The Mathematics Enthusiast

Volume 9
Number 1 *Numbers 1 & 2*

Article 4

1-2012

From conic intersections to toric intersections: The case of the isoptic curves of an ellipse

Thierry Dana-Picard

Nurit Zehavi

Giora Mann

From conic intersections to toric intersections: The case of the isoptic curves of an ellipse

Thierry Dana-Picard
Jerusalem College of Technology
ndp@jct.ac.il

Nurit Zehavi
Weizmann Institute of Science
nurit.zehavi@weizmann.ac.il

Giora Mann
Beit Chanan
gioramann@013.net.il

Abstract
Starting from the study of the orthoptic curves of parabolas and ellipses, we generalize to the case of isoptic curves for any angle, i.e. the geometric locus of points from which a parabola or an ellipse are viewed under a given angle. This leads to the investigation of spiric curves and to the construction of these curves as an actual intersection of a self-intersecting torus with a plane. The usage of a Computer Algebra System facilitated this investigation.

Keywords: Computer algebra systems; algebraic curves; conics; torics

I. Orthoptic curves.

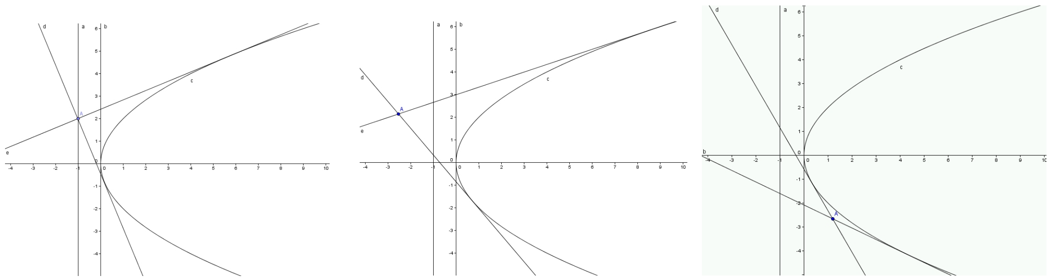
Given a plane curve *C*, the *orthoptic curve* of *C* is the geometric locus of points from which *C* can be viewed under a right angle, i.e. the locus of points through which passes a pair of perpendicular tangents to the curve *C*. For example, the orthoptic curve of a parabola *P* is a line, called the directrix of *P*, and the orthoptic curve of an ellipse is a circle whose center is the center of the ellipse.

Take a parabola with the canonical equation $y^2 = 2px$. We wish to find whether there exist tangents to the parabola through a given point with coordinates (X,Y) in the plane. For that purpose we explore the set of solutions of the following system of equations: $\begin{cases} y^2 = 2px \\ y = mx - mX + Y \end{cases}$.

We find the following classification (see Figure 1):

- For a point out of the parabola (i.e. for which the inequality $y^2 > 2px$ holds), there exist two tangents;
- Through a point on the parabola, there exists a single tangent;
- There is no tangent to the parabola through an interior point (i.e. a point for which the inequality $y^2 < 2px$).

As already mentioned, the directrix of the parabola is the geometric locus of the points from which the parabola is seen in a right angle (Figure 1a). We check graphically and justify geometrically that the directrix divides the exterior of the parabola into two regions. We illustrate this in Figure 1 for the ellipse whose equation is $x^2 + 4y^2 = 1$. One of them is the locus of points from which the parabola is seen under an acute angle (Figure 1b), the other one is the locus of points from which the parabola is seen under an obtuse angle (Figure 1c).



(a) **(b)** **(c)**

Figure 1: Viewing angles of the parabola

The last result intrigued the authors and the in-service teachers who attended a professional development course in Analytic Geometry. In order to obtain more details, we explored the locus of points from which the parabola is seen under a given angle. To our surprise, for specific angles we found branches of hyperbolas (see Figure 2, for the same ellipse as above): an unfamiliar relationship between parabolas and hyperbolas!

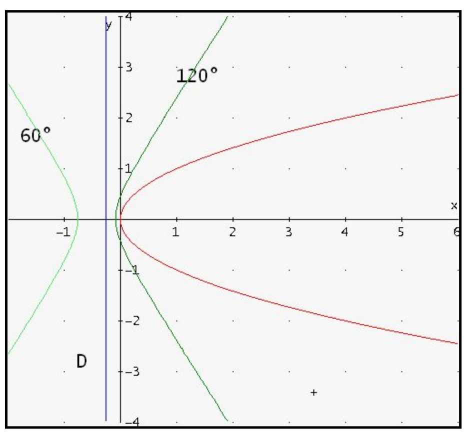
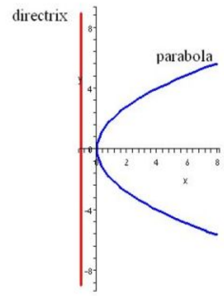


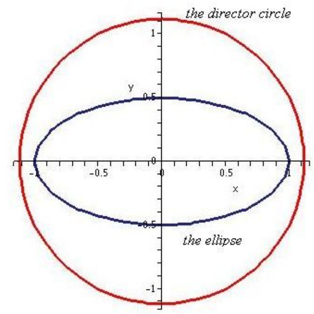
Figure 2: An unfamiliar relationship between parabolas and hyperbolas

Conic sections are an important domain in classical Mathematics. Within this domain, there exist topics which drew little attention in the past. We wish to shed light on one of these topics. Specifically, we are interested in geometric loci of points from which a given conic section is viewed under given angles. This means to look for points from which originate pairs of rays which are tangent to the conic and create a given angle θ.

The orthoptic curve of a parabola, i.e. its directrix, is displayed on Figure 3a, and in Figure 3b the orthoptic curve of an ellipse is shown. Actually, the orthoptic curve of an ellipse is the whole of a circle concentric with the ellipse. In the case of a hyperbola, finer tuning is necessary: first, an orthoptic curve may not exist and second, when it exists, holes appear.



(a) Parabola



(b) Ellipse

Figure 3: Orthoptic curves.

Given a curve C and an angle θ, the geometric locus of points through which passes a pair of tangents to the curve C making a angle of θ is a curve called a *θ-isoptic curve*. This curve can adopt very different forms. It happens that a *θ-isoptic curve* is given by a polynomial equation of higher degree (up to degree 4 for conics) in two real variables x and y.

