics as

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[2] Isaac Asimov-**"The Red Queen's Race"**, in Astounding Science-Fiction, January 1949. Reprinted in **"The Complete Stories II"**, Doubleday, 1990.

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connected to the larger world, that is, we are asking what Mathematics means outside of its own world, in answering our question we can largely ignore many of the details of Mathematics[3].

Plato tries to clarify a position when indicating that the mathematical objects have their own existence, beyond the mind.Aristotle saw the mathematics like one of the divisions of the knowledge that was different from the physical knowledge and the theological one. He denied that the mathematics were a theory of an external knowledge, independent and unnoticeable. It associated to the mathematics with a reality where the knowledge obtains by experimentation, observation and abstraction. This position joint party that the construction of the mathematical ideas occurs through idealizations realized by the mathematicians like a result of its experience with objects in a specific context.The points of view of Plato and Aristotle have represented the great poles where the discussion has oscillated about the nature of the mathematics.

But the absolutist vision entered crisis due to the discovery of some contradictions found in certain theorems that comprised of mathematical systems considered rigorous. For example, Russell demonstrated that the logical system of Frege was inconsistent. The paradox of the property of *being an element of itself* (a set is element of itself if and only if it is not element of itself) did collapsed its law number 15. But the mathematic one is certain and if all theorems are true, how can contradictions exist between their theorems? Something must be mistaken in the foundation of the mathematics. Paul Ernest proposes a socio-constructivist vision of the mathematical one[4]. In this vision it is considered that the mathematical truth is fallible and correctable, and that is the overhaul always open. This theory takes from the conventionalism the idea that the human language with their rules and agreements plays an important role in the establishment and justification of the mathematical truths. Also he takes from quasi-empiricism, its epistemology of the fallibility of the mathematics and the principle of which the mathematical concepts and knowledge change by means of conjectures process and refutations.
Mathematics is the subject where answers can definitely be marked right or wrong, either in the classroom or at the research level. Mathematics is the subject where statements are capable in principle of being proved or disproved, and where proof or disproof bring unanimous agreement by all qualified experts—all who understand the concepts and methods involved.

Reasoning about mental objects (concepts, ideas) that compels assent (on the part of everyone who understands the concepts involved) is what we call "mathematical". This is what is meant by *mathematical certainty*. It does not imply infallibility![5]

History shows that the concepts about which we reason with such conviction have sometimes surprised us on closer acquaintance, and forced us to re-examine and improve our reasoning.

Ah, but on the library shelves, in the math section, all those formulas and proofs, isn't that math? No, as long as it just sits on the shelf, it's just ink on paper. It becomes

[3] An interesting review of various postures on the nature of Mathematics may be found in **"Lectures on the Foundations of Mathematics"** of John L. Bell.
[4] Ernest, P. (1991)-**"The Philosophy of Mathematics"**, London, The Falmer Press, also cf. Morris Kline (1980)-"**Mathematics: The Loss of Certainty"**, Oxford University Press.
[5] http://www.math.unm.edu/~rhersh/Definition%20of%20mathematics.doc

mathematics; it comes alive, when somebody starts to read it. And of course, it was alive when it was being thought and written by some mathematician.

Mathematical conclusions are decisive. Just as physical or chemical knowledge can be independently verified by any competent experimenter, an algebraic or geometric proof can be checked and recognized as a proof by any competent algebraist or geometer. Saunders MacLane, among others, said, *"What characterizes mathematics is that it's precise"*. But what, precisely, should be meant here, by *precise*? Not *numerical* precision. A huge part of modern mathematics, including MacLane's contribution, is geometrical or syntactical, not numerical. Should *precise* mean formally explicit, expressed in a formal symbolism? No. There are famous examples in mathematics of conclusive *visual* reasoning, accepted as *mathematical proof* prior to any *post hoc* formalization. Several famous mathematicians have said *"You don't really understand a mathematical concept until you can explain it to the first person you meet in the street"*.

Probably the correct interpretation of *precise* should be simply, *subject to conclusive, irrefutable reasoning*. So I am accepting the familiar claim, *"Mathematics is characterized above all by precision"*, but only after *unpacking* what we should mean by *precise*.
In the past 25 or 30 years, it has come to be recognized that mathematics is not a fixed, unitary, absolute body of knowledge that changes only by growth at the periphery. Advances in the history and philosophy of mathematics, the sociology of knowledge, and post-modernist thought[6] have shown that the myth of the unchanging nature of mathematics is probably held in place by the use of single term *mathematics* for several diverse domains of knowledge and discursive practice. School mathematics and the research mathematician's pure mathematics are wholly different areas of study. Controversy has erupted over the natures of both of these domains: the first is the subject of political contestation; the latter of philosophical dispute[7].
In this work we present some reflections on mathematics and mathematicians motivated by a recent work of Pan Shengliang[8]. Special emphasis is placed on the questions what is mathematics? And what is a mathematician? Some open questions related to these topics are posed at the end of the work.

1. What is mathematics?
One of the oldest of all fields of study is that now known as Mathematics. Often referred to, used, praised, and disparaged, it has long been one of the most central components of human thought, yet how many of us could describe what mathematics really is?
Searching with Google for "definitions of mathematics" gives approximately 8.620.000 hits[9]. From a quite traditional and very general view mathematics is often seen as (...) *"a science (or group of related sciences) dealing with the logic of quantity and shape and arrangement"*[10]. However such a characterisation only describes what, not how (or why). Hence methodological aspects that might be of significance are not mentioned. A description that combines what and how (underlined in the quote by me) is found in Wikipedia where mathematics is seen as (...) *"the body of knowledge centred on concepts such as quantity, structure, space, and change, and also the academic discipline that studies them"*. Benjamin Peirce called it *"the science that draws necessary conclusions"*. Other practitioners of mathematics maintain that

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[6] See for example Ernest, P. Ed. (1994)-**"Mathematics, Education and Philosophy: An International Perspective"**, London, The Falmer Press.
[7] Ernest, P. (1991)-Op. Cit.
[8] Shengliang, P. (2003)-**"Some reflections on mathematics, mathematics education and mathematicians"**, The China Papers, July, 95-99.
[9] November 2009.
[10] http://www.thefreedictionary.com/science

mathematics is the science of pattern, that mathematicians seek out patterns whether found in numbers, space, science, computers, imaginary abstractions, or elsewhere. Mathematicians explore such concepts, aiming to formulate new conjectures and establish their truth by rigorous deduction from appropriately chosen axioms and definitions[11].
The common people belief among many students is that Mathematics is about numbers, formulas and cranking out computations. It is the unconsciously held delusion that Mathematics is a set of rules and formulas that have been worked out by God knows who for God knows why, and the student's duty is to memorize all this stuff. This position can take to diverse mistaken answers to the question that heads this section.

Kasner and Newman's point of view is that, *"Mathematics is the science which uses easy words for hard ideas"*[12]. According to Kant, *"the science of mathematics presents the most brilliant example of how pure reason may successfully enlarge its domain without the aid of experience"*.
Said Feynman *"To those who do not know Mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty of nature. ... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in"*.

In fact, like other sciences, mathematics reflects the laws of the material world around us and serves as a powerful instructional tool for understanding Nature. Mathematics reveals the hidden patterns that empower us to understand better the information-laden world in which we live. As a science of abstract objects, Mathematics relies on logic rather than on observation for the purpose of stating truths, yet employs observation, simulation, and even experimentation as a means of discovering truth. Through its results, mathematics offers science both a foundation of truth and a standard of certainty. Mathematics offers distinctive modes of thought which are both versatile and powerful; including modeling, abstraction, optimization, logical analysis, inference from data, and use of symbols. Mathematics enables us to read critically, to identify fallacies, to detect bias, to assess risk, and to suggest alternatives. The resolution of mathematical problems supplies people with techniques, which can be used in different areas; even to everyday problems mathematical thinking is logical and strict, intuitive and creative, dynamic and changing.
By other hands, an opposite position is due to Russell *"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true"*. It is clear that the definition of Russell does not help us much.

When mathematics is understood in the broadest sense, not overstepping the thresholds to neighbouring academic disciplines, the field embraces 97 different specific kinds or sub-branches of mathematics according to MSC (of which for instance ordinary differential equations is just one)[13].

In a principle inquiry of definitions of mathematics Bonnie Gold identifies and discusses critically nine major claims[14]. As a result of the inspection she outlines 13 criteria for *good definitions*. Taken collectively these criteria seem to have a dual function, to describe (valid) internal cohesions within the discipline of mathematics and to relate what one could call

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[11] http://en.wikipedia.org/wiki/Mathematics#Mathematics_and_physical_realityA
[12] See Blank, B. E. (2001)-**"What is mathematics? An elementary approach to ideas and methods"**, Notices of the AMS, December, 1325-1329; a review of the classic book of Richard Courant and Herbert Robbins, Oxford University Press, USA; 2 edition (July 18, 1996).
[13] Rusin, D. (2004)-**"The Mathematical Atlas. A gateway to modern mathematics"**, in http://www.math-atlas.org/welcome.html, http://www.math.niu.edu/~rusin/known-math/index/tour.html and http://www.math.niu.edu/~rusin/known-math/index/tour.html
[14] Gold, B. (2003)-**"What is mathematics? I: The question"** Monmouth University, in http://www.math.utep.edu/Faculty/pmdelgado2/Math1319/Philosophy/Bonnie.doc .

mathematically to other disciplinarities. These two concerns are of course often closely related. Of the nine types of descriptions of mathematics there are hardly any that does not play some role in other disciplines. It is therefore not likely to find one *single* aspect that makes mathematics unique, and which can be used solely to define every former, present and future kind of mathematics.

As pointed to above a philosophical challenge for mathematics is that during its historical purification process, becoming an academic discipline, it tends to obliterate its own foundations. At the heart of the discipline as *established* there seems to be a kind of safety-game where a *universal given's* of mathematics makes a critical questioning of the discipline irrelevant and inadequate. This intellectual *laziness* (or this sensible pragmatic taken for granted attitude) is transmitted to mathematics education because mathematics of course here normally is based on and focuses the stability and not the slow development of the discipline. This tendency consolidates the idea that mathematics is given rather than developed and thus may function as another set of blinkers for how disciplinarily is generated.

Gold dismisses the claim that *mathematics is what mathematicians do*. Although she admits that one (...) *could modify it by saying that it is what mathematicians do when acting as mathematicians,* she doubts that one can avoid circularity when specifying what it is to act as a mathematician. However if one looks at this definition in the light of pragmatics (which Gold does not), it could be further refined. Mathematics as discipline could be described by the full set of practical and intellectual acts that are at work when doing mathematics (but not only). In other words, even mathematics needs to be seen, not just as products, but as processes. This will obviously accumulate into a long list, at least containing activities such as theorising, doing inductions and deductions, defining, arguing, calculating, giving premises, concluding, etc. This implies a *pragmatic* understanding of language and communication.
In discussions there at this point often tends to appear an opposition between applied and pure mathematics[15], where the kind of acts related to these types of doing mathematics are said to be qualitatively different (cf. paragraph C in Gold's paper). In any case the question of which mental and practical activities that are involved can not be finalised without a valid description of the content of mathematics (to the degree this is practically and principally possible). Gold finds that listing sub-fields is the most common way of defining mathematics.

Even if this gives some kind of concreteness to the question there are several dangers: (...) *"such definitions risk becoming dated by the evolution of mathematics; even if we make our list include all the current Mathematics Reviews subject classifications, new subjects are being added all the time. Second, they emphasize the separateness of the different branches of mathematics, whereas if there has been any lesson from the development of mathematics in the last 50 years, it is the unity of mathematics, the complex web of interconnections between the supposedly different fields, even those which seem to have very different flavors (more on this in section IV). Third, they give no assistance in recognizing a new kind of mathematics when it appears"*[16].

In other words, from our perspective one should combine a *synchronic* and a *diachronic* view of the discipline, a conclusion which of course is close to the former that one needs to differentiate between *products* and *processes*. Nevertheless it takes into account the interplay between *stability* and *dynamics*.

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[15] See Stewart, I.-**"Letters to a young mathematician"**, Basic Books, 2006; mainly the Letter 15 "Pure or Applied".
[16] Gold, B. (2003)-Op. Cit., p.4.

Gold further claims that the difficulty with (...) *"finding a common subject has caused people to turn to the methodology of mathematics to find its unifying theme, mathematics being unique among the sciences in making deductions from axioms the cornerstone of its reasoning"*[17]. The crucial role of axioms in mathematics is agreed upon in mathematics. Mathematics is built and continues to be built upon this particular genre. Metaphorically an axiom functions as a humming top in a supposedly eternal spin, so that it will never fall. From a (pragmatic) speech act perspective it can simplistically be described by an utterance beginning with *Given that...* It is the final preciseness, creativity and relevance of the description of the set of axioms that will bring mathematics further, closer to the cutting edge of its disciplinarily. But it is by the same token the continuous growth of (interrelated) axioms that makes mathematics stable. Paradoxically, using language to create a fixed point of departure is also what gives mathematics the imaginative freedom and makes *pure* mathematics possible (and even free, fresh and fascinating). According to Gold, Nevanlinna expresses a similar sentiment *"Mathematics combines two opposites, exactitude and freedom"*[18].

Surprisingly, and for some, provokingly, this view makes language and mathematics to rather inherited (semiotic) phenomena. Hence while language in general and fiction in particular can be seen (with Umberto Eco) as the tool with which one in principle *can* lye, the regime of axioms in mathematics leads to the opposite, a position which is at the heart of Benjamin Peirce's famous definition of mathematics as the science which draws necessary conclusions.

In this perspective one of the foundations of mathematics is a purification of a particular kind of speech act where lying is made impossible. You can make mistakes, but not lye, once given the axioms that close the mathematical entities. Consequently, if you are lying or cheating deliberately, what you do is not (according to) mathematics. The main reason for that this is possible is the axiomatic closing of open signs. According to semiotic theory signs in *natural* language are under the law of semiotic, a never-ending growth in the meaning of all concepts over time. In mathematics however such concepts/objects can not be part of an axiomatic act/definition.

How said before, one popular definition of mathematics is the discipline that studies *patterns*. Gold argues that this view does not distinguish structures found in mathematics from other structures[19]. Mathematics is for instance not interested in the patterns of atoms or molecules, rather, (...) *"mathematics is concerned with the properties of patterns, the general relationships between patterns, how they behave, and so on"*. To see mathematics as the science of patterns implies a structuralist perspective[20]. Reuben Hersh, famous for advocating the (implicit pragmatic) view that mathematics is what mathematicians do, writes critically in

"What Is Mathematics, Really?".
The definition, *science of patterns* is appealing[21]. It's closer to the mark than *"the science that draws necessary conclusions"* (Benjamin Peirce), *"the study of form and quantity"*[22] or *"The*

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[17] Idem.
[18] See page 456 of Nevanlinna, R. (1966)-**"Reform in Teaching Mathematics"**, *Monthly*, 73: 451-464.
[19] Gold, B. (2003)-Op. Cit.
[20] Cf. **"Taming the infinite. The story of Mathematics"** of Ian Stewart, published by Quercus Publishing PLC, UK, 2007 and Devlin, K. J. (2000)-**"The language of Mathematics: making the invisible visible"**, W. H. Freeman and Company.
[21] See Letter 3 of Stewart-Op.Cit.
[22] Webster's Unabridged Dictionary. Also cf. the page http://www.mathacademy.com/pr/quotes/index.asp?ACTION=AUT&VAL=Steen or the Lynn Arthur Steen's Home Page http://www.stolaf.edu/people/steen/

mathematics is the study of the true thing of the hypothetical situations" (Charles S. Peirce)[23]. This one is its essence and its definition. Unlike formalism, structuralism allows mathematics a subject matter. Unlike Platonism, it doesn't rely on a transcendental abstract reality. Structuralism grants mathematics unlimited generality and applicability. Structuralism is valid as a partial description of mathematics, an illuminating comment. As a complete description, it's unsatisfactory[24].

The Marxist case. Many Marxist historians maintain the Engels' definition *"Pure mathematics deals with the space forms and quantity relations of the real world -that is, with material which is very real indeed"*[25] and they continue insisting on the existence of objective laws of the development of the mathematics, without showing which these laws are and how they work to predict its development.

Kolmogorov[26] in a very famous paper for the Marxists says *"In the continuing relationship with the requirement of the technical and scientific knowledge, the wealth of quantitative relationships and forms space studied by the Mathematics, is constantly expands, so the general definition of Mathematics is filled with a content increasingly rich."*[27]

"In conclusion", says Sánchez, "*the definition of the object of the Mathematics given by Engels, continues being actual"*[28].

These Marxist philosophers, have created what I call a *Metaphilosophy of the Mathematics*, discussing and analyzing questions of the "classics of Marxism" (Marx, Engels and Lenin of course¡¡¡¡), and the social practice as the bases of the development of Science, underestimating other causes and factors systematically and, what is worse, subordinating mathematics researches to certain ideological goals and placed under a strict ideological scrutiny.[29]

On the matter they are sufficient the following affirmations[30]:

"As was the case with all the work of the classics, the Manuscript Mathematicians from Karl Marx, were a need for its general plan to fight." …

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23 "The Essence of Mathematics", ch. 3 of his unpublished **"Minute Logic"** online in http://www.unav.es/gep/EssenceMathematics.html (Spanish).

24 Hersh, R. (1997)-**"What Is Mathematics, Really?"**, Oxford: Oxford University Press and Yiparaki, O. (1999)-**"Another General Book on Mathematics?"**, Complexity, Vol. 4/4, pp 55-60.

25 Engels, F. (1975)-**"Anti-Duhring"**, La Habana, Editorial Pueblo y Educación (Spanish) available online in http://www.marxists.org/archive/marx/works/1877/anti-duhring/ch01.htm

26 See Vucinich, A. (2000)-**"Soviet Mathematics and Dialectics in the Stalin Era"**, Historia Mathematica, Vol. 27, Nº 1, 54-76 and Vucinich, A. (2000)-**"Soviet Mathematics and Dialectics in the Post-Stalin Era: New Horizons"**, Historia Mathematica, Vol. 29, Nº 1, 13-39, for a characterization of the soviet Mathematic.

27 See the Kolmogorov's article "Mathematics" of Bolshaya Sovietskaya Entsiklopedja, Great Soviet Encyclopedia (1936) online in http://www.kolmogorov.pms.ru/bse-mathimatic.html (Russian)

28 Sánchez F., C. (1987)-**"Conferences on philosophical and methodological problems of the Mathematic"**, Universidad de la Habana (Spanish).

29 See, for example, Alexandrov, A. D. (1964)-"Mathematics", in **"Philosophical Encyclopedia"**, t. III, p. 329, Sovietskaia Enziklopedia, Moskva (Russian); Alexandrov, A. D., A. N. Kolmogorov, M. A. Lavrentiev (eds) (1999)-**"The mathematics: its content, methods and meaning"**, Dover; Alekxeev, B. T.-"Dialectic of the mathematic knowledge" in F. B. Konstamtinov (1983)-**"Materialist Dialectic"**, Thought Editorial, T. 3 (Russian); Arkadi, U. (1981)-**"The dialectic and the methods scientific generals of investigation"**, La Habana, Social Sciences Editorial, pp.190-191; Casanovas, G. (1965)-**"The Mathematic and the Dialectic Materialism"**, La Habana, National Editorial of Cuba and Ruzavin, G. I. (1967)-**"The nature of mathematical knowledge"**, Thought Editorial, Moskva (Russian).

30 To see for additional details Szekely, L. (1990)-**"Motion and the dialectical view of the world"**, Studies in Soviet Thought 39, 241-255.

"From the point of view philosophical Marx was proposed penetrate to dialectic materialism in the contradictions of Infinitesimal Calculus." … *"and considered that this ultimately should be settled with the implementation of dialectic method to the mathematic"* …

"Marx considered the Calculus as a new degree or stage in the development of Mathematics, qualitatively superior."

"The principle of the unity of the logical and historic as a method of cognition was a key factor in the arrival in this conclusion."[31]

By other hand, research was subject to censorship. Hence, scientists and researches were denied access to some publications and research of the Western scientists, or any others deemed politically incorrect; access too many others sources was restricted. Their own research was similarly censored, some scientists were forbidden from publishing at all, many others experienced significant delays or had to agree to have their works published only in closed journals, to which access was significantly restricted. But this not only happened in the Soviet Union, in Cuba the phrase *"he is off the track ideologically"* meant a danger for the academic race.

Finally, are all considerations of Marxists are wrong? No, many works are useful and brilliants guide, but the *super valuations* of "classics" they make into platonic and absolutist, don't forget the Mathematics is a human construction, no a legacy of certain people[32].

As a mathematician, I would further say that mathematics is an awe-inspiring science, filled with mystery and wonder, and brimming with opportunities to make triumphant intellectual discoveries. It is truly one of the highest points of humankind's achievements. It offers everyone the chance to get a glimpse of the nature of the universe around us, and to learn and understand something more about the human condition. It is the most universal of our languages and the most useful of our tools; it is the most beautiful of our music and the most elegant of our poems; it is silent harmony and form; it is to some people an art.

As an undergraduate student I found the following results very beautiful:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\frac{1}{n^2}+\dots=\frac{\pi^2}{6}\text{ [33]},\quad 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots\frac{(-1)^n}{n}+\dots=\frac{\pi^2}{12}.$$

What do you feel about them? What about $e^{\pi i}=1$. A tale of three wonderful numbers. Arguably, that formula is the most beautiful single formula in all mathematics.

Finally, I beliefs that *mathematics is the science which deals with magnitudes (variables and constants, qualitative and quantitative), forms (abstract and concretes), patrons and rules, that it uses general methods and own techniques for study, understand and modify social, naturals and human systems and phenomena. The mathematics is a collective activity of the mathematic community, consolidated gradually in the time.*

2. What is a mathematician?

In my opinion, a mathematician is a person who not only studies mathematics but also does research in mathematics. Some mathematicians do research as well as teach Mathematics.

The ideas people entertain regarding what happens in mathematicians' heads when they are engaged in practicing their science originate no doubt from their own personal

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31 Matute P., M.; A. Soldatov and others (1987)-**"Philosophical and methodological problems of Mathematics"**, Universidad de Oriente, Santiago de Cuba (Spanish), pp.35-36.

32 See the introduction of Reuben Hersh in **"18 Unconventional Essays on the Nature of Mathematics"**, Springer, 2006.

33 Cf. Capítulo 1 "A história como elemento unificador na educaçao matemática" in **"A História como un agente de cogniçao na Educaçao Matemática"**, Porto Alegre: Editora Sulina, 2006, of Fossa, J.; J. Nápoles and I. Abreu.

mathematical experiences. For no mathematicians these will mostly be confined to math classes in school or at university, where mathematics appears wearing the hat of an ancillary science. This type of experience is unfortunately prone to lead to fundamental misunderstandings that give rise to completely mistaken ideas as to what mathematics is all about: it is most emphatically not a machine-translatable aptitude for calculating according to formulae and rigid precepts that do not allow space for individual freedom. The reason for the wide prevalence of this travestied image of mathematics is arguably the fact that exams cast in this ostensibly 'objective' form are easier to implement both for preparation and assessment.

Lack of experience on the part of teachers and examiners will often lead to an aggravation of the misunderstanding[34].

By opposite mathematicians are typically interested in finding and describing *significant* which may have originally arisen from problems of calculation, but have now been abstracted to become their own problems. From much published research work of mathematicians, it may look as if the primary approach of a mathematician is to start with some given assumptions, often called axioms, and then proceed to prove other facts which follow from the assumptions according to exact rules of logic. That, however, is the finished product that gets published; it is not work in progress.

Contrary to popular belief, mathematicians are not typically any better at adding or subtracting numbers, or figuring the tip on a restaurant bill, than members of any other profession, in fact, some of the best mathematicians are notoriously bad at these tasks!
A mathematician uses numbers and symbols in many ways, from creating new theories to translating scientific and technical problems into mathematical terms. There are two types of researching mathematicians: the theoretical mathematicians, who work with pure mathematics to develop and discover new mathematical principles and theories without regard to their possible applications; and applied mathematicians, who use mathematical methods to solve practical problems in diverse areas.

To some extent, people give differing definitions of the mathematician, probably owing to the nature of their own work. We cite some examples.

- *"A mathematician is a machine for turning coffee into theorems"*, P. Erdos (1913-1996)[35].
- *"A person who can, within a year, solve* $x^2 - 92y^2 = 1$, *is a mathematician"*, Brahmagupta (598-668)[36].
- *"I have hardly ever known a mathematician who was capable of reasoning"*, Plato (429-347 b.C.[37]
- *"To be a scholar of mathematics you must be born with talent, insight, concentration, taste, luck, drive and the ability to visualize and guess"*[38].
- *"Mathematics is a dangerous profession; an appreciable proportion of us go mad, and then this particular event would be quite likely"*[39].
- *"Mathematicians are like lovers. Grant a mathematician the least principle, and he will draw from it a consequence which you must also grant him, and from this consequence another"*, B. B. Fontenelle (1657-

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[34] See Reinhard Winkler-**"What is mathematics? – A subjective approach"**, http://www.dmg.tuwien.ac.at/winkler/pub/manfred-englisch.pdf
[35] Rose, N. (1988)-**"Mathematical Maxims and Minims"**, Raleigh NC, Rome Press Inc.
[36] Ernest, P. (1991).
[37] Rose (1988).
[38] Halmos, P. R. (1985)-**"I Want To Be A Mathematician"**, Washington: MAA Spectrum.
[39] Littlewood, J. E. (1953)-**"A Mathematician's Miscellany"**, Methuen and Co. Ltd.

1757)[40].

• *"It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done."*[41]

In other words, mathematicians are interested not only in what happens when you adopt a particular set of rules, but also in what happens when you change the rules. For example, Lobatchevski, Bolyai, Gauss and Riemann started with Euclid's geometry, but asked *"What if parallel lines could intersect each other? How would that change things?"* And they ended up inventing an entirely new branch of geometry, which turned out to be just what Einstein needed for his theory of general relativity.

So, we can distinguish three types of mathematical activity:
1. To solve problems.
2. To demonstrate theorems or to refute conjectures[42].
3. To apply, broadly speaking, the ideas, methods, algorithms, etc. obtained in the preceding steps.

As becomes apparent from studying truly great mathematicians, the real motivations to get involved with mathematics are these activities. Only those who have some knowledge of this allure of mathematics at least from hearsay can hope to do justice to the science[43].

3. Conclusion.

Mathematics is an old, broad, and deep discipline (field of study). I think that people working to improve math education need to understand *"What is Mathematics?"* It's clear if we take in account to Philip J. Davis[44] when say *"The inter-interpretability exhibited in Mathematics Elsewhere is often cited as evidence for both unity and universality. But this depends on a definition of mathematics that is sufficiently restricted to exclude the cultural underlay. My own definition of mathematics is that it includes everything that makes its core comprehensible. (And I'm not sure how to define the core.) I see unity only in a weak sense."*

The proper stuff of mathematics is ideas and concepts. Mathematicians are called upon to describe these as accurately as possible and to ascertain whether they are categories inherent to the process of thought or whether other options are available. As opposed to empirical sciences that explore the world as it is, mathematics charts the world under the double aspects of necessity and freedom.

This is why mathematics is the least restricted and most universal science. This also accounts for its applicability too many other branches of science, this applicability, make the math a special case: *"Mathematics occupies a special position among the sciences and in the educational system. This position is determined by the fact that mathematics is an a priori science building on ideal elements abstracted from sensory experiences, and at the same time mathematics is intimately connected to the experimental sciences, traditionally not least the natural sciences and the engineering sciences. Mathematics can be decisive when formulating theories giving insight into observed phenomena, and often forms the basis for further conquests in these sciences because of its power for deduction and calculation.*

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[40] Larney, V. H. (1975)-**"Abstract Algebra: A First Course"**, Boston, Prindle, Weber and Schmidt.

[41] Hardy, G. R. (1941)-**"A Mathematician's Apology"**, London Cambridge University Press, p. 1.

[42] Thurston, W. P. (1994)-**"On proof and progress in mathematics"**, Bull. Amer. Math. Soc. 30, 161-177.

[43] An interesting point of view is Eisenberg, T. (2008)-**"Flaws and Idiosyncrasies in Mathematicians: Food for the Classroom?"** The Montana Mathematics Enthusiast, 5(1), 3-14.

[44] **"Book Review"** of SIAM News, Volume 36, Number 2, March 2003 refer to Marcia Ascher (2002)-**"Is Mathematics a Unified Whole? Mathematics Elsewhere: An Exploration of Ideas Across Cultures"**, Princeton University Press, Princeton, New Jersey, and Oxfordshire, UK.

The revolution in the natural sciences in the 1600s and the subsequent technological conquests were to an overwhelming degree based on mathematics. The unsurpassed strength of mathematics in the description of phenomena from the outside world lies in the fascinating interplay between the concrete and the abstract."[45]
Mathematics is the highest form of symbiosis between intuition and scientific precision. For it to be taught adequately requires the most holistic form of communication, i.e. communication between individuals in terms of states of mind.

Any discipline (an organized, formal field of study) such as mathematics tends to be defined by the types of problems it addresses[46], the methods it uses to address these problems, and the results it has achieved. One way to organize this set of information is to divide it into the following three categories (of course, they overlap each other):

1. Mathematics as a human endeavor[47]. For example, consider the math of measurement of time such as years, seasons, months, weeks, days, and so on. Or, consider the measurement of distance, and the different systems of distance measurement that developed throughout the world. Or, think about math in art, dance, and music. There is a rich history of human development of mathematics and mathematical uses in our modern society.
2. Mathematics as a discipline. You are familiar with lots of academic disciplines such as archeology, biology, chemistry, economics, history, psychology, sociology, and so on. Mathematics is a broad and deep discipline that is continuing to grow in breadth and depth. Nowadays, a Ph.D. research dissertation in mathematics is typically narrowly focused on definitions, theorems, and proofs related to a single problem in a narrow subfield in mathematics.
3. Mathematics as an interdisciplinary language and tool. Like reading and writing, math is an important component of learning and *doing* (using one's knowledge) in each academic discipline. Mathematics is such a useful language and tool that it is considered one of the *basics* in our formal educational system.

To a large extent, students and many of their teachers tend to define mathematics in terms of what they learn in math courses, and these courses tend to focus on instrumental view of mathematics. The instructional and assessment focus tends to be on basic skills and on solving relatively simple problems using these basic skills. As the three-component discussion given above indicates, this is only part of mathematics.

Even within the third component, it is not clear what should be emphasized in curriculum, instruction, and assessment. The issue of basic skills versus higher-order skills is particularly important in math education. How much of the math education time should be spent in helping students gain a high level of accuracy and automaticity in basic computational and procedural skills? How much time should be spent on higher-order skills such as problem posing, problem representation, solving complex problems, and transferring math knowledge and skills to problems in non-math disciplines?

I take as mathematics that which in the course of history has evolved as the product of the activity of mathematicians and has to a great extent been standardized, conventionalized and corroborated by extended experience and manifold practical usages.

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[45] Hansen, V. L. (2003)-"**Popularizing Mathematics: From Eight to Infinity"**, ICM 2002·Vol. III, ·1–3, in arXiv:math/0305019v1 [math.HO] 1 May 2003.

[46] In the Lecture **"The History of Mathematics across yours problems"** (in Spanish) held in the VIII SNHM, Belen do Pará, Brazil, April 5 to 8 of current year, we present the different types of problems resolved and how they have been changing throughout history. In a next work we will give more details. Also cf. Grinin, L. E.-"**Periodization of History: A theoretic-mathematical analysis"**, in History & Mathematics: Analyzing and Modeling Global Development, Grinin, L. E., de Munck V., Korotayev A. (eds.), pp. 10–38. Moscow: KomKniga.

[47] Manin, Yu. I. (2007)-**"Mathematical knowledge: internal, social and cultural aspects, in Mathematic and Culture"**, M. Emmer (Ed.), Ch.2, Springer, 2004, preprint arXiv:math/0703427v.

It is the concepts, methods, notations, basic assumptions, etc. which rather unanimously are considered to be mathematical that make up *mathematics*. I admit that possibly those concepts and methods might differ depending on basic views about the nature of mathematics[48].

A fallible perspective provides a powerful additional source of arguments for the social responsibility of both mathematics and its teaching. It also fits well with the emerging constructivist views of learning in mathematics and science education. But all of these benefits can be had without this philosophical commitment.

Finally, the Mathematical one should be considered as a class of mental activity, a social construction that contains conjectures, tests and refutations whose results are subjected to revolutionary changes and whose validity, therefore, it can be judged with relationship to an it pierces social and cultural, contrary to the absolutist vision (platonic) of the mathematical knowledge.

The affirmed thing takes previously us to the following open questions that will be treated in other works[49]:

- What about *applied mathematics*?[50]
- What about the *mathematical proof*?[51]
- What is the *history of mathematics*?[52]
- What is *relevant* and/o *necessary* for the mathematics education?[53]

[48] Dörfler, W. (2000)-**"Mathematics, Mathematics Education and Mathematicians: An Unbalanced Triangle"**, International Commission on Mathematical Instruction http://www.emis.de/mirror/IMU/ICMI/bulletin/49/Mathematics.html

[49] In other direction cf for example "**What is good Mathematics?"** of Terence Tao in arXiv:math/0702396v1 [math.HO] 13 Feb 2007.

[50] Maybe start with Wigner, E. P. (1960)-**"The Unreasonable Effectiveness of Mathematics in the Natural Sciences"**, Communications in Pure and Applied Mathematics 13: 1-14.

[51] Reid, D. A. (2002)-**"What is proof?"**, International Newsletter on the Teaching and Learning of Mathematical Proof, June, available online in http://www-didactique.imag.fr/preuve

[52] Many historians deal prefer with this question, see works of Heiede, Hersh, Bagni, Furinghetti, Garciadiego, Grattan Guinnes, etc.

[53] **"Mathematical Education"** in Notices of the AMS 37 (1990), 844– 850, of William P. Thurston, online in arXiv:math/0503081v1 [math.HO] 4 Mar 2005.

If not, then what if as well? *Unexpected Trigonometric Insights*

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Abstract

In performing an exercise of "What if not", one can end up with a paucity of structure. Adding alternative structure can be a rich source of discovery, as we present here. The framework of this presentation is the original voyage of discovery, from a trivial geometric problem to the derivation of some unexpected trigonometric formulae based on regular polygons. The original "voyage" has been changed only sufficiently to make the text readable.

Keywords: Trigonometric identities; Problem posing; Euclidean Geometry

Introduction

Consider the following construction problem: *ABC* is a triangle inscribed in its circumscribing circle. Using a compass, take the measure of one of the triangle's sides, which is also a chord of the circle, and mark off identical chords around the circle, starting at either vertex on the selected chord. Can you characterize the triangles for which the construction will return exactly to the starting vertex? Can you compute how many revolutions are required?

In figure 1, starting from *C*, using chord length *BC*, the suggested construction goes through *CD* = *BC*, *DE* = *BC* and *EF* = *BC*. Since *F* is beyond *B* it is clear that this

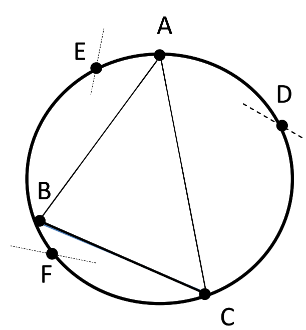

Figure 1

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construction will not return to *C*.

A construction that returns to the starting point in one revolution will follow the vertices of a regular polygon since the chords are all equal and are circumscribed in a circle. If the construction returns to the starting vertex after several revolutions, an obvious next question is whether the points on the circumference that are visited will constitute points on a regular polygon. We can, in fact, pose a general question: Is there an explicit property of the angles or sides (or both) that will *characterize* which triangles support constructions, of the above type, that return to their starting point in a discrete number of revolutions?

The answer to both questions is yes and the characterizing property is surprisingly simple.

The question, however, did not arise spontaneously. It came at the end of a surprising voyage of discovery whose source was an innocuous what-if-not exercise (Brown & Walter, 1990). This what-if-not exercise took an interesting route I have labeled "If not, then what-if-as-well".

What follows, documents my personal voyage and of course answers the question raised above. Some curious by-products of the answer are also presented.

The Starting Point

The starting point for this voyage of discovery was a desire to compose a small research activity for students in the 10th or 11th grade who are studying Euclidean geometry. In doing so, a common textbook problem serves as a convenient starting point.

The original textbook problem is illustrated in Figure 2:

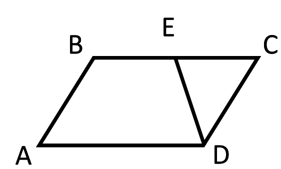

Figure 2

ABCD is a parallelogram. *E* is a point on *BC*.

1. Prove $\frac{S_{ECD}}{S_{BEDA}} = \frac{BC - BE}{BC + BE}$.

2. Then ask yourself: what if not?

Proof:

1. Suppose the distance between parallel sides *BC* and *AD* is *h*. Then

$$\frac{S_{ECD}}{S_{BEDA}} = \frac{EC \cdot h/2}{BC \cdot h - EC \cdot h/2} = \frac{EC}{2 \cdot BC - EC} = \frac{BC - BE}{BC + (BC - EC)} = \frac{BC - BE}{BC + BE}$$

2. For the what-if-not exercise, I started by writing down problem attributes one might consider changing. There are many alternatives, of which the following are some examples:

    1. *E* is on *BC*. Suppose *E* is not on *BC*:

        a) *E* is on the extension of *BC* to the right or to the left.

        b) *E* is on *AB*

        c) *E* is interior to the parallelogram.

    2. The problem is about areas. Suppose the problem is not about areas:

        a) Consider some relationship between perimeters.

    3. *E* is connected to *D*. Suppose *E* is not connected to *D* but to *F*:

        a) *F* is on the extension of *CD*.

        b) *F* is on *AD* and the triangle *ECD* becomes the trapeze *ECDF*.

        c) *F* is on an extension of *AD* to the left or to the right.

    4. *ABCD* is a parallelogram. Suppose it is not:

        a) *ABCD* is an arbitrary quadrilateral.

        b) *ABCD* is an arbitrary polygon with more than 4 sides.

An initial run through the above what-if-not scenarios raised some moderately interesting situations but nothing really "meaty". Having made no real headway, I decided I would at least extract a minor victory by writing up the formulae for the general case of 4b, as illustrated in Figure 3.

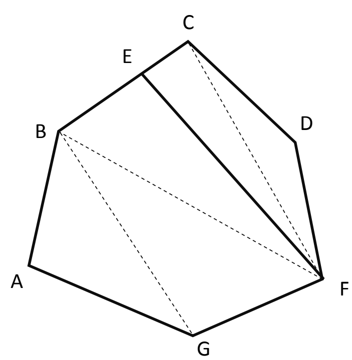

Figure 3

The correspondence to the original problem is that *E* is a point on *BC* of *polygon ABCDFG*, which is connected to vertex *F* by a line, and we are interested in the area proportion $\frac{S_{ECDF}}{S_{EFGAB}}$.

Counting included triangles reveals that $\frac{S_{ECDF}}{S_{EFGAB}} = \frac{S_{CDF} + S_{ECF}}{S_{EFB} + S_{BFG} + S_{BGA}}$. Can we simplify or find an interesting rule to compute that? The answer seemed to be a disappointing no. There is just too little structure, as there was with other alternative what-if-nots I tried earlier.

However*, if not, what if* ***as well***? Suppose we *add* some structure to the problem and see if something interesting comes up. Suppose the polygon to be a *regular* polygon. Can we find an expression for the area proportion that will be a function of, say, only the length *d* of a side and the distance *EC*? I decided to examine an octagon (it seemed easiest to draw!), as a representative of a generic regular polygon with *k=8* sides and side length *d*, illustrated in Figure 4 below. Note that although the illustration is for an octagon, the calculations are written in a general form that applies to a regular polygon with any number of sides.

