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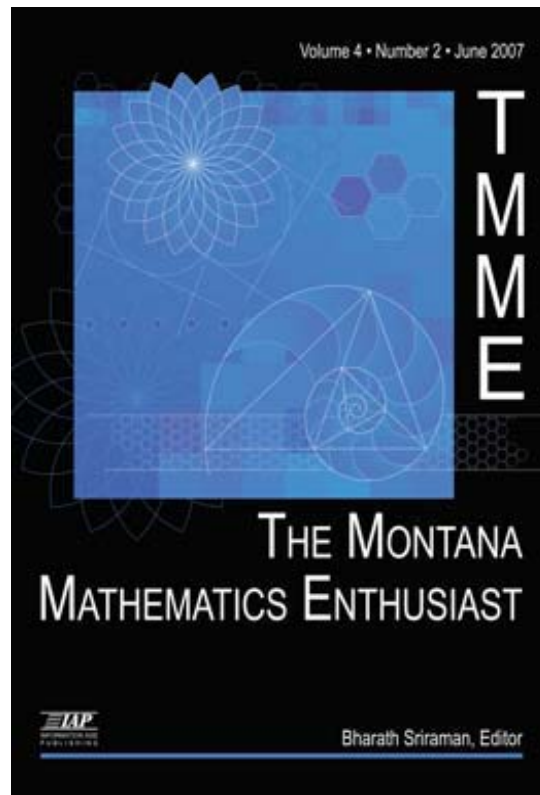
Editor-in-Chief
Bharath Sriraman, The University of Montana

"Was man so verständig nennt, ist oft mehr Eitelkeit und Kurzsinn "
- Goethe

AIMS, SCOPE AND EDITORIAL INFORMATION

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Editorial: Opening 2011's Journal Treasure Chest

Bharath Sriraman, Editor

Over the years, people have described different journal issues of *The Montana Mathematics Enthusiast*, as pulling a rabbit out of a hat, always surprising readers by its variety and content. One wishes that putting together an issue were as simple as that! Most TMME issues are planned 10-12 months in advance and require a convergence of numerous elements- first and foremost of which is a healthy submission rate. I have been proactive about promoting the journal at institutions in numerous countries I have lectured at as a visiting Professor, as well as relied on the support of collaborators and colleagues for suggesting the journal as an avenue of publication at research conferences and symposia.

In 2010 the journal received 109 manuscripts, out of which 27 were accepted for publication, 70 were rejected, 12 required reworking and resubmission. Out of the 70 manuscripts that were rejected, 9 were found inappropriate in terms of the aims and scope of the journal. Another 6 of the rejected manuscripts eventually found other outlets for publication. The acceptance rate is thus around 25%. Some editorial board members suggested increasing the frequency of the journal from 3 to 4 issues per year. However, due to very limited resources the journal is unable to do so. *The Montana Mathematics Enthusiast* is a grass roots enterprise in which the bulk of the work is done by the Editor, with help from the editorial assistant [Linda Azure], hundreds of ad-hoc reviewers, and a cadre of language checkers and copy editors scattered around the world. It operates very much like a complex system, with numerous lower order agents unknowingly acting in chorus and contributing to the end product, namely journal issues.

Instead of thinking of a journal issue as a rabbit pulled out of a hat, it is much more like an anthill requiring hundreds of hours of slow and steady work. It is also fascinating to watch articles from the journal and monographs cited in other mathematics education journals, books and conference proceedings in addition to myriad disciplines such as pure mathematics, exact sciences, history, philosophy, physics, cultural studies and even aesthetic plastic surgery! The skeptical reader can go into Google Scholar or Google Books and enter "The Montana Mathematics Enthusiast" and verify for themselves. As a testimony to the eclectic nature of the journal and its ability to publish mathematical articles of interest and quality, Princeton University Press selected Wagner's (2009) "If mathematics is a language, how do you swear in it?" (from TMME, vol6, no3) in its *The Best writing on Mathematics 2010*.

The journal continues to remain free online with the option of purchasing print copies from Information Age Publishing. The Montana Monograph series also continues to thrive and released its 11th volume entitled "*Interdisciplinarity for the 21st Century*" in the Fall, and two additional monographs are in development for release in 2011 and 2012.

I wish to thank several colleagues who have been in the art of editing longer than I have from whom I have learned a lot. First and foremost, Lyn English (Australia), the editor of *Mathematical Thinking and Learning* for teaching me to identify potential and quality in submissions; Ian Winchester (Canada), the editor of *Interchange: A Quarterly Review of Education*, and Don Ambrose (USA), the editor of *Roeper Review*, individuals from whom I have learned the art of eclectics, challenging philosophical assumptions and interdisciplinary initiatives; and last but not least Gabriele Kaiser (Germany), the editor of *ZDM-The International Journal of Education* from whom I have learned the art of patience and revision to improve quality of manuscripts, and compiling theme issues.

Indeed I feel fortunate to be able to roam in a variety of disciplinary circles and combine the editorial styles and approaches of different journals to make the journal what it is.

The current double issue that opens up 2011 is a veritable treasure chest – I will not elaborate on this and simply let the table of contents speak for itself! Not too many journals can claim a line up with articles from Hyman Bass, Reuben Hersh, Wolff-Michael Roth, Klaus Hoechsmann, Thomas O'Brien, and many other illustrious colleagues from Canada, Israel, Iran and other countries. Indeed the journal may be one of the few places where one notes a synergy between colleagues from Iran and Israel, untainted and far different from the vitriol and rhetoric of the popular media and the continuation of *Realpolitik* in the world today.

Finally, I wish to pass my condolences to family, friends and colleagues who mourn the passing of Thomas O'Brien on December 6. We hope the present article that was in the pipeline but sadly appears posthumously in this issue carries on his *Polya-esque* vision for the teaching and learning of mathematics.

I have received requests to start a Letters from Readers section in 2012 and urge those interested in seeing this happen to send their correspondence to me. I hope the 9360+ readers of the journal from 110+ countries enjoy the New Year offering of The Montana Mathematics Enthusiast.

Reference

Pitici, M. (Ed) (2011). *The Best Writing on Mathematics 2010*. Princeton University Press.

***A Vignette of Doing Mathematics: A Meta-cognitive Tour of the
Production of Some Elementary Mathematics***

Hyman Bass
University of Michigan

I. Introduction

What is this about?

Mathematics educators, including some mathematicians, have, in various ways, urged that the school curriculum provide opportunities for learners to have some authentic experience of *doing* mathematics, opportunities to experience and develop the practices, dispositions, sensibilities, habits of mind characteristic of the generation of new mathematical knowledge and understanding – questioning, exploring, representing, conjecturing, consulting the literature, making connections, seeking proofs, proving, making aesthetic judgments, etc. (Polya 1954, Cuoco et al 2005, NCTM 2000 - Standard on Reasoning and Proof). While this inclination in curricular design has a certain appeal and merit, its curricular and instructional expressions are often contrived, or superficial, or no more than caricatures of what they are meant to emulate. One likely source of the difficulty is that most mathematics educators have little or no direct experience of doing a substantial piece of original mathematics, in part because the technical demands are often too far beyond the school curriculum. Studying the history and evolution of important mathematical developments can be helpful, but provides a less immediate and direct experience.

This paper is written from the ambivalent space that I inhabit, as a practiced mathematician who is also seriously inquiring into the problems of teaching and learning at the school level. It exploits my experience and sensibilities as a mathematician, but it is addressed to some of the challenges and concerns of school mathematics teaching and learning. It tells a story that happened in the sometimes conflicted, but potentially fruitful zone between those two worlds.

My intention is to offer the reader a first hand and accessible account of the generation of an interesting and elementary piece of new mathematics. The mathematics itself, while of some modest interest, serves here mainly as context, or backdrop. The main story is the meta-cognitive narrative of the mathematical trajectory of the work. Several features of the event recommend it for this purpose. First, the initial question grew from a topic in the elementary mathematics curriculum, in the teaching of fractions. The mathematical work illustrated here is launched by asking a "natural question" that is precipitated by this elementary context. From that start, explorations, discoveries, and new questions proliferate, some within easy reach of the standard repertoire of the school curriculum, perhaps mobilized in some novel ways, and others seeming to demand some new idea or perspective or method. But, importantly for our present purposes, the ideas and methods invoked never transcend the reach of a secondary learner who is prepared to think flexibly about some less familiar ways of combining elementary ideas.

In summary then, what is presented here is a narrative of a small mathematical journey, meant to give the reader a palpable and authentic, yet accessible, image of what it means to *do* mathematics. I have tried to scaffold the mathematical work to ease the reading as much as possible, but it would be foolish to pretend that this will be an "easy read." That cost is perhaps

inevitable in an undertaking like this, which is therefore, in a way, a part of the message that this is meant to convey. While I am uncertain of the natural audience for this, I would hope that at least it might be of interest to mathematics educators, to mathematics teachers, elementary as well as secondary and perhaps to undergraduate mathematics majors.

Many authors have written about the nature of mathematics, and of mathematical practice. Some have focused on the psychological aspects of creative mathematical discovery (Poincaré, Hadamard). Polya has insightfully articulated much of the craft and heuristics of creative problem solving. Others (Lakatos, Davis and Hersh, Cuoco et al,) have provided some images or descriptions of the nature of mathematical practice and experience. This paper can be viewed as a reflective case study in this general tradition, but with an orientation toward knowledge for instruction.

Some of the things entailed in doing mathematics

It will be helpful to name and (at least partially) specify some of the things – practices, dispositions, sensibilities, habits of mind – entailed in doing mathematics, and to which we want to draw attention in our story. These are things that mathematicians typically do when they do mathematics. At the same time most of these things, suitably interpreted or adapted, could apply usefully to elementary mathematics no less than to research. Though we offer them as a list, it must be emphasized that they interweave and mutually interact in practice.

Also I must make it clear that this is a personally constructed list. Other mathematicians would likely come up with somewhat different categories and descriptions, but I would expect there to be much in common. The first person plural "we" in this discussion refers to "mathematicians."

1. <u>Question</u>: We ask what we like to call "natural questions" in a given mathematical context.

Here is a partial repertoire of frequent questions. The most basic question we ask is "Why?," whenever we see some claim, or witness an interesting phenomenon. Given a well-posed problem, we ask questions like: Does it have a solution? (Existence) Is the solution unique, or are there others? (Uniqueness) Can we find/describe all of them? Can we *prove* that we have all of them? If the number of solutions is large, perhaps even infinite, does the *solution set* have some natural (for example geometric or combinatorial or algebraic) structure? Which solutions optimize some property (for example being largest, if the solutions are numbers)? Do the answers to any of these questions generalize, to broader contexts? How are the answers to these questions affected by variation in the parameters of the context? Etc. Which of these questions is most appropriate, or most interesting, in a given context is in part a matter of mathematical judgment and sensibility, which develop with practice and experience.

2. <u>Explore</u>: We explore and experiment with the context.

Initially, this may be relatively unguided but eyes-open playing around with the context. If the context is arithmetic or algebraic, one may experiment with numerical or algebraic calculations, to get a feel for the size and shape of things, looking for patterns. Hand drawn diagrams and pictures can often be helpful as well. If the context can be modeled and manipulated on a computer, this may allow for some visual exploration, using graphs or dragging figures in dynamic geometry.

3. <u>Represent</u>: We find ways to mathematically model or represent the context, and we examine the representation. We may choose alternative representations, to highlight or foreground particular aspects or features of the context.

This is a particularly important process. We need some way to look at, examine, manipulate, transform the problem at hand, and we need ways of portraying, or representing the problem to enable this. For example, a rational number might be written as a fraction, if you are a number theorist, or as a decimal if you are an analyst or statistician. A portrait might be a picture, a graph, a diagram, an equation, or even some general kind of mathematical structure. Or the representation may be symbolic, formally naming key variables and relationships in a problem. Typically, more than one representation will be deployed, for each one will make certain features visible, and leave others obscure. Some will be amenable to certain kinds of manipulation, for which others may be more cumbersome. Judicious choice of representations can be crucial to successful analysis and understanding. This is the site of some of the most artful aspects of problem solving (and of teaching).

4. <u>Structure</u>: We look for some kind of organizing structure or pattern or significant feature. This may lead to conjectures (or new questions).

Mathematics is not merely a descriptive science. It seeks simple, general, unifying principles that provide insight and explanatory power for phenomena or data of great variety or complexity. These principles, sometimes called "patterns," or "structures," might take the form of a formula (like a closed form expression of a partially or recursively defined function, or like the Pythagorean formula, $c^2 = a^2 + b^2$). Or they might express some (hidden) symmetries or other relations in a data set or geometric object. Or they may provide a structured way (for example linear or Cartesian) of representing some data set. If such patterns or structures are only suspected, but not verified, they take, once precisely formulated, the form of <u>conjectures</u>.

5. <u>Consult</u>: If we get stuck, or are not sure about something, we can consult others (expert friends or professionals), or the literature. Often Google (or Advanced Google, or Google Scholar) can be quickly helpful for this. It can often expedite some otherwise long library searches.

In doing mathematical research, unlike school work, we don't want to expend great effort trying to solve a problem that has already been solved, (unless our intention is to find a simpler solution or proof). So, once a question we confront resists our first serious efforts, it is wise to consult the literature, or expert colleagues, to find out what is already known about the problem. This is also appropriate in school mathematics if working on an open-ended and long-term mathematical project. Mathematics is a hierarchical subject, and we don't want to constantly reinvent the wheel. But of course this means learning to interrogate and learn from the expert knowledge of others. Google provides a remarkably effective and congenial instrument for such inquiry, and it tolerates very informal versions of your questions. But be prepared for (and welcome) some interesting but time consuming scientific browsing. You will find more things than you sought, but surprisingly many of these will eventually turn out to be fruitful. And you will likely learn to see your problem in a larger context than first envisaged, and the potential for applications and ramifications of a possible solution. Mathematicians <u>learn</u> much new mathematics this way.

6. <u>Connect</u>: Such searching, or perhaps just reflection, may help us see connections, or analogies, with other mathematics (questions or results) that we know, that may suggest useful ways to think about the problem at hand.

Some of the most powerful, and satisfying, mathematical insights and discoveries arise from seeing some significant connection established between two *a priori* unrelated mathematical situations. Mathematicians are disposed to be alert to finding such connections, and they develop the sensibilities to see and value them when they are present. For example, these might take the form of finding two fundamentally different representations of the same mathematical context. Or, the situation of the problem you are working on may remind you of a similar situation you encountered in some previous problem, and the way you dealt with that problem might suggest useful ways of treating the one at hand.

7. <u>Proof seeking</u>: We seek proofs, or disproofs (counterexamples) of our conjectures. Often this proceeds by breaking the task into smaller pieces, for example by formulating, or proving, related, hopefully more accessible, conjectures, and showing that the main conjecture could be deduced from those.

Once faced with a well-articulated mathematical claim or conjecture, we or course seek to show whether, and why, it is true. All of the above processes can be mobilized in the search of evidence, an explanation, and, eventually, a proof. Or, failing that, we may come to doubt the truth of the claim, and seek a counterexample, or disproof. There are no general algorithms for this. Otherwise, the question would already have been answered, and there would be no adventure to the enterprise.

8. <u>Opportunism</u>: Sometimes the mathematics seems to be leading you, rather than the other way around. Mathematicians will often take a cue from this, and follow these inviting trails with unknown destination.

For example, the quest for a proof may seem to be making good progress, but, on close examination, it appears to be answering a different question than the one you started with. It is a good idea to "listen to the math." The new question may be more interesting or natural than the original. Lots of good math is fallen upon by such serendipity. Mathematicians are disposed to welcome this when it happens, and seize the opportunity that it presents.

9. <u>Proving</u>: Writing a finished exposition of the proof (if one is found), using illuminating representations of the main ideas, meeting standards of mathematical rigor, and crafted to be accessible to the mathematical expertise of an intended audience.

If one finds, or believes one has found, a proof of the claim, there remains the task of providing a precise and compelling exposition of the argument that can convince – oneself, one's expert friends, impartial experts (peer review), and, eventually, one's students or the profession or some public. The "granularity" of the exposition will depend on the audience and purpose of the communication.

10. <u>Proof analysis</u>: Proofs are conceived of as a means to an end (a theorem). But the proof itself is a product worthy of note and study, since the theorem typically distills only a small part of what the proof contains.

First, of course, proofs must be examined for their correctness. But also, study of the proof may show that the full strength of a hypothesis was never used, and that a weaker form of the hypothesis suffices. Making that substitution gains added generality to the theorem with no extra work. In fact there have been cases where a hypothesis in a theorem is never used in the proof. If one knows, for external reasons, that the hypothesis is essential, then that is a signal that the proof is faulty.

If examination of a number of results shows a strong similarity in their proof methods, then that raises the suspicion that they are all special cases of one general result, which a synthesis of the proof methods may uncover.

> 11. <u>Aesthetics and taste</u>: As in any profession, mathematicians are diverse in their styles and tastes. Still, in mathematics, there is a remarkable degree of shared aesthetic sensibility – associated with words like elegance, precision, lucidity, coherence, unity, … – that affects not only how they appreciate, but even how they do mathematics.

There are many ways in which this shows up concretely. For example, the statement of a theorem may involve a hypothesis that seems extraneous to the conclusion, and which is therefore seen to 'disfigure' the statement, and invite the suspicion that it is not really necessary. Or, in dealing with geometric reasoning, there is a natural desire to have some visual image of the claims and processes used. This creates an urge to provide geometric interpretations of highly algebraic or analytic arguments. In choosing representations of mathematical situations, mathematicians will aim for something that resolves the need to capture important information with the desire for simplicity and manipulability or for conceptual transparency.

Now we proceed to the mathematics of our story. The 'meta-discussion' will be interspersed, indented and in italics.

II.   THE MATHEMATICAL STORY – PART 1: CAKE DISTRIBUTIONS

The initial mathematical problem, and first explorations

Division is often introduced in school in the context of sharing problems, say some students want to (equally) share some cookies, or cakes; we'll talk here about cakes, just to fix ideas. At first, in the whole number world, say 2 students want to share six cakes. Then each student gets 3 cakes, the 3 being the answer to 6 ÷ 2. Later, when introducing fractions, we first ask how 2 students might share 1 cake; each receives ½ cake, which is accomplished by cutting the cake in half. But 3 students sharing 2 cakes is already a bit more complicated. Each student receives 2/3 of a cake. But how is that to be distributed? Children generally come up with these two ways to do this. One is to cut each cake into thirds, and to give each student a third of each cake. But a more efficient (fewer pieces) way to do this is to cut 1 third from each cake, and give these 2 thirds to the first student, and then give the remaining (2/3)-cake pieces to the remaining 2 students. The first distribution involves 6 cake pieces, and the second involves 4.

[Insert pie charts illustrating the 2 cakes for 3 students distributions]

What about other cases? Say 3 cakes for 5 students, or 5 cakes for 7 students, or for 12 students? (We shall look below at 5 cakes for 7 students.) In general, suppose that c cakes are to be equally shared by s students. One general way to do this is to cut each cake into s equal pieces, and then give one piece from each cake to each student. This requires c•s cake pieces, and, when s is large, will pretty much physically ravage the cakes. What is a less invasive way of cutting up the cakes for this distribution? More precisely,

> *If c cakes are to be equally shared by s students, what is the smallest number, call it p = p(c, s), of cake pieces needed to make this distribution?*

> *This is our first "natural <u>question</u>." It has been formulated right away for general c and s, though it might well have been first explored for small numerical values of c and s. At*

Bass

The distribution described above shows that that p ≤ c•s. Also p ≥ s, since each student gets at least one piece. So we have right away,

$$s \quad \le \quad p(c, s) \quad \le \quad c•s$$

If c = 1, then we can cut the one cake into s equal pieces for the distribution, and so

$$p(1, s) \quad = \quad s$$

Let's look at a more interesting case – 5 cakes shared by 7 students: (c, s) = (5, 7)
So each student receives 5/7 of a cake. What is an efficient way to distribute these shares? …
After a bit of reflection and experiment you might come up with one or both of the following methods.

The "Linear Distribution:" Line up the cakes, and the students. From the first cake, cut out a full share (5/7 of the cake) for the first student. Give the remaining 2/7 of the first cake to the second student, and then cut 3/7 of the second cake to complete the second student's share. Then give the remaining 4/7 of the second cake, plus 1/7 of the third cake, to the third student. Etc. Here is a picture of this distribution, where the 7 student shares are identified by colors.

<u>5 circular cakes</u>.   <u>7 student shares</u>: Red, blue, tan, purple, black, yellow, green

The Linear Distribution
Pieces:
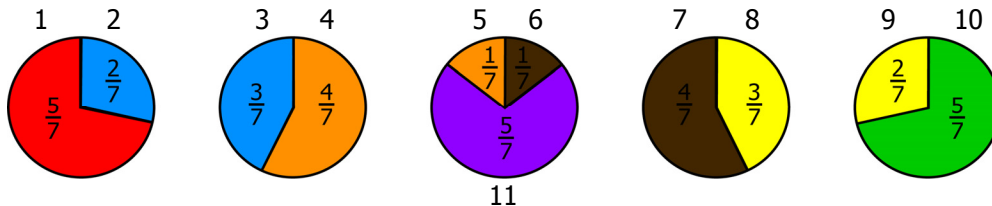
The "Euclidean Distribution:" In this case we start by removing a full share (5/7) from each of the 5 cakes, and we distribute these full shares to 5 of the students. What remains are 5 small cakes (of size 2/7 of the original) to be equally shared by the remaining 2 students. Thus, the (5, 7) distribution problem has been reduced to a (5, 2) distribution problem. We start the latter by giving each of the 2 students 2 of the (small) cakes. There remains 1 small cake that we cut in half to be equally shared by the 2 students.

Here is a picture of this distribution;

"Euclidean Distribution"

Pieces:
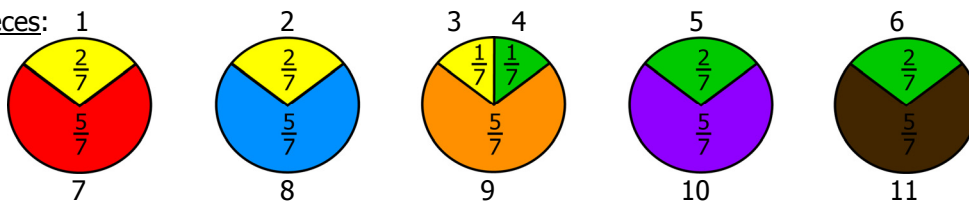
Notice that, though these distributions are quite different, they both lead to 11 cake pieces. Is this a coincidence? Is 11 pieces the minimum possible? In other words, is p(5, 7) = 11? Do these two distribution methods make sense for any (c, s)? If so, how could one describe them in general?

> *Ok, there are several important things to notice here. First, we identified two fairly natural methods to distribute the cake shares, resulting from an initial <u>exploration</u>. And we invented a <u>representation</u> scheme to make visible these distribution processes that might be less clear from a purely verbal description. We used colors to visually identify the different student shares. Student names would have been somewhat more cumbersome, and numbers might have conflicted with the numbering of the cake pieces, which we wanted in order to be able to count them. Finally, we asked several "natural <u>questions</u>" precipitated by examination of the two representations. In particular, the appearance of 11 pieces for both distributions may hint at a general <u>pattern</u>. We <u>experiment</u> with these ideas below.*

To check what this pattern might be, we could examine some smaller cases. For example, starting with the first two cases we considered, we find that

$$p(1, s) = s$$

$$p(2, 3) = 4$$

$$p(3, 5) \leq 7$$

For, in the case of 2 cakes for 3 students, the Linear and Euclidean Distributions coincide and give 4 pieces, and it is clear that the two cakes cannot be cut into 3 equal pieces (all of size 2/3); so p(2, 3) = 4.

In the case of 3 cakes for 5 students, the Linear and Euclidean Distributions both give 7 pieces.

If we believe that p(3, 5) = 7, and also that p(5, 7) = 11, then what might we guess is a general formula for p(c, s)? We tried, optimistically, the nice formula:

$$p(c, s) = c + s - 1 ?$$

This was quickly defeated already in the case of sharing 4 cakes among 6 students, when 4 + 6 − 1 = 9. In this case we can split the problem into two 2-cakes-among-3-students problems. Each of these produces 4 pieces, and so altogether 4 + 4 = 8 < 9 pieces.

To better understand what is going on it will be convenient to choose a more illuminating representation of our distributions.

From round to rectangular food

In the Linear Distribution we are measuring off successive pieces of the cakes, lined up one after another, so the cakes are functioning mathematically like successive intervals on a number line. To capture this aspect and yet keep them "cake-like" we can take our cakes to be long thin rectangles. Since we are only interested in the length (they are functioning as 'thickened intervals') we can simply assume that they have width 1. As for the length, it will be convenient, as we shall see, to assume that they have length s, the number of students. In other words, we

can assume that the units of length are chosen so that each cake is a (1 x s)-rectangle. And the cake pieces will again be sub-rectangles of width 1.

Now for the Linear Distribution, we place the cakes end to end to form a long (1 x c•s)-rectangle of cake, where the boundaries between successive cakes occur at the multiples of s. Let's look at the case (c, s) = (5, 7) studied above.

The Linear Distribution of 5 cakes for 7 students

First we line up the 5 cakes.

Cake 1	Cake 2	Cake 3	Cake 4	Cake 5

Next we ignore the cake separations, and view this as one long cake (of length 5•7 = 35) to be shared equally by the 7 students. The cuts to create their (equal) shares will occur at the multiples of c = 5.

Cake 1		Cake 2		Cake 3		Cake 4		Cake 5	
S1	S2	S3	S4	S5		S6		S7	

Finally, we combine the cake separations with the student share cuts to obtain the combined division of the cakes into pieces for the distribution.

Cake 1		Cake 2		Cake 3		Cake 4		Cake 5	
S1	S2	S3	S4	S5		S6		S7	
1	2	3	4	5 6		7 8		9	10 11
5/7	2/7	3/7	4/7	1/7 5/7		1/7 4/5		3/7	2/7 5/7
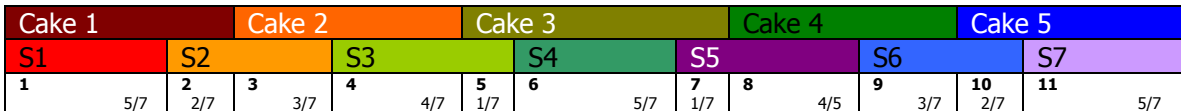
Row 1: Cakes (5).   Row 2: Student shares (7).   Row 3: Cake pieces (11).

pieces   =   11   (=   7 + 5 - 1)

The Linear Distribution of c cakes for s students

In general, the cake separations occur at multiples of s: s, 2s, 3s, … , (c-1)s. There are c-1 of these. The student share cuts occur at multiples of c: c, 2c, 3c, … , (s-1)c. There are s-1 of these. So this makes altogether (c-1) + (s-1) cuts, except that some of the two sets of cuts coincide. The common cuts occur at common multiples of c and s. These are just multiples of m = lcm(c, s), the least common multiple of c and s. We have the greatest common divisor,
          d = gcd(c, s) =   cs/m,
so the cuts common to the two sets are: m, 2m, … , (d-1)m. There are d-1 of these. Thus the total number of cuts is:

          # cuts   =   (c-1) + (s-1) - (d-1)   =   c + s - d - 1,

and so the total number of cake pieces for this "Linear Distribution" is one more:

          # cake pieces   =   c + s - d,   where d = gcd(c, s)

Of course we see here a significantly new (rectangular) <u>representation</u> of the Linear Distribution, one that better coordinates the geometry of the representation with the arithmetic of the distribution. Moreover, this representation makes easily available (and visualizable) an analysis of the number of pieces, as a function of c and s. We could see the <u>structure</u> in the (5, 7) case, and this guided the analysis in the general case. (Notice also that, from the point of view of this analysis, there is a certain <u>symmetry</u> in the roles of c and s.) And it raises the "natural next <u>question</u>:" "Can we do something similar for the Euclidean Distribution?"

<u>The Euclidean Distribution of 5 cakes for 7 students</u>

For the Euclidean Distribution of 5 cakes for 7 students, we began by cutting off 5/7 of each cake. To do this all at once, it would be convenient to arrange the cakes not end-to-end, but rather side-by-side, so as to form, this time, a (5 x 7)-rectangle of cake. This done, the Euclidean Distribution looks as follows:

The students: S1, … , S7; their shares are color coded.
The pieces: P1, … , P11
The fractions indicate fractions of a cake; each cake is one of the 5 rows of the rectangle.

So, while the Linear Distribution is an essentially 1-dimensional (length) representation, we see here that the Euclidean Distribution appears to exhibit something more like a 2-dimensional (area) phenomenon. Moreover, a little reflection suggests that this is closely related to the Euclidean Algorithm (for finding the gcd of two numbers, using successive division with remainder). Explicitly, the Euclidean Algorithm for calculating gcd(5, 7) (= 1) looks like:

$$7 \quad = \quad 1 \cdot 5 + 2 \qquad (1 \ 5\text{x}5 \text{ square consisting of } 1 \cdot 5 = 5 \text{ pieces})$$
$$5 \quad = \quad 2 \cdot 2 + 1 \qquad (2 \ 2\text{x}2 \text{ squares consisting of } 2 \cdot 2 = 4 \text{ pieces})$$
$$2 \quad = \quad 2 \cdot 1 + 0 \qquad (2 \ 1\text{x}1 \text{ squares consisting of } 2 \cdot 1 = 2 \text{ pieces})$$

$$\text{\# pieces} \quad = \quad 5 + 4 + 2 \quad = \quad 11$$

In fact, (see the picture below) *we can interpret the Euclidean algorithm (for finding gcd(c, s)) geometrically as successively filling up the (c x s)-rectangle with maximal size squares so that what remains at each stage is still a rectangle.* And so we can interpret the result as a "square tiling" of the rectangle, in the sense that the rectangle is covered by the squares, and any two square intersect at most along an edge of each. And in fact, the Euclidean Algorithm is a kind of "greedy algorithm" for producing a square tiling of a (c x s)-rectangle, in the sense that, at each stage, it inserts a square 'tile' of maximum possible size.
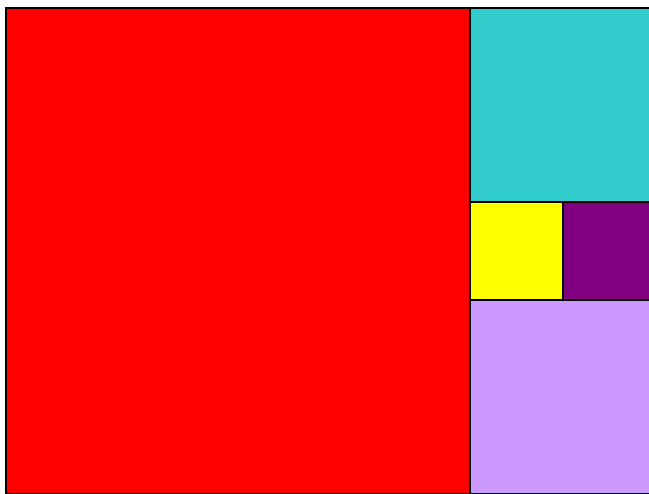
$$\text{\# square tiles} \quad = \quad 1 \ (5\text{x}5) \ + \ 2 \ (2\text{x}2) \ + \ 2 \ (1\text{x}1) \quad = \quad 5$$

A natural (side) question here is,

Is the "Euclidean tiling" of a (c x s)-rectangle optimal in some sense?
For example, does it produce a square tiling with the smallest possible number of tiles?

We'll come back to this question later.

> *So several interesting things happened here. First we found a new (area model) representation of the Euclidean Distribution which makes visible its connection with the Euclidean Algorithm, and also exhibits the geometric connection of the latter with 'square tilings' of rectangles. This new context in turn suggested new natural questions about the "Euclidean tiling," albeit pointing in a direction somewhat orthogonal to our original interest. Such "side tracks" are not uncommon when doing mathematics, and some of them turn out to be helpful, or independently interesting, in unexpected ways. But first we return to our initial question.*

A closer look at the Euclidean Distribution, and the number of pieces it produces

The Euclidean Algorithm applied to a pair of whole numbers, (c, s) (not both = 0), proceeds as follows: Take the larger of the two numbers, divide it by the smaller, and replace the larger one

by the remainder in this division. After a finite number of such steps, one of the two numbers will be zero, and then the non-zero remaining number is the gcd(c, s). More explicitly, and with interpretation for the cake distribution, we have the following cases:

> If $c \geq s$, write $c = qs + r$, with $0 \leq r < s$. (Euclidean division with remainder; q, the quotient, is the number of times you can remove s from c, and $r < s$ is the remainder.) Then we give q cakes to each student, making qs pieces distributed, and then continue by applying the Euclidean Distribution to (r, s): r cakes among s students. If s divides c, then $r = 0$, and we are done.

> If $c < s$, write $s = qc + r$, with $0 \leq r < c$ (Euclidean division again). In this case we cut off q pieces, each of size c/s of a cake, from each cake, and distribute one each of these (full) shares to qc of the students. There remain c small cakes, each of size r/s of the original, to be equally shared among the remaining r (= $s - qc$) students. Thus we are reduced to a distribution of c (small) cakes among r students, with $r < c$, to which we apply the first step above. (In case c divides s, then $r = 0$, and we are done.)

To count the number of cake pieces that the Euclidean Distribution produces, let us denote this number by E(c, s). We <u>claim</u> that, just as for the Linear Distribution,

$$E(c, s) = c + s - d, \quad \text{where } d = \gcd(c, s)$$

To prove this claim, note first that this is true if there are no cakes. For then there are no pieces, i. e.

$$E(0, s) = 0 = 0 + s - \gcd(0, s)$$

In the first case above, $c \geq s$, we have

$$c = qs + r, \quad \text{with } 0 \leq r < c, \quad \text{and then we see that}$$
$$E(c, s) = qs + E(r, s)$$

Since $r < c$, we can assume *by (mathematical) induction* that $E(r, s) = r + s - \gcd(r, s)$. But it is easily seen that $\gcd(r, s) = \gcd(c - qs, s) = \gcd(c, s) = d$, and so

$$E(c, s) = qs + E(r, s) = qs + r + s - d$$
$$= c + s - d$$

In the second case above, $c < s$, we have

$$s = qc + r, \quad \text{with } 0 \leq r < c, \quad \text{and then we see that}$$
$$E(c, s) = qc + E(c, r)$$

Since $r < s$, we can apply induction to conclude that $E(c, r) = c + r - \gcd(c, r)$. Just as above, we see that $\gcd(c, r) = \gcd(c, s - qc) = \gcd(c, s) = d$. Thus

$$E(c, s) = qc + c + r - d = c + s - d$$

This completes the proof, by induction, that

> *The Euclidean Distribution, just like the Linear Distribution, produces*
> $c + s - d$ *pieces, where* $d = \gcd(c, s)$

What we have just seen, though a bit technical, is a rather straightforward inductive <u>analysis</u> of the number of pieces produced by the Euclidean Distribution. The inductive method here is quite natural, since the Euclidean Algorithm is itself an inductive (or recursive) procedure. In particular, this offers a <u>proof</u> of the remarkable, and perhaps unexpected, fact that the Linear and Euclidean Distributions, though quite different,

produce the same number of pieces, c + s - d, thus establishing an interesting connection. This makes the number c + s – d seem quite special to the cake distribution problem, and strongly tempts us to make the:

Main Conjecture: $p(c, s)$ = $c + s - d$,   where $d = \gcd(c, s)$

In other words, the smallest number, $p(c, s)$, of cake pieces you can use to share c cakes among s students is $c + s - d$. We have already seen, with the Linear and Euclidean Distributions use exactly $c + s - d$ pieces, and so

$$p(c, s) \quad \leq \quad c + s - d$$

Side comment on the Euclidean Algorithm: The school curriculum often gives diminished attention to 'long division' (here called Euclidean division), and therefore also small attention (if any) to the Euclidean Algorithm for finding the gcd(c, s) = d of two whole numbers c and s, which is based on Euclidean division. The method generally offered is to first find the prime factorizations of c and s, and then simply inspect these to find d. And in fact, for small numbers, this is likely most efficient. However, if nothing more is said, this deprives students of the awareness, in comparing the two methods – Euclidean Algorithm vs. prime factorization – in general, that for large numbers (say > 6 digits), the problem of prime factorization becomes an intractably difficult computation, whereas the Euclidean Algorithm, despite appearances, is relatively straightforward and can be done in practical ('polynomial') computational time relative to the size of c and s.   This phenomenon is fundamentally important in cryptography. Thus, ironically, neglecting long division, often done on the grounds that we have calculators to do such computations, will deprive students of exposure to an important idea about complexity of computations that is central to modern computer science.