Our study demands the usage of a couple of general tools from the theory of plane algebraic curves. A central tool used is Bezout's theorem (Kirwan 1992, page 54). For the situations described in the paper, the theorem states that the intersection of a line (a plane curve of degree 1) and a conic (a plane curve of degree 2) contains at most two points, possibly identical. The point of contact of a tangent to a conic with the conic itself is of multiplicity 2, thus there is nothing left for another point of intersection. As the points of intersection of a conic and a line are determined by the solutions of a quadratic polynomial, the fact that a line is a tangent to a conic is determined by the vanishing of a certain discriminant.

Most of the computations and the drawings in this paper have been performed using a Computer Algebra System (generally denoted by the initials CAS), either Derive or Maple. Figure 1 has been drawn using GeoGebra[1].

II. Isoptic curves of an ellipse.

We will work with the one-parameter equation $x^2 + k^2 y^2 = 1,\ k > 0$. No loss of generality occurs because of this decision, as it can be easily shown that any ellipse is similar to one of the ellipses in the above one-parameter family. The algebraic computations will be easier than with the canonical 2-parameter presentation.

We address now the following question: **Given an angle θ, what is the geometric locus of all the points in the plane from which the ellipse is viewed under the angle θ?** In other words what is the geometric locus of all the points in the plane through which passes a pair of tangents to the ellipse E making an angle of θ?

[1] http://www.geogebra.org

For $\theta = 90^0$, the answer can be found in the literature, but we prefer to expose this situation as a particular example. First note that four pairs of perpendicular tangents to the given ellipse are trivially found, namely pairs of tangents parallel to the coordinate axes. In Figure 4 we show the ellipse corresponding to $k = 2$ and its four tangents parallel to the axes. The points A, B, C, D belong to the geometric locus we are interested in.

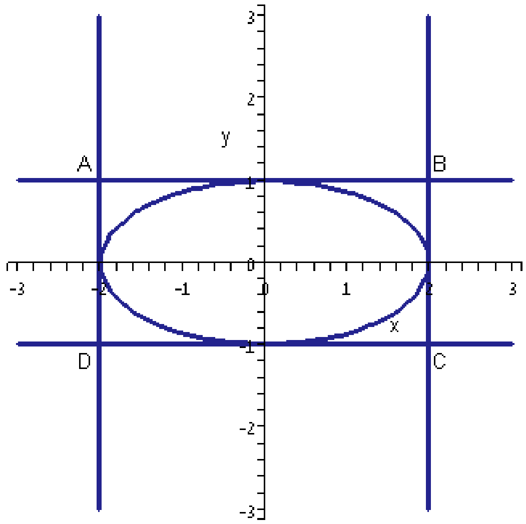


Figure 4: Tangents to the ellipse, parallel to the coordinate axes.

We consider now the general case: none of the tangents in the pair is parallel to the *y*-axis, therefore both have a slope. Take a point $T(x_0, y_0)$; a line L through T and non-parallel to the *y*-axis has an equation of the form $y = m(x - x_0) + y_0$, where m is the slope of L. An ellipse has no singular point, thus Bezout's theorem (Berger 1996, section 16.4) ensures that the line L is tangent to the ellipse E if, and only if, it has a "double" point of intersection with the ellipse. The possible slopes are the following:

(9) $$m_1 = \frac{\sqrt{x_0^2 + y_0^2 k^2 - 1} - kx_0 y_0}{k\left(1 - x_0^2\right)} \quad \text{and} \quad m_2 = -\frac{\sqrt{x_0^2 + y_0^2 k^2 - 1} + kx_0 y_0}{k\left(1 - x_0^2\right)},$$

where $x_0^2 + y_0^2 k^2 > 1$. The tangents are perpendicular if, and only if, $m_1 m_2 = -1$, i.e.

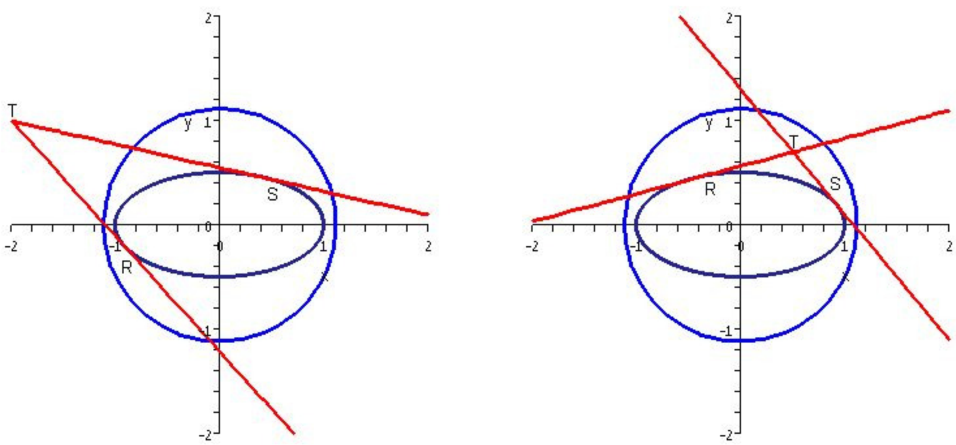
(10) $$\frac{k^2 y_0^2 - 1}{k^2\left(x_0^2 - 1\right)} = -1.$$

This equation is equivalent to

(11) $$\begin{cases} x_0^2 + y_0^2 = 1 + \dfrac{1}{k^2} \\ x_0 \neq -1, 1 \end{cases}.$$

It follows that the geometric locus of points from which the ellipse is viewed under a right angle without a side being parallel to the y-axis is a subset of the circle C_k whose center is at the origin and whose radius is equal to $\sqrt{1+1/k^2}$. Conversely, through every point on the curve defined by Equation (11) passes a pair of perpendicular tangents. Moreover, through the four points A, B, C, D one of the tangents to the ellipse is parallel to the *y*-axis, therefore it cannot be described by Equations (10) and (11), but these four points complete the curve given by Equation (11) to be the circle C_k. This circle is called *the* ***director circle*** or ***the orthoptic circle*** of the ellipse *E* (Spain 1963, page 79). Actually the director circle of the given ellipse is the circumcircle of the rectangle ABCD``. Figure 3 shows the ellipse and its director circle for $k = 2$.

Note that from a point exterior to the orthoptic circle, the ellipse is viewed under an acute angle, and from a point interior to the circle, the ellipse is viewed under an obtuse angle. The proof is easy to write; examples are displayed in Figure 5.