The Main Derivation

In the general case of the problem, as illustrated in Figure 4, we wish to find a simple expression for the ratio of the areas of the two polygons, here ECDFG and EGHIAB: $\frac{S_{ECDFG}}{S_{EGHIAB}} = \frac{S_{CDF} + S_{CFG} + S_{CGE}}{S_{EGB} + S_{BGH} + S_{BHI} + S_{BIAI}}$

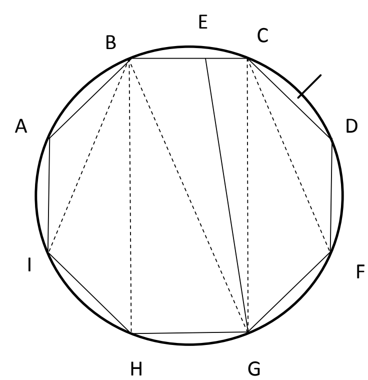

Figure 4

Consider triangle ΔCDF and let us compute its area.

$$\angle D = 180° \cdot \frac{(n-2)}{n} = \pi - \frac{2\pi}{n} \qquad (1)$$

This is the value for all angles between adjacent sides of a regular polygon.

Since ΔCDF is an isosceles triangle and the sum of the angles of a triangle sum to π:

$$\angle DCF = \angle DFC = \frac{1}{2} \cdot \left[\pi - \angle D\right] =$$

$$= \frac{1}{2} \cdot \left[\pi - \left(\pi - \frac{2\pi}{n}\right)\right] = \frac{\pi}{n} \qquad (2)$$

Using the sine law we can deduce the length of CF:

$$\frac{d}{\sin\frac{\pi}{n}} = \frac{CF}{\sin\left(\pi - \frac{2\pi}{n}\right)} = 2R \qquad (3)$$

where R is the radius of the circumscribing triangle. Therefore $CF = \dfrac{d\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}}$ and the area of ΔCDF is obtained from the sine rule for area: $S_{\Delta CDF} = \dfrac{a\cdot c\cdot \sin B}{2} = \dfrac{d^2\cdot\sin\frac{2\pi}{n}}{2}$

Continuing with computing the area to the right of EG, let us now compute the area of triangle ΔCFG:

$\angle CFG = \angle DFG - \angle DFC = \left(\pi - \frac{2\pi}{n}\right) - \frac{\pi}{n} = \pi - \frac{3\pi}{n}$. Application of the sine rule to triangle ΔCFG yields $\dfrac{CG}{\sin\angle CFG} = 2R$. We now use the convenient and elegant fact that *all* triangles that can be constructed by connecting three vertices of a regular polygon have the *same value of R* for their circumscribed circle because *they are all the same circle!*

So $\dfrac{CG}{\sin\angle CFG} = 2R = \dfrac{d}{\sin\frac{\pi}{n}}$ from which it follows that

$CG = \dfrac{d\cdot\sin\left(\pi - \frac{3\pi}{n}\right)}{\sin\frac{\pi}{n}}$. To determine the remaining angles of ΔCFG we apply the sine rule once again:

$\dfrac{d}{\sin\frac{\pi}{n}} = 2R = \dfrac{FG}{\sin\angle FCG} = \dfrac{d}{\sin\angle FCG}$. Since $n \geq 2$, $\angle FCG$ is an acute angle and therefore $\angle FCG = \dfrac{\pi}{n}$.

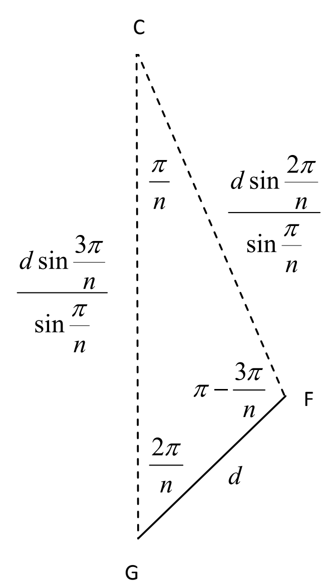

Figure 5

Finally $\angle FGC = \pi - \left(\pi - \frac{3\pi}{n}\right) - \frac{\pi}{n} = \frac{2\pi}{n}$.

The parameters of ΔCFG are summarized in Figure 5:

By the sine rule for area, and using the fact that $\sin\alpha = \sin(\pi - \alpha)$, the area of ΔCFG is:

$$S_{\Delta CFG} = \frac{FG \cdot CG \cdot \sin\frac{2\pi}{n}}{2} = \frac{d^2 \sin\frac{2\pi}{n} \sin\frac{3\pi}{n}}{2\sin\frac{\pi}{n}}$$

The expression looked so tidy that I asked myself if it might be an instance of a *generic* formula for the area of *any such triangle* in the regular polygon.

The generic expression would be of the form:

$$S = \frac{d^2 \sin\frac{k\pi}{n} \sin\frac{(k+1)\pi}{n}}{2\sin\frac{\pi}{n}} \qquad (4)$$

where k is the number of sides between the given side and the given vertex. In the case of the previous triangle (ΔCFG), *k=2*.

The first test of its validity would be to see if the area of triangle ΔCDF is of the same form. To test, I set $k = 1$ in equation (4) and found:

$$S_{\Delta CDF} = \frac{d^2 \sin\frac{1\cdot\pi}{n} \sin\frac{(1+1)\cdot\pi}{n}}{2\sin\frac{\pi}{n}} = \frac{d^2 \sin\frac{\pi}{n} \sin\frac{2\pi}{n}}{2\sin\frac{\pi}{n}} = \frac{d^2 \sin\frac{2\pi}{n}}{2}$$, so the generic expression applies to ΔCDF as well.

I now had reason to believe in the following claim:

Lemma: In a regular polygon with n sides, the angles of the triangle constructed from a given side and vertex of the polygon are $\frac{\pi}{n}, \frac{k\pi}{n}, \frac{(n-(k+1))\pi}{n}$ where *k* is the number of sides between the given side and the given vertex.

Proof: The proof is by mathematical induction for *k=1, 2, ..., n-2*. These cover all the *n-2* triangles in a regular polygon of *n* sides.

For *k=1*, we have $\frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-(1+1)\pi}{n} = \pi - \frac{2\pi}{n}$ which is correct as shown in (1) and (2) above.

Suppose the claim is true for $k = j, j < n-2$, i.e. the sizes of the angles of triangle ΔABC

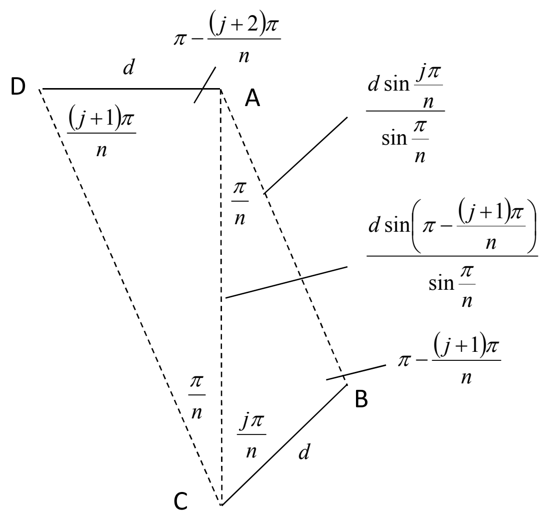

Figure 6

are: $\frac{\pi}{n}, \frac{j\pi}{n}, \frac{(n-(j+1))\pi}{n} = \pi - \frac{2\pi}{n}$ as illustrated in triangle ΔABC in Figure 6.

The sides of triangle ΔABC are determined using the sine law, as was shown above.

We show that the claim holds for $k = j+1$, i.e. that the triangle ΔACD has angle sizes $\frac{\pi}{n}, \frac{(j+1)\pi}{n}, \frac{(n-(j+2))\pi}{n}$.

By the sine law applied to triangles ΔABC and ΔACD,

$$2R = \frac{d}{\sin \frac{\pi}{n}} = \frac{d}{\sin \angle ACD} \Rightarrow \angle ACD = \frac{\pi}{n}.$$ Also,

$$\frac{AC}{\sin\angle ADC}=\frac{d\sin\left(\pi-\frac{(j+1)\pi}{n}\right)}{\sin\frac{\pi}{n}\cdot\sin\angle ADC}=\frac{d}{\sin\frac{\pi}{n}}\Rightarrow\sin\angle ADC=\sin\left(\pi-\frac{(j+1)\pi}{n}\right).$$

Since the angle $\angle ABC$ is not equal to the angle $\angle ADC$ (if it were, the polygon would not be regular), angle $\angle ADC=\frac{(j+1)\pi}{n}$.

It follows that angle $\angle DAC=\pi-\frac{(j+1)\pi}{n}-\frac{\pi}{n}=\pi-\frac{(j+2)\pi}{n}$.

By the principle of mathematical induction on the set *k=1,.., n-2*, the claim holds for all the triangles of the defined type in the regular polygon of size n. Since n is arbitrary, the lemma holds for all finite size n. *Q.E.D.*

Corollary: Using a version of the sine rule for area, the area $S_{\Delta k}$ of a generic polygonal triangle can be written:

$$S_{\Delta k}=\frac{a^2\cdot\sin B\cdot\sin C}{2\cdot\sin A}=\frac{d^2\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}{2\sin\frac{\pi}{n}}.$$

Going back to the question of the ratio between the areas to the right and left of a diagonal line connecting a side to a vertex of a regular polygon, as illustrated in Figure 7, we can now write down the parametric solution.

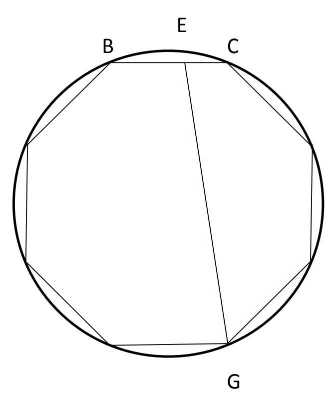

Figure 7

Solution:

$$\frac{S_{E\ldots G(clockwise)}}{S_{E\ldots G(anticlockwiase)}}=\frac{\sum_{j=1}^{k-1}\frac{d^2\sin\frac{j\pi}{n}\sin\frac{(j+1)\pi}{n}}{2\sin\frac{\pi}{n}}+\frac{EC}{d}\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}{\sum_{j=k+1}^{n-2}\frac{d^2\sin\frac{j\pi}{n}\sin\frac{(j+1)\pi}{n}}{2\sin\frac{\pi}{n}+}+\left(1-\frac{EC}{d}\right)\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}$$

where *k* is the number of sides counting anticlockwise from *G* to the side containing *E*.

For the octagon illustrated above, *k=3* and we obtain:

$$\frac{S_{ECDFG}}{S_{EGHIAB}}=\frac{\sum_{j=1}^{2}\frac{d^2\sin\frac{j\pi}{n}\sin\frac{(j+1)\pi}{n}}{2\sin\frac{\pi}{n}}+\frac{EC}{d}\sin\frac{3\pi}{n}\sin\frac{4\pi}{n}}{\sum_{j=k+1}^{6}\frac{d^2\sin\frac{j\pi}{n}\sin\frac{(j+1)\pi}{n}}{2\sin\frac{\pi}{n}+}+\left(1-\frac{EC}{d}\right)\sin\frac{3\pi}{n}\sin\frac{4\pi}{n}}$$

So we have solved the original "what if not" problem for the ratio of the two areas in a regular polygon formed by a dividing line from a point on one side to an opposing vertex.

However, the reader may still recall that the opening question to this paper was about how to characterize triangles that can support a construction where a series of chords of identical length to one of the sides, when counted off along the circumference of the circumscribing circle, returns to the starting vertex.

It turns out that using the lemma above we can now answer that question. A sufficient condition for a triangle to support the construction is given in the following theorem:

Theorem: ABC is a triangle inscribed in a circle. Using a compass, take the measure of the smallest chord and mark off identical chords around the circle, starting at the vertex at either end of the chord. It is a sufficient condition for the construction to return exactly to the starting vertex that the angles of the triangle satisfy the relation $\angle A : \angle B : \angle C = 1 : p : q$ where p,q are natural numbers with no common divisor.

Proof: To prove sufficiency, let *1 + p + q = n*. Since the sum of the angles in the triangle is π, the angles are in proportion $\frac{\pi}{n}, \frac{p\pi}{n}, \frac{(n-(p+1))\pi}{n}$. From the lemma, we know that this is a triangle that can be embedded in a regular polygon of n sides such that the smallest side (subtending the smallest angle) is a side of the polygon. We can construct the remainder of the polygon from the inscribed triangle by marking off the sequence of identical-sized chords around the circumference of the circumscribing circle. Since the chords of the construction are the sides of the polygon, the construction will return to the starting vertex. *Q.E.D.*

The condition is not a necessary condition because it is easy to change the two longer sides of the triangle so they still intersect on the circle but do not fulfill the stated proportions. The construction using the smallest chord will still return to the starting vertex.

But what if we demand that the construction hold for *all* sides of the triangle, where we allow any discrete number of revolutions to return to the starting vertex.

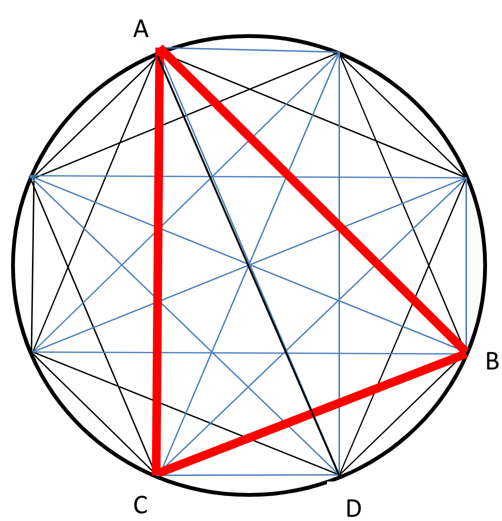

Figure 8

This situation can be observed in Figure 8, which shows an octagon in a circumscribed circle with all pairs of vertices connected by straight lines and a trangle constructed from three of the lines. The ratios of the angles in *ΔABC* is not *1:p:q.* In fact, as we will see, it is

2:3:3. However, side CB can be counted off around the circumference starting from vertex C and will return to its starting vertex C in one revolution.

Careful contemplation of the above construction (where all pairs of vertices are connected), helped reveal the full characterization as an extension to the previous theorem.

Theorem: ABC is a triangle inscribed in a circle. Using a compass, take the measure of any of the chords and mark off identical chords around the circle, starting at the vertex at either end of the selected chord. It is a necessary and sufficient condition for the construction to return exactly to the starting vertex, that the angles of the triangle satisfy the relation: $\angle A : \angle B : \angle C = p : q : r$ where p, q, and r are natural numbers with no common divisor. If p is associated with the angle subtending the selected chord, then the number of circumferences required to return to the starting vertex is the smallest l for which $2 \cdot l \cdot (p+q+r) = 0 \pmod{p}$.

Proof: The proof of sufficiency relies on a construction around the given triangle, of the type illustrated in Figure 8.

First, circumscribe the triangle with a circle. For each angle of the triangle, divide the arc it subtends into k sub-arcs, where k equals p, q or r, according to the relative size of the angle. (This is not a compass and edge construction but a virtual construction, based on the fact that the angle is divisible by a natural number.) Connect the start and end points of each sub-arc with a chord. The result is a regular polygon of n sides where $p + q + r = n$. This follows from the fact that chords that subtend the same angle are equal in length and from the claim stated in the lemma. Each triangle created from the subtended chord of the sub-arc and the lines enclosing the subtended angle, is a triangle of the type described in the lemma. Hence all the chords subtend an angle of size $\frac{\pi}{n}$.

Figure 8 illustrates the result for a polygon with $n = 8$. Observe that angle A subtends two sub-arcs and angles B and C subtend 3 sub-arcs. as specified by the construction. (In Figure 8 we have also connected the other vertices in order to appreciate the symmetry).

The proof that the construction returns to the starting vertex follows with the aid of some algebraic manipulation. Without loss of generality, assume the selected side of the given triangle is a chord that subtends the angle $\frac{p\pi}{n}$. If we mark off similar size chords from a starting vertex on the source chord, then the construction will return to the starting vertex if the number of revolutions *k*, when multiplied by the angle is a multiple of 2π. Let the multiple of 2π be *l*. Then we can write:

$$k \cdot \frac{p\pi}{n} = k \cdot \frac{p\pi}{p+q+r} = l \cdot 2\pi \quad \text{or} \quad k = \frac{2 \cdot l \cdot (p+q+r)}{p}$$

Taking $l = p$ proves sufficiency since it leads to a natural number $k = 2 \cdot (p+q+r)$. However, the lowest number of revolutions is achieved at the smallest *l* for which $2 \cdot l \cdot (p+q+r)$ is divisible by p, that is, when $2 \cdot l \cdot (p+q+r) = 0 \ (mod\ p)$.

The proof of necessity is obtained with the aid of similar algebraic manipulation. Since the sides of the triangle fulfill the conditions of the construction and all return to their starting vertex after a discrete number of revolutions, there exist constants l_A, l_B, l_C and k_A, k_B, k_C, such that:

$$\begin{aligned} k_A \cdot 2\pi &= l_A \cdot \angle A \\ k_B \cdot 2\pi &= l_B \cdot \angle B \\ k_C \cdot 2\pi &= l_C \cdot \angle C \\ \angle A : \angle B : \angle C &= \frac{k_A}{l_A} : \frac{k_B}{l_B} : \frac{k_C}{l_C} \end{aligned}$$

Since proportions do not change when multiplied by a constant, we can write:

$$\angle A : \angle B : \angle C = \frac{(l_A \cdot l_B \cdot l_C) \cdot k_A}{l_A} : \frac{(l_A \cdot l_B \cdot l_C) \cdot k_B}{l_B} : \frac{(l_A \cdot l_B \cdot l_C) \cdot k_C}{l_C}$$
$$\angle A : \angle B : \angle C = (l_B \cdot l_C \cdot k_A) : (l_A \cdot l_C \cdot k_B) : (l_A \cdot l_B \cdot k_C)$$

After dividing by the highest common factor of the three numbers in the last equation, we arrive at the required condition $\angle A : \angle B : \angle C = p : q : r$. *Q.E.D.*

The construction in Figure 8 reveals a further intriguing fact: all possible triangles whose angles are a multiple of $\frac{\pi}{8}$ can be found among the constructed angles. Similarly, a regular polygon of 360 sides with all pairs of vertices connected contains all possible triangles whose angles are drawn to an accuracy of 1º!

I leave it to the reader to contemplate the construction and convince himself of the truth of these statements.

Finite trigonometric identities

At this point, pleased with the progress of the what-if-as-well exploration, it seemed appropriate to stop. But the temptation was too great: Suppose we examine the expression for the full area of the polygon, rather than a ratio of two areas?

The area of a regular polygon is usually derived by locating the center of the circumscribed circle and dividing the polygon into n congruent triangles as illustrated in Figure 9a. Applying the sine rule to each triangle finds a relationship between the radius *R* and the side *d* and enables computing the area of a single triangle and therefore the sum of the areas of all n triangles.

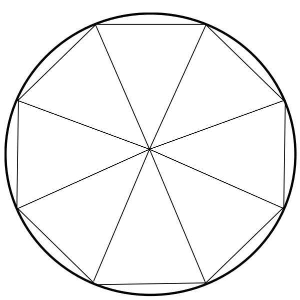

Figure 9a

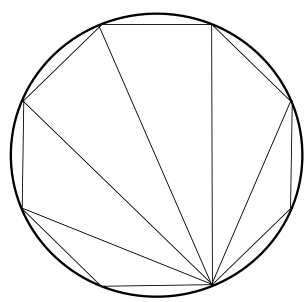

Figure 9b

The area of a single triangle in Figure 9a is $S = \frac{R^2 \cdot \sin\frac{2\pi}{n}}{2}$. Using the fact that the radius *R* of the circumscribed circle is the same *R* as in equation (3), it follows that:

$$R = \frac{d}{2\sin\frac{\pi}{n}}$$

Summing all the triangles in Figure 9a, the area of a regular polygon with *n* sides of length *d* is: $S = n \cdot \dfrac{R^2 \cdot \sin\frac{2\pi}{n}}{2} = n \cdot \dfrac{\dfrac{d^2}{4\sin^2\frac{\pi}{n}} \cdot \sin\frac{2\pi}{n}}{2} = \dfrac{n \cdot d^2 \cdot \sin\frac{2\pi}{n}}{8\sin^2\frac{\pi}{n}}$

Using the trigonometric identity for the sine of a double angle we obtain

$$S = \frac{n \cdot d^2 \cdot \cos\frac{\pi}{n}}{4 \cdot \sin\frac{\pi}{n}}.$$

But we have already expressed the area of the polygon as the sum of the triangles constructed from connecting vertices of the polygon, as illustrated in Figure 9b or in Figure 4 above. Equating the two results leads to the equation:

$$S = \sum_{k=1}^{n-2} \frac{d^2 \sin\frac{k\pi}{n} \sin\frac{(k+1)\pi}{n}}{2\sin\frac{\pi}{n}} = \frac{nd^2 \cos\frac{\pi}{n}}{4\sin\frac{\pi}{n}} \qquad (5)$$

Since the sum is over *k*, we extract terms not dependent on *k* from the sum, cancel equal terms from both sides of the equation, and arrive at:

$$\boxed{\sum_{k=1}^{n-2} \sin\frac{k\pi}{n} \sin\frac{(k+1)\pi}{n} = \frac{n}{2}\cos\frac{\pi}{n}}$$

Substituting for *n* we obtain a family of trigonometric identities. The following are the identities for the first few values of *n*:

n = 3 (equilateral triangle): $\sin\frac{\pi}{3}\sin\frac{2\pi}{3} = \frac{3}{2}\cos\frac{\pi}{3}$

n = 4 (square): $\sin\frac{\pi}{4}\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\sin\frac{3\pi}{4} = 2\cos\frac{\pi}{4}$

n = 5 (pentagon): $\sin\frac{\pi}{5}\sin\frac{2\pi}{5} + \sin\frac{2\pi}{5}\sin\frac{3\pi}{5} + \sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{2}\cos\frac{\pi}{5}$

The last identity is not an obvious one. Of course, the really intriguing property of this family of identities is that they have an elegant constructive proof!

Infinite trigonometric identities

Having seen two ways of dividing a regular polygon into triangles, it became irresistible to try and find a third way.

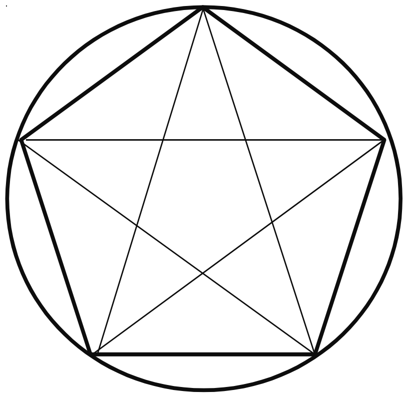

Figure 10

Consider the perimeter triangles created by connecting all pairs of vertices spaced one vertex apart, as illustrated in Figure 10 for an even and odd number of sides.

Notice that the inner polygon circumscribed by all the perimeter triangles is again a regular polygon *of the same order* as the original (outer) polygon, with reduced side length.

Let us derive the relevant quantities:

The areas of all the perimeter triangles are equal. Each is an isosceles triangle with equal side d and angles $\frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-2)\pi}{n}$. The area of overlap between two perimeter triangles is also an isosceles triangle (the "overlap" triangle) with base d and base angles $\frac{\pi}{n}$. These quantities follow directly from the constructions in Figures 5 and 6.

Using the sine rule on the overlap triangle we find $\frac{d}{\sin\left(\pi - \frac{2\pi}{n}\right)} = \frac{l}{\sin\frac{\pi}{n}}$ where l is the short side of the overlap triangle, which is also the side of the inner regular polygon. So the inner polygon has a side length of $l = \dfrac{d\sin\frac{\pi}{n}}{\sin\frac{2\pi}{n}}$. Using equation (5), the area of the inner polygon is then

$$S_{inner} = \sum_{k=1}^{n-2} \frac{l^2 \sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}{2\sin\frac{\pi}{n}} = \frac{\sin^2\frac{\pi}{n}}{\sin^2\frac{2\pi}{n}} \cdot \sum_{k=1}^{n-2} \frac{d^2 \sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}{2\sin\frac{\pi}{n}}.$$

So $S_{inner} = \dfrac{\sin^2\frac{\pi}{n}}{\sin^2\frac{2\pi}{n}} S_{outer}$. From this we can compute the area of the difference between the two polygons: $S_{diff} = \left(1 - \dfrac{\sin^2\frac{\pi}{n}}{\sin^2\frac{2\pi}{n}}\right) S_{outer}$.

If we continue to construct perimeter triangles and sum the differences between outer and inner polygons in an infinite progression, we will attain the area of the original outer polygon. Therefore

$\sum_{k=1}^{\infty} S_{diff} = \sum_{k=1}^{\infty} \left(1 - \dfrac{\sin^2\frac{\pi}{n}}{\sin^2\frac{2\pi}{n}}\right) \cdot S_{outer} = S_{outer}$. Cancelling the common factor and rearranging terms, we obtain: $\sum_{k=1}^{\infty} \left(\dfrac{\sin^2\frac{2\pi}{n} - \sin^2\frac{\pi}{n}}{\sin^2\frac{2\pi}{n}}\right) = 1$

At this point I called it a day!

Conclusion

The voyage has brought us a long way from the original problem. The process taught me several lessons.

Firstly, to get an interesting result requires a certain amount of structure. Too little structure yields dilute results and too much structure is likely to be stifling. An ideal amount goes a long way. Using what-if-as-well seems to be an effective means for enhancing the what-if-not technique.

Secondly, the domain of regular polygons appears to be a goldmine of interesting structures. I have scratched the surface. I leave it to the reader to discover more. I should add that not all the proofs fell out immediately. Some of them were originally haphazard, inaccurate, and not entirely understood. I returned to them later to add rigor.

Thirdly, I experienced very vividly what can be gained by setting a challenge to your students using simple problems. Moreover, in "What if not" exercises, we should encourage the students to change assumptions and add novel structure. They may well become as interested as I was in the process of discovery itself.

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Mathematical Competitions in Hungary: Promoting a Tradition of Excellence & Creativity

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Abstract: Hungary has long been known for its outstanding production of mathematical talent. Extracurricular programs such as camps and competitions form a strong foundation within the Hungarian tradition. New types of competitions in recent years include team competitions, multiple choice competitions, and some exclusively for students who are not in a special mathematics class. This study explores some of the recent developments in Hungarian mathematics competitions and the potential implications these changes have for the very competition-driven system that currently exists. The founding of so many new competitions reflects a possible shift in the focus and purpose of competitions away from a strictly talent-search model to a more inclusive "enrichment" approach. However, it is clear that in Hungary, tradition itself remains a strong motivating factor and continues to stimulate the development of mathematically talented students. The involvement of the mathematical community in the identification and education of young talents helps perpetuate these traditions.

Keywords: Hungary, mathematics education, mathematics competition, Olympiads, international comparative mathematics education, problem solving, creativity, mathematically talented students.

Introduction

World-renowned for its system of special schools for mathematically talented students, Hungary also has a long tradition of extracurricular programs in mathematics. Extracurricular activities play an important role in the overall education of mathematically talented students (Koichu & Andzans, 2009). Some of the longest-standing extracurricular mathematics activities in Hungary can be traced back to the 1890s and the efforts of Lorand Eötvös (Weischenberg, 1984). The Bolyai Society

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continues to coordinate many of the traditional extracurricular programs established at the end of the 19th century, including the journal KöMaL and the Eötvös Competition (now called the Kürschák Competition), along with several new competitions and programs for mathematically talented students. This article focuses on mathematical competitions in Hungary and reviews recent developments, new competitions, and modifications to traditional offerings over the past twenty years.

Methods

This article is an excerpt from a larger study on the changes in the Hungarian mathematics education system for mathematically talented students over the past twenty years (Connelly, 2010). During the study a series of in-depth interviews were conducted with current Hungarian secondary school teachers, mathematicians, professors, and other educators. Interviews were conducted by the researcher in Budapest, Hungary, with the aid of a native-Hungarian-speaking research assistant when necessary. Commentary on the history, purpose, and impact of mathematics competitions was gathered through interviews with competition directors, organizing committee members, journal editors, and past participants. The interviewees will be referred to as respondents A-U in the remainder of the article.

Historical background information was gathered from sources such as ministry of education publications, mathematical and pedagogical professional journals, and earlier dissertations in the field. Information on competition structures and rules was obtained primarily from each competition's official website and contest documentation. Example problems from a variety of Hungarian mathematics competitions are presented throughout. For larger compilations of Hungarian competition problems published in English, see Berzsenyi & Olah, 1999; Kürschák, 1963; Liu, 2001; and Székely, 1995.

Traditional Competitions

One of the most famous Hungarian mathematics competitions, the Eötvös Competition, is considered "the first mathematical Olympiad of the modern world" (Koichu & Andzans, 2009, p. 287). Founded in 1894, it was designed for students who had just graduated from secondary school; the competition consisted of three questions based on the mathematics of the secondary school curriculum. However, the competition was designed to test problem-solving ability and mathematical creativity more than factual recall. As one winner of the prize explained, "the problems are selected, however, in such a way that practically nothing, save one's own brains, can be of any help… the prize is not intended for the good boy; it is intended for the future creative mathematician" (Rado, as quoted in Wieschenberg, 1984, p. 32). Winners of the competition were awarded a monetary prize and granted automatic admission to the university of their choice. Such a reward was particularly valuable during the times of "numerus clausus" restrictions on university attendance for pupils of Jewish descent or those from the "wrong class" under communist rule. For many Hungarian students, winning such a spot may have been their only path to higher education (*D*, personal communication, 2009). From its inception, the Eötvös Competition was designed to accomplish two related goals – identification of mathematically talented students and stimulation of more creative mathematics teaching and learning (Reiman, 1997).

The prominence of this competition soon led to the development of supporting activities to help prepare students for the competition, including the publication of new types of problems each month in the journal KöMaL (*Középiskolai Matematikai és Fizikai Lapok* – the Mathematical and Physical Journal for Secondary Schools), the initiation of KöMaL's own year-long competition, and the formation of after-school "study circles" for interested students to spend additional time working on problems and practicing for the competition. Later, more competitions were founded at the school, local, regional, national, and international levels – with local competitions often serving as "feeders" into the larger, nation-wide competitions. Competitions were also established for younger students than those in secondary school, so that the students could gain experience with the competition environment prior to participation in the largest and most prestigious competitions. When the first special mathematics class was founded in 1962, students were invited to the class on the basis of their results in a local Budapest competition. In this sense, the traditional Hungarian system for educating mathematically talented students could be considered "competition-driven" – competitions were used to determine the input to the system, they drove the development of the content of the system both in the school curriculum and in supporting extracurricular activities, and competitions were used to measure the output of the system. If we take mathematical creativity to be defined as "the process that results in unusual and insightful solutions to a given problem, irrespective of the level" (Sriraman, 2008a, p. 4), then the network of mathematics competitions developed in Hungary seems to have been designed particularly to promote creativity in problem solving. The Hungarian approach served as a model for many other countries in Eastern and Central

Europe, the former Soviet Union, and for the United States (Koichu & Andzans, 2009, p. 289).

Competitions have continued to play a central role in the identification and recruitment of mathematically talented students in Hungary since the time of Eötvös. Some of the competitions for Hungarian students, such as the Kürschák and Schweitzer competitions, date back to the beginning of the 20th century. OKTV (Országos Középiskolai Tanulmányi Verseny), the "National Secondary School Academic Competition" administered by the Hungarian Ministry of Education, now holds competitions in more than 15 subjects but first began as a mathematics competition in 1923. While each competition has its own set of rules and procedures, on average a traditional Hungarian mathematics competition would consist of 3 - 6 challenging questions for students to solve individually in a time span of approximately three hours. These questions typically require a detailed proof as the solution, rather than a simple numerical answer. The results are typically scored not just on correctness but also on the quality and conciseness of the explanation, with possible additional points awarded for elegance of the proof or presentation of multiple solutions. In general, like the original Eötvös competition, most Hungarian mathematics competitions are designed to test problem solving skills and creative thinking; the goal is to identify future talented mathematicians at a young age.

One exception to this standard format among traditional competitions is the Miklós Schweitzer competition for college students, established in 1949 in memory of a young Hungarian mathematician killed during the siege of Budapest in World War II. The Schweitzer competition consists of ten questions, covering the range of topics of the classic undergraduate mathematics curriculum – analysis, algebra, combinatorics, number theory, set theory, probability theory, topology, and more. The questions are to be completed over a period of ten days, during which contestants are allowed to use any books or notes (Székely, 1996). An example question from the 2009 Schweitzer competition is given below (from Maróti, 2009):

> **7.** Let H be an arbitrary subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(M)$ of a differentiable manifold M. We say that a $\mathcal{C}^\infty$ vector field X is *weakly tangent* to the group H, if there exists a positive integer k and a $\mathcal{C}^\infty$-differentiable map $\varphi :]-\varepsilon, \varepsilon[^k \times M \to M$ such that
>
> (i) for fixed $t_1, \ldots, t_k$ the map
> $$\varphi_{t_1,\ldots,t_k} : x \in M \mapsto \varphi(t_1, \ldots, t_k, x)$$
> is a diffeomorphism of M, and $\varphi_{t_1,\ldots,t_k} \in H$;
>
> (ii) $\varphi_{t_1,\ldots,t_k} \in H = \mathsf{Id}$ whenever $t_j = 0$ for some $1 \le j \le k$;
>
> (iii) for any $\mathcal{C}^\infty$-function $f : M \to \mathbb{R}$
> $$Xf = \left.\frac{\partial^k (f \circ \varphi_{t_1,\ldots,t_k})}{\partial t_1 \ldots \partial t_k}\right|_{(t_1,\ldots,t_k)=(0,\ldots,0)}.$$
>
> Prove, that the commutators of $\mathcal{C}^\infty$ vector fields that are weakly tangent to $H \subset \mathsf{Diff}^\infty(M)$ are also weakly tangent to H.

As an open-book, multi-day exam, the Schweitzer competition maintains the Hungarian tradition of testing more than just regurgitation of facts (even high-level facts, as the SAT II or GRE may be said to test). The problems are not ones whose proofs are contained in standard textbooks, but rather require mathematical creativity coupled with advanced knowledge in order to solve.

New Styles of Competition

In recent years, more competitions have been developed that stray from the traditional format, such as the new Gordiusz competition and its corresponding competition for younger students, the Zrínyi competition. Unlike traditional Hungarian competitions and standardized exams, which often require written responses and detailed proofs in order to receive full credit, these two competitions have a multiple-choice format. Similar to the SAT in the United States, there is a "penalty for guessing" built in to the scoring, such that students receive 4 points for a correct answer, 0 for leaving the question blank, and lose 1 point for each wrong answer (Mategye, 2010). Although growing in popularity and easier to administer and score than traditional exams, the format still appears to be looked-down-upon by many mathematics educators within Hungary, who question the value of using multiple-choice exams to identify mathematically talented students (*J*, personal communication, 2009).

While there are too many regional competitions to describe each one, two new competitions in Budapest are presented here as examples of a new approach to competitions in Hungary: Mathematics Without Borders and the Kavics Kupa competition. Both are team-style competitions that originated outside of Hungary (Mathematics Without Borders started in France, Kavics Kupa in Italy) and were introduced in Hungarian schools by teachers who had spent time in the other country and brought back with them the idea for a new style of competition. Mathematics Without Borders was first established in 1989 and now operates in 46 countries around the world. This competition is focused on promoting mathematics as something fun, interesting, and accessible to the average student. The Hungarian branch of the competition began in 1994, and while still mostly limited to schools within Budapest, participation has increased significantly in recent years (Ökördi, 2008). In this competition, 9th grade classes compete to solve a set of 13 problems in 90 minutes. The students are responsible for organizing themselves, assigning someone to hand in the solution, deciding how to divide up the work, etc. On the day of the competition, students in all participating countries work on the same set of problems at exactly the same time. There is even a foreign language problem at the start of each competition, in which the problem is stated in a series of languages other than the students' own and they have to submit their solution in a foreign language as well. According to one of the competition organizers, the students find these features exciting and motivating, impressing on them the nature of mathematics as a field that crosses language and cultural barriers (*C*, personal communication, 2009).

It is particularly interesting to note that special mathematics classes are *not* permitted to compete (they can participate, but not receive scores or prizes). In this way, the organizers of the competition are trying to spark an appreciation of mathematics among all levels of students, and to "level the playing field" so as to keep otherwise interested but perhaps less-advanced students from becoming discouraged. By drawing on a variety of skills including mathematical talent and creativity, linguistic ability, paper folding or drawing skills, the competition offers the opportunity for all members of the class to feel they have contributed to the group's success. A few sample questions are presented below:

> 1. Peter has to read a book during his holidays. He calculates that he must read 30 pages a day to succeed. The first days of holidays, he doesn't respect the rhythm: he reads 15 pages a day. Anyway Peter thinks that he can keep this rhythm until he reaches half of the book, if he reads 45 pages of the second half every day. What do you think of the way he reasons? Explain. (Matematika Határok Nélkül, February 2009)
>
> 2. It is a dark and moonless night. Juliet, Rob, Tony, and Sophie are being chased by dangerous bandits. In order to escape they have to cross a precipice on a footbridge which is in a very bad state. It can hold the weight of two persons only. A light is absolutely needed to cross. The four friends have only got one lantern which will go out in half an hour. Juliet is quick; she can cross the footbridge in one minute. Rob needs two minutes to do that. Tony is slow: ten minutes will be necessary. Sophie is

even slower: she will need twenty minutes. If two friends cross together, they will move according to the rhythm of the slowest. The four of them managed to cross in less than thirty minutes. Explain their strategy.
(Matematika Határok Nélkül, February 2008)

The shared credit of a team competition presents a very different model from the traditional Hungarian competitions, which have typically focused on individual results as a means of talent identification. The Kavics Kupa competition is another team competition, originating in Italy and first held in Hungary in 2005. Teams are formed by the students themselves, and must consist of 7 total students: at least 2 boys and at least 2 girls, no more than 3 special mathematics class students, and at least one member from each grade (9-12). The organizers have also recently started a "Little Kavics Kupa" competition for 7^{th} and 8^{th} grade students. Teams work together to solve 20 questions in a period of 90 minutes. Rather than requiring detailed proofs as other competitions do, the solution for each of the twenty Kavics Kupa questions is a single numerical value between 0000 and 9999 (Pataki, 2009). This format allows for on-the-spot judging – student runners submit solutions as the team progresses, with the option to resubmit again (with a slight penalty) if the first response is incorrect. Having a numerical answer rather than requiring a written proof may also be a more appropriate format for a group competition attracting multiple grade and talent levels. Topics include number theory, algebra, geometry, and logic puzzles. Some of the questions from the 2009 competition are presented below:

6.) The sum of three numbers is 0, their product is different from zero and the sum of their cubes is equal to the sum of their fifth powers. Find the one hundredth multiple of the sum of their squares. **(45 points)**

8.) For given integers n and k denote the multiple of k closest to n by $(n)_k$. Solving the simultaneous system $(4x)_5 + 7y = 15$, $(2y)_5 - (3y)_7 = 74$ on the set of integers write, as your answer, the difference $x - y$. **(30 points)**

13.) 7 dwarfs are guarding the treasure in the cellar of a castle. There are 12 doors of the treasury with 12 distinct locks on each door, making hence 144 distinct locks altogether. Each dwarf is holding some keys and the distribution of the keys secures that any three of the dwarfs are able to open all the doors. At least how many keys are distributed among the guards, altogether? **(25 points)**

15.) The lengths of the sides of the triangle *ABC* are whole numbers, and it is also given that $A\angle = 2B\angle$ and $C\angle$ is obtuse. Find the smallest possible value of the triangle's perimeter. **(45 points)**

These two competitions represent a significant break from tradition within the Hungarian mathematics competition system, most notably in their design as team competitions but also in the way they have been set up to intentionally limit or preclude participation by students from the special mathematics classes. Traditional competitions in Hungary have served specifically as talent identification and recruitment tools, hence the importance of awarding prizes to individuals rather than groups. The success of these early competitions then drove the development of the rest of the education system for mathematically talented students, including an extensive network of supporting extracurricular activities to help students prepare for the highest competitions.