Seeking a proof of the Conjecture: A side trip into graph theory

It remains to show (in order to prove the Conjecture above) that we can't do better, i.e. distribute c cakes to s students with fewer than $c + s – d$ pieces. In other words, it remains to show that,

$$p(c, s) \quad \geq \quad c + s - d$$

How can we possibly show this?   It is here that we shall push the envelope of school mathematics a bit. So far, we have been using fairly basic, though substantial, mathematical ideas and tools of High School mathematics. I think it is fair to say that most mathematicians who spent some serious time thinking about this question would arrive eventually at the point we are at now. But the next steps seem less predictable. At this point, after considerable reflection, I had to reach for a new connection.

The graph of a cake distribution

The problem now is that we have to consider <u>any</u> possible distribution D of c cakes to s students, and show that D must consist of at least $c + s – d$ pieces. In contrast with our discussion of the Linear and Euclidean Distributions, we have no special information about D. So let's think a bit about what D is. D distributes cake pieces to students. So one way to picture this schematically is as follows. For each cake piece, draw a line from the cake from which it came, to the student

to whom it is given. If we forget that the cakes are cakes, and that the students are people, and simply represent them abstractly as dots, then what we have is a collection of dots, together with some lines (corresponding to the cake pieces) connecting various pairs of these dots. This is in fact a familiar kind of mathematical object, called a (combinatorial) *graph*. We shall call this the graph of the distribution D, and denote it $\Gamma(D)$. To see what this looks like, consider the graphs of the Linear Distribution D_L and the Euclidean Distribution D_E, for c = 5 and s = 7. We shall represent the students by dots, and the cakes by short horizontal line segments instead of dots, just to be a bit more suggestive of the context.

Graph of the Linear Distribution, $\Gamma(D_L)$:
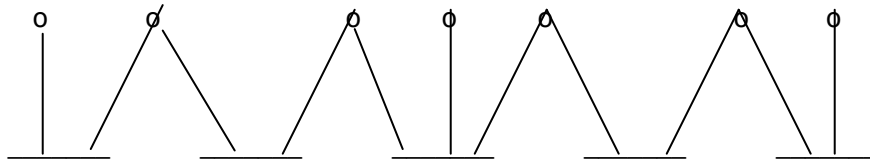
Graph of the Euclidean Distribution, $\Gamma(D_E)$:
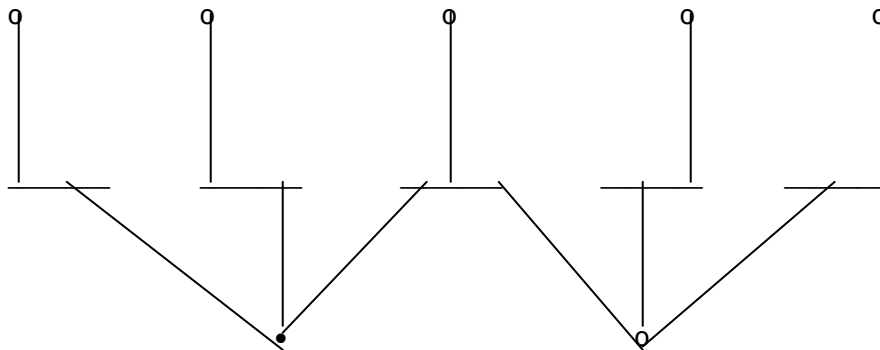
Here the graph of a distribution brings into play a dramatically new <u>representation</u> of our problem. What are its pros and cons? Well, it captures rather well, and elegantly, the "combinatorial structure" of a cake distribution. But it loses the geometric and metric aspects. For example, in the graph, a cake piece becomes an undifferentiated line segment, independent of the size of the piece. So, what does this graph do for us? At first we're not sure. But at least this is a familiar and widely used kind of mathematical object, so we can 'consult graph theory' to see if it has anything useful to offer.

<u>A tip-toe into graph theory</u>

Mathematically, a <u>graph</u> Γ is <u>defined</u> to consist of a set V (called <u>vertices</u>, or nodes), a set E (called <u>edges</u>), and a specification of a pair of <u>endpoints</u> (which are vertices) for each edge. The vertices are generally depicted as dots, and the edges as line segments joining their endpoints. (These lines do not have to be drawn straight; they may be curved. All that is essential is specifying the vertices that they connect.) Here is an example, from our School of Education, with 16 vertices and 16 edges.
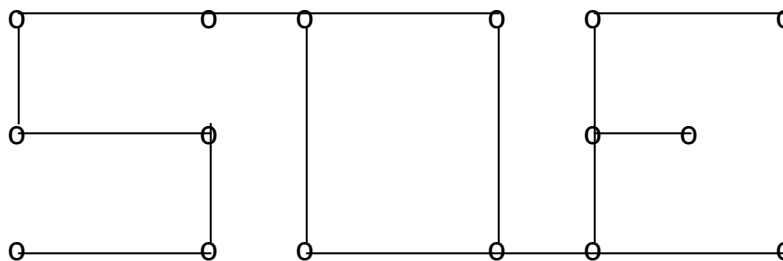
This graph is said to be *connected*, since you can get from any vertex to any other along an edge-path. In general, a graph is a disjoint union of connected sub-graphs, called its *connected components*. A graph is called a *tree* if there is a *unique* edge path (without backtracking) from any vertex to any other. In particular, a tree is connected. The above graph is <u>not</u> a tree, since you can go around the "O" in two ways.

We are going to make use of one basic fact from graph theory: What does it take to make a graph connected? Well, if there are lots of vertices, then you will need lots of edges to connect them all. How many edges do you require?

PROPOSITION. (The "Basic Inequality") If a graph Γ is connected then

$$\#E \quad \geq \quad \#V - 1,$$

with equality if and only if Γ is a tree.

This is easy enough to prove, inductively, as follows. We can build a connected graph by starting with a single vertex, and then successively attach edges, by either one or both of their endpoints, to what we already have. (You might try to picture doing this on the graph displayed above.)

If Γ consists of a single vertex and no edges, then

$$\#E \quad = \quad 0 \quad = \quad \#V - 1,$$

and Γ is a tree.

Next suppose that Γ is obtained from a connected graph Γ' (with vertices V' and edges E') by attaching a new edge e. We assume, by induction on $\#E$, that

$$\#E' \quad \geq \quad \#V' - 1, \qquad \text{with equality if an only if } \Gamma' \text{ is a tree}$$

Case 1: We attach only one end point of e to Γ'. Then

$$\#E = \#E' + 1 \quad \text{and} \quad \#V = \#V' + 1, \quad \text{so}$$
$$\#V - \#E \quad = \quad \#V' - \#E' \quad \leq \quad 1$$

and Γ clearly remains a tree if Γ' was one.

Case 2: We attach both end points of e to Γ'. Then

$$\#E = \#E' + 1, \quad \text{but} \quad \#V = \#V', \quad \text{so}$$
$$\#V - \#E \quad = \quad \#V' - \#E' - 1 \quad < \quad 1.$$

Moreover, Γ is not a tree, because we can connect the end points of e either using e itself, or using a path in the (connected) graph Γ'.

We shall see next that the Basic Inequality above can be applied to the graph of a cake distribution to get the lower bound we seek on the number of pieces in a cake distribution.

Here we have '<u>consulted</u> graph theory' to find some resource that can give us new traction on our cake distribution problem. Also we have provided an accessible <u>proof</u> of

the basic graph theoretic inequality that we will need. In doing this we needed to give precise mathematical definitions to the graph theoretic concepts being used. The representation of a cake distribution by its graph gives us the bridge of access to this resource. Of course it took some exploration and experimentation (lengthy, but not described here) to discover what from graph theory might be useful for this purpose. But with this in hand, we are now in a position to finish the proof of the main conjecture.

Proof that: p(c, s)   ≥   c + s - d

Suppose that D is a 'minimal' distribution of c cakes to s students, i.e. one that involves the *least possible number* p = p(c, s) of cake pieces. Let Γ = Γ(D) be the graph of the distribution D. Then its vertex set is
          V   =   {cakes} $\cup$ {students},
and so
          #V   =   c + s
Its edges are just the set
          E   =   {cake pieces},
and so
          #E   =   p

We would like to apply the Basic Inequality above to Γ. However, we are not entitled to do this since we do not know that Γ is connected. So, instead, let's look at a *connected component,* call it Γ', of Γ. Now the vertex set V' of Γ' consists say of c' cake vertices and s' student vertices, and its edges E' are just the cake pieces taken from cakes in V' and given to students in V'. However, the fact that Γ' is a *connected component* of Γ implies that every piece taken from a cake in V' is given to a student in V', and, conversely, students in V' receive pieces only from cakes in V'. It follows that
          Γ' *is itself the graph of a distribution* D' *of* c' *cakes to* s' *students*.
Moreover, D' must also be minimal, i.e. involve the minimal number p' = p(c', s') of pieces; otherwise we could replace D' by something using fewer pieces, and this could be embedded in D to reduced the number of pieces in D, contrary to our assumption that D was already minimal.

Ok, now we are in a position to deploy all that we have learned. Let d' = gcd(c', s'). Then the Linear and Euclidean Distributions (for (c', s')) show us that

(1)   p'   ≤   c' + s' - d'

On the other hand, since Γ' is connected, the Basic Inequality of graph theory tells us that

          p'   =   #E'   ≥   #V' - 1,   i.e.

(2)   p'   ≥   c' + s' - 1

Combining (1) and (2) we see that

          d'   =   1,   i.e.   c' and s' are relatively prime,
and
          p'   =   c' + s' - 1,   and Γ' is a tree.

Now the students in V' each get c'/s' of a cake. But they must receive the same share, c/s, as all of the other students. Thus

        c'/s'   =   c/s,   which is independent of the connected component Γ'

Let

        c_0/s_0 =   the reduced form of the fraction c/s,

so that

        c   = dc_0 and   s   = ds_0,   where d = gcd(c, s)

Then the discussion above shows that c' = c_0 and s' = s_0, independent of Γ'. Moreover it follows that

        Γ is a disjoint union of d trees, each with $c_0 + s_0$ vertices and $c_0 + s_0 - 1$ edges,

and so

        p   =   #E   = $d(c_0 + s_0 - 1)$ =   c + s - d

This completes the proof of our main conjecture, which is now a theorem.

CAKE DISTRIBUTION THEOREM Let D be an equal distribution of c cakes among s students. Then

$$\text{\# (cake pieces in D)} \quad \geq \quad c + s - d, \quad \text{where } d = \gcd(c, s)$$

For the Linear Distribution and the Euclidean Distribution, we have equality above in place of $\geq$.

We have presented here a reasonably formal, yet I hope accessible, <u>proof</u> of this result. The argument combines information coming from different sources (different representations) and so can be viewed as establishing some interesting <u>connections</u>. Moreover, the graph theory even gives us a bonus, in the way of more detailed information about the combinatorial <u>structure</u> of a minimal cake distribution. It is also worth noting how the imported concepts and language of graph theory ('connected,' 'connected components,' 'trees') fit so comfortably and conveniently with our cake distribution context.   With our new theorem in hand, it is "<u>natural to ask</u>:" What is the significance of this result? What might it be good for? This is a kind of 'debriefing' stage of the reasoning.

III. THE MATHEMATICAL STORY – PART 2: SQUARE TILINGS OF RECTANGLES

Square tiling of rectangles

In our analysis of the Euclidean Distribution (of c cakes for s students) we saw that the Euclidean Algorithm, on which it is based, could be interpreted geometrically as producing a "square tiling" of the (c x s)-rectangle. We raised, in passing, the question of whether this "Euclidean tiling" is optimal in some sense, for example whether it uses the smallest possible number of square tiles.

Let's pause here to say more precisely what we mean by a *square tiling* T *of a rectangle* R.   By T we understand a set (here assumed to be finite) of squares in the plane such that their union is exactly R, and any two of them intersect at most along an edge of each one. (Here we are treating squares and rectangles as two-dimensional regions, not just their one-dimensional boundaries.)

In the course of thinking about the above questions, I did a Google search under the heading "Square tilings of rectangles." This produced a wealth of references, showing that there is in fact a minor industry around this and related topics. In particular, one of the references (Kenyon, 1994) shows that the answer to the above question is negative. (In special cases the Euclidean tiling <u>is</u> minimal for the number of tiles, for example when c and s are consecutive terms in the Fibonacci sequence.) To see that the Euclidean tiling is not minimal in general we can take s = c + 1, in which case the Euclidean tiling consists of 1 (c x c)-square together with a column of c (1 x 1)-squares, for a total of c + 1 = s tiles. Consider the case c = 8, so s = 9.

The Euclidean tiling of the (8 x 9)-rectangle, with 9 tiles
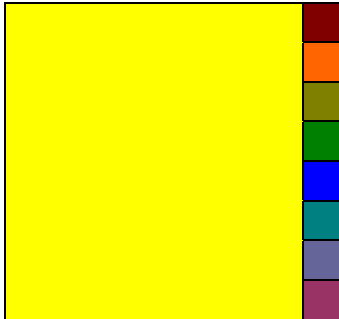
A square tiling of the (8 x 9)-rectangle with 7 tiles
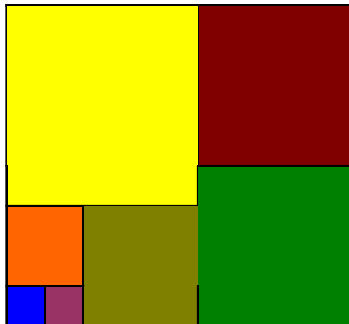
So this 'wishful thinking' guess did not pan out. Still, since, as we have shown above, the Euclidean Distribution minimizes the number of pieces for cake distributions, we have the feeling that the corresponding Euclidean tiling of the (c x s)-rectangle should also be minimal, in some sense to be determined. Well, a natural approach to this might be to:

Find a <u>geometric interpretation</u> of the minimal number
p = p(c, s) = c + s - d
of cake pieces in the Euclidean distribution of c cakes to s students.

> *Here we are <u>opportunistically</u> picking up on some side issues that appeared in the course of the work, but were not central to it. The interest here, beyond the fact that these are interesting new <u>questions</u> in their own right, is that the <u>connections</u> noticed earlier might lead the way to some possible elaborations or <u>applications</u> of the result proved above. Also note that, as we engaged more seriously with these ideas, it was important to give a precise <u>mathematical definition</u> of the main terms (like 'square tilings') being used.*

In fact it is not so hard to see a geometric interpretation of the number of cake pieces. Imagine the rectangular area picture of the Euclidean distribution. We reproduce below the illustration for

c = 5 and s = 7. Each cake piece is a horizontal slice of one of the squares in the tiling, and the number of these slices in a given square is clearly just the side length of that square. Thus, for each square of side length ℓ, we get ℓ pieces, and so the total number of pieces will be the sum of the side lengths of all the squares in the Euclidean tiling of the rectangle.

S1 P1 5/7	**S6** P6 2/7
S2 P2 5/7	**S6** P7 2/7
S3 P3 5/7	**S6** P10 1/7 **S7** P11 1/7
S4 P4 5/7	**S7** P8 2/7
S5 P5 5/7	**S7** P9 2/7

The students: S1, … , S7; their shares are color coded.
The pieces: P1, … , P11
The fractions indicate fractions of a cake; each cake is a row (of width 1) of the
        rectangle.

This leads us to <u>define</u> the following quantity associated with <u>any</u> tiling T of a (c x s)-rectangle. Here T is understood to be a set of squares whose union is exactly R and such that any two of them intersect at most along an edge of each. If σ is one of these (square) tiles, i.e. $\sigma \in$ T, we shall write s(σ) for its side length. Then we define p(T) to be the sum of these side lengths.

$$p(T) \quad = \quad \sum_{\sigma \in T} \quad s(\sigma)$$

With this notation, our observation about the Euclidean tiling, T_E , can be expressed by the formula,

$$p(T_E) \quad = \quad p(c, s) \quad = \quad c + s - d$$

So we might thus be led to make the following:

Conjecture. For <u>any</u> square tiling T of a (c x s)-rectangle, p(T) $\geq$ c + s - d.

This passage illustrates some important kinds of 'mathematical moves.' We are <u>*navigating between two mathematical worlds,*</u> *one the world of cake distributions, the other the world of square tilings of rectangles. We saw (earlier) that the Euclidean*

Distribution established a bridge between these two worlds, the Euclidean Distribution at one end, the Euclidean tiling at the other. We proved that the Euclidean Distribution has a minimizing property in the cake distribution world, so we were tempted to ask if (or suspect that) the Euclidean tiling has some analogous minimizing property. This is a kind of <u>reasoning by analogy</u> that mathematicians often use, to guess what might be true, by developing a relation of some new situation to an old one, about which we already know something. It can be viewed as another kind of pattern seeking. The procedure we followed was to try to build up the dictionary of translation from the cake world to the tiling world. Given that [Euclidean Distribution] translates to [Euclidean tiling], we ask, [# pieces] translate to [???]. What we seek here is something that we can measure geometrically for all tilings in the tiling world, and so that, when applied to the Euclidean tiling, gives something closely related to the number (c + s − d) of pieces. We found p(T) as the answer to that question, and accordingly we gave it a <u>name</u>, p(T), so that we could talk about and work with it.

The Conjecture above, if true, would indeed show that the Euclidean tiling minimizes p(T), and so it is geometrically optimal among tilings, in this sense. Can we prove this Conjecture? The geometric statement is not so obvious. Perhaps, instead of directly attacking it geometrically, we can use our Cake Distribution Theorem to help. In other words, perhaps we can interpret <u>any</u> square tiling T of a (c x s)-rectangle as arising somehow from a cake distribution of c cakes among s students, and in such a way that p(T) is the number of cake pieces. If we can do that, then we will have proved the above conjecture by reducing it to the Cake Distribution Theorem.

So here we are proposing to show that our dictionary is (at least partly) reversible; in other words we can go back from a square tiling to a cake distribution. In this way, we can use our dictionary to import our theorem on cake distributions to the tiling world, where it translates into a geometric theorem.

Making a cake distribution from a square tiling

For this argument, let us <u>assume that not only c and s, but also the side lengths of all of the square tiles in T, are integers</u>. To help follow the argument, let's illustrate what happens with the square tiling of the (8 x 9)-rectangle that we saw above:
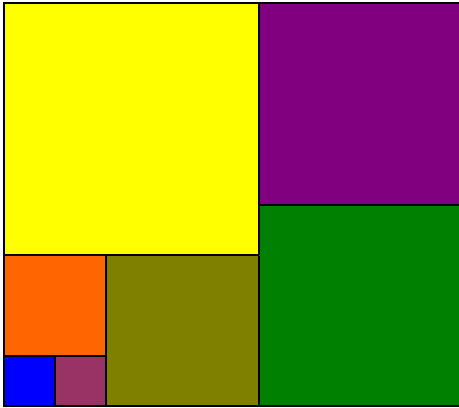
Let us cut the rectangle into c horizontal (1 x s)-rectangles, that we consider to be the 'cakes.' Then the vertical sides of the square tiles can be viewed as cuts through some of these cakes. The result is that each square tile σ, say of side length s(σ), will consist of s(σ) horizontal cake pieces, each of size 1 x s(σ).
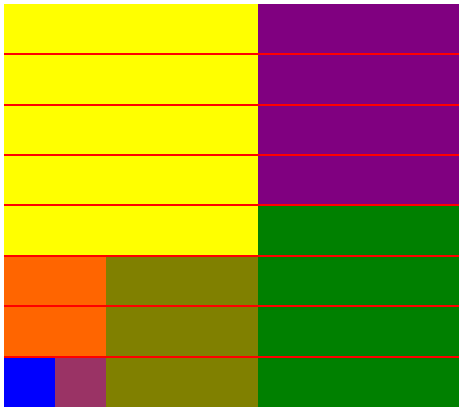
It remains to explain the distribution of these pieces to the s students. For this let us label the size-(c x 1)-columns of the big (c x s)-rectangle R, by the numbers 1, … , s, one for each student. So student j corresponds to column j.

1 2   3 4   5 6 7   8   9
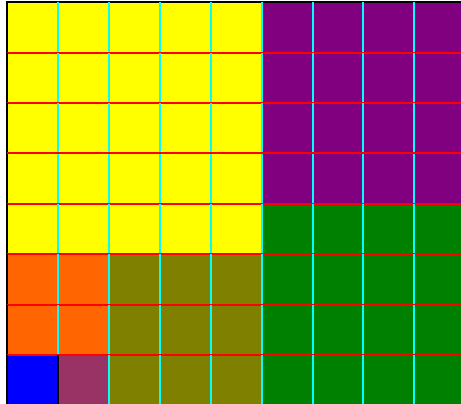
For each tile σ through which column j passes, give student j one of the cake pieces from σ. (In the following picture, the numbers indicate the student to whom that piece is given.)

Since exactly s(σ) columns pass through σ, and since σ is composed of exactly s(σ) cake pieces, this distribution of the cake pieces from σ is possible. Now we have distributed all of the pieces to the s students. To see that this is an *equal* distribution, we need to see that each student receives the same share, c/s of a cake. In other words, student j should receive an amount of cake equivalent to that cut out by the (c x 1) column j. But, for each square σ through which column j passes, student j receives a horizontal cake piece of size 1 x s(σ), while the intersection of column j with σ is a rectangle of size s(σ) x 1, of the same area. Thus, the area of column j, being the sum of the areas of its intersections with the squares through which it passes, also equals the total share received by student j. And this is what we needed to show.

We have thus proved:

SQUARE TILING THEOREM. If T is a tiling of a (c x s)-rectangle by squares of integer side length, then

$$p(T) \quad \geq \quad p(c, s) \quad = \quad c + s - d, \quad \text{where } d = gcd(c, s)$$

This is an equality for the Euclidean tiling T_E.

So this is a <u>satisfying outcome</u>, but with the one <u>caveat</u> that we had to restrict attention to square tiles of integer side length. We'll come back to that issue later, but just take note of it now. The proof has, I think, a very nice 'fit' to it. It shows I think a close <u>structural relation</u> between square tilings and cake distributions, so that results about the latter have applications to the former. The proof above seems 'natural enough,' even though it is a bit tricky to explain (especially without the pictures). The key was finding the idea for the proof, not its execution. I have not found a direct geometric proof of the theorem above.

The "complete perimeter"

One geometrically <u>un-aesthetic</u> feature of the theorem is the fact that $p(T)$ is not a 'visually obvious' quantity. For example, if we look at a square tiling,
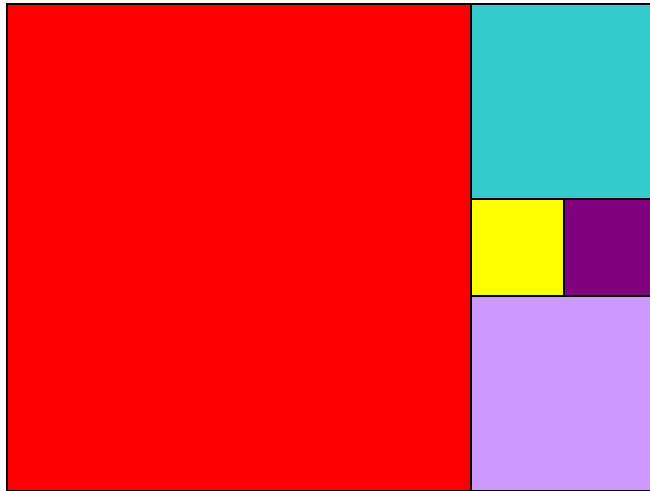
we can't 'see' $p(T)$. Of course we can just add up all of the side lengths of the squares, but many geometrically visible pieces of this are counted twice, and this happens in slightly complicated ways. A more visually obvious geometric quantity is the total length of all of the boundary lines seen in this picture, viewed as a partition or (cartographic) 'map,' of the rectangle (with the squares as "countries"). Let's call this the "*complete perimeter*" of the tiling T, the sum total of the lengths of all the boundaries, and denote it CP(T). A more precise, but less intuitive, definition could be given as follows:

$$CP(T) \quad = \quad \text{the total length of the (set theoretic) union of the sides of all of the square tiles in T}$$

This union is exactly the set of line segments that we see in the picture. An intuitive way to think of CP(T) is that it measures "the amount of ink needed to draw the picture of the tiling." Then, with this more geometrically natural quantity, we can ask,

Does the Euclidean tiling also minimize CP(T)?

Put another way, does the Euclidean tiling, among all square tilings of R, minimize the 'boundary'? In this form, question reminds us of what are called "isoperimetric problems," which are about enclosing a given area with minimum perimeter.

The motivations in play here are partly <u>aesthetic</u>. The cake distribution world is primarily algebraic/combinatorial, while the tiling world is primarily geometric. But when we translated the number of pieces into the quantity p(T), the p(T) was still mainly an algebraic expression, with no <u>visible</u> geometric meaning. So there was a <u>mathematical impulse</u> to seek some more visibly geometric quantity that we could relate to the number of pieces in a cake distribution. This would make the theorem more interesting or natural from a purely geometric point of view. We shall see now in what follows that this is easily achieved from what we have already done.

Instead of trying to directly answer the question of whether the Euclidean tiling minimizes CP(T), let's first just try to <u>calculate CP(T)</u>. One way to do this is to first sum the perimeters of all the square tiles, and then compensate for things we have counted twice. So we begin with

$$4p(T) \quad = \quad \Sigma_{\sigma \in T} \; 4s(\sigma)$$

$$= \quad \text{the sum of the perimeters of all of the square tiles}$$

The sides that are not counted twice are those on the boundary of R, and their lengths add up to the perimeter of R, which is 2(c + s). All of the other square side lengths are effectively counted twice. It follows that

$$CP(T) \quad = \quad 2(c + s) \; + \; \tfrac{1}{2}\,[4p(T) \; - \; 2(c + s)], \qquad \text{so}$$

$$\boxed{CP(T) \quad = \quad (c + s) \; + \; 2p(T)}$$

It is worth noting in passing here that this calculation was purely geometric, and did not require c, s, and the square side lengths to be integers. They could be any real numbers > 0.

The formula above shows that, for a fixed (c x s)-rectangle R, CP(T) is a linear function (with slope 2) of p(T), as T varies over all square tilings of R. Thus, a tiling T minimizes CP(T) if and only if it minimizes p(T). In particular therefore, the Euclidean tiling T_E minimizes CP(T), in which case we have

$$CP(T_E) \quad = \quad (c + s) \; + \; 2(c + s \; - \; d), \qquad \text{so}$$

$$\boxed{CP(T_E) \quad = \quad 3(c + s) \; - \; 2d}$$

So we have proved the geometric result that we sought:

PERIMETRIC SQUARE TILING THEOREM For any tiling T of a (c x s)-rectangle R by squares with integer side lengths, we have

$$CP(T) \quad \geq \quad 3(c + s) \; - \; 2d, \qquad \qquad \text{where } d \; = \; \gcd(c, s)$$

For the Euclidean tiling, T_E, we have equality in place of $\geq$ above.

Misgivings, new questions, and generalizations

While the theorem above seems to offer a pretty happy state of affairs, there remain some issues in the background that are puzzling, if not troubling. First of all, it seems <u>mathematically unpleasant</u> that we had to assume that our tilings used only squares of integer side lengths, while the statement of the conclusion requires only that c and s be integers. What happens if a tiling T involves squares not of integer side length? Is it still true that

$$CP(T) \quad \geq \quad 3(c + s) - 2d?$$

And, more generally, the notion of square tiling is purely geometric and makes perfectly good sense for <u>any</u> rectangle, say c x s, where c and s can be any real numbers > 0, not necessarily integers. What is the story for these? In this case, CP(T) above still makes sense, but what about d = gcd(c, s)? How could that possibly be interpreted? In fact this raises in turn an <u>existence question</u>: If c and s are not integers, how do we know that there even exists any tiling of R by squares?

> *In short, we are here asking questions about the "natural <u>mathematical boundaries</u>" of what we have done, and about ways to frame our results in their "natural mathematical generality." These are the kinds of <u>questions</u> that a mathematician would typically be <u>disposed</u> to ask, before even thinking hard about their likely outcome. Such <u>questioning repertoires</u> are an important resource in mathematical practice (just as in teaching).*

Let's begin with the last question:

Which rectangles can be tiled by squares?

First observe that this is a property that is invariant under rescaling. If we change everything by a scaling factor, then a square tiling gets transformed into another one (of a different size). Now if a (c x s)-rectangle has <u>rational</u> side lengths, c and s, then we can scale up by a common denominator of c and s to get a rectangle with integer side lengths, which can clearly be tiled, for example by (1 x 1) squares, and thus so also can R be square tiled, after scaling back down. More generally, if a (c x s)-rectangle admits a square tiling, then so also does a (rc x rs)-rectangle, for any real number r > 0, as we see by rescaling with the factor r. (So the side lengths don't even have to be rational numbers.) Thus, a (c x s)-rectangle R can be square tiled if, for some number r > 0, rc and rs are both rational. But then the ratio c/s = rc/rs is also rational. Conversely, if c/s is rational, say c/s = a/b with a and b integers, then, setting r = a/c = b/s, we have rc = a and rs = b, which are both rational. Two non-zero real numbers c and s are said to be *commensurable* if the ratio c/s is a rational number. With this terminology, the discussion above shows that,

A rectangle can be square tiled if its side lengths are commensurable.

I wondered if the converse might be true, believing that it is. I asked some colleagues, and finally was led to the answer, in the (old) literature.

THEOREM (Max Dehn, 1903)

A rectangle can be square tiled if and only if its side lengths are commensurable.

In fact, more can be said:

THEOREM   If the side lengths of a rectangle R are rational numbers, then a square tiling

of R must involve only squares of rational side length.

HISTORICAL NOTE. **Max Dehn** (1878-1952) was a German mathematician who studied under David Hilbert at Gottingen. Dehn did deep and fundamental work in geometry, topology, and group theory. He was the first to solve one of Hilbert's famous list of 23 problems. Giving a negative solution to Problem #3, Dehn showed that a cube and a regular tetrahedron of the same volume could not be cut into polyhedra that are pairwise congruent. This contrasts with what happens in the plane, where two polygons of the same area can be decomposed into triangles that are pairwise congruent.

In 1938 Dehn, a Jew, was forced by the Nazis to leave his professorship in Frankfurt. In 1945 he became the unique math professor at Black Mountain College in North Carolina, where he remained till his death. There was no opportunity there to teach advanced mathematics, but he also taught Latin, Greek, and Philosophy. The Black Mountain faculty included such figures as John Cage, Merce Cunningham, Willem de Kooning, Buckminster Fuller (of whom Dehn became a close friend), Walter Gropius, and many other artists.

Consulting the literature in pursuit of the questions above was the occasion for learning some very interesting mathematics (old, but much of it new for me), and I welcomed the opportunity to thereby gain new knowledge and techniques, as well as culturally broaden my mathematical horizons. I did not hesitate to take in more than was needed for the questions that motivated my search. I'll report on some of the highlights below, providing mathematical details only when they are within reach of high school mathematics.

If we import Dehn's Theorem from the literature for our use, then we can give a version of our theorem on square tilings of rectangles in more natural mathematical generality. First we need to interpret gcd(c, s) when c and s are any real numbers.

A generalized meaning of gcd and lcm

Let c be any real number. By a multiple of c we shall mean a number of the form q•c, where q is an *integer*. A number d is called a divisor of c if c is a multiple of d. Now let s be another real number. Then a common multiple of c and s is just that; it is a number that is a multiple of both c and s. We similarly define a common divisor of c and s. Note that these definitions agree with those we already know when c and s are integers. Here are some exercises that we leave to the reader.

EXERCISES. Let c and s be real numbers, not both 0, and let r be a real number > 0.

1.   0 is a common multiple of c and s.

2.   m is a common multiple of c and s   if and only if   rm is a common multiple of rc and rs.
        d is a common divisor of c and s   if and only if   rd is a common divisor of rc and rs.

3.   The following conditions are equivalent.

        (a) c and s are commensurable, i.e. rc and rs are rational for some r > 0
        (b) c and s have a common multiple ≠ 0
        (c) c and s have a common divisor

4.   Under the equivalent conditions of #3, c and s have a greatest common divisor, denoted
        d = gcd(c, s), and a least common multiple > 0, denoted m = lcm(c, s). Moreover,
                        c•s   =   d•m

5.   We have gcd(rc, rs) = r•gcd(c, s) and lcm(rc, rs) = r•lcm(c, s).
        (This follows from #2 and #4.)

With these definitions we can now state our theorem in its natural generality.

PERIMETRIC SQUARE TILING THEOREM (GENERALIZED) Let R be a (c x s)-rectangle, and let T be a

square tiling of R. Then c and s are commensurable, and

        CP(T)   ≥   3(c + s) - 2d,   where   d = gcd(c, s),

with equality when T is a 'rescaling' of the Euclidean tiling.

The discussion above was designed just to give meaning to the quantity "gcd(c, s)" in the theorem. The definitions and exercises are a fairly typical example of how a mathematician may try to find a natural <u>general framework</u> for some mathematical concept. With some elementary concepts from "group theory" (out of bounds in the present discussion) one could give a more conceptual and more precise formulation to these ideas.

The proof of the Generalized Perimetric Square Tiling Theorem goes as follows. The commensurability of c and s is just Dehn's Theorem. So, after rescaling R and T, we can assume that c and s are rational. Then the sequel to Dehn's theorem tells us further that the tiles in T all have rational side length as well. Choosing a common denominator for c and s and all the side lengths of tiles in T, we can use this to rescale the situation again and arrange that c and s are integers, as are the side lengths of all the tiles in T. Now we are in a position to quote the Perimetric Square Tiling Theorem we proved above under these conditions. Finally, we scale back to the original R and T. Exercise #5 above is used to see that gcd(c, s) behaves consistently in each of these rescalings.

Dehn's Theorem tells us that square tileable rectangles are commensuarable, i.e. their side lengths are rational after rescaling. A further rescaling makes the side lengths integers, where we can apply the earlier Perimetric Square Tiling Theorem. To scale back to the original rectangle and tiling, we need to know how to give meaning to a rescaling of the gcd(c, s) that appears in the earlier theorem. That is what we worked to accomplish in the discussion preceding the generalized theorem. So finding the "<u>mathematical boundary</u>" of our result had two ingredients. First, Dehn's Theorem <u>restricts</u> the <u>geometric boundary</u> of the set of rectangles for which it is meaningful to discuss square tilings. Second, we <u>conceptually expanded</u> the algebraic notion of gcd(c, s) so that it has meaning in the full geometric context defined by Dehn's Theorem.

The only 'gap' in our story now, i.e. the only component that we have not mathematically derived from essentially High School level mathematics, is Dehn's Theorem itself. Can we make that also accessible?

Proofs of Dehn's Theorem

There are several proofs of Dehn's Theorem, but I have not found one that stays within the mathematical bounds that I have tried to maintain here. Dehn's original proof (Dehn, 1903) was quite complicated. Later proofs (see for example, Freiling and Rinne, 1994) are short and elegant, but make use of some abstract linear algebra, and the Axiom of Choice. An ingenious proof was devised by Brooks et al, (1940). From a square tiling of a rectangle, they constructed an electrical circuit, and used Kirchoff's Laws to deduce Dehn's Theorem, as well as many interesting generalizations. This method is also described in Blackett's book on Elementary Topology (1982).

For mathematical completeness, but outside the framework of the exposition above, we provide here a proof of Dehn's Theorem as used here. First a preliminary on "area functions."