(a) Acute angle **(b) Obtuse angle**

Figure 5: Acute and obtuse angle for viewing the ellipse.

For tangents non parallel to the *y*-axis whose respective slopes are m_1 and m_2, the condition is equivalent to $\dfrac{m_1 - m_2}{1 + m_1 m_2} = \tan\theta$.

The requested geometric locus is determined by the equation

$$\frac{2k\sqrt{k^2 y_0^2 + x_0^2 - 1}}{1 - k^2 (x_0^2 + y_0^2 - 1)} = \tan\theta . \tag{13}$$

Denote $t = \tan\theta$ and square both sides of this equation. We obtain the following equation:

$$\frac{4k^2 (k^2 y_0^2 + x_0^2 + 1)}{\left(1 - k^2 (x_0^2 + y_0^2 - 1)\right)^2} = t^2 . \tag{14}$$

Actually, the vanishing points of the denominator are points though which pass a suitable pair of tangents, one of the tangents being parallel to the *y*-axis. Multiplying both sides by the common denominator, we obtain the following polynomial equation of degree 4:

$$k^4t^2x^4 + 2k^4t^2x^2y^2 - 2k^2(k^2t^2 + t^2 + 2)x^2 + k^4t^2y^4 - 2k^2(k^2t^2 + 2k^2 + t^2)y^2 + k^4t^2 + 2k^2(t^2 + 2) + t^2 = 0 \tag{15}$$

Equation (15) determines a plane curve called a *spiric curve*. See Wassenaar (2003a) and Ferréol (2001). A spiric curve is the intersection of a plane with a torus; see Wassenaar (2003b) and the appendix, at the end of the present paper.

The geometric locus of points from which the given conic ellipse E is viewed under a given angle θ is a called the ***θ-isoptic curve*** of the ellipse E for the given angle θ; we denote this curve by $OPT(k,\theta)$. The orthoptic curve of E is $OPT(k,90)$. In Figure 6, we show the curves $OPT(2,45)$, $OPT(2,135)$ and $OPT(2,90)$. The equation describing together the first one and the last one is $16x^4 + 32x^2y^2 + 16y^4 - 56x^2 - 104y^2 + 41 = 0$. This equation can be written under the form

$$\left(x^2 + y^2\right)^2 - \frac{7}{2}x^2 - \frac{13}{2}y^2 + \frac{41}{16} = 0\,. \tag{16}$$

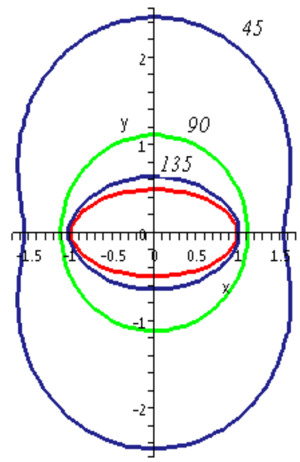


Figure 6: Examples of isoptic curves of an ellipse.

Note that the squaring before Equation (14) has an important consequence: the angles θ and 180^o-θ are studied at the same time. Therefore Equation (16) describes in fact the union of two isoptic curves, namely $OPT(k,\theta)$. and $OPT(k,180-\theta)$. We will call this union a ***bi-isoptic curve***, and will denote it by $OPT_2(k,\theta)$. In Figure 6, the union of $OPT(2,45)$ and $OPT(2,135)$ is $OPT_2(2,45)$, which can be also denoted by $OPT_2(k,135)$.

As an example, let us consider the case where $k = 4$, i.e. a case with greater eccentricity than what we had previously. In Figure 7, we show the ellipse whose equation is $x^2 + 16y^2 = 1$ and the bisoptic curve for $\boldsymbol{\theta} = 45^\circ$ and $\boldsymbol{\theta} = 135^\circ$ (recall that they are obtained simultaneously). Its equation is

$$x^4 + 2x^2y^2 + y^4 - \frac{19}{8}x^2 - \frac{49}{8}y^2 + \frac{353}{256} = 0\,. \tag{17}$$

Here too the isoptic curve is a spiric curve. In fact Equation (15) shows that it is this is a general situation.

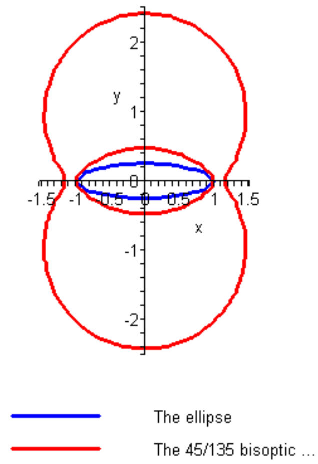


Figure 7: An ellipse with two isoptic curves

Additional remark: take a variable ellipse, with fixed vertices on the major axis. When the length of the minor axis tends to 0, then the ellipse "tends to" a line segment (in a sense to be defined of course), and the isoptic curve "tends to" a form built as the union of two symmetric arcs of circles, recalling a well known theorem: let *PQ* be a segment in the plane and let α be a given angle. The locus of points *M* in the plane such that $\angle PMQ = \alpha$ is the union of two symmetric arcs of circles whose endpoints are *P* and *Q*.

III. Reconstruction of the bisoptic curve as a toric section.

The equations we obtained in previous section for the bisoptic curves of an ellipse showed that these bisoptic curves are actually spiric curves, i.e. intersection a torus with a plane parallel to the torus axis. In this section we wish to reconstruct the torus and the plane from the knowledge of the spiric curve.

The general equation of a torus whose axis is the *z*-axis is as follows:

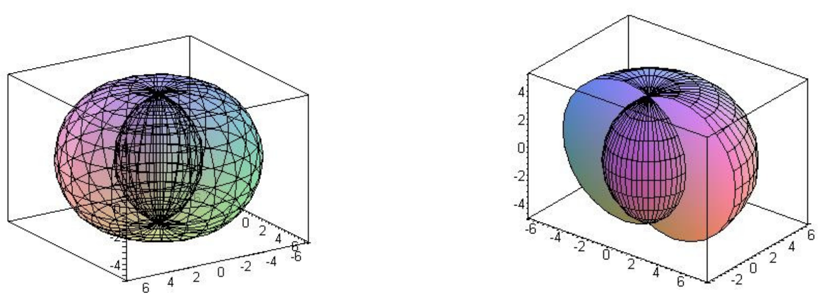
(18) $$\left(x^2 + z^2 + R^2 - r^2 + y^2\right)^2 - 4R^2\left(x^2 + y^2\right) = 0,\ R > 0, r > 0.$$

If $0 < R < r$, then the torus will be called a ***self-intersecting torus***. See Figure 8 for two examples.