A chart summarizing some of the major national competitions and a sampling of international and regional competitions that Hungarian students may participate in, as of 2009, is given in Table 1. The full spectrum of competitions Hungarian students may participate in is not limited to those in this chart; those listed were chosen as a representative sampling to cover the most well-known national competitions and some of the newer types developed in recent years. The chart includes the name of the competition; the date it was established; the intended grade level of participants; whether it is a local, regional, national, or international competition; and a brief overview of the format of the competition, types of questions, and scoring system. In terms of geographic region, the scope of the intended, average participant pool is listed – some competitions, such as KöMaL, while primarily national in scope, do allow international entries – but since these do not make up a significant portion of the participant pool, the competition is listed here as “national”.

Table 1: Selection of Mathematics Competitions for Hungarian Students as of 2009

Competition	Est.	Grade Level	Geographic Level	Format
Kürschák (formerly Eötvös)	1894	12	National	3 questions, 5 hours; detailed proofs for solutions.
KöMaL	1894	9-12	National	Traditional section ‘B’: 10 questions per month, with 6 best questions scored; Also: ‘K’ section for 9th graders only (6 questions); ‘C’ section for grades 1-12 with 5 easier, practice exercises per month; ‘A’ section with 3 advanced problems, for Olympiad preparation. Competition runs for 9 months; solutions submitted electronically or by mail; Detailed proofs for solutions; additional points for multiple solutions.
OKTV (Országos Középiskolai Tanulmányi Verseny – “National Secondary School Academic Competition)	1923	11 & 12	National	3-5 questions in 5 hours; three categories of competition, each with their own set of questions – I. Vocational secondary school, II. Standard secondary school (non-special mathematics class), III. Special mathematics classes
Arány Daniel	1947	9 & 10	National	Three categories of competition, based on hours of mathematics per week in the students’ school (with category III reserved for special mathematics classes).

Schweitzer	1949	College	National	10 questions in 10 days, open-book. Detailed proofs required for solutions.
International Mathematics Olympiad	1959	9-12	International	6 questions, 9 hours over 2 days; detailed proofs for solutions
Competition	**Est.**	**Grade Level**	**Geographic Level**	**Format**
Kalmár László	1971	3-8	National	3^{rd} & 4^{th} grade: 6 questions. 5^{th} & 6^{th} grade: 4 questions. 7^{th} & 8^{th} grade: 5 questions. Different set of questions per grade; written answer with explanation required for solution.
Varga Tamás	1988	7-8	National	Three rounds of competition: school, county, & national. 5 questions in 2 hours, per round. Two categories per grade, based on number of hours of mathematics per week.
Hungary-Israel Olympiad	1990	9-12	Bi-national	6 questions over a period of 2 days (similar to IMO format).
Abacus	1994	3-8	National	Journal competition, similar to KöMaL.
Gordiusz	1994	9-12	National	Multi-round competition leading up to national final. 30 multiple choice questions in 90 minutes.
Mathematics Without Borders	1994	9^{th}	International / Local	Teams are whole 9^{th} grade classes (special mathematics classes may not compete); 13 questions in 90 minutes.
Zrínyi Ilona	1994	3-8	National	Multi-round competition leading up to national final. 3^{rd} & 4^{th} grade: 25 multiple choice questions in 60 minutes. 5^{th} & 6^{th} grade: 25 multiple choice questions in 75 minutes. 7^{th} & 8^{th} grade: 30 multiple choice questions in 90 minutes.
Kavics Kupa	2005	9-12	Local	20 questions in 90 minutes for a team of 7 students; solutions are numerical values.
Middle European	2007	9-12	International	Day 1: Individual competition (4

Mathematical Olympiad				questions in 5 hours). Day 2: Team competition (8 questions in 5 hours). Detailed proofs required for solution.

Competitions and Education

Competitions and their historical legacy also serve as a key source of curricular material for teachers and students; published compilations of competition problems are widely used throughout Hungary, both in classrooms and in study circles or camps focused on preparing for future competitions. More recently, a number of internet resources have been developed, providing students and teachers with searchable databases of problems going back more than 100 years. In fact, the widespread availability and active use of older problems has had an impact both on the mathematical curriculum in Hungary and on the competitions themselves. As students have access to more resources and spend more time specifically preparing for these competitions, it becomes increasingly difficult to generate new, challenging problems that have not been used in previous competitions. (*D,* personal communication, 2009).

> 30 years ago a problem appeared at the Kürschák competition for high school graduates, and now the same question can be given to 12 year olds. So certain mathematical ideas are much more widespread on one hand, and students are familiar with them at a much younger age. (*E*, personal communication, 2009)

In general, the number of mathematics competitions in Hungary continues to rise. According to one interviewee, this may be in part because of the restriction on establishing any new special mathematics classes; as a result, motivated teachers who want to provide some outlet for talented and creative students will now develop a competition for their school or region instead (*E*, personal communication, 2009). The emergence of new types of competitions in Hungary that are specifically designed to downplay the participation of the most talented students also raises some questions about the value of traditional competitions. Are they perceived as being too stressful for students? Are they no longer successful means of identifying and recruiting talented students? Perhaps the changing competition styles emphasizing teamwork over individual achievement reflect a transition in the cultural values within Hungarian education – one downside to a system with such a tradition of excellence is the potentially exclusive atmosphere established around those highly-selective programs. While the selection and encouragement of individual talents continues, these new team competitions have a different purpose – their goal is to make mathematics exciting and accessible for the average Hungarian student. Organizers of both traditional and new mathematics contests cite the motivating power of competitions and the value they add to the mathematics education system by promoting student interest and involvement in mathematics (*A*, personal communication, 2009).

However, participating in competitions is not necessarily a positive experience for all students. Some teachers mentioned the potential damaging power of a competition if a student receives a poor result and interprets that to mean he or she should not continue to pursue mathematics, as well as the risk that a student could "burn out" from the stress of too many competitions. Other concerns include the potential failure to identify mathematically talented students if their talents do not align well with the structure of the competitions (Reiman, 1997, p. 8). One interviewee highlighted an additional concern about the emphasis placed on competitions in Hungary, which was that it created a cultural expectation about the nature of mathematics that does not line up with the requirements of advanced mathematics study at the college level and beyond:

> So (for regular competitions) you must be very creative, very ingenious, very well prepared, very quick! But this is a false aspect of mathematics. So you should leave it behind before developing wrong habits. Mathematical ability can be exercised by study. This is something which must be the college experience, by the way, but here [in Hungary] they kind of overload the kids with very intensive mathematical experience for the younger generation. It's kids' play to be busy with math competitions. Just it is so trendy or this is everywhere… it is just a narrow aspect of mathematics. And there is a devaluation of [studying mathematics]… it has an effect on the image on what's happening in math classes in Hungary. Because if you enter a math class, what do you find? Solving problems. Problem solving can mean you solve a quadratic equation or that you are thinking of an IMO problem. You ask a student what is a mathematician doing? "solving problems". But this is just… it is a very important aspect of mathematics, but studying mathematics, mathematization, how do you evolve a theorem, is missing. It's almost exclusively problem solving on various levels, in Hungary. (*E*, personal communication, 2009)

Despite these concerns over the apparent trivialization of mathematics that competitions may present, researchers have identified problem solving as the aspect of mathematics most directly related to the activity of research mathematicians and stress its importance in the development of mathematical talent (Pólya, 1988; Schoenfeld, 1983; Sriraman, 2008b; Sternberg, 1996). Although there may be growing discomfort with the competition-driven mindset of the Hungarian mathematics education system and concern that it is becoming too much of a "sport" rather than studying mathematics, the continued emergence of new contests suggests that competition itself remains a key way to engage students and motivate their interest in mathematics. The founding of team competitions and competitions for students from outside the special mathematics classes reflect a possible shift in the focus and purpose of competitions away from a strictly talent-search model to a more inclusive "enrichment" approach. There remains one competition, however, which must be considered in a separate category from all the rest. It emphasizes careful thinking and stamina over speed, lasting an entire school year rather than a few hours. This is the competition run by the journal KöMaL.

KöMaL

KöMaL is a journal for secondary school students (and teachers) published once a month throughout the academic year. It includes articles by teachers and mathematicians about new, interesting topics in mathematics; a series of mathematics problems for students to solve; and the results from other national and international competitions along with a sampling of problems and their solutions. There are sections on physics and informatics as well. The KöMaL journal and competition have played a significant role in the development of mathematically talented students in Hungary over the past century, in no small part because of the prominence of previous winners:

> This can be stated for sure: Almost everyone who became a famous or nearly famous mathematician in Hungary, when he or she was a student, they took part in this contest. I actually personally do not know anyone among them who would be a counterexample to that statement.
> (Peter Hermann, KöMaL editor, in Webster, 2008)

KöMaL's competition differs from the typical mathematics competitions because it is not a single, timed event, but rather takes place over the course of an entire school year. New problems are published each month, and students have until the middle of the next month to submit their solutions. The best solutions are published in the following issue. Students accrue points over the course of the year and the winners are announced in August. Since 2000, KöMaL has operated under its current format, which offers four separate mathematics competitions: K, C, B, and A. The K section is a joint contest with their partner journal, ABACUS, offered for 9th grade students only, and consists of 6 questions per month. The C contest was established in 1984 to provide a forum for students from outside the special mathematics classes and for younger students just getting started in problem-solving competitions; this contest has five easier, so-called "practice" exercises each month. Special mathematics class students are encouraged not to compete in this section, but rather in the B section, which is KöMaL's traditional competition. The B competition publishes ten questions per month; students can submit as many solutions as they like but only the top six are scored. An additional, newer section is the A competition, started in 1993, which has three advanced problems each month. These problems are designed for students who are preparing for the IMO or other high-level mathematics competitions. A few questions from the February 2010 competition, highlighting the differences between the four levels of competition, are included below:

A. 501. Let $p > 3$ be a prime. Determine the last three digits of $\sum_{i=1}^{p}\binom{i\cdot p}{p}\binom{(p-i+1)p}{p}$ in the base-p numeral system. (5 points)

(Based on the proposal of *Gábor Mészáros,* Kemence)

A. 502. Prove that for arbitrary complex numbers $w_1, w_2,\ldots,w_n$ there exists a positive integer $k \le 2n + 1$ for which $\mathrm{Re}(w_1^k + w_2^k + \ldots + w_n^k) \ge 0$. (5 points)

B. 4242. Is there an n, such that it is possible to walk the $4\times n$ chessboard with a knight touching each field exactly once so that with a last step the knight returns to its original position? What happens if the knight is not required to return to the original position? (4 points)

B. 4247. Two faces of a cube are *ABCD* and *ABEF*. Let *M* and *N* denote points on the face diagonals *AC* and *FB*, respectively, such that *AM=FN*. What is the locus of the midpoint of the line segment *MN*? (3 points)

B. 4251. Let $p > 3$ be a prime number. Determine the last two digits of the number

$$\sum_{i=1}^{p} \binom{i \cdot p}{p} \binom{(p-i+1)p}{p}$$

written in the base-*p* numeral system. (5 points)

(Based on the proposal of *Gábor Mészáros,* Kemence)

C. 1020. The members of a small group of representatives in the parliament of Neverland take part in the work of four committees. Every member of the group works in two committees, and any two committees have one member in common from the group. How many representatives are there in the group? (5 points)

C. 1021. *P* is a point on side *AC*, and *Q* is a point on side *BC* of triangle *ABC*. The line through *P*, parallel to *BC* intersects *AB* at *K*, and the line through *Q*, parallel to *A*C intersects *AB* at *L*. Prove that if *PQ* is parallel to *AB* then *AK*=*BL*. (5 points)

K. 241. The road from village *A* to village *B* is divided into three parts. If the first section were 1.5 times as long and the second one were 2/3 as long as they are now, then the three parts would be all equal in length. What fraction of the total length of the road is the third section? (6 points)

K. 246. Four different digits are chosen, and all possible positive four-digit numbers of distinct digits are constructed out of them. The sum of the four-digit numbers is 186 648. What may be the four digits used? (6 points)

Hungarian IMO team leader István Reiman highlighted the value KöMaL provides in preparing students for advanced mathematics work as well:

> The journal plays two main roles: it helps close the gap between mathematics and physics in the real world and that taught in school, and it also serves to awaken the students' interest. From the point of view of the Olympiads, the journal is most helpful in teaching the students how to write mathematics correctly. (Reiman & Gnädig, 1994, p. 17)

As László Miklós Lovász, son of Fields Medalist László Lovász, described, "in KöMaL, it matters how hard working you are. Some people are very good at competitions, but too lazy to do KöMaL. So it's very close to what a mathematician does, as far as I can tell" (in Webster, 2008). The connection between the KöMaL competition and professional mathematics can also be understood in the context of Renzulli's three-ring model of giftedness: the challenging questions attract students with above-average intellectual ability; both the untimed and multi-month nature of the competition require significant levels of task commitment; and the awarding of extra points for submitting multiple solutions to the same problem promotes mathematical creativity (Renzulli, 1979).

In the online introduction for *C2K: Century 2 of KöMaL* (1999), one of KöMaL's special English-language issues, the editor shares a particularly appropriate story about the value of tradition:

> There is a joke about an American visitor, who, wondering about the fabulous lawn of an English mason asks the gardener about the secret of this miracle. The gardener modestly reveals that all has to be done is daily sprinkling and mowing once a week.
> - So very simple?
> - Yes. And after four hundred years you may have this grass.
>
> (Berzsenyi)

In fact, the story of the English gardener reflects not just the more than 100 year heritage of KöMaL, but also the value of engaging in mathematics on a regular, sustained basis, which is one of the key features of the KöMaL competition. The long duration and continuous effort required make the KöMaL competition quite different from other national and international competitions, and also make it one of the cornerstones of Hungary's approach to encouraging and identifying mathematically talented students. As Csapo pointed out in *Math Achievement in Cultural Context: The Case of Hungary* (1991), the expectation of success based on previous success has created a kind of self-fulfilling prophecy in Hungarian mathematics education. In other words, the tradition of excellence breeds excellence.

Conclusion

Competitions have always played a central role in the Hungarian mathematics education system for talented students, but the nature of that role is changing. Traditional competitions, such as the Kürschák, Schweitzer, and KöMaL competitions, are attracting fewer participants. This may be because winning these competitions is no longer quite as meaningful as it once was. In the past, winners were granted automatic admission to the university of their choice; this was a highly valuable prize during the Communist regime when there were a number of restrictions on admission, but is perhaps a less relevant prize in the present system. New types of competitions have been developed in recent years, including multiple choice exams and team competitions, numerous school and local competitions, and competitions for students of all ages and ability levels. This variety may explain another reason for the decline in participation rates of some of the competitions - there are now many more competitions but the same number of contestants (or fewer[2]) to go around. Still, talented students seem to be participating in as many, if not more, competitions than students did 20 or 40 years ago. Given that the Hungarian ranking system for secondary schools depends largely on students' competition results, it is clear that competitions remain an integral part of the Hungarian mathematics education system. It is clear that in Hungary, tradition itself remains a strong motivating factor and continues to stimulate the development of mathematically talented students.

[2] Demographic trends in Hungary indicate a decreasing number of school-age children

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From conic intersections to toric intersections: The case of the isoptic curves of an ellipse

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Abstract

Starting from the study of the orthoptic curves of parabolas and ellipses, we generalize to the case of isoptic curves for any angle, i.e. the geometric locus of points from which a parabola or an ellipse are viewed under a given angle. This leads to the investigation of spiric curves and to the construction of these curves as an actual intersection of a self-intersecting torus with a plane. The usage of a Computer Algebra System facilitated this investigation.

Keywords: Computer algebra systems; algebraic curves; conics; torics

I. Orthoptic curves.

Given a plane curve *C*, the *orthoptic curve* of *C* is the geometric locus of points from which *C* can be viewed under a right angle, i.e. the locus of points through which passes a pair of perpendicular tangents to the curve *C*. For example, the orthoptic curve of a parabola *P* is a line, called the directrix of *P*, and the orthoptic curve of an ellipse is a circle whose center is the center of the ellipse.

Take a parabola with the canonical equation $y^2 = 2px$. We wish to find whether there exist tangents to the parabola through a given point with coordinates (X,Y) in the plane. For that purpose we explore the set of solutions of the following system of equations: $\begin{cases} y^2 = 2px \\ y = mx - mX + Y \end{cases}$.

We find the following classification (see Figure 1):

- For a point out of the parabola (i.e. for which the inequality $y^2 > 2px$ holds), there exist two tangents;
- Through a point on the parabola, there exists a single tangent;
- There is no tangent to the parabola through an interior point (i.e. a point for which the inequality $y^2 < 2px$).

As already mentioned, the directrix of the parabola is the geometric locus of the points from which the parabola is seen in a right angle (Figure 1a). We check graphically and justify geometrically that the directrix divides the exterior of the parabola into two regions. We illustrate this in Figure 1 for the ellipse whose equation is $x^2 + 4y^2 = 1$. One of them is the locus of points from which the parabola is seen under an acute angle (Figure 1b), the other one is the locus of points from which the parabola is seen under an obtuse angle (Figure 1c).

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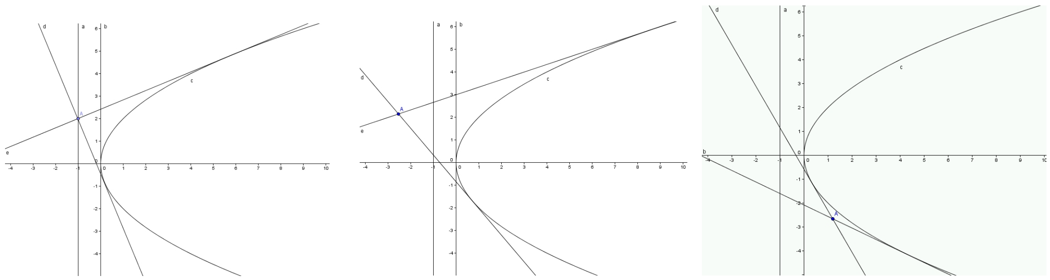


(a) **(b)** **(c)**

Figure 1: Viewing angles of the parabola

The last result intrigued the authors and the in-service teachers who attended a professional development course in Analytic Geometry. In order to obtain more details, we explored the locus of points from which the parabola is seen under a given angle. To our surprise, for specific angles we found branches of hyperbolas (see Figure 2, for the same ellipse as above): an unfamiliar relationship between parabolas and hyperbolas!

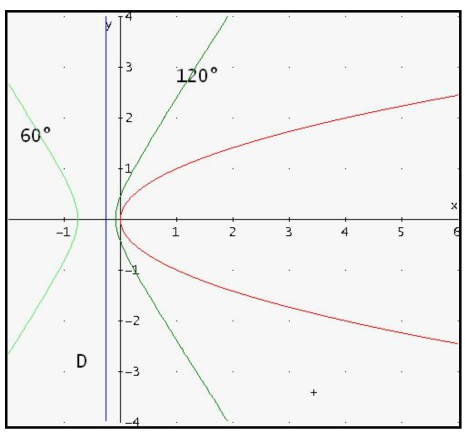
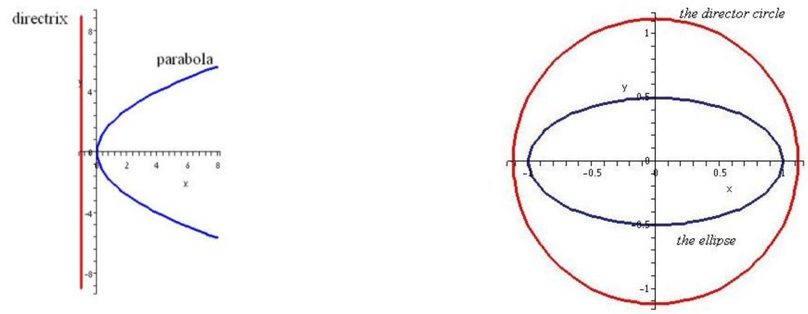


Figure 2: An unfamiliar relationship between parabolas and hyperbolas

Conic sections are an important domain in classical Mathematics. Within this domain, there exist topics which drew little attention in the past. We wish to shed light on one of these topics. Specifically, we are interested in geometric loci of points from which a given conic section is viewed under given angles. This means to look for points from which originate pairs of rays which are tangent to the conic and create a given angle θ.

The orthoptic curve of a parabola, i.e. its directrix, is displayed on Figure 3a, and in Figure 3b the orthoptic curve of an ellipse is shown. Actually, the orthoptic curve of an ellipse is the whole of a circle concentric with the ellipse. In the case of a hyperbola, finer tuning is necessary: first, an orthoptic curve may not exist and second, when it exists, holes appear.



(a) Parabola (b) Ellipse

Figure 3: Orthoptic curves.

Given a curve C and an angle θ, the geometric locus of points through which passes a pair of tangents to the curve C making a angle of θ is a curve called a *θ-isoptic curve*. This curve can adopt very different forms. It happens that a *θ-isoptic curve* is given by a polynomial equation of higher degree (up to degree 4 for conics) in two real variables x and y.

Our study demands the usage of a couple of general tools from the theory of plane algebraic curves. A central tool used is Bezout's theorem (Kirwan 1992, page 54). For the situations described in the paper, the theorem states that the intersection of a line (a plane curve of degree 1) and a conic (a plane curve of degree 2) contains at most two points, possibly identical. The point of contact of a tangent to a conic with the conic itself is of multiplicity 2, thus there is nothing left for another point of intersection. As the points of intersection of a conic and a line are determined by the solutions of a quadratic polynomial, the fact that a line is a tangent to a conic is determined by the vanishing of a certain discriminant.

Most of the computations and the drawings in this paper have been performed using a Computer Algebra System (generally denoted by the initials CAS), either Derive or Maple. Figure 1 has been drawn using GeoGebra[1].

II. Isoptic curves of an ellipse.

We will work with the one-parameter equation $x^2 + k^2 y^2 = 1,\ k > 0$. No loss of generality occurs because of this decision, as it can be easily shown that any ellipse is similar to one of the ellipses in the above one-parameter family. The algebraic computations will be easier than with the canonical 2-parameter presentation.

We address now the following question: **Given an angle θ, what is the geometric locus of all the points in the plane from which the ellipse is viewed under the angle θ?** In other words what is the geometric locus of all the points in the plane through which passes a pair of tangents to the ellipse E making an angle of θ?

[1] http://www.geogebra.org

For $\theta = 90^0$, the answer can be found in the literature, but we prefer to expose this situation as a particular example. First note that four pairs of perpendicular tangents to the given ellipse are trivially found, namely pairs of tangents parallel to the coordinate axes. In Figure 4 we show the ellipse corresponding to $k = 2$ and its four tangents parallel to the axes. The points A, B, C, D belong to the geometric locus we are interested in.

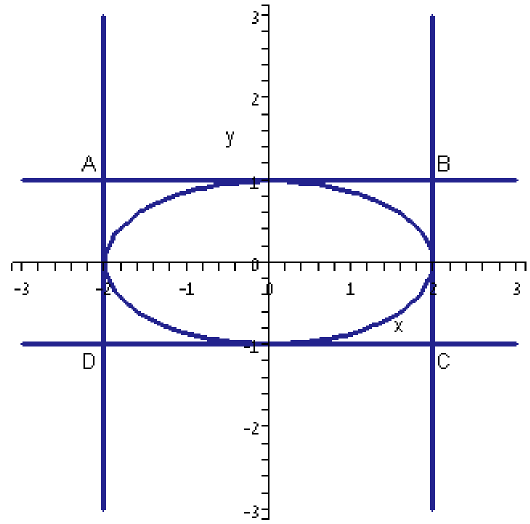


Figure 4: Tangents to the ellipse, parallel to the coordinate axes.

We consider now the general case: none of the tangents in the pair is parallel to the *y*-axis, therefore both have a slope. Take a point $T(x_0, y_0)$; a line L through T and non-parallel to the *y*-axis has an equation of the form $y = m(x - x_0) + y_0$, where m is the slope of L. An ellipse has no singular point, thus Bezout's theorem (Berger 1996, section 16.4) ensures that the line L is tangent to the ellipse E if, and only if, it has a "double" point of intersection with the ellipse. The possible slopes are the following:

(9) $$m_1 = \frac{\sqrt{x_0^2 + y_0^2 k^2 - 1} - k x_0 y_0}{k\left(1 - x_0^2\right)} \quad \text{and} \quad m_2 = -\frac{\sqrt{x_0^2 + y_0^2 k^2 - 1} + k x_0 y_0}{k\left(1 - x_0^2\right)},$$

where $x_0^2 + y_0^2 k^2 > 1$. The tangents are perpendicular if, and only if, $m_1 m_2 = -1$, i.e.

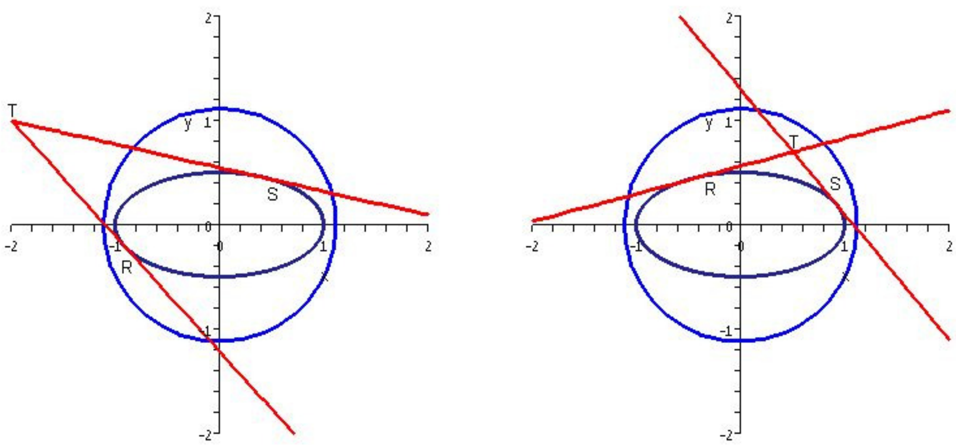
(10) $$\frac{k^2 y_0^2 - 1}{k^2\left(x_0^2 - 1\right)} = -1.$$

This equation is equivalent to

(11) $$\begin{cases} x_0^2 + y_0^2 = 1 + \dfrac{1}{k^2} \\ x_0 \neq -1, 1 \end{cases}.$$

It follows that the geometric locus of points from which the ellipse is viewed under a right angle without a side being parallel to the y-axis is a subset of the circle C_k whose center is at the origin and whose radius is equal to $\sqrt{1+1/k^2}$. Conversely, through every point on the curve defined by Equation (11) passes a pair of perpendicular tangents. Moreover, through the four points A, B, C, D one of the tangents to the ellipse is parallel to the *y*-axis, therefore it cannot be described by Equations (10) and (11), but these four points complete the curve given by Equation (11) to be the circle C_k. This circle is called *the* ***director circle*** or ***the orthoptic circle*** of the ellipse *E* (Spain 1963, page 79). Actually the director circle of the given ellipse is the circumcircle of the rectangle ABCD``. Figure 3 shows the ellipse and its director circle for $k = 2$.

Note that from a point exterior to the orthoptic circle, the ellipse is viewed under an acute angle, and from a point interior to the circle, the ellipse is viewed under an obtuse angle. The proof is easy to write; examples are displayed in Figure 5.



(a) Acute angle **(b) Obtuse angle**

Figure 5: Acute and obtuse angle for viewing the ellipse.

For tangents non parallel to the *y*-axis whose respective slopes are m_1 and m_2, the condition is equivalent to $\dfrac{m_1 - m_2}{1 + m_1 m_2} = \tan\theta$.

The requested geometric locus is determined by the equation

$$\frac{2k\sqrt{k^2 y_0^2 + x_0^2 - 1}}{1 - k^2 (x_0^2 + y_0^2 - 1)} = \tan\theta . \tag{13}$$

Denote $t = \tan\theta$ and square both sides of this equation. We obtain the following equation:

$$\frac{4k^2 (k^2 y_0^2 + x_0^2 + 1)}{\left(1 - k^2 (x_0^2 + y_0^2 - 1)\right)^2} = t^2 . \tag{14}$$

Actually, the vanishing points of the denominator are points though which pass a suitable pair of tangents, one of the tangents being parallel to the *y*-axis. Multiplying both sides by the common denominator, we obtain the following polynomial equation of degree 4:

(15)
$$k^4t^2x^4 + 2k^4t^2x^2y^2 - 2k^2(k^2t^2 + t^2 + 2)x^2 + k^4t^2y^4$$
$$- 2k^2(k^2t^2 + 2k^2 + t^2)y^2 + k^4t^2 + 2k^2(t^2 + 2) + t^2 = 0$$

Equation (15) determines a plane curve called a *spiric curve*. See Wassenaar (2003a) and Ferréol (2001). A spiric curve is the intersection of a plane with a torus; see Wassenaar (2003b) and the appendix, at the end of the present paper.

The geometric locus of points from which the given conic ellipse E is viewed under a given angle θ is a called the ***θ-isoptic curve*** of the ellipse E for the given angle θ; we denote this curve by $OPT(k,\theta)$. The orthoptic curve of E is $OPT(k,90)$. In Figure 6, we show the curves $OPT(2,45)$, $OPT(2,135)$ and $OPT(2,90)$. The equation describing together the first one and the last one is $16x^4 + 32x^2y^2 + 16y^4 - 56x^2 - 104y^2 + 41 = 0$. This equation can be written under the form

(16)
$$\left(x^2 + y^2\right)^2 - \frac{7}{2}x^2 - \frac{13}{2}y^2 + \frac{41}{16} = 0\,.$$

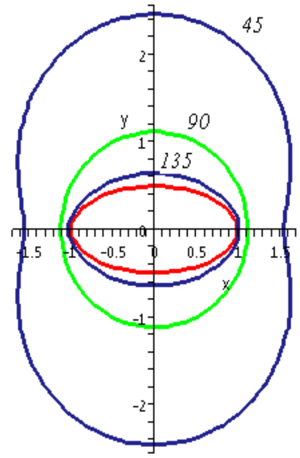


Figure 6: Examples of isoptic curves of an ellipse.

Note that the squaring before Equation (14) has an important consequence: the angles θ and 180^{o}-θ are studied at the same time. Therefore Equation (16) describes in fact the union of two isoptic curves, namely $OPT(k,\theta)$. and $OPT(k,180-\theta)$. We will call this union a ***bi-isoptic curve***, and will denote it by $OPT_2(k,\theta)$. In Figure 6, the union of $OPT(2,45)$ and $OPT(2,135)$ is $OPT_2(2,45)$, which can be also denoted by $OPT_2(k,135)$.

As an example, let us consider the case where $k = 4$, i.e. a case with greater eccentricity than what we had previously. In Figure 7, we show the ellipse whose equation is $x^2 + 16y^2 = 1$ and the bisoptic curve for $\mathbf{\theta} = 45^\circ$ and $\mathbf{\theta} = 135^\circ$ (recall that they are obtained simultaneously). Its equation is

(17)
$$x^4 + 2x^2y^2 + y^4 - \frac{19}{8}x^2 - \frac{49}{8}y^2 + \frac{353}{256} = 0\,.$$

Here too the isoptic curve is a spiric curve. In fact Equation (15) shows that it is this is a general situation.

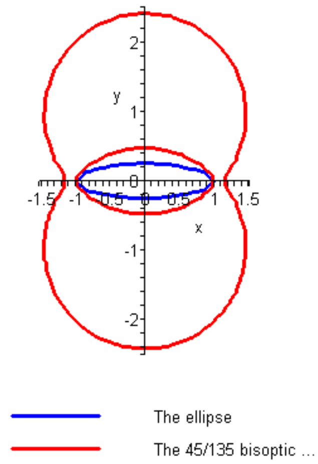


Figure 7: An ellipse with two isoptic curves

Additional remark: take a variable ellipse, with fixed vertices on the major axis. When the length of the minor axis tends to 0, then the ellipse "tends to" a line segment (in a sense to be defined of course), and the isoptic curve "tends to" a form built as the union of two symmetric arcs of circles, recalling a well known theorem: let PQ be a segment in the plane and let α be a given angle. The locus of points M in the plane such that $\angle PMQ = \alpha$ is the union of two symmetric arcs of circles whose endpoints are P and Q.

III. Reconstruction of the bisoptic curve as a toric section.

The equations we obtained in previous section for the bisoptic curves of an ellipse showed that these bisoptic curves are actually spiric curves, i.e. intersection a torus with a plane parallel to the torus axis. In this section we wish to reconstruct the torus and the plane from the knowledge of the spiric curve.

The general equation of a torus whose axis is the z-axis is as follows:

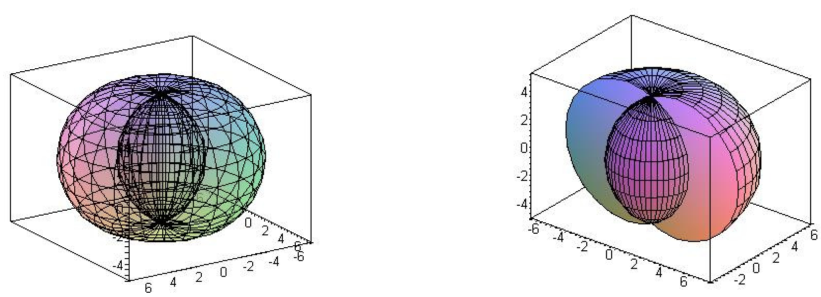
(18) $$\left(x^2 + z^2 + R^2 - r^2 + y^2\right)^2 - 4R^2\left(x^2+y^2\right) = 0,\ R > 0, r > 0.$$

If $0 < R < r$, then the torus will be called a ***self-intersecting torus***. See Figure 8 for two examples.

(a) Non self intersecting **(b) self-intersecting**
R=5, r=2 ***R=4, r=5***

Figure 8: Two tori.

A self intersecting torus is a surface which comprises not only what is visible on Figure 8b, but also an internal part, as shown in Figure 9. As any torus, it is a surface of revolution generated by a circle being revolved about a line; in the case of a self-intersecting torus, the line intersects the circle, whence the internal part, as shown on Figure 9b.



(a) **The torus** **(b)View from inside**

Figure 9: Self-intersecting torus.

For the reader's convenience, we presented here tori and self-intersecting tori with the *z*-axis as their axis of revolution. In what follows we will change our point of view in order for the 3D-geometry we explain to be coherent with the plane configuration that we studied in the previous sections.

Note that any ellipse in the *xy*-plane is similar to an ellipse whose equation is $x^2 + k^2 y^2 = 1,\ k > 1$. Therefore, WLOG, we consider an ellipse E with this equation. As $k > 1$, the torus we are looking for has the x-axis as its axis or revolution. Thus, the equation of the torus has the following form:

$$(19) \qquad \left(x^2 + y^2 + z^2 + R^2 - r^2\right)^2 - 4R^2\left(y^2+z^2\right) = 0 .$$

Expanding the left-hand side, and multiplying by a suitable constant, w show that Equation (19) is equivalent to

$$(20) \qquad \begin{aligned} &k^4t^2x^4 + 2k^4t^2x^2y^2 + 2k^4t^2x^2\left(R^2 - r^2 + z^2\right) + k^4t^2y^4 \\ &- 2k^4t^2y^2\left(R^2 + r^2 - z^2\right) + k^4t^2\left(R^4 - 2R^2\left(r^2 + z^2\right) + r^4 - 2r^2z^2 + z^4\right) = 0. \end{aligned}$$

Equation (20) is equivalent to Equation (15) if, and only if, the following system of equations holds:

$$(21) \qquad \begin{cases} -2k^2\left(k^2t^2 + t^2 + 2\right)x^2 = 2k^4t^2x^2\left(R^2 - r^2 + z^2\right) \\ 2k^2\left(k^2t^2 + 2k^2 + t^2\right)y^2 = 2k^4t^2y^2\left(R^2 + r^2 - z^2\right) \\ k^4t^2 + 2k^2\left(t^2 + 2\right) + t^2 = k^4t^2\left(R^4 - 2R^2\left(r^2 + z^2\right) + r^4 - 2r^2z^2 + z^4\right) \end{cases} .$$

We simplify the first two equations, obtaining thus:

$$(22) \qquad \begin{cases} -\left(k^2t^2 + t^2 + 2\right) = k^2t^2\left(R^2 - r^2 + z^2\right) \\ \left(k^2t^2 + 2k^2 + t^2\right) = k^2t^2\left(R^2 + r^2 - z^2\right) \end{cases}.$$

By sidewise addition we obtain finally:

$$(23) \qquad R^2 = \frac{k^2 - 1}{k^2t^2} .$$

By a sidewise substraction of the first equation from the third one in (21), we obtain after simplification

$$(24) \qquad k^2t^2\left(r^2 - z^2\right) = k^2t^2 + k^2 + t^2 + 1 .$$

Let $p = r^2$ and $q = z^2$. Thus Equation (24) can be written under the following form

$$(25) \qquad k^2t^2(p - q) = k^2t^2 + k^2 + t^2 + 1 ,$$

Whence

$$(26) \qquad q = \frac{k^2\left(pt^2 - t^2 - 1\right) - t^2 - 1}{k^2t^2} .$$

A substitution into the third equation in (21) yields

$$(27) \qquad k^4t^2 + 2k^2\left(t^2 + 2\right) + t^2 = k^4t^2\left(R^4 - 2R^2(p + q) + p^2 - 2pq + q^2\right).$$

We substitute for q using Equation (26) and finally we obtain

$$(28) \qquad p = \frac{k^2\left(t^2 + 1\right)}{t^2\left(k^2 - 1\right)},$$

which is equivalent to

$$(29) \qquad r^2 = \frac{k^2\left(t^2 + 1\right)}{t^2\left(k^2 - 1\right)}.$$

It follows easily that

$$(29) \qquad z^2 = \frac{t^2 + 1}{k^2t^2\left(k^2 - 1\right)}.$$

In conclusion we have:

$$\left\{\begin{array}{l} R = \dfrac{\sqrt{k^2-1}}{kt} \\ r = \dfrac{k\sqrt{t^2+1}}{t\sqrt{k^2-1}} \\ z = \dfrac{\sqrt{t^2+1}}{kt\sqrt{k^2-1}} \end{array}\right. . \tag{30}$$

It follows that

$$\left\{\begin{array}{l} r - R = \dfrac{k^2\left(\sqrt{t^2+1}-1\right)+1}{kt\sqrt{k^2-1}} \\ r - R - z = \dfrac{\sqrt{k^2-1}\cdot\left(\sqrt{t^2+1}-1\right)}{kt} \end{array}\right. \tag{31}$$

Therefore $r - R > 0$ and $r - R - z > 0$. This shows that the given bisoptic curve is the intersection of a self-intersecting torus with a plane parallel to the *y*-axis at distance *z*, such that we have a union of two loops.
If we substitute now the values for the squares of *R*, *r* and *z* in the original equation of the torus we get a quartic equation showing that the bisoptic curve under study is actually a spiric curve:

$$\left(x^2 + y^2 - \frac{k^2t^2 + t^2 +^2}{k^2t^2}\right)^2 = \frac{4\left(k^2-1\right)}{k^2t^2}y^2 + \frac{4\left(t^2+1\right)}{k^4t^4}. \tag{32}$$

For example, if t=1 and k=2, we have the following system of equations:

$$\left\{\begin{array}{l} x^2 + 4y^2 = 1 \\ \left(x^2 + y^2 - 1.75\right)^2 = 3y^2 + 0.5 \end{array}\right. \tag{33}$$

The first equation represents the ellipse and the second equation represents the bisoptic curve for $\theta = 45^o/135^o$ in Figure 10.