Area functions on rectangles

Consider a plane rectangle

$$R = [x, x'] \times [y, y'],$$

with vertices the points (x, y), (x, y'), (x', y) and (x', y'); here $x < x'$ and $y < y'$. We call these "coordinate rectangles" (the sides are parallel to the coordinate axes), and assume that all rectangles in what follows are such.

Let $f(x,y)$ be any function on $\mathbf{R}^2$. We define the "f-area" of R to be

$$A(R) \ \ (\text{or } A_f(R)) \qquad = \qquad f(x', y') - f(x, y') - f(x', y) + f(x, y)$$

LEMMA. *If a rectangle R is partitioned by a line parallel to one of its sides into two rectangles R' and R", then*

$$A(R) \quad = \quad A(R') + A(R").$$

Proof. We show this in the case that the dividing line is vertical. The horizontal dividing line case is similar.

(x, y')	(x', y')	$(x", y')$
R'	R"	
(x, y)	(x', y)	$(x", y)$
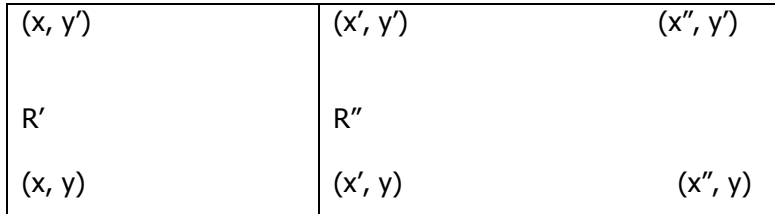
We have

$$A(R') + A(R") \quad = \quad \begin{aligned} & f(x', y') - f(x, y') - f(x', y) + f(x, y) \\ & + f(x", y') - f(x', y') - f(x", y) + f(x', y) \end{aligned}$$

$$= \quad f(x", y') - f(x, y') - f(x", y) + f(x, y)$$

$$= \quad A(R)$$

5.3 PROPOSITION. *If a rectangle R is tiled by rectangles $R_1, R_2, \ldots, R_n$ then*

$$A(R) \quad = \quad A(R_1) + A(R_2) + \ldots + A(R_n)$$

Proof. Say $R = [a, a'] \times [b, b']$. If the tiling is the coordinate tiling resulting from partitions of the intervals $[a, a']$ and $[b, b']$, then the result follows easily from the Lemma, for example first summing over the tiles in a given row, to replace the row of tiles by a single row tile, and then summing over the rows.

In general, we can extend the edge lines of all the tiles to refine the tiling to a coordinate tiling, and note that, by the Lemma, the sum of the areas in the refined tiling agrees with the sum over the original tiles, as well as with A(R).

Bilinear area functions. Suppose now that the function $f(x, y)$ is bilinear, in the sense that $f(x+x', y) = f(x,y) + f(x', y)$, and $f(x, y+y') = f(x,y) + f(x, y')$ for all numbers x, x', y, y'. Then for a rectangle

$$R = [x, x + a] \times [y, y + b]$$

we have

$$A(R) = f(x+a, y + b) - f(x, y + b) - f(x + a, y) + f(x, y)$$

$$
\begin{aligned}
= \quad & f(x, y) + f(x, b) + f(a, y) + f(a, b) \\
& - f(x, y) - f(x, b) \\
& - f(x, y) \qquad\qquad - f(a, y) \\
& + f(x, y)
\end{aligned}
$$

$$= f(a, b)$$

Thus, when f is bilinear, the Proposition above can be formulated as:

PROPOSITION. *Suppose that f is bilinear. If a rectangle R of side lengths (a, b) is tiled by rectangles with side lengths $(a_1, b_1) , \ldots , (a_n, b_n)$, then*

$$
\begin{aligned}
A(R) \quad = \quad & f(a, b) \\
= \quad & f(a_1, b_1) + \ldots + f(a_n, b_n).
\end{aligned}
$$

DEHN'S THEOREM (GENERALIZED). *Let R be a rectangle of height c and base s, and let T be a finite set of square tiles that tile R.*

> *(a) (Dehn) c/s is a rational number.*

> *(b) Suppose that c and s are rational (which we may achieve by rescaling, thanks to (a)). Then all squares in T have rational side lengths.*

Proof of (a) (See Freiling and Rinne, p. 549): If c/s is not rational, choose a **Q**-vector space basis of the real numbers, **R** (a "Hamel basis") containing c and s. Then there exists a **Q**-linear function $g:\mathbf{R} \to \mathbf{Q}$ such that $g(c) = 1 = -g(s)$. Put $f(x, y) = g(x)g(y)$, a bilinear function on $\mathbf{R}^2$, and use f to define an area function $A = A_f$ as above. Then (Proposition above)

$$A(R) = f(c, s) = g(c)g(s) = -1$$
$$= \Sigma_{\sigma \in T} \, g(s(\sigma))^2 > 0,$$

which is a contradiction. (Here, for $\sigma \in T$, $s(\sigma)$ denotes the side length of σ.)

Proof of (b): Decompose **R** as a **Q**-vector space – $\mathbf{R} = \mathbf{Q} \oplus W$. Take a **Q**-basis B of **R** consisting of 1, followed by a **Q**-basis of W. Let $g(x,y)$ be a symmetric **Q**-bilinear form (inner product) on **R** for which B is an orthonormal basis. Hence g is positive definite. For $x \in \mathbf{R}$, we can write $x = x_0 + x'$, uniquely, with $x_0 \in \mathbf{Q}$ and $x' \in W$. Choose a real parameter t, define the **Q**-bilinear function

$$f(x, y) = x_0 y_0 + t \bullet g(x', y'),$$

and let $A = A_f$ be the corresponding "area function."

We are given a finite set T of squares that tile the rectangle R with rational base c and height s. Then, as above, we have

$$A(R) \quad = \quad f(c, s) \quad = \quad cs \quad > \quad 0$$

$$= \quad \sum_{\sigma \in T} f(s(\sigma), s(\sigma)) \quad = \quad \sum_{1 \leq i \leq r} f(s(i), s(i)),$$

where $s(1), s(2), \ldots, s(r)$ is the list of side lengths of the square tiles in T. We can write
$$s(i) \quad = \quad s(i)_0 + s(i)', \qquad \text{with} \qquad s(i)_0 \in \mathbf{Q} \qquad \text{and} \quad s(i)' \in W.$$
Then
$$f(s(i), s(i)) \quad = \quad s(i)_0^2 + t \bullet g(s(i)', s(i)')$$

These $f(s(i), s(i))$ are linear functions of t, with t-coefficient ≥ 0, and > 0 if $s(i)$ is irrational. Since their sum, A(R), is a constant (independent of t) it follows that none of the $s(i)$ can be irrational.

IV. CONCLUSION

I have tried to provide a vivid image of a small piece of 'mathematics in the making,' accessible (apart from this last section on Dehn's Theorem) with only a base of High School level mathematics. The main agenda, carried by the interleaved meta-discussion, was to make explicit some of the moves, dispositions, and motivations that guided the mathematical work. My hope is that this can help illuminate some of the resources that mathematicians deploy in the course of their work, and that many of these will resonate with and prove helpful to teachers and learners of school mathematics.

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Mathematical Intuition (Poincaré, Polya, Dewey)

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Summary: Practical calculation of the limit of a sequence often violates the definition of convergence to a limit as taught in calculus. Together with examples from Euler, Polya and Poincare, this fact shows that in mathematics, as in science and in everyday life, we are often obligated to use knowledge that is derived, not rigorously or deductively, but simply by making the best use of available information—plausible reasoning. The "philosophy of mathematical practice" fits into the general framework of "warranted assertibility," the pragmatist view of the logic of inquiry developed by John Dewey.

Keywords: intuition, induction, pragmatism, approximation, convergence, limits, knowledge.

In Rio de Janeiro in May 2010, I spoke at a meeting of numerical analysts honoring the 80th anniversary of the famous paper by Courant, Friedrichs and Lewy. In order to give a philosophical talk appropriate for hard-core computer-oriented mathematicians, I focused on a certain striking paradox that is situated right at the heart of analysis, both pure and applied. (That paradox was presented, with considerable mathematical elaboration, in Phil Davis's excellent article, "The Paradox of the Irrelevant Beginning.") In order to make this paradox cut as sharply as possible, I performed a little dialogue, with help from Carlos Motta. With the help of Jody Azzouni, I used that dialogue again, to introduce this talk in Rome.

To set the stage, recall the notion of a convergent sequence, which is at the heart of both pure analysis and applied mathematics. In every calculus course, the student learns that whether a sequence converges to a limit, and what that limit is, depend only on the "end" of the sequence--that is, the part that is "very far out"—in the tail, so to speak, or in the infinite part. Yet, in a specific instance when the limit is actually needed, usually all that is considered is the beginning of the sequence—the first few terms--the finite part, so to speak. (Even if the calculation is carried out to a hundred or a thousand iterations, this is still only the first few, compared to the remaining, neglected, infinite tail.)

In this little drama of mine, the hero is a sincere, well-meaning student, who has not yet learned to accept life as it really is. A second character is the Successful Mathematician--the Ideal Mathematician's son-in-law. His mathematics is ecumenical: a little pure, a little applied, and a little in-between. He has grants from federal agencies, a corporation here and there, and a private foundation or two. His conversation with the Stubborn Student is somewhat reminiscent of a famous conversation between his Dad, the Ideal Mathematician, and a philosophy grad student, who long ago asked, "What is a mathematical proof, really?"

The Successful Mathematician (SM) is accosted by the Stubborn Student (SS) from his Applied Analysis course.

- -

SS: Sir, do you mind if I ask a stupid question?
SM: Of course not. There is no such thing as a stupid question.
SS: Right. I remember, you said that. So here's my question. What is the real definition of "convergence"? Like, convergence of an infinite sequence, for instance?

SM: Well, I'm sure you already know the answer. The sequence converges to a limit, L, if it gets within a distance epsilon of L, and stays there, for any positive epsilon, no matter how small.

SS: Sure, that's in the book, I know that. But then, what do people mean when they say, keep iterating till the iteration converges? How does that work?

SM: Well, it's obvious, isn't it? If after a hundred terms your sequence stays at 3, correct to four decimal places, then the limit is 3.

SS: Right. But how long is it supposed to stay there? For a hundred terms, for two hundred, for a hundred million terms?

SM: Of course you wouldn't go on for a hundred million. That really *would* be stupid. Why would you waste time and money like that?

SS: Yes, I see what you mean. But what then? A hundred and ten? Two hundred? A thousand?

SM: It all depends on how much you care. And how much it is costing, and how much time it is taking.

SS: All right, that's what I would *do*. But when does it *converge*?

SM: I told you. It converges if it gets within epsilon—

SS: Never mind about that. I am supposed to go on computing "until it converges," so how am I supposed to recognize that "it has converged"?

SM: When it gets within four decimal points of some particular number and stays there.

SS: Stays there how long? Till when?

SM: Whatever is reasonable. Use your judgment! It's just plain common sense, for Pete's sake!

SS: But what if it keeps bouncing around within four decimal points and never gets any closer? You said *any* epsilon, no matter how small, not just point 0001. Or if I keep on long enough, it might finally get bigger than 3, even bigger than 4, way, way out, past the thousandth term.

SM: Maybe this, maybe that. We haven't got time for all these maybes. Somebody else is waiting to get on that machine. And your bill from the computing center is getting pretty big.

SS: (mournfully) I guess you're not going to tell me the answer.

SM: You just don't get it, do you? Why don't you go bother that Reuben Hersh over there, he looks like he has nothing better to do.

- -

SS: Excuse me, Professor Hersh. My name is---

RH: That's OK. I overheard your conversation with Professor Successful over there. Have a seat.

SS: Thank you. So, you already know what my question is.

RH: Yes, I do.

SS: So, what is the answer?

RH: He told you the truth. The real definition of convergence is exactly what he said, with the epsilon in it, the epsilon that is arbitrarily small but positive.

SS: So then, what does it mean, "go on until the sequence converges, then stop"?

RH: It's meaningless. It's not a precise mathematical statement. As a precise mathematical statement, it's meaningless.

SS: So, if it's meaningless, what does it mean?

RH: He told you what it means. Quit when you can see, when you can be pretty sure, what the limit must be. That's what it means.

SS: But that has nothing to do with convergence!

RH: Right.

SS: Convergence only depends on the last part, the end, the infinite part of the sequence. It has nothing to do with the front part. You can change the first

hundred million terms of the sequence, and that won't affect whether it converges, or what the limit is.

RH: Right! Right! Right! You really are an A student.

SS: I know…. So it all just doesn't make any sense. You teach us some fancy definition of convergence, but when you want to compute a number, you just forget about it and say it converges when common sense, or whatever you call it, says something must be the answer. Even though it might not be the answer at all!

RH: Excellent. I am impressed.

SS: Stop patronizing me. I'm not a child.

RH: Right. I will stop patronizing me, because you are not a child.

SS: You're still doing it.

RH: It's a habit. I can't help it.

SS: Time to break a bad habit.

RH: OK. But seriously, you are absolutely right. I agree with every word you say.

SS: Yes, and you also agree with every word Professor Successful says.

RH: He was telling the truth, but he couldn't make you understand.

SS: All right. *You* make me understand.

RH: It's like theory and practice. Or the ideal and the actual. Or Heaven and Earth.

SS: How is that?

RH: The definition of convergence lives in a theoretical world. An ideal world. Where things can happen as long as we can clearly imagine them. As long as we can understand and agree on them. Like really being positive and arbitrarily small. No number we can write down is positive and arbitrarily small. It has to have some definite size if it is actually a number. But we can imagine it getting smaller and smaller and smaller while staying positive, and we can even express that idea in a formal sentence, so we accept it and work with it. It seems to convey what we want to mean by converging to a limit. But it's only an ideal, something we can imagine, not something we can ever really do.

SS: So you're saying mathematics is all a big fairy tale, a fiction, it doesn't actually exist?

RH: NO! I never said fairy tale or fiction. I said imaginary. Maybe I should have said consensual. Something we can all agree on and work with, because we all understand it the same way.

SS: That's cool. We all. All of you. Does that include me?

RH: Sure. Stay in school a few more years. Learn some more. You'll get into the club. You've got what it takes.

SS: I'm not so sure. I have trouble believing two opposite things at once.

RH: Then how do you get along in daily life? How do you even get out of bed in the morning?

SS: What are you talking about?

RH: How do you know someone hasn't left a bear trap by your bedside that will chop off your foot as soon as you step down?

SS: That's ridiculous.

RH: It is. But how do you know it is?

SS: Never mind how I know. I just know it's ridiculous. And so do you.

RH: Exactly. We know stuff, but we don't always know how we know it. Still, we do know it.

SS: So you're saying, we know that what looks like a limit really is a limit, even though we can't prove it, or explain it, still we know it.

RH: We know it the same way you know nobody has left a bear trap by your bedside. You just know it.

SS: Right.

RH: But it's still possible that you're wrong. It is possible that something ridiculous actually happens. Not likely, not worth worrying about. But not impossible.

SS: Then math is really just like everything else. What a bummer! I like math because it's not like everything else. In math, we know for sure. We prove things. One and one is two. Pi is irrational. A circle is round, not square. For sure.

RH: Then why are you upset? Everything is just fine, isn't it?

SS: Why don't you admit it? If you don't have a proof, you just don't know if L is the limit or not.

RH: That's a fair question. So what is the answer?

SS: Because you really want to *think you know* L is the limit, even if it's not true.

RH: Not that it's not true, just that it *might* not be true.

- -
End of dialogue
- -

Thanks for your kind attention. What is supposed to be the meaning of this performance? What am I getting at? In this talk I am NOT attempting to make a contribution to the "problem of induction." Therefore I may be allowed to omit a review of its 2,500-year literature. I am reporting and discussing what people really do, in practical convergence calculations, and in the process of mathematical discovery. I am going into a discussion of practical knowledge in mathematics, as a kind of real knowledge, even though it is not demonstrative or deductive knowledge. I try to explain why people must do what they do, in order to accomplish what they are trying to accomplish. I will conclude by arguing that the right broader context for the philosophy of mathematical practice is actually the philosophy of pragmatism, as expounded by John Dewey.

But first of all, just this remarkable fact. What we do when we want actual numbers may be totally unjustified, according to our theory and our definition. And even more remarkable—nobody seems to notice, or to worry about it!

Why is that? Well, the definition of convergence taught in calculus classes, as developed by those great men Augustin Cauchy and Karl Weierstrass, seems to actually convey what we want to mean by limit and convergence. It is a great success. Just look at the glorious edifice of mathematical analysis! On the other hand, in specific cases, it often is beyond our powers to give a rigorous error estimate, even when we have an approximation scheme that seems perfectly sound. As in the major problems of three-dimensional continuum mechanics with realistic nonlinearities, such as oceanography, weather prediction, stability of large complex structures like big bridges and airplanes….And even when we could possibly give a rigorous error estimate, it often would require great expenditure of time and labor. Surely it's OK to just use the result of a calculation when it makes itself evident and there's no particular reason to expect any hidden difficulty.

In brief, we are virtually compelled by the practicalities to accept the number that computation seems to give us, even though, by the standards of rigorous logic, there is still an admitted possibility that we may be mistaken. This computational result is a kind of mathematical knowledge! It is practical knowledge, knowledge sound enough to be the basis of practical decisions about things like designing bridges and airplanes—matters of life and death.

In short, I am proclaiming that in mathematics, apart from and distinct from so-called deductive or demonstrative knowledge, there is also ordinary, fallible knowledge, of the same sort as our daily knowledge of our physical environment and our own bodies. "Anything new that we learn about the world involves plausible reasoning, which is the only kind of reasoning for which we care in everyday affairs." (Polya, 1954). This sentence of his makes an implicit separation between mathematics and everyday affairs. But nowadays, in many different ways, for many different kinds of people, mathematics blends into every-day affairs. In these situations, the dominance of plausible over demonstrative reasoning applies even to mathematics itself, as in the daily labors of numerical analysts, applied mathematicians, design engineers… Controlling a rocket trip to the moon is not an exercise in mathematical rigor. It relies on a lack of malice on the part of that Being referred to by Albert Einstein as *der lieber Gott*.

(For fear of misunderstanding, I explain—this is **not** a confession of belief in a Supreme Being. It's just Einstein's poetic or metaphoric way of saying, Nature is not an opponent consciously trying to trick us.)

But it's not only that we have no choice in the matter. It's also that, truth to tell, it seems perfectly reasonable! Believing what the computation tells us is just what people have been doing all along, and (nearly always) it does seem to be OK. What's wrong with that?

This kind of reasoning is sometimes called "plausible," and sometimes called "intuitive." I will say a little more about those two words pretty soon. But I want to draw your attention very clearly to two glaring facts about this kind of plausible or intuitive reasoning. First of all, it is pretty much the kind of reasoning that we are accustomed to in ordinary empirical science, and in technology, and in fact in everyday thinking, dealing with any kind of practical or realistic problem of human life. Secondly, it makes no claim to be demonstrative, or deductive, or conclusive, as is often said to be the essential characteristic of mathematical thinking. We are face to face with mathematical knowledge that is not different in kind from ordinary everyday commonplace human knowledge. Fallible! But knowledge, nonetheless!

Never mind the pretend doubt of philosophical skepticism. We are adults, not infants. Human adults know a lot! How to find their way from bed to breakfast—and people's names and faces--and so forth and so on. This is real knowledge. It is not infallible, not eternal, not heavenly, not Platonic, it is just what daily life depends on, that's all. That's what I mean by ordinary, practical, everyday knowledge. Based not mainly on rigorous demonstration or deduction, but mainly on experience properly interpreted. And here we see mathematical knowledge that is of the same ordinary, everyday kind, based not on infallible deduction, but on fallible, plausible, intuitive thinking.

Then what justifies it in a logical sense? That is, what fundamental presupposition about the world, about reality, lies behind our willingness to commit this logical offense, of believing what isn't proved?

I have already quoted the famous saying of Albert Einstein that supplies the key to unlocking this paradox.

My friend Peter Lax supplied the original German, I only remembered the English translation.

> *Raffinniert ist der lieber Gott, aber boshaft ist Er nicht.*
> *The Lord God is subtle, but He is not malicious.*

Of course, Einstein was speaking as a physicist struggling to unravel the secrets of Nature. The laws of Nature are not always obvious or simple, they are often subtle. But we can believe, we *must* believe, that Nature is not set up to trick us, by a malicious opponent. God, or Nature, must be playing fair. How do we know that? We really *don't* know it, as a matter of certainty! But we must believe it, if we seek to understand Nature with any hope of success. And since *we do have some success* in that search, our belief that Nature is subtle but not malicious is justified.

This problem of inferring generalizations from specific instances is known in logic as "the problem of induction." My purpose is to point out that such generalizations in fact are made, and must be made, not only in daily life and in empirical science, but also in mathematics.

That is, in the practice of mathematics also we must believe that we are not dealing with a malicious opponent who is seeking to trick us. We experiment, we calculate, we draw diagrams. And eventually, using caution and the experience of the ages, we see the light. Gauss famously said, "I have my theorems. Now I have to find my proofs."

But is it not naïve, for people who have lived through the hideous twentieth century, to still hope that God is not malicious? Consider, for example, a people who for thousands of years have lived safely on some atoll in the South Pacific. Today an unforeseen tsunami drowns them all. Might they not curse God in their last breath?

Here is an extensive quote from Leonhard Euler, by way of George Polya. Euler is speaking of a certain beautiful and surprising regularity in the sum of the divisors of the integers.

> This law, which I shall explain in a moment, is in my opinion, so much more remarkable as it is of such a nature that we can be assured of its truth without giving it a perfect demonstration. Nevertheless, I shall present such evidence for it as might be regarded as almost equivalent to a rigorous demonstration….anybody can satisfy himself of its truth by as many examples

as he may wish to develop. And since I must admit that I am not in a position to give it a rigorous demonstration, I will justify it by a sufficiently large number of examples…I think these examples are sufficient to discourage anyone from imagining that it is by mere chance that my rule is in agreement with the truth…The examples that I have just developed will undoubtedly dispel any qualms which we might have had about the truth of my formula….it seems impossible that the law which has been discovered to hold for 20 terms, for example, would not be observed in the terms that follow. (Polya, 1954).

Observe two things about this quote from Euler. First of all, for him the plausible reasoning in this example is so irresistible that it leaves no room for doubt. He is certain that anyone who looks at his examples is bound to agree. Yet secondly, he strongly regrets his inability to provide a demonstration of the fact, and still hopes to find one.

But since he is already certain of the truth of his finding, why ask for a demonstrative proof? The answer is easy, for anyone familiar with mathematical work. The demonstration would not just affirm the truth of the formula, it would show why the formula MUST be true. That is the main importance of proof in mathematics! A plausible argument, relying on examples, analogy and induction, can be very strong, can carry total conviction. But if it is not demonstrative, it fails to show why the result MUST be true. That is to say, it fails to show that it is rigidly connected to established mathematics.

At the head of Chapter V, Polya (1954) placed the following apocryphal quotation, attributed to "the traditional mathematics professor": "When you have satisfied yourself that the theorem is true, you start proving it." (Polya 1954)

This faith---that experience is not a trap laid to mislead us--is the unstated axiom. It lets us believe the numbers that come out of our calculations, including the canned programs that engineers use every day as black boxes. We know that it can sometimes be false. But even as we keep possible tsunamis in mind, we have no alternative but to act as if the world makes sense. We must continue to act on the basis of our experience. (Including, of course, experiences of unexpected disasters.)

Consider this recollection of infantile mathematical research by the famous physicist Freeman Dyson, who wrote in 2004:

One episode I remember vividly, I don't know how old I was; I only know that I was young enough to be put down for an afternoon nap in my crib…I didn't feel like sleeping, so I spent the time calculating. I added one plus a half plus a quarter plus an eighth plus a sixteenth and so on, and I discovered that if you go on adding like this forever you end up with two. Then I tried adding one plus a third plus a ninth and so on, and discovered that if you go on adding like this forever you end up with one and a half. Then I tried one plus a quarter and so on, and ended up with one and a third. So I had discovered infinite series. I don't remember talking about this to anybody at the time. It was just a game I enjoyed playing. (Dyson 2004)

Yes, he knew the limit! How did he know it? Not the way we teach it in high school (by getting an exact formula for the sum of *n* terms of a geometric sequence, and then proving that as *n* goes to infinity, the difference from the proposed limit becomes and remains arbitrarily small.) No, just as when we first show this to tenth-graders, he *saw* that the sums follow a simple pattern that *clearly* is "converging" to 2. The formal, rigorous proof gives insight into the *reason* for a fact we have already seen plainly.

Can we go wrong this way? Certainly we can. Another quote from Euler.

There are even many properties of the numbers with which we are well acquainted, but which we are not yet able to prove; only observations have led us to their knowledge…the kind of knowledge which is supported only by observations and is not yet proved must be carefully distinguished from the truth; it is gained by induction, as we usually say. Yet we have seen cases in which mere induction led to error. Therefore, we should take great care not to accept as true such properties of the numbers which we have discovered by observation and which are supported by induction alone. Indeed, we should use such a discovery as an opportunity to investigate more exactly the properties discovered and to prove or disprove them. (Polya 1954, p. 3)

Notice how Euler distinguishes between "knowledge" and "truth"! He does say "knowledge," not mere "conjecture."

There is a famous theorem of Littlewood concerning a pair of number-theoretic functions PI(x) and Li(x). All calculation shows that Li(x) is greater than PI(x), for x as large as we can calculate. Yet Littlewood proved that *eventually* PI(x) becomes greater than Li(x), and not just once, but infinitely often! Yes, mathematical truth can be very subtle. While trusting it not to be malicious, we must not underestimate its subtlety. (PI(x) is the prime counting function and Li(x) is the logarithmic integral function.)

<u>Mathematical</u> <u>Intuition</u>

We are concerned with "the philosophy of mathematical practice." Mathematical practice includes studying, teaching and applying mathematics. But I suppose we have in mind first of all the discovery and creation of mathematics—mathematical research. We start with Jacques Hadamard, go on to Henri Poincare, move on to George Polya, and then to John Dewey.

Hadamard had a very long life and a very productive career. His most noted achievement (shared independently by de la Vallee Poussin) was proving the logarithmic distribution of the prime numbers. I want to recall a famous remark of Hadamard's. "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there never was any other object for it." (Polya 1980)

From the viewpoint of standard "philosophy of mathematics," this is a very surprising, strange remark. Isn't mathematical rigor—that is, strict deductive reasoning—the most essential feature of mathematics? And indeed, what can Hadamard even mean by this word, "intuition"? A word that means one thing to Descartes, another thing to Kant. I think the philosophers of mathematics have pretty unanimously chosen to ignore this remark of Hadamard. Yet Hadamard did know a lot of mathematics, both rigorous and intuitive. And this remark was quoted approvingly by both Borel and Polya. It seems to me that this bewildering remark deserves to be taken seriously.

Let's pursue the question a step further, by recalling the famous essay "Mathematical Discovery," written by Hadamard's teacher, Henri Poincare. (Poincare 1952) Poincare was one of the supreme mathematicians of the turn of the 19th and 20th century. We've been hearing his name recently, in connection with his conjecture on the 3-sphere, just recently proved by Grisha Perelman of St. Petersburg. Poincare was not only a great mathematician, he was a brilliant essayist. And in the essay "Mathematical Discovery," Poincare makes a serious effort to explain mathematical intuition. He tells the famous story of how he discovered the Fuchsian and Theta-Fuchsian functions. He had been struggling with the problem unsuccessfully when he was distracted by being called up for military service:

> At this moment I left Caen, where I was then living, to take part in a geological conference arranged by the School of Mines. The incidents of the journey made me forget my mathematical work. When we arrived at Coutances, we got into a bus to go for a drive, and, just as I put my foot on the step the idea came to me, though nothing in my former thoughts seemed to have prepared me for it, that the transformations I had used to define Fuchsian functions were identical with those of non-Euclidean geometry. I made no verification, and had no time to do so, since I took up the conversation again as soon as I had sat down in the bus, but I felt absolute certainty at once. When I got back to Caen, I verified the result at my leisure to satisfy my conscience. (Poincare 1952)

What a perfect example of rigor "merely legitimizing the conquests of intuition"! How does Poincare explain it? First of all, he points out that some sort of subconscious thinking must be going on. But if it is subconscious, he presumes it must be running on somehow at random. How unlikely, then, for it to find one of the very few good combinations, among the huge number of useless ones! To explain further, he writes:

> If I may be permitted a crude comparison, let us represent the future elements of our combinations as something resembling Epicurus's hooked atoms. When the mind is in complete repose these atoms are immovable; they are, so to speak, attached to the wall…On the other hand, during a period of apparent repose, but of unconscious work, some of them are detached from the wall and set in motion. They plough through space in all directions, like a swarm of gnats, for instance, or, if we prefer a more learned comparison, like the gaseous molecules in the kinetic theory of gases. Their mutual impacts may then produce new combinations. (Poincare 1952)

The preliminary conscious work "detached them from the wall." The mobilized atoms, he speculated, would therefore be "those from which we might reasonably expect the desired solution….My comparison is very crude, but I cannot well see how I could explain my thought in any other way." (Poincare 1952)

What can we make of this picture of "Epicurean hooked atoms," flying about somewhere—in the mind? A striking, suggestive image, but one not subject even in principle to either verification or disproof. Our traditional philosopher remains little interested. This is fantasy or poetry, not science or philosophy. But this is Poincare! He knows what he's talking about. He has something important to tell us. It's not easy to understand, but let's take him seriously, too.

To be fair, Poincare proposed his image of gnats or gas molecules only after mentioning the possibility that the subconscious is actually more intelligent than the conscious mind. But this, he said, he was not willing to contemplate. However, other writers have proposed that the subconscious is less inhibited, more imaginative, more *creative* than the conscious. (Poincare's essay title is sometimes translated as "Mathematical Creation" rather than "Mathematical Discovery.") David Hilbert supposedly once said of a student who had given up mathematics for poetry, "Good! He didn't have enough imagination for mathematics." Hadamard (1949) carefully analyzes the role of the subconscious in mathematical discovery and its connection with intuition. It is time for contemporary cognitive psychology to pay attention to Hadamard's insights. See the reference below about current scientific work on the creative power of the subconscious

Before going on, I want to mention the work of Carlo Cellucci, Emily Grosholz and Andrei Rodin. Cellucci strongly favors plausible reasoning, but he rejects intuition. However, the intuition he rejects isn't what I'm talking about. He's rejecting the old myth, of an infallible insight straight into the Transcendental. Of course I'm not advocating that outdated myth. Emily Grosholz, on the other hand, takes intuition very seriously. Her impressive historical study of what she calls "internal intuition" is in the same direction as my own thinking being presented here. Andrei Rodin has recently written a remarkable historical study of intuition (Rodin 2010). He shows that intuition played a central role in Lobachevski's non-Euclidean geometry, in Zermelo's axiomatic set theory, and even in up-to-date category theory. (By the way, in category theory he could also have cited the standard practice of proof by "diagram chasing" as a blatant example of intuitive, visual proof.) His exposition makes the indispensable role of intuition clear and convincing. But his use of the term "intuition" remains, one might say, "intuitive," for he offers no definition of the term, nor even a general description, beyond his specific examples.

Polya

My most helpful authority is George Polya. I actually induced Polya to come give talks in New Mexico, for previously, as a young instructor, I had met him at Stanford where he was an honored and famous professor. Polya was not of the stature of Poincare or Hilbert, but he was still one of the most original, creative, versatile and influential mathematicians of his generation. His book with Gabor Szego (Polya-Szego 1970) made them both famous. It expounds large areas of advanced analytic function theory by means of a carefully arranged, graded sequence of problems with hints and solutions. Not only does it teach advanced function theory, it also teaches problem-solving. And by example, it shows how to teach mathematics by teaching problem-solving. Moreover, it implies a certain view of the nature of mathematics, so it is a philosophical work in disguise.

Later, when Polya wrote his very well-known, influential books on mathematical heuristic, he admitted that what he was doing could be regarded as having philosophical content. He writes, "I do not know whether the contents of these four chapters deserve to be called philosophy. If this is philosophy, it is certainly a pretty low-brow kind of philosophy, more concerned with understanding concrete examples and the concrete behavior of people than with expounding generalities." (Polya 1954 page viii) Unpretentious as Polya was, he was still aware of his true stature in mathematics. I suspect he was also aware of the philosophical depth of his heuristic. He played it down because, like most mathematicians (I can only think of one or two exceptions), he disliked controversy and arguing, or competing for the goal of becoming top dog in some cubbyhole of academia. The Prince of Mathematicians, Carl Friedrich Gauss, kept his monumental discovery of non-Euclidean geometry hidden in a desk drawer to avoid stirring up the Boeotions, as he called them,—meaning the post-Kantian German philosophy professors of his day. (In ancient Athens, "Boeotian" was slang for "ignorant country hick.") Raymond Wilder was a leading topologist who wrote extensively on mathematics as a culture. He admitted to me that his writings

implicitly challenged both formalism and Platonism. "Why not say so?" I asked. Because he didn't relish getting involved in philosophical argument.

Well, how does Polya's work on heuristic clarify mathematical intuition? Polya's heuristic is presented as pedagogy. Polya is showing the novice how to solve problems. But what is "solving a problem"? In the very first sentence of his preface Polya (1980) writes, "Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable. Solving problems is the specific achievement of intelligence, and intelligence is the specific gift of mankind: solving problems can be regarded as the most characteristically human activity." "Problem" is simply another word for any project or enterprise which one cannot immediately take care of with the tools at hand. In mathematics, something more than a mere calculation. Showing how to solve problems amounts to showing how to do research!

Polya's exposition is never general and abstract, he always uses a specific mathematical problem for the heuristic he wants to teach. His mathematical examples are always fresh and attractive. And his heuristic methods? First of all, there is what he calls "induction." That is, looking at examples, as many as necessary, and using them to *guess* a pattern, a generalization. But be careful! Never just believe your guess! He insists that you must "Guess and test, guess and test." Along with induction, there is analogy, and there is making diagrams, graphs and every other kind of picture, and then reasoning or guessing from the picture. And finally, there's the "default hypothesis of chance"—that an observed pattern is mere coincidence.

(Mark Steiner has the distinction among philosophers of paying serious attention to Polya. After quoting at length from Polya's presentation of Euler's heuristic derivation of the sum of a certain infinite series, Steiner comes to an important conclusion: in mathematics we can have knowledge without proof! Based on the testimony of mathematicians, he even urges philosophers to pay attention to the question of mathematical intuition.)

I have two comments about Polya's heuristic that I think he would have accepted. First of all, the methods he is presenting, by means of elementary examples, are methods he used himself in research. "In fact, my main source was my own research, and my treatment of many an elementary problem mirrors my experience with advanced problems." (Polya, 1980, page xi). In teaching us how to solve problems, he's teaching us about mathematical practice: How it works. What is done. To find out "What is mathematics?" we must simply reinterpret Polya's examples as descriptive rather than pedagogical.

Secondly, with hardly any stretching or adjustment, the heuristic devices that he's teaching can be applied for any other kind of problem-solving, far beyond mathematics. He actually says that he is bringing to mathematics the kind of thinking ordinarily associated with empirical science. But we can go further. These ways of thinking are associated with every kind of problem-solving, in every area of human life! Someone needed to get across a river or lake and had the brilliant idea of "a boat"—whether it was a dugout log or a birch bark canoe. Someone else, needing shelter from the burning sun in the California Mojave, thought of digging a hole in the ground. And someone else, under the piercing wind of northern Canada, thought of making a shelter from blocks of ice.

How does anyone think of such a thing, solve such a concrete problem? By some kind of analogy with something else he has seen, or perhaps been told about. By plausible thinking. And often by a sudden insight that arises "from below."

Intuitively, you might say.