(a) Non self intersecting **(b) self-intersecting**
R=5, r=2 ***R=4, r=5***

Figure 8: Two tori.

A self intersecting torus is a surface which comprises not only what is visible on Figure 8b, but also an internal part, as shown in Figure 9. As any torus, it is a surface of revolution generated by a circle being revolved about a line; in the case of a self-intersecting torus, the line intersects the circle, whence the internal part, as shown on Figure 9b.



(a) **The torus** **(b)View from inside**

Figure 9: Self-intersecting torus.

For the reader's convenience, we presented here tori and self-intersecting tori with the *z*-axis as their axis of revolution. In what follows we will change our point of view in order for the 3D-geometry we explain to be coherent with the plane configuration that we studied in the previous sections.

Note that any ellipse in the *xy*-plane is similar to an ellipse whose equation is $x^2 + k^2 y^2 = 1,\ k > 1$. Therefore, WLOG, we consider an ellipse E with this equation. As $k > 1$, the torus we are looking for has the x-axis as its axis or revolution. Thus, the equation of the torus has the following form:

(19) $$\left(x^2+y^2+z^2+R^2-r^2\right)^2-4R^2\left(y^2+z^2\right)=0\,.$$

Expanding the left-hand side, and multiplying by a suitable constant, w show that Equation (19) is equivalent to

(20) $$\begin{aligned}&k^4t^2x^4+2k^4t^2x^2y^2+2k^4t^2x^2\left(R^2-r^2+z^2\right)+k^4t^2y^4\\&-2k^4t^2y^2\left(R^2+r^2-z^2\right)+k^4t^2\left(R^4-2R^2\left(r^2+z^2\right)+r^4-2r^2z^2+z^4\right)=0.\end{aligned}$$

Equation (20) is equivalent to Equation (15) if, and only if, the following system of equations holds:

(21) $$\begin{cases}-2k^2\left(k^2t^2+t^2+2\right)x^2=2k^4t^2x^2\left(R^2-r^2+z^2\right)\\ 2k^2\left(k^2t^2+2k^2+t^2\right)y^2=2k^4t^2y^2\left(R^2+r^2-z^2\right)\\ k^4t^2+2k^2\left(t^2+2\right)+t^2=k^4t^2\left(R^4-2R^2\left(r^2+z^2\right)+r^4-2r^2z^2+z^4\right)\end{cases}.$$

We simplify the first two equations, obtaining thus:

(22) $$\begin{cases}-\left(k^2t^2+t^2+2\right)=k^2t^2\left(R^2-r^2+z^2\right)\\ \left(k^2t^2+2k^2+t^2\right)=k^2t^2\left(R^2+r^2-z^2\right)\end{cases}.$$

By sidewise addition we obtain finally:

(23) $$R^2=\frac{k^2-1}{k^2t^2}\,.$$

By a sidewise substraction of the first equation from the third one in (21), we obtain after simplification

(24) $$k^2t^2\left(r^2-z^2\right)=k^2t^2+k^2+t^2+1\,.$$

Let $p=r^2$ and $q=z^2$. Thus Equation (24) can be written under the following form

(25) $$k^2t^2\left(p-q\right)=k^2t^2+k^2+t^2+1\,,$$

Whence

(26) $$q=\frac{k^2\left(pt^2-t^2-1\right)-t^2-1}{k^2t^2}\,.$$

A substitution into the third equation in (21) yields

(27) $$k^4t^2+2k^2\left(t^2+2\right)+t^2=k^4t^2\left(R^4-2R^2\left(p+q\right)+p^2-2pq+q^2\right).$$

We substitute for q using Equation (26) and finally we obtain

(28) $$p=\frac{k^2\left(t^2+1\right)}{t^2\left(k^2-1\right)},$$

which is equivalent to

(29) $$r^2=\frac{k^2\left(t^2+1\right)}{t^2\left(k^2-1\right)}.$$

It follows easily that

(29) $$z^2=\frac{t^2+1}{k^2t^2\left(k^2-1\right)}.$$

In conclusion we have:

$$(30)\qquad \begin{cases} R = \dfrac{\sqrt{k^2-1}}{kt} \\ r = \dfrac{k\sqrt{t^2+1}}{t\sqrt{k^2-1}} \\ z = \dfrac{\sqrt{t^2+1}}{kt\sqrt{k^2-1}} \end{cases} .$$

It follows that

$$(31)\qquad \begin{cases} r - R = \dfrac{k^2\left(\sqrt{t^2+1}-1\right)+1}{kt\sqrt{k^2-1}} \\ r - R - z = \dfrac{\sqrt{k^2-1}\cdot\left(\sqrt{t^2+1}-1\right)}{kt} \end{cases}$$

Therefore $r - R > 0$ and $r - R - z > 0$. This shows that the given bisoptic curve is the intersection of a self-intersecting torus with a plane parallel to the *y*-axis at distance *z*, such that we have a union of two loops.

If we substitute now the values for the squares of *R*, *r* and *z* in the original equation of the torus we get a quartic equation showing that the bisoptic curve under study is actually a spiric curve:

$$(32)\qquad \left(x^2+y^2-\frac{k^2t^2+t^2+{}^2}{k^2t^2}\right)^2 = \frac{4\left(k^2-1\right)}{k^2t^2}y^2+\frac{4\left(t^2+1\right)}{k^4t^4}.$$

For example, if t=1 and k=2, we have the following system of equations:

$$(33)\qquad \begin{cases} x^2+4y^2=1 \\ \left(x^2+y^2-1.75\right)^2 = 3y^2+0.5 \end{cases}$$

The first equation represents the ellipse and the second equation represents the bisoptic curve for $\theta = 45^o/135^o$ in Figure 10.

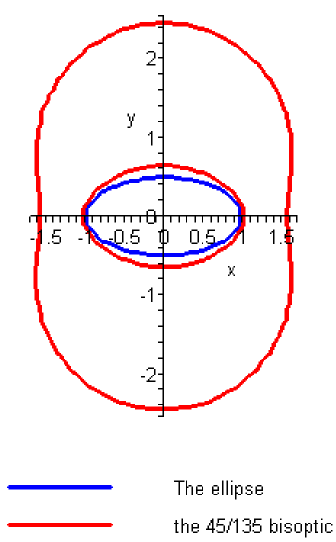


Figure 10: An ellipse and one of its bisoptic curve: plane configuration

We show two views of the 3D-configuration in Figure 11. Note the internal part of the torus (consequence of the self-intersection).