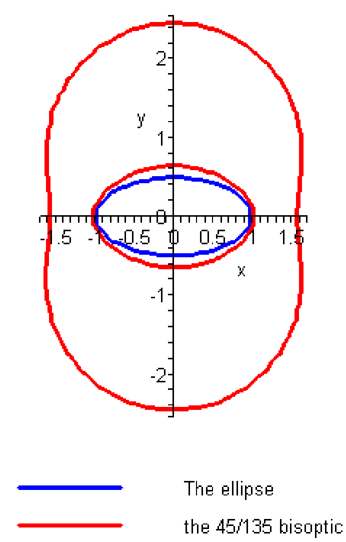


Figure 10: An ellipse and one of its bisoptic curve: plane configuration

We show two views of the 3D-configuration in Figure 11. Note the internal part of the torus (consequence of the self-intersection).

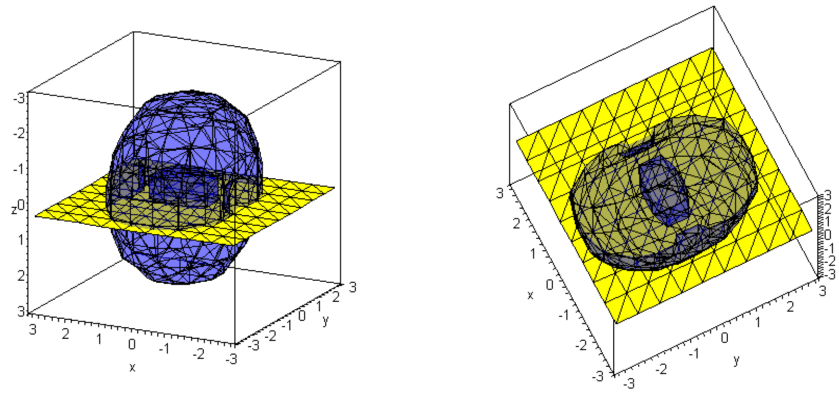


Figure 11: The 45/135 bisoptic of an ellipse as a toric intersection

IV. Limiting cases.

In this section we wish to show the internal coherence of what has been studied in the previous section. For this purpose, recall that the bisoptic curve under study is determined by two parameters: the positive parameter k determines the shape of the ellipse (namely its eccentricity), and the parameter t encodes the angle under which the ellipse is viewed. We study the behavior of a bisoptic curve when one of the parameters, either t or k , is fixed and the second one tends to infinity.

1. Fixed ellipse and variable angle.

We recall that an ellipse E whose equation is $x^2 + k^2 y^2 = 1,\ k > 0$, is viewed under an angle θ such that $\tan\theta = t$ from all the points on a curve whose equation is

$$k^4t^2x^4 + 2k^4t^2x^2y^2 - 2k^2(k^2t^2 + t^2 + 2)x^2 + k^4t^2y^4$$
$$- 2k^2(k^2t^2 + 2k^2 + t^2)y^2 + k^4t^2 + 2k^2(t^2 + 2) + t^2 = 0$$

as shown in Equation (15).

The case t=0 corresponds to points through which passes only one tangent to the ellipse, i.e. the points of the ellipse itself. For the general case, i.e. $t \neq 0$, we can divide out the left-hand side of Equation (15) by t^2. We obtain:

(34)
$$k^4x^4 + 2k^4x^2y^2 - 2k^4x^2 - \frac{2kx(t+2)}{t^2} + k^4y^4$$
$$- \frac{2k^4y^2(t^2+2)}{t^2} - 2k^2y^2 + k^4 + \frac{2k^2(t^2+2)}{t^2} = -1$$

If t tends to infinity, i.e. the angle tends to 90^o (and note that 90^o=180^o-90^o, whence the two components of the bisoptic tend to the same curve), then the equation becomes

(35) $$k^4x^4 + 2k^4x^2y^2 - 2k^4x^2 - 2k^2x^2 + k^4y^4 - 2k^4y^2 - 2k^2y^2 + k^4 + 2k^2 = -1$$

which is equivalent to:

(36) $$\left(k^2\left(x^2 + y^2\right)\right) - \left(k^2 + 1\right)^2 = 0\,.$$

This equation can be written as follows:

(37) $$x^2 + y^2 = \frac{k^2+1}{k^2},$$

which is the equation of the director circle (v.s. Equation (11)).

Figure 12 shows the bisoptic curves (always in red) for the ellipse E whose equation is $x^2 + \frac{4}{9}y^2 = 1$ (always in blue) and $t = 1,5,10$ from left to right, the rightmost figure showing the limiting case for t going to infinity, i.e. the ellipse with its director circle. We can see how the two components get closer and look more and more circular. At the limit, they coalesce into one circle.

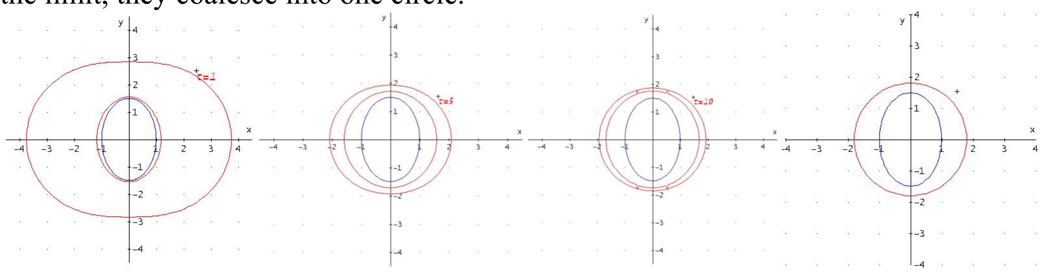


Figure 12: Fixed ellipse and variable angle

2. Variable ellipse and fixed angle.

If $k\to\infty$, then the limit configuration for the ellipse is the segment AB on the x-axis, where A has coordinates (-1,0) and B has coordinates (1,0). A well known result of plane geometry is that the bisoptic becomes the union of two circles from which that segment AB is seen by an angle whose tangent is t, the two points A and B being excepted. We can check this with Equation (15), when $k\to\infty$.
Following a method similar to the previous subsection, we divide out the left-hand side of Equation (15) by k^2; we obtain:

(38)
$$t^2x^4 + 2t^2x^2y^2 - \frac{2t^2x^2(k^2+1)}{k^2} - \frac{4x^2}{k^2} + t^2y^4 - \frac{2t^2y^2(k^2+1)}{k^2} - 4y^2 + \frac{t^2(k^4+2k^2+1)}{k^4} + \frac{4}{k^2} = 0$$

Now, if $k\to\infty$, we obtain the equation:

(39) $$t^2x^4 + 2t^2x^2y^2 - 2t^2x^2 + t^2y^4 - 2t^2y^2 - 4y^2 + t^2 = 0$$

which is equivalent to:

(40) $$x^4 + 2x^2y^2 - 2x^2 + y^4 - \frac{2y^2(t^2+2)}{t^2} + 1 = 0.$$

The left-hand side of Equation (40) can be written as the product of two quadratic polynomials, namely Equation (40) is equivalent to the following equation:

(41) $$\left[x^2 + \left(y - \frac{1}{t}\right)^2 - \left(1 + \frac{1}{t}\right)^2\right]\cdot\left[x^2 + \left(y + \frac{1}{t}\right)^2 - \left(1 + \frac{1}{t}\right)^2\right] = 0$$

This is the equation of the union of two symmetric circles going through the points A

(-1,0) and B (1,0) and having the points $C_1(0,1/t)$ and $C_2(0,-1/t)$ as their centers, respectively.
Figure 13 shows two examples for $t=1$, i.e. $\theta=45^o$, and $k=2,5$, and at the rightmost the limiting case for infinite k.

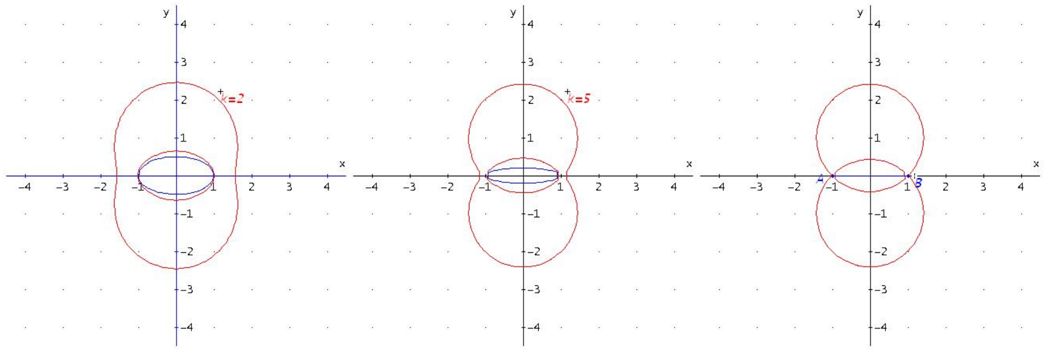


Figure 13: Fixed angle and variable eccentricity

V. Final remarks.

The work in this paper originated in a course in Analytic Geometry for in-service teachers, based on the usage of technology. The study of isoptic curves of ellipses led to the study of plane curves of higher degree, a topic which is not studied in a regular curriculum for teacher trainees. A byproduct was the development of mathematical activities based on the usage of a Computer Algebra System and of other kinds of mathematical software. The usage of technology enabled the participants to work according to an experimental method in order to develop new mathematical knowledge.
The consideration of these curves yielded the participants in the course a more profound insight into the geometry of plane curves and more understanding of the interplay between different mathematical fields, such as 2D geometry, 3D geometry, algebra and computer algebra. Introducing the spiric curves in the context of locus of viewing angles opens up opportunities for students to view and explore analytically curves of degree higher than 2.
With a broader scope than what has been presented in the course, the present study helps to enhance the understanding of isoptic curves and of spiric curves, as it introduces a spiric curve as the union of two connected components appearing together as the intersection of a self-intersecting torus with a plane parallel to its axis of revolution. The visualization provided by a Computer Algebra System gave a strong added value to the topic, and is an important component of the revival of the topic in recent years.
The isoptic curves present sometimes points of inflection, but not always. This can be studied by letting the angle θ vary for a fixed value of the parameter k, or by enquiring the influence of variations of parameter k for a fixed angle θ. The authors address this issue in a companion paper.
Other teams work in this field, from another point of view; for example, see (Miernowski and Mosgawa, 2001), (Szałkowski, 2005) and the papers referenced there. Together with the study of bisoptic curves of ellipses, the authors worked on

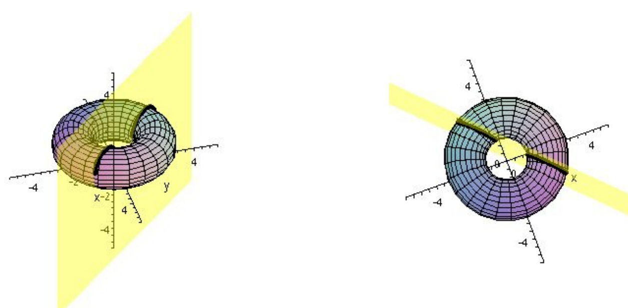
bisoptic curves of hyperbolas. This case is more complicated and new phenomena appear. This is the topic of a subsequent paper.

Appendix: the spiric of Perseus

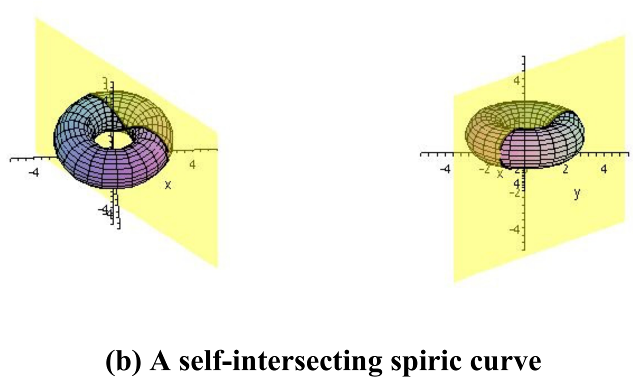
The intersection of a torus with a plane parallel to the axis of the torus is called a spiric of Perseus. The first reference to Perseus is in the writings of Proclus, where he says that Perseus found the spiric in the same way Apollonius studied conics[2] (see MacTutor). A spiric curve can have different forms according to the respective positions of the torus and the plane. It can be the union of two disjoint loops (Figure 16 (a)), one self intersecting loop (Figure 16 (b)), one non intersecting loop without a point of inflection (Figure 16 (c)) or a non intersecting loop with four points of inflection (Figure 16 (d)). Figure 16 shows the intersections of the planes whose respective equations are $x+y=1$, $x+y=\sqrt{2}$, $x+y=2$ and $x+y=3$ with the torus given by the following parametric representation:

$$\begin{cases} x(u,v)=(2+\cos u)\sin v \\ y(u,v)=(32+\cos u)\cos v \ , \ u,v\in[0,2\pi]. \\ z(u,v)=1+\sin(u) \end{cases}$$

We begin with a 3D presentation. For each case, two different views of the torus and of the spiric curve are shown.

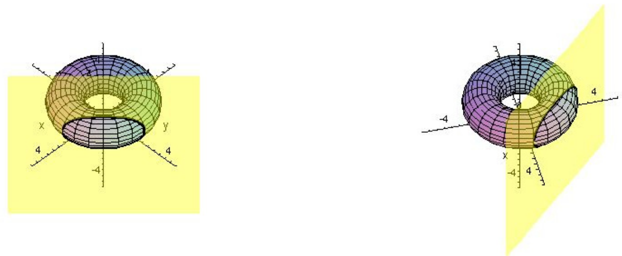


(a) The union of two disjoint loops

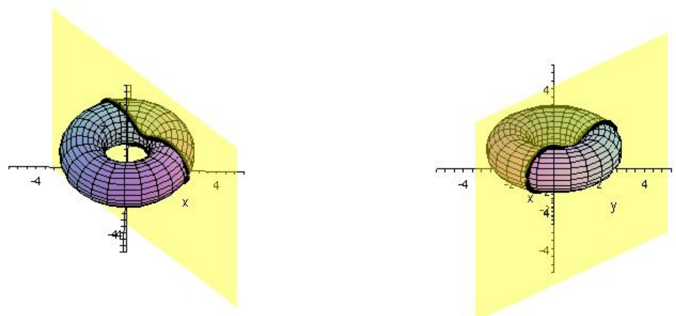


(b) A self-intersecting spiric curve

[2] Other sources refer to Menaechmus as the discoverer of conic sections (see http://www-groups.dcs.st-and.ac.uk/~history/Curves/Spiric.html).



(c) A non self-intersecting spiric curve without a point of inflection



(d) A non self-intersecting non convex spiric curve

Figure 13: Spiric curves

The authors can also provide to the interested reader files of animations built using DPGraph. Please ask by email.

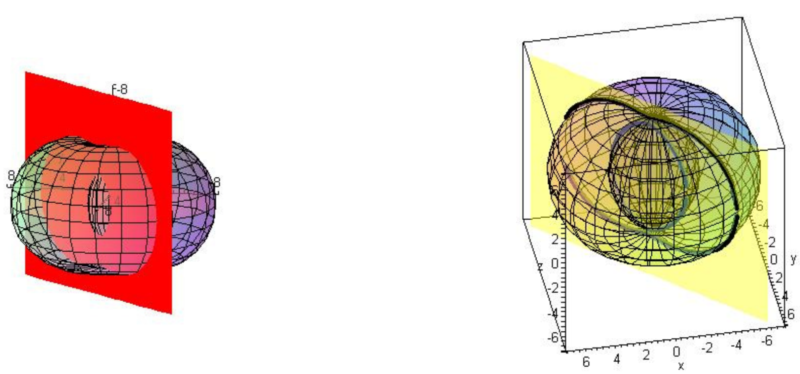


Figure 14: Plane cut of a self-intersecting torus

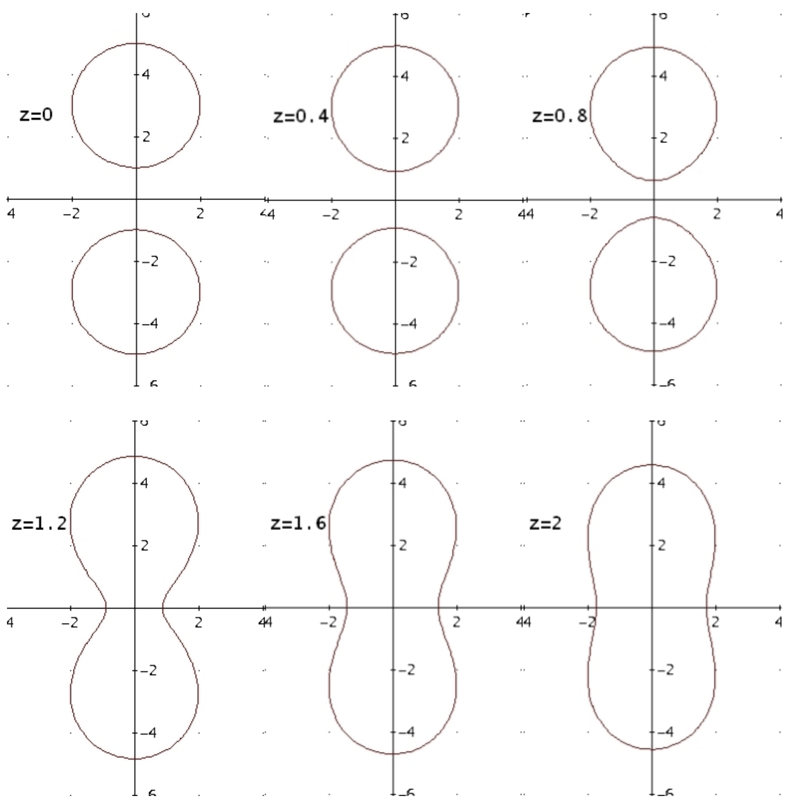
Now we present the 2D situation. In the literature, spiric curves are generally presented as plane sections of a "regular" torus, i.e. a non self-intersecting one. In this paper, we show that the bisoptic curves of ellipses are plane sections of self-intersecting tori, extending somehow the notion of a spiric curve. We include now the case where the curve has two disjoint components. In what follows we show spiric curves, according to our new point of view. The notations are those of Section III.

The equation of a torus with the *x*-axis as its axis of revolution is (Equation (15) above):

$$\left(x^2 + y^2 + z^2 + R^2 - r^2\right)^2 - 4R^2\left(y^2 + z^2\right) = 0$$

We show spiric curves obtained as plane sections of two different tori: T_1 whose characteristics are R=3 and r=2, and T_2 given by R=2 and r=3.

The torus T_1 is a regular one. His equation is $\left(x^2 + y^2 + z^2 + 5\right)^2 - 36\left(y + z\right) = 0$.
The curves shown in Figure 15 are the spiric curves corresponding to $z = 0, 0.4, 0.8, 1.2, 1.6, 2, 2.4, 2.8, 3.2$.



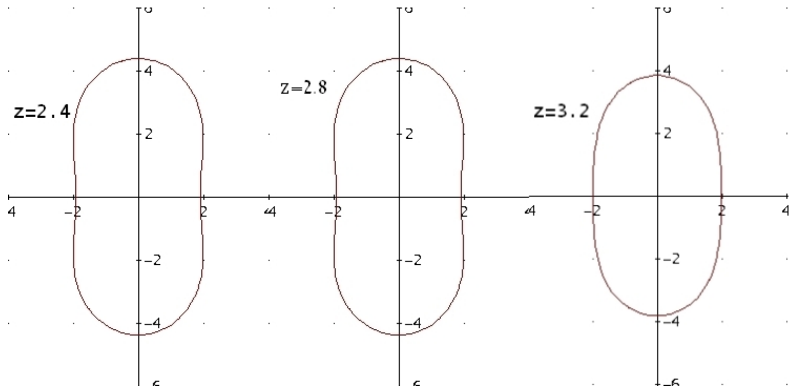
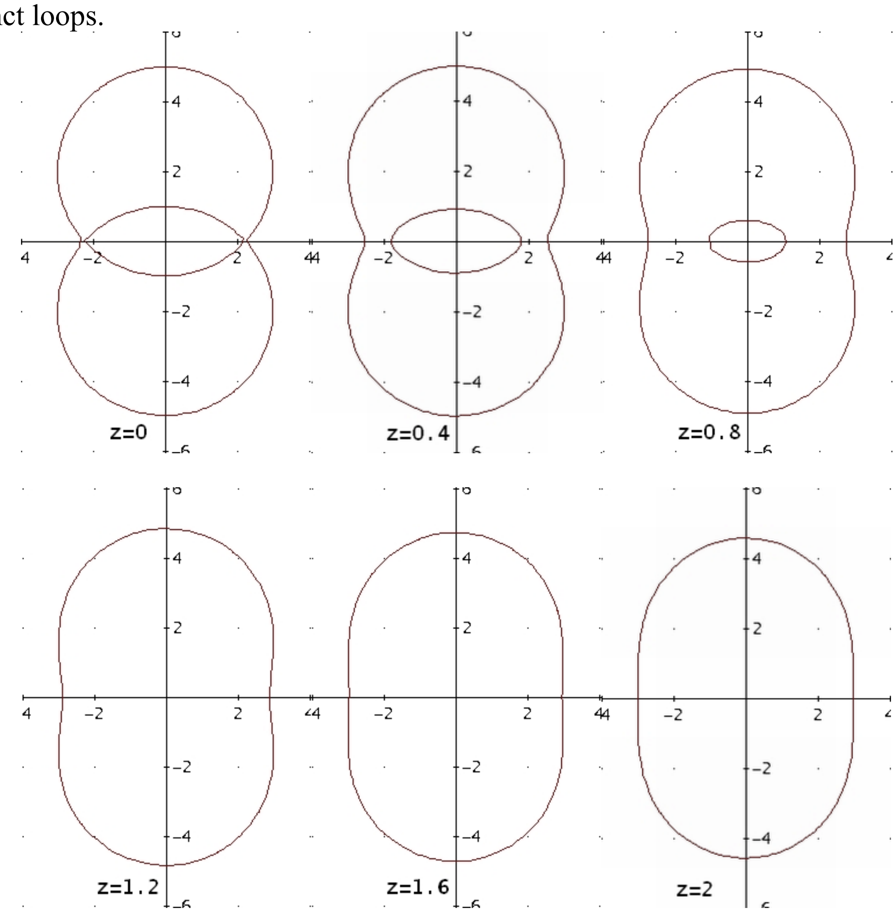


Figure 15: Spiric curves – plane sections of a regular torus

The torus T_2 is self-intersecting. His equation is $\left(x^2+y^2+z^2-5\right)^2-16\left(y+z\right)=0$. The curves displayed in Figure 16 are the spiric curves corresponding to the values z= 0, 0.4, 0.8, 1.2, 1.6, 2, 2.4, 2.8, 3.2. If z = 0 then the spiric curve is the union of two symmetric intersecting circles. If 0 < z < r-R, the spiric curve is the union of two distinct loops.



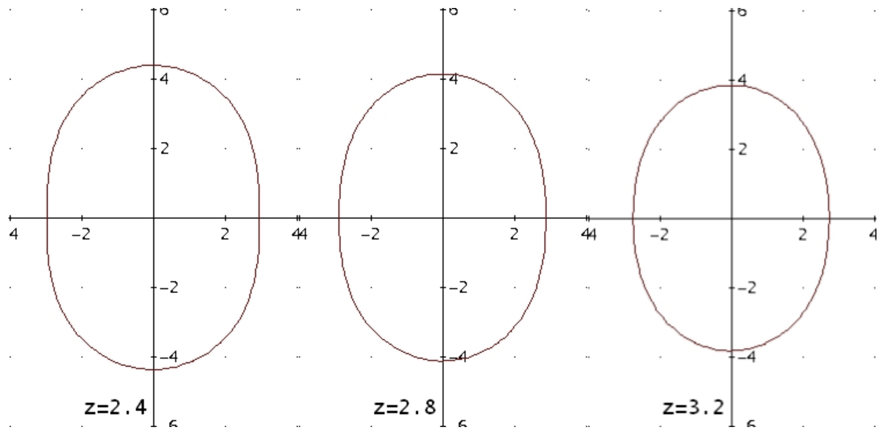


Figure 16: Spiric curves – plane sections of a self-intersecting torus

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Increasing the Use of Practical Activities through Changed Practice

A case-study examination of the influence of a value-based intervention on two teachers' use of practical activities in mathematics teaching

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Abstract: *This study sets out to examine the influence of a value-based intervention on two elementary school teachers' use of practical activities in mathematics teaching. The intervention was a "Values and Knowledge Education" (VaKE)-based in-service course that introduced the two teachers to a value-based approach to mathematics teaching. The introduction included examples that were supported by use of practical activities. Interviews prior to the intervention made the teachers aware of an inconsistency between the desired and actual practice of their own teaching. The intervention provided them with a possibility of narrowing the gap between vision and practice by changing practice. Qualitative data show how the VaKE approach offered an alternative that opened up for increased use of practical activities in the teaching of mathematics, but also showed how good intentions of changing practice might be restrained or hindered by beliefs and previous experience.*

Keywords: Mathematics teaching; Beliefs; Practice; Practical activities; Values and Knowledge Education.

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Today the educational policy in Norway (KD, 2006) encourages the use of practical approaches to mathematics teaching.[2] Using practical activities[3] is one way of doing this. However, Norwegian research shows that teachers find it difficult to change existing practice (Kjærnsli, Lie, Olsen, Roe & Turmo, 2004; Klette, 2003) and that teachers of mathematics do not necessarily acknowledge the theoretical consensus supporting practical activities (Alseth, Breiteig & Brekke, 2003; Haara & Smith, 2009). If a teacher is going to use more practical activities, the teacher has to believe that such an approach supports student learning.

Values and Knowledge Education (*VaKE*)

VaKE is a teaching approach that emphasizes developing students' moral and ethical values[4] through the acquisition of new disciplinary knowledge within a constructive learning environment (Patry, Weyringer & Weinberger, 2007). Based on a constructive theory of learning with a foothold in both sociocultural learning theory and radical constructivism, and influenced by Kohlberg's theory on moral development through social interaction (Kohlberg, 1976), the teacher who wants to follow the *VaKE* paradigm teaches through the introduction of a moral dilemma. This implies that the students have to choose between two possible decisions. Two factions of students are then formed, based on the students' decisions. This is followed by a moral viability check through discussion, first within each faction and then

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[2] From a mathematical didactical perspective, traditional teacher-dominated teaching has been challenged by the influence of theories of teaching and learning, ethno-mathematics and realistic mathematics education. In addition, the development of mathematics teaching in Norway is influenced by societal factors. Norwegian society needs to increase the numbers of students entering higher education in mathematics and science, a realization which has given extra weight to the political and societal demands for the development of additional, or even change of, working methods in the teaching of mathematics in elementary school. This is a longitudinal and manifold process that has brought about an increased focus on the practical relevance and use of practical activities in school mathematics as one domain of development. For a more thorough introduction to the background of changes in mathematics teaching for educational policy reasons, see Haara, Stedøy-Johansen, Smith and Kirfel (2009).

[3] In Haara and Smith (2009), we define a practical activity to include all forms of engagement where the pupil uses physical objects while carrying out the activity at hand. That means including the opportunity for physical activity, and not just the use of artefacts or material found in nature.

[4] The term *Values* in *VaKE* refers to the emphasis given to moral and ethical aspects through the use of dilemmas that challenge the students' opinion of right and wrong. Hence, in *VaKE* there is no explicit element of value regarding the application of mathematics (Skovsmose, 2002).

between the two factions. The need for new disciplinary knowledge to better illuminate different aspects of the topic and provide more coherent arguments through the collecting of new knowledge, is revealed. Rounds of discussion, and content viability checks on arguments are then possible, until both factions are ready to present their conclusions as the final moral and content viability checks.[5] The teacher and the class close the sequence by capitalizing on the whole process. Accordingly, the teaching aims to develop students' critical thinking, basic values and ethical principles.

Research Question

In this article we examine the influence of the introduction to a value-based intervention on two teachers' use of practical activities in mathematics teaching, based on the following two assumptions. First, elements of value and viability with regard to the application of mathematics are not commonly used to increase the use of practical activities in school mathematics. It might therefore offer a new approach to the use of practical activities in mathematics teaching and initiate reflective processes regarding beliefs (Lerman, 2002) about using practical activities in mathematics teaching. Experience with a different setting for practical activities might stimulate reflection on one's own beliefs, which is essential for a lasting change of practice (Wilson & Cooney, 2002). Second, the introduction of new mathematical content in a *VaKE*-based learning environment entails a sociocultural approach. In sociocultural learning theory, the construction of knowledge takes place through interaction or activities of a social and cultural kind (Dysthe, 2001). Conversation and joint activities are crucial to learning, and each individual's development is recognized by changed participation in the practical situation. Communities of practice are important for the development of knowledge, and social factors become more than a frame surrounding the learning situation

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[5] See Patry, Weyringer and Weinberger (2007) for a detailed review of each step of the *VaKE* methodology.

(Wenger, 1998). Such features characterize an encouraging environment for practical activity-based teaching (Bell, 1993; Meira, 1995; Wæge, 2007). Therefore, an unmodified application of the *VaKE* method can be applied when introducing new mathematical content, supported by practical activities, in an attempt to influence the teacher's use of practical activities. Based on the described prevailing situation, and the assumptions presented, our research question is as follows.

How does the introduction to a VaKE-based teaching approach, supported by practical activities, influence two elementary school mathematics teachers' use of practical activities in mathematics teaching?

Theoretical Background

Beliefs

The *Teachers Matter* report (McKenzie, Santiago, Sliwka & Hiroyuki, 2005) confirms the important role teachers play in students' learning. According to the work of Shulman (1987) and Handal and Lauvås (1987), teachers' professional knowledge, which combines disciplinary knowledge, didactical knowledge and beliefs, is regarded as the most fundamental impact factor on teachers' professional choices. Furthermore, beliefs, values and attitudes can be seen as part of an individual belief system, where the conviction about an issue or task often develops into "values, which house the evaluative, comparative, and judgemental functions of beliefs and replaces predispositions with an imperative to action" (Rokeach, as cited in Pajares, 1992, p. 314). Such views imply that teachers' beliefs are fundamental factors influencing teachers' practice, and that they influence disciplinary and didactical choices made by each teacher. Factors that make an impact on teachers' professional knowledge are dynamic features (e.g. Korthagen & Vasalos, 2005), but a teacher's beliefs are seen as an impact factor that have been found to be difficult to challenge and to change (Borasi, Fonzi, Smith & Rose, 1999; Chin, Leu & Lin, 2001; Pehkonen, 2003; Philipp, 2007; Thompson, 1992; Wilson & Cooney, 2002). Furthermore, the change of all other impact factors is more or less regarded as superficial and temporary if they are not in accordance with the teacher's prevailing beliefs (Day, 2004; Lloyd, 1999; Pehkonen, 2003). It seems that if teachers are to make a sustainable change in teaching practice, their beliefs need to be challenged (Wilson & Cooney, 2002).

Rokeach (1968) and Pehkonen (2003) look at different degrees of knowledge as subsumed in personal beliefs. Beliefs that are in accordance with an objective coherence in the surroundings are established as knowledge. Beliefs that remain as subjective knowledge are disputable, and therefore susceptible to be influenced by feelings (Grelland, 2005) and personal evaluation of good or bad consequences (values) when transformed into action. In a review of research on teachers' beliefs, Pajares (1992) identifies several commonalities concerning beliefs, summed up by Beijaard, Verloop, Wubbels and Feiman-Nemser (2000, p. 262) who suggest three common features of beliefs:

1. They are highly individual, deeply personal, and seem to persist.
2. They are formed by past experiences.
3. They represent an individual's understanding of reality enough to guide thought and behavior and to influence learning.

The understanding of beliefs as subjective knowledge influenced by feelings materialized through actions, and thereby defined as values, seems to be recognized as the way beliefs are visualized (Bishop, 2001). Moreover, through the fundamental influence that beliefs have on the interpretation of impressions and new knowledge, Pajares (1992) ascribes to beliefs a filtrating effect on new impulses. This is in accordance with the fundamental position of beliefs emphasized in the research literature on beliefs in mathematics teaching

(Pehkonen, 2003). Beliefs are influenced by new impulses and make an impact on how impulses are interpreted (Feiman-Nemser & Remillard, 1996).

Changing Practice and Changing Beliefs

According to Kerem Karaağac and Threlfall (2004, p. 137), with reference to Lerman (2002), the assumption within research on teachers' beliefs about mathematics teaching and learning has been "that awareness of a difference between beliefs and practice would result in some attempt to change". Within this area of study, however, there is a growing body of research that reports cases where the teacher either does not try to change even though he/she is aware of a difference between beliefs and practice (Kerem Karaağac & Threlfall, 2004) or simply does not become aware of such a discrepancy (Raymond, 1997). Hence, a discrepancy between beliefs and practice does not always initiate an attempt to change.

However, a change in beliefs increases the possibility of developing practical knowledge (Beijaard et al., 2000), but because of the presence of feelings, beliefs are found to be resistant to change. Independently of the content of presented arguments or experiences, efforts are made to interpret the impressions so as to support prevailing beliefs. Should that prove impossible, the arguments or experiences are ignored or rejected as a result of the influence of feelings, such as irritation or even anger (Pehkonen, 2003). Pehkonen (2003) further states that if a person's beliefs are supposed to change, it is a long process demanding personal engagement. Based on Shaw, Davis and McCarty (1991), Pehkonen (2003) suggests that the teacher must accept being challenged with a problem, doubt or an inconsistency between attitude and practice, and feel responsible to do something about it. The teacher must also have a vision of how teaching ought to be and prepare a plan for how the vision may be realized.

Shulman (1987) and Handal and Lauvås (1987) see the development of teaching practice as a cyclic process based on the impression that all impact factors are dynamic.

According to Kolb (1984), teachers' practical experiences generate observation and reflection and are based on general notions that are tested and developed in new situations. This provides the teacher with experiences at a higher level. The developmental process (*experiential learning*) is cyclic (Kolb, 1984), in the form of a helix. The process alternates between reflection and action (Korthagen & Wubbels, 2001). Korthagen and Vasalos (2005) develop this further by focusing specifically on teachers' reflections and actions attached to fundamental beliefs and views (*core reflections*). If one is supposed to change practice, both beliefs and actions must be changed. Such an impression about change of beliefs is also presented by Handal and Lauvås (1987, p. 12): "we experience our own practical efforts very much in the light of structures, concepts and theories transmitted to us in such a way that this may even lead us to change our values and beliefs to some extent". Teachers' *pedagogical content knowledge* (Shulman, 1987) and teachers' professional development are influenced during and by practice.

In the essay "The Logical Categories of Learning and Communication", Bateson (1972) links *learning* to the element of *change*. According to Bateson, a logical hierarchy of learning and communication can be identified and applied to suggest what priorities are relevant for change of teacher practice. The hierarchy consists of different levels of influence, with the levels of the hierarchy labeled 0, 1, 2 and so forth. With regard to change of practice and beliefs, level 0 in the hierarchy is about receiving and developing actions (here, practice) based on internal or external signals received by the teacher. Level 1 relates to how the teacher acts to change actions in accordance with responses to experienced practice. Level 2 focuses on the teacher's internal responses to the experiences at level 1. Level 2 then relates to changes of beliefs based on experiences initiated by practice (level 0) and change of practice (level 1). Hence, existing beliefs need to be challenged to create a permanent change of practice.

Independently of the chicken-and-egg discussion about what comes first, practice or beliefs, we agree with Pehkonen (2003) and Shaw, Davis and McCarty (1991) that the impact must stem from an experienced inconsistency between vision and practice. Transferred to the mathematics classroom, this means that teachers must be given the opportunity to initiate change in teaching practice if change of beliefs is to be facilitated.

Methods

In this article we examine the influence of the introduction to a value-based intervention on two teachers' use of practical activities in mathematics teaching. Since we wanted to focus on this particular excerpt of what might influence teachers' use of practical activities, we decided to apply a "two-case" comparative case study (Flick, 2006; Yin, 2003) to collect qualitative data. This approach was chosen because of its appropriateness when investigating "a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (Yin, 2003, p. 13). We find that "case studies of teachers can be used intentionally to prompt teachers to reflect upon and examine their own beliefs and practices" (Thompson, 1992, p. 143).

The data were collected from two teachers over a period of about 18 months. Data collection instruments were multiple: interviews, video-recorded observations of teaching together with the teachers' own reactions and impressions about the content of the recorded lessons, log-writing and a questionnaire based on open-ended questions. This is in accordance with Yin (2003, p. 14), who states that "the case study relies on multiple sources of evidence, with data needing to converge in a triangulating fashion". The importance of multiple sources of evidence offered by a case-study approach is also emphasized by research reviews on the change of mathematics teachers' beliefs about mathematics and mathematics teaching (Philipp, 2007; Thompson, 1992; Wilson & Cooney, 2002).

The two teachers, *Vivian* and *Walter* (pseudonyms), were recruited to the study by their respective principals upon our request for a teacher from their respective schools. We contacted these two schools because they were supposed to participate in an EU-FP7 project that aimed to try out *VaKE* in science teaching, but which did not make it to the final stage in competing for an EU-FP7 grant.[6] We asked the school principals to find a teacher recognized as an acknowledged teacher by the work environment (Haara & Smith, 2009),[7] and who was interested in developing his/her teaching of mathematics. Vivian has been teaching mathematics and other subjects in the Norwegian upper primary school (students 9 to 13 years old) system for 10 years, and Walter has been teaching mathematics and other subjects in the Norwegian lower secondary school (students 13 to 16 years old) system for 5 years. They are about the same age, and both have 30 ETCS (*European Credit Transfer and Accumulation System*) in mathematics from their Norwegian teacher education.

An intervention was designed for the case studies (Lane, Weisenbach, Little, Phillips & Wehby, 2006). The intervention was a 20-hour-long in-service course in *VaKE* held by one of the two researchers responsible for the research project, focusing on applying *VaKE* when teaching mathematics. The course consisted of two gatherings of two five-hour-long course days each, and focused on *VaKE*, areas on which *VaKE* is based (constructivism, value education, moral dilemmas in teaching), and on the professional development of teachers. In between the two gatherings the course participants prepared suggestions for themes and dilemmas for mathematics lessons based on the *VaKE*-method and how practical activities could be included in the mathematics lessons. The first gathering consisted of lectures presenting the course literature, and there was an emphasis on practical examples allowing for

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[6] EU-FP7 is EU's 7th framework program for research and technological development, and the *VaKE* project was one of the eight finalists for the grant (Patry et al., 2007).
[7] In Haara and Smith (2009), acknowledged teachers of mathematics are defined to be "teachers who are viewed as competent mathematics teachers by the principal and earn respect from colleagues, pupils and other groups of relevance within the working environment".

teaching of mathematics through moral dilemma supported by a practical activity. An example of this focused on airline overbooking policies or salary payments for completed work. The second gathering focused on change of practice using themes and practical activities suggested by the two participating teachers, for instance, on choosing between refurbishing the playground at the school and expanding the computer facilities for the students of one class, or on delivering a tender for a house building contract.

The data collection period started when Vivian and Walter were interviewed about 6 months prior to the intervention. The interviews focused on their opinions on mathematics and school mathematics in general and their present and future teaching practice. Each semistructured interview lasted for approximately 75 minutes and was recorded and transcribed. Essences of meaning were extracted from the transcriptions (Kvale, 2006) and interpreted through a hermeneutical approach. The interpretation process contributed to the planning of the forthcoming intervention since it offered impressions of how beliefs about mathematics and teaching in general, and more specifically about practical activities in mathematics teaching, were part of Vivian's and Walter's visions of teaching. These impressions also served as references for comparison in the analysis of data produced after the intervention.