Mental Models

It often happens that a concrete problem, whether in science or in ordinary daily life, is pressing on the mind, even when the particular materials or objects in question are not physically present. You keep on thinking about it, while you're walking, and when you're waking from sleep. Productive thought commonly takes place, in the absence of the concrete objects or materials being thought about. This thinking about something not present to sight or touch can be called "abstract thinking." Abstract thinking about a concrete object. How does that work? How can our mind/brain think productively about something that's not there in front of the eyes? Evidently, it operates on something mental, what we may call a mental image or representation. In the current literature of cognitive psychology, one talks about "a mental model." In this article, I use the term "mental model" to mean a mental structure built from recollected facts (some expressed in words), along with an ensemble of sensory memories, perhaps connected, as if by walking around the object in question, or by imagining the object from underneath or

above, even if never actually seen in these views. A rich complex of connected knowledge and conjecture based on verbal, visual, kinesthetic, even auditory or olfactory information, but simplified, to exclude irrelevant details. Everything that's helpful for thinking about the object of interest when the object isn't here. Under the pressure of a strong desire or need to solve a specific problem, we assemble a *mental model* which the mind-brain can manipulate or analyze.

Subconscious thinking is not a special peculiarity of mathematical thinking, but a common, taken-for-granted, part of every-day problem-solving. When we consider this commonplace fact, we aren't tempted to compare it to a swarm of gnats hooking together at random. No, we assume, as a matter of course, that this subconscious thinking follows rules, methods, habits or pathways, that somehow, to some extent, correspond to the familiar plausible thinking we do when we're wide awake. Such as thinking by analogy or by induction. After all, if it is to be productive, what else can it do? If it had any better methods, then those better methods would also be what we would follow in conscious thinking! And subconscious thinking in mathematics must be much like subconscious thinking in any other domain, carrying on plausible reasoning as enunciated by various writers, above all by George Polya. This description of subconscious thinking is not far from Michael Polanyi's "tacit dimension."

When applied to everyday problem solving, all this is rather obvious, perhaps even banal. My goal is to clarify mathematical intuition, in the sense of Hadamard and Poincare. "Intuition" in the sense of Hadamard and Poincare is a fallible psychological experience that has to be accounted for in any realistic philosophy of mathematics. It simply means guesses or insights attained by plausible reasoning, either fully conscious or partly subconscious. In this sense it is a specific phenomenon of common experience. It has nothing to do with the ancient mystical myth of an intuition that surpasses logic by making a direct connection to the Transcendental.

The term "abstract thinking" is commonplace in talk about mathematics. The triangle, the main subject of Euclidean geometry, is an abstraction, even though it's idealized from visible triangles on the blackboard. Thinking of a physical object in its absence, like a stream to be crossed or a boat to be imagined and then built, is already "abstract" thinking, and the word "abstract" connects us to the abstract objects of mathematics.

Let me be as clear and simple as I can be about the connection. After we have some practice drawing triangles, we can think about triangles, we discover properties of triangles. We do this by reasoning about mental images, as well as images on paper. This is already abstract thinking. When we go on to regular polygons of arbitrarily many sides, we have made another departure. Eventually we think of the triangle as a 2-simplex, and abstract from the triangle to the n-simplex. For n = 3 this is just the tetrahedron, but for n = 4 or 5 or 6, it is something never yet seen by human eye. Yet these higher simplexes also can become familiar, and, as it were, concrete-seeming. If we devote our waking lives to thinking about them, then we have some kind of "mental model" of them. Having this mental model, we can access it, and thereby we can reason intuitively—have intuitive insights—by which I mean simply insights not based on consciously known reasoning. An "intuition" is then simply a belief (possibly mistaken!) arising from internal inspection of a mental image or representation—a "model." It may be assisted by subconscious plausible reasoning, based on the availability of that mental image. We do this in practical life. We do it in empirical science, and in mathematics. In empirical science and ordinary life, the image may stand for either an actual object, a physical entity, or a potential one that could be realized physically. In mathematics, our mental model is sometimes idealized from a physical object—for example, from a collection of identical coins or buttons when we're thinking about arithmetic. But in mathematics we also may possess a mental model with no physical counterpart. For example, it is generally believed that Bill Thurston's famous conjectures on the classification of four-manifolds were achieved by an exceptional ability, on the part of Thurston, to think intuitively in the fourth dimension. Perhaps Grisha Perelman was also guided by some four-dimensional intuition, in his arduous arguments and calculations to prove the Thurston program.

To summarize, mathematical intuition is an application of conscious or subconscious heuristic thinking of the same kind that is used every day in ordinary life by ordinary people, as well as in empirical science by scientists. This has been said before, by both Hadamard and Polya. In fact, this position is similar to Kurt Godel's, who famously wrote, "I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception." Why, indeed? After all, both are fallible, but both are plausible, and must be based on plausible reasoning.

For Godel, however, as for every writer in the dominant philosophy of mathematics, intuition is called in only to justify the axioms. Once the axioms are written down, the role of mathematical intuition is

strictly limited to "heuristic"—to formulating conjectures. These await legitimation by deductive proof, for only deductive proof can establish "certainty." Indeed, this was stated as firmly by Polya as by any analytic philosopher. But what is meant by "mathematical certainty"? If it simply means deductive proof, this statement is a mere circular truism. However, as I meant to suggest by the little dialog at the beginning of this paper, there is also *practical* certainty, even within mathematics! We are certain of many things in ordinary daily life, without deductive proof, and this is also the case in mathematics itself. Practical certainty is a belief strong enough to lead to serious practical decisions and actions. For example, we stake our lives on the numerical values that went into the engineering design of an Airbus or the Golden Gate Bridge. Mainstream philosophy of mathematics does not recognize such practical certainty. Nevertheless, it is an undeniable fact of life.

It is a fact of life not only in applied mathematics but also in pure mathematics. For example, the familiar picture of the Mandelbrot set, a very famous bit of recent pure mathematics, is generated by a machine computation. By definition, any particular point in the complex plane is inside the Mandelbrot set if a certain associated iteration stays bounded. If that iteration at some stage produces a number with absolute value greater than two, then, from a known theorem, we can conclude that the iteration goes to infinity, and the parameter point in question is *outside* the Mandelbrot set. What if the point is *inside* the Mandelbrot set? No finite number of iterations in itself can guarantee that the iteration will never go beyond absolute value 2. If we do eventually decide that it looks like it will stay bounded, we may be right, but we are still cheating. This decision is opportunistic and unavoidable, just as in an ordinary calculation about turbulent flow.

Computation (numerics) is accepted by purists only as a source of conjectures awaiting rigorous proof. However, from the pragmatic, non-purist viewpoint, if numerics is our guide to action, then it is in effect a source of knowledge. Dewey called it "warranted assertibility." (Possibly even a "truth." A "truth" that remains open to possible reconsideration.)

Another example from pure mathematics appeared on John Baez's blog (Baez 2010) where it is credited to Sam Derbyshire. His pictures plot the location in the complex plane of the roots of all polynomials of degree 24 with coefficients plus one or minus one. The qualitative features of these pictures are absolutely convincing— i.e., impossible to disbelieve. Baez wrote, "That's 2^{24} polynomials, and about 24×2^{24} roots — or about 400 million roots! It took Mathematica 4 days to generate the coordinates of the roots, producing about 5 gigabytes of data." (Figure 1 shows the part of the plot in the first quadrant, for complex roots with non-negative real and imaginary parts.)
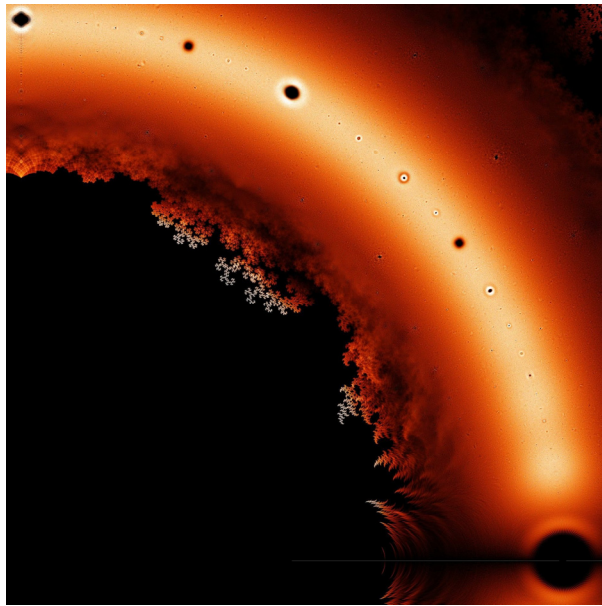
Figure 1.

There is more information in this picture than can even be formulated as conjectures, let alone seriously attacked with rigor. Since indeed we cannot help believing them (perhaps only believing with 99.999%

credence) then (pragmatically) we give them "warranted assertibility," just like my belief that I can walk out my door without encountering sudden death in one form or another. The distinction between rigorous math and plausible math, pure math and applied math, etc, becomes blurred. It is still visible, certainly, but not so sharp. It's a little fuzzy. Purely computational results in pure mathematics, when backed up by sophisticated checking against a relevant theory, have a factual status similar to that of accepted facts from empirical science. The distinction between what is taken to be "known," and what is set aside as merely guessed or "conjectured", is not so cut and dried as the usual discussions claim to believe.

Mental Models Subject to Social Control

"Plausible" or "heuristic" thinking is applied, either consciously or subconsciously, to mental models. These mental models may correspond to tangible or visible physical objects in ordinary life and empirical science. Or they may not correspond to any such things, but may be pure mental representations, as in much of contemporary analysis, algebra, and even geometry. By pure mental models I mean models not obtained directly by idealization of visual or other sensual experience.

But what controls these mental models? If they have no physical counterpart, what keeps them from being wildly idiosyncratic and incommunicable? What we have omitted up to this point, and what is the crux of the matter: mathematical images are not private, individual entities. From the origin of mathematics in bartering, buying and selling, or in building the Parthenon and the Pyramids, this subject has always been a social, an "inter-subjective" activity. Its advances and conquests have always been validated, corrected and absorbed in a social context—first of all, in the classroom. Mathematicians can and must talk to each other about their ideas. One way or the other, they do communicate, share and compare their conceptions of mathematical entities, which means precisely these models, these images and representations I have been describing. Discrepancies are recognized and worked out, either by correcting errors, reconciling differences, or splitting apart into different, independent pathways. Appropriate terminology and symbols are created as needed.

Mathematics depends on a mutually acknowledging group of competent practitioners, whose consensus decides at any time what is regarded as correct or incorrect, complete or incomplete. That is how it always worked, and that is how it works today. This was made very clear by the elaborate process in which Perelman's proposed proof of the Thurston program (including the Poincare conjecture) was vetted, examined, discussed, criticized and finally accepted by the "Ricci flow community," and then by its friends in the wider communities of differential geometry and low-dimensional topology, and then by the prize committees of the Fields Medal and the Clay Foundation.

Thus, when we speak of a mathematical concept, we speak not of a single isolated mental image, but rather of a family of mutually correcting mental images. They are privately owned, but publicly checked, examined, corrected, and accepted or rejected. This is the role of the mathematical research community, how it indoctrinates and certifies new members, how it reviews, accepts or rejects proposed publication, how it chooses directions of research to follow and develop, or to ignore and allow to die. All these social activities are based on a necessary condition: that the individual members have mental models that fit together, that yield the same answers to test questions. A new branch of mathematics is established when consensus is reached about the possible test questions and their answers. That collection of possible questions and answers (not necessarily explicit) becomes the means of accepting or rejecting proposed new members.

If two or three mathematicians do more than merely communicate about some mathematical topic, but actually collaborate to dig up new information and understanding about it, then the matching of their mental models must be even closer. They may need to establish a congruence between their subconscious thinking about it as well as their conscious thinking. This can be manifested when they are working together, and one speaks the very thought that the partner was about to speak.

And to the question "What is mathematics?" the answer is "It is socially validated reasoning about these mutually congruent mental models."

What makes mathematics possible? It is our ability to create mental models which are "precise," meaning simply that they are part of a shared family of mutually congruent models. In particular, such an image as a line segment, or two intersecting line segments, and so on. Or the image of a collection of mutually interchangeable identical objects (ideal coins or buttons). And so on. To understand better how that ability exists, both psychologically and neurophysiologically, is a worthy goal for empirical science. The current interactive flowering of developmental psychology, language acquisition, and cognitive

neuroscience shows that this hope is not without substance. (See, e.g., Carey, Dehaene, Johnson-Laird, Lakoff/Nunez, Zwaan.)

The existence of mathematics shows that the human mind is capable of creating, refining, and sharing such precise concepts, which admit of reasoning that can be shared, mutually checked, and confirmed or rejected. There are great variations in the vividness, completeness, and connectivity of different mental images of the "same" mathematical entity as held by different mathematicians. And, also great variations in their ability to concentrate on that image and squeeze out all of its hidden information. Recall that well-known mathematician, Sir Isaac Newton. When asked how he made his discoveries in mathematics and physics, he answered simply, "By keeping the problem constantly before my mind, until the light gradually dawns." Indeed, neither meals nor sleep were allowed to interrupt Newton's concentration on the problem. Mathematicians are notoriously absent-minded. Their concentration, which outsiders call "absent-mindedness," is just the open secret of mathematical success.

Their reasoning is qualitatively the same as the reasoning carried out by a hunter tracking a deer in the Appalachian woodland a thousand years ago. "If the deer went to the right, I would see a hoof print here. But I don't see it. There's only one other way he could have gone. So he must have gone to the left." Concrete deductive reasoning, which is the basis for abstract deductive reasoning.

To sum up! I have drawn a picture of mathematical reasoning which claims to make sense of intuition according to Hadamard and Poincare, and which interprets Polya's heuristic as a description of ordinary practical reason, applied to the abstract situations and problems of mathematics, working on mental models in the same way that ordinary practical reasoning *in absentia* works on a mental model. (We may assist our mental images by creating images on paper—drawing pictures—that to some extent capture crucial features of the mental images.)

<u>Dewey</u> <u>and</u> <u>Pragmatism</u>

Before bringing in John Dewey, the third name promised at the beginning, I must first mention Dewey's precursor in American pragmatism, Charles Saunders Peirce, for Peirce was also a precursor to Polya. To deduction and induction, Peirce added a third logical operation, "abduction," something rather close to Polya's "intelligent guessing."

The philosophy of mathematics as practiced in many articles and books is a thing unto itself, hardly connected either to living mathematics or to general philosophy. But how can it be claimed that the nature of mathematics is unrelated to the general question of human knowledge? There has to be a fit between your beliefs about mathematics and your beliefs about science and about the mind. I claim that Dewey's pragmatism offers the right philosophical context for the philosophy of mathematical practice to fit into. I am thinking especially of *Logic—the Theory of Inquiry*. For Dewey, "inquiry" is conceived very broadly and inclusively. It is "the controlled or directed transformation of an indeterminate situation into one that is so determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole." So broadly understood, inquiry is one of the primary attributes of our species. Only because of that trait have we survived, after we climbed down from the trees. I cannot help comparing Dewey's definition of inquiry with Polya's definition of problem solving. It seems to me they are very much pointing in the same direction, taking us down the same track. With the conspicuous difference that, unlike Dewey, Polya is concise and memorable.

Dewey makes a radical departure from standard traditional philosophy (following on from his predecessors Peirce and William James, and his contemporary George Herbert Mead). He does not throw away the concept of truth, but he gives up the criterion of truthfulness, as the judge of useful or productive thinking. Immanuel Kant made clear once and for all that while we may know the truth, we cannot know for certain that we do know it. We must perforce make the best of both demonstrative and plausible reasoning. This seems rather close to "warranted assertibility," as Dewey chooses to call it. But Polya or Poincare are merely talking about mathematical thinking, Dewey is talking about human life itself.

What about deductive thinking? From Dewey's perspective of "warranted assertibility," deductive proof is not a unique, isolated mode of knowledge. A hunter tracking a deer in the North American woodland a thousand years ago concluded, "So it must have gone to the left." Concrete deductive reasoning, the necessary basis of theoretical deductive reasoning. And it never brings certainty, simply because any particular deductive proof is a proof in practice, not in principle. Proof in practice is a human artifact, and so it can't help leaving some room for possible question, even possible error. (And that remains true of machine proof, whether by analog, digital, or quantum computer. What changes is the

magnitude of the remaining possible error and doubt, which can never vanish finally.) In this way, we take our leave, once and for all, of the Platonic ideal of knowledge—indubitable and unchanging—in favor, one might say, of an Aristotelian view, a scientific and empirical one. And while deductive proof becomes human and not divine or infallible, non-deductive plausible reasoning and intuition receive their due as a source of knowledge in mathematics, just as in every other part of human life. Dewey's breadth of vision-- seeing philosophy always in the context of experience, that is to say of humanity at large--brings a pleasant breath of fresh air into this stuffy room.

Nicholas Rescher (2001) writes,

> The need for understanding, for 'knowing one's way about,' is one of the most fundamental demands of the human condition….Once the ball is set rolling, it keeps on going under its own momentum—far beyond the limits of strictly practical necessity….The discomfort of unknowing is a natural aspect of human sensibility. To be ignorant of what goes on about us is almost physically painful for us…The requirement for information, for cognitive orientation within our environment, is as pressing a human need as for food itself. (Rescher 2001)

The need for understanding is often met by a story of some kind. In our scientific age, we expect a story built on a sophisticated experimental-theoretical methodology. In earlier times, no such methodology was available, and a story might be invented in terms of gods or spirits or ancestors. In inventing such explanations, whether in what we now call mythology or what we now call science, people have always been guided by a second fundamental drive or need. Rescher does not mention it, but Dewey does not leave it out. That is the need to impart form, beauty, appealing shape or symmetry to our creations, whether they be straw baskets, clay pots, wooden spears and shields, or geometrical figures and algebraic calculations. In *Art as Experience* Dewey shows that the esthetic, the concern for pleasing form, for symmetry and balance, is also an inherent universal aspect of humanity. In mathematics this is no less a universal factor than the problem-solving drive. In "Mathematical Discovery" Poincare takes great pains to emphasize the key role of esthetic preference in the development of mathematics. We prefer the attractive looking problems to work on, we strive for diagrams and graphs that are graceful and pleasing. Every mathematician who has talked about the nature of mathematics has portrayed it as above all an art form. So this is a second aspect of pragmatism that sheds light on mathematical practice.

Rescher's careful development omits mathematical knowledge and activity. And Dewey himself doesn't seem to have been deeply interested in the philosophy of mathematics, although there are interesting pages about mathematics in *Logic*, as well as in his earlier books *The Quest for Certainty* and *The Psychology of Number*. He may have been somewhat influenced by the prevalent view of philosophy of mathematics as an enclave of specialists, fenced off both from the rest of philosophy and from mathematics itself.

But if we take these pragmatist remarks of Rescher's seriously and compare them to what mathematicians do, we find a remarkably good fit. Just as people living in the woodland just naturally want to know and find out about all the stuff they see growing—what makes it grow, what makes it die, what you can do with it to make a canoe or a tent—so people who get into the world of numbers, or the world of triangles and circles, just naturally want to know how it all fits together, and how it can be stretched and pulled this way or that. "Guess and test," is the way George Polya put it. "Proofs and refutations" was the phrase used by another mathematically trained Hungarian philosopher, following up an investigation started by Polya. Whichever way you want to put it, it is nothing more or less than the exploration of the mathematical environment, which we create and expand as we explore it. We are manifesting in the conceptual realm one of the characteristic behaviors of *homo sapiens*.

Even though we lack claws or teeth to match beasts of prey, or fleetness to overtake the deer, or .swimming, paddling and sailing, cooking and brewing and baking and preserving, and we expanded our social groups from families to clans to tribes to kingdoms to empires. All this by "inquiry," or by problem-solving. Dewey shows that this inquiry is an innate specific drive or need of our species. It was manifested when, motivated by practical concerns, we invented counting and the drawing of triangles. That same drive, to find projects, puzzles, and directions for growth, to make distinctions and connections, and then again make new distinctions and new connections, has resulted in the Empire of mathematics we inhabit today.

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Hersh

Transcriptions, Mathematical Cognition, and Epistemology

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Abstract: The epistemologies researchers bring to their studies mediate not only their theories but also their methods, including what they select from their data sources to present the findings on which claims are based. Most articles reduce mathematical knowing to linguistic/mathematical structures, which, in the case of embodiment/enactivist theories, undermines the very argument about the special nature of mathematical knowing. The purpose of this study is to illustrate how different transcriptions of mathematics lessons are generally used to support different epistemologies of mathematical knowing/competence. As part of our third illustration, we provide embodiment/enactivist researchers with an innovative means of representing classroom interactions that are more consistent with their theoretical claims. We offer a comprehensive transcription, which, when treated by readers in the way musicians treat their scores, allow them to enact and feel the knowledge that the article is about.

Keywords: Transcribing • Epistemology • Enactivism • Performance

1. Introduction

1.1. The problematic: theories and research data

Our theories about knowing and learning mediate how we look at the world generally, and at the data sources we collect as part of mathematics education research more specifically. The currently most dominant theories have come to us through a lineage of work from Kant to Piaget and (radical, social) constructivism. In these theories, knowing is thought of in terms of a mind that constructs itself (e.g., von Glasersfeld, 1991), or as a "collection of minds" that first construct knowledge together before constructing it individually (e.g., Cobb, 1999). More recently, embodiment (Lakoff & Núñez, 2000) and enactivist theories (Davis, 1995) have been proposed to mathematics educators. In these theories, knowing is not supposed to be reduced to the mind that constructs itself but is to be considered in terms of mind that arises from intentional bodily engagements with the world.[1] Embodiment theorists tend to focus on the relation between sensorimotor schemas – e.g., the source-path-goal schema – and similar structures in language. The transition between the two, that is, the transformation, is said to occur by metaphorization processes. Empirical support for each of these theories is provided by particular data produced in and

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[1] It has been shown that the very framing of embodiment/enactivist theories in terms of *intentions*, *material body*, and *world* gets us further into metaphysics and body mind distinctions rather than out of it (Henry, 2003). A way of framing a non-metaphysical theory of mathematical cognition has been proposed (Roth, 2010a, in press).

through mathematics education research, presented in the form of transcriptions of communicative situations – e.g., clinical interviews, classroom conversations, or written tests. In this article, we show that some of these transcription forms do not support the theories they are intended to support and other forms of transcriptions contain interactional detail that some but not other theories can explain. In the following section, we provide an example of enactivist/embodiment theories.

1.2. Data and epistemology: the case of enactivist/embodiment theories

Enactivist scholars tend to encapsulate their theories around the diction *knowing is doing*. Many mathematics educators do not buy into enactivist/embodiment theories. Thus, for example, one critic (rightfully) questions the sources of the metaphors offered by Lakoff and Núñez: "Do they really form a natural basis for our thinking, or are they the logical creations of the authors, who are trying to develop a consistent epistemology" (Dubinsky, 1999, p. 557). For embodiment/enactivist theories to become reasonable alternatives to going conceptualizations of mathematical knowing – those fundamentally based in Kant's analyses – they have to show that there is a *necessary* link between moving about (and sensing) in the world, on one hand, and understanding mathematical concepts, on the other. However, the nature of their data and way in which embodiment/enactivist mathematics educators present these works against them. This idea constitutes the starting point of the present article.

To sharpen the problematic of the relation between data and theory, consider the following example. The paper that introduced many mathematics educators to embodiment presents the mathematical idea of continuity as a case study (Núñez, Edwards, & Matos, 1999). Paradoxically, their article consists entirely of text and mathematical formalism – e.g., the statements "$\lim_{x \to a} f(x) = L$" and "if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$." In that article, therefore, knowing mathematical continuity is reduced to language and language-like formulations. That is, despite the rhetoric about the embodiment of mathematics, the authors only appeal to our mind and obliquely point to the embodied dimensions of knowing without *directly* addressing or appealing to them. Moreover, it may be that culturally and historically these formulations have been derived from embodied experiences; but this does not *necessitate* similar experiences on the part of mathematics learners who live today (Husserl, 1997).

It is not surprising, therefore, that mathematics educators ask what embodiment theories – to take but one example – have to offer to the teaching and learning of mathematics (Dubinsky, 1999). Children may learn about *cylinders* without having had the same experiences as early Greek mathematicians and mathematics learners, for whom the concept arose from the experience of rolling objects metaphorically extended to the concept "cylinder." The ancient Greek used this experience, associated with the word *kúlindros*, roller, derived from the verb *kulínd-ein,* to roll to develop the mathematical-deal *concept* of the cylinder. In fact, the Greek word has even more ancient roots in the Proto-Indo-Germanic *(s)kel-*, to bend, crooked. That is, for the Greek, the word *kúlindros* (cylinder) was an active rather than a dead metaphor, a term that has been carried (Gr. *férein*) across (Gr. *meta*) from the everyday experience of rolling things to the mathematical entity.

In our viewpoint, the main argument of embodiment/enactivist researchers would be much stronger if the data they produce actually forced readers to mobilize forms of

knowing that cannot be reduced to linguistic/mathematical structures. Similarly, perception constitutes a form of consciousness that reflects reality differently than intellectual (verbal) consciousness, leading to the fact that the former cannot be reduced to latter (Merleau-Ponty, 1945; Vygotsky, 1986). A verbal transcription of an event, therefore, never renders those aspects in which perceptual consciousness differ from intellectual consciousness. On the other hand, more advanced forms of transcriptions just might exhibit structures that (radical, social) constructivist can no longer explain, or for which they need to develop extensions of their theory so that it continues to provide a viable account of mathematical knowing.

1.3. Purpose

In this article, we present different approaches to representing mathematical communication (knowing) and we show how the resulting transcriptions offer different forms of data that support some but not other epistemologies. Besides, , and most relevant to our own work and theoretical commitments, we develop a means for embodiment/enactivist mathematics educators to show which aspects of the body are *necessary* for understanding formal mathematics. Our representations of lesson fragments relate to knowing mathematics as musical scores relate to the performance of a symphony. That is, we suggest that if someone is capable to read a score, this does not mean that the person knows, or knows how to play, the music with an instrument. This reader does not inherently know what the person referred to in the score has exhibited in his/her performance. Just as the (practical) performance of the music cannot be reduced to the symbols of the score (notes, figures, etc.), the mathematical performance cannot be reduced to the words that appear in transcriptions.

2. Knowing and representations thereof

Historians (e.g., Kuhn, 1970) and sociologists of scientific and mathematical knowing (e.g., Barnes, Bloor, & Henry, 1996) have shown that there exists an interactional relationship between theories and observation. This relationship has been captured in the diction that "If observation is 'theory-laden,' theory is 'observation-laden'" (p. 92). Such is not only the case for mathematics and science but also for research in mathematics (and science) education. Our (authors') own commitments are to embodiment and enactivist theories of cognition. But we have realized only of late that the real issue in the debate may be due to the nature of the data: enactivist/embodiment researchers do not produce the kind of data that would show the *necessity* of the body in and to mathematical knowing. We therefore present the background to the present problematic of data and theory in terms of our own theoretical commitments.

2.1. Practical understanding and formal knowledge

On both cultural-historical and ontogenetic scales, knowing-how in (practical understanding of) the world precedes formal theories. Thus, everyday understandings and the measurement of objects and places preceded and constituted the grounds of formal geometry in ancient Greece (Husserl, 1939). Children learn to speak their mother tongue without knowing any formal grammar whatsoever. High-performance athletes, such as football or soccer players, do not have to know an ounce of physics to make a successful pass even under the most adverse, weather-related conditions. Practical mastery generally

does not require symbolic mastery. However, when tennis or golf players do want to change the way in which they play their balls, then they often seek a different form of understanding. They think *about* their play; and this thinking requires signs for a mediated access to their practical understanding. Yet it is also widely known that while they are conscious of their play, these athletes tend to play worse than they have done before or will afterward. That is, symbolic (conscious) access interferes with the playing itself, which tends to be based on unmediated relations between players and their lifeworld. However, the symbolic access to practice is required to *think about* what one is doing.

In the history of human practices, these symbolic forms of knowing – i.e., symbolic mastery – began to separate from the practical understanding of the world. Thus, for example, formal architecture began to develop and separate from master craftsmanship around the time that the great Gothic cathedrals were built (Turnbull, 1993). Prior to the separation, the craftsmen had no plans or knowledge of structural mechanics. The cathedrals were built based on the bodily embodied design skills of the master artisans, working with templates, strings, and embodied geometry in the context of a community of artisans. From the occupation of master craftsmen evolved architects, and craftsmen no longer did design. The new architects concentrated on designing buildings, including the ways in which the strength and stability of the walls had to be increased to make them larger and larger. There is therefore a separation between practical mastery of building cathedrals and symbolic mastery underlying the construction thereof. In a similar way, the peoples around the world developed and played different forms of music before developing means of representing music in a formal way (Treitler, 1982). The point that enactivist/embodiment and practice theorists make is that formal mastery requires some form of practical (embodied and enacted) understanding of the world that is *always present* and in fact *required* by formal mastery. However, it is precisely this latter part that scholars in the field *do not* make apparent and evident in their presentations.

In the theory of textual interpretation, it is well known that explanation *requires* practical understanding of the world (e.g., Ricœur, 1991). Thus, the practice of textual interpretation involves two moments that mutually constitute each other. On one hand, there is practical understanding that we evolve while and through participating in the world. For example, children learn to speak a language and to count before knowing grammar or arithmetic. On the other hand, there is explanation. The point theorists of hermeneutics make is that explanation *cannot* occur without practical understanding, which precedes, accompanies, and concludes explanation. That is, practical understanding completely envelops explanation; but it is through explanation that practical understanding is developed. Thus, children already have to speak language before they can engage in explaining how language works – that is, before they learn grammar. It is evident that to know formal grammar, one has to know language – without language, there would be no need to theorize something like language, there would be no way of asking the question of formal *versus* practical understanding, and so forth.

The same point has been made in a study of categorization in the social sciences (Garfinkel, 1967). Graduate students in sociology had been asked to categorize medical records according to a set of criteria that the supervisors of the research project had created. The purpose of the project was to find out how hospitals worked based on the records that the various personnel created in the course of a patient's trajectory. It turned out that the graduate students, in their classification work, drew on the very type of

knowledge that the study was to yield from an analysis of the hospital records. That is, the graduate students drew on their practical understanding of hospital work and organization to classify the records such that the researchers could find out about the practical understanding that makes hospitals work the way they do. The medical records simply constitute formal representations; and to understand them, the practical understanding of how hospitals work is required.

2.2. Mathematical representation and mathematical work

The relationship between practical understanding and formal representation thereof has been conceptualized as the relation between practical action – i.e., *work* – and its formal representation – i.e., the ways in which it is *accounted for* (Garfinkel & Sacks, 1986). Formally, this relation, for the proof of the sum of the interior angles of a triangle, is represented in the form of "doing [proofing that the sum of the internal angles of a triangle is 180 degrees]." Here, "doing" designates the *work* for which "proofing that the sum of the internal angles of a triangle is 180 degrees" are the notational particulars. Take the diagram in Figure 1. It can be taken as the notational particulars of a proof that the sum of the internal angle of a triangle is 180 degrees. But these notational particulars constitute only the formal representation. They do not denote the actual work of *doing* the proof. That is, the formal representations stand in as accounts of the work but do not denote the work itself, and, therefore, they do not denote the knowing underlying the production of the account (Garfinkel, 1996). Knowledgeable readers will easily show, using Figure 1, why the sum of the internal angles of a triangle has to be 180 degrees. And it is precisely this bodily and embodied *work* they do in such a showing that constitutes practical understanding of mathematics (geometry). It is precisely this work that embodiment/enactivist mathematics educators do not sufficiently analyze, show the structure off, and theorize. If this work requires forms of knowing that are not present in the account (e.g., Figure 1), especially, if it involves embodied forms of knowing (e.g., sensorimotor knowing) that have to be enacted in the process of doing, then there exists the *necessary* condition for formal mathematics. But these are precisely the kinds of data lacking in current enactivist/embodiment accounts of mathematical knowing because the transcriptions offered do not point readers to or require the enacting of the *work*. It is only in doing such work that a person can *feel* what it means to *do* mathematics.
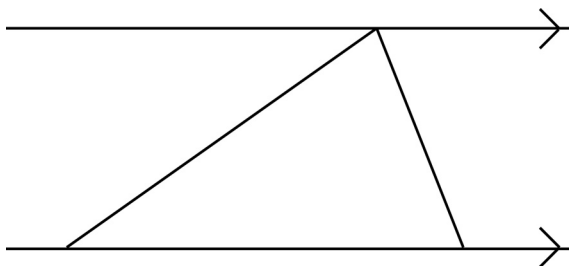
Figure 1. Account of proof that the sum of the internal angles of a triangle (on the Euclidean plane) is 180°.

This way of thinking about mathematics also allows us to understand the debate between Núñez (e.g., 2009) and his critics (e.g., Goldin, 2001). The former points out that the structures of mathematics – e.g., the different notions of continuity – are the results of cultural-historical contingent metaphorization processes whereby practical, bodily and embodied understandings of continuity lead to formal, objective mathematics that anyone can reproduce anywhere in the world. The critics however focus on the formal representations, the diagram (Figure 1) and the fact that the sum of the internal angle of a triangle is 180 degrees. This representation is objective in the sense that the proof can be reproduced over and over again, and each time the result is 180 degrees. This constitutes the objective part of geometrical science (Husserl, 1939). For Núñez it is the embodied work that matters; but it is precisely the work that is not represented in or pointed to by his transcriptions. Thus, we (authors) find that embodiment/ enactivist mathematics educators have by and large failed to provide accounts in which the nature of this work has become available. They have failed because they offer up *formal* properties (e.g., Núñez and colleagues on continuity) and verbal descriptions rather than the non-formal properties of mathematical communication that underlie and ground the formal ones. What such scholars must offer to be more convincing are representations of mathematical activity that allows access to and shows the necessity of the practical, bodily and embodied dimensions of mathematical *work*.

The purpose of this article is to exhibit a form of transcribing mathematical communication that provides readers with access to the bodily and embodied work that one can feel when *doing* mathematics. We propose a kind of transcription that is something like a recipe, which does not in itself represent the work but provides guidance for action. In doing what the transcription denotes, through, and with their own embodied performances, readers perform the mathematical communication presented in the transcription. Whether they have successfully followed the transcription can be established only after the fact. That is, like with any recipe or musical score, the formal representation is not a causal antecedent of the work, though it is a resource in and for the practical action (Suchman, 1987). A simple word-by-word transcript of a lesson may not be sufficient to exhibit what students in a mathematics classroom actually know. It will exhibit even less the didactical skill of a teacher, who may know, just because of the way a student speaks in an interaction, whether the student speaks with certainty, whether she likely or unlikely knows, and so on. This, then, is precisely our point of departure for developing transcripts, which we suggest should be used as scores that readers have to enact rather than just read – much like a musician who picks up the instrument and plays a tune rather than read sheet music and much in the way a (hobby) cook actually makes a dish rather than just read a recipe book and marvel at the accompanying images.

3. Representing mathematical communication/knowing

In this section, we provide a fragment from a second-grade geometry lesson to exemplify the kinds of data that different forms of transcriptions make available. We provide sample analyses that the proposed transcription supports and that the analyses can explain. We show, for example, how a particular kind of transcription supports constructivist claims about stable knowledge structures; we also show that this requires particular reductions where any temporality is removed from the transcription. We are specially interested in producing transcription and transcription use that lead to a better

understanding on the part of researchers of precisely *what* the students' knowledge consists in. Our contention is that if researchers only focus on what can be presented in text, they know very little about what precisely the interaction participants know.