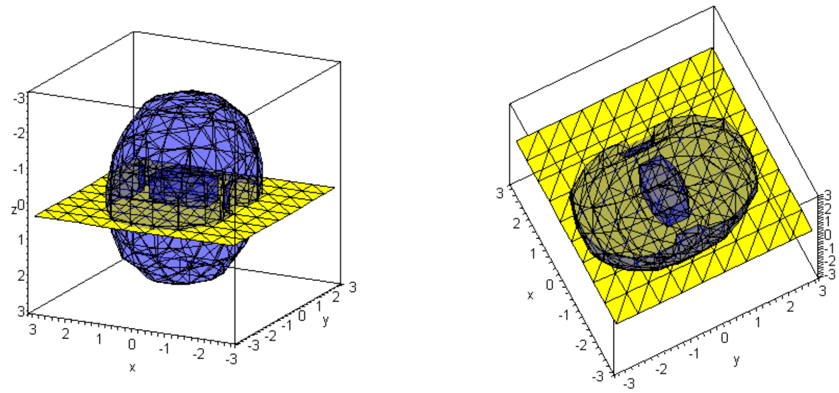


Figure 11: The 45/135 bisoptic of an ellipse as a toric intersection

IV. Limiting cases.

In this section we wish to show the internal coherence of what has been studied in the previous section. For this purpose, recall that the bisoptic curve under study is determined by two parameters: the positive parameter k determines the shape of the ellipse (namely its eccentricity), and the parameter t encodes the angle under which the ellipse is viewed. We study the behavior of a bisoptic curve when one of the parameters, either t or k , is fixed and the second one tends to infinity.

1. Fixed ellipse and variable angle.

We recall that an ellipse E whose equation is $x^2 + k^2y^2 = 1,\ k > 0$, is viewed under an angle θ such that $\tan\theta = t$ from all the points on a curve whose equation is

$$k^4t^2x^4 + 2k^4t^2x^2y^2 - 2k^2(k^2t^2 + t^2 + 2)x^2 + k^4t^2y^4$$
$$- 2k^2(k^2t^2 + 2k^2 + t^2)y^2 + k^4t^2 + 2k^2(t^2 + 2) + t^2 = 0$$

as shown in Equation (15).

The case t=0 corresponds to points through which passes only one tangent to the ellipse, i.e. the points of the ellipse itself. For the general case, i.e. $t \neq 0$, we can divide out the left-hand side of Equation (15) by t^2. We obtain:

(34)
$$k^4x^4 + 2k^4x^2y^2 - 2k^4x^2 - \frac{2kx(t+2)}{t^2} + k^4y^4$$
$$- \frac{2k^4y^2(t^2+2)}{t^2} - 2k^2y^2 + k^4 + \frac{2k^2(t^2+2)}{t^2} = -1$$

If t tends to infinity, i.e. the angle tends to 90^o (and note that 90^o=180^o-90^o, whence the two components of the bisoptic tend to the same curve), then the equation becomes

(35) $$k^4x^4 + 2k^4x^2y^2 - 2k^4x^2 - 2k^2x^2 + k^4y^4 - 2k^4y^2 - 2k^2y^2 + k^4 + 2k^2 = -1$$

which is equivalent to:

(36) $$\left(k^2\left(x^2+y^2\right)\right) - \left(k^2+1\right)^2 = 0\,.$$

This equation can be written as follows:

(37) $$x^2 + y^2 = \frac{k^2+1}{k^2},$$

which is the equation of the director circle (v.s. Equation (11)).

Figure 12 shows the bisoptic curves (always in red) for the ellipse E whose equation is $x^2 + \frac{4}{9}y^2 = 1$ (always in blue) and $t = 1,5,10$ from left to right, the rightmost figure showing the limiting case for t going to infinity, i.e. the ellipse with its director circle. We can see how the two components get closer and look more and more circular. At the limit, they coalesce into one circle.

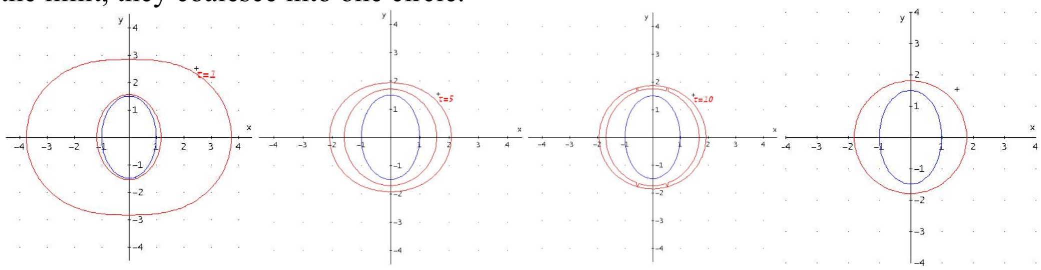


Figure 12: Fixed ellipse and variable angle

2. **Variable ellipse and fixed angle.**

If $k \to \infty$, then the limit configuration for the ellipse is the segment AB on the x-axis, where A has coordinates (-1,0) and B has coordinates (1,0). A well known result of plane geometry is that the bisoptic becomes the union of two circles from which that segment AB is seen by an angle whose tangent is t, the two points A and B being excepted. We can check this with Equation (15), when $k \to \infty$.
Following a method similar to the previous subsection, we divide out the left-hand side of Equation (15) by k^2; we obtain:

(38)
$$t^2x^4 + 2t^2x^2y^2 - \frac{2t^2x^2(k^2+1)}{k^2} - \frac{4x^2}{k^2} + t^2y^4 - \frac{2t^2y^2(k^2+1)}{k^2} - 4y^2 + \frac{t^2(k^4+2k^2+1)}{k^4} + \frac{4}{k^2} = 0$$

Now, if $k \to \infty$, we obtain the equation:

(39) $$t^2x^4 + 2t^2x^2y^2 - 2t^2x^2 + t^2y^4 - 2t^2y^2 - 4y^2 + t^2 = 0$$

which is equivalent to:

(40) $$x^4 + 2x^2y^2 - 2x^2 + y^4 - \frac{2y^2(t^2+2)}{t^2} + 1 = 0.$$

The left-hand side of Equation (40) can be written as the product of two quadratic polynomials, namely Equation (40) is equivalent to the following equation:

(41) $$\left[x^2 + \left(y - \frac{1}{t}\right)^2 - \left(1 + \frac{1}{t}\right)^2\right] \cdot \left[x^2 + \left(y + \frac{1}{t}\right)^2 - \left(1 + \frac{1}{t}\right)^2\right] = 0$$

This is the equation of the union of two symmetric circles going through the points A

(-1,0) and B (1,0) and having the points $C_1(0,1/t)$ and $C_2(0,-1/t)$ as their centers, respectively.
Figure 13 shows two examples for $t = 1$, i.e. $\theta = 45^o$, and $k = 2,5$, and at the rightmost the limiting case for infinite k.