Vivian and Walter were observed and filmed in 3 mathematics lessons each. The observations took place within a two-week period starting about a month after the intervention. Observational data were collected when Vivian taught mathematics in 4^{th} grade (students 9 to 10 years old), and Walter taught mathematics in 8^{th} grade (students 13 to 14 years old). Respectively, the first lesson was typical for the kind of mathematics teaching that Vivian and Walter traditionally practiced, and the other two were based on the introduction of new mathematical content in a *VaKE*-based environment supported by a practical activity opportunity. Immediately after each lesson the teacher and the researcher who video-recorded the lesson, watched it together. During these sessions, Vivian and Walter were free to comment on what they saw (Jacobs & Morita, 2002). This gave access to Vivian's and Walter's reflections and observations on the recent teaching experience. Comments and evolving discussions were recorded and transcribed.

The transcribed comments from the video sessions were coded. From the comments made by the teachers we created units (Grønmo, 2004) that were then categorized as "positive", "negative" or "neutral" (Jacobs & Morita, 2002). Units including discussion of practical activities, isolated or within the progress of the *VaKE*-methodological structure, were divided into five subcategories and given an interpretation according to the teacher's comments: "positive – unconscious", "positive – conscious", "neutral", "negative – conscious", "negative – unconscious". This is in accordance with how people are conscious about some reactions and prevented from being conscious about other exhibited reactions. Unconscious reactions are difficult to explain. In other words, the observing teachers' reactions could be separated similar to the distinction between conscious and unconscious values (Bishop, 2001; Grelland, 2005).

Vivian and Walter wrote personal logs. They started on the day they received the in-service course information and reading list. The logs cover the last approximately 12 months of personal impressions about mathematics teaching, the in-service course, and experiences in accordance with both observed and independently conducted *VaKE* lessons. The same categorizing system as with the video sessions was used in the analysis of the two logs, but based on systematic extraction of meaning of sequential content organized in a matrix (Grønmo, 2004), structured by a timeline, and the participants.

Exactly 12 months after the intervention started, Vivian and Walter responded to an open-ended questionnaire focusing on beliefs regarding factors with influence on their use of practical activities in mathematics teaching. The questionnaire was validated by 3 researchers and 3 mathematics teachers in elementary school, who commented on the relevance and clarity of the questions. The questions did not focus on *VaKE*, but were developed based on interpretations stemming from the analysis of the preintervention interviews, observations and video sessions. The collected data were analyzed in the same way as the logs, but the matrix was structured by the questions and participants.

Based on the analysis of the logs and questionnaires and in accordance with the interpretations of the prequestionnaire analysis, Vivian and Walter were interviewed once more at the end of the project, about 1 month after responding to the questionnaire. The logs and questionnaires served as data-producing devices in a triangulation quest for points of refutation and confirmation of prequestionnaire interpretations. The interviews were structured, and the interview guide was divided into three main parts.

- The teacher's beliefs about mathematics and practical activities in mathematics.
- The teacher's response to the value-based intervention.
- The influence of the intervention on the teacher's teaching of mathematics.

From a hermeneutical perspective, our interpretations in the analysis have probably been affected by our unconscious prejudices, although the triangulation process and validation by Vivian's and Walter's interpretations strengthened the viability of our conjectural suggestions and the subsequent discussion of how the intervention influenced the teachers' use of practical activities. Hence, in the analysis we used both a phenomenological approach and a hermeneutical approach (Grønmo, 2004). The phenomenological approach is recognized in the use of Vivian's and Walter's experience with the intervention program as a basis for the analysis. The hermeneutical approach is reflected in the comparison of the influence of the intervention with the preintervention situation, as well as similarities and discrepancies between the two teachers' beliefs about the teaching of mathematics.

Findings

The findings are reported through a description of beliefs Vivian and Walter had about mathematics and practical activities in mathematics, their response to the value-based intervention and the use of *VaKE* supported by practical activities in teaching. This follows the pattern of four phases for teacher change, reported by Shaw, Davis and McCarty (1991) and Pehkonen (2003):

- experiencing personal inconsistency
- feeling responsible for doing something about the inconsistency
- developing a vision of how teaching ought to be
- making a plan for how the vision can be realized

Vivian

Experiencing personal inconsistency. Vivian was fairly open about her own lack of understanding of generalized mathematics in the preintervention interview and indicated that she did not always see the application of theoretical dimensions to real-life situations. She was more focused on mathematics in a strictly real-life context, with an emphasis on the practical application of mathematics. Furthermore, she was an active teacher, who enjoyed being the focus of attention and explaining the mathematical content at hand, as she explained during the preintervention interview:

Vivian: I think I am very present … and very active. In a mathematics lesson which could actually be boring, I still feel that I am creative, and I feel … I think that my problem maybe is that I am too … ehh … active. So what happens … especially in mathematics … what happens when I am about to explain something … then it is like Oh yes! (changes her voice), and then I like to use things which they know. Imagine!

> (changes her voice again) … and then I tell a little story about something ….

Vivian used narratives and relied on the students' imagination when using examples in teaching. In her opinion, the teacher had to explain the mathematical content to the students, and then the students had to do quite a lot of exercises to internalize the content. Kuhs and Ball (1986) refer to this "as content-focused with emphasis on conceptual understanding". The students' understanding of ideas and processes is emphasized through the instruction of the mathematical content, and the lessons might vary considerably from lesson to lesson. It was important to Vivian that the students both have fun and learn, and that they are offered some exiting experiences when learning mathematics. In accordance with Ernest's (1989) recognized pattern for an *Explainer's* use of curricular materials, this meant to Vivian that the textbook approach was enriched through her introducing additional examples, problems and activities of real-life relevance.

The preintervention interview revealed that Vivian was confident that her students learned mathematics, but she was not satisfied with her own organizing priorities. She felt that the lessons ought to be more varied, and she wanted to be more attuned to what Kuhs and Ball (1986) refer to as "learner-focused", in the sense of focusing the teaching more on the students' active involvement. She therefore experienced an inconsistency between her teaching and her beliefs about how mathematics ought to be taught.

Feeling responsible for doing something about the inconsistency. Vivian was clear about her bad conscience for what she experienced as a lack of variation in her teaching. In her opinion, the content-based teaching of mathematics for which she had been an exponent, with emphasis on the progress and approaches suggested by the textbook, ought to be supported by an expanded organizational repertoire, as she stated during the final interview in the project:

Vivian: My mathematics teaching ought to consist of exercises which the students master, exercises which challenge the students, use of the textbook, use of different tools and props, collaboration among the students, individual work, work through theoretical approaches, work through practical approaches, and so forth. I would like my teaching to be varied.

Developing a vision of how teaching ought to be. Vivian wanted her teaching to be more varied and student focused. She also wanted to make her instructive *Explainer* role less dominant. The introduction of practical activities supported by a *VaKE*-based approach provided her with an opportunity to change her practice, as she concluded during the observation of one of the video-recorded *VaKE*-based lessons:

Vivian: I have missed such an approach in mathematics …. I have needed something to change my teaching of mathematics with, and this is what I have been missing!

Making a plan for how the vision can be realized. On 2 occasions, 3 weeks after the in-service course, Vivian used dilemmas, which she found relevant to the students' real-life interests. The first dilemma depended on, in terms of mathematics, economics calculations related to choosing between computer accessories for the students involved and a new climbing frame area for all students in school. The second dilemma involved economics and volume calculations related to choosing between a party for the entire school to celebrate the new climbing frame area, and refurbishing the school entrance. The students had access to props. In the first lesson it was fake money, and in the second lesson it was drinking glasses, deciliter and liter measures and free access to water. The dilemmas required the students to work with the four arithmetical operations, money values, estimation, measuring and geometrical figures. The props made it possible to systematize information practically and carry out operations that initiated, simplified and confirmed or refuted the students' calculations.

Vivian was conscious of her neutrality while applying the *VaKE* approach, but she was very focused on setting "a conflict zone". The competitive organization appealed to her. She reorganized the classroom before the lessons, initially grouping the students on the floor. Vivian clarified the moral dilemma and each student made a written, initial decision on the dilemma. Based on the students' decisions she then divided them into two groups, separated by a front line. "It is you against them!", she said several times to each group, referring to the students in the other group.

When observing the video recording of her own teaching, she reported that she could see that the *VaKE* approach introduced a new organizational possibility to her mathematics lessons:

> Vivian: And that is just what this math builds on. That you actually do not only sit and work on some numbers, you actually go into yourself a bit … because when you start to tear at something inside yourself, you automatically become more motivated, and then you approach the problem in another way than you would do if you just sat there.

At the same time she claimed that the new method occasionally resembled her regular approach:

> Vivian … and I have got something of a revelation by entering this project, and I now feel that one of my strengths is that I have motivated students … and that the reason for that maybe is because I challenge them in relation to themselves to some extent ….

She was familiar with challenging the students and pitting them against each other, but not in such a planned and structured way. This was supported by the video recordings, which showed that she was comfortable with the organizational demands of the *VaKE* method and that she was able to let the students and the method set the pace of the lesson.

In the interview at the end of the project, Vivian revealed that she believed that her teaching of mathematics and the use of practical activities in the teaching had changed:

Researcher: Did your use of practical activities change after you were introduced to VaKE?

Vivian: Yes, it is much more … it is no longer so structured. Now I start trying to make the students curious, investigative and uncertain for a while. I give them a challenge which involves them, and then … they can get a feeling of solving, and I can focus on challenges which occur. So it is a bit different now.

Walter

Experiencing personal inconsistency. Whereas Vivian was content with focusing on practical applications, Walter found in the preintervention interview that is was important to emphasize both the theoretical dimension and the practical applications of theoretically based results:

Walter: Well, it is a theoretical subject, but at the same time one can approach it in a practical way, and I feel that is very important.

Furthermore, Walter and Vivian held different views about how mathematics ought to be taught. In the preintervention interview he emphasized, as did Vivian, that the teacher should explain the mathematical content and that this should be followed by the students' work on exercises. However, the observation of lesson 1 showed that Walter taught in a more traditional way than Vivian. He explained the new content and examples to the students before they worked on exercises. Finally, Walter gave a summary of the lesson. Whereas Vivian focused on motivating the students, Walter to a larger extent wanted mathematics as a subject to be self-motivating, as he reveals through his description of his mathematics lessons in the preintervention interview:

> Walter: … and traditionally school mathematics is kind of a mix between a theoretical review, usually using the blackboard, and a conversation with the students, and then this is combined with solving exercises in the textbook. That is in a way how I have experienced mathematics myself through my own schooling, and how I to a large extent teach myself … although I sometimes perhaps would have wished that I could vary my teaching more.

Walter's teaching seems to be in accordance with a "content-focused view with emphasis on conceptual understanding" (Kuhs & Ball, 1986), but it is, to a larger extent than Vivian's teaching, "content-focused with emphasis on performance" (Kuhs & Ball, 1986) . In this approach it is assumed that acquiring the content motivates further studies and practical applications.

In the preintervention interview, Walter expressed beliefs about mathematics as a general education subject:

> Walter: Everybody needs mathematics. That is, a certain basic mathematical knowledge … in order to make reasonable, good choices. And one will be confronted with it no matter what … regardless of profession … if

not with pure, formal mathematics, then certainly with a mathematical way of thinking.

Researcher: Are you thinking about the terms which you used earlier [in the interview], like problem solving, logical reasoning, and structuring …?

Walter: Yes! Because I think that mathematics is an educational subject which structures one's thoughts … which I often miss among the students. If they are given some kind of problem or exercise or something, they are not able to see logical flaws, and in my opinion that has to do with mathematical thinking ….

In Walter's opinion, the educational subject dimension of mathematics seems to vanish as an argument for maintaining interest in learning mathematics when compared with the legitimacy of the general education dimension in mathematics that he remembered from his own time as a student. He sees mathematics as an educational subject based on concepts such as curiosity, logic and persistence, but mathematics proves not to be as self-motivating to the students as he would expect it to be. In fact, he reveals that he always has a bad conscience for his lack of practical activity-based teaching. A more varied lesson structure would hopefully increase the students' interest in mathematics, as he reveals in this sequence from the preintervention interview:

Walter: I do have to say … I have always had an ambition to use practical activities in mathematics because I think it is a very useful approach if you can combine it … with another kind of mathematics teaching, so that the students are given a balance towards … well, solving of exercises and such. And I must admit that I have always had a bad conscience for my lack of practical activity-based teaching.

Walter seemed to be influenced by both a “content-focused view with emphasis on conceptual understanding” and a “learner-focused view” (Kuhs & Ball, 1986), but suppressed the influence because of bewilderment about how to change his teaching, which becomes apparent during the preintervention interview:

> Walter: It is a bit about … that I am not used to using it, and I spend much more time in preparing such activities. And, obviously, you did not get trained in such teaching during teacher education. And that … and that puts you … and the textbooks do not emphasise such teaching either, and that leaves you to … to your own … oh, what is the word I am looking for? … That is, my own … you have to rethink, maybe be a bit creative, and that … is maybe a bit time consuming in a … well, in the hectic school day.

Walter’s traditional teaching is a compromise between his beliefs about how mathematics ought to be taught and his awareness of the advantage of emphasizing structure, performance and textbook applications when teaching mathematics, a phenomenon previously shown by, for instance, Cooney (1985), Lloyd (1999) and Raymond (1997). Hence, Walter experienced a personal inconsistency between his beliefs about mathematics teaching and his actual teaching, since his teaching lacked variety and did not prioritize practical activities in the way he wanted.

Feeling responsible for doing something about the inconsistency. As with Vivian, Walter expressed a kind of guilt for lacking variety in his teaching. Moreover, in the preintervention interview he was not entirely willing to accept the students’ prevailing opinion, who saw mathematics from a utility perspective only:

Walter: For instance, I remember compared to my own schooling, I thought it was really funny to get some practical … the daily puzzle or things like that to work on. But when I try such problems with students … they do not seem to see any point in it … Well, what is this then? Are we supposed to wo … (changes his voice). Often they do not understand the problem at all. They are not used to think in a … in a mathematical way.

He therefore felt that instead of the rather traditional teaching, he should teach more in accordance with a "learner-focused view" (Kuhs & Ball, 1986), and include more practical activities in his teaching.

Developing a vision of how teaching ought to be. Walter did not have the same starting point regarding the *VaKE* approach as Vivian, and based on the organization of his regular teaching, Walter's vision implied a more radical change of practice. The intervention introduced Walter to an approach that he believed could challenge the present suppression of his mathematics teaching beliefs, as seen on separate occasions in his log during the in-service course:

Walter: Making teaching more realistic is a massive challenge, especially when compared to one's own view about what teaching is, and ought to be. I believe the VaKE project to be useful in this respect.

Walter: I especially approve of using such a methodology as an approach to teaching mathematical content, and then later on concentrate on the theoretical approach to the mathematical topic at hand. I believe that the students are more easily able to see that what we are supposed to learn

> is relevant to learn, that this is something which they actually may find useful.

Making a plan for how the vision can be realized. As with Vivian, Walter on 2 occasions about 3 weeks after completing the in-service course, taught by introducing 2 dilemmas that he found relevant to the students' real-life interests. The first dilemma depended on economics calculations and the calculation of an area of a planned house where a compound area consisting of different geometrical shapes represented the new mathematical content. In terms of mathematics, the students worked on calculating construction costs. The second dilemma was about a nonregular pyramid-shaped box of chocolate pudding and a lack of coherence between the quantity of pudding stated on the package and the measured quantity of pudding in the package. The new mathematical content was represented by a pyramid-shaped polyhedron, the theorem of Pythagoras, and the connection between cubic centimeter and deciliter. In the first lesson the equipment for the practical activity consisted of the traditional compass, protractor and ruler, but in the second lesson, these were accompanied by an actual package of the chocolate pudding polyhedron.

The observations of the lessons show that Walter experienced some challenges. He struggled to find his position in the context, and the students were not sure what was expected of them. They seemed curious and interested at first, but the lessons did not work out the way Walter had planned. The dilemma discussions did not develop as planned for two reasons. One of the discussion groups was outnumbered in both *VaKE*-based lessons, and Walter did not succeed in pushing the two groups to find arguments in favor of the group's point of view. In the end Walter found the lessons to be rather boring and worthless, an impression that he states explicitly in his log after both *VaKE*-based lessons:

Walter: I had the first VaKE-session today, and it was done pretty much the way I had planned. I would perhaps have hoped for more engagement from the students, but it turned out to be rather boring.

Walter: I had the chocolate pudding session today, and I have to say that the lesson was not a success. I felt that the dilemma at hand engaged only a few of the students.

As the observations of the video recordings proceeded, Walter expressed doubts about his loyalty to the *VaKE* approach in mathematics teaching. In his opinion, he did not seem to be able to make the students aware of the moral aspects of the dilemmas. In fact, he changed his view on the *VaKE* approach as he gained more experience with it. When observing himself and the class on video in the second lesson he stated that the *VaKE* approach would be appropriate to use after the mathematical content had been introduced in another way, instead of combining the introduction of mathematical content and the value emphasis:

Walter: In general I think that such approaches … VaKE-approaches in relation to mathematics, would be best to have when you have finished a mathematical topic. Because then you can use the knowledge, put it into a setting which in a way creates engagement and shows that you need mathematics in daily life.… Because … if you do it when you are introducing a mathematical topic, I believe … that the students will find it difficult to do the necessary calculations, and then the foundations disappear for some of the arguments which they may put forward ….

In his opinion, the calculations that the students would need to do in order for the dilemma discussion to become active, were too complicated, a situation that is also described by Lloyd (1999). It would therefore be better to revisit the mathematics they had learned in a traditional manner by applying it in a *VaKE*-based context. In the end, Walter argued for his usual teaching to be a way to make the students better prepared or disciplinary-skilled enough in a mathematical theme before relying on mathematical arguments in discussions focusing on moral dilemmas, as shown in this concluding comment from the same video observation session:

> Walter: As a way of teaching it obviously brings along more noise, and it becomes a bit more difficult to see what each student actually does. If they are seated at separate desks it gives me a much better overview … what each student does, if he is disturbing others or not …. Students who work in groups often make teaching more complicated than when students work individually.

Discussion

The preintervention interviews revealed that both Vivian and Walter claimed that they believed in using practical activities in mathematics teaching, and that they were interested in changing their practice in order to increase the use of practical activities. They did not find their current teaching to be in accordance with personal visions, and they struggled to find personal acceptance for increased reliance on practical activities in mathematics teaching. A change of practice towards an increased use of practical activities would therefore only be temporary or superficial unless the change made an impact on their beliefs and didactic knowledge (Bateson, 1972; Wilson & Cooney, 2002).

Vivian was enthusiastic about the theoretically supported approach to mathematics teaching provided by the value-based intervention. It acknowledged elements of her previous teaching, and she referred both to how she was influenced and how she experienced excitement among the students. "I have probably never seen the students this engaged!", she said during the observation of the first *VaKE*-based lesson. Vivian became more aware of her own role and about making the students engaged without her direct involvement and guidance. Hence, her role as a facilitator became more important (Ernest, 1989), and she personally felt that she had experienced a kind of revelation by participating in the study. Finally, her impression of the students' work with mathematical content was also influenced. She experienced the group work as coherent with her opinion that students were active learners, an opinion which she reported she had not been able to include in mathematics teaching in the same way as she had done when teaching other subjects.

Walter was also enthusiastic at first, but developed a resistance towards the thought of introducing new mathematical content through the *VaKE* approach supported by practical activities, as the experience with the approach increased. Walter experienced that the positive expectations that followed the in-service course disintegrated when he applied the *VaKE* approach in his own teaching. This feeling was reinforced by watching video recordings of his lessons, all of which proved a setback regarding his vision of how to change practice. Making his suppressed beliefs about how mathematics ought to be taught explicit once more seemed to capitulate to the prevailing and familiar way of teaching mathematics. Similar situations are described by Kerem Karaağac and Threlfall (2004) and Raymond (1997), but the case of Walter refers to a situation where the teacher actually attempted to change practice. He experienced constraints that prevented him from further consideration of the new approach as a possible way to learn mathematics through a new perspective and increase the use of practical activities. His rather modest level of didactic knowledge of mathematics, revealed in the preintervention interview through his bewilderment about how to arrange for appropriate use of practical activities, and the response from the students to his new approach to teaching strengthened this impression. Hence, he withdrew to the established form of teaching familiar to himself and the students.

Vivian and Walter experienced the *VaKE* approach in different ways, which led to different outcomes. Vivian maintained her enthusiasm about a value-based approach supported by practical activities. Walter did not. The main reason for this, in our opinion, is found in the different starting points of the two teachers. Vivian's beliefs were not challenged to the same extent as Walter's beliefs were. Her vision of teaching proved to be within an approachable reach. The discrepancy between Walter's beliefs and experiences of constraints given by his teaching practice of mathematics and the actions used in the value-based approach was too wide, and in a way he "broke" the cycle of reflection and action necessary to change practice and beliefs (Kolb, 1984; Korthagen & Vasalos, 2005). In the in-service course, a community of learning was created for Vivian and Walter (Wenger, 1998). Vivian entered a productive moderation process since her beliefs were not severely challenged and her students did not meet a teaching approach that was totally different from what they had experienced before. Vivian and her students were able to explore the new approach together. Walter's beliefs were deeply challenged and his students met a teaching approach that was quite alien to them. Walter therefore lacked the moderation process from which Vivian so successfully benefitted. Having said this, though, professional growth can take the form of maintaining present beliefs after having had the courage to challenge them. Walter tried to change his practice and had the courage to challenge his beliefs about using practical activities for teaching mathematics, but this did not lead to change because of the influence from what he experienced as restraining constraints.

Conclusions

Changing beliefs about the teaching of mathematics is an extensive and longitudinal process (e.g. Pehkonen, 2003; Wilson & Cooney, 2002). Change of beliefs and change of practice can be independent of each other, i.e. they are not synonymous. However, change of beliefs and change of practice are often tangled in such a way that when one is changed it will cause change to the other. In this study we aimed to examine the influence of the introduction to a value-based intervention on two teachers' use of practical activities in mathematics teaching. The two teachers, Vivian and Walter, were introduced to a value-based approached to teaching mathematics that opened up practical activity support opportunities, which implied a change of practice for them both. From this study, we can note that Vivian approved of the alternative practice, both as a teaching approach, and as a possibility to increase the use of practical activities, while Walter did not. A more thorough examination of the study reveals, however, that the change of practice challenged both Vivian's and Walter's beliefs about how to teach mathematics and the possibilities for using practical activities. It is a common impression that beliefs have a filtrating effect on new impulses (Beijaard et al., 2000; Pajares, 1992; Philipp, 2007), and since the applied change of practice was not too controversial in relation to Vivian's prevailing beliefs, her positive attitude towards an increased use of practical activities and student involvement was strengthened. Walter found the change of practice to be too controversial in relation to his prevailing beliefs, and instead of maintaining the positive attitude towards increasing the use of practical activities through applying the value-based approach nurtured by the offered intervention, he returned to the previously established teaching practice as the preferred way of teaching mathematics.

Regardless of the tangled question of whether a change of practice implies change of beliefs, or if change of beliefs implies change of practice (e.g. Bateson, 1972; Kolb, 1984), we are left with the impression that Vivian managed to offer the new teaching approach to the students in a way that appealed to them, whereas Walter did not. There might be several encouraging or restraining constraints that paved the way for such a course of events, and the impact from different constraints are not necessarily similar for Vivian and Walter. Nevertheless, we find that three constraints on this occasion need to be mentioned on behalf of both teachers. First, we would like to mention the two teachers' beliefs about teaching mathematics and their didactic knowledge as crucial impact factors. Second, the impact of the intervention in which Vivian and Walter participated must be acknowledged. Third, the students' response to the new teaching approach probably also played a role in forming Vivian's and Walter's acceptance of the use of a value-based teaching approach supported by practical activities. To maintain a changed practice, it seems that the changed practice must also lead to a change of beliefs. If not, practice will eventually drift back to its initial pattern or to something less radical than the alternative practice. The isolated findings in this study show that Vivian entered a process that might lead to increased use of practical activities in her future teaching, whereas Walter in the end found his traditional way of teaching to suit him better. For Walter, this return implied staying faithful to the explicit practice he upheld when entering this study, as his professional conscience did not allow for increased use of practical activities.

In this article, we base our cautious suggestions on interpretations of data stemming from the cases of two teachers' experiences with the introduction to a value-based approach to changed practice in mathematics teaching. The interpretations have been validated by the 2 teachers through a triangulating process. Our temporary interpretations were tested and reformulated in the light of their logs and responses to an open questionnaire. Finally, the interpretations were validated by conducting individual interviews with the 2 respondents, which allowed for their personal interpretations. Hence, the contextual interpretations are based on multiple data sources and we believe the interpretations to be well justified, despite the limitations of basing a study on a relatively small and narrow empirical source (Yin, 2003).

We want to conclude that the in-service course, which emphasized the use of practical activities in mathematics teaching through a value-based approach to new mathematical content, influenced one of the participating teacher's beliefs about teaching mathematics and increased the space given to practical activities in her teaching. Furthermore, we add another study to the body of research that confirms that awareness of a difference between beliefs and practice will result in some attempt to change (Kerem Karaağac & Threlfall, 2004; Lerman, 2002). However, the case of Walter shows that the influence from restraining constraints might result in an aborted attempt to change. We hope that Vivian's and Walter's reported struggles and challenges with the correspondence between beliefs and practice will bring about further research on persistent change of teachers' practice.

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Student Enrollment in Classes with Frequent Mathematical Discussion and Its Longitudinal Effect on Mathematics Achievement

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Abstract:

Mathematical discussion has been identified as being beneficial to students' understandings of mathematics (Goos, 1995; Lee, 2006). Students in classrooms with more effective math discussion have been observed to engage more frequently in discussion (e.g. Hiebert & Wearne, 1993), but the converse is not necessarily true (e.g. Manouchehri & St. John, 2006). Utilizing hierarchical linear modeling, the present study examined student enrollment in classes with more and less frequent discussion and such enrollment's effect on mathematics achievement over time. Results indicated that students enrolled in classes that discuss math "almost every day" consistently have higher math achievement than students enrolled in classes that discuss math "never or hardly ever." These results and their implications are discussed in depth.

Keywords: Mathematical discussion, mathematics achievement, hierarchical linear modeling.

Introduction

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According to Silver, Kilpatrick, and Schlesinger (1990), "mathematics deepens and develops through communication" (p. 15). Students gain a better understanding of the meaning of mathematics when they communicate with others about it. Mathematics discussion in the classroom involves students in describing, explaining, defending, and justifying their ideas about mathematics. By doing so, mathematics discussion deepens students understanding of mathematics (Goos, 1995; Lee, 2006; Pimm, 1987) and has been shown to have a positive impact on mathematical achievement (D'Ambrosio, Johnson, & Hobbs, 1995; Grouws, 2004; Hiebert & Wearne, 1993; Koichu, Berman, and Moore, 2007; Mercer & Sams, 2006). Yet, there is evidence that discussion does not always have a positive impact on mathematics achievement (Shouse, 2001), which may imply that either discussion is not consistently effective in deepening mathematical understanding or that it is not consistently implemented to maximize its effectiveness. Perhaps this inconsistency in the research concerning the effectiveness may explain why Pimm (1987) identifies mathematical discourse as a topic that is continuously advocated by researchers but rarely implemented by teachers.

In addition to the contradictory results of research on the impact of mathematics discussion on math achievement (e.g. Mercer & Sams, 2006; Shouse, 2001), there are few examples of such research. Of the studies that focus on discussion's impact on math achievement, all are cross-sections of the samples evaluated. To date, the author has yet to find a longitudinal study to investigate the compound effects of discussion on mathematics achievement. Given that many teachers do not engage their students in mathematical discussion (Pimm, 1987), it may very well be that positive benefits of more frequent discussion may not be statistically evident within one iteration of its implementation.

The current study seeks to investigate whether students' presence in a classroom with frequent mathematical discussions has any longitudinal effect on their mathematics achievement. Students who are in classrooms where their peers frequently talk about mathematics should be more likely to be able to meaningfully and skillfully discuss mathematics than students who have not been in such classroom environments. Likewise, this ability should have a positive influence on their mathematics achievement. Therefore, students who are more exposed to classroom environments with frequent discussion about mathematics should demonstrate more growth in mathematics achievement than students who are less exposed to such classroom environments.

Mathematical Discussion and Mathematics Achievement

Describing effective teaching strategies for increasing mathematics achievement, D'Ambrosio et al. (1995) suggested that engaging students in discussions about mathematics would improve their mathematical understanding of it. One study supporting this claim was conducted by Hiebert and Weane (1993) with second grade students and teachers. Hiebert and Wearne observed six classrooms and found that two teachers observed asked students to explain and justify their mathematics significantly more than the other four teachers. In addition to being engaged in mathematical discussion more frequently, students of these two teachers had statistically significant higher gains in content knowledge than the students of the four teachers who engaged in mathematical discussion less often.

A study in Great Britain conducted by Mercer and Sams (2006) compared teachers who received training in setting up mathematical discourse environments to those who did not. Students in the treatment group engaged more frequently in self-directed discussions about mathematics, while students in the control did so less often and to a lesser degree. Similar to the findings of Hiebert and Wearne (1993), Mercer and Sams found that students in the treatment had significantly higher gains in math achievement than students in the control.

Other studies support the two previously mentioned studies' claims of mathematical discussions' positive impact on student math achievement (e.g. Koichu et al., 2007; Stigler & Hiebert, 1997). However, Shouse (2001) found that more frequent mathematical discussion had a negative impact on high school student math achievement. Shouse used a regression analysis with the 1988 National Education Longitudinal Study (NELS) dataset. The resulting coefficient was small but negative, providing a contrast to the findings of other studies. However, Kosko and Miyazaki (in press) found that the impact of the frequency that 5th graders discuss mathematics varies significantly (statistically and meaningfully) between classrooms and schools. In some schools the impact of discussion was overwhelmingly positive while in others the impact was largely negative. Additionally, the frequency students' 3rd grade classrooms engaged in math discussion increased the effect of 5th grade discussion frequency (Kosko & Miyazaki, 2009).

The results found by Kosko and Miyazaki (in press) suggest that more frequent discussion does not necessarily mean better discussion, but student exposure to more frequent discussion in the previous grade suggested that more frequent discussion may have a positive effect on math achievement over time (Kosko & Miyazaki, 2009). Hiebert and Wearne (1993) and Mercer and Sams (2006) both observed more frequent discussion on the part of students who had larger gains in math achievement. Additionally, since mathematical discussion is argued to increase math achievement (D'Ambrosio et al., 1995), it is logical to expect that more frequent student involvement in discussion is necessary for students to see the benefits of discussion.

Observed Implementation of Mathematics Discussion

There are several qualitative studies where teachers identified as implementing effective mathematical discourse are observed (e.g. Truxaw & DeFranco, 2007; Williams & Baxter, 1996; Wood, 1999). Teachers in these studies typically emphasize the characteristics of dialogic discourse when engaging students in mathematical discussion. Dialogic discourse involves both students and the teacher in developing the course of a discussion. Students are encouraged to justify and explain their reasoning while the teacher creates a positive atmosphere lacking social penalties for incorrect math answers. Additionally, students were informed why they were being asked to explain and justify mathematical ideas as well as how to go about doing it (Williams & Baxter, 1996; Wood, 1999). As encouraging as the teachers in these studies may be, it does not paint an accurate picture of how many other teachers implement mathematical discussion.

Manouchehri and St. John (2006) compared two episodes of classroom talk where there was a large degree of student participation. On the surface the two episodes appeared to be similar in that the teachers actively engaged students in the topic discussed. Yet the teachers in both classrooms acted differently in how material was explained. In one classroom the teacher explained and justified mathematical positions where in the other classroom it was the students who did so. Kazemi and Stipek (2001) found similar results in studying fourth and fifth grade classrooms. On the surface, all teachers seemed to have similar levels of discussion in their classrooms and a positive social environment for students to learn in. Results showed, however, that one set of teachers was more likely to require explanation and justification from their students than the other set of teachers. Characteristic of the teachers studied by Kazemi and Stipek (2001) was that while all four teachers asked their students to describe how they solved problems, some teachers had students discuss such descriptions while other teachers simply asked whether the class agreed or not. The lesson to be learned from these two studies is that while some teachers may seem like they are actively implementing mathematical discussion more frequently, it such discussions not contain the elements that make the discussion effective.

Contextual Effects of Mathematical Discussion

> In trying to make sense of what someone says, we never rely only on our knowledge of the basic meanings of words, or our familiarity with the grammatical constructions they use. As listeners, we always access some additional, contextual information, using any explicit guidance or hints provided by a speaker and drawing on any remembered past experience which seems relevant (Mercer, 2000, p. 44).

The above quote by Mercer (2000) demonstrates the importance of context on an individual and their discourse-related decisions. The very context a student is in not only helps define the student's interpretations of what others say but also defines the social and sociomathematical norms that the student abides by (Yackel & Cobb, 1996). The student also contributes to both social and sociomathematical norms, whether knowingly or not. This reflexive relationship is a key ingredient in what Yackel and Cobb described as the development of intellectual autonomy. "Students who are intellectually autonomous in mathematics are aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgments as they participate in these practices" (Yackel & Cobb, 1996, p. 473). Students without such autonomy rely on "pronouncements of an authority" (p. 473), such as the teacher or a textbook. Further, Yackel and Cobb emphasized the necessary involvement of students' co-development of sociomathematical norms as instrumental in their development of intellectual autonomy in the mathematics classroom. Yet it is important to note the teacher's role in guiding the development of sociomathematical norms within a classroom.

Investigating and comparing the development of sociomathematical norms in two different classrooms, Lopez and Allel (2007) noted that the way teachers go about having explanations and solution strategies validated can influence how students participate in the classroom. Lopez and Allel found that by providing students with opportunities to evaluate their peers' mathematical explanations, students became more self-regulated to engage in such actions. Similar to the findings of Lopez and Allel, McClain and Cobb (2001) found that established sociomathematical norms can provide "directionality to the students' learning…" (p. 264). Additionally, prior experiences in contexts with facilitative soicomathematical norms were found to support students' autonomous conjecture.

The appropriate development of sociomathematical norms facilitates students' development of mathematical dispositions. Yet, simply providing opportunities for students to discuss mathematics does not always yield productive or rich discussions. Sfard (2007) observed 12 and 13 year olds discussing mathematics with their teacher. During one discussion, a conflict between the teacher and students emerged but the students failed to make any attempts to rectify this conflict. Sfard characterized this breakdown of discourse as being caused by the students' lack of properly developed sociomathematical norms. Sfard provides other characterizations of "students' unawareness of what kind of argument counts as legitimate in a mathematics classroom" (p. 594).

Both Yackel and Cobb (1996) and McClain and Cobb (2001) characterized how, over time, the development of certain sociomathematical norms facilitates students' intellectual autonomy, or mathematical dispositions. Lopez and Allel (2007), along with McClain and Cobb, further characterize how exposure to such norms influences student actions. Sfard (2007) provided a description of what a lack of developed sociomathematical norms looks like in mathematical discussions. For the purposes of the current study, the literature presented here is meant to emphasize the importance of students being in such discourse environments where sociomathematical norms *can* develop.

The Current Investigation

The present study used a national dataset collected by the U.S. Department of Education. The benefits of using such a dataset include its relative size and generalizability, its taking into account of the nested nature of educational data, and the reliability of its measures. The main drawback is that the items asked of teachers, parents, administrators, and students were not items specifically tailored for a specific research interest or area. Yet, often the benefits outweigh the drawbacks and, if such drawbacks are taken into consideration, these datasets can answer research questions that could not otherwise be evaluated with smaller samplings.

Such is the case with the present study. The dataset used here was the Early Childhood Longitudinal Study (ECLS). The study collected, among other variables, the frequency teachers' classes engaged in mathematical discussion and discourse-related actions, but did not assess questions that would provide a description of these classrooms' sociomathematical norms. However, "to understand the role of any given sociomathematical norm, it [is] necessary to analyse how it [is] related to the other norms" (Lopez & Allel, 2007, p. 263). As Lopez and Allel noted, sociomathematical norms are complex and the role one sociomathematical norm plays in one classroom can be quite different in another classroom. Therefore, any quantitative assessment of sociomathematical norms would likely be an exercise in and of itself, and beyond the scope of this study. What the current study seeks to examine is not the impact of sociomathematical norms on student math achievement, but the effect of a certain, general context has on math achievement. This context can be described as classrooms where frequent discussion takes place.

Classrooms in which frequent discussion takes place may or may not have properly developed sociomathematical norms. This is evidenced by the contrasting descriptions provided by Sfard (2007) and McClain and Cobb (2001). However, student exposure to classroom environments where discussion occurs frequently may, over the course of time, allow the student to develop competencies in mathematical discussion. The development of such competencies would, undoubtedly, benefit from the teacher's purposeful guidance in co-constructing sociomathematical norms with the students. Yet, it is equally logical to conclude that given enough exposure to contexts with frequent mathematical discussion, most students will develop some level of competence in mathematical discussion. Students with higher levels of mathematical discussion ability should also have higher mathematics achievement (Mercer & Sams, 2006). This line of logic leads to the following research questions:

1. Does student presence in classrooms with frequent discussion have a general, longitudinal effect on their mathematics achievement growth?
2. Does the effect of classroom discussion frequency differ from one grade to the next?
3. Do different frequencies of classroom discussion have more positive effects on individual student math achievement than other frequencies?

Methods

Sample and Data

Data collected from the Early Childhood Longitudinal Study (ECLS) was used in this study. ECLS was designed as a longitudinal study collecting data from kindergarten students in the 1998-1999 school year through their eighth grade enrollment in 2006-2007 (NCES, 2009). In all, data was collected in kindergarten, first grade, third grade, fifth grade, and eighth grade. The current study uses data from each grade level, which included different sample sizes each year due to attrition. Items selected from teacher questionnaires of students in the sample were a primary source of data, as were student math achievement scores. Due to missing data on questionnaires and attrition, some sample sizes were reduced. The sample sizes for each year are displayed in Table 1.

Table 1

Sample Sizes for Each Year of Data.

Grade & Year	Students	Teachers
Kindergarten 1998 – 1999	11,461	1,778
1st Grade 1999 – 2000	8,939	2,276
3rd Grade 2001 – 2002	7,336	2,713
5th Grade 2003 – 2004	3,358	1,763
8th Grade 2006 – 2007	2,832	1,641

Note: Samples presented here are effective sample sizes for analysis.

As can be observed from Table 1, the students per teacher ratio decreased each year to a mere 1.73 students per teacher in grade 8. An examination of the longitudinal effect of frequent classroom discussion was therefore conducted at the individual level. This decision and its implications are discussed in the following section.

Measures

Dependent variable.

The dependent variable, or outcome variable, in the current study is student mathematics achievement as measured with a standardized cognitive domain test (NCES, 2009). Versions of this assessment were administered in each year of data collection of ECLS and included a variety of math content. The mathematics cognitive domain test scores were standardized using Item Response Theory scale scores (IRT scores). IRT scores utilize student item response patterns to obtain a scale score that represented their content knowledge and, therefore, their ability. One of the advantages of IRT scale scores is their comparability to other achievement measures observed at different time points. IRT scores measured in the spring of each year in data collection were used as the outcome measure of the longitudinal analysis (*Math_IRT*) and the cross section analyses (*GK_IRT, G1_IRT, G3_IRT, G5_IRT, G8_IRT*), representing grades K through 8, respectively. Descriptive statistics of these variables can be found in Table 2.

Table 2

Descriptive Statistics for Students Math Achievement

	Range	Mean	S.D.	*N*
GK_IRT	11.57 – 112.51	35.98	11.83	11,461
G1_IRT	13.44 – 132.49	60.56	18.32	8,939
G3_IRT	37.47 – 166.25	97.12	25.14	7,336
G5_IRT	50.86 – 170.66	119.87	26.11	3,358
G8_IRT	66.26 – 172.20	138.05	23.75	1,641

Note: Statistics were weighted using appropriate cross-section weights

Independent variable.

Each spring, the teachers of students enrolled in the ECLS study completed a questionnaire asking questions regarding teacher background, instructional practices, observations of the child participant, and observations of the child participant's class. Of interest in the current investigation were items regarding mathematical discussion. The question asked for each grade level assessed is shown in Table 3.

Table 3.

ECLS Items Assessing Discussion Frequency.