The fragment was randomly selected from 30 hours of recordings in a second-grade mathematics class in the process of completing a unit on three-dimensional geometry. It derives from a lesson in which children were provided with a shoebox containing a "mystery object." The object could be reached and touched through a hole in the shoebox but not seen, as there was a plastic bag taped to the inside. That is, the children could only touch/feel the object by sticking their hand through the hole and into the plastic bag, which separated their hands from the object. The video shows the three girls – Sylvia (S), Jane (J), and Melissa (M) – at a large, round table on which their shoebox is placed (Figure 2). The research assistant Lilian (L) videotaping this group also participates in the conversation transcribed. From the beginning of the modeling task, Melissa has repeatedly said that she feels a cube; and she has built a cube from her lump of plasticine. Jane and Sylvia have formed rectangular prisms of similar shape from their respective plasticine lumps. But the teacher explicitly has instructed the students to produce one and the same model and, if there is disagreement about its shape, to discuss until they reach agreement. The fragment picks up when Melissa asserts once again that she "thinks it is a cube" just as she pulls her right hand back from the shoebox after another trial of feeling the mystery object. In the following, we provide three takes on the fragment leading up to a different form of representing the events with consequences for the kinds of conclusion that can be made and are supported by the fragment.

Figure 2. Sylvia, Jane, and Melissa (from left to right) are in the process of building models of the mystery object inside the shoebox.

3.1. Take 1: logocentrism

Most transcriptions that appear in mathematics education journals reduce events – lesson, interviews, or problem-solving sessions – to the transcription of the words said, augmented by ethnographic descriptions of actions and context where necessary. Moreover, the words are not taken for and by themselves but rather as indices pointing to something else not directly present: "meaning," "conception," or "idea." It is precisely these two strategies that lead to the separation of body and mind and lend themselves to Kantianism and other constructivist theories (Henry, 2003; Nancy, 2007).

Transcribing videotape by using only words flattens the observed events into language. The ancient Greek originally used the term *logos* for language and word; they later also used it to denote reason, a use that has survived to the present day sedimented in the term "logic" (Heidegger, 2000). By transcribing events into words, we obtain a representation thereof where everything that exists is named and, being in the form of words, is reduced to the form of intellect and reason. In the philosophical critique of metaphysics, this tendency to reduce everything to words and reason (i.e., *logos*) has come to be denoted by the term *logocentrism* (Derrida, 1967), a way of thinking about *being* that has its origin in the ancient Greek culture and has shaped the Western way of relating to the world. That is, the idea of rational thought apart and independent from the material world, metaphysics, is bound up with the practice of reducing complex situations to words and verbal description.

3.1.1. Producing the transcription

To produce transcriptions of this first type requires little else than playing a video and noting the words heard. Generally, we produce such transcriptions using a digital video file (.mov format) and then transcribe the words we hear directly into a word processing program. Where transcribers hear someone speaking but without being able to make out specific words, question marks are used to indicate the approximate number of words (e.g., <??> to indicate two words). The transcriber also inserts verbal descriptions of actions where appropriate or necessary. Many transcribers/researchers also insert punctuation that follows common grammatical practices. That is, where the transcriber hears a question, a question mark will be inserted at the end of the sentence independent of the fact how participant listeners have heard the current speaker as evidenced in their subsequent turns.

Transcript 1
```
01 M: ((after putting her hand in the box for a while)) I still think it is a
      cube.
      ((The whole group pauses))
02 S: Let me check ((puts her hand into box)).
03 L: Why do you think it is a cube?
04 M: Because it's the same; it's the same ((turns her model over in her
      hands)).
```

3.1.2. Reading, analyzing, and theorizing the transcription

Characteristic of this form of transcript is the removal of temporality of all dimensions of participants' action, not only regarding the production of their talk but also regarding their physical behavior (e.g., gestures, body position, transactions with physical object/s, gaze orientation). As readers can see, the transcript presented above is reduced to the order in which words have been pronounced. The verbal description of the hand/arm movement no longer renders the temporality of the movement and is not coordinated with the temporal unfolding of the speech. Because temporality has been removed, the forms of thought said to be "behind" the utterance are taken to be relatively constant over the length of a typical lesson or interview. Such a description, by and large static, facilitates making claims about "conceptions" and "conceptualizations" that can be sampled unproblematically in an interview. Researchers tend to make no difference between some word used at the beginning, in the middle, or at the end of an interview.

Most mathematics education researchers take such transcriptions and infer "meanings" and "mental structures" that somehow are in the speakers' minds and that have led them to say what they said. For example, a mathematics educator interested in our work took the video and transcript, concluding from the episode that "Melissa (initially) conceptualizes the mystery object as a cube. She bases her conclusion on the tactile observations she makes by turning the object over and 'checking the sizes' of its faces." Here, the verbal articulations and descriptions of movements become indices for something that is not directly available. On one hand, there is Melissa saying, "I still think it is a cube," and on the other the mathematics educator claims that "Melissa (initially) *conceptualizes* the mystery object as a cube." The relation between word and thought (mind) is taken to be as a rather simple one, the former providing access to the latter. Thus, in mathematics education research, verbal transcriptions of interviews and classroom videotapes are regularly used to find out what and how students think, how they solve problems, or how they "construct" their mathematical mental structures (or, conceptions, representations, or even identities).

3.1.3. Discussion

Nearly 80 years ago,, it has already been suggested that "thought is not merely expressed in words . . . the structure of speech does not simply mirror the structure of thought (Vygotsky, 1986, p. 218–219). All three – speech, thought, and the relation between the two – are *processes*. We do not see any evidence for a conceptualization, unless simple word use is taken to be synonymous with conceptualizing something. Instead, there is evidence for the fact that students and adult talk about phenomena even before they have thought about and reflected upon some idea (phenomenon, topic), and, therefore, *could not have formed* (i.e., "constructed") a concept (Roth et al., 2008). Rather, thought is the *consequence of* speech, comes to existence through speech. Moreover, whereas it might be appropriate to say that Melissa "turned over the cube," the simple description of this action in words may overstate the issue. For Melissa may have turned the cube in the way we walk or scratch an itchy spot: it does not require our conscious intentional thought. We also do not know whether Melissa was intentionally "'checking the sizes' of its faces." Rather, we observe her using the thumb and index of the right hand in apparently the same or slightly changing configuration along three different edges of the cube while articulating that some "it" – which we do not know whether it is an edge, a face, her cube, or the mystery object – "is the same." That is, as soon as something is articulated in words, it is moved from the realm of Being, presence, and presentations into that of beings, present of

the present, and re-presentations (Heidegger, 2000). Moreover, in this realm, it is subject to verbs that inherently embody intentionality (Henry, 2003).

This kind of transcript is consistent with a constructivist approach, which, at least since Kant, is concerned with abstractions and abstract thought. In Piaget's theory, we find this gesture((what gesture??)) in the development from concrete operations that lead to formal thought as embodied sensorimotor schema are abstracted and become the pattern for logical thought. It is also a description that runs counter to the epistemologies of embodiment and enactivism because it emphasizes a conscious mind and mental structures in situations that may not be appropriate. Thus, whereas it is evident that we would not characterize a person as *consciously* placing feet in walking, there is a tendency in mathematics education research to use an *intentionalist* discourse when it comes to describe what children/ students do in the mathematics classroom: "construct meaning," "develop conceptions," "acquire knowledge," "position themselves," "construct identity," and so on. Interestingly, though, scholars interested in mathematical cognition from both embodiment and enactivist camps, too, make use of such transcriptions, thereby doing a disservice to their argument. It is not surprising then that many mathematics educators opposed do not buy into embodiment and enactivist theory, as everything there is made available in such transcription is at the verbal level itself an image of the concepts thought of in metaphysical, linguistic terms.

3.2. Take 2: sequential analysis of turn taking

The afore-described constructivist inferences are inconsistent with social/cultural-historical theory that theorizes speech (communication) and thought as continuously developing processes that mediate their respective developments (Vygotsky, 1986). That is, thought and speech are different, incompatible expressions of some higher order unit; and they are processes. Thus, from such a perspective we have to take Transcript 1 as a temporal event in which not only speech unfolds from top to bottom but thought as well. Moreover, in such a theory, gesture and speech are dialectically related; they are manifestations of a higher order communicative unit rather than precisely corresponding to each other (McNeill, 2002). That is, as speech unfolds so do gestures; and speech and gesture mediate their mutual development in the same way as speech (communication) and thought. In this section, we provide a form of transcription and approach that lends itself to viewing thinking, speaking, and their relation as processes.

We begin this second take by representing the fragment in an augmented way typical of conversation analysis. This transcription form includes all the sounds produced, pauses, hesitations, respiration, prosodic information, and emphases (see Transcript 2). The approach is grounded in a history of ideas of language philosophy that what matters to understand language are not "meanings" but the ways in which words are used (Wittgenstein, 1958). Subsequent developments in language philosophy focused on speech acts (Austin, 1962). A speech act consists of three parts: *locution, illocution*, and *perlocution*. Locution refers to the act of saying something, illocution to the intent (asking, ordering, responding), and perlocution to the effect. In any concrete analysis, the effect that a locution has on others in the setting is available only in and through their subsequent acts. Consequently, to understand a speech act, researchers have to take the *turn pair* as the minimal unit of analysis. That is, it is no longer possible to attribute speech to an individual because a speech act is inherently spread across multiple participants, across

speakers/audiences. This is consistent with a conceptualization of discourse in which any utterance straddles speaker and listener, where any word – spoken for the benefit of another – belongs to both speaker and listener (Bakhtine [Volochinov], 1977; Derrida, 1996). This way of approaching transcription and its interpretation therefore focuses on understanding this event as unfolding event, as something living and lived, rather than on purported structures of individual minds whose contributions to the conversation are independent of those of others.

3.2.1. Producing the transcription

Notice how Transcript 2 adds features that were not present in the first transcription. (The differences in the text itself derive from the fact that the original transcription was done by someone else, and subsequent enhancements revealed problems in the original hearing.) For example, pauses within speaking turns and between speaking turns are measured and indicated to 1/100th of a second. The transcription also marks emphases (capitalization), partial sounds ("sti"), mispronunciations ("cob")[2], extended sounds (colons), and trends of the pitch (punctuation). Thus, the transcription renders aspects of the real time production of speech; that is, it contains the mumbles, stumbles, stutters, breathings, malapropisms, metaphors, and tics characteristic of everyday speech. Conventions to produce this kind of transcripts can be found in Appendix A.

```
Transcript 2
01 M: ((pulls rH out of box, pushes it away)) I sti (0.18) I s::TILL think it
      is a cube.
02    (1.66)
03 S: ((S picks the box, turns it, reaches in)) LET me CHECK.
04 L: WHY do you think its a CJOB (.) CUBE.
05    (0.20)
06 M: CAUSE like (0.31) the SAME ((turns cube and has caliper grip with
      thumb/index)) (1.13) its the SA::ME shape.
07    (1.55)
08 S: WHERE i:s IT; ((reaches into the box))
```

The production of such transcriptions begins with word-by-word renderings such as those in Transcript 1 but with punctuation removed, as it is used to mark the pitch tendency within the locution. We export the sound from the video into an audio format (.aif) so that it can be imported into a program for linguistic analyses. A freely downloadable, multi-platform package frequently used by linguists is PRAAT (www.praat.org). It allows precise timing of pauses in speech, measurement of speech intensities (volume), pitch (F0) levels, and speech rates. Speech emphases can be heard and – because these are produced by means of changing intensity, pitch, or rate – can be verified by visual inspection of the PRAAT display. The display also allows identification of pitch jumps and within-word movements, which are indicated in the transcript using specific signs. The conventions used follow published conversation analytic conventions that are enhanced for the analysis of prosody (Selting et al., 1998).

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[2] The research assistant, Lilian, is a native Portuguese speaker. In that language, cube is *cubo*. An interference might have occurred between the pronunciations of cube (IPA: kju:b) and *cubo* (kubó).

In those instances where visual information is relevant, screen prints or drawings are imported into the transcription or provided in an accompanying figure (see below). The precise timing of the visual information with the speech is indicated in the transcription. When drawings are used instead of screen prints – which may be to implement confidentiality or to feature only essential information while dropping gratuitous detail – the off print is imported into Photoshop. A second layer is created and an outline copy of the essential information is produced using the "paint brush" (see Figure 2, 3). To make essential elements stand out even further than they would in a pure line drawing, different degrees of shading may be used.

3.2.2. Reading, analyzing, and theorizing the transcription

Focusing on the second transcript presented, we first note that the locution in turn 01 is not fluent. There is a beginning "I sti," a pause, another beginning with drawn out "s" before the remainder of the word "still" is completed followed by "think it is a cube" that will have completed the locution. (We never know whether some word constitutes the end of a locution or speaking turn until some next speaker begins to speak, or until the same speaker takes another turn at talk.) Both the repeated articulation of the personal pronoun "I" and the second part of the word "sTILL" are articulated with emphasis (as indicated by the capitalization). This utterance cannot be understood on its own because, from a conversation analytic and speech act theoretic perspective, it is only the second part of a unit, the first part of which is not available in this transcription. In a fuller consideration of the entire episode, a researcher would focus on the emphases, which produce contrasts to the different claims that Sylvia and Jane have made and which make salient that Melissa already has repeatedly made statements about the mystery object as a cube.

Melissa's turn is the first part of what turns out to be two turn pairs. Sylvia says, "Let me check," which allows us to hear the pair of turns as a constative/verification speech act. In fact, Sylvia not only says "let me check," but also pulls the shoebox over close to herself and sticks her hand into it. Her verbal articulation is a *formulation* of the action: Sylvia not only reaches into the box but she formulates for others what she is doing, that is, the she articulates the *intent*. She makes explicit and available to her audience a verbal description of the illocutionary act. Her reaching into the box is formulated as an action that has the intent of checking. Because of the pairing of turns, the checking is heard with respect to the constative "it is a cube."

The second turn pair exists in the sequence with Lilian, the research assistant, who is also acting as the teacher of this small group of students. We can hear turns 01 and 04 to constitute a sequence, because Lilian's locution "Why do you think it is a cube" picks up on and repeats the contents of Melissa's utterance. Interestingly, the transcription indicates that the pitch is falling toward the end of the locution, which is typical for constative phrases. But the fact that the interrogative adverb "why" is articulated with emphasis allows us to hear a constative/request-for-justification speech act: "I still think it is a cube" is followed by "Why do you think it is a cube?" This hearing is consistent with the next turn sequence, which we can hear as a question/response pair: "Why do you think it is a cube" is followed by a coordinating conjunction "[be]cause," which introduces a reason, "like the same . . . it's the same shape."

This form of transcript in the hands of conversation analytically informed researchers, therefore, allows readings that focus on the unfolding nature of the event. Such researchers

also focus on analyzing pairs of turns, that is, on the effect a locution has on the other participants as their actions make it available to everyone else. There is a focus on the sequential enchainment of locutions (utterances), where turn pairs constitute the minimal unit. This kind of analysis is *process oriented*, allowing us to understand the constitution of this segment. What matters is – consistent with Wittgenstein's (1958) language philosophy – *how* words are used rather than purported and never accessible "meanings" behind the word. Moreover, from a discursive psychological perspective, Melissa's and Lilian's reference to thought processes ("I still think," "Why do you think?") are taken to be everyday ways of reasoning where psychological concepts are invoked for the purposes at hand. Such researchers are little interested in purported contents of the mind; instead, they focus on the mobilization of psychological discourses for the purposes of the situation at hand (Edwards & Potter, 1992).

From a conversation analytic perspective, Melissa's "I think" is taken to be a formulation of the work she is/has been doing at the instant, and Lilian is taking up the self-description as a way of referring to the same work description. It is not the researcher who imputes thought processes – as in the preceding section, where a mathematics educator imputes conceptions – but it is one of those ongoing descriptions that interaction participants provide to articulate the situation together and *for one another* with the content. Here, the content is the nature of the model Melissa has built, and its relation to the mystery object. It is the situation itself that suggests the use of the "thinking" as a description, and the available language form to describe what she has been doing is that she is "thinking." An alternative might have been to say, "I feel it to be a cube" or "I believe it to be a cube."

In this transcript, because the gestures are described in words, their contribution to the communication comes to be evaluated purely in terms of the linguistic sense ("meaning") that researchers attribute to them. In classical conversation analysis, gestures were not attended to – in part because the research was based on audio-recorded conversations on the telephone. But many conversation analytically and ethnomethologically oriented studies of this nature focusing on mathematics – following the ground-breaking work of the applied linguistic Charles Goodwin (e.g., 2000) – now include precise studies of gesture. In our own work on the role of gestures in science learning, we precisely coordinated information about gestures with speech because, as it turned out, the changes were related to familiarity and expertise of the speaker within the domain talked about (e.g., Roth, 2000). These studies included transcriptions such as the following rendering of turn 06, in which vertical lines indicate at which point a particular hand/arm configuration occurred. (Vertical bars coordinate speech and image.)

```
06 M:  CAUSE like (0.31) the
```
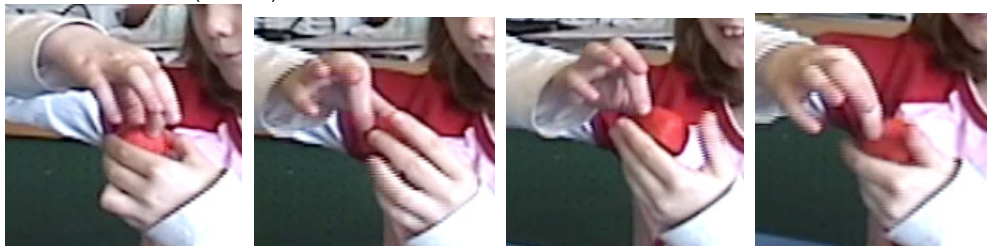


```
       SAME |       (0.66)      |      (0.47)   |        its the | SA::ME shape.
```

In this transcript, we observe the rotation of the cube held in the left hand and an associated movement of the right hand, the thumb and index finger of which grab the plasticine "cube."[3] The transcription clearly shows that three bodily configurations precede the articulation of the predicate "it's the same shape," and the fourth configuration also precedes the second, key part of the predicate "same shape." This key part is further of interest, as the word "same" is drawn out (see colons in transcription), which might be heard – depending on context – as an emphasis or as a delay in the verbal performance. Psycholinguists often focus on the relation between gestures and the contents of speech that is said to correspond to the former (Roth, 2003). It turns out that developmental studies of mathematics, for example, show that gestures expressing a new developmental level *precede* verbal expressions at the same conceptual level (Alibali, 1999). That is, words and gestures manifest *very different forms* of knowing. In fact, when the conceptual content of the gestures is different from those of speech, it is taken as an indicator of developmental readiness (Church & Goldin Meadow, 1986); and without training even teachers and undergraduate students glean information from children's hands (Alibali, Flevares, & Goldin-Meadow, 1997). Using words instead of images to depict children's communication *falsifies* what they are communicating to the teacher or researcher. Moreover, studies in science education show that the alignment between gestures and corresponding speech during conceptual transitions, which may be out by up to three seconds, decreases with students' familiarity in the domain (Roth, 2002). When alignment is achieved, observers tend to assess as competent the explanations of the phenomena that are the current topic. It matters that language and gesture are different in nature, have different content and form, and that they may contradict each other.[4] This form of transcription therefore provides support to theoretical approaches that assume the continuous development of both speaking and thinking at the moment-to-moment and ontogenetic scales, but they are inconsistent with those approaches that theorize stable mental structures.

In this instance, the hand movements may actually not be purely symbolic. The left hand holds the cube rather than gesturing a cube, and the right hand produces a configuration that is applied with little change to the cube that turns underneath it. The situation does not symbolically represent the events that have occurred just seconds before while Melissa has had her right hand in the shoebox, but her left hand remained outside. We do not know what happened inside the shoebox, how and even whether the mystery object has been turned. This is of particular importance later given that the mystery object turns out not to be a cube. But in the present instance, the configuration is repeatedly applied to the different dimensions (x, y, z) of the plasticine model ("cube"). The configuration, therefore, especially when it occurs the first time, constitute an epistemic (knowledge-seeking) movement designed to "check the faces," as the mathematics educator referred to above suggested to us. During the same and other lesson of this geometry curriculum, we did observe purely symbolical movements when the same hand configurations were used in communication in the absence of cubes.

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[3] It is a cube but not in the sense of geometry, which only deals with ideal objects. Rather, it is a figure of the kind that preceded geometry (Husserl, 1997).

[4] In dialectical psychology and philosophy, speech and gesture *inherently* are contradictory, each manifesting the communicative content in a one-sided way (e.g., Roth & Lee, 2007).

3.2.3. Discussion

Transcription 2 exhibits temporal features characteristic of human interactions; it also features some of the details of the actual production of communication, including hesitations, false starts, emphases, and so on. This type of transcription – embodied in our conversation analytic reading above – lends itself to theories that include temporal features between thinking and speaking and to theories that focus on the interactional nature of human life and its continually unfolding nature where subsequent states are unavailable to the actors. Moreover, theories that take the actor perspective on social events find such transcription useful, as these contain implicit and explicit information that participants use in the pragmatic conduct of social/societal events, including interviews and mathematics lessons.

One of the questions one might ask is this: Is there something *behind* these performances, some structures, that drive/cause what we observe? In other words, is there knowledge of some kind in the brain that causes the vocal track and the hands/arms to do what they do in order to externalize something that is hidden from direct observation in the brain? Or should we take the verbal and gestural performances as the knowing itself? If the second is the case, as researchers informed by embodiment and enactivist theories claim, then this and the preceding form of transcription are insufficient in two ways. First, because these contain too little information about the communicative productions and expressions themselves; and, second, the relation between knowing as represented and knowing-how of what the representation refers to is the same as knowing to read a recipe and knowing-how to make the dish. We contend that mathematics educators who read transcripts do not (necessarily) have the know-how of these performances; someone who reads a musical score does not (necessarily) know how to play the tune on the musical instrument it was intended for. And it takes precisely the cooking or playing to *know what it feels to cook or play*. In the following section, we address the first of these questions and then make a proposal about how to address the second.

3.3. Take 3: interaction rituals

Recent developments in philosophy and sociology (of emotions) focus on temporality, periodicity, and resonance as fundamental phenomena for the constitution of (common) sense (Collins, 2004; Nancy, 2007). Thus, we can observe an increasing alignment of prosody across speakers within turn pairs among teachers who are working together over several months; and these alignments are coextensive with the sharing of sense in and of the situation (Roth et al., 2005). For example, pitch misalignment is associated with conceptual dissociation and conflict; and rhythmic alignment across speakers and listeners can be observed even when listeners cannot see the speaker's rhythmic body movements (e.g., Roth, 2010b). These rhythmic alignments are sources of emotional alignment and a sense of solidarity (Collins, 2004). Pitch and rhythm are of interest because speakers are not conscious of it. That is, these features of speech and body movement *determine* sense, but, because consciousness is not involved, words only one-sidedly represent the content of communication. This also tends to be the case for speech intensity, though under certain circumstances speakers are conscious of their speech intensity and increase or decrease their volume. In contrast, as part of outbursts of anger, they do not voluntarily control

speech intensity. Because these are non-conscious features of communication, these cannot be theorized in the same way as verbal consciousness. Transcriptions including these features therefore lend themselves to provide support to embodiment and enactivist theories and to theories that track the real-time evolution of events from the perspective of the participants (Roth & Pozzer-Ardenghi, 2006).

Our recent work in mathematics classrooms also exhibits the importance of prosody and rhythmic features in the voice, gestures, and body movements. In Figure 3, we provide a more extensive transcription. In the following, we articulate the possible readings it affords consistent with a radical approach to embodiment that has been termed "incarnation" (Roth, 2010a). The following dimensions are represented in the transcript: intensity and pitch of the participants' talk, duration of their utterances (see black boxes), the sounds/words they pronounce, and other relevant embodied dimensions that emerged during the entire episode such as hand gestures performed with the object, body position, and gaze orientation. Because the variable "time" is the main criteria to display our empirical evidence, we suggest below that this transcript is to be treated in the way musicians treat a musical score: as an occasion for playing a particular tune in a particular way. In this way, the rate and total time of playing themselves become performative aspects. As a result, readers will *feel* the type of knowing observed when they re-play the transcript rather than merely look at and read it.
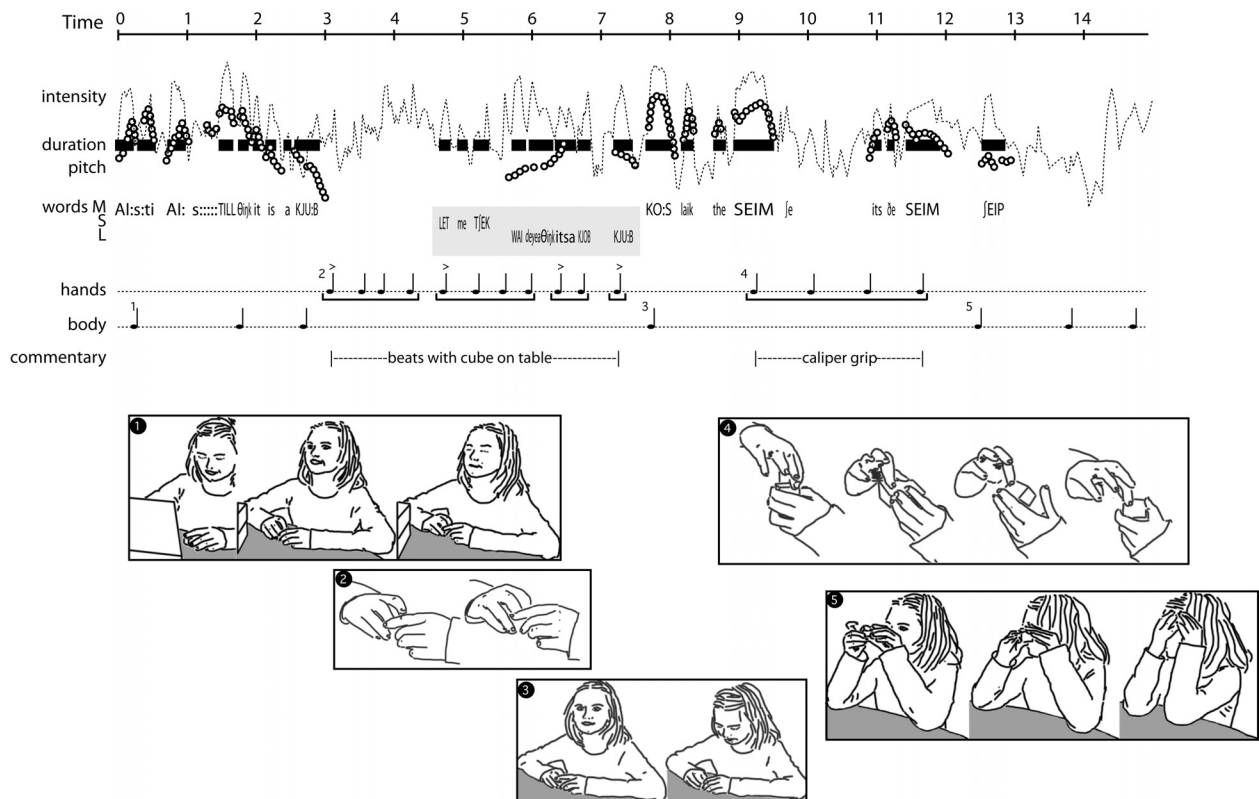
Figure 3. The extensive "transcription" includes prosodic features (pitch, volume, rate), rhythm, verbal, and visual information. The "words" are transcribed using the conventions of the International Phonetics Association

3.3.1. Producing the transcription

As can be observed, this type of transcription uses information that was presented in the preceding types of inscriptions (e.g., words). In addition, the transcription directly maps the sound (phonemes), using the conventions of the International Phonetics Association, onto the prosodic information (Figure 3). Because the phonemes are directly mapped against the prosodic information, changing speech rates, emphases, and rhythms also become visible. We used a graphics program into which the PRAAT display was imported. Using horizontal black bars, the length of the phonemes is indicated. Each word is typed at a specific font size and then changed in horizontal extension until the transcribed phoneme has the same length as the black bar. Moreover, as in a musical score, the melodic line (pitch) and changes in intensity – indicated in musical terms (e.g., piano, pianissimo, forte, diminuendo) in the second type of transcriptions – are given quantitative expression. In addition to the coordination of visual information already present in the augmented version of Transcript 2, these now are associated with the information about repeat patterns. This, therefore, allows exhibiting the rhythmic aspects of a performance, which also would be available in a musical score.

3.3.2. Reading, analyzing, and theorizing the transcription

This transcription (Figure 3) exhibits some striking differences with respect to the preceding Take 1 and Take 2. First, it makes explicit the temporality of all the dimensions of the students' and the teacher's verbal/physical action. Not only is speech in time, it makes time as "words," phonemes, and even individual letters are drawn out or speed up; there are pauses; and there are emphases that punctuate what is being said. For example, Melissa stresses "I," "still," "cube," "cos," "same," "same," and "shape." These stresses with the interspersed more rapid deliveries punctuates the utterance as it unfolds in time; it gives it a particular rhythm. In actual listening, (a) perceiving the rhythm requires a consciousness very different from intellectual consciousness and (b) perceiving the rhythm means producing the rhythm (Abraham, 1995). In Lilian's utterance, the "words" run together making out of "do you think it's a cube" one single sound complex.

We note that the pitch moves up and down, sometimes producing spikes with individual words (e.g., ""cos," "like, "same") and producing overall tendencies (e.g., the pitch drops with the production of "still think it is a cube." Such information is important, as research shows that in harmonious exchanges, speakers tend to latch onto the pitch of the preceding speakers, whereas in conflictual situations, the pitches tend to be significantly apart. In fact, in conflict, the pitch levels tend to rise, each speaker "trumping" over the preceding one so that both may be speaking with fundamental frequencies three to four times above their normal pitch (e.g., Goodwin, Goodwin, Yaeger-Dror, 2002). Thus, for example, one study in a science classroom showed such a phenomenon as a teacher and her student argue about chemical valences, and their argument over conceptual differences come to be reflected in the differing pitch levels; appeasement was associated with falling pitch levels across a number of speaking turns also involving other students (Roth, 2010b). Speech intensity, too, contributes to the way we understand what and how someone else speaks, as interaction participants tend to hear much louder than normal speech as "shouting," in many situations heard as an expression of anger. Much lower than normal

speech intensity, in the case of a student who also speaks slowly, may be heard as a sign of timidity, not knowing the answer, or as a tentative exploration of ideas. Teachers use such hearings routinely in their assessments of teaching, yet at present, mathematics education research does not account for these embodied features.

The transcript includes visual information similar to the one we presented in the preceding subsection. For example, the fourth image sequence exhibits the same four hand/finger configurations introduced previously. Here, however, we also mark with a " " on the temporal axis the precise instant when the configuration is produced. The musical notation exhibits the highly rhythmic feature of the gestural production. That is, the four configurations that exhibit mathematical features – sameness of the length of the edges – are produced in a highly rhythmic fashion, which constitutes a very different manifestation of sameness across the dimensions. Melissa is transacting with a solid characterized by the idea of even number (such as 4 and 2, as demonstrated in the stresses of the beats she produces on the table), and vice versa – the object is transacting with her as well. To a certain extent, it might be argued that the idea of "evenness" emerges from Melissa's physical action while she transacts with the plasticine model.

Comparison with the verbal production shows that the first gestural beat falls together with the emphasized "same"; the second beat falls at the beginning of the pause which in speech, as in music, is an important feature; the third gestural beat coincides with the restart of the verbal "melody"; and the forth beat falls on the second "same." We might expect another beat corresponding to the verbal production of "shape"; but, as our transcription shows in the change of the bodily configuration where the gaze, heretofore exclusively oriented to the hands and cube, now is raised to meet that of Lilian, the person who has requested the justification Melissa has just ended producing. Melissa then turns to gaze at Jane, and finally appears to complete her presentation by enclosing her cube in a gathering movement that also brings the elbows close together. This, therefore, constitutes a continuation of the rhythm but in a different modality, that is, on a different "stave" of our "score of mathematical communication."

Returning to the beginning of the transcription, we note that the changing orientations constitute a rhythmic phenomenon as Melissa orients from her cube to others and back to her cube (image sequences 1, 3, and 5). Between these sequences there are long pauses of speech. The second of these "pauses," as shown above, occurs when Melissa rhythmically produces the four gestures that constitute an integral aspect of the (unconscious) embodied/enacted justification why the mystery object is a cube. The first "pause" in the shift of orientation is associated with a pause in Melissa's speech. There is a long pause, which Sylvia breaks announcing that she is going to check, followed in turn by Lilian's request for a reason. During this pause in speech, Melissa hits the table repeatedly with her plasticine model (in the sound wave, there are spikes that mark the precise instant that the cube hits the table). As our transcription shows (Figure 3), there is a rhythmic beat that is produced and that we can perceive. Not only is this performance rhythmic, but the transcription shows that the beats fall together with the beats in Sylvia's talk; it also coincides with the beginnings of the major segments in Lilian's talk as exhibited by the speech intensity profile (i.e., where she says "deya [do you]," "it's a," "cob," and "cube"). That is, the same rhythm can be perceived in all three speakers, or, if Melissa were to be taken as the main figurant in this instance, the others would be found to have aligned

themselves with the beat she has initiated. But, because perception of rhythm means production of rhythm, all of these rhythmic features produce interactive interference that leads to entrainment into the same rhythm. This is precisely what we have observed both in mathematics (Roth, in press) and in science classrooms (Roth, 2010b) where there are rhythmic features in speech and other bodily productions across individuals; and these beat frequencies change across individuals. Thus, it is not that the same beat occurs by chance. Rather, when the speaker changes the beat, others follow, sometimes imitating it and sometimes improvising on the original beat. This is so even though the beat is not accessible to verbal consciousness but constitutes a very different form of consciousness (Abraham, 1996; Nancy, 2007). The perception of beat is a form of active resonance that allows for the alignment through entrainment.

The rhythmic aspects together with the prosody emphasize ritualistic aspects of human interactions. Our transcription therefore is consistent with social theories that focus on interaction *rituals* (Collins, 2004) and sense as a resonance phenomenon (Nancy, 2007). Sense cannot be reduced to words, as integral aspects of sense manifest themselves in and are expressed by non-verbal means. Moreover, the ritualistic moments also are tied to emotion, finding both their expression in the performance and driving this performance.

3.3.3. Discussion

Readers unfamiliar with such analyses might ask why this is important. It is because these changes in rate and intensity are associated with what we hear as main and subsidiary clauses of a sentence (Roth, in press). Whether something is a main or subsidiary clause goes right to the heart of competence in mathematical communication and mathematical understanding. Thus, the prosodic and rhythmic aspects, which appear to have nothing at all to do with the mathematical content – they do not appear in mathematics textbooks – nevertheless are integral and irreducible aspects of mathematical communication and the practice of mathematics. That is, the difference between mathematical content and purely performative dimensions of communicative production is undecidable. They constitute one and the same phenomenon. These analyses therefore are important for those who adhere to embodiment/enactivist perspectives on mathematics education. Mathematics is not embodied because bodily gestures (hands, hand/arm, other body parts) exhibit logical structures that may be seen as parallel to and exhibiting the same verbal-conceptual content. Rather, mathematics is embodied because there are features in mathematical communication and practice that play integral and central role of producing mathematical distinctions, but they are not part of the verbal-linguistic register. More importantly, the two registers are irreducible to each other, each constituting a one-sided and therefore partial manifestation of a higher-order phenomenon of mathematics and mathematical communication. And it is precisely this irreducibility of mathematical linguistic features and purely embodied features (prosody, rhythm, bodily gestures) that support enactivist/embodiment theories.

We propose taking our transcription differently than transcriptions normally are taken in the literature. We suggest that our transcription relates to the performative of mathematical communication as a cookbook recipe relates to cooking or in the way a musical score relates to a musical performance. That is, to really feel the knowing and understanding in Melissa's communication, readers need to perform our "score." Such performances relate to Melissa's in the way one musician's rendering relates to that of

another; this relation is different from the one between score and performance. This is especially so because the performative dimensions (such as prosody and the rhythmic performances) are irreducibly involved in the mathematical sense even though they cannot be rendered in terms of linguistic consciousness. Rhythm has to be performed to involve and make it accessible to rhythmic consciousness in the same way that the visual aspects (e.g., hand gestures) require a form of consciousness different from and irreducible to verbal consciousness (Vygotsky, 1986). Performing the transcription, therefore, amounts to a process of *reterritorialization* (Deleuze & Guattari, 1991/2005), whereby something said to be transcendent and metaphysical comes to return to the real world. This very same thematic exists in the biblical literature under the phenomenon of incarnation with its image of the word (a representation) becoming flesh. It is precisely this idea of incarnation that we have recently offered as a way out of the problematic presentation of the enactivist/embodiment literature (e.g., Roth, in press).