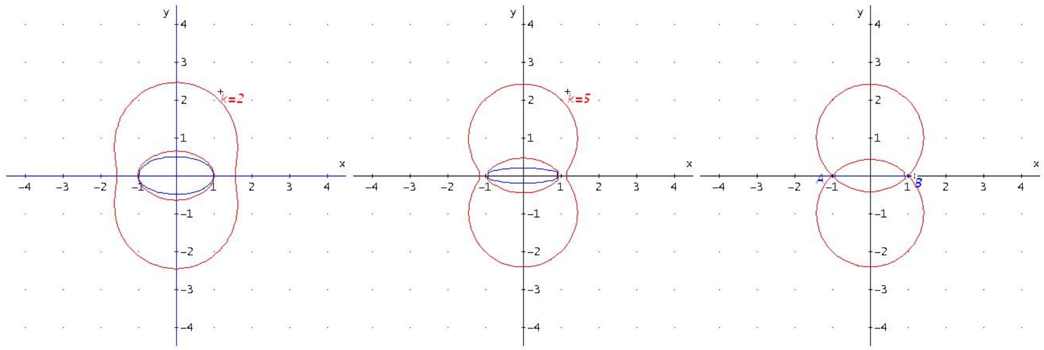


Figure 13: Fixed angle and variable eccentricity

V. Final remarks.

The work in this paper originated in a course in Analytic Geometry for in-service teachers, based on the usage of technology. The study of isoptic curves of ellipses led to the study of plane curves of higher degree, a topic which is not studied in a regular curriculum for teacher trainees. A byproduct was the development of mathematical activities based on the usage of a Computer Algebra System and of other kinds of mathematical software. The usage of technology enabled the participants to work according to an experimental method in order to develop new mathematical knowledge.
The consideration of these curves yielded the participants in the course a more profound insight into the geometry of plane curves and more understanding of the interplay between different mathematical fields, such as 2D geometry, 3D geometry, algebra and computer algebra. Introducing the spiric curves in the context of locus of viewing angles opens up opportunities for students to view and explore analytically curves of degree higher than 2.
With a broader scope than what has been presented in the course, the present study helps to enhance the understanding of isoptic curves and of spiric curves, as it introduces a spiric curve as the union of two connected components appearing together as the intersection of a self-intersecting torus with a plane parallel to its axis of revolution. The visualization provided by a Computer Algebra System gave a strong added value to the topic, and is an important component of the revival of the topic in recent years.
The isoptic curves present sometimes points of inflection, but not always. This can be studied by letting the angle θ vary for a fixed value of the parameter k, or by enquiring the influence of variations of parameter k for a fixed angle θ. The authors address this issue in a companion paper.
Other teams work in this field, from another point of view; for example, see (Miernowski and Mosgawa, 2001), (Szałkowski, 2005) and the papers referenced there. Together with the study of bisoptic curves of ellipses, the authors worked on

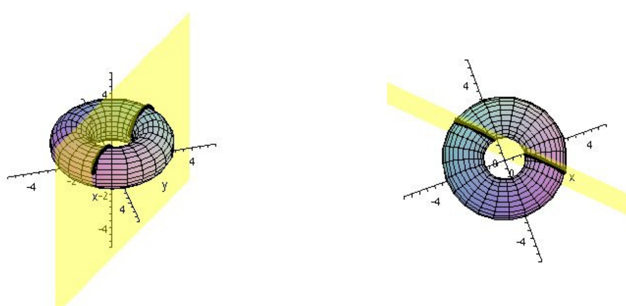
bisoptic curves of hyperbolas. This case is more complicated and new phenomena appear. This is the topic of a subsequent paper.

Appendix: the spiric of Perseus

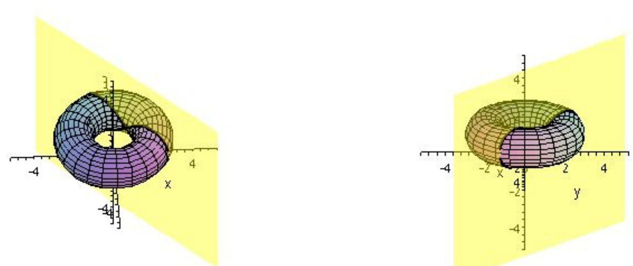
The intersection of a torus with a plane parallel to the axis of the torus is called a spiric of Perseus. The first reference to Perseus is in the writings of Proclus, where he says that Perseus found the spiric in the same way Apollonius studied conics[2] (see MacTutor). A spiric curve can have different forms according to the respective positions of the torus and the plane. It can be the union of two disjoint loops (Figure 16 (a)), one self intersecting loop (Figure 16 (b)), one non intersecting loop without a point of inflection (Figure 16 (c)) or a non intersecting loop with four points of inflection (Figure 16 (d)). Figure 16 shows the intersections of the planes whose respective equations are $x+y=1$, $x+y=\sqrt{2}$, $x+y=2$ and $x+y=3$ with the torus given by the following parametric representation:

$$\begin{cases} x(u,v)=(2+\cos u)\sin v \\ y(u,v)=(32+\cos u)\cos v\ ,\ u,v\in[0,2\pi]. \\ z(u,v)=1+\sin(u) \end{cases}$$

We begin with a 3D presentation. For each case, two different views of the torus and of the spiric curve are shown.