Grade	Question	Responses
K, 1	How often do children in this class do each of the following MATH activities? -Explain how a math problem is solved. (NCES, 1999, p. 15; NCES, 2000, p. 20)	Never; Once a Month; Two or Three Times a Month; Once or Twice a Week; Three or Four Times a Week; Daily
3	How often do children in your class engage in the following?	Never or Hardly Ever;

	-Discuss solutions to mathematics problems with other children (NCES, 2002, p. 20)	Once or Twice a Month; Once or Twice a Week; Almost Everyday
5	How often does the child identified on the cover of this questionnaire engage in the following as part of mathematics instruction? -Discuss solutions to mathematics problems with other children. (NCES, 2004, p. 6)	Never or Hardly Ever; Once or Twice a Month; Once or Twice a Week; Almost Everyday
8	How often do the students in this class engage in the following? -Discuss their solutions to mathematics problems. (NCES, 2007, p. 11)	Never or Hardly Ever; Once or Twice a Month; Once or Twice a Week; Almost Everyday

As shown in Table 3, responses to discussion items in third, fifth, and eighth grades were on a 4 point scale. Therefore the Kindergarten and first grade responses were recoded to match the outcomes of later grades. “Never” was recoded to match “never or hardly ever;” “once a month” and “two or three times a month” were recoded to match “once or twice a month;” “Once or twice a week” was matched to “once or twice a week;” and “three or four times a week” and “daily” were matched to “almost everyday.”

The fifth grade item was assessed for the participating student, whereas all other discussion related items were assessed of the participating student’s class. Thus, this item was recoded to reflect classroom frequency rather than individual frequency. An aggregate variable was created for each classroom and then these means were rounded to the nearest whole number to match the 4 point scale of the other items.

After the recoding of discussion items outlined in the previous paragraphs was conducted, it was decided that frequencies of discussion would be dummy coded with *Never or Hardly Ever* as the reference group. *Almost Everyday* became *disc_daily* (1 = Almost Everyday, 0 = all other frequencies); *Once or Twice a Week* became *disc_weekly* (1 = Once or Twice a Week, 0 = all other frequencies); and *Once or Twice a Month* became *disc_monthly* (1 = Once or Twice a Month, 0 = all other frequencies). These new variables were formatted as within-student variables for longitudinal analysis and student-level variables for cross sectional analyses. Descriptive statistics of the discussion variables are displayed in Table 4.

The variables *disc_daily*, *disc_weekly*, and *disc_monthly* were defined so that they represented student enrollment in classes with more or less frequent discussion. This is an important distinction to make. The variables, as defined in this study, do not represent student frequency of discussion or classroom frequency of discussion. Since it was defined as a within-persons variable and student-level variable (for longitudinal and cross section analysis respectively), the dummy coded variables are characterized as enrollment. A similar type of variable assignment can be likened to student enrollment in a specific content level course. Such was done by Ma and Wilkins (2008) who used math course type as a student level variable which represented student enrollment in the course rather than the items' original description of course type.

Table 4

Descriptive Statistics for Recoded Discussion Variable by Grade.

Grade	Level of Discussion	Enrollment Frequency	Weighted Statistics*
K	Never or Hardly Ever	1052 (9.2%)	$\bar{X}$ = 1.93
	Once or Twice a Month	3047 (26.6%)	S.D. = .99
	Once or Twice a Week	3204 (28.0%)	*n* = 11,461
	Almost Everyday	4158 (36.3%)	
1	Never or Hardly Ever	65 (0.7%)	$\bar{X}$ = 2.55
	Once or Twice a Month	913 (10.2%)	S.D. = .70
	Once or Twice a Week	2244 (25.1%)	*n* = 8,939
	Almost Everyday	5717 (64.0%)	
3	Never or Hardly Ever	518 (7.1%)	$\bar{X}$ = 2.03
	Once or Twice a Month	1510 (20.6%)	S.D. = .90
	Once or Twice a Week	2835 (38.6%)	*n* = 7,336
	Almost Everyday	2475 (33.7%)	
5	Never or Hardly Ever	233 (6.9%)	$\bar{X}$ = 1.97
	Once or Twice a Month	540 (16.1%)	S.D. = .99
	Once or Twice a Week	1338 (39.8%)	*n* = 3,358
	Almost Everyday	1247 (37.1%)	
8	Never or Hardly Ever	60 (2.1%)	$\bar{X}$ = 2.61
	Once or Twice a Month	140 (4.9%)	S.D. = .66
	Once or Twice a Week	757 (26.7%)	*n* = 2,832
	Almost Everyday	1875 (66.2%)	

**Cross-Section Weights C2CW0, C4CW0, C5CW0, C6CW0, & C7CW0 were used for each respective grade level. Means are based off of the following coding scheme (0 = Never or Hardly Ever; 1 = Once or Twice a Month; 2 = Once or Twice a Week; 3 = Almost Everyday).*

Covariates.

Covariates included at the individual level for both longitudinal and cross section analyses included student gender (*dFemale*) and race/ethnicity (*dBlack*, *dHispanic*, *dAsian*, *dOther*). Socio-economic status (*SES*) was included at the within-student level for the longitudinal analysis since it can vary from year to year, but was included at the student level for the cross section analyses. *SES* was calculated at the household level and included the following components: father/male guardian's education; mother/female guardian's education; father/male guardian's occupation; mother/female guardian's occupation; and household income. For further details on how SES was computed, see NCES, 2009, p. 7-23.

Gender and race/ethnicity variables were included at the student level for both the longitudinal and cross section analyses. Gender (dFemale) was dummy coded to compare to male studentds (0 = male, 1 = female). Each race/ethnicity variable (*dBlack*, *dHispanic*, *dAsian*, *dOther*) were similarly dummy-coded as to compare to white students. For example, *dBlack* was coded such that 1 = Black, and 0 = non-Black. Descriptive statistics for all covariates for each grade level are presented in Table 5.

Table 5.

Descriptive Statistics of Covariates.

	Kindergarten	1st Grade	3rd Grade	5th Grade	8th Grade
dFemale	$\bar{X}$ = .49 SD = .50	$\bar{X}$ = .49 SD = .50	$\bar{X}$ = .49 SD = .50	$\bar{X}$ = .49 SD = .50	$\bar{X}$ = .49 SD = .50
dBlack	$\bar{X}$ = .18 SD = .38	$\bar{X}$ = .16 SD = .37	$\bar{X}$ = .15 SD = .35	$\bar{X}$ = .14 SD = .35	$\bar{X}$ = .12 SD = .32
dHispanic	$\bar{X}$ = .10 SD = .30	$\bar{X}$ = .09 SD = .29	$\bar{X}$ = .09 SD = .28	$\bar{X}$ = .10 SD = .30	$\bar{X}$ = .08 SD = .28
dAsian	$\bar{X}$ = .05 SD = .21	$\bar{X}$ = .04 SD = .21	$\bar{X}$ = .04 SD = .20	$\bar{X}$ = .05 SD = .21	$\bar{X}$ = .04 SD = .20
dOther	$\bar{X}$ = .05 SD = .21	$\bar{X}$ = .04 SD = .20	$\bar{X}$ = .04 SD = .20	$\bar{X}$ = .05 SD = .21	$\bar{X}$ = .05 SD = .20
SES	$\bar{X}$ = .04 SD = .78	$\bar{X}$ = .04 SD = .79	$\bar{X}$ = .03 SD = .78	$\bar{X}$ = .03 SD = .78	$\bar{X}$ = .04 SD = .78

Analysis

A two level hierarchical linear model (HLM-2) was used in the current study. HLM can be conceptually described as a "hierarchical system of regression equations" (Hox, 2002, p. 11).

HLM was employed both for the cross section analyses of each grade level and for the longitudinal analysis across grade levels. For the cross section models, I considered students as nested within classrooms (or teachers). HLM-2 allows us to explain this nested relationship by having separate regression equations for each classroom and an additional regression equation that examines the classroom-level data. For the longitudinal model, we considered variables measured at differing grade levels as being nested within the individual. Certain variables (e.g. *discuss*, *SES, Math_IRT*) change for the individual over time and are therefore nested aspects of the individual. For longitudinal models, HLM-2 allows us to examine the slope of growth as attributed to time and other factors. While the regression equations in the cross section models allow for interpretations of effect and/or impact on math achievement, the regression equations in the longitudinal HLM-2 model allows for the interpretation of coefficients as general changes over time in the effect/impact itself.

Specification of the Cross-Section Models

Five separate HLM-2 cross-section models were run using HLM6 software (Raudenbush, Bryk, & Congdon, 2007). Students were considered nested within classrooms (or teachers). Therefore, level-1 represented student level variables and level-2 represented classroom level variables. For the purpose of comparison, the model specifications were the same for each model:

$$\begin{aligned}\left(Math_{IRT_{K,1,3,5,8}}\right)_{ij} = \beta_{0j} + \beta_{1j}\left(disc_{daily_{K,1,3,5,8}}\right)_{ij} + \beta_{2j}\left(disc_{weekly_{K,1,3,5,8}}\right)_{ij} + \\ \beta_{3j}\left(disc\_monthly_{K,1,3,5,8}\right)_{ij} + \beta_{4j}\left(Prior\_Math\_IRT\right)_{ij} + \\ \beta_{5j}\left(dFemale\right)_{ij} + \beta_{6j}\left(SES_{K,1,3,5,8}\right)_{ij} + \beta_{7j}\left(dBlack\right)_{ij} + \\ \beta_{8j}\left(dHispanic\right)_{ij} + \beta_{9j}\left(dAsian\right)_{ij}\beta_{10j}\left(dOther\right)_{ij} + r_{ij}\end{aligned}$$

In the level-1 model displayed above, $(Math\_IRT_{K,1,3,5,8})_{ij}$ represents the Math IRT score student *i* achieved in classroom *j* each spring given the specific grade level (i.e. K, 1, 3, 5, 8). β_{0j} represents the average grade-specific math IRT score for white male students enrolled in classrooms that discuss mathematics *Never or Hardly Ever*, adjusted for prior achievement and *SES*. β_{1j}, β_{2j}, and β_{3j} represent the effect of student enrolment type on their mathematics IRT score for that particular grade. β_{4j} represents the association of prior achievement with students' spring math IRT scores. Spring math IRT scores from the previous measure in the data were used as the prior achievement measure for grades 1, 3, 5, and 8. For Kindergarten, a math IRT score obtained in the Fall of 1998 was used as the measure for prior achievement. β_{5j} is the gender effect and β_{6j} represents the effect of SES. Finally, the coefficients β_{7j}, β_{8j}, β_{9j}, and β_{10j} represent the effects of race/ethnicity.

The main reason for our use of cross-section analyses in each grade level was to have an additional perspective on the effect of students being enrolled in courses with frequent discussion from grade to grade. Therefore, classroom level variables were not examined. However, the slopes of β_{1j}, β_{2j}, and β_{3j} were set to vary randomly at level-2. This allowed for differences in the effect of student enrollment to vary between classrooms.

Specification of the Longitudinal Model

HLM-2 was used for the longitudinal analysis. Students were the level-2 grouping factor and level-1 was specified as within-student measures (i.e. *disc_daily, disc_weekly, disc_monthly, SES, Math_IRT*). Additional interaction effects between grade level and the discussion variables were also included. Therefore, level-1 was a set of separate regression equations, one for each student (Hox, 2002). Students' longitudinal math IRT scores were regressed, at level-1, onto their grade level, class discussion enrollment, and their socio-economic status.

Level-1:

$$
\begin{aligned}
Math_{IRT\,ti} = {} & \pi_{0i} + \pi_{1i}(grade)_{ti} + \pi_{2i}(disc_{daily})_{ti} + \pi_{3i}(disc_{weekly})_{ti} \\
& + \pi_{4i}(disc_{monthly})_{ti} + \pi_{4i}(grade \times disc_{daily})_{ti} \\
& + \pi_{4i}(grade \times disc_{eekly})_{ti} + \pi_{4i}(grade \times disc_{monthly})_{ti} \\
& + \pi_{3i}(SES)_{ti} + R_{ti}
\end{aligned}
$$

Level-2:

$$
\begin{aligned}
\pi_{0i} = {} & \beta_{00} + \beta_{01}(dFemale)_i + \beta_{02}(dBlack)_i + \beta_{03}(dHispanic)_i + \beta_{04}(dAsian)_i \\
& + \beta_{05}(dOther)_i + \mu_{0i} \\
\pi_{1i} = {} & \beta_{10} + \beta_{11}(dFemale)_i + \beta_{12}(dBlack)_i + \beta_{13}(dHispanic)_i + \beta_{14}(dAsian)_i \\
& + \beta_{15}(dOther)_i + \mu_{1i} \\
\pi_{2i} = {} & \beta_{20} \\
\pi_{3i} = {} & \beta_{30} \\
\pi_{4i} = {} & \beta_{40} \\
\pi_{5i} = {} & \beta_{50} + \beta_{51}(dFemale)_i + \beta_{52}(dBlack)_i + \beta_{53}(dHispanic)_i + \beta_{54}(dAsian)_i \\
& + \beta_{55}(dOther)_i + \mu_{5i} \\
\pi_{6i} = {} & \beta_{60} + \beta_{61}(dFemale)_i + \beta_{62}(dBlack)_i + \beta_{63}(dHispanic)_i + \beta_{64}(dAsian)_i \\
& + \beta_{65}(dOther)_i + \mu_{6i} \\
\pi_{7i} = {} & \beta_{70} + \beta_{71}(dFemale)_i + \beta_{72}(dBlack)_i + \beta_{73}(dHispanic)_i + \beta_{74}(dAsian)_i \\
& + \beta_{75}(dOther)_i + \mu_{7i} \\
\pi_{8i} = {} & \beta_{80}
\end{aligned}
$$

π_{0i} represents the math IRT score of student *i* at the initial measurement, spring 1999. π_{1i} represents the effect of grade level on students' math IRT score. In other words, π_{1i} can be viewed as the natural effect of time on increasing math achievement. π_{2i}, π_{3i}, and π_{4i} are the general effects of student enrollment in classrooms with different levels of frequent math discussion. However, each of these variables disregards the effect of time and individual. Therefore, while these variables are statistically necessary for a rigorous analysis, they are not meaningfully useful for the research questions posed in the current study. Therefore, the interaction effects π_{5i}, π_{6i}, and π_{7i} where included. Each of these represents the general change enrollment in classes with more or less frequent math discussion had on student growth in math achievement as they progressed in grade level. π_{3i} represents the effect of SES on math achievement over time.

The slopes of *grade*, $grade \times disc\_daily$, $grade \times disc\_weekly$, and $grade \times disc\_monthly$ were set to vary randomly at level-2. Setting *grade* to vary randomly between individuals is a logical decision since different individuals will likely experience different rates of growth in their mathematics achievement over their schooling. Setting $grade \times disc\_daily$, $grade \times disc\_weekly$, and $grade \times disc\_monthly$ to vary randomly was done to see if the interaction effect between enrollment and grade level varied between individuals. In the model equation displayed above, only the level-1 variables that were set to vary randomly were modeled at level-2. Additionally, each of these slopes were regressed on gender and race/ethnicity to examine whether there were significant effects for these factors.

Results

Cross-Section Results

While it is customary to present baseline results for all HLM models constructed, for the sake of brevity, only the final models for the cross-section analyses are presented here. Since the main purpose of this study was to examine the longitudinal effect of enrollment in classrooms with different frequencies of discussion, the cross-section results provide additional information to the longitudinal results, while not being a major point of focus. Results from cross-section analysis of each grade level yielded varying results. These results are presented in Table 6, with the shaded rows representing the effects of the independent variables of interest. In general, the intercept for each grade level was statistically significant from zero and was larger than the preceding grade level. Enrollment in classes with daily math discussion was found to be positive in every grade level and statistically significant from *Never or Hardly Ever* in each grade but 3rd grade. Enrollment in classes with weekly discussion was statistically significant only in grades K and 1 but was near significant in grade 5 (p = .085). Enrollment in classes with monthly discussion was only statistically significant in kindergarten. An overall look at the impact of enrollment by grade level illustrates that in grades K and 1 a generally positive relationship between the frequency a class discusses mathematics and the enrolled student's math achievement was found. However, this trend becomes convoluted by grade 3. In 3rd grade, all enrollment types are statistically similar, but weekly discussion had a larger effect than daily discussion. In 5th grade, monthly discussion had a larger effect than weekly and in 8th grade monthly discussion had a larger effect than weekly and daily discussion. Yet, enrollment in classes with monthly discussion was not statistically significant in grades 5 or 8. This appears to be due to large standard error for 5th grade (S.E. = 2.78) and 8th grade (S.E. = 2.65) at level-1. A similar finding appears for enrollment in classes with both daily and weekly discussion. In 3rd grade, the standard error for each enrollment type was larger than the level-1 coefficients. Typically, such large level-1 standard errors would indicate a possibly significant amount of variability at level-2 as well. Due to loss in degrees of freedom, setting the dummy coded variables' slopes as random at level-2 would not allow the model to converge. Therefore, such a possibility was not able to be examined in detail. What can be concluded from the cross section analysis is that student enrollment in classes with daily discussions about mathematics will generally predict higher levels of math achievement. This is true in grades K, 1, 5, and 8. Enrollment in classes with other levels of discussion can predict higher math achievement than enrollment in classes with little or no discussion, but such effects appear to vary since the standard errors for these effects were often quite large in later grades.

Table 6.

Results of Final Models by Grade for Cross-Section Analyses.

Estimated Fixed Effects

	K	1	3	5	8
β_{0j}, intercept	36.22**	59.36**	101.70**	120.09**	137.48**
β_{1j}, disc_daily	1.81**	5.53**	1.09	5.54*	3.89*
β_{2j}, disc_weekly	1.71**	4.51*	1.11	4.48	3.47
β_{3j}, disc_monthly	1.08**	3.40	0.13	4.55	4.42
β_{4j}, Prior IRT	1.04**	1.07**	0.97**	0.82**	0.75**
β_{5j}, dFemale	-0.34*	-0.82**	-3.61**	-1.16*	1.50
β_{6j}, SES	0.83**	2.06**	4.26**	2.54**	1.78**
β_{7j}, dBlack	-1.59**	-2.86**	-4.58**	-3.98**	-0.61
β_{8j}, dHisp	-0.87**	-0.34	-0.22	1.43	-1.69
β_{9j}, dAsian	0.62	0.46	0.89	2.76	1.90
β_{10j}, dOther	-0.65	-2.76**	-1.49	-1.56	1.39

Estimation of Variance Components

	K	1	3	5	8
μ_{00}	2.16**	3.71**	6.40**	6.27**	6.11**
r_{ij}	6.12	10.34	12.79	10.39	9.09

**p<.05, **p<.01*

Prior achievement and SES were found to be statistically significant for each grade level. The effect of prior achievement decreased for each grade level, while the effect of SES increased to a maximum at grade 3 but then decreased substantially after that point. Students who were black had a statistically significant negative effect in each grade except 8th grade. Students of other ethnicities had a statistically significant negative effect in grade 1, but were otherwise statistically similar to Caucasian students. The effect of female gender was statistically significant in every grade level and was negative through 5th grade. However, in 8th grade, girls appeared to outperform boys in general. While the findings of the covariates are interesting in and of themselves, it is the effect of *disc_daily*, *disc_weekly*, and *disc_monthly* that are of primary interest in the current investigation.

Longitudinal Results

Baseline model results.

Results of the baseline model are displayed in Table 7. Students were found to vary significantly on their initial math achievement ($var(\rho_{0j})$ = 165.78, $p < .01$), but students were not found to vary significantly in their rate of growth from grade to grade ($var(\rho_{1j})$ = 1.24, $p > .50$). However, the growth rate was found to be statistically significant from zero (π_{10} = 13.14, $p < .01$), meaning that on average, students' math IRT scores increased 13.14 points every grade level.

Table 7

Results of Baseline Model

Estimation of Fixed Effects

Fixed Effect	Coefficient	Std. Error	t-ratio	df	p-value
Intercept, π_{00}	48.46	.19	260.54	7305	.000
grade, π_{10}	13.14	.03	453.71	7305	.000

Estimation of Variance Components

Random Effect	SD	Variance Component	df	Chi-Square	p-value
Intercept, ρ_{0j}	12.88	165.78	7327	15654.76	.000
grade, ρ_{1j}	1.24	1.55	7327	7250.00	>.50
Level-1 effect, r_{tj}	15.23	231.82	-	-	-

Final model results.

The results of the final model are presented in Table 8 and Table 9. Similar to the baseline model results, the intercept was found to vary significantly between students ($var(\rho_{0j})$ = 82.85, *p* < .01), but now the effect of grade level was found to vary significantly as well ($var(\rho_{1j})$ = 4.47, *p* < .01). Similar to the cross section analyses, the HLM model would not converge with any of the dummy coded interactions (*grade*, $grade \times disc\_daily$, $grade \times disc\_weekly$, $grade \times disc\_monthly$) set as random at level-2. Therefore, these coefficients were fixed at level-2.

Table 8.

Level-1 Results of Longitudinal Final Model

Estimation of Fixed Effects.

Fixed Effect	Coefficient	S.E.
π_{00}, intercept	41.71**	.60
π_{01}, dFemale	-1.41**	.34
π_{02}, dBlack	-8.52**	.52
π_{03}, dHispanic	-6.97**	.59
π_{04}, dAsian	-1.20	.86
π_{05}, dOther	-6.40**	.83
π_{10}, grade	17.48**	.25
π_{11}, dFemale	-.26	.30
π_{12}, dBlack	-1.03*	.46
π_{13}, dHispanic	-.76	.50
π_{14}, dAsian	1.89*	.74
π_{15}, dOther	-2.11**	.68
π_{20}, disc_daily	9.18**	.59
π_{30}, disc_weekly	7.49**	.61
π_{40}, disc_monthly	1.29*	.63
π_{50}, grade*daily	-3.72**	.26
π_{51}, dFemale	-.09	.30
π_{52}, dBlack	-.48	.46
π_{53}, dHispanic	1.05*	.51
π_{54}, dAsian	-.90	.74
π_{55}, dOther	1.43*	.69
π_{60}, grade*weekly	-1.85**	.27
π_{61}, dFemale	-.26	.31

π_{62}, dBlack	-1.00*	.47
π_{63}, dHispanic	.80	.52
π_{64}, dAsian	-1.38	.76
π_{65}, dOther	.28	.71
π_{70}, grade*monthly	.40	.39
π_{71}, dFemale	-.04	.42
π_{72}, dBlack	-1.82	.55
π_{73}, dHispanic	1.15	.85
π_{74}, dAsian	-1.76	1.14
π_{75}, dOther	-.44	.72
π_{30}, SES	5.62**	.21

Table 9

Estimation of Variance Components for Longitudinal Final Model.

Random Effect	Variance	df
ρ_{00}, intercept	82.85**	6785
ρ_{10}, grade	4.47**	6785
r_{ij}, level-1 error	190.35	-

Discussion variables.

The variables *disc_daily*, *disc_weekly*, and *disc_monthly* were each found to be statistically significant (π_{20} = 9.18, *p* < .01; π_{30} = 7.49, *p* < .01; π_{40} = 1.29, *p* < .05), indicating that, disregarding time, students enrolled in classes with more frequent discussion had higher math IRT scores than students enrolled in classes that never have discussions about mathematics. While it is tempting to regard these specific results as promoting the use of math discussion, these results should not be interpreted as such. Since *disc_daily*, *disc_weekly*, and *disc_monthly* do not acknowledge differences in math achievement due to grade level, some of the differences in the effect of enrollment in this regard may be due to certain students being enrolled in such classes in, say, 8th grade while others may be enrolled in classes with little or no discussion in, say, kindergarten or 1st grade. While the distribution of frequencies displayed in Table 4 suggest the reverse may be more typical, we cannot make the assumption that this is the case. Neither can we make the assumption that the differences found for *disc_daily*, *disc_weekly*, and *disc_monthly* are due simply to differences in grade level. The variables themselves, quite simply, cannot be interpreted in a way meaningful to the questions addressed in this study. Their purpose in the model are to moderate effects of the variables that can be more meaningfully interpreted.

Interaction variables for discussion and grade level.

The interaction effects were found to have intriguing results. $grade \times disc\_daily$ was found to be statistically significant (π_{50} = -3.72, $p < .01$), as was $grade \times disc\_weekly$ (π_{50} = -1.85, $p < .01$). $grade \times disc\_monthly$ was not found to be statistically significant (π_{70} = .40, $p = .30$). These results should be interpreted with care. The negative coefficient found for $grade \times disc\_daily$ indicates that, in comparison to enrollment in classes with little or no math discussion, the effect of enrollment in classes with daily math discussions decreases as students progress in school. Therefore, enrollment in classes with daily discussion is more associated with student math achievement in earlier grades than it is in later grades. A similar interpretation can be made for $grade \times disc\_weekly$. The impact of enrollment in classes with monthly discussion over time does not appear to be statistically different from that of enrollment in classes with little or no discussion. Again, the coefficients found for $grade \times disc\_daily$, $grade \times disc\_weekly$, and $grade \times disc\_monthly$ describe how the impact of enrollment changes over time, not the impact itself. Results from the cross section analyses indicate that enrollment in classes with daily discussion typically predicts higher math achievement scores than enrollment in classes with little or no math discussion. The results described in this longitudinal analysis suggest that while enrollment in classes with daily math discussion is more effective in increasing students' math achievement, this effectiveness decreases as students progress through school.

Level-1 and level-2 covariates.

SES, which was added as a covariate at level-1, was found to be statistically significant (π_{30} = 5.62, $p < .01$), indicating that students with higher SES improved their math IRT scores at a higher rate than other students. Examination of student level variables show that females tended to have lower initial math IRT scores than boys (π_{01} = -1.41, $p < .01$). Blacks, Hispanics, and students of other ethnicities tended to have lower initial math IRT scores than white students (π_{02} = -8.52, $p < .01$; π_{03} = -6.97, $p < .01$; π_{05} = -6.40, $p < .01$). Black students and students of other ethnicities tended to have slower natural growth in achievement than white students (π_{12} = -1.03, $p < .05$; π_{15} = -2.11, $p < .01$), while Asian students tended to have higher natural growth than white students (π_{14} = 1.89, $p < .05$). Hispanic students and students of other ethnicities tended to see less of a decrease in $grade \times disc\_daily$ than white students ($\pi_{33} = 1.05, p < .05$; π_{35} = 1.43, $p < .05$), but this relationship did not hold for enrollment in classes with other frequencies of math discussion. Black students tended to see more of a decrease in $grade \times disc\_weekly$ and $grade \times disc\_monthly$ than white students (π_{62} = -1.00, $p < .05$; π_{72} = -1.82, $p < .05$). While these results are intriguing, it is difficult to say what may or may not cause the differences found here. It may be that black students experience some form of social inequality when enrolled in classes with less frequent discussion, but it is strange that a similar relationship would not be found of other minority students. The results for Hispanic students and students of other ethnicities is equally perplexing.

Overview of Cross Section and Longitudinal Results

Figure 1 illustrates the growth, by grade, of student math achievement by enrollment in classes with more or less frequent discussion. As one can observe, classes that never or hardly ever discuss mathematics consistently perform less well than classes that have discussion of any frequency. The differences between grade level appear to be between the actual frequency of engaging or not engaging in discussion. This graphical analysis could provide clues for the different results found in both the cross-sectional and longitudinal analyses.

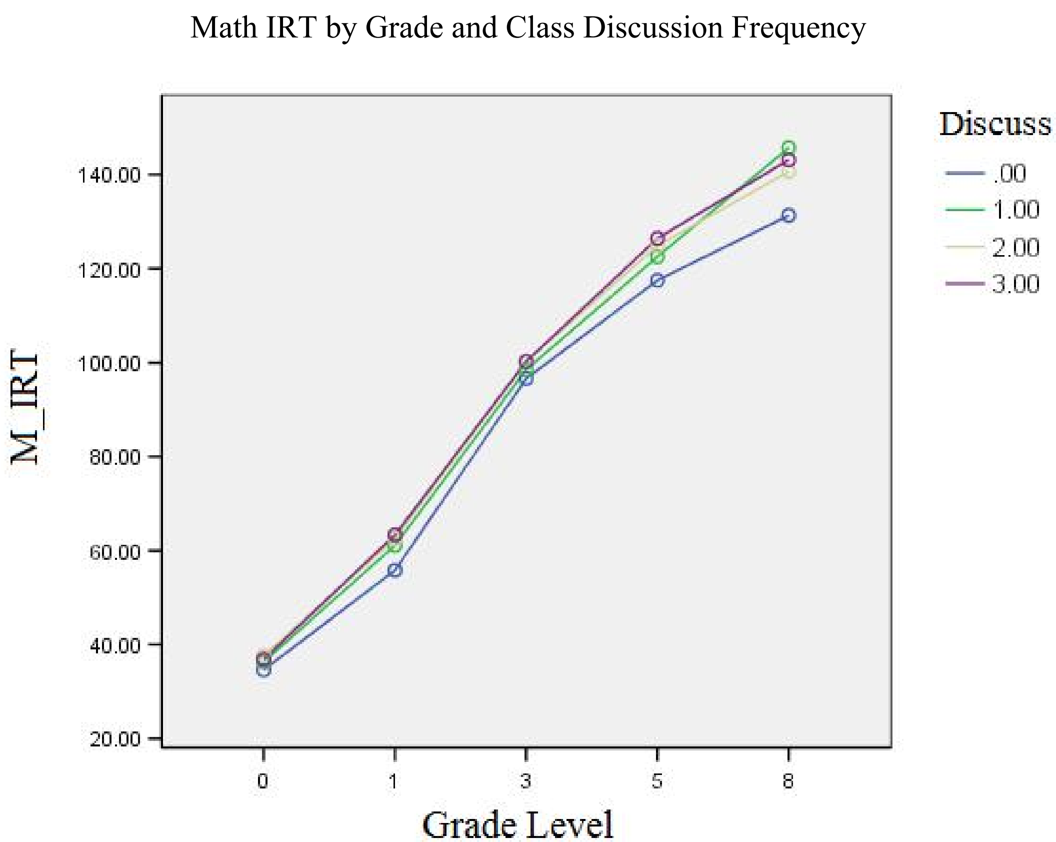

Figure 1. Graph of Math Achievement and Class Discussion Frequency by Grade Level.

The cross section analyses illustrated that the standard error found in each grade level had an effect on whether certain means were found to be statistically significant from others. For example, even though students who were enrolled in 8th grade classes with monthly discussions tended to have higher math IRT scores than students in classes with daily math discussions (see Figure 1), the degree of variance for *disc_monthly* characterized it as statistically similar to *disc_never*. This suggests that *disc_daily* is a more consistent predictor of higher math IRT scores than *disc_monthly* in 8th grade, even though *disc_monthly* had a higher mean. Similar results were found in 5th grade and 3rd grade. These results suggest that while enrollment in classes that discuss math more frequently is often more beneficial than enrollment in classes with no discussion, there is a large amount of variability in the impact of such enrollment.

The longitudinal analysis found that, over the course of time in school, the impact of *disc_daily* and *disc_weekly* affects math achievement growth at a slower rate than enrollment in classes that never discuss mathematics. When examining these results in combination with those of the cross sectional analyses, an intriguing picture begins to form. Students' enrollment in classes with more frequent discussion generally has a more positive impact on their math achievement than enrollment in classes with little to no discussion. This was found to be true in each grade level examined except 3rd grade, and these results took into account prior achievement, race/ethnicity, gender, and SES. Yet, even though enrollment in classes with daily discussion was found to consistently be a better predictor of higher math achievement than enrollment in classes with no discussion, the effect associated with *disc_daily* tended to slow down as students progressed in schooling. In other words, enrollment in classes with more frequent math discussion was more beneficial at earlier grades than later grades. Therefore, enrollment in classes with daily math discussions was found to be generally more beneficial to student achievement than enrollment in classes with little or no discussion about math, but such enrollment is relatively more effective in earlier grades than in later grades.

Discussion

The results of the present study are significant in two distinctive ways. First, cross sectional analyses found that, in general, student enrollment in classes with daily discussions about mathematics consistently outperformed students in classes with little or no discussion. A high standard error in 3rd grade prevented a statistically significant result in this regard, but in grades K, 1, 5, and 8 this trend held true. Second, the effectiveness of enrollment in classes with frequent discussion decreased as students progressed to later grade levels. Therefore, while student math achievement scores tended to benefit from enrollment in classes with daily or weekly math discussions in any grade, the size of this effect decreased when compared to the effect of being enrolled in classes with little to no discussion. While these findings are highly significant in and of themselves, it is important to remember that it was student enrollment in classes with frequent math discussion that was evaluated; not student discussion itself. Therefore, it was the effect of a context for mathematical discussions that was examined in the current study. This distinction should be considered at the forefront of the discussion that follows.

The classroom context a student is present in is an important contributor to a student's discourse-related decisions (Mercer, 2000). The context examined in the current investigation was classrooms with different frequencies of mathematical discussion. Classrooms with effective mathematical discussion practices have been observed to have more frequent discussions about mathematics (e.g. Truxaw & DeFranco, 2007; Williams & Baxter, 1996; Wood, 1999). However, the converse is not necessarily true (Kazemi & Stipek, 2001; Manouchehri & St. John, 2006). The present study took this into consideration, but given the nature of the data used, I used the frequency of discussion as a measurement rather than quality of discussion. This is not a weakness in the study's design, but rather a distinctive perspective of a certain population. Despite several qualitative studies emphasizing that mere frequency of discussion is not an indicator of effective discussion, many teachers undoubtedly use this approach. The findings of Kosko and Miyazaki (2009; 2011) suggest this may be the case. Kosko and Miyazaki (2009) observed that a significant amount of variance between classrooms and schools in the impact of more frequent student discussion on 5th grade math achievement could be explained by enrollment in 3rd grade classes with higher frequencies of math discussion. The results suggested that discussion was more effective in some classrooms and schools than others. Yet, when one examines Table 3 in the present study, we can see that in each grade well over half of classrooms have discussions about math more than once a week. This prevalence in frequency of discussion, which was evaluated in the current study, suggests that a large number of students are in classrooms with less effective discussion practices.

The results presented here suggest that even with a large amount of variability, student exposure to contexts with daily math discussions has a large and positive impact on their math achievement. Described another way, whether discussion practices are likely to be more or less effective, in general, a student enrolled in a class with daily math discussions will have larger gains in math achievement than a similar student enrolled in a class with little or no discussions about mathematics. This relationship was found to be true and statistically significant in grades K, 1, 5, and 8, and is considered to be highly significant. The potential implications of this finding suggests that while more frequent discussion does not equate to better quality discussion (Kazemi & Stipek, 2001), the context of classrooms with frequent math discussions indirectly or directly improve students' math achievement.

When one conjectures about what types of social or sociomathematical norms may exist in the classrooms with more frequent discussion, we might consider that such classrooms would be likely to have more caring and supportive environments and would involve students in the co-creation of sociomathematical norms. However, we know this is not always the case (e.g. Kazemi & Stipek, 2001; Sfard, 2007). In fact, the large amount of variance in the impact of frequent discussion found in certain grade levels in the current study suggests that this is not the case in a large number of classrooms and schools. Additional findings from other studies on discussion and math achievement suggest that the way students engage in mathematical discussion is important (Hiebert & Wearne, 1993; Mercer & Sams, 2006). However, both these studies also suggested that students who engaged in discussion about math more frequently also showed higher gains in math achievement. The current study supports these latter findings.

The findings of the current study related to the decreased impact of *disc_daily* and *disc_weekly* were surprising. Additionally, the author is at a loss for a possible reason to explain this decreasing impact. There are a number of possibilities that might be explored. First, do teachers in earlier grades facilitate mathematical discussions differently than teachers of later grades? If so, is one method better than another? It is quite possible that teachers of earlier grade levels scaffold student engagement in mathematical discussions better than teachers of later grade levels, thereby accounting for the decrease in impact found in the current study. Another possibility is that younger students may simply be more receptive to learning how to discuss mathematics than older students who may have internalized a more traditional view of mathematics and the mathematics classroom. No matter what the cause in the decrease in impact is, the results of the current study suggest two implications in this regard. First, this decrease in impact should be further studied to see what possible causes it has. Second, mathematical discussion should be encouraged early and often, as it is relatively more effective in earlier grades and generally effective in improving math achievement throughout schooling.

D'Ambrosio et al. (1995) outlined mathematical discussion as a means of increasing math achievement. Certain studies uphold this claim (Hiebert & Wearne, 1993; Mercer & Sams, 2006). The results presented here suggest that mathematical discussion does have a positive effect on students' mathematics achievement, but this effect is higher in earlier grades than in later grades. Additionally, there is a large degree of variability in the effect of enrollment in classes with more or less frequent discussion, which could infer that discussion in some classrooms has less of an impact on students' math achievement than discussion in other classrooms. Therefore, while more frequent mathematical discussion in the math classroom appears to have a generally beneficial effect on mathematics achievement, the practical implications of this study suggest that any incorporation of mathematical discussion should be accompanied with appropriate mathematical discourse practices.

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Translating and Adapting the Mathematical Knowledge for Teaching (MKT) Measures: *The Cases of Indonesia and Norway*

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Abstract

This paper examines issues of translating and adapting an instrument that aims at measuring mathematical knowledge for teaching in Indonesia and Norway. The instrument was created for use in the U.S., and we discuss problematic and challenging issues of translation and adaptation. Two items from the released items pool were translated using a common framework modified from a previous study to exemplify critical issues that need to be resolved prior to using such instrument in another country. Themes identified in this study include a) minor challenges due to cultural differences; b) the use of technical language in schools; c) incommensurable contexts across countries; and d) the use of mathematical models.

Keywords: cross-national comparison, instrument translation, instrument adaptation, mathematical knowledge for teaching

Introduction

There is a growing interest and need to develop valid and reliable instruments to measure teachers' knowledge of mathematics due to a climate of increased accountability. Few scholars will dispute that teachers' knowledge of mathematics is one of the most important influences on teaching practices and eventually on what students learn (Ball, 1990; Ball, Lubienski, & Mewborn, 2001; Hill, Ball, & Schilling, 2008; Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008). Thus, the availability of measures to reliably assess what teachers know holds promises for further understanding factors contributing to this knowledge and thus inform teacher education programs. Moreover, with the increased attention to comparative studies in mathematics education in the past decades, examining the quality of teachers' mathematical knowledge in different countries may provide insights on improving students' achievement (An, Kulm, & Wu, 2004; Cai, 2005; Ma, 1999). However, the scope of cross-national studies on teachers' mathematical knowledge has been limited to a few countries, and these selective countries perform well when compared to the United States on international comparison (e.g., An et al., 2004; Cai, 2005; Ma, 1999; Stigler & Hiebert, 1999; Zhou, Peverly, & Xin, 2006). Widening the range of these studies to incorporate more countries, including developing countries, may be useful to reach a greater understanding of the teaching and learning of mathematics. This article reports an initial stage of such endeavor by examining issues of translation and adaptation of a U.S. based instrument for measuring teachers' mathematical knowledge in Indonesia and Norway, where the focus is on the challenges faced and problems encountered when using this instrument in different cultural settings. To illustrate the underlying complexity of the translation process of such instrument, a case study was conducted in which two items were translated and adapted for use in the two countries, and are discussed in this article.

Bradburn and Gilford (1990) suggest that using existing test instruments for international comparative studies is beneficial in that there is linkage to other ongoing studies. However, Emenogu and Childs (2005) remark that even when rigorous processes of translation, verification, and field-testing are followed, translation may introduce measurement non-equivalence. Differences may occur, not only due to language differences, but also variability in teaching practices. For instance, curriculum differences such as the sequence of mathematics courses, the time spent on topics, availability of textbooks and other materials may cause differences in the relative item difficulty of measures (Emenogu & Childs, 2005). Understanding the context of the intended country where the measures are to be used is therefore deemed necessary.