4. General discussion
There is a close relationship between the format in which researchers present the data (e.g., transcription) they extract from the data sources (e.g., videotape) and the theories they use to interpret or (try to) explain these data. Some data are such that they cannot be explained by particular theories. In such cases, researchers of the standard paradigm likely do not accept the data as valid, explain unwanted effects away, or introduce hidden variables to the theory (Kuhn, 1970). Here, we present the case of different forms of transcriptions that use classroom video as their source that researchers collect to develop their findings. Such transcriptions stand in a mutually constitutive relation with the claims that researchers (can) make. On one hand, the transcription is the source material from which claims are (inductively) developed. On the other hand, in research publications, the transcriptions function as evidence in support of the claims made.

In this study we show how different forms of transcription render visible different aspect of mathematical communication and therefore support different kinds of claims and the associated theories. We show that transcriptions that make use of words only and omit all information about the actual production of communication (*Take 1*) lend themselves to support constructivist arguments that make claims about stable knowledge (structures) in the mind somehow abstracted from the physical world. As soon as gestures and other perceptual aspects, for example, are rendered in terms of verbal descriptions, they no longer constitute embodied dimensions. Aspects of a situation produced and recognized by perceptual consciousness have been reduced to the verbal consciousness. Even talk about sensorimotor schemas does not get us any further because this talk is consistent with a Kantian position that makes mind a metaphysical entity – the embodiment theorist Johnson (1987) acknowledges having borrowed his conception of the schema from Kant – to the point that there is nothing outside (verbal) understanding (Henry, 2003). Because "the presuppositions of the Kantian ontology remain closed to the being of life" (p. 45), no constructivist account of knowing is able to capture the essence of embodiment/enactivist theory.

The preceding sort of claims are impossible if a researcher takes the stance that we present in *Take 2* as the production of communication that can no longer be reduced to individuals. The minimum unit of analysis is the turn pair, which means – consistent with a range of theories – that each word pertains both to the speaker and to the listener.

Moreover, in this second kind of approach the temporality of the production matters, because what is said at some time takes into account what has been said before but may be entirely inconsistent with what is said thereafter. The approach therefore is consistent, for example, with Vygotskian (1986) theory, which stipulates communicating and thinking to be continually changing processes. Any word uttered therefore no longer is the same when it is uttered again. Even an individual word repeated once or more no longer has the same function and therefore cannot be analyzed in terms of a constant sense or "meaning." In fact, researchers taking this stance no longer worry about "meaning" that somehow is indicated but not really present because the only thing that counts, consistent with Wittgenstein's position, is word-use and how consecutive speakers employ, re-employ, or change employment of words. Because temporality and time are important, this second approach much better than the preceding one can account for the continual changes that we observe in language and culture in a mathematics classroom over time, even though individual students and teachers do not think about or are conscious of such changes.

If it is the case that others are entrained into the collective pitch and bodily rhythms – as our example here shows consistent with other research (Auer & Couper-Kuhlen, 1994; Szczepek Reed, 2010) – then the production of the individual locution no longer is reducible to the speaker. Thus, more so than articulated in the context of the second case, where the word is a feature common to speakers and listeners, the *production* of the locution no longer is independent from other productions in the setting. Each locution then has to be theorized as an integral *part* of a more complex situation. This situation that cannot be reduced to its parts, for the parts are produced as a function of the whole, and this whole only exists in and through the production of the parts. In this manner, our work also suggests a link between the individual and the collective through completely embodied phenomena inaccessible by and irreducible to mental phenomena (mind). Other than articulated by the enactivist theorists, bodily phenomena are *collective* rather than the result of individual sensorimotor actions.

The most difficult phenomenon to explain with (radical, social) constructivist theories is real-time production of mathematical communication. This is so because there are aspects that are central to the sense that participants mark and re-mark in and through their communicative contributions but that have no place in mathematics in the form we can find articulated in a textbook. But Kant (1964) did realize that the separation between the purely mental and the purely bodily may be impossible. Thus, at the very end of his life he wrote an analysis of jokes where the intellectual recognition of the pun occurs at the same time and indistinguishable from laughter.[5] His explanation involves both: The tension within the set up of the joke that addresses the senses creates a disequilibrium of the innards, which, when released, creates laughter. The two aspects are irreducible because the mind does not need laughter because it could simply analyze the pun (and perhaps find nothing funny about the story). A "joke" that is not funny is not a joke and is not associated with laughter. We suggest that precisely the same irreducible aspects between the conceptual and the purely bodily come to be sensed and experienced when readers perform our transcription (score). This transcription then is nothing other than an *account*

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[5] Actually, in the early part of his work, Kant (1960) thinks of wit only in mental terms and uses the example of the well-known mathematician and founder of the mathematics curriculum in Germany, Christopher Clavius, to suggest that someone can be intelligent but dull (no wit).

(recipe, plan; manual of instructions); the performance involves the actual mathematical *work*. After the fact the performance can be judged to be a more-or-less adequate rendering of the account/score/plan – much like we might judge a musical performance to be inconsistent with the score or the dish to be inconsistent with the recipe. As a result, knowing to perform what the transcription *refers* to, readers are enabled to *feel* the work of mathematics that leads participants in an episode to produce what we see. But the transcription itself does not get us to this feel. The purpose of the present article is precisely to provide "scores" for the performance of the mathematical knowing that researchers write about in their studies. This is especially important for those mathematics educators adhering to enactivist/embodiment theories, which require very different forms of data than the alternative constructivist-cognitive accounts.

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Appendix A
Typical conventions used for transcriptions such as those presented in Take 2

Notation	Description	Example
(0.14)	Time without talk, in seconds	`more ideas. (1.03) just`
()	Pause of less than 0.10 seconds;	`kay. () bert`
((turns))	Verbs and descriptions in double parentheses are transcriber's comments	`((nods to Connor))`
::	Colons indicate lengthening of phoneme, about 1/01 of a second per colon	`si::ze`
[]	Square brackets in consecutive lines indicate overlap	`S: s[ize ]` `T:  [colby]`
<u>this one</u>	Underlined part coordinates with a gesture described; lH and rH indicating left and right hand, respectively	`this ones:? ((rH moves down, up, down right face, Fig 4.1b))`
<u>((:B))</u>	Colon prior to letter in double parentheses: The speaker directly addresses another person "B"	`57 T: ((:B)) hOW did`
<<p> >	Piano, words are uttered with less than normal speech volume	`<<p>um>`
<<pp> >	Pianissimo, words are uttered with very low, almost inaudible volume	`<<pp>this>`
<<f> >	Forte, words are uttered with greater than normal speech volume	`<<f>that> makes`
<<ff> >	Fortissimo, much louder than normal speech volume	`<<ff>hU::::::::ge.>`
<<all> >	Allegro, faster than normal speech rate	`<<all>[whawould]> that`
<<len> >	Lento, slower than normal speech rate	`<<len, drawn out>but () its like a flA:Tcube.>`
<<confidently> >	Ethnographic description of speech that is enclosed in brackets	`<<confidently>because its like a sort of (0.60) vertex>`
ONE bert	Capital letters indicate louder than normal talk indicated in small letters.	`no? okay, next ONE bert.`
.h, hh	Period before "h" indicates in-breath; "h" without period is out-breath	`.hhi, hh hh`
(?cular)	Question mark with whole or part word in parentheses indicate	`(serial?), (?cular)`

	possible hearings of words or missing sound	
(??)	Question mark(s) in parentheses: Inaudible word(s), the approximate number given by number of marks	`i (??)`

,?;.	Punctuation is used to mark movement of pitch toward end of utterance, flat, slightly and strongly upward, and slightly and strongly downward, respectively	`T: so can we tell a`
		`   shape by its color?`
		`T: does it 'belong to`
		`   another 'group`
		`   (0.67) O:r.`

=	Phonemes of different words are not clearly separated	`loo::ks=similar`
↑↓	Arrow up, down: Significant jump in pitch up or down	`is ↑sort, ↓<<all>so` `thats`
ˋˊ^˅	Diacritics indicate movement of pitch within the word that follows—down, up, up-down, and down-up, respectively	`ˋum; 'sai:d;` `^Cheyenne; ˇsquare`

Seeking More Than Nothing: Two Elementary Teachers' Conceptions of Zero

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Abstract: *Zero is a complex and important concept within mathematics, yet prior research has demonstrated that students, pre-service teachers, and teachers all have misconceptions about and/or lack of knowledge of zero. Using a hermeneutic approach based upon Gadamer's philosophy, this study examined how two elementary mathematics teachers understand zero and how and when zero enters into their teaching of mathematics. The results of this study add new insights into the understandings of teachers and students related to zero and the origins, relationships between, and consequences of those understandings. Significant gaps and misconceptions within both teachers' understandings of zero suggest the need for pre-service education programs to bring attention to the development of a more complete and meaningful understanding of zero.*

Key words: Zero; In-service teachers; Elementary teachers; Mathematics teachers; Prospective teacher development; Teacher research

What is zero? When asked, many teachers and students will tell you that zero is "nothing" (Wheeler, 1987; Leeb-Lundberg, 1977; Wilcox, 2008; Crespo & Nicol, 2006; Wheeler & Feghali, 1983). For being nothing, though, it has gotten a lot of attention in mathematics education research. Since the 1960s, various researchers have explored how students understand zero (Inhelder & Piaget, 1964, Pasternack, 2003; Evans, 1983; Baroody, Gannon,

Berent, & Ginsburg, 1983; Wheeler, 1987; Neuwirth Beal, 1983; Reys & Grouws, 1975; Allinger, 1980; Leeb-Lundberg, 1977; Whitelaw, 1984; Kamii, 1981; Crespo & Nicol, 2006) with findings including that students confuse zero with the letter "O", believe that zero is not a number , believe zero is just a part of the symbol for the 'digit' ten (believe that zero is 'nothing' and therefore can be ignored, and have difficulties within arithmetic calculations (including, but not limited to, division by zero) when zero is involved. Much research has also focused on teacher's and prospective teacher's understanding of division by zero (Crespo & Nicol, 2006; Ball, 1990; Evan, 1993; Wheeler & Feghali, 1983; Even & Tirosh, 1995). What is remarkable and disheartening is that the same errors in thinking and understanding about division by zero are noted in Crespo & Nicol's (2006) work as were noted originally by Wheeler & Feghali (1983) in their research. Given the extensiveness of student and teacher misunderstanding about zero, it is also notable that, other than Wheeler & Feghali's (1983) research, no research designed to explore specifically teachers understandings of zero, other than those studies related to division by zero, has been done. The research upon which this article is based was intended to help to begin filling in some of that void.

This article reports about a qualitative research study designed to explore how teachers conceive and preconceive of zero, both personally and within their classrooms. Through the use of Gadamer's (1989) hermeneutic philosophy of understanding, two teachers were engaged though dialogue and explorations in the consideration of the questions: "How do you understand zero," and "When and why does zero become a part of teaching and learning in your classroom". Many of the results of this study parallel those found by Wheeler & Feghali, but because of the study's qualitative design, the results also help to paint a picture of the thinking and reasoning behind those results for these two teachers. These insights could be used to begin developing

aspects within teacher education programs that would help prospective teachers to learn both the "that" and "why" knowledge (Shulman, 1986). related to zero and how to share that knowledge with future students

1. BACKGROUND

This study is informed by the history of the development and evolution of zero as a mathematical concept, research into student understanding of zero, and research related to teacher understanding of zero. These topics are discussed in the following sections.

1.1 Evolution of zero

The history of zero is one that has been well documented. Around the turn of the last century, and the millennium, three authors, Barrow, Kaplan, and Seife, each wrote detailed accounts of that history, and it is upon those three authors' works that this section is based. The development of the mathematical concept of zero (including its roles as a place holder, a number, and a symbol) is one that happened quickly in some societies, such as India and the Mayan culture. In Greece, however, the acceptance of zero took more than 1000 years (Barrow, 2000). These variances in timing were the result of differences in the religious and philosophical beliefs of the societies themselves. Within India's Hindu religion, there were many gods that represented different dualisms in life, one being that of the void and the infinite. Thus, when zero as a placeholder reached India, the extension of the concept to a quantity was natural because a related idea (the void) already existed in the religion. In Greece, however, the notion of zero as a number contradicted a mathematical proof that God existed (Kaplan, 1999). In addition, Greek philosophers were unwilling to accept that a symbol could be used to represent a void, since a

void is nothing and one cannot represent something that does not exist. Even the Greek mathematicians found zero challenging because it could not be represented by a shape as Pythagoras' philosophy said that all numbers could (Seife, 2000).

Once zero found its place in different societies, however, it quickly became integrated into mathematical thought and logic, and soon opened doors to formalized arithmetic, algebra, and calculus, to name just a few areas. Even mathematicians' understanding and conceptualization of numbers began to evolve through their explorations of zero and its meaning, leading Newton, for one, to conclude that "…mathematical quantities [are]… not … consisting of very small parts, but as described by a continuous motion" (Kaplan, 1999, p. 156). Thus, zero had evolved from being an arbitrary symbol used to denote a blank space to a quantity of such complexity and depth that it allowed mathematics to move into a realm that embraced asymptotic and limiting behaviors.

Despite the discrepancies related to how zero was welcomed into the number and mathematical systems of different societies, zero has become a foundational, complex, and multi-functional concept in modern mathematics (Seife, 2000; Barrow, 2000; Kaplan, 1999). As such, it can be argued, a robust understanding of zero is a necessary part of students and teachers' abilities to think and work mathematically with confidence and competence.

1.2 Student understanding

Given the revolutionary impact of zero on mathematics, it is surprising, yet understandable, to realize the limited understanding and many misconceptions that students have related to zero. Inhelder & Piaget (1964) found that children under the age of 10 or 11 do not recognize the null set (i.e., a set whose defining characteristic is the lack of a characteristic, such as the set of pictures with no birds versus the set of pictures with one bird) when sorting.

Research also reveals that students do not recognize zero as a number; they see it as only a part of the symbol for ten (Pasternack, 2003; Evans, 1983; Baroody et. al., 1983). In fact, many students believe that "[Zero] isn't really a number … it is just nothing" (Wheeler, 1987, p. 42), with the implication that it can be ignored whenever it occurrs. As an alternative, some students recognize zero as a number that "develops and exists separately from other number rules" (Evans, 1983, p. 96). Similar results are also found in the research of Neuwirth Beal (1983) and Reys & Grouws (1975). All of these notions of zero were found to support students' misunderstandings related to computations involving zero (Wheeler, 1987; Neuwirth Beal, 1983; Anthony & Walshaw, 2004; Evans, 1983).

Students have also been shown to struggle with the mathematical concept of zero because of the inconsistent use of oral and written language related to zero within society. Baroody, et. al. (1983), Allinger (1980), and Whitelaw (1984) all report that students frequently confuse the the number "0" and the letter "O". Society's frequent use of "oh" when stating area codes, license plates, phone numbers, and room numbers is cited as a common source for the students equating of zero and the letter "O" by these researchers.

Even the naming of numbers in the English language, zero is not mentioned within the name (e.g., 203 is read as two hundred three, and not two hundred zero three). This convention of mathematics can cause students to misunderstand and misuse zero. Students try to "spell" numbers in the same way they "spell" words, thus one hundred twenty is often written as 10020 (Kamii, 1981). This "ignoring of zero" convention in the naming of numbers supports the student belief that zero is "nothing" and thus it can be ignored (Wheeler, 1987).

Two researchers (Leeb-Luneberg, 1977; Wilcox, 2008), did demonstrate, however, that young students can develop an understanding of zero. In Leeb-Luneberg's (1977) research, the

students concluded that zero must be "… nothing - of something!" (25), while Wilcox's grade 1 daughter told her mother: "No [zero] means something. It means you don't have anything" (204). In both studies, the children were able to resolve the Greek dilemma that zero cannot exist because it is nothing by refining their understanding of zero to be that it represents none of some item. Cockburn and Littler's (2008) *Mathematical misconceptions: A guide for primary teachers* includes an opening chapter (Chapter 0) that specifically addresses students' misunderstandings about zero and how to correct and prevent them, with a number of the activities suggested mimicking those used by Leeb-Luneberg and Wilcox .

1.3 Teacher understanding

Consider first the area of research into teachers' understandings of zero that is most prominent in the literature – that of division by zero. Even & Tiroch (1995), found that when asked what 4 divided by zero was, most answered "undefined", however, when the same teachers were asked to explain why "most could not supply any appropriate explanation" (9) beyond stating that it was a mathematics rule. A number of researchers, however, did not find the same results (Crespo & Nicol, 2006; Ball, 1990; Wheeler & Feghali, 1983). Instead, these researchers found that most pre-service teachers did not even know that the answer should be undefined, let alone why. In some cases, the pre-service teachers recalled learning that anything divided by zero was zero, and in some others they reasoned out the answer of zero by thinking of zero as "nothing".

Wheeler & Feghali's (1983) study of pre-service teachers' understandings of zero, as noted earlier, is the only research that has explored this topic in breadth (beyond division by zero). The study revealed that pre-service teachers have many of the same misunderstandings and lack of knowledge as the above-mentioned research indicated for students. One such

similarity between teacher and student understandings of zero is in relation to the exchange of the word "oh" for the number "zero" (Baroody, et. al., 1983; Allinger, 1980; Whitelaw, 1984). Also like the students, many of the pre-service teachers that Wheeler and Feghali (1983) worked with believed that zero is not a number (Wheeler & Feghali, 1983), referring to it as 'nothing'. The pre-service teachers in this research explained that it did not meet their criteria for what a number is, namely that a number represents 'something' and therefore it must be countable. This interplay between zero and numbers, and 'nothing' and 'something' is remarkably similar to some of the arguments made by Greek philosophers almost 2000 years ago (Kaplan, 1999), , and plays a large role in both Leeb Luneberg's (1977) Wilcox's (2008) studies of how young children can come to understand zero. .

Wheeler & Feghali (1983) also repeated Inhelder &Piaget's (1964) earlier testing involving the null set with the pre-service teachers and found that, like the children decades before, the participants were not inclined to sort cards into sets that included a set of cards *without* particular characteristics. In following up with the pre-service teachers after the test, Wheeler & Feghali (1983) also found that some of the participants would not accept a null set even when it was presented as a possible solution to consider.

The only other place that demonstrations of teachers' understandings of zero (other than those related to division by zero), can be found is hidden within other research topics. One example of such research is Ma's (1999) study, which compared the mathematical understandings of teachers in the US and Singapore. Ma demonstrated through this research that the teachers from Singapore possessed a much higher level of "profound understanding" of mathematics than did the US teachers. When considered through the lens of "what's happening with zero", her research also revealed that, in particular, the US teachers had misconceptions

about zero within the concepts of place value and number decomposition. These misconceptions become evident through Ma's study of the teachers' understanding of two-digit subtraction and multi-digit multiplication. With respect to subtraction, the data collected revealed that the teachers did not understand the role of the decomposition of numbers into different groupings of tens and ones within the subtraction algorithm that they taught. As a result, these teachers also lacked understanding of place value and the role of zero within place value and the decomposition and subtraction of numbers. With respect to multiplication, the teachers were shown a students' solution to a multi-digit multiplication question which included a common error made by students – that of failing to account for the place value of the individual digits in the multiplicand and multiplier (or "forgetting to put zeros at the end" of each partial product). Ma asked the teachers how they would correct the student. Although the teachers noted that the student "did not understand the rationale of the algorithm" (p. 29), the teachers' own explanations of the student's errors revealed that the teachers did not understand place value and its role in multiplication themselves. Some of the teachers even suggested that rather than putting zeros in the partial products, the student should be encouraged to use Xs to hold the places. These teachers argued that by using zeros as the place holders, the students would be led to believe that the partial products were actually larger than what they really were (e.g., 4920 versus 492 which comes from multiplying 123 by the 4 in 645). Thus, in both subtraction and multiplication, the US teachers lacked understanding of number decomposition, place value, and zero.

1.4 Understanding of zero

Whether the understanding of zero be considered from the perspective of students or teachers, there is clear evidence within mathematics education research that there is cause for

concern, not only in the upper middle-level grades of mathematics, when teachers are introducing students to the complexities of dividing by zero for the first time, but even within the earliest elementary grades, and with respect to the understanding of teachers in general. This study was designed to explore the understandings of zero that elementary teachers have, how they came to have those understandings, and how they engage their students in understanding zero.

2. METHODOLOGY

Given the nature of the research questions for this study, a qualitative approach was required in order to explore the nuances and contextualizations for the different participants' understandings of zero. In this section, the methodology, method of data collection, and methods for analysis of the data are described.

2.1 Gadamer's hermeneutic philosophy

In order to explore the understandings, often hidden, that teachers have of zero, and how these ideas developed, were supported by experiences, and the issues those ideas raised for the teachers, a qualitative methodology based upon Gadamer's (1989) hermeneutic philosophy was used to frame the collection of data. In this hermeneutic approach, what and how one knows about an idea or concept is defined by one's past and present horizons of understanding related to the idea. In Gadamer's theory, the past horizon is a cognitive construct that contains the historical knowledge and resulting traditions that define a concept. As well, every person also has a present horizon of understanding that encompasses everything that one believes and understands about the concept at a particular point in time. Gadamer argues that the past horizon easily influences one's present horizon (with or without intention or recognition) and thus needs

to be exposed so that the present horizon can be better understood and evaluated. Through dialogue, the two horizons fuse together, with the past horizon remaining fixed, but the present horizon "continually … being formed because we are continually having to test all our prejudices" (Gadamer, 1989, p. 306). By engaging in a hermeneutic dialogue, Gadamer maintains that these prejudices are exposed and evaluated individually as the participants in the dialogue ask questions and seek to clarify their own understanding.

Gadamer (1989) proposes that one cannot develop a rich understanding by one's self. Rather, it is through dialogue with others that one becomes not only aware of one's own horizons of understanding, but also of the understandings of others. Anyone involved such a dialogue is not expected to outright reject their own horizons, nor reject those of others. Instead, the dialogue is intended to help each person understand the horizons of others and, as a result, their own horizons expand with this understanding. The goal is not to seek an ultimate truth, which Gadamer argues cannot be attained, but rather to play with the possibilities of understanding.

Experiences play a large role in the defining of one's horizons of understanding, so within dialogue it is important for those involved to share their own experiences, and to engage in new experiences. It is through openness to experience and consideration of other's horizons of understanding that meaning can be clarified, sought, and expanded (Gadamer, 1989). The environment within which dialogue and experiences occur must be structured to be open and non-judgmental (Silverman, 1991).

Gadamer (1989) also emphasizes that at any given moment, and in relation to any given context, participants in dialogue access only a limited portion of their horizons of understanding. Thus, dialogue plays a final role to expand the portion of the horizons of understanding being engaged by the participants so that both speaker and listeners are engaging in the discussion with

broader, better defined, and more closely aligned horizons of understanding which include awareness of the beliefs and knowledge of everyone involved (Silverman, 1991).

2.2 Collection and analysis of data

With Gadamer's hermeuntic philosophy as the guiding methodology for this research, it was important for data to be collected dialogues and experiences that the participants were engaged in. These interactions were designed to allow the researcher to explore and come to understand the past and present horizons of understanding of zero held by the participants. In addition, the data collection also had to allow for the two teachers to help in giving direction to the next dialogues and experiences as a response to their own changing awareness and curiousity about their understandings of zero.

Working within the above noted framework for the data collection, the study was comprised of three three-hour meetings of the two teachers and the researcher, an in-class teaching session in each of the teachers' classrooms for one hour (during which the researcher explored the teachers' students' understandings of zero while the teacher observed and kept anecdotal records), and an interview with the teachers immediately following the in-class session with her students. Each of these interactions focused on questions and activities that sought to reveal more of and challenge each of the teachers' present horizons of understanding zero and the relationship between those horizons to their past horizons of understanding. To help expose some of the nuances of their past horizons, the two teachers were also encouraged to keep a journal of their memories of learning about zero, as well as of any experiences or dialogues that they had outside of our scheduled meeting times with colleagues, friends, or students regarding zero. These journals contributed greatly to helping direct the study's conversations and explorations. Although the researcher was explicitly involved in the directing and redirecting of

the conversations and activities, great care was taken to not reveal aspects of her own present or past horizons. It must be acknowledged however, that the understandings of the researcher very likely influenced choices regarding activities and directions taken with the participants.

The specific research questions for this study were: "How do you understand zero" and "When and why does zero become a part of teaching and learning in your classroom"? These questions provided the initial direction for the group meetings, but with the acknowledgement that the questions might be modified or replaced in order to be responsive to the dialogue, and hence the horizons of understandings of the two teachers.

Each of the three meetings was tape-recorded, transcribed, and the transcripts verified by the two teachers. In addition, the one post-classroom session interview (only one was done due to time restraints for the one participant) was also tape-recorded, transcribed, and verified by the teacher. The classroom visits were not recorded, but rather both the teachers and researcher made anecdotal notes of the experiences and these were discussed and reflected upon during the follow-up interview and the third group meeting.

Prior to the first group meeting, the two teachers were asked to reflect upon what they knew about zero, when they learned about it, and what language was used with respect to zero. These memories were a source of much of the dialogue during the first meeting, as was an exploration of works of children's literature and their inclusion/exclusion of zero. Although much of the children's literature used was familiar to the two teachers, the exploration of zero within the literature was a new experience for them and was a rich source for the ongoing dialogue. The dialogue that resulted from the teachers' memories of learning about zero and from their analysis of the role in the children's literature both sought to expose not only parts of their present horizon of understanding, but also to have the teachers consider what some of

society's past horizon understandings of zero are and to reflect upon the validity and relevance of those understandings to their individual situations.

The second meeting began with the sharing of experiences and recalled memories the teachers had had related to zero since the previous meeting. This meeting also focused on what the teachers believed about students and the number zero, including what students know about zero and what they should know about zero and why. The question of why students should learn particular facts or ideas about zero again helped to expose some of the past horizons of understanding for zero held within our society, while also engaging the teachers in revealing more of their present horizon of understanding zero. This information helped to inform the choice of activities that the researcher used in each of the classroom sessions with the participants' students.

The final group meeting involved the teachers sharing new ideas, recalled memories, and reflections on their observations of the in-class sessions, as well as a discussion of what they felt to be most important for elementary students and teachers to understand about zero and how those understandings might be developed. It was an opportunity to bring together many of the different facets of their present and past horizons of understandings of zero to develop a broader perspective of what each teacher knew and believed about zero, while also allowing for new ideas and connections to be made. The meeting ended with the researcher sharing some of her present horizon understandings of zero and teaching and learning about it, as well as how the dialogues and experiences had helped to reveal, clarify, change, and expand the researcher's own horizons of understanding zero.

The in-class teaching sessions were included in this study to provide a new experience for the teachers that could engage them in further reflection, discussion, and exploration of their

horizons of understanding. The activities used in each session were different, and were designed to engage with the students within the contexts that their current mathematics study was focused. In one classroom, the students were focusing on place value and its role in addition and subtraction, while in the other classroom, students were learning about two-digit multiplication. The activities in each session were designed to probe how the students in each classroom understood zero and to explore how the teachers interpreted and understood the students' engagement and responses to the activities. The interview that was held immediately following one of the in-class session provided the researcher with an opportunity explore how that teacher placed the experience within her present horizon of understanding zero as it related to the understandings of both herself and her students. For the second session, this dialogue occurred in the final group meeting and involved both teachers.

The analysis of the data involved the recognizing and codifying of common themes of agreement and/or disagreement within the two teachers' horizons of understanding zero. Once the themes were defined, the researcher then compared those understandings with the historical development of zero and to the prior research findings related to student, pre-service teacher, and teacher understandings of zero.

3. RESULTS

The analysis of the collected data revealed a number of links between the history of zero, past research findings related to student, pre-service teachers, and teachers understanding of zero and the two teachers horizons of understanding zero. However, there were ideas generated by the two teachers that had not previously been referenced in the literature. The main six themes that emerged from the data analysis are described in this section. In this discussion of the results, the pseudonyms Elaine and Nora will be used for the two teachers.

3.1 The start of knowing

When first questioned about when and how they first learned about zero, both Nora and Elaine had very few memories, which resulted in much speculation and misgivings about what and how they had been taught. Elaine spoke of learning that zero was the starting point of numbers, but not a number itself. Then, in middle school, Elaine was told that zero was the "middle of the integers", like a type of physical divider. These two meanings were irresolvable for Elaine as a student. Without knowing of the social construction and evolution of zero and the integers, as well as not knowing that zero defines a quantity, Elaine's learning about zero, as well as about whole numbers and integers, was relegated to points of trivia to be remembered, but not, necessarily, understood. Elaine's view of zero as not being a number correlates with findings from Wheeler & Feghali's (1983) study of pre-service teachers and with research involving students (Evans, 1983; Anthony & Walshaw, 2004; Wheeler, 1987), however, the emphasis on zero being viewed as a starting point is one that was not put forward in prior research.

Nora quickly came to the conclusion that her memories of learning about zero were in fact memories of not learning about zero. As had been the case for many students in Wheeler's (1987) study and for many of the pre-service teachers in Wheeler and Feghali's (1983) research, all that Nora could recall was being told that "zero is nothing". Nora struggled with this definition of zero. She frequently spoke of how it gave rise to her belief that zero was not important and as a result could be ignored. Throughout our discussions, Nora regularly returned to this definition and trying to remedy it was a major motivation for her seeking of a philosophical and theoretical understanding of the concept of zero. Eventually, Nora expanded her definition to "nothing of *some thing*". This modification to her definition parallels the conclusion by the students in Leeb-Lundberg's (1977) research: "Zero is nothing – of

something!" (25) and that of Wilcox's (2008) daughter's statement that: "… [zero] means something. It means you don't have anything" (204). This modification to her definition allowed Nora to later create a philosophical understanding of zero that made sense of the technical roles and rules for zero that she had learned as a student.

The students in both teachers' classes also perceived zero as "nothing" and said it could be ignored. This concerned both Nora and Elaine, and Nora was particularly troubled to hear her declare zero was not a number. The students reasoned that if zero was a number, then they would have been taught it when they were taught about one to ten and that it would be said in number names (e.g., we say "twenty" and "twenty-one", but if zero was a number, we would say "twenty-zero"). Although some of the past research had demonstrated students' confusion over the naming of numbers containing zeros (Kamii, 1981; Baroody et. al., 1983), these explanations about their thinking about number naming revealed that the students were generalizing patterns and ideas from the mathematics that they had learned which were causing them to come to invalid conclusions (zero is not a number).

3.2 Memories related to computations

With respect to zero within computations and computational procedures, Nora's memories again focused on the lack of inclusion of zero. Her memories were primarily about addition and subtraction, and specifically being taught about "carrying" and "borrowing" the "1" (see Figure 1).

$$\text{carry the one} \quad \nearrow \quad \begin{array}{r} {}^{1}38 \\ \underline{27} \\ 65 \end{array}$$

Figure 1. Carrying the one

As a student, Nora was perplexed by this procedure because she did not understand where the "1" came from. As an adult, Nora realized that the "1" actually represented "10", but she questioned why the procedure had been described as "carry the 1" and not "carry the 10". She argued that by dropping the zero off the ten, this procedure had confirmed her belief that zero could be ignored. Nora questioned what she really understood about the role of zero in any computation, and if she was ignoring it everywhere.

Similarly, Elaine spoke of learning to divide questions such as $20\overline{)340}$ and being told to "just knock off the zeros". Elaine did not know why she could do this, only that ignoring the zeros made things easier.

Nora and Elaine's descriptions of their limited and misconstrued understandings of computations involving zero reflect many of those by Ma (1999) with the teachers from the United States' understandings of subtraction and multiplication and of those identified in Neuwirth Beal's (1983), Evans' (1983), Anthony & Walshaw's (2004), and Wheeler's (1987) research involving students. Both teachers had learned to do computations involving zero as procedures without any basis of understanding. Over time, they had come to theoretically understand some of the computational procedures they had been taught, but many of the procedures had remained unquestioned until our meetings.

3.3 Zero Outside the Classroom.

Elaine and Nora both spoke of how much of their understanding of zero developed outside the classroom. Elaine recalled having an aunt who used the word 'aught' in place of zero, and both teachers spoke of how 'oh' was frequently used when saying the number zero in different contexts, such as postal codes. Such uses of "oh" was also frequently noted in other research (Wheeler & Feghali, 1983; Allinger, 1980; Baroody et. al., 1983; Whitelaw, 1984). This avoidance of the word zero had reinforced many of Nora and Elaine's mathematically incorrect beliefs about zero, including that it can be ignored.

Elaine and Nora also talked about how, while regularly avoided in numerical contexts, the word "zero" was frequently used in non-numerical contexts. Elaine remembered zero being associated with people in ways that indicated they were defective or substandard, and Nora provided the example of the novel *Holes* (Sachar, 2000) and its central character, Zero, as verification. Both teachers agreed that these experiences gave them the belief that "zero" somehow indicated a deficiency or disapproval and that they did not have such a pessimistic (or emotional) attitude towards other individual numbers. Although Allinger (1980) mentions the occurrence of this type of use of the word zero, Nora and Elaine proposed that this non-mathematical contextualization of zero influenced how they view and think about the mathematical concept of zero. Whether it be the casual replacing of the word "zero" with "oh" or the use of zero as a qualitative descriptor, their everyday encounters with the quantity zero and the word "zero" presented the two teachers with incomplete and often misleading facts that contributed to their lack of theoretical understanding of zero within the context of mathematics.

3.4 Marginalization and legitimatization of zero

Throughout this study, Elaine and Nora struggled with their understanding of zero in terms of what they had been taught as a student, in making sense of what they teach to their question that emerged through our dialogues: "why do I think I was taught about zero in the way that I was," led Nora and Elaine to seek for and identify ways in which zero had been marginalized, not only for themselves, but within society as a whole. The natural consequence was that they both then sought ways in which this marginalization could be corrected and zero could become a legitimate and valued number. In both cases, Elaine and Nora returned to their earliest memories of knowing zero to define where and how this legitimatization could take place.

Nora began by looking for where zero was ignored, with her first concern emerging from students' learning of oral and symbolic representation of numbers and letters. She spoke of how, "in public it's acceptable to switch 'zero' and 'oh'," and that "when kids are starting to read, they get confused with "1" and "l" … but it's not viewed acceptable to confuse them." Nora said that she felt that little or no attention was given to students not confusing the letter "O" and the numeral "0". This inconsistent use of words and the failure to give equal status and time to the number zero as was given to other numbers and letters bothered Nora. This concern regarding equity for all numbers in the classroom became a frequent perspective that she brought to the discussion. In order to legitimatize zero within the context of numbers, Nora felt that it was important that a deliberate attempt be made at using the correct vocabulary when speaking in contexts involving zero, and to put an emphasis on distinguishing between the written letter "O" and the numeral "0".

During the in-class session with Nora's students, another representational confusion emerged that surprised both Nora and the researcher. When asked to identify objects that they had zero of, Nora's students pointed to the objects as they said "clock", "fan", and "my earrings", all three of which were circular in shape and were present in the room, if not in the direct possession of the student identifying the object.. After some discussion, it became evident that the students thought that "zero" meant "having the shape of a circle". For Nora, this confusion of the shape of a circle with the meaning of zero added an additional source of marginalization of zero in that zero. She argued that this too was an aspect of teaching and learning about zero that must be prevented. Although there were references to the shape of the letter "O" and the digit "0" in some of the research related to students (Baroody, et. al., 1983; Allinger, 1980), Nora's students identified a more extensive misunderstanding and point of confusion, that of extending beyond the letter "O" to any object that is circular, which has not been discussed prior in the research.