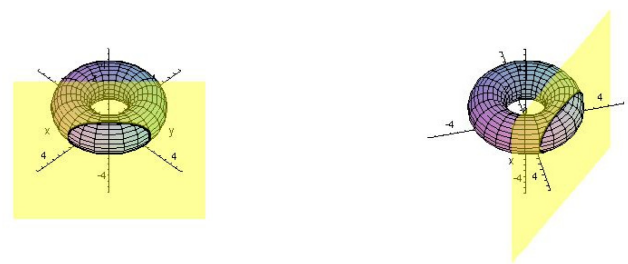


(a) The union of two disjoint loops

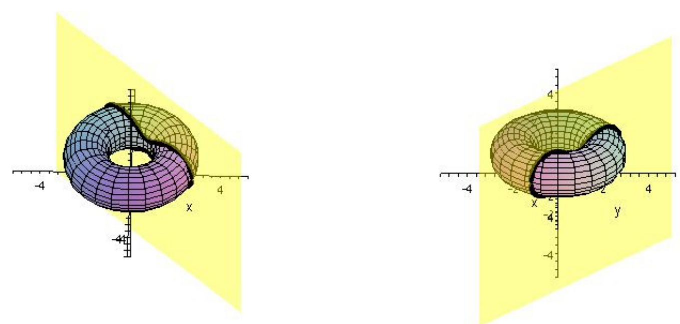


(b) A self-intersecting spiric curve

[2] Other sources refer to Menaechmus as the discoverer of conic sections (see http://www-groups.dcs.st-and.ac.uk/~history/Curves/Spiric.html).



(c) A non self-intersecting spiric curve without a point of inflection



(d) A non self-intersecting non convex spiric curve

Figure 13: Spiric curves

The authors can also provide to the interested reader files of animations built using DPGraph. Please ask by email.

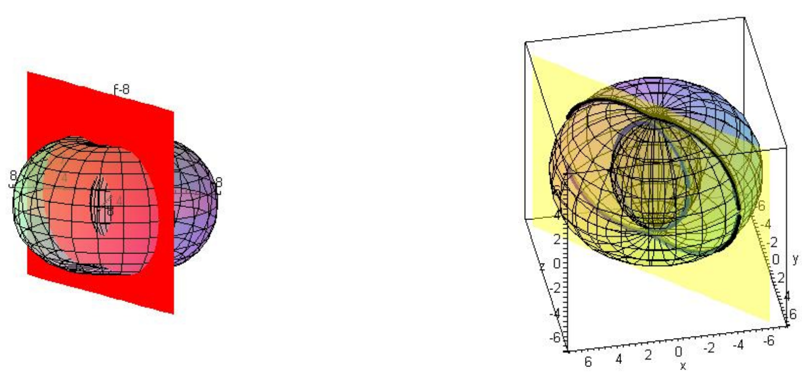


Figure 14: Plane cut of a self-intersecting torus

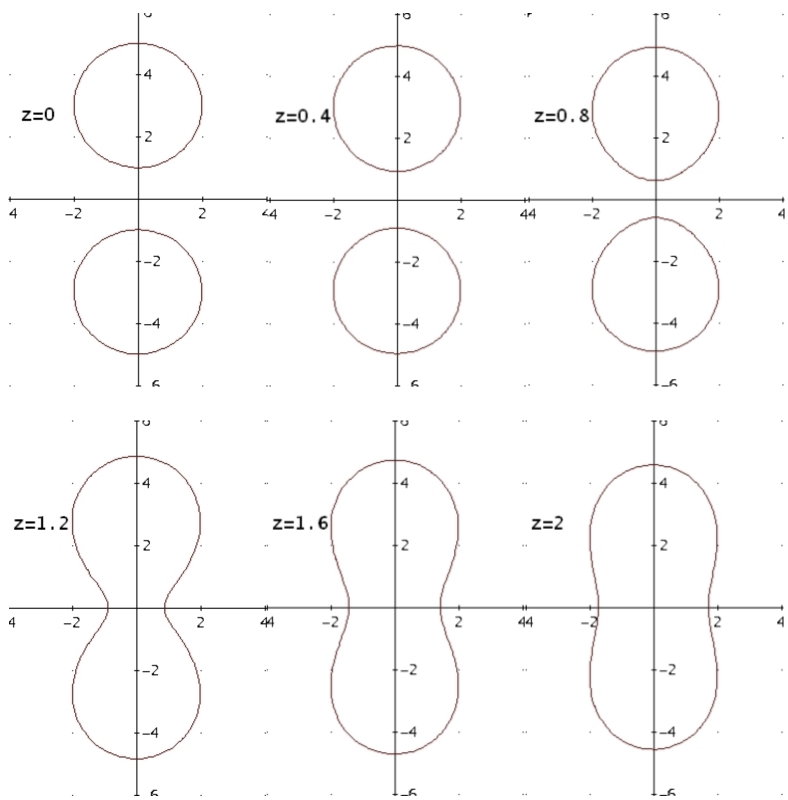
Now we present the 2D situation. In the literature, spiric curves are generally presented as plane sections of a "regular" torus, i.e. a non self-intersecting one. In this paper, we show that the bisoptic curves of ellipses are plane sections of self-intersecting tori, extending somehow the notion of a spiric curve. We include now the case where the curve has two disjoint components. In what follows we show spiric curves, according to our new point of view. The notations are those of Section III.

The equation of a torus with the *x*-axis as its axis of revolution is (Equation (15) above):

$$\left(x^2+y^2+z^2+R^2-r^2\right)^2-4R^2\left(y^2+z^2\right)=0$$

We show spiric curves obtained as plane sections of two different tori: T_1 whose characteristics are R=3 and r=2, and T_2 given by R=2 and r=3.

The torus T_1 is a regular one. His equation is $\left(x^2+y^2+z^2+5\right)^2-36\left(y+z\right)=0$.
The curves shown in Figure 15 are the spiric curves corresponding to z = 0, 0.4, 0.8, 1.2, 1.6, 2, 2.4, 2.8, 3.2.