One set of measures for teachers' mathematical knowledge that has been widely studied and shown to be successful in the United States is the Mathematical Knowledge for Teaching (MKT) measures (Learning Mathematics for Teaching Project, 2006)[i]. Researchers from other countries have been interested in using these measures, and attempts to adapt the MKT instrument have been conducted in Ireland where Delaney (2008) points to some possibilities as well as some problematic issues. The process of translating such items is far from straightforward, and there are several issues to be aware of when attempting such endeavor (cf. Delaney et al., 2008). The Irish project has lately been followed by similar attempts in Ghana (Cole, 2009), South Korea (Kwon, 2009), Indonesia (Ng, 2009) and

Norway (Mosvold & Fauskanger, 2009; Mosvold, Fauskanger, Jakobsen, & Melhus, 2009). The MKT measures were not built for the purpose of comparing the knowledge of teachers in the U.S. with that of teachers in other countries, say Indonesia or Norway. However, investigating the adaptability of these measures in other countries would be worthwhile for future studies to compare teachers' knowledge across nations. Such attempts would have to pay close attention to the possible challenges and pitfalls in the process of translation and adaptation. If there are significant differences, it is important to figure out whether these differences are related to the translation process, to cultural differences, or to other aspects.

Such challenging questions were raised in a symposium session at the 2009 Annual Meeting of the American Educational Research Association, in which the authors of this article took part. We discovered that although our projects so far had been carried out without any direct contact or co-operation, the challenges that we had faced were quite similar. As a follow-up to the above mentioned symposium session, we decided to go deeper into a discussion of the experiences that we had gained in our projects, with a particular focus on issues related to translation and adaptation of the MKT items. Through the discussions in this article, we hope to contribute to the field with recommendations and suggestions for future research. Although our discussions and recommendations concern the use of the MKT measures in particular, we believe that our findings might influence other similar projects in different areas of research as well.

The following is the research question that we will discuss in this article:

> What challenges were encountered in the process of translating and adapting the MKT measures for use in Indonesia and Norway?

Theoretical background

Mathematical knowledge for teaching (MKT)

Research from the last 15 years indicates that "the mathematical knowledge of many teachers is dismayingly thin" (Ball, Hill, & Bass, 2005, p. 14). When analyzing 700 first and third grade teachers (and almost 3000 students), researchers found that the teachers' knowledge had an effect on the students' knowledge growth (Hill, Rowan, & Ball, 2005). Stigler and Hiebert (1999) claim that: "Although variability in competence is certainly visible in the videos we collected, such differences are dwarfed by the differences in teaching methods that we see across cultures" (p. 10). But even though research indicates that teachers' knowledge might have a positive influence on students' learning, it is not obvious what the content of this knowledge is.

Both studies referred to in this article made use of the MKT items that were developed by researchers at the University of Michigan in relation to the LMT project. Theoretically, the MKT construct follows Shulman's (1986) efforts to define the theories concerning subject matter knowledge (SMK) and pedagogical content knowledge (PCK). The categorization of the various components of teacher knowledge has evolved from Shulman's original proposal, where he distinguished between SMK, PCK, and knowledge of curriculum. In the LMT project, this model evolved into a model of MKT (Figure 1).

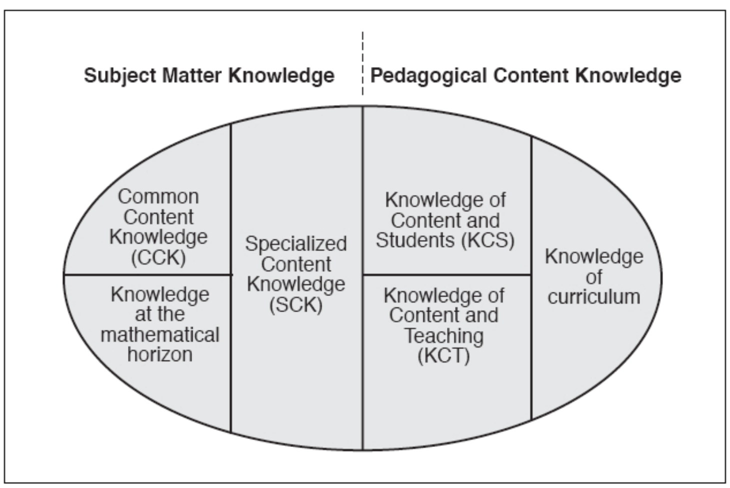

Figure 1. Domains of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008, p. 403).

These domains were identified through psychometric analyses, but the MKT items were developed based on teaching practices (studies of videos from classrooms) in the United States. Although the items focus on tasks of teaching, which is supposed to be of a universal nature, the items may not translate to other countries. Consider what Ball and colleagues (2008) identify as mathematical tasks of teaching in the U.S.:

- Presenting mathematical ideas
- Responding to students' "why" questions
- Finding an example to make a specific mathematical point
- Recognizing what is involved in using a particular representation
- Linking representations to underlying ideas and to other representations
- Connecting a topic being taught to topics from prior or future years
- Explaining mathematical goals and purposes to parents
- Appraising and adapting the mathematical content of textbooks
- Modifying tasks to be either easier or harder
- Explaining the plausibility of students' claims (often quickly)
- Giving or evaluating mathematical explanations
- Choosing and developing usable definitions
- Using mathematical notation and language and critiquing its use
- Asking productive mathematical questions
- Selecting representations for particular purposes
- Inspecting equivalencies

(Ball, Thames, & Phelps, 2008, p. 400)

Upon close examination, some of these tasks may be foreign to teachers in other countries. In some Asian countries, for instance, student questioning is not endorsed, and thus, responding to students "why" questions might therefore not be a relevant task of teaching. As another example, many countries have a national curriculum and there may not be many variations among textbooks. In some countries, teachers are expected to use the textbook as a prescriptive manual, and thus there is no room for *appraising or adapting mathematical content from the textbooks*. These examples indicate that the translation and adaptation of measures such as the MKT is not straightforward and requires careful scrutiny in order to be used successfully in another setting. There may also be differences within each task of teaching. *Presenting mathematical ideas* might be a task of teaching that applies worldwide, but the choice of words may be different for one setting than another. This makes the translation of MKT items challenging.

Issues of translation and adaptation of the MKT measures

Historically, translation of texts between cultures was a matter of substituting each word in the original language with an equivalent word in the new language, and this is often referred to as interlinear translation. Such ideas about translation have been important for translators for centuries, in particular with reference to translation of sacred texts like the Bible, where faithfulness to the original text has been of utmost importance. These ideas have changed, however, and translation has become more focused on preserving the functional equivalence of a text. Also, there is an agreement that different strategies of translation have to be used for different types of text (Lefevere & Bassnet, 1998). This is true for translation of test items as well, where it is not only a matter of finding equivalent word in an interlinear manner. Translation errors are known to be a major reason why some items function poorly in international tests of students' knowledge (see Adams, 2005). At the same time, studies normally provide little information as to how measurement instruments are translated and adapted for use from one country to another (e.g., Ma, 1999). Publications resulting from such studies typically present little to no description about translation issues arising in the research, particularly in the case of measures of teachers' knowledge (Delaney et al., 2008).

The process of translating the MKT measures into a different language is not only a matter of word choice. It is also of importance to adapt the measures for use in a cultural context that is quite different from the original intended setting. This is particularly crucial with the MKT items, since they were not originally created for use outside the U.S. Although the items aim at covering tasks of teaching (supposedly to be of a universal nature), they are strongly grounded in the practice of teaching mathematics, which may vary across countries. Therefore, it is necessary to include experts of teaching in the process of translating and adapting the items.

After the translation process, an instrument should continue to measure the same characteristics that it originally intended to measure (Geisinger, 1994). The idea of ensuring construct equivalence is therefore an important methodological goal for translating the MKT measures into another language. Singh (1995) suggests six steps in establishing construct equivalence. The first three of these (functional equivalence, conceptual equivalence, and instrument equivalence) should be established prior to using the measures for data collection (Singh, 1995; Delaney, 2008). The first, functional equivalence, relates to whether or not a construct serves the same function in all the countries where the instrument will be used (Singh, 1995). Conceptual equivalence refers to the question of whether or not a construct means the same across cultures (Delaney, 2008). Finally, instrument equivalence is related to both the format and the content of the items. Instrument equivalence is established when the items in the instrument are equally interpreted in other cultural settings as well as the initial targeted population (Singh, 1995).

Delaney and colleagues (2008) argue that since the MKT is defined as “the mathematical knowledge needed to carry out the work of teaching mathematics” (Ball, Thames, & Phelps, 2008, p. 395), the notion of MKT serves the same function in every country where mathematics is taught. Using logical argument, they deducted that since teachers anywhere will require some forms of mathematical knowledge in order to teach their students, it is self-evident that the construct of MKT satisfies the requirement of having functional equivalence, despite possible differences in curricula, teaching traditions, or expectations from education systems (Delaney et al., 2008). In terms of conceptual equivalence, the Irish research team examined the MKT construct more closely by studying the work of teaching in Ireland. They compared their work to conceptions of the work of teaching that informed the development of MKT. These researchers also studied literature about the construct, and they analyzed items based on the construct. They found relatively minor differences in their analysis. Finally, for instrument equivalence in the Irish study, a focus group consisting of Irish teachers and mathematicians scrutinized the items and proposed changes to make the items culturally fit (Delaney et al., 2008). Based on the result of this focus group and subsequent interviews with respondents, only two items were identified to cause differences in interpretation by the Irish teachers.

Therefore, judging from the functional, conceptual, and instrument equivalences, the adaptation of the U.S. based MKT instrument in Ireland was relatively successful. One possible explanation might be that these two countries share similarity in language. This proximity of language may have facilitated the smooth exchange of ideas and conceptions about teaching between the U.S. and Ireland. However, the adaptation of the MKT construct may face more challenges when it comes to other countries where English is not the primary language of instruction. Further studies are necessary to examine equivalence of the MKT across countries where there exist differences in language. This article describes an initial effort to examine the translation of the MKT items in Indonesia and Norway where languages other than English are used.

Delaney and colleagues (2008) contribute significantly in the area of instrument adaptation by developing categories of changes for translating the MKT instruments for use in another setting. These categories are 1) changes related to general cultural context; 2) changes related to the school context; and 3) changes related to mathematical substance; and 4) other changes. General cultural context changes included changing people's names to make them familiar to teachers in the new setting, adapting non-mathematical language, and changing culturally specific activities to be familiar to the teachers in that particular country or cultural contexts. The second category refers to changes in language used related to the cultural context of school or to the education system in general. Although these two categories of changes do not affect the mathematical substance of the items and therefore were unlikely to compromise the validity of the measures, they are important so that teachers in another setting are not distracted by terms or contexts to which they are not accustomed

(Delaney et al., 2008). The third category of change is related to the mathematical substance of the items, and consisted of changes of units of measurement, changes to the mathematical language that is used in schools, changes to representations commonly used in schools, and changes to anticipated student responses. Finally, other changes that do not fit the previous three categories are placed under "other changes".

In addition to the above four categories, two additional categories came out of the Norwegian study: (1) Changes related to the translation from American English into Norwegian (also applicable to Indonesian) and (2) Changes related to political directives (see Mosvold et al., 2009). The first of these additional categories could also fit well within the fourth category ("other changes") above, but it was included as its own category to emphasize the added complexity of translating to a different language rather than only adjusting to a different cultural setting, which was the challenge for Delaney and his colleagues (2008) in the Irish study. Some items include words and phrases that are simply problematic to translate, and such changes were placed in this category. The second additional change was especially related to the Norwegian context, and it was included to emphasize a challenge that was special to the Norwegian study. The Norwegian Department of education decided that the organization of students in traditional classes should no longer be the norm, and the word "group" replaced the word "class" in most official documents. This was not simply a challenge related to school culture, since some teachers continued to talk about classes of students, whereas some were very conscious about the change.

In order to understand the cultural influence on item translation and adaptation, we present the education contexts for the two countries involved in our study below.

Educational contexts of Indonesia and Norway

Indonesia and Norway are different in many respects. Norway, being a relatively small country with a population of 4.7 million, is among the wealthiest countries in the world.

Indonesia, on the other hand, is a larger country with a population of 223 million, and it is often categorized as a developing country (Mullis et al., 2008). In international studies like TIMSS and PISA, students from both Indonesia and Norway are low performing in mathematics. Despite differences in terms of socioeconomic and political structures, the two countries have somewhat similar challenges when it comes to students' performance in mathematics.

Like many countries, Indonesia and Norway are undergoing efforts to improve their education. The instructional practice in Indonesian classrooms is characterized as mechanistic, with teachers tending to dictate formulas and procedures to their students (Armanto, 2002; Fauzan, 2002; Hadi, 2002). The prevailing method of teaching-as-telling creates a passive learning atmosphere, where misconceptions frequently emerge (Armanto, 2002). In Norway, some similar issues seem to be prevalent. Although the 1997 curriculum reform focused more on projects, group work and guided discovery (KUF, 1996), classroom instruction remained traditional, focusing on review of previously taught issues, presentation of new theories with corresponding examples, and a strong focus on solving textbook tasks (Alseth, Breiteig, & Brekke, 2003).

Indonesian students' performance in mathematics on national examinations is poor, with an average of below 5 on a 10-point scale, making it consistently the lowest-scoring subject of all those taught in school (Depdikbud, 1997). In international comparative studies like TIMSS and PISA, Indonesian students performed below most other participating countries. Norwegian students performed somewhat better than their Indonesian peers, but the Norwegian students were still below average (Gonzales et al., 2008).

In the area of curriculum development, there have been efforts to develop exemplary curriculum materials for teaching school mathematics in Indonesia (Armanto, 2002; Fauzan, 2002; Hadi, 2002; Sembiring, Hadi, & Dolk, 2008). However, unlike the United States, where efforts to reform mathematics instruction have proceeded for a considerably long period of time, Indonesia is just beginning to initiate reform in mathematics education through a standard based curriculum introduced in the 2009/2010 academic year.

Both Indonesia and Norway adopted national curricula where mathematics is required at all grades at both the elementary and secondary levels. Unlike the United States, where there are five content strands (Number and Operation, Algebra, Geometry, Measurement, and Data Analysis and Probability) (NCTM, 2000), Indonesian elementary curriculum consists of only four strands (Algebra is not included at the elementary grades) (Departemen Pendidikan Nasional, 2003). Moreover, Geometry and Measurement are also treated as one strand, and Probability is excluded in the elementary grades. In Norway, students are supposed to work on the subject areas: Numbers (becomes Numbers and algebra in years 5-10), Geometry, Measuring, Statistics (becomes Statistics and probability in years 5-7, and then Statistics, probability and combinatorics in years 8-10), and Functions (only in years 8-10) (Utdanningsdirektoratet, 2008).

When compared with the content of the MKT items that are used in our studies, it is important to notice that algebra does not appear as a main content area in first grade through fourth grade. In the previous curriculum guideline, algebra only appeared in years 8-10. Functions only appear in years 8-10. For Indonesian students, algebra is not included at all in elementary school.

The most recent Norwegian curriculum (UFD, 2005) has increased the focus on achievement goals, and it does not include descriptions of processes or methods, or materials that could be used. The Indonesian curriculum is somewhat more similar to the previous Norwegian curriculum, in that it includes all these aspects. Both countries publish the national curriculum as an official publication. There are, however, differences when it comes to textbooks and teacher guides. In Norway, there are no longer mandated or recommended textbooks for mathematics. According to Mullis and colleagues (2008), there are no official instructional or pedagogical guides, but this has been developed more recently. In Indonesia, there are both official guides and textbooks.

The language of instruction in Indonesia is Bahasa Indonesia, whereas Norwegian is the official language of instruction in Norway (Mullis et al., 2008). Although students in the two countries are taught English in school, neither of the countries have English as an official language. The MKT items therefore had to be translated into Norwegian and Indonesian respectively.

Method

The Indonesian and Norwegian projects in focus in this article were part of larger projects with different purposes. The purpose of the Indonesian project was to use the MKT instrument to examine factors that may contribute to Indonesian elementary teachers' mathematical knowledge for teaching geometry and to evaluate the effectiveness of a professional development program focusing on knowledge of geometry in the context of teaching elementary mathematics (Ng, 2011). Thus, only the geometry scales were used. On the other hand, a whole set of items were used in the Norwegian project since the main purpose was to ensure that the construct of MKT meant the same to Norwegian teachers as to U.S. teachers, and to address issues related to the format as well as the content of the items.

The two projects also differed in the translation procedure used. The Norwegian project used a double translation procedure (Adams, 2005), where the translation of the items took about half a year. Towards the end of this period, a working seminar was conducted where pairs (or sometimes three) of researchers translated all the items. Two groups of two researchers would work separately on the same set of items. In the Indonesian project, the researcher translated the items, and the results were examined by a team consisting of a mathematics educator, a TESOL professor, and two staffs from a professional development provider.

Throughout the translation process, all changes that were made to the items were carefully documented according to a common framework. Both the Indonesian and the Norwegian projects used a modified version of Delaney and colleagues' (2008) categories of change as a framework for translating the MKT instrument for use in another cultural setting:

1. Changes related to the general cultural context
2. Changes related to the school cultural context
3. Changes related to mathematical substance
4. Changes related to the translation from American English into Norwegian and Indonesian

The translation of the MKT items were conducted by mathematics educators, rather than professional translators because we believe that there are many issues that need to be resolved other than making sure that the texts mean the same in both contexts, which is what will be discussed in this paper.

This article reports on a case study where two items were translated and adapted for use in Indonesia and Norway. The two items were selected for the particular purpose of illustrating the challenges that came up in the process of translating and adapting items in our separate projects. After selecting two items from the released items pool that would best exemplify the goal of this paper, these items were translated and adapted according to the same principles that were used in the main studies. All changes were categorized according to the above-mentioned framework from Delaney and colleagues (2008). For the purpose of discussions and analyses in this article, the changes were translated back to and explained in English. The challenges of translation and adaptation were then analyzed and discussed in relation to this common framework.

Results

Item 1

As mentioned earlier in the article, we are going to discuss two sample items in this article, in order to shed light on issues related to translation and adaptation. The first item asks teacher to evaluate which story problem could be used to illustrate division by a fraction.

7. Which of the following story problems could be used to illustrate $1\frac{1}{4}$ divided by $\frac{1}{2}$? (Mark YES, NO, or I'M NOT SURE for each possibility.)

	Yes	No	I'm not sure
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?	1	2	3
b) You have $1.25 and may soon double your money. How much money would you end up with?	1	2	3
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$cup) will you need?	1	2	3

Figure 2. One item from the released items pool.

This item involves two story problems where food is a vital part of the context, and food is a culturally specific entity. Thus, any item that uses food as a context needs to be changed in order to be familiar to the audience. Some changes are straightforward because of the availability of the same food in the other culture, such as butter. Others, such as pies, are not common in Norway and Indonesia. In the Indonesian translation of alternative a), cake was used instead of pie. In the Norwegian translation, pies were replaced by pizzas, even though pies are sometimes used in Norwegian textbooks (Boye Pedersen, Andersson &

Johansson, 2005). The choice of pizzas over pies was made because pizzas are more widely used in Norwegian textbooks to illustrate fractions (for example in Alseth, Nordberg & Røsseland, 2006). Although this change seems to be rather trivial, such alteration brings out some interesting new issues into the item. Where pies have circular shapes, cakes might be squares and rectangles too, and homemade pizzas may also be rectangular in Norway. In Indonesian textbooks, if cakes were to be used as a context, a pictorial representation of a circular cake would always be provided to avoid confusion. Despite possible differences in the shape of the food, this variation does not change the intended mathematical substance of this problem. In this case, the amount of fractional pieces will be the same.

Alternative c) also involves food as a context, and this one is even more challenging. In the Norwegian context, it is uncommon to make homemade taffy, so this would have to be replaced with something else. Similarly, taffy is not commonly found in Indonesia. Also "cups of butter" and "sticks of butter" are not standard units of measurement in either the Norwegian or Indonesian contexts. Instead, weight measurements such as grams are used for recipes. Although it is possible to make a literal translation of this part of the item, the result would be something which teachers in these two countries would find quite unfamiliar, and this could potentially make the item more confusing and difficult to answer. If the context of homemade taffys was to be kept and the units of measure were changed into grams, the problem would change completely. Therefore, this particular alternative would have to be completely rewritten and the context changed.

The item also includes a change that relates to the school cultural context, but this alteration is a minor issue in this particular item. "Story problem" would in this connection be translated into the Norwegian word "regnefortelling" rather than the more direct translation. Although this change is minor, it is important in order to provide the teachers with the most sensible context, and it is an example of a challenge that demands knowledge of the school cultural context (and also knowledge of teaching mathematics) that goes beyond knowledge of translation alone.

Alternative b) in this item involves an example of a type of change pertaining to the mathematical substance of the item, and this is relevant in both countries. Where a decimal point is used in the U.S. context, both the Norwegian and Indonesian context calls for a decimal comma here. Such a change is not problematic in this particular example. However, the difference of currency makes it more complicated, and this is related to the general cultural context of the countries. Indonesia and Norway both have different currencies, and neither use dollars. Quarters are commonly used in the U.S., and although Rp. 25 do exist in Indonesia, due to its extremely small nominal value (to date $1 is approximately Rp. 9000) and are therefore rarely used or even available in circulation anymore. The mathematical substance is also altered considerably in the translated version of the item because the whole is no longer one ($1) but one hundred (Rp. 100), which changes the original problem. In the context of Norway, no such thing as a quarter exists (although it did several years ago); 1,25 NOK (Norwegian crowns) would therefore not make sense in a daily context. An alternative might be to introduce the context of bank accounts, because 1,25 NOK might exist in a bank account, although no such coins exist in the Norwegian currency. Still, the context would provide little meaning, because most people do not really care if 1,25 NOK will double or not. If that were the amount of money they had in their bank account, it would imply that there were practically no money there. The context of money could therefore not be used to illustrate this kind of mathematical problem in a Norwegian context.

When it comes to other changes, there is one change that might be important in the Norwegian setting. The initial formulation of the question in the item stem was “Which of the following story problems could be used...” In many items similar formulations were used. When translating the word “which” into Norwegian, you have to make a choice between the words "hvilke" and "hvilken". The first alternative indicates that there is more than one solution, whereas the second indicates that there is only one solution. In this particular item there are no indications of whether or not only one solution is correct, and a minor difference like the choice of words in the Norwegian translation here could potentially help the teacher. An alternative would be to translate it with "hvilke(n)", indicating that there might be one or more solutions, and this was often done when such items were translated into Norwegian. The challenge is that the original set of items was not made with the purpose of being translated into other languages, and it is unlikely to know if the creator of the item intended to provide an indication of the number of correct solutions or not. The worst scenario would be if the intention of the item is that only one solution is correct, and the translation of the item confuses the teacher into thinking that several correct answers exist, or the other way round.

Item 2

The second item is interesting in many respects, and it contains examples of changes that relate to all the four categories. This item is also a geometry item, and the Indonesian study (Ng, 2009) made use of the geometry items in particular whereas the Norwegian study (Mosvold & Fauskanger, 2009) used a complete form including numbers, geometry and algebra items.

27. Mrs. Davies' class has learned how to tessellate the plane with any triangle. She knows that students often have a hard time seeing that any quadrilateral can tessellate the plane as well. She wants to plan a lesson that will help her students develop intuitions for how to tessellate the plane with any quadrilateral.

Which of the following activities would <u>best</u> serve her purpose? (Circle ONE answer.)

a) Have students cut along the diagonal of various quadrilaterals to show that each can be broken into two triangles, which students know will tessellate.

b) Provide students with multiple copies of a non-convex kite and have them explore which transformations lead to a tessellation of the plane.

c) Provide students with pattern blocks so that they can explore which of the pattern block shapes tessellate the plane.

d) These activities would serve her purpose equally well.

Figure 3. A second item from the released items pool.

In this item, and many similar items, teachers (and sometimes students) are referred to with their names. When translating these items into a different language, a change of names was often necessary in order to preserve the familiarity of the context. Mrs. Davies would therefore be translated into Ibu Dariah in the Indonesian version, where Dariah is a more familiar Indonesian name and Ibu is the title used to address a female adult. Similarly, Mr. Davies would have been translated into Bapak Dariah. In the Norwegian context, however, the change of name is not so trivial in this particular context. In Norwegian classrooms, particularly in elementary schools, teachers are almost exclusively addressed by their first name rather than by their family name and a title. Mrs. Davies would therefore be translated with Dorthe, or a similar first name, which is more common in the Norwegian context. Such a change is a little less trivial, for several reasons. First, it might sometimes make it harder to distinguish between the teacher and the students in items like these, and a misunderstanding of parts of the context is possible. In some items, we had to include some kind of explanation or clarification of who is the teacher. In this item, a Norwegian translation would read something like "Dorthe's class", with the possibility of confusing Dorthe with a student. The context of the item eventually makes it clear that Dorthe is the teacher in this case, but there is still the possibility of producing a slightly more confusing text. The use of Mrs. Davies or Ibu Dariah leaves no doubt as to who is the teacher in this class.

Another issue with this item, which is related to the school cultural context, is that the direct translation of the word "students" would not be used in the Norwegian context, although it exists. In the Norwegian school system students are referred to as pupils (or "elever" in Norwegian) in school, and the term student is only used when they enter university or college. This is a trivial change, but if it had not been made, it would result in a somewhat more unfamiliar context for a Norwegian teacher.

The word “tessellation” is also challenging in the translation of this item. In Indonesian, there is no equivalent word for tessellation, and instead a more general term like “tiling” (“pengubinan” in Indonesian) would be used. The word tessellation does exist in Norwegian, however, but it is considered a more technical term and seldom used by teachers or students in an elementary school classroom. A term like tiling might also be used in a Norwegian setting, but then again the terms tiling and tessellation are not precisely equal terms. One solution in the translation of this item might be to use the word tessellation and include a short explanation of the term to avoid confusion, but this would probably make the item easier for teachers in Indonesia and Norway than it was intended.

The original formulation in alternative a) that quadrilaterals can be “broken into” two triangles had to be rewritten in both countries. In both Indonesian and Norwegian, the translation “divided into” makes more sense because “broken” refers to many pieces.

Alternative b) was problematic in both the Norwegian and the Indonesian context. The text refers to “non-convex” kites, and this is a technical term that would not be used in Indonesian or Norwegian elementary classrooms. The text would therefore have to be translated into a somewhat more everyday language. The problem here is that we do not really know if the term was supposed to be technical, and possibly even teachers in the U.S. might experience this as a technical term. If a decision is made to rewrite the term into a more everyday language, the result might be that the item becomes easier than it was intended. If, on the other hand, one decides to keep the technical term in a more direct translation, there is a possibility that teachers in a certain culture are more familiar with this term than teachers in another culture, and the difficulty of the item might vary across cultures.

“Pattern blocks” are referred to in alternative c), and this is an example of concrete materials or representational tools that might vary across cultures. A direct translation would therefore not be good enough in neither the Norwegian nor the Indonesian context. One possibility might be to include a picture or an explanation, but that would imply a rather substantial change of the item. Furthermore, in the context of Indonesia, physical manipulatives such as pattern blocks are not widely available commercially, and a majority of the teachers will be unfamiliar with these concrete materials.

There are also examples of phrases that are hard to translate for other reasons, which are partly language related and partly culturally related, like the sentence in alternative d). The phrase "would serve her purpose equally well" would not be used in Indonesian or Norwegian, and it would have to be rewritten to make sense. Although this should be a minor issue, care needs to be taken to avoid adding unnecessary clutter to the item.

Discussion

The two items presented in this article were selected because they represent many of the issues that came up when translating the MKT items into Norwegian and Indonesian. Not all items were this complicated to translate, but most items included some of the elements that are discussed in this article. Our discussion here is not meant to be exhaustive in nature. However, our idea was to use these items as examples in order to explain the kinds of difficulties that might occur, and these are difficulties or challenges that researchers (or translators) need to be aware of when they work with similar kinds of projects. Some of the themes we found during the process of translating these two items include a) cultural differences that present minor challenges but do not affect the mathematical substance of the items; b) differences in the use of technical language in schools; c) incommensurable contexts across countries; and d) differences in instructional practice specifically in the use of mathematical models. Different countries might have somewhat different challenges, but we believe that several of these types of challenges are as relevant for many countries or languages as they were for the Norwegian and Indonesian translation.

Some cultural aspects such as language and food present challenges when translating the MKT items. The use of food in mathematical problems serves as a familiar context for the audience. However, when adapting to a different country, the food in context may not be common for the audience in that setting. The challenge becomes finding similar food without changing the mathematical substance of the problem. In our case, exchanging pies with pizza or cake, although they pose a problem because these substitutes exist in other shapes such as square or rectangular, did not change the representation of the fraction.

The difference in languages may result in minor issues as evidenced in the Norwegian context. Such issue does not necessarily change the validity and difficulty of the items, but because additional explanations need to be provided so as to avoid confusion, which may make the descriptions of the item wordy and cluttered.

The way teachers are addressed may differ across cultures. Although this issue seems to be trivial, it poses problem in the Norwegian culture where teachers are normally addressed by their first names. Also, the way that students are organized in classes or groups, and how these are referred to, might differ between countries. Quite recently, there was an official convention in Norway that “class” should not be used when referring to groups of students (cf. Mosvold et al., 2009; Mosvold & Fauskanger, 2009). This was related to a government initiative to change the more formal organization of schools. Lately, however, this initiative is about to be reversed, and this would apparently not be a problem in the future. Still, the use of the concept of class might be confusing in cultures where students are organized in other types of group structures.

Mathematical language may be considered universal in terms of the symbolic expressions used. However, when it is related to definitions or terminologies, variations may exist across cultures. As exemplified in both the Indonesian and Norwegian context, although the more technical terms such as “non-convex” and “tessellation” exist in both countries, they are not used in the grade schools. Instead, a more general term such as “tiling” is used. Such change may undermine the integrity of the item in measuring teachers’ mathematical knowledge for teaching, and would instead measure teachers’ familiarity of certain technical vocabulary. In other instances, the more familiar terminology may be more descriptive than the technical term. For example, Ng (2009) found that the term “polygon” needed to be translated into a more familiar term to Indonesian teachers, “bangun datar segi-banyak,” which literally means “multi-sided flat shape”. This more descriptive term thus may compromise the difficulty of the item when assessing teachers’ knowledge of the definition of polygon. This difference in mathematical language might also imply distinction in the school cultures of our countries when compared with the U.S., but this is difficult to conclude because we do not have any data on how teachers in the U.S. conceive these issues.

While adapting the MKT measures in Ireland did not face major problems in terms of context (Delaney et al., 2008), in the case of Indonesia and Norway, we found instances where the context of the problem could not be translated without majorly change the mathematical substance of the items. The use of measurement units, such as “cups” and “sticks” of butter, is not standard in the two countries. Changes to grams or similar units are possible, but then the context of the problem differs significantly because the whole in the fraction needs to be changed. So is the case with the context where money is involved. The currencies in Norway and Indonesia, although different from each other, differ enough from the U.S. currency to provide a serious problem. Using equivalent values of money does not make sense mathematically to represent the intended fraction. These two contexts would be almost impossible to translate into either of our languages without changing the entire context. This issue of incommensurable context introduces a serious threat to the instrument equivalence.

Teaching practices vary greatly across cultures (Stigler & Hiebert, 1999), especially the use of concrete materials as tools or models for representing mathematical ideas. Unlike the U.S., many developing countries may not have commercially available physical manipulatives. Any mention of these objects, for example in our case "Pattern Blocks," needs to be clarified either by providing an explanation or a picture, or both. Such an attempt, however, may make the item easier because of the availability of multiple modality of representations.

Conclusion

The purpose of this article is to shed light on some of the challenges that might arise when translating test items from one language into another and adapting them to be used in a different setting than they were intended. Both the projects that are described and discussed in this paper have made use of the MKT items, and it might be argued that these items are special in that they were written particularly to be used among teachers in the United States. Researchers have argued that there are cultural variations in teaching practice (cf. Stigler & Hiebert, 1999), and it can therefore be argued that items that are developed based on the teaching practice in one culture are not necessarily applicable in another. Still, the MKT items were created in order to describe and measure issues related to the tasks of teaching rather than teaching practice, and the tasks of teaching are considered to be of a more universal nature per se (cf. Ball, Thames, & Phelps, 2008). The discussions in this article indicate, however, that although the construct might be of a more universal nature, actual items that are developed for use in measures of this kind of construct would necessarily have to be dependent on some kind of representations, tools, or other kinds of artifacts that by nature will be more culturally specific. We do not, however, attempt to dismiss the MKT items (or any similar types of measures for that matter) or the use of such measures in other countries altogether, but we want to direct attention to some of the issues and challenges that are necessarily going to be part of the process.

As Delaney and colleagues (2008) and others have done, we would also like to call for an inclusion of these types of discussions whenever research of this kind is reported. We would also like to suggest a closer investigation of how cultural issues actually might influence the results in measures like these. Since the MKT measures are now being used by researchers in other countries, and this appears to be a growing tendency, we suggest a common effort to create a framework where these issues can be investigated in a more scientific way. Such investigations might even result in a further development of the theoretical construct of MKT and possible cultural differences in relation to this construct as well.

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[i] For more info about LMT, see http://sitemaker.umich.edu/lmt/home.

Reflections on Teaching with a *Standards*-Based Curriculum: A Conversation Among Mathematics Educators

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Many teachers and researchers have written about the challenges inherent in adopting new teaching practices in mathematics classrooms (e.g., Chazan, 2000; Clarke, 1997; Heaton, 2000). The authors of this article, all with secondary mathematics teaching experience, are convinced by research suggesting that *Standards*-based mathematics curricula are beneficial for student learning.[1] However, the first three authors had not used such curriculum materials in their own classrooms, and we desired experience using a *Standards*-based mathematics curriculum with secondary students. To this end, we

[1] For our purposes, *Standards*-based mathematics curriculum materials are those developed to align with the National Council of Teachers of Mathematics' standards (NCTM, 1989, 1991, 1995, 2000) with funding provided by the National Science Foundation.

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taught a week-long summer course with a focus on linear functions to high school students who had previously struggled with algebra and volunteered to participate.

The aim of our course was to improve students' understanding of linear functions through the use of an inquiry-based learning environment and a *Standards*-based curriculum. This was not a typical classroom setting since there were three instructors for less than 20 students. Nonetheless, it was useful for both teachers and students to experience participating in an inquiry-based curriculum for the first time. The purpose of this article is to stimulate thinking and conversation among teachers by sharing our own conversations about learning to teach mathematics using a *Standards*-based curriculum.

The curriculum selected for the course was the second edition of the *Connected Mathematics Project* (*CMP*) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006), specifically Investigations 1 and 2 of *Moving Straight Ahead* which focused on linear relationships. The *CMP* curriculum (similar to other *Standards*-based curricula) uses a Launch-Explore-Summary model of instruction. During the *Launch*, the teacher poses a mathematical task or provides information that is intended to provide a "hook" to get students interested in the investigation, while making connections to their prior knowledge and experiences. During the *Explore*, students collaborate in small groups on a mathematical task that requires them to construct important mathematical ideas. The teacher's role during the *Explore* is to monitor groups' mathematical conversations and begin to plan a sequence of presentations for the *Summary*. Finally, the *Summary* provides an opportunity for students to present their work to the class and discuss the emerging mathematical ideas with one another. During the *Summary*, the teacher serves as a facilitator to synthesize students' ideas and draw out the key mathematical concepts and/or procedures from students' work.

Upon completion of the course, the instructors agreed that *Standards*-based teaching requires a great deal of on-the-spot intellectual work. The pay-off, however, which we learned through teaching the course, was that students seemed to be able to make sense of the mathematics rather than following prescribed procedures. For example, in addition to calculating the slope, the students could often understand what the slope represented in each of the application problems (e.g., walking rate). Seeing students' pride - and sometimes surprise - when they realized that they were able to write an equation to model a situation or solve a challenging question without the explicit direction of a teacher, was priceless. The week ended leaving us with much to think about (e.g., how to manage mathematical discourse). Teaching using an inquiry-based model with a Launch-Explore-Summary structure highlighted many dilemmas that we were able to experience firsthand. We composed reflections based on our experiences into a series of questions. We subsequently sent them to Lisa, the fourth author, who responded to our questions using her many years of experience teaching with *Standards*-based curricula. Our questions fell into three overlapping categories: (1) questions about mathematical discourse, (2) questions about facilitating the *Summary*, and (3) questions about general pedagogy. We have posed our questions (Q) in regular font, with the answers (*A*) in italicized font.

Questions about Mathematical Discourse

Q: It was often challenging to know how to respond to students' ideas during any part of the *Launch*, *Explore* or *Summary* when their answers were mathematically incorrect. I did not want to make them feel like the idea they shared was not useful, and I wanted them to feel safe sharing conjectures even before they had a chance to test and refine them. We all worried about what to do if the students said a "wrong" answer and the class agreed. This happened several times, but each time other students questioned the idea and it did not stand up to the rest of the class' judgment. What do you do during the *Summary* if a group presents a solution that has mathematical flaws that are obvious to you, and no student challenges it?

A: When you are circulating around the room while the students are working, this is an opportunity to see those students who have mathematically correct responses and those that are incorrect. If most of the students think the incorrect answer is correct, I would say something like, "Carmen, I noticed when I looked at the work that you and your partner did, you did not come up with this answer, can you explain to us what you were thinking?. This allows a different response to surface, and then I would pursue the differences between the two responses by asking the students to think about why the answers are different and what may account for the differences.

Q: I often found myself playing ping-pong in the conversation. I would ask a mathematical question, someone would make a remark or give an answer, and I would typically be the next person to speak again, to question the speaker further or summarize what he/she said. How do I de-center myself from the class discussion so that more mathematical discussion can occur between students?

A: When students would make a claim, I found that asking follow-up or extending questions such as "What do you think?" "What is your evidence?" "Does anyone disagree?" "Is there another way to look at this?" was helpful to promote productive *discourse and place the mathematical authority among the students. As a classroom teacher, it's important to model for students how to respectfully disagree, and encourage students to talk to each other, not the teacher.*

Q: As is always the case, some students volunteered to provide answers or explain their mathematical reasoning and others did not (or did so less often). Since *Standards*-based instruction is meant to engage all learners in mathematics, is it okay to randomly select students to present their work or answer questions or should it usually be on a volunteer-basis?

A: It's important as a classroom teacher to develop a classroom culture in which students feel comfortable sharing ill-formed ideas and students are willing to take risks. Students need to know that every student in the classroom is responsible for contributing to the discussion. I think though it's okay for a student to say, "I pass" when they really aren't comfortable explaining a specific problem. If this becomes a habit with particular students, teachers need to figure out what will work to keep these students engaged. Again, I think it's important that teachers from the onset of the school year develop a culture where students are motivated, engaged, and know it's okay to be wrong, and they will be supported.

Q: I noticed that if I responded to a students' mathematical answer or reasoning in a way that indicated I agreed with what they were saying, even in a subtle way, then when I asked, "Anyone disagree?" it produced nothing but silence. I began to wonder if because my students see me as the authority of correct answers in my classroom, they adjust or change their thinking to match what they think I want, regardless of what they were thinking previously. When students are sharing mathematical ideas with you, either one-on-one or as a class, and then they look at you with eager eyes, waiting for you to affirm or evaluate their idea, how do you respond?

A: I wouldn't agree with the student. When a student presents a response, instead of validating you could ask, "what do you think?" to other students in the class. I would also be very explicit saying to students, "I am not the only mathematical authority in this classroom, it's important that we hear from everyone . . .we are working on these problems together, and we are all going to contribute to building our new mathematical knowledge together.

Q: When two conflicting mathematical ideas are presented in the class, how do you facilitate a conversation about these without indicating which side you are on? Is it appropriate to "vote" or is it better to stay at a level of asking the student to logically "argue" the two cases?

A: These conflicting ideas are significant mathematical learning opportunities. I think it's important for students to logically argue the two cases by asking students to clarify and justify their thinking. This notion speaks to the NCTM Reasoning and Proof process standard. Students need opportunities to evaluate others' mathematical claims as well as formulate mathematical arguments.