When considering children's literature related to numbers, Nora was concerned that in many of the works we explored, the only inclusion of zero was through the numeral "10" and not as a number in its own right. Nora also felt that in those works where zero was presented, the ways in which it was shown and described was inconsistent with that of other numbers. As an example, in *Zero is not enough* (Zimmermann, 1990)*,* Nora was very concerned because although zero was defined in words, it was not on any of the placards held up by the monsters in the illustrations, whereas, all of the numbers from 1 to 10 were on placards. Nora argued that because the book did not present zero in exactly the same way it did all other numbers that the book wasn't doing zero justice. In order to correct this injustice, and give zero equal standing and

importance with the other numbers in the book, Nora suggested the addition of a monster holding a placard with zero on it.

Nora raised the concern that teachers marginalize zero in their teaching of mathematics. Nora began the discussion by admitting that, "honestly – I [have] never taught zero," adding that she felt that "teachers just aren't… aware of incorporating zero into lessons". Nora went on to reflect that, "I think maybe I don't understand zero the way I should… If I can't explain it myself, how am I explaining it to the children?" This marginalization of zero through the teaching and learning process was one that Nora felt was continuing from when she had been a student. She argued for more emphasis on zero in university education methods classes and in resources as ways to legitimatize and bring zero into the classroom.

Nora noted that zero was not included as a number in the hundreds charts used in elementary classrooms. Although there do exist commercially produced charts that include the numbers of 0 to 99 rather than 1 to 100, Nora proposed a unique change to the chart in which zero would be added above the column containing multiples of tens. Nora argued that by placing the 0 above the 10, the students would see the connection between the two quantities in terms of both having zero-ones. In this way, zero was being placed in its equitable position in relation to the other numbers.

Nora's focus on where zero is ignored also brought her to consider many situations and contexts that sit outside the realm of theoretical mathematics. For example, Nora was concerned that in naming streets and avenues, "you have First Avenue, Second Avenue, Third Avenue, but no Zero Avenue". Again, Nora felt that this was an error that could and should be corrected.

Nora also struggled with recording time. She noted that between 12:59 and 1:00 there should be a time called 0 to signify starting the cycle over again: "zero would be the millisecond

between 12:59 and 1:00." Similarly, Nora proposed that there should be a year 0 between 1 BC and 1 AD. In both cases, Nora was suggesting that these infused zeros were more of a movement to 1 than a quantity. Although fundamentally different, it is interesting to note how this new conceptualization that Nora had for zero was similar in some respects to the understanding of zero as a process, motion, or change that emerged during the development in calculus (Kaplan, 1999). In her suggested inclusion of zero into time on clocks and the calendar years, Nora did not recognize that her approaches were not giving zero the same status or role as other numbers and thus did not help in her search for that particular type of legitimatization of zero.

For Elaine, the marginalization of zero occurred in situations in which there was a starting point, but no zero. Her attempts at legitimatizing zero began with her proposal of a solution to the issue of zero not being a starting point for integers. By viewing the integers as two separate sets of numbers, positive and negative, she argued that zero was in fact the starting point for both sets. Elaine was not aware that by doing this, she was taking the stance that the integers can only be viewed as two distinct sets, and not as one cohesive set of numbers. There is some question as to whether she viewed the zero she defined (as the starting point for positive and negative numbers) as a single entity or two distinct ones.

Elaine then moved the conversation to changes outside of the specific realm of mathematics that needed to be made in order to incorporate zero in meaningful contexts as a starting point. First, Elaine spoke of baseball, and of changing "home" base to "zeroth" base: "Its home base, but you're going to first – so where are you starting from? … They're not calling it zero, but it's your staring point". Elaine did not recognize that, by the same logic, the home base would then have to become "fourth" base at the desired completion of a batter's run.

Elaine also argued that timers should always count up from zero, rather than down from a specified time because zero is where timing starts. Elaine explained that she tended to interpret the passage of time "from a moment in the future" back to the present time rather than from the present to a point in the future. The history of the Mayan society includes a similar notion of counting back in time to zero (Seife, 2000).

Next, Elaine asked "does zero gravity exist?" to which Nora replied that zero gravity was "buoyancy – it's when you float". Together, the two teachers concluded that zero must be the lowest value for a measurement of gravity. The discussion of the two teachers around gravity demonstrated that their understanding and conceptualization of zero had a definite link and impact on other understandings outside the context of mathematics.

Elaine's final recommendation was to change the role of zero with respect to temperature, She argued that zero should be the starting point of, or lowest possible, temperature. Nora disagreed with this proposal, because she valued $0°$ C as the freezing point of water. Nora's argument can be seen as supporting her goal to make zero something that cannot be ignored, while Elaine's argument supported her belief that zero should always be a starting point. Interestingly, Elaine did not argue for zero being the starting point of both the positive and negative temperatures as she had done for the role of zero in integers. Both teachers were unaware of the Kelvin system of measuring temperature.

In all these cases, Nora and Elaine were attempting to bring zero to the attention of the public eye, to make it "something" rather than "nothing", and to bring consistency to the "world" of zero. Their arguments were almost exclusively based upon and in reaction to the technicalities that they remembered learning about zero and were rarely supported by an understanding of the theoretical and socially constructed aspects of zero.

3.5 Representing zero

As Elaine and Nora formed a sense of importance for zero, they began to argue for purposeful and meaningful teaching about zero to students. Both teachers felt that students should learn about zero as soon as they started learning about whole numbers, but Elaine struggled with finding a way that students so young could understand a concept as abstract as zero. Nora, on the other hand, proposed three different types of activities that young children could engage in to begin their understanding of zero: through the absence of specified objects, as an empty set that proceeds one, and as the result of subtraction.

One of Nora's suggestions was to highlight the absence of a specific type of object on a page in a literature book. For example, in the book *Ten friends* (Goldstone, 2001) the description of an illustration as showing "2 teachers, 2 trolls, and 2 tycoons…" could be modified to read "2 teachers, 2 trolls, 0 frogs…". Nora explained that she specifically chose frogs because, although they did not appear on the current page, they were found on other pages in the book and thus had been a possibility for the situation and had connections to the students understandings of quantity. As an extension activity, Nora suggested that students could create their own pictures with statements describing the quantity of objects or things present; including what the students noticed that there was zero of. With respect to the historical development of the concept of zero, Nora's suggested activities intended to engage students in understanding zero as a quantity, much in the same way that the mathematicians of India came to understand zero (Seife, 2000).

A second type of representation for zero that Nora introduced was intended to help students understand zero as the quantity before one. She described how she could use manipulatives to represent the narrative of, "first I have zero, now I have one, now I have two…". She explained that, by having no manipulatives present for the first part of the narrative,

students would come to understand that zero represents a set of objects that is empty. Although she did not refer to this notion as the "empty set", her activity was an attempt to have student construct an understanding of zero as the "null set" as emerged through the history of zero (Kaplan, 1999). Based upon the research of Inhelder and Piaget (1964), as well as Wheeler and Feghali (1983), the understanding that Nora sought to have students gain was one that students and pre-service teachers would have great difficulty in attaining; however, according to Leeb-Luneberg's (1973) and Wilcox's (2008) research would indicate that in fact young students could understand zero in this way.

Nora also extended this notion of the empty set to understanding place value. In *Anno's counting book* (Anno, 1977)*,* Nora noted how the inclusion of an empty set on the page about 10 could be used to support the students' learning of place value. In the book, 10 x 1 grids are shown on each of the pages associated with a number between 0 and 11. For example, on the zero page, none of the grid is coloured in, while on the page for 11, two grids are shown, one completely coloured in and one with only one square coloured. On the page for 10 Nora noticed that only one grid was shown and it was completely coloured in. To connect the students' understanding of zero being a null set to its role in the place value system, Nora argued that the page for 10 should in fact have two grids on it – one completely coloured in and one with no colour. In this example, Nora was brining together the roles of zero as a quantity and as a place holder, just as the development of the Hindu-Arabic number system historically did (Seife, 2000).

Finally, Nora reasoned that students could represent and understand zero in relation to subtraction. In this case, Nora described an activity she would use in which students would tell the subtraction story that could have resulted in a particular picture that she shows to them. For

example, the story for a picture of trees without birds might be that "five birds were there and then flew away". This understanding of zero is one that also emerged in India, and became very important centuries later in the advent of the two-column bookkeeping system (Kaplan, 1999). As well, Leeb-Lundberg's (1977) research demonstrated that elementary students were capable of understanding zero as the subtraction of equal quantities.

Elaine liked Nora's suggestions as activities that students could do, but questioned whether the students would actually understand zero from these experiences. Elaine spoke of young students being at a concrete stage, yet all of Nora's forms of representation required students to abstract the notion of zero from concrete representations of quantities that are not zero. Elaine wondered how students could learn about zero concretely, in which they were able to "see" and "touch" zero. This quandary was one of the root causes for Greek society's struggle with accepting zero as a quantity for hundreds of years (Barrow, 2000; Kaplan, 1999; Seife, 2000). Leeb-Luneberg's (1977) and Wilcox's (2008) research demonstrates instances in which young students could understand the abstract nature of zero, and Cockburn's and Littler's (1977) chapter 0 includes a number of activities and ideas for teachers to use that are similar in intention to those proposed by Nora.

3.6 Zero and student learning

When contemplating the current role of zero in elementary students' mathematical learnings, Nora and Elaine focused on zero within place value and number compositions, and within computations. With respect to place value and number composition, Nora reflected that if she gave her students the numbers 45, 54, and 37, "they'll see the 7 and think that the 37 is the bigger number". Nora explained that this showed her that the students did not understand place value. This discussion prompted Elaine to add that when showing students numbers on a place

value mat "If [it was] a number in which there was something in every place value [the students had] no problem. But as soon as I removed something off the matt," then the students believed that nothing existed there. Elaine explained that it was because the students do not understand zero as a quantity that they don't know how to deal with zero in place value in numbers. Nora agreed and said that this was the root of her students not being able to order the numbers correctly. Pasternack (2003), Evans (1983), and Baroody et. al. (1983) all provided similar evidence related to, but gained through different tasks, students' misunderstanding of zero in place value.

Nora and Elaine hypothesized that their students' lack of understanding about the role of zero in place value was also the source of their problems with naming numbers. They stated that the students were merely relying on procedures that they had memorized to name numbers. Elaine spoke of how many of her students "just lose [the zero]" when naming numbers, such as 204 being called twenty-four. Interestingly, Nora's students had said in the in-class session that zero was not a number, and one of the arguments that they provided was "you don't say it when you read numbers, but you say all the other number". One student then provided the example of 20 (twenty) and 21 (twenty-one). This student argued that if zero was a number, then it would be called 20 "twenty-zero". Previous research, such as that of Baroody et. al. (1983) and Kamii (1981), spoke of students ignoring the zero in naming numbers as Elaine had noticed, but the previous research did not indicate that students' reasoning in this regard might actually be a result of our naming conventions for numbers and the way that it treats zero differently.

Elaine and Nora generated a number of suggestions for what they believed to be important in students' development of an understanding of zero. First, Elaine stated that students must learn that although zero can act as a place-holder, it cannot be ignored because it indicates

something about the size of the quantity. Nora agreed and added that place value should receive more emphasis in grade one. Both teachers felt that there was too much emphasis in the earlier grades on addition with not enough emphasis on number decomposition. Elaine commented "9 + 1 = 10 or ten plus one more is 11 … we're not teaching [place value] – we're teaching addition". Elaine continued to provide examples, circling the addition sign (+) in every statement she wrote, explaining that the emphasis was on that sign and not on knowing the numbers.

In discussing the four operations on whole numbers, both Elaine and Nora argued for the standard procedures and algorithms to be deemphasized and left until later in the students learning. Instead, the two teachers felt that it was important that the students use their understanding of place value and number decomposition, as well as of the operations, to develop strategies for performing different types of calculation. Their concerns about student and teacher misconceptions were reflective of the findings in Ma's (1999) study involving US and Shanghai teachers. Repeatedly throughout their discussions about the teaching and learning of the operations, the two teachers kept revisiting the importance of students being flexible in their understanding of the decomposition of numbers (e.g., recognizing 204 as 20 tens and 4 ones, 19 tens and 14 ones, etc) and the role of zero in place value. Both teachers also emphasized the importance of never giving the students the impression that zeros were being ignored or dropped off.

4. DISCUSSION

The past and present horizons of understanding that emerged from the dialogues and experiences that Nora and Elaine engaged in during this study had many parallels as well as some departures from what previous literature had noted for students, pre-service teachers and

teachers. This section summarizes those similarities and differences and then proposes a framework that could be applied to future research that is analyzing one's understanding of zero.

4.1 Results summary

Through the dialogues and experiences the uncovering of the past and present horizons of understanding of zero for both Nora and Elaine highlighted understandings that were both correct and incorrect. Just as had been previously noticed for students (Baroody, et. al., 1983; Allinger, 1980; Whitelaw, 1984) and pre-service teachers (Wheeler & Fegahli, 1983) Nora struggled with her understanding of zero as "nothing", but she was able to adjust her definition to "nothing of something" as had also been done by Leeb-Lunburg's (1977) students. Alternatively, Elaine sought to make sense of zero being a starting point in all situations, resulting in zero not necessarily being a number, just as pre-service teachers argued in Wheeler & Feghali's (1983) research.

For both teachers, zero in computations was not well understood and was a source of frustration in trying to teach students. Many of these same issues had been raised by Ma (1999), Neuwirth Beal (1983), Evans (1983), Anthony & Walshaw (2004), and Wheeler (1987) with respect to teachers and students involved in their research. Nora and Elaine's emphasis on the importance of understanding zero as part of place value and number decomposition was also a finding of Pasternack (2003), Evans (1983), Baroody et. al (1983), and Kamii (1981).

Elaine and Nora brought forward the use of the word "oh" in the place of "zero" and the resulting confusion between the two concepts by students which was a finding in the research of Wheeler and Feghali, (1983) for pre-service teachers as well as for students in the research of

Allinger (1980), Baroody et. al. (1983) and Whitelaw (1984). The negativity associated with the word zero that Nora and Elaine noted can also be found within the research of Allinger (1980).

The two teachers engagement in the dialogues and experiences also revealed insights and ways of perceiving zero that did not emerge in previous reported research. The use of a hermeneutic approach allowed the teachers to explore the reasons why they held the beliefs that they did about zero, and as a result caused them to also question the validity of what they knew and had been told. The result was that the teachers sought to identify cases where zero had not been included, both within their learning as a student and in society in general, and to propose ways to rectify the situation. In many of these instances, the teachers were not aware of the cultural history that had led to the marginalizations of zero that they perceived. Elaine and Nora also grappled with how zero could be represented and understood by elementary students, taking them into an exploration of their beliefs about cognitive readiness and pedagogy with relation to zero.

Finally, Nora's students brought forward an understanding of zero which had not been reported previously – that zero is the same thing as any circle. Although some research (Allinger, 1980' Baroody et. al., 1983; Whitelaw, 1984) mentions the exchange of the word "oh" for "zero", there is no discussion of students assuming that because the symbol for zero was circular in nature that zero must be in itself a circle.

4.2 A way to conceptualize the understandings

As the group meetings and in-class sessions proceeded, it became evident that the teachers were exploring and struggling with knowledge that had evolved over time and that had resulted in two different, yet related categories of understandings: procedural and technical understandings of zero, and philosophical and theoretical understandings of zero (see Figure 2).

The philosophical and theoretical understandings of zero are the result of both societal conventions related to zero as well as the theories and axioms of meaning given to zero by past and contemporary academic mathematicians. On the other hand, the technical and procedural understandings of zero are the routines carried out in mathematical situations that involve zero. These two categories are related to each other in that the technical and procedural understandings of zero can be directly explained by the philosophical and theoretical understandings of zero. For example, why one puts zeros in when doing multi-digit multiplication is a direct consequence of the theoretical definition of the place value of the quantities being multiplied.
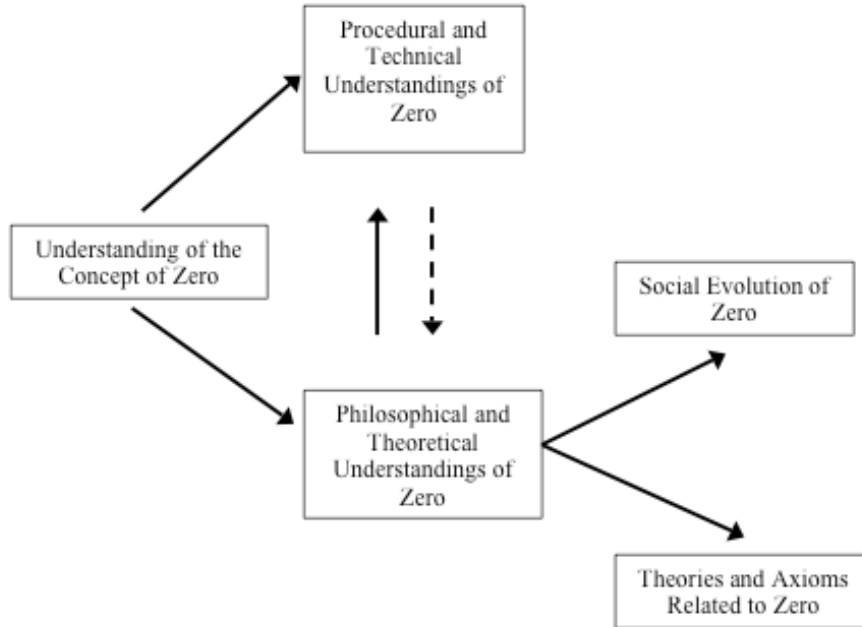
Figure 2. A Conceptual Framework for Viewing Teachers' Conceptions of Zero

Initially, it was the researcher's view that one's procedural and technical understandings would be supported by one's philosophical and theoretical understandings of zero, however, Nora and Elaine tended to have procedures and techniques that they used, but they did not have the understanding of why they should use those procedures and techniques. Consider even where

the two teachers began in their memories of learning about zero. Nora believed it was nothing (something to be ignored) and Elaine believed it was the starting point. Both conceptions of zero were based on nothing more than a single "fact" that had been stated without any evidence of reasoning or support.

As adults, they had begun to construct possible philosophical and theoretical underpinnings for their procedures and techniques. In many cases, such as Nora's desire to write "10" rather than just "carry the 1" in addition, and Elaine's emphasis on number decomposition before starting in on operations, can be seen to be developing conceptualizations of the theoretical and philosophical foundations of understanding zero. However, there are many cases where the two teachers' attempts at building theoretical and philosophical reasoning for their technical and procedural understandings conflicted with the true theoretical and philosophical understandings of zero within mathematics. This often caused the two teachers to venture into "reforms" to the way zero is used and known that, although interesting, are neither practical nor informative from a mathematical perspective. Elaine, believing that zero must be a starting point, which it need not be, becomes trapped in an exploration to "convert" the world around her, while Nora, believing that zero should be everywhere where any other number is, seems ready to begin a crusade to bring zero to its rightful place of respect. Neither teacher is aware of the conventions, defined by society over time, that have brought rise to these situations that they desire to change. As a result, their efforts in trying to get to understand zero better become derailed by their imaginations and misconceptions. Thus, it would seem that the interplay, or lack thereof, between the two categories within understanding of zero, philosophical and theoretical, and technical and procedural, can impact the depth and accuracy of understanding that one has about zero..

5. CONCLUSION

The concept of zero is a complex mix of social evolution, theoretical mathematics, and procedures. As a result, the interplay between these aspects of the understanding of zero is of great importance in developing a cohesive and mathematically correct understanding of the concept. Reflecting once more upon the research related to students' and teachers' understanding of zero done prior to this study, it appears that Nora and Elaine's abundance of procedural and technical understandings, without strong philosophical and theoretical understandings of zero, may be the status quo for students and teachers alike.

Understanding zero is foundational to understanding mathematics. Whether it be place value and whole number computational situations as Nora and Elaine discussed, or other topics such as locating fractions on a number line or understanding division by zero and asymptotes of functions, misconceptions about zero can lead to students failing to learn key ideas in mathematics and teachers struggling to try to correct the situation without understanding it themselves. Research such as Leeb-Luneberg (2003) and Wilcox (2006), and this study have shown that given interactive, contextualized, and meaningful learning experiences, both students and teachers can learn to better understand zero. Zero needs to become more than "nothing" within the classroom and in pre-service education programs.

The research that this article is based upon considered a very limited sample, only two teachers; however, it does reveal some findings that are new, promising, and as such, warrant further investigation. Such future research, involving a larger and similar population, may very well provide insights into teachers' philosophical and theoretical as well as procedural and technical understandings of zero, which in turn could help to better inform pre-service teacher

education programs and in-service teacher professional development in relation to the creation of robust understandings of zero by pre-service and in-service teachers.

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Revisiting Tatjana Ehrenfest-Afanassjewa's (1931) "Uebungensammlung zu einer geometrischen Propädeuse": A Translation and Interpretation

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EXERCISES IN EXPERIMENTAL GEOMETRY

BY

TATJANA EHRENFEST-AFANASSJEWA

1931

TRANSLATOR'S PREFACE

This remarkable booklet first appeared in 1931 under the title "Uebungensammlung zu einer geometrischen Propädeuse", in a charmingly eccentric form of German — a well-learnt foreign language to the author. In 1901, Tatyana Alexeyevna Afanasyeva and Paul Ehrenfest were studying mathematics at Göttingen, were married in Vienna three years later, and after several years in St.Petersburg moved to the Netherlands in 1912, to stay. These details are recalled only to explain the quality of Tatjana's German: lucid, even eloquent, but not quite standard. The translator sometimes had to guess, and it can only be hoped that he usually guessed right. Square brackets denote additions made by the translator in an effort to clarify meanings.

<div align="right">Klaus Hoechsmann, March 2003</div>

AUTHOR'S PREFACE

The present exercises differ from those usually found in propaedeutic geometry books in one essential respect: they do *not* aim at visually and intuitively grasping the *theorems* of a systematic course, but at developping the most important geometric *notions* while acquiring a repertory of spatial images and mental representations, which is as large as possible. Therefore the *qualitative* aspect of spatial relations will be emphasised.

Creating this collection was guided by the conviction, that fostering imaginative ability by means of the pictorial proofs in the traditional preparatory courses was, on the one hand, insufficient – and on the other, unnecessary – unnecessary when the imagination is sufficiently developped by the time it encounters any theorems; and that the systematic course would be more valuable in every regard, if the fuzzy preliminaries in previous classes had not made it inevitably tedious.

For the rest, the following introductory sketch may throw some light on the author's position relative to the predominant ideas about teaching geometry, and explain her own conception of the most appropiate approach to the subject.

INTRODUCTION

§1. As is well known, there are two directions in teaching geometry. The first, rather old one, is based on the assumption that the fundamental notions and axioms of geometry are innate to human beings, and that it therefore suffices – starting there – to unfold the whole system of theorems and proofs for the students, in the spirit of Euclid.

The other direction – which has lately gained much ground – begins with the insight, that students often show a far greater understanding of what the theorems say if these are illustrated by concrete examples in real situations, and are more likely to believe in their correctness if they are corroborated by inspection or even measurement – rather than being demonstrated by a rigorous discursive proof.

For the representatives of both directions, the "geometry" which the student must know is the contents of the same *theorems* – they differ only in what they see as valuable in these theorems. For those of the first direction – I shall take the liberty to call "logicians" hereafter – the essence lies in the educational value of the flawless argument; for those of the second direction – I might perhaps be allowed to call them "technicians" — it lies in the agreement of these theorems with physical experience, and in the possibility of using them for practical purposes and for the investigation of other scientific fields.

These two uses of geometry are respectively held in contempt by the most extreme representatives of the two directions, and – by the less extreme – acknowledged only sparingly and even more sparingly employed.

It seems to me that the inability of both parties, to convince each other of the value of what each of them holds dear in geometry, is due to their too narrow understanding of geometry – too narrow in the same way. For: they both forget that the systematic geometry course represents only an excerpt of the inexhaustible complex of spatial problems.

What impact this situation has on teaching, I shall presently try to explain.

§2. What is the "educational value" of a school subject? – Probably this: that the method of treating problems in this subject becomes so much the student's own that he also transfers it to other areas of his thinking.

We shall therefore focus on the following questions:

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1) What is most typical in the method by which geometry problems are handled?

2) Why is it desirable – and is it desirable – to transfer it to other areas?

3) Is it possible to transfer it to other areas?

4) Do the logicians do everything necessary to ingrain it in the students?

Of course, we shall here only comment on those aspects which are apt to shed some light on our collection of exercises.

1) The geometric method is adorned by the words "logical thinking", and it is thereby suggested that geometric truths are to be sought and found by certain recipes which are available a priori.

However, by a prescribed recipe one cannot find anything essentially new, since the new is always unknown, one does not know where it is, why it is needed, and often not even *that* it exists. With all our skill in proving the theorems of Euclid, we possess, for instance, no recipe for the direction to choose toward new research projects, in order to discover further relations of importance to us. For such really new things, even geometry contains no search engine.

One cannot speak of "logical seeking" and should not speak of "logical thinking": this combination of words is misleading and has caused actual damage in education, because it has diverted the teachers' thinking from an attentive examination of the thought-process!

If the word "logical" is taken to mean that which is free from contradiction, well arranged, and transparent, it may be used only in connection with the Euclidean *style of presentation*, which is so characteristic for geometry.

If, however, it is asked whether there is not something typical to emphasize in the process by which the geometric system arose, it can certainly be said that there is such a feature: it is *the relentless striving for utmost clarity.*

The "geometric method" would thus be: not to let up until a given problem has become entirely transparent, to formulate the result in the most vivid manner, and to arrange all obtained results so that they form a harmonious, well laid out system.

2) That harmony and order are the ideals of any research need probably not to be discussed [does it?] — at any rate, we shall herewith regard our second question as dealt with.

3) Can we achieve that our pupils enhance their knowledge in various other areas by striving for the utmost clarity? Personally I am convinced that better results are attainable in this regard, if only the pupils are given every opportunity, in their geometry lessons, to experience that geometric

method.

4) The question, however, of whether this is being accomplished by the logicians, must decidedly be answered in the negative. Indeed, they do not see the most responsible task hidden in the geometric system — because they think only of the finished theorems already in the books.

§3. Building a system means creating order out of chaos. Before one can spread out the system in front of the admiring public, one has — darting about in a somewhat unsystematic way — brought together the following things:

a. One has taken *cognisance* of the *existence* of the research area in question — by having isolated the features belonging to it in diverse phenomena, and joined them in a common category.

b. One has formulated the first *problems with sufficient clarity.*

c. Gingerly brought out those *concepts*, to which the other concepts of the area can be reduced (let us recall the hopeless state of mechanics, before the concept of "mass" was introduced by Newton.)

d. Determined those elementary relations between these concepts — the *axioms* — from which all others can be deduced as logically necessary consequences (in many areas of natural science they are still being sought today.)

e. Invented the most appropriate *style* for the *presentation* of the system.

For every science, items (a) and (b) seem to be brought about in a rather long semi-conscious progress, by many people, under the influence of life itself. In geometry (as opposed to mechanics, for instance), this seems to be the case for (c) as well. On the other hand, (d) and (e) might well be considered as the wonderful and immensely valuable achievement of one man — Euclid. (In this context, the imperfections later discovered in his system pale, of course, into insignificance!)

What do the logicians think of all this? — They believe that the geometric axioms are incarnated in the newborn ready-made, and therefore the most important task of the Ancient Greeks was this: to doubt everything, even the most evident, as much as possible, in order subsequently — with the help of these axioms — to remove the doubts; and that precisely this constitutes the "scientific" aspect of the geometric method.

And they place this in the foreground so persuasively, that the technicians too, even when they are becoming teachers of geometry, remain under the spell of this notion of what science is about. Since, however, they abhor proving the obvious, they maintain a hostile attitude toward the whole

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Euclidean style of presentation.

The logicians do not show the *progress from chaos to system*, which comes with the *systematic treatment* of the subject, and thus "geometry" appears through them to be a mental game cut loose from anything material, and instead of operating with *concepts* — which are, after all, obtainable only by a personal act of abstracting from one's own live experience — pupils deal with *names* and *diagrams* which often fail to remind them of anything they know.

Nevertheless: there are always those rare pupils who, without the teacher's help or knowledge, somehow fill some gaps in the instruction and come to appreciate the geometric method. But even among these are very few who show the courage to create order in some other area; for *this* they certainly did not learn from the logicians!

However, it is not true that an untrained person has the geometric axioms — or even the elementary concepts — at his or her disposal. It is not even necessary to look into pre-euclidean history in order to see that space — just like any other area of exploration — shows itself to naive humanity in a rather chaotic form: evidence for this can be found every day by observing how helpless many people are when, in their activities, they must take into account the properties of space (packing, moving furniture, carrying it through a door, cutting cloth, making something with their own hands, etc.). It certainly will not occur to them to reduce the relevant spatial relations to planar problems, and to consider the appropriate lengths and angles in the plane. Even tradespeople, who through the kind of their trade have acquired some facility in the practical treatment of certain geometric problems, are only partially conscious of spatial laws and concepts, or not even that. This does not prevent the axioms of Euclidean geometry from appearing obvious to almost every person, as soon as he or she has occasion to begin thinking about them at all. But the *capacity of seeing* them as evident is not the same as *having them at one's disposal!*

§4. In order for a formative energy to emanate from a method, it is necessary that its success in mastering a given area be experienced personally, but more than that: this mastery must be seen as desirable.

How do logicians and technicians demonstrate the importance of geometry to their pupils? For the logicians, the question does not really exist: once there were these Greeks who had geometry growing out of their heads, and this was so scientific that now every pupil must know geometry. — Such a

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position is, of course, hardly apt to make the majority of pupils undergo the logico-geometric treatment with confidence.

In many pupils, enough vivid interest in geometry could be aroused, if they were shown how *everywhere densely* spatial problems penetrate everythything we do or know.

If, for the logicians, the agreement between geometric theorems and experience is only an accidental affair, which they cannot place in the centre of attention, one would expect a fundamentally different attitude from the technicians: for them geometry is, after all, the study of things in the physical world.

But — unhappily — the entire geometric aspect of this world is, for them, inscribed within the few dozen theorems of the geometry-book; all they do to establish the connection between the book and the world is this: to show practical applications for a part of *these* theorems.

Unfortunately these applications are not among the most interesting things one can find in Nature. Let us remember that the system which makes up the contents of the textbook aims mainly at the derivation of certain *formulas*, by means of which a *quantitative* mastery of space might be reached. — The Theorem of Pythagoras and the volume of the sphere are certainly noteworthy, and no other part of the exact sciences can do without them. But do we often have occasion to apply them in practical life? what is so particularly pleasing about them? why should we, come what may, teach exactly *these* formulas to pupils at an early stage, when they cannot yet critically judge their correctness? — The interest which these theorems can arouse in us does not so much lie in each of them individually, as in the possibility of fitting them into such a straightforward system, as well as in the use we can make of them in such diverse investigations of importance to us in so many ways. The Theorem of Pythagoras is *fascinating* only for those who find joy in *mental effort*. But this sort of joy is nipped in the bud by heuristic proofs, because the pupil can no longer come to the subsequent systematic exposition with fresh curiosity.

Considering the narrowness of the geometric material in propaedeutic geometry-books, one cannot avoid the impression that for the technicians, too, the whole plethora of spatial problems surrounding us remains closed off.

§5. In spite of this, it cannot be denied that "practical" teaching has more pupils reacting to the taught material with a certain liveliness: at any

rate, they see more clearly what it is *about*, and they are better at picturing spatial objects. The development of the imaginative capacity is also often mentioned in connection with the practical method of instruction.

However, this development is much weaker and more one-sided than it could have been. The following is also overlooked: in order to internalize a pictorial proof completely, one already needs a sufficiently nimble imagination; without this, even such a proof is taken over rather mechanically; and by itself contributes little to that development, as it is quickly followed by an application of what was proved — and that usually has a *computational* rather than pictorial character.

There is yet another matter: technicians often do not distinguish what the *visual* investigation of a theorem yields for the development of the imagination and what the manipulation of *measurement* contributes to the verification of the theorem; indeed the measurement can often proceed without stimulating the appropriate visualization in any way. [1]

§6. We come to the conclusion that neither logicians nor technicians can, by their own methods, make what is important to them prevail, and that the cause of this is: the neglect of that preliminary work which must precede the study of the geometric system.

Our task would therefore be: to stimulate this preliminary work in the pupil's mind, and to guide it purposefully.

In whatever we do, our actions encounter spatial problems; even our stud-

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[1] As an example, take the experimental method for finding the ratio of perimeter to diameter of a circle: a string is wrapped around an object with a circular cross-section (say, a tin can), then it is straightened and its length measured with a ruler. Likewise the diameter is measured. This is repeated, as means permit, with several round objects of various diameters. Then the respective ratios (which include some smaller than 3) are recorded, and their average is determined. Thereupon the teacher declares that, with more accurate measurements, this average would be roughly 3.14 (which it seldom is when the pupils do the measuring!).

Such a procedure has little to do with visualizing the lines involved here: a ratio is determined after the rounded string is pulled *straight*, and *in the picture itself no compelling reason is found* that the ratio must be this and no other. — How much more do the means of the discursive proof speak to the imagination! After a regular hexagon is inscribed in the circle, the pupils cannot only prove but also *see*, that its perimeter is equal to six radii, but smaller than the circular perimeter. From that moment on, it will simply be impossible for them to forget that π is greater than 3, but not by much!

On this point, as on various others, I have spoken more extensively in my brochure *"Wat kan en moet het Meetkundeonderwijs an een Niet-wiskundige geven"* (1924, J.B. Wolters, Groningen)

ies in various other school subjects, such as geography, mechanics, physics, etc., involve questions which simply cannot be grasped without a sufficiently developed sense of space. For some of us, these very problems are sufficient motivation to work on our geometric insight, and they help us master the geometry course with ease. For many others, this is usually too complicated; on their own, they do not have the energy to analyse the geometric aspects (this is often also the reason for their lacking manual dexterity and seeing certain school subjects as unsurmountable obstacles), and thus they come to their geometry lesson completely unprepared.

It is the teacher's role to sort out the most appropriate examples from all this material, and to present them to the pupil in the most effective way — then the following results become attainable:

1. A sufficiently diverse reservoir of geometric images is acquired, along with the ability of mentally varying sizes and positions of the figures.

2. The elementary concepts of the geometry course will grow from concrete experience, and the corresponding terminology will be tied in with live images.

3. The diagrams in the geometry book will be seen as schematic, generalizing pictures of much that has been observed on real things, and not as objects of study sufficient onto themselves.

4. Pupils will learn how strongly the geometric properties of things shape the aspect of physical phenomena, and how important geometry therefore is for the natural sciences and technology.

5. It will be possible to confront the pupils with problems whose geometric content is perfectly accessible to them, but where they notice that a sure solution requires critical investigation: they will be impelled to *prove theorems!* And they will thereby experience how many things fall into place, when they formulate with utmost clarity everything they can regard as proved: they will be led in a natural way to the *geometric manner of presentation.*

Our collection should convince teachers that there is an abundance of problems for whose comprehension no theorems of the systematic course are required — but which, inversely, are apt to prepare a live understanding of those theorems.

Admittedly — in whatever question — it will rarely be avoidable to make tacit use of one rather sophisticated theorem or the other (e.g., Jordan Curve Theorem) and many a teacher might be inclined to accuse this method of the same misuse of images which taint that of the technicians. There is,

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however, an essential difference between the two methods: I emphasize that these preparatory exercises do not uncover, formulate, or prove *general theorems*, that the relations encountered in any particular case remain considered episodical, and that only the *the naming of concepts* be retained at this stage. However, the subconscious intake of impressions, which in the course of these exercises will lead to the recognition of theorems, is truly a necessary stage for the future command of the latter: this is the way the human mind works in building a science, there is no other way; even in number theory, a great many special examples usually precede the discovery of a general theorem.

USER'S GUIDE

§7. The questions are ordered by geometric categories and not in the sequence they must necessarily be put to the pupils. I thought this way would be the easiest for teachers to find appropriate exercise material; the judgment of what may be easier or harder for a given group of pupils depends anyway so much on the whole prehistory of their development that it cannot be fixed in a book once and for all.

It is not intended that each exercise should be taken as an "exercise for the geometry lesson." Many of them are meant to be tied into other kinds of the pupils' activities: drawing, sculpting, fashioning school-models, toys, tools, clothes, classroom decorations, etc. from all sorts of materials; visiting factories, doing nature excursions, learning physics, etc. The mathematics teacher should thus be attentive to whatever his pupils are busy at and also be in close contact with other teachers, in order to strive together for the common goal!

These problems differ greatly in their level of difficulty: some of them could and should already be put forward at the pre-school age; since, however, this was not done for most of the pupils we usually get in geometry class, it is — as experience shows — not entirely superfluous to go through them in middle school, too.

The collection is, of course, too large to be exhausted in a particular preparatory period. Many of them can, by the way, be usefully employed in the systematic course — as illustrations or to motivate the formulation of various theorems — where they would, of course, be treated in style different from what which is possible in a propaedeutic course.

The questions should be arranged in such a way that, at first, the very simplest concepts are teased out. However, it should not be forgotten that

eventually there will be a transition to the systematic course; at the end of the preparatory period, questions should appear which can suggest the formulation of theorems. For this purpse, one may, of course, use problems which have been previously solved by intuitive *action*, and now draw the pupils' attention to relations which they had used as something obvious though they are not particularly self-evident (e.g., the similarity of triangles and quadrilaterals used in a cardboard construction; the drawing of a triangle with three given sides, where nobody had worried whether certain circles would always intersect).

§8. In discussing these problems with the pupils, it is advisable not to rush into showing them visual support material, but to let them have the opportunity of using their own ability to picture things: even if they do not get very far in this, the effort alone is useful, if only to heighten their awareness as they finally look at the object. On the other hand — especially at the beginning — such visual support should not be skipped, even in the case of immediate right answers: the correct answer could turn out to be a pretty illusion, when you watch the pupil subsequently handling the object; certainly not all pupils will have answered; and finally, the significance of repeated sensual exposure for the vividness of our mental imagery can hardly be exaggerated!

The visual support material in the pupils' hands is an unequalled means for offering a glimpse into the life of their imaginations. Let the teacher allow them to perform as much as possible with their own hands — he will be witness to revelations, and only then will he know exactly what the pupils are missing in order to follow the seemingly simplest arguments with lucidity.

It is very valuable to ask for schematic drawings of whatever is being dicussed: in those very moments, the act of abstraction is accomplished by many a pupil.

One should never try to save time on the necessary manipulations: one will eventually be able to proceed much faster in the systematic course, and only then with real success!

§9. If, after all this, you proceed to the systematic course, taking it literally from the usual textbooks, you will still experience a serious disappointment. In fact, almost all pupils will still fail to see the point of "logical" proofs: for, as you know, the usual textbooks begin with the proofs of totally obvious theorems!

At this stage it is customary to say: "the simplest things are the most difficult". It is really high time to go beyond that platitude and to ask onseslf

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why "simplicity" is so difficult. One will come to the conclusion that it is not simplicity which makes those theorems difficult, but something so unsimple that even most teachers have no inkling of it, namely the *purpose* of these proofs!

In the sweat of their faces, many teachers try to act as if they could really find something to doubt in those theorems. Of course, this is a lie. No one has proved the equality of cross angles because he had previously doubted it. In reality *not the truth of theorem is being proved* but rather *its logical dependence on other theorems*, which are assumed already established and are no more evident than this theorem itself.

Of course, a beginner cannot grasp this kind of a problem: he is satisfied to know a theorem and to know he can rely on it, and not linger on the question of whether this theorem is held to be true as an independent axiom or as the logical consequence of other axioms.

Let us remark, that most people — *whatever their age* — are indifferent to this distinction (*unreasonably*, it would seem) — even if they have learnt their geometry from logicians.

It should be decided to skip the proofs of ovbious theorems for the first little while.

This will not damage the *clarity* of the presentation, as long as these theorems are not smuggled in tacitly, but are well formulated and clearly labelled as being assumed without proof.

Determining which theorems are true axioms (i.e., a set of statements which is logically independent and permits the deduction of all further theorems describing a given subject) — is a serious task, whose importance for science is very great, though not even acknowledged by all researchers. — If one wishes to acquaint some, and be it only the most talented, pupils with this question, postponing it till the end of the systematic course will achieve a greater success.

If both of these measures are taken:

1. the pupils are prepared for the systematic study of space, perhaps in the way sketched here,

2. the proofs of self-evident theorems are eliminated,

it will be seen that the pupils not only understand geometry, but will be able, under the teacher's guidance, to take an *active* part in building the system of geometric theorems — and only this will lead to a real mastery of the geometric method, so dear to the logicians, and also of the factual contents of geometry, so important to the technicians.

I. LENGTHS

For the pupils these exercises may be allowed to appear as a sport-like practice of visual sizing up. The teacher, however, should keep in mind that, apart from the — really very valuable ability of estimation — these exercises will oblige the pupils to take in a series of geometric forms, become aware of their existence, and to learn the terms they will find at the beginning of their systematic course, such as: being larger or smaller, setting off, comparing, measuring, including, being proportional to, etc., as abstractions from living experience.

1. Take a good look at this table, and tell me how many such tables could be put, close together, along that wall. (Record the different answers of the pupils as well as the number of votes received by each one.)

How could we test which of these answers is the closest? Do we really need that many tables and put them along the wall in order to check the answers?

a. Mark the length of the table on a string, and set off off the latter on the wall (the *integer* part of the desired number results *immediately*. In the beginning, deal only with whole numbers "we don't want it more exactly").

b. With strides or — less ambiguously — with footprints, determine the lengths of table and wall, and divide one of these numbers by the other. (The correct integer can be different this time — if the footprints on the table do not yield a whole number, and we round up or down. In the further course of the exercises, this must be discussed in detail; the first time around, it should only by noticed). "In the preceding case, we had measured the wall with the table: *the table* was our unit of length".

c. Measure table and wall in decimeters and point out the imprecision of the measuring process; if the decimeter does not clearly fit a whole number of times into the table or the wall, discuss which pair of whole numbers one would need to take in order to obtain a result which is surely too large or too small.

d. Measure with objects which do fit the table-length a whole number of times. What are the advantages of this method over the previous one?

e. Look for the common measure of both lengths. (Of course, during these *first exercises* one should not introduce all methods and all scruples, they are touched on here in order to be gradually developed by the pupils in

the course of the exercises).

Compare the various estimated answers with the final result of measurement. Compute the average answer and its deviation from that result.

2. By what part of its own length (more than 1/2? than 1/3? than 2/3?) is this wall longer than that one? From where can you most easily see this? Estimate and then measure. Where to measure — on this wall or the one opposite? If there are protrusions, how do we get around them? Where to lay the yardstick: along the floor or at some convenient height — where do we have less to worry about?

As in the previous case, determine the average error of the answers obtained.

By what part is one wall longer than the other? How many parts — and which ones — does this wall contain of the other? How many parts does the other contain of this one? "Ratio" as a fraction.

3. Estimate diverse lengths variously oriented in space, and then measure them. Compare the average errors of estimation for horizontal and vertical lengths.

4. The same outside the school building for larger distances.

5. The same for very small objects.

6. The same for the diameters of round objects: table-tops, plates, glasses, coins; treetrunks, pencils, balls, etc. With which type of coin can you entirely cover the six-place number on a paper money bill?

7. Compare the average errors of estimation on small objects versus big ones.

8. Relative errors. Influence of absolute size on relative error. Effective arrangement length-measurements of small objects (with the help of instruments): measure the the total length of a series of small objects touching one another, and divide by their number.

9. Estimate and later measure certain lengths on the human body: distance between the pupils of both eyes; length of the middle finger; span of the spread hand, of both arms extended; stride, footprint. — Fix measurements on your own body — keeping in mind that it is growing! — in order to have measures with you at all times.

10. What part of the height of a medium-tall man would be the average height of a 3-, 6-, 10-year old child? Check it on various examples.

11. When do you say "narrow", when do yo say "thin"? — "thick" or "broad"? — a "short wall" or a "low wall"? Which is the "length" of brick,

and which are its "breadth" and "thickness"? When do you say "height", when do you say "depth"?

12. Which animals have a long tail, which have short one? Which has a longer tail: a mouse or an elephant?

II. ANGLES

13. Can you tell the time by looking at a clock-dial where the numbers are illegible? What features would you use? What is *equal* on the dial of a watch and that of a clock tower, when they show the same time?

14. Draw, free-hand, the angle by which the minute hand turns in *ten minutes*. Cut this angle out of your paper and put it on the clock-dial for comparison. Correct it, and save the corrected "ten minute angle" as an angle measure.

15. Use two pens to form the angle through which the minute hand runs in twenty minutes. Put them on paper and draw that angle with a pencil. Check the result with the paper angle saved before.

!6. Using the same paper angle, draw the angles which the minute hand makes in 10, 20, 30, 40, 50, 60, 70 minutes.

17. Naming things: vertex, leg; straight, convex, concave angle. Which of the angles drawn are concave, convex, straight — how can you characterize them in words? With what symbols shall we distinguish the drawing of a straight angle from that of a straight line? Make paper models for concave angles.

18. Cut out four angles of 20 minutes each and glue them together — model of an angle which is greater than that of a full rotation.

19. Using the ten minute angle, make a five minute angle; then a 15 minute angle.

20. Draw a big triangle on the blackboard, then copy it on paper — "smaller" — but as similar as possible to the original. By how much have the sides become smaller? What about the angles?

21. On all kinds of objects, try to find angles which are equal to the 15 minute one — "right angles". Make a right angle, without using the clock dial, by simply folding paper.

22. Draw two angles which have only the vertex in common. How many non-overlapping angles have been created? — Let us use letters as labels [for points], so that we talk about these angles more easily. Naming them

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carefully: of the three letters [naming an angle] always put the one belonging to the vertex *between* the other two.

23. Draw two angles which have one leg and the vertex in common.

The pupils are to try and define the difference between any such pair of angles and a "supplementary" pair drawn by the teacher. The teacher draws supplementary and non-suppementary pairs in various positions, and points out the ones he calls "supplementary". The definition is to result from an analysis of what is seen.

How many angles supplementary to given one can you draw? Compare their sizes,

24. Define the concept "right angle" on the basis of its creation through paper-folding.

25. Namings: acute, right, obtuse angle. If two supplementary angles are not right angles, what can be said about each of them?

26. Assign numbers to the previously drawn angles, using the right angle as a unit. How big is a straight angle?

27. How big is our previous unit [10 minutes on the clock] in terms right angles?

"Degree" as a new unit. How many degrees are in the angle by which the minute hand turns in one minute?

"Degree" — "minute" — "second". How does this new minute [of angle] relate to the angle by which the minute hand turns in one minute of time?

28. A pupil is to stand in front of the class facing the window, then turn to the blackboard. By what angle has he turned? Let him turn about one, two, four, five right angles; about -1, -2, $+3$, -4, ... right angles. By what symbol should we distinguish $+a$ and $-a$ on a drawing?

29. Find a general formula for all angles whose legs are positioned the same way as those of a given angle ABC (taking into account the direction of rotation). Make models of such angles. The expression "congruent modulo 4 right angles" may appropriately be used.

30. By what angle has a man turned, from the moment he stood at the foot of the stairs, ready to climb them, to the moment he has just reached the next floor? ... or the following one? (Careful!)

31. Someone walks along the edge of a square-shaped field, starting in the middle of one of its sides. About what angle has he turned by the time he arrives at the starting point? What if the shape of the field is triangular, pentagonal, or round? What if he does a figure eight through the middle of the field?

32. On a piece of cardboard, the pupil is to sketch successively all angles he turns through, especially at street corners, on his way to school. From this, try to figure out the angle between the front walls of his and the school. Check it on a city map. If some parts of his path are not straight, think about possible errors. As a preliminary exercise, do a similar experiment inside the school building.

33. The distance between two successive windings of a screw is 4 mm. It has been driven 20 cm deep into a board. Through what angle has it been turned?

34. Which angles — formed by which straight lines — were we actually dealing with in the preceding exercises? Let the pupil again turn from the window to the blackboard: once with his right arm stretched out horizontally, again with that arm lifted a little, finally with it completely vertical. In all three cases, determine the angle between starting and finishing position of the arm. Which one of these three angles correponds to what we had — unanalysed — called the "angle of rotation"?

35. How do you have to attach a straight line to a screw, in order that the angle of rotation of this straight line is also the angle of rotation of the screw? — *Simply show it.*

36. How much faster than the hour-hand does the minute hand rotate on the clock-dial? What has a greater angular velocity: the earth turning on its axis or the hour-hand?

37. What arc-lengths (meant naively: measured with a string) are described by various points on a minute-hand in ten minutes? Can the lengths of the minute and hour hands be chosen in such a way that their end-points describe the same arc-lengths in the same time?

38. By how many right angles per second does a wheel turn, if its circumference is one meter and it is rolling down a straight path at the speed of three meters per second? Does it turn slower or faster than the second-hand on a clock?

39. Draw a tilted straight line on the blackboard. The pupil is to draw a "perpendicular" to it, which goes through a given point on the original line. Through the same point he is to draw a "vertical". (Can he do this if the blackboard itself is tilted?)

Two pupils are to stretch a string in the room somehow. Two others are to stretch a second string so that it goes through a certain point on the first one and is perpendicular to it. In how many ways can the second string be drawn?

III. THE STRAIGHT LINE AS AXIS OF ROTATION

40. Study the structure of a simple machine. Which parts rotate against each other? Does every rotating part turn about a rod going through it?

41. Fasten a solid between two immobile spikes. How can its various points still move after that? Are there immobile points other than the tips of the spikes? *"Axis of rotation."* What happens if we fix *one more* point? (On or off the axis).

42. Choose a solid which is such that part of the axis of rotation can be observed if the spikes are suitably positioned. For easy visibility, paint various places on its surface with lively colors.

43. Discuss questions (34) and (35) using the concept of "axis of rotation" and note at what angle to the latter the straight line [or arm] mentioned there should be oriented.

44. What kind of surface is cut out by the rotation of a straight line (knitting needle, knife-edge, . . .) in sand, butter, etc., if this line goes through the axis of rotation at various angles; or does not even go through the axis?

45. Make string models for surfaces of rotation (cylinder, cone, one-sheet hyperboloid, plane). Watch them rotate on a potter's wheel.

46. As it is being screwed into [the wall], does the screw perform a simple rotation? Does it have points which remain immobile? How is its axis characterized?

47. On which principle are lathe and potter's wheel constructed? Forming clay on the potter's wheel. Contrast with oval vessels.

48. Checking whether a knife blade is straight.

49. Testing a ruler: a. Slide it along two fixed points on the paper; b. Turn it by 180° in the plane of the paper; c. Turn it by 180° in space around those two fixed points. — Critique of each of those methods. (a — can also be [an arc of] a circle; b — can be any centrally symmetric figure; c — the ruler has thickness, and turning it over brings the second edge onto the paper). Combine the first two methods.

50. Analogy between a straight line on the plane and a great circle on a sphere: the "straightest" line! Distinguish experimentally, by means of methods analogous to (49), between a great circle and any other line on the sphere. (Very desirable: a black globe for chalk drawings. Diameter about 50 cm).

IV. THE SHORTEST DISTANCE BETWEEN TWO POINTS

51. How long is the way from a point A at the foot of a mountain, dune, or hill, known to the pupil, to the station B situated high above? How should you walk along the slope to make it as short as you can? Would this be the very shortest distance between A and B?

52. Find the distance between Berlin and Moscow according to the railway tables. Estimate the distance by air between the same cities according to the globe. Would this be the very shortest spatial distance?

53. In which direction by compass should an airplane leave Berlin in order to reach Moscow by the shortest route? The same question for Berlin and Java. — Determine it on the globe by means of a taut string. — Is the arc of a circle of latitude the shortest line on the sphere between its two ends?

54. A worm crawls from a point A on one wall to the point B on the adjacent wall. Find its shortest path.

55. Draw the shortest path between two arbitrary points on (1) a cylinder, (2) a cone. Make these surfaces out of paper.

56. Which form does a stetched thread take? What about a thick rope? Observe this on various uses of ropes by workers. How must the two ends of the rope lie so that it can be streched straight?

V. STRAIGHT LINES FROM LIGHT RAYS

57. Three pupils are asked to stand along a straight line in front of the class — without using instruments of any kind; a fourth pupil is to check the line-up, again without aids. — On what basis is this task feasible?

58. Make an opening in each of two pieces of cardboard, and put up these screens one in front of the other, in such a way that a certain point in the classroom can be seen by looking through both of them. Check that from this point a string can be pulled straight through the two openings.

59. A pupil is to hold his finger in front of his eye at some distance so as to cover up a certain point; other pupils should convince themselves that the finger, the eye and that point lie on a straight line.

60. In a mirror, a pupil observes a point reflected in it; he then puts his finger on the mirror so as to cover the point. The positions of his eye, his finger, and that point are somehow marked. Let a thread be strung between these three positions, in order to probe the law of reflection. Vary the position of the point, keeping that of the eye the same. The direction of

a perpendicular [to the mirror] can be indicated by fixing a pencil, say, with wax, so that it forms a straight line with its own reflection.

61. Let schematic drawings be made for the last three experiments.

62. Determine at what distance from the eye an object (say, a plate, a coin) must be held, in order to cover a more distant object just barely. Make a schematic drawing.

63. Why do objects appear smaller to us, when we move away from them? *What* is really getting smaller in the process?

Make a schematic drawing — say, of a tree being observed by a person from various distances. What should be included in the drawing, which features of the objects are relevant for *our question*?

64. Estimate the angles under which the blackbord appears from various points in the classroom. Check with a home-made tool for measuring such angles: a tube attached to a piece of cardboard and turnable parallel to it; mark the two positions of the tube by making lines on the cardboard, then find the angle between them.

Later on, a finer instrument will be very desirable! — At the present stage of acquaintance with angles, it is very important to focus on their *legs*. Only later should this be supplanted by measuring arcs.

65. Estimate and then measure the angular distance between different points in space — between two stars when they are high in the sky and when they are near the horizon.

66. When the sun or the moon, near the horizon, is about to disappear behind an object (say, a tree-trunk or a chimney), estimate whether it will be hidden completely. Wait to see what happens; explain which quantity you were really estimating.

67. Which one do we see under a larger angle: the sun or the moon?

68. Into the picture of a landscape, draw the moon as big as it would appear to us in nature. Compare this with similar photograhic images of the moon. (This experiment was carried out by the magazine "Nature").

69. Why does the moon run after us? Why do near objects pass us more rapidly than distant ones when we ride, say, in a train? — Make a schematic drawing.

70. Stereoscopic vision.

VI THE PLANE AND RULED SURFACES.

71. Using a taut string, check whether the table-top is *plane*.

72. You have two flat plates [slabs, tiles] and no other aids. Can you check whether they are *plane*? What kind of misunderstandig could arise? How many plates are necessary to remove any doubt? (Compare with no.49). In making flat glass surfaces by grinding them against one another, why are three of them required? Why would two not be sufficient?

73. Is the plane the only surface which allows drawing a straight line through every point? — Diverse surfaces — with or without names — which can be obtained by bending paper. In particular: cylinder, cone, helix [2] (circular or other), Möbius Band [3]. Discover such surfaces on nearby objects; build models.

74. What chracterises the *plane* among all *"ruled surfaces"*?.

75. Define cylinder and cone.

76. Ruled surfaces which cannot be made from paper. Models: a twisted pile of books or deck of cards; spiral staircase; one-sheeted hyberboloid; hyperbolic paraboloid. Make them from string.

VII. THE ENDLESS STRAIGHT LINE. PARALLEL LINES.

77. Through two points fixed somewhere in the middle of the classroom, mentally draw a straight line, and imagine in which points it penetrates the boundary of the room, and where it goes from there.

78. Fix a third point, and through it draw a second line meeting the first one. Demonstrate with two taut strings. Determine the angle between the two lines. Make a schematic drawing on the board. Now turn the second line around the third point in such a way that its point of intersection with the first line moves always in the same direction. Watch how the angle between the two lines changes. Mark a number of positions of the second line on the blackboard drawing. Imagine to what limit position the second line can be turned without stopping to intersect the first. — Where is the point of intersection of the two lines *in* the limit position? — Now the same thing while turning in the opposite direction. Again imagine the limit position and draw it on the board. How many limit positions are there? *Without influencing the pupils* let each of them decide, and make a statistic of the answers.

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[2]A piece of this surface is obtained by radially cutting the region between two concentric circles, and lifting one side of the cut above the other until the surface is part of a spiral.

[3]See footnote for no.193

79. What kind of surface does the second line describe in the preceding example?

Look for *"parallel"* and non-parallel lines in the classroom, the street. — How should one define "parallel lines"?

80. Make a perspective drawing — even if only a sketch — of the front wall of the classroom, the two adjacent walls, the ceiling, and the floor.

81. Through a rectangular opening in a vertical screen, look at two rods perpendicular to it. Keeping the eye near the centre of the opening, fix the pair of rods successively at different heights: at the lower edge, at 1/3 of the height, in the middle, at the upper edge. For each position make a perspective drawing. The *"vanishing point"*.

VIII RELATIVE POSITIONS OF LINES AND PLANES.

82. Measure the angle between a slanted stick fixed to the floor and some straight line running along the floor from its base. Find another line running along the floor from the base of the stick and making the same angle. Is there a third such line? How does the angle between line and stick change as one varies the direction of the line on the floor? Which one of all these lines makes the largest, and which one the smallest, angle? How are they positioned to each other?

83. How do these two extremal angles change when the slant of the stick is varied?

84. Can you always draw a line on the floor which is perpendicular to the stick? Where is the second perpendicular? Can you place the stick in such a way that yet another line on the floor is perpendicular to it? What are the two extremal angles in this case? The *"line perpendicular to the plane."* (Preparation for the exact theory.)

85. In your surroundings, find examples of lines and planes which are perpendicular to each other. Examine a pencil held at a slant against a mirror, and its image.

86. Draw a straight line in some plane. Fix a point outside that plane and draw through it a straight line which intersects the first one. Turn the second line around the fixed point until it is parallel to the first one. Will it then still have points in common with the plane?

87. Draw parallel lines for planes in various positions. Find examples in the classroom.

88. How many straight lines *parallel* to a certain plane go through the same point? How many planes *parallel* to a certain straight line go through the same point?

89. The same question for *perpendicular*.

90. Through a straight line parallel to a certain plane, draw a second plane. Rotate the latter around the straight line, and watch how the line of intersection of the two planes is displaced. *"Parallel planes"* — which do not intersect. How is the line of intersection placed with respect to the first straight line (when the planes do intersect)?

IX. POSITIONS OF LINES AND PLANES WITH RESPECT TO A HORIZON.

91. A *"vertical"* line — the direction of a string with a weight hanging on it. *"Horizontal"* (line or plane) — as perpendicular to that. Draw the surface of water in a glass held at a slant (cross-section). The water-surface on the ocean: look at a globe, draw a cross-section.

92. The line of the horizon and a horizontal line. What is their relation?! What form does the line of the horizon have (say, for an observer on the high sea) and why?

93. *"Vertical plane"*. — The pupils, who, by feeling and habit, are able to distinguish a vertical plane from a slanted one, are now to determine the characteristics of a vertical plane.

94. How many vertical planes can be drawn through a given point? Examples: (door, window, etc.) — look for them yourself!

95. Can a vertical plane be drawn through any straight line? How many? Can a horizontal plane be drawn through any straight line?

96. Does every plane contain a horizontal (a vertical) line?

97. Examples of objects which rotate around a horizontal; a vertical; a slanted line. Look for them yourself!

X. PLANE SECTIONS OF SURFACES. SHADOWS. PERSPECTIVE PICTURES.

98. What form does a surface of water have in a cylindrical glass held at different angles? — Imagine it, draw it, look at it.

99. The same for a funnel.

100. The same for containers of various shapes.

101. Hold a glass in such a way that its rim appears as a straight line (look with *one* eye!) Then move it downwards gradually. How does the rim appear now? How should it be drawn? — Put up a glass-pane between the eye and the glass and draw on it the rim just as you see it.

102. Is there a solid which can be drawn as a circle no matter how it is oriented?

103. What part of a spherical surface do we see when we look at it from various distances? From which distance would we see exactly half of it? — Experiment with the black globe. — Make schematic drawings.

104. What forms can the shadow of a sphere on a wall take — when the source of light is point-like? — Imagine, then experiment. Compare this with the water-surface in the funnel (which cases are lost there?).

105. Let there be a red and a green point-like source of light. Imagine the shadows when only the red or only the green are lit; when both sources are on. Check by experiment. Many point-like sources of light. Discuss the penumbras [partial shadows], draw them schematically, check eperimentally.

106. What breadth can the shadow of a narrow cylinder have (ignoring any penumbra)?

107. Why does a shadow in sunlight become sharper [less fuzzy] as the object moves closer to the wall?

108. A point-like source of light produces a circular shadow on a wall. What is the form of the object that makes the shadow?

109. Imagine the shadow-space of a given object — in diverse orientations toward a point-like source of light. Vary the distance of the object to the light source. Vary the distance and orientation of the object to the wall.

XI. ANGLES BETWEEN TWO AND THREE PLANES.

110. Open two books in such a way that the front and back covers respectively have the same inclination to each other — equal *"two-flat angles"*. What two-flat angles could be called "straight", "right", "acute", "obtuse", "concave"? — Look for such angles on nearby objects. One *"degree"*. Planes which are *"perpendicular"* with respect to each other.

111. What angles are formed by the edges of the book-covers when these form a right two-flat angle? — "Line angles" of the two-flat angle — how should they be defined? Convince yourself: the equality of two-flat angles corresponds to the equality of certain line angles — the line angle as a measure of the two-flat angle.

112. How can three planes be situated with respect to one another? — Imagine all cases, construct them from cardboard. — Surrounding examples.

113. In the classroom look for *"three-flat angles"* which can be see from their "inside" or from their "outside".

114. From three pairs of respectively equal planar angles, make two three-flat angles neither of which can be made to fit into the other. (Use cardboard). A pair of gloves. An object and its mirror-image. Two spherical triangles whose respective sides are equal but arranged in different order. — Compare with planar triangles.

115. How many rooms can be adjacent to that [inner] corner of the ceiling. How many for the second floor, how many for the top floor. — How many non-overlapping three-flat angles are formed by three interecting planes?

116. Mentally vary the mutual inclinations of the planes. Construct models: from three pieces of cardboard, which are mutually slanted to one another, from three wire circles — as intersections of just those planes with the sphere; draw these circles on the black globe.

117. Are there, among those eight three-flat angles, pairs of equal ones? — "Congruent". — "Symmetric". ("Equal" should be reserved for *magnitudes*).

XII. SYMMETRY.

118. Out of folded paper, cut figures having some symmetry. What do you obtain if the paper is folded just once, what if it is folded twice or more?

119. Anticipate the results of the cutting and draw them. Reflect a given figure at a given straight line. *"Line of symmetry" (not axis* of symmetry).

120. Draw lines of symmetry on a figure so as to make it lose (resp. keep) its symmetry with respect to colour.

121. In the classroom, find surfaces with lines of symmetry. On the fronts of houses, or parts of them, look for lines of symmetry, or determine which features destroy a symmetry.

122. Formulate what kind of figure has a line of symmetry.

123. An object and its mirror image. Define "plane of symmetry". Look for one in diverse objects. Symmetry of the human body, of animals. Are all fish symmetric as seen from the outside?

124. *"Centre of symmetry"* of a planar figure — define it for the pupils and ask them for examples.

125. Define regular polygons and ask the pupils to decide what kinds of symmetry they have.

126. Complete a given figure to a centrally symmetric one, with a given centre of symmetry.

127. Figures with (1) a line of symmetry, (2) a centre of symmetry, on the surface of the globe. In which cases can one half be made to cover the other?

128. "Second order axis of symmetry" as an extension of the concept of "centre of symmetry" of planar figures. Does every parallelepiped have second order axes of symmetry? — Find various nearby objects with second order axes of symmetry.

129. Third, fourth, ..., n-th order axes of symmetry. Find all second, third, fourth order symmetry axes of a cube. Symmetrical features of regular pyramids.

130. Does a skewed parallelepiped have any symmetries? *"Centre of symmetry"*. Remember centres of symmetry in planar (or spherical) figures — and compare. Can a skewed parallepiped be divided into two halves which can be made to coincide with each other? Does a regular tetrahedron have a centre of symmetry?

XIII. THE GEOMETRY OF MECHANISMS.

131. Make a schematic drawing of the treadle [crank] for a lathe or sewing machine; sketch, free hand, the paths taken by the end-points of the connecting-rod.

132. How do the various other points of the connecting-rod move? From strips of cardboard, make a model of this mechanism; lay it on white paper, and through holes in the strip, draw paths of various points of the rod. Imagine a continuous deformation of one path into the other as the observed point slides down the rod.

133. Vary the length relationships of the strips, and study the influence of this variation on the paths taken by end-points of the connecting-rod. The transformation of circular into rectilinear motion. Where is the fourth point of the corresponding quadrilateral? — Observe the various shapes of such quadrilaterals on machines.

134. Gears — cylindrical and conical. What is the purpose of the cone-shape? Find it in mechanisms.

135. What kind of paths can be described by various points of a plane which is fixed in one point and can move only in itself? — a sheet of paper tacked to the table. Similar question for a spherical shell fixed to a globe in one point. Are there further fixed points in that case?

136. Can a piece of paper move on a table in such a way that all its points describe circles of *equal size*? Are there immobile points in that case? Can a similar thing be done with the shperical shell? Centrifugal sieves. Why does one not rather allow a sieve to rotate around a fixed axis?

137. What paths can be described by the different points of a solid, if two of its points are held fixed? — what if only one point is fixed?

138. Give a general characterisation of translating motion and find examples of it.

139. What kind of path is described by a point on a screw which is being screwed in — if it lies on the axis, off the axis?

XIV. DEGREES OF FREEDOM.

140. How many data does one need to know in order to find a point on a surface? (where a treasure is buried on a field, where a tree is to be planted, where on the surface of the earth a city is located).

141. How many data determine the position of a point in space? (Exact position of an airplane, of a bird in the air, of an object in the room).

142. Different kinds of data. Geometric loci. The pupils are to reveal the data for the location of an object chosen by them, and the other pupils should then decide which object it is.

143. How many data does one need to know in order to determine the location and spatial orientation of a rod? The same for a strip of cardboard on the surface of the table.

144. How many data determine the location of a point on the surface of globe? — the orientation of a globe whose centre is fixed?

145. How many data (*"degrees of freedom"*) can be chosen arbitrarily to determine completely the location and spatial orientation of the following objects: a point in space; a point on the plane, on the surface of the globe, on the inside of the globe; a strip of cardboard on a table; the same if it is fixed in one point; and with the additional constraint of two extremal directions forming a given angle; a globe whose equator can move within a given plane; a pair of rods, one of which is fixed in one point while the other is attached to the first by a fixed joint allowing arbitrary turns; a wheel whose axle is fixed

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in space, a wheel which can roll along a given rail, a wheel on a stationary car; a sewing machine.

146. Degrees of freedom of various parts of our bodies.

XV. DETERMINATE, INDETERMINATE, AND OVERDETERMIDED PROBLEMS.

The exercises of the following three parts are apt to suggest how a systematic discussion can help to find exact answers, and thus could serve as an introduction to the systematic course, and also as illustrations of it.

147. Three pupils are to stand in such a way that every pair of them stands at the same distance to each other. Use string and chalk as aids. Make a schematic drawing on the board.

148. Can this exercise be extended to four pupils? To four birds in the air? to five birds?

149. With ruler and compass, draw a triangle with three given sides. Let the pupils themselves prescribe the three lengths — are they completely arbitrary? Same exercise on the black globe.

150. Construct a quadrilateral with four given sides. Compare this exercise with the preceding one. How many degrees of freedom did the triangle have once its sides were fixed? How many does our quadrilateral have? Find additional data to make this exercise determinate (so that the quadrilateral, too, will no longer have any degrees of freedom).

151. Construct a triangle given one side and an adjacent angle. The geometric locus of the third vertex. Is the triangle completely determined by the choice of the third vertex? By what kind of data can this be accompished? — Similar things with spherical triangles. Is the triangle then still completely determined by *every* choice of the third vertex? — Similar things on a cylindrical surface (say, a circular one).

152. In several ways, presribe two data for a triangle and discuss the geometric locus of the third vertex. Analogously for spherical triangles.

153. On the sphere, prescribe two angles and, varying the length of the side between them, examine the tird angle; prescribe a certain value for the third angle, and — *groping* with your hands — determine the corresponding triangle; convince yourself that this is a *determinate* problem.

154. Analogous things with planar triangles. Why is the case of the planar triangle inderminate?

XVI. THE PARALLEL AXIOM AND ITS CONSEQUENCES.

155. A large floor is to be paved (ignoring its edges) with tiles of a single shape and size. What forms can the tiles have? Can they be triangles, quadrilaterals, pentagons? Can they have arbitrary angles?

156. Can the whole surface of the globe be divided into an arbitrary number of equilateral triangles of the same size?

157. Which regular polyhedra are possible?

158. What do we mean by an "enlarged picture" of a planar figure, or a "scale model" of a spatially extended object? Examples of figures whose respective angles are equal, but without proportional sides; with proportional sides, but without equal angles.

159. Can two triangles, quadrilaterals, pentagons of different sizes on the surface of the globe be similar?

160. In the plane and on the sphere, obtain a figure from a triangle by "multiplying" [all three sides by the same factor]. Can the figure thus obtained on the sphere be "rectilinear" (i.e., bounded by arcs of great circles)?

161. Determine the distance to an unreachable point (e.g, a tree on the other side of the river; a corner of the ceiling; the top of the tree).

162. Schematic drawings for a method of determining the radius of the earth; the distance to the moon, knowing the radius of the earth; the distance to the sun, knowing that to the moon; the distance to Sirius, knowing that to the sun.

163. All sides of a triangle are enlarged in the ratio 1:n. In what ratio is its area enlarged?

164. All edges of a parallelepiped are enlarged 2,3,4,... times. By how much is its total surface area, its volume, enlarged?

165. Why is it thought more economical to build *one* house with 8 apartments of equal size than to build 8 houses with one apartment each?

166. Eggs of two different sizes are for sale. One kind has a diameter one and a half time as long as the other, but is twice as expensive. Which one is a better deal (from whose point of view)?

167. A box is to be filled with as much as possible of a certain substance available in the form of balls of two different sizes: smaller ones and larger ones. Which kind would you choose?

168. Somebody has derived the formula $S = 3ab + bc$ for the "size" of a certain kind of solid, and it is known that a, b, c — are certain *linear*

dimensions of such a solid, while S — is either its surface or its volume. Can you tell from the formula, which of these two quanities is really given by S?

XVII. THE CIRCLE

169. A string is to be tied around a parcel having the form of a circular cylinder. How long should it be? — judged by sight! One side of a cylindrical box is to be covered with paper — what form and length should the piece of paper have?

170. You are to sew a skirt and a pair of trouseres having the same length. Which one requires more material? — make a rough estimate by schematically thinking of the skirt as one, the trousers as two, cylinders.

171. You have three cylindrical boxes of equal height, one being twice as wide as each of the two others. The side of the wide one is to be covered with red paper, the sides of the thin ones with green paper. Estimate of which paper you need more.

172. Around the equator of a ball as big as the earth, imagine a string. Now imagine this string lengthened by one metre, but still slung around the ball in a circle concentric with the equator. Estimate how far removed it will be from the surface of the ball.

173. What kind of buttons work better: flat ones or thick ones? — looking more like coins or more like marbles? For which kind would you have to make bigger button-holes, and bigger in which ratio, for buttons of the same diameter?

174. Through a circular opening in a piece of *paper* pass a coin which is larger than the opening. Estimate how small an opening would still work.

XVIII. SURFACES AND SECTIONS

175. From a piece of cardboard, cut out the shell of a cube.

176. Cut away pieces from the corners of a rectangular piece of