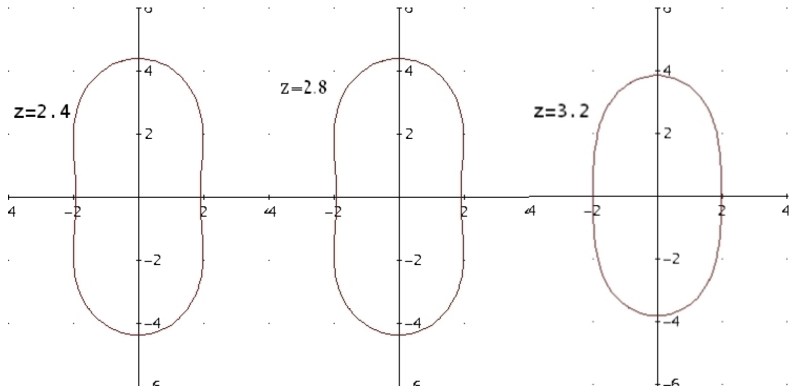
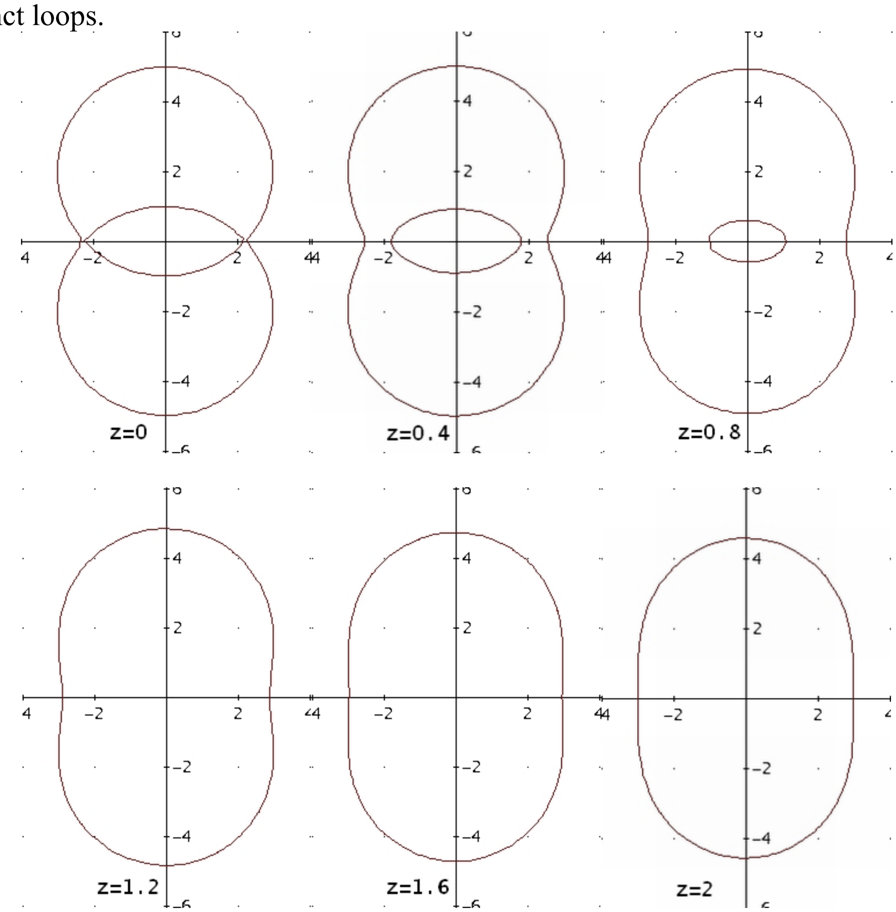


Figure 15: Spiric curves – plane sections of a regular torus

The torus T_2 is self-intersecting. His equation is $(x^2+y^2+z^2-5)^2-16(y+z)=0$. The curves displayed in Figure 16 are the spiric curves corresponding to the values z= 0, 0.4, 0.8, 1.2, 1.6, 2, 2.4, 2.8, 3.2. If z = 0 then the spiric curve is the union of two symmetric intersecting circles. If 0 < z < r-R, the spiric curve is the union of two distinct loops.



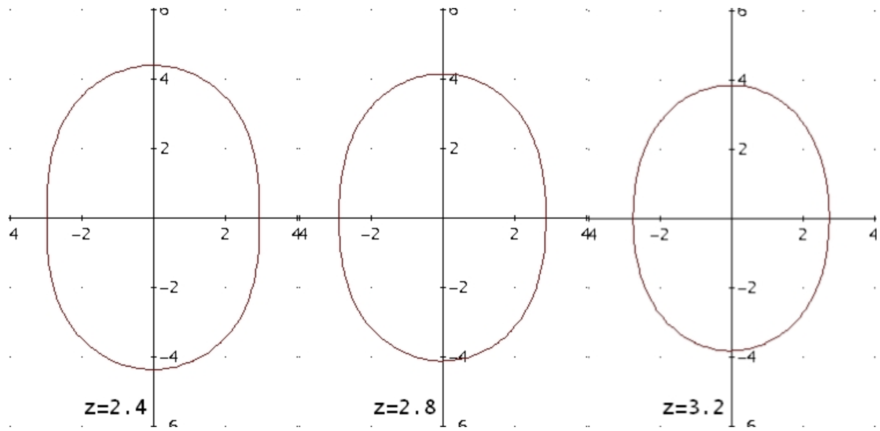


Figure 16: Spiric curves – plane sections of a self-intersecting torus

References

Berger, M. (1996). *Geometry II*, Universitext, Springer Verlag.

Dana-Picard Th., Mann G. and Zehavi N. (2011). Bi-isoptic curves of hyperbolas, Preprint.

Ferréol R. (2001) Quartiques bicirculaires, available:

http://www.mathcurve.com/courbes2d/quarticbicirculaire/quarticbicirculaire.shtml

Kirwan F. (1992). *Complex Algebraic Curves*, London Mathematical Society Student Texts 23.

Mann G., Dana-Picard Th. and Zehavi N. (2006), Technological Discourse on CAS-based Operative Knowledge, *International Journal of Technology in Mathematics Education* 14 (3), 113-120.

Miernowski A. and Mosgawa W. (2001), Isoptics of Pairs of Nested Closed Strictly Convex Curves and Crofton-Type Formulas, *Beiträge zur Algebra und Geometrie Contributions to Algebra and Geometry* 42 (1), 281-288.

Szałkowski D. (2005), Isoptics of open rosettes, *Annales Universitatis Mariae Curie-Skłodowska, Lublin – Polonia* LIX, Section A, 119–128.

Spain B. (1963). *Analytical Geometry*, Oxford: Pergamon Press.

Wassenaar J. (2003a) Bicircular quartic, available: http://www.2dcurves.com/quartic/quarticbc.html

Wassenaar J. (2003b): Plane spiric curve, available: http://www.2dcurves.com/quartic/quarticsp.html.

MacTutor History of Mathematics, Definition of curves, available: http://www-groups.dcs.st-and.ac.uk/~history/Curves/Definitions2.html#Orthoptic_curve