Questions about Facilitating the *Summary*

Q: The *Summary* was by far the most challenging part of the lesson for us to implement. We often asked students to share their mathematical results with the class. Of course, this takes some time and I wondered if this is always the way to go. Should students always present their work during the *Summary*? If not, how might a teacher decide when it is necessary or helpful and when another strategy might suffice? What other strategies would you suggest as options for getting the main ideas on the table during the *Summary*?

A: The summary is very difficult to orchestrate. This is where the important mathematical ideas are coalesced. However, it seems to be the segment of the lesson that teachers tend to short-change. I think that while the students are working, it is important that the teacher looks for student work that they wish to highlight during the summary segment. Teachers should highlight not only correct mathematical responses, but incorrect responses as well. The summary takes practice, but it's important for teachers to understand that students need this summary, as this is where the important mathematics is made explicit.

Q: Students work at different paces in mathematics classrooms and this presented a challenge for deciding when to start the *Summary*. Can you provide any pointers for making this decision?

A: When the teacher is circulating around the room, they get a sense of how many students are near completion. I would often say, "let's take 5 more minutes to finish up." I just ask the students that are not finished to put their pencils down and listen. This is another reason why the summary is so important. Even if a student did not complete the problems, the summary should help them understand the important mathematical concepts.

Q: When a student or a group of students present their mathematical findings to the class, I would often want them to explain their thinking further or share something additional, and so I would ask them follow up questions to get them to share what I felt was mathematically important. How do I encourage students to ask their own follow up questions of their peers?

A: I would often look for students that looked confused. Then I would say, "Johnny you look confused, why don't you ask Kendra to explain her thinking again, or in a different way." Once you start doing this, it becomes part of the classroom culture and students know that they are expected to ask each other for clarification.

Q: The *Summary* seems to be where the important mathematics is highlighted. Teaching with this style seems to keep the *Summary* informal and more conversational, rather than formal involving traditional lecture and taking notes. How do you make sure that you emphasize the important mathematics without making it teacher-centered? Is there a way to create notes during the *Summary*, having students write down the formal math involved in the section? Or do you find that note-taking isn't as useful as simply discussing the mathematics?

A: Yes, the summary is where the important mathematics is highlighted. The summary is very informal and conversational. As mentioned earlier it's important to clear up any misconceptions before leaving the room—at least that's what I thought. I would have my students take mathematical notes from time to time, but these notes were often the work of other students on the board, and they would write notes to themselves about the mathematical work. For example, if a student solved a problem in a way that helped another student think about the mathematics, the student would write this in their notebook (we called them Toolkits). They would also write about anything in the explanation that would help them make sense of the mathematics.

Questions about General Pedagogy

Q: We had the luxury of three teachers in one classroom! If I were doing this alone, how would I go about helping all groups and listening to what all groups are doing? Is it necessary to get around to all groups? How would teaching it alone affect the speed/overall effectiveness of the lesson?

A: Teachers need to understand that teaching with Standards-based curricula is exhausting but also extremely rewarding. I didn't always get around to every group, but would make sure at least twice a week I spent time with each group. I don't think teaching alone affects the speed or the overall effectiveness. The reality is that it is very difficult to teach like this every single day. I would give myself a break once in a while, and have the students work on the unit projects, mathematical reflections, or additional practice provided with the curriculum. You need to give yourself permission to have a breather!

Q: Lecturing in mathematics classrooms seems to have a nice neat beginning and ending. Teaching mathematics with a *Standards*-based curriculum felt better (i.e., students were engaged in learning), but messier. Often the temptation to tidy things up (e.g., tell procedures, have the conversation go through me) was irresistible! Is this messiness normal?

A: When it appeared that we would run out of time for an adequate summary, I would have a mini-summary, but ask students to continue to think about the mathematical ideas, and we would resume the discussion the next day. In theory wrapping up the lesson at the end of the class period is ideal, but in reality that doesn't always happen.

Q: I worried about letting students go for a period of time believing an incorrect answer. Some educators say that getting the wrong answer isn't a big deal, as long as students are engaged in rich mathematical discussions and reasoning they will still improve and understand the mathematical concepts more deeply. But as a high school math teacher, I'm responsible for my students getting the "correct" answers according to the state exams, so allowing students to be "wrong" is something I find quite problematic. How do you know when or how long to let wrong answers go? For how long do you let them go? How do you correct them? Do you correct them? What are your thoughts about how this affects students, if at all, on state exams?

A: I always allowed incorrect responses to come out in discussions. I would purposely ask students who had incorrect responses to share their answers, and then I would ask students if they agreed or if someone thought about it differently. I would make sure that any mathematical misconceptions were reconciled before the end of the class period. I thought it was important that they didn't leave the classroom with misconceptions, especially if I assigned homework that evening.

Q: Some students seem to have locked themselves into using one mathematical representation and although they are becoming proficient in working with one type of representation, they're not learning to use others. For example, one student will always graph a function to understand it. Another student might always rely on a table. Another will attempt to write a mathematical equation. Will students typically voluntarily choose to learn to master another representation after some time? Do you just encourage them to use a variety of representations? Do you make it mandatory to vary their mathematical representations?

A: *I think that it is important for students to be able to move fluently among mathematical representations (e.g., graphs, tables, symbols). I would always require students to show at least two ways. Also, if I recognized that a student tended to gravitate toward one type of representation (e.g., always wanting to graph), I would suggest that they might try a table and an equation for the problem.*

Q: It seems difficult to determine, in advance, what to assign for homework when using a *Standards*-based mathematics curriculum, and it seems even more difficult to make multiple classes end at the same place each day in order to assign the same homework. Sometimes it seems like it would be impossible to plan homework ahead of time at all! How do you deal with this situation? How can you have homework planned when you don't know where students will end? What do you do when you don't get through an activity or even to a point where students could attempt the homework problems on their own?

A: *If I felt like students were not at a place where they would be somewhat successful with the planned homework assignment, I would consider just assigning a few of the less complex homework problems. If it didn't seem as though they had enough information and knowledge to be successful with any of the homework, I would readjust and assign it the following day. Part of being able to teach this way is to be flexible! For my own sanity, I did try to keep all the classes at the same spot, otherwise it would be too difficult to manage although students are very helpful when it comes to remembering what you did the day before.*

Using the Launch-Explore-Summary model or *Standards*-based practices is no easy task, particularly because this is not the way that most of us learned mathematics.

However, it has been said that many places worth going are not easy to get to (e.g., Great Barrier Reef, Mt. Everest, Antarctica). In these cases, the experiences (both the journey and the destination) make the difficult trip worthwhile. The same philosophy applies here. Excellence in mathematics teaching is not an easy place to go. In fact, it is more difficult to reach than the faraway destinations mentioned here because it is a challenging journey each day. We have not yet reached our destination, but will continue to move forward on the journey as we develop our practice. It is, without question, worth the trouble as students' engagement in and love for mathematics depends on it.

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Number Theory and the Queen of Mathematics

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Introduction

Geometry is an integral component of secondary mathematics curriculum. From my experience in a mathematics teacher preparation program, I have seen a real push to connect geometry to other areas of mathematics. Secondary geometry can often be presented without clear or any connections to other areas of mathematics. One main purpose of this paper is to explore geometry and its rightful connection to other areas of mathematics, specifically number theory. Such strong emphasis is placed on drawing connections to number theory because of its intrinsic value in enhancing understanding of mathematical concepts. Learning number theory has positive ramifications for students, making the transition from arithmetic to an introduction to algebra. It helps "students develop better understandings of the abstract conceptual structure of whole numbers and integers," and it has important algebraic characteristics, which relate to variables and mathematical reasoning (Campbell & Zazkis, 2006, p.28). Another purpose of this paper is to explore not only number theory as it relates to geometry, but also the basic history of number theory. Number theory has a beauty, accessibility, history, formal and cognitive nature, and intrinsic merits all to its own (Campbell & Zazkis, 2006, p. 13). For the sake of all the intrigue of number theory, I have a desire to learn more about it to fuel my own pursuit of a better understanding of effectively teaching mathematics, but also to use it to encourage and engage students in their personal pursuit of mathematical understanding.

Because of the purposes of this paper, this research was compiled and organized from a mathematics education perspective. Although, in order to meet the purposes of this paper, the history of number theory and its history of interaction of geometry (This is more of the history of the acknowledgment of number theory and its interaction of geometry.) will play an important role. Despite this, however, this research was not compiled strictly from a historical perspective, but rather from a mathematics education perspective, with a historical perspective that is inherently a part of it.

Ancient Geometry—An Early look at the Union of Geometry and Number Theory

Geometry consistently played an important role in the mathematics of early civilizations. The discovery and study of approximately 500 clay tablets from the area once known as Mesopotamia indicate that the Babylonians were no exception. Babylonians' interest in

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geometry is evident. One tablet, Plimpton 322[1]—a tablet of what appears to be pythagorean triples—indicates that the Babylonians knew of the relationship between the sides of a right triangle. Plimpton 322, dated to the mid-18th century B.C., has been the subject of research and study for several decades (Robson, 2001, p. 170). Plimpton 322 appears to be incomplete; there is an evident break along its left edge. Also, this tablet does not include any scratch work that would shed light on the methods of the Babylonians' pythagorean triple generation. The question on the minds of researchers is *how did the Babylonians generate this list of pythagorean triples*?

The significance of answering this question relates to the study of the origins of Babylonians generation of pythagorean triples. To study these origins, it is imperative to unveil our eyes of our modern and cultural views embedded into our mathematical understanding and try to see Babylonian mathematics through their cultural veil. In answering this question, it will be important to acknowledge Babylonian culture. There are significant distinctions between the Babylonians' culturally view of mathematics and our modern view. Also, in answering this question, an important union of two mathematical branches, geometry and number theory, will reveal itself. Modern mathematical views and Babylonian mathematical views are distinct without question. There needs to be the use of a common medium, which will facilitate understanding of how Babylonians generated pythagorean triples. This medium is the theory of numbers, after all numbers belong to a universal language, which has been used by researchers to formulate theories, trying to answer the question of how Babylonians generated the list of pythagorean triples included on Plimpton 322. Three theories on Plimpton 322 have received the most attention. The formulation and deciphering of these theories illustrate the importance of taking cultural factors into consideration and using number theory.

The trigonometric table theory is not so much a theory on the generation of the pythagorean triples, but it is rather a theory/interpretation of Plimpton 322's contents. This interpretation originates from the fact that the first column appears to be tan^2 of the angle opposite of the short side of the triangle. Calculating this angle, θ, shows that θ decreases about 1^o row to row, showing some type of order to the arrangement of the rows. Also, the calculations show that θ is between 30 degrees and 45 degrees (Buck, 1980, p. 344). This theory, however, violates the importance of cultural considerations. Through her study of Babylonian tablets, Robson has deduced that Babylonians used the circumference of a circle, not the radius, to define circles and find their areas. Instead of using $A = \pi r^2$, they appear to have used $A = \frac{c^2}{4\pi}$, where π is approximately equal to 3 (Robson, 2002, p.111). According to Robson, there is no evidence of the Babylonians' rotating of radii, and without this evidence, there is no "conceptual framework" for trigonometry (2002, p. 112). The researchers that authored this theory did not look through the veil of Babylonian culture, so this theory is deemed invalid.

The second theory to be discussed in this section is the generating pair method. Otto Neugebauer and Abraham J. Sachs introduced this theory in their 1945 book, *Mathematical*

Cuneiform Texts. The generating pair theory is essentially taken from Book X of *The Elements*, which presents an understanding of rational and irrational lines using the concepts of commensurable and incommensurable[2] lengths and squares (Roskam, 2009, p. 277). The generating pair theory advocates that the Babylonians used a formula, comparable to Euclid's formula, for generating pythagorean triples, as seen in Book X. With this formula, triples are produced with an m and n, satisfying the following conditions:

- $m > n$, gcd(m,n)=1
- m,n are not both odd
- $a = mn$, $b = m^2 - n^2$, $c = m^2 + n^2$.

This theory is dismissed on a few grounds. No evidence has been found to suggest that Babylonians knew the concepts of odd and even numbers and coprime numbers (Robson, 2001, p. 177). This violates the importance of looking at Babylonian mathematics through the veil of their cultural view of mathematics; since there is no evidence of these concepts, they cannot be assumed to be a part of their mathematics. Another reason for this dismal is the fact that Plimpton 322 has a clear, purposeful order. Other Babylonian tablets indicate that order was important to Babylonians. This theory and the number theory behind this theory do not support the formulation of a tablet with structure with such order. According to Robson, the scribe would have hundreds of pairs from the standard reciprocal table from which to choose, so creating Plimpton 322, as it is, would be exceedingly difficult (2001, p. 177-178; 2002, p. 110-111). The third of the theories, discussed in this paper, is revered as the theory with the most merit because of its unveiled view and the use of the medium of the theory of numbers, respecting the cultural context of the Babylonians.

The reciprocal theory was introduced in 1949 by E.M. Bruins. The reciprocal theory advocates that the tablet was constructed with the use of reciprocals and cut-and-paste geometry. Another Babylonian clay tablet, YBC 6967, offers evidence that these techniques were indeed used regularly by the Babylonians. According to this theory, Babylonians used reciprocals to concretely compete squares to generate integral pythagorean triples (Robson, 2001, p. 183-185). This theory not only views the mathematics of Plimpton 322 within the cultural context of Babylonians, but it also uses number theory effectively to interpret the method of Plimpton 322's generation of pythagorean triples; therefore, the reciprocal theory has been given the most merit among all the theories of Babylonian pythagorean triple generation, and the consensus of the mathematical community is that this theory is most plausible.

The important thing to take away from the study of Plimpton 322 is that it illustrates that mathematics is not culture-free; however, most importantly, it illustrates a powerful application of uniting geometry and number theory. Plimpton 322, itself,—without deciphering the method of pythagorean triple generation— is an artifact of ancient "modern" number theory. It, after all, makes use of a particular case of Fermat's Last Theorem, $a^n + b^n = c^n$ when $n = 2$. The Babylonians' work with pythagorean triples and this case of Fermat's Last Theorem can be viewed as a prelude to modern number theory. The discovery of Plimpton 322 has been called one of the "most surprising discoveries in twentieth century archaeology" because it shows that the Babylonians were working with this type of problem for centuries before Diophantus, Euclid, and Fermat (Edwards, 1977, p. 4). It appears that number theory and geometry have always had some connection, but this connection has not always been known or acknowledged. In fact, in the 17th century, geometers vehemently insisted that geometry be untainted by arithmetic (Mahoney, 1994, p. 3). Pierre de Fermat entered the mathematics scene in 17th century Europe. His work indicates that he had a similar fascination with the particular case of his last theorem of when $n = 2$ to that of the Babylonians. Fermat is credited as being the father of modern number theory, the queen of mathematics. His time spent working with mathematics is marked by his efforts to end the segregation of number and geometry.

Mathematical Climate in 17th Century Europe

To better understand Fermat and his contributions to mathematics, it is important to become familiar with the mathematical climate in which he worked. Mathematics was a fragmented system. The lack of a unifying term for the work of mathematics contributed to a fragmented mathematical community. The term mathematician was not used in reference to those that work with mathematics (although for the purposes of this paper, the term *mathematician* will be used to refer to those working in mathematics). "*Mathematicus* retained in the sixteenth and seventeenth centuries the meaning it had for the Middle Ages; it meant; 'astrologer' or 'astronomer'" (Mahoney, 1994, p. 14). Geometers called themselves *geomatre*. Germans called themselves *Rechenmeister*. Also, mathematics was divided into six distinct branches with distinct philosophies on mathematics, further fragmenting the mathematical community.

Most individuals working in mathematics identified themselves as one or more of the following: classical geometers, cossist algebraists, applied mathematicians, mystics, artists and artisans, and analysts. The philosophies on the purpose and nature of mathematics and the mathematical styles of each of these groups mostly contrasted (Mahoney, 1994, p. 2). Classical geometers viewed Greek tradition as the superior model for behavior, and thus exclusively used techniques and developments that had Greek roots. They were more interested in presentation style than in new and unique results. Limiting themselves to a purely Greek style, they refused to adopt new and useful non-Greek methods (Mahoney, 1994, p. 3-4). Cossist algebraists valued efficiency and novelty in problem-solving, oftentimes at the expense of presentation style. When they wrote their solutions for the public, they would do so in a complicated manner, which only other cossists could understand, and for the purpose of announcing their triumphs to their peers. "A cossist's ability to solve problems his competitors could not solve gave him an advantage he was loath to surrender through publication" (Mahoney, 1994, p. 5-6). Applied mathematicians valued Greek methods, but they did not exclusively seek these methods. They were most interested in developing operational mathematics for uses such as business and navigation. They placed more emphasis on computation than proof, which sharply contrasted with most of the mathematicians in the other branches (Mahoney, 1994, p. 9-10). Mystics were interested in reviving ancient number theory. They searched for "the secrets of integers." They saw mathematics as a means to reveal the secrets of the universe (Mahoney, 1994, p. 11). The final branch of 17th century European mathematics was the group of which Fermat was a member, the analysts. They borrowed philosophies from the other five branches. They valued methods with Greek heritage, borrowed from the geometers. From the cossists, they borrowed the advocacy of algebra as a powerful tool for solving problems. They viewed Greek models as a foundation on which other non-Greek mathematics could build (applied mathematicians). Finally, they desired to unite mathematics with a system of symbolic reasoning (mystics) (Mahoney, 1994, p.12).

Some other important characteristics of this time to consider are the university setting in relation to mathematics and the communication of mathematical inquiry and results. Most universities in early 17th century Europe had curricula that promoted very little mathematics. The focus of these universities was mainly law, medicine, and theology. Toward the middle of the century, mathematics was more readily integrated into university curricula. Even with this shift of mathematics' presence in the universities, most mathematical influence was outside higher education; most mathematical discovery and advancement was outside universities. Most mathematicians of that time may have been university graduates, but they received their mathematical training from their peers or they were self-taught. René Descartes, John Wallis, Sir Isaac Newton, Christiaan Huygens, and Fermat are among this self-taught/peer-taught community (Mahoney, 1994, p. 13). There were no mathematical journals in the 17th century, so mathematical discoveries or inquiries were shared through the correspondence of letters. Marin Mersenne was unofficially the overseer of letters shared among mathematicians. Mathematicians wrote to Mersenne about their discoveries, and then Mersenne relayed this information to other mathematicians through written letters (Dudley, 2008, p. 60). This correspondence adds an intriguing element to this period of math. Fermat and his works were shaped by this mathematical climate and these characteristics of the 17th century.

Background on Fermat

Pierre de Fermat (1601-1665) was not a mathematician by profession; he was trained in law. Mathematics was a pursuit of leisure for Fermat as is was for most mathematicians. Working as an amateur mathematician allowed Fermat freedom. He was a "free agent." If another mathematician rejected his methods or work, it affected nothing more than his self-esteem. Once again, mathematics was a fragmented system, so there was a lot of rejection of others' works. Since mathematicians' professional careers were not a stake, he could feud as he wanted to with others because even if his opposition gained the favor of all other mathematicians, it did not have a real effect on his career (Mahoney, 1994, p. 21). Of course, he may have taken offense at his opposition's negative opinion of his work (and he did let his opposition frustrate him, which will be addressed later), but his professional life was safe from the impacts of disagreement. In fact, Mahoney claims that Fermat gravitated toward mathematics because it was a safe haven from the disputes and controversies that he saw in his career of law. Controversies in mathematics seemed less intense, which Mahoney thinks is interesting, since many mathematical disputes were, in fact, very intense (1994, p. 23).

Fermat wrote to several mathematicians. He started correspondence with Mersenne in 1636. It was not until about 1662 that this correspondence ended; Pierre de Carcavi took over Mersenne's role of mathematical mediator after Mersenne's death in 1648 (Weil, 1984, p. 41-42). Even though neither of these men were "creative mathematicians," themselves, they enthusiastically relayed information to and from the most prominent mathematicians of the day (Goldman, 1998, p.13). Correspondence played an especially important role in Fermat's leisurely study of mathematics. The only known personal contact that Fermat had with another mathematician was a three-day visit with Mersenne in 1646. Fermat corresponded with men including the following: Bernard Frénicle de Bessy, (a fellow "number lover"), Descartes, Étienne Pascal, Blaise Pascal, Gilles Personne de Roberval, and Wallis (Weil, 1984, p. 41, 53, 81). Letters took the place of personal contact. Fermat's correspondence with Frenicle was very valuable. Frenicle, interested in number theory, challenged Fermat's discoveries, seeking reasoning behind his number theory discoveries. This questioning of his discoveries led Fermat to reveal a few of his "carefully guarded secrets." This correspondence of Fermat and Frenicle quite possibly yielded some of the most important information on Fermat's number theory (Mahoney, 1994, p.293). Letters from correspondence, in general, have played a significant role in unveiling Fermat's work because of his shying away from the publication of his work.

Fermat chose not to publish much of his work. In fact, there are no formal publications of any of his work in number theory (Goldman, 1998, p. 12). There are several reasons that he may have chosen to not publish much of his work. Publishing was a stressful endeavor. If the mathematician handed over his work to be printed, there was a substantial risk that, if the printer was not familiar with the mathematician's notation and style, errors would be made. "Only too often, once the book had come out, did it become the butt of acrimonious controversies to which there was no end" (Weil, 1984, p. 44). Perhaps out of this fear, Fermat declined to publish his work. Also, Fermat struggled with writing up his proofs. He especially was plagued with awkwardness in writing up his proofs concerning number theory. This is mainly because he had no models of number theory publications to emulate (Weil, 1984, p. 44). It was the fear of Fermat's admirers that Fermat's work would be simply lost and forgotten, if his work was not published. After his death, some of his writing on geometry, algebra, differential and integral calculus were published posthumously. Also, many of the letters that Fermat wrote to fellow mathematicians have been published. Samuel de Fermat is responsible for publishing much of his father's work. In fact, it appears that only one of Fermat's proofs in number theory has been published. This proof was published by Samuel posthumously as Observation 45 on Diophantus.

This proof is of the proposition that "the area of a right triangle cannot be a square" (Edwards, 1977, 9.10-11). This proof will be further discussed in a later section of this paper. It is because of Samuel that Fermat's Last Theorem was published for the world to see (Weil, 1984, p.44; Edwards, 1977, p.1-2).

Fermat's Interest in Number Theory

Fermat's interest in number theory was fostered by the works of Diophantus. At this time, the only sources on number theory were Diophantus' *Arithmetic* and Books VII-IX of *The Elements* (Kleiner, 2005, p. 4). Ironically, Fermat meant to revive old, classical mathematical traditions, but he actually ended up laying the foundation for a "new, modern tradition," modern number theory (Mahoney, 1994, p. 283). He did return to one ancient tradition that had been discarded by his peers. This ancient tradition was the belief that arithmetic was "the doctrine of whole numbers and their properties" (Mahoney, 1994, p. 283-284). Plato advocated this ancient tradition. In *The Republic*, Plato states, "Good mathematicians, as of course you know, scornfully reject any attempt to cut up the unit itself into parts..." (Mahoney, 1994, p. 284). *Arithemtic* contains about 200 problems, which require the use of one or more indeterminate equations to solve. Diophantus[3] sought rational solutions to these equations (Kleiner, 2005, 4). Although Fermat was captivated by Diophantus' work in *Arithmetic*, he rejected a lot of his work because it allowed rational solutions. Motivated by his intention to renew arithmetic's commitment to integers, he felt that the only solutions sought ought to be integral solutions (Mahoney, 1994, p. 284). Nevertheless, Fermat was truly inspired by *Arithmetic*, as illustrated in *Observations on Diophantus*, which is the publication of the abounding notes that Fermat wrote in the marginalia of *Arithmetic* (Mahoney, 1994, p. 286). His particular fascination with indeterminate equations is evident in much of his work.

One proposition of Diophantus' *Arithemtic* which piqued Fermat's interest and undoubtedly, significantly impacted much of Fermat's work with number theory. This proposition, "to write a square as the sum of two squares," is one of mathematics' oldest problems (Edwards, 1977, p. 3); after all, it dates back to the ancient mathematics of the Babylonians, as discussed earlier in this paper. Edwards claims that this proposition is a great inspirer of Fermat's Last Theorem (1977, p. 3); clearly, this proposition is the particular case of Fermat's Last Theorem that has been referenced throughout this paper. A geometric expression of Diophantus' proposition is "find right triangles in which the lengths of the sides are commensurable, that is, in ratio of whole numbers" (Edwards, 1977, p. 4). Fermat worked with the geometric expression of Diophantus' proposition throughout his career. It not only inspired Fermat's Last Theorem, but it also led Fermat to discover intriguing details relating to pythagorean triples and to pen some theorems on right-angled triangles. Most importantly, its inspiration led Fermat to blend his work with geometry and number theory and to pave the way for the idea that geometry and number theory could live harmoniously in the realm of mathematics.

Fermat's Use of Number Theory in Geometry

It is only natural that Fermat worked with pythagorean triples, given his fascination with indeterminate equations. His work with pythagorean triples illustrates his interest in joining number theory and geometry. He posed and solved several problems involving right-angled triangles. As briefly mentioned earlier, pythagorean triples relate to Fermat's Last Theorem. This theorem states that it is impossible for "any number which is a power greater than the second to be written as a sum of two like powers" (Edwards, 1977, p. 2). The algebraic representation of this theorem is the following: $a^n + b^n = c^n$ has nontrivial, positive integral solutions[4] only if $n \leq 2$. This, of course, bears resemblance to the Pythagorean Theorem. Fermat never presents a proof for this theorem. "I have a truly marvelous demonstration of this proposition, which this margin is too narrow to contain" (Edwards, 1977, p.2). Even though the margin simply provided insufficient room for his proof and he never presented this "marvelous demonstration," his work shows that he was very comfortable using this particular case of Fermat's Last Theorem of when $n = 2$ and finding integers *a, b*, and *c*, which satisfy this case.

In 1643, he posed the following problem: "To find a pythagorean triangle in which the hypotenuse and the sum of the arms are squares" (Sierpiński, 2003, p. 67). Fermat wrote to Mersenne and claimed that he had found the smallest such pythagorean triangle. The triangle that Fermat found was triangle $(4565486027761, 1061652293520, 4687298610289)$. This solution was significant because obviously Fermat did not find this triangle by guessing or simply stumbling upon it (Sierpiński, 2003, p. 67). Fermat did not reveal his method of finding this triangle, but Sierpiński offers an explanation of an approach that Fermat may have used. Sierpiński explains that finding a pythagorean triangle in which the hypotenuse and the sum of the arms are squares is equivalent to finding positive rational values for x, y, u, v satisfying the following equations: $x^2 + y^2 = u^4$, $x + y = v^2$. Once these positive rational values are determined, a common denominator, *m*, can be found; then, the first equation is multiplied by m^4 and the second equation is multiplied by m^2. Because the denominator of x and y is m^2, the solution to the triples satisfying: $x^2 + y^2 = u^4$ are the integers m^2x, m^2y, and m^2u^4 (Sierpiński, 2003, p. 67-69).

Fermat's work with pythagorean triples resulted in several theorems on right-angled triangles. One theorem is that "there are no pythagorean triangles of which at least two sides are squares" (Sierpiński, 2003, p. 48- 49). An implication of that theorem is that there are no pythagorean triples, where each side within a triple is a square. Algebraically, this is represented as $a^4 + b^4 = c^4$. This is clearly, the case of Fermat's Last Theorem when $n = 4$; thus, further highlighting Fermat's efforts to unite number theory and geometry (Sierpiński, 2003, p. 55).

Fermat also sought to determine whether, given some number A, there is a pythagorean triangle with arms, whose sum is equal to A. In order to find a pythagorean triangle, whose arms sum to a given number A, A must meet the necessary and sufficient condition that A must be "divisible by at least one prime number of the form $8k \pm 1$ (Sierpiński, 2003, p. 34-35). For example, since $41 = 8(5) + 1$, there must be a pythagorean triple, where the sum of a and b is 41. If a, b, and c are such that, $a = 20$, $b = 21$, and $c = 29$, then is such a pythagorean triple satisfying this proposition.

Fermat also observed some properties of the areas of right-angle triangles, and he authored a few theorems on this topic, including the theorem, mentioned earlier as the only known number theory theorem that is accompanied with a proof of Fermat's. The proposition, whose proof was published after Fermat's death, is "the area of a right triangle cannot be a square" (Edwards, 1977, p.10-11). Algebraically, this means that there does not exist a pythagorean triple, whole numbers a, b, c satisfying $a^2 + b^2 = c^2$, such that the area of the triangle formed by this triple, $\frac{1}{2}\boldsymbol{ab}$, is a square[5] (Edwards, 1977, p. 11). Fermat essentially shows that, for this to be true, then a and b would be squares. Then, he likens this proposition to the $n = 4$ case of his last theorem, which he shows is impossible. He uses a proof by infinite descent to prove this proposition. His proof ends with "the margin is too small to enable me to give the proof completely and with all detail" (Edwards, 1977, p. 11-12; Mahoney, 1994, p. 352). The preservation of Fermat's work sure had an on-going battle with margins. Another theorem that involved areas of pythagorean triangles is "for each natural number *n* there exist *n* pythagorean triangles with different hypotenuses and the same areas" (Sierpiński, 2003, p. 37). As a result of this theorem, three triangles with different hypotenuses but with the same area exist. (Of course, this is an arbitrary example, and it could really be any *n* number of triangles.) The area of these triangles would be large because it takes large side lengths to achieve equal area but different hypotenuses. In fact, the smallest area common to three such primitive pythagorean triangles with different hypotenuses is 13123110. These triangles with the common area are $(3059, 8580, 9109)$, $(4485, 5852, 7373)$, and $(19019, 1380, 19069)$ (Sierpiński, 2003, p.37, 40). Another theorem from Fermat is the Fundamental Theorem on Right-Angled Triangles. This theorem states that every prime number in the form $4m + 1$ is the hypotenuse of one and only one primitive pythagorean triple (Vella, Vella, & Wolf, 2005, p. 237). Fermat went on to prove by infinite descent that every prime number of the form $4m + 1$ is composed of two squares and thus, is, geometrically, the hypotenuse of a pythagorean triangle (Mahoney, 1994, p. 349).

Fermat's mixing of geometry and number theory—his work with pythagorean triples and pythagorean triangles—had a significant impact on other elements of his work in number theory. Working with pythagorean triples gave Fermat plenty of experience in working with decomposition of squares. His work with pythagorean triples was the gateway to his work with decomposition of squares, in general (Mahoney, 1994, p.287). Much of his later work in number theory is concerned with the decomposition of squares—there is certainly a connection of the decomposition of squares to Fermat's Last Theorem (Mahoney, 1994, p. 303). Another significant contribution of Fermat's work with pythagorean triangles is Fermat's initial use of infinite descent. He introduced the concept of infinite descent in proving theorems, such as "no number of the form $3k - 1$ can be composed of a square and the triple of a square, or no right triangle has a square area" (Mahoney, 1994, p. 348). Fermat shows this by infinite descent by saying if such a triangle with a square area existed, then there would have to be another triangle smaller than the original that also has a square area; then, there would have to be another triangle smaller than the previous triangle with the same characteristic. He states that smaller and smaller triangles with this property would be found, "descending *ad infinitum*" and since, natural numbers are bounded below, this cannot be. Therefore, "no number of the form $3k - 1$ can be composed of a square and the triple of a square, or no right triangle has a square area" (Mahoney, 1994, p. 348). Fermat's successes and developments in number theory left most of his contemporaries unenthused.

17th Century Perception of Number Theory

If number theory were an island, Fermat would have been its only inhabitant. Although mathematicians like Mersenne and Frenicle were "number lovers," none of his peers were true number theorists (Weil, 1984, p. 51). The rest of the mathematical community showed less interest in number theory. Fermat sent problems to several mathematicians, in hopes of fostering interest in what he was so interested in. He sent some number theory problems to mathematicians in England, including Wallis. When Wallis sent rational solutions and thus, disregarding Fermat's criterion of integral solutions, Fermat rejected Wallis' solutions. This rejection did nothing but reinforce Wallis' view of the unimportance of number theory (Mahoney, 1994, p. 63). As with the Wallis incident, other mathematicians were not inclined to show much interest because of how Fermat treated them and their inquiries. Even Frenicle[6] got angry with Fermat. Fermat would send Frenicle problems, and Frenicle would ask for details, but Fermat would never send more details. At one point, Frenicle accused Fermat of sending problems that were impossible to solve (Mahoney, 1994, p. 56). Edwards makes the argument that Fermat's habit of rarely sharing methods or further explaining problems, whether or not he was aware of it, is an indication that he was a jealous, competitive, and secretive mathematician, like most of his peers (1977, p.11). Regardless if that is true, Fermat wanted his number theory to speak for itself and attract the attention and admiration of others, and he apparently did not see creating positive relationships as a way of promoting his ideas. The lack of interest in number theory was something that upset Fermat. "…his failure to convince Wallis and others of the beauty andchallenge of number theory was a source of anguish and frustration to Fermat" (Mahoney, 1994, p. 22). Toward the end of his life, Fermat, hoping to stir someone to continue his work in number theory, wrote to Huygens about "handing on the torch." Huygens, in reference to Fermat's letter, wrote to Wallis, "There is no lack of better things for us to do" (Weil, 1984, p. 119-120). No one picked up "the torch" until the 18th century, when the "rebirth" of number theory took place. This rebirth came through the works of number theorist, Leonhard Euler (1707-1783) (Kleiner, 2005, p.4; Edwards, 1977, p. 39; Weil, 1984, p. 2).

[6]Frenicle, one of Fermat's few allies, sadly never contributed anything of any significance to number theory. Fermat finally found an admirer of his work, but he was unable to contribute much of anything to Fermat's cause of promoting number theory (Mahoney, 1994, p. 340).

Looking to the Future (18th Century and Beyond)

Euler ran with this "torch." Even though he was born over forty years about Fermat's death, Euler, fascinated with Fermat's work, worked with Fermat's theorems and other propositions. Euler proved the cases of $n = 3$ and $n = 4$ of Fermat's Last Theorem. In a 1753 letter to fellow mathematician Christian Goldbach, Euler said that "the general case still seemed quite unapproachable" (Edwards, 1977, p. 59). If only Euler knew how truly "quite unapproachable" this proof really was, and that a complete proof would not be presented for over a century after his death. In 1995, nearly 350 years after Fermat introduced this theorem, Andrew Wiles, with the help of Richard Taylor proved Fermat's Last Theorem (Goldman, 1998, p. 15). Euler also worked with sum of squares, in general. He proved Euler's proposition that "every prime of the form $4n + 1$ is a sum of two squares" (Edwards, 1977, p.46). Unfortunately, Fermat's hope that his admiration for and intrigue of number theory would be shared was not fulfilled under after his death, but it was shared. Sophie Germain, a female mathematician of the late-18th century, early-19th century, like Euler, shared Fermat's interest in number theory. She discovered a result of Fermat's Last Theorem, which bears her name. Sophie Germain's Theorem relates to solutions to cases of Fermat's Last Theorem and different divisibility properties of these solutions (Edwards, 1977, p. 64). She is worth mentioning because of her significance as a woman that "[overcame] the prejudice and discrimination which have tended to exclude women from the pursuit of higher mathematics…" (Edwards, 1977, p. 61). The queen caught the attention of Germain, and she caught the attention of the mathematical community as a woman in a male-dominated field.

Conclusion

Fermat faced significant opposition to his work in number theory, opposition that he was unable to overcome during his lifetime. Firstly, most mathematicians were enthralled in a love affair with calculus. Number theory, a newcomer on the mathematics scene, had a difficult time competing for the spotlight (Weil, 1984, p. 119). Secondly, Fermat was trying to revive an idea that others had deemed as archaic. The idea of placing the constraint of only accepting integral solutions in arithmetic was unappealing to many, who saw no reason to reject rational solutions. Thirdly, even though number theory offered another dimension to geometry, many mathematicians would not entertain the idea of mixing arithmetic and geometry. They believed that arithmetic and geometry were to be two entirely separate things. Even though modern number theory was valued by few in 17th century Europe, it gained more attention and became more valuable to mathematicians in the 18th century, with Euler's emergence as a prominent number theorist. His number theory built upon Fermat's foundation, which included a relationship forged between geometry and number theory.

This forged relationship between geometry and number theory is important, as illustrated in Fermat's work, especially his extensive work with pythagorean triples, but also as illustrated by the study of Plimpton 322. Pythagorean triangles are not merely triangles, whose sides satisfy $a^2 + b^2 = c^2$. As a result of Fermat's work, pythagorean triangles almost seem like a mystical triangle. Fermat brought the study of these triangles to an entirely different level. It helps to illustrate the myriad of sources of intrigue that pythagorean triples offer to number theory because of all the unique characteristics of these triples. Also, this new relationship between geometry and numbers makes possible the study of ancient artifacts, like Plimpton 322. It has allowed for the theories of the origin of the triples on Plimpton 322 to be soundly and thoroughly developed. These theories use number theory to revisit the time of the Babylonians. If Bruins or Robson did not have the tools to discover that reciprocals and using cut-and-paste geometry could yield pythagorean triples, they would not have a theory to help them reconstruct this time in the history of mathematics. Whether or not the attempt to reconstruct the history of mathematics is acceptable or honorable is irrelevant in this paper. The fact is that number theory is a medium that can be used to visit the time of ancient mathematics. It allows for understanding of ancient civilizations' mathematics even when there is a strong cultural element embedded in their mathematics. With the Babylonians, their generation of pythagorean triples, and the discovery of Plimpton, it is clear that the combination of geometry and number theory allows for insight and formulation of probable theories.

Also, the extensions of Fermat's integration of number theory and geometry cannot be forgotten. Both square decomposition and the use of infinite descent played important roles in his career and overall contributions to number theory. Based on all of Fermat's work with pythagorean triangles and triples and his work with square decomposition and the use of infinite descent justify the rightful connection of geometry and number theory. In the 17th century, it was a matter of the mathematical community refusing to embrace or even acknowledge this connection. Regardless of attitude, the connection between the queen and geometry has been and is present. It is important that geometry courses in secondary mathematics make this connection known and take advantage of this connection and its positive attributes, which is bound to intrigue students more than if geometry is simply an interlude between algebra courses. The history of the queen is quite the tale, which offers intrigue and beauty, matching the intrigue and beauty of number theory, itself. The story of number theory also offers inspiration, in particular, inspiration to female mathematics students. Sophie Germain's success in such a male-dominated world offers hope to female students that they, too, have a place in mathematics.

Fermat was a rebel. He founded this thing called modern number theory, which bucked the trends and attitudes of the 17th century. This queen was not taken out by a rebellion. She started her own rebellion and gained a kingdom.

ENDNOTES

1 Plimpton 322 is number 322 in the G.A. Plimpton Collection at Columbia University; this is the origin of its name (Joseph, 2000, p.115).

2 The concept of incommensurability was deeply embedded in Greek mathematics and its roots are seen in *The Elements*. Babylonian tablets indicate that this concept has been known since 1800-1500 BC. These tablets "supposedly demonstrate knowledge of the fact that some values cannot be expressed as ratios of whole numbers" (Roskam, 2009, p. 277). The Greek discovery that the hypotenuse of a right-angled triangle (with the congruent sides equal to a length of 1) is $\sqrt{2}$, an irrational quantity, led to their understanding of incommensurability. "Prior to this inexorable discovery, the Pythagoreans viewed numbers as whole number ratios…" (Roskam, 2009, p. 277; Edwards, 1977, p. 4).

3 Ironically, Diophantus never restricted himself to solutions with integers; he was concerned about rational numbers, in general. However, in modern terminology, "Diophantine" is practically synonymous with "integer," as in Diophantine equations (Edwards, 1977, p. 26).

4 Negative and trivial solutions were unacceptable because they were treated with suspicion in 17th century mathematics (Edwards, 1977, p. 3).

5 Since a and b cannot both be odd (this is a necessary condition for the formulation of a primitive pythagorean triple), $\frac{1}{2}ab$ is always an integer (Edwards, 1977, p. 11).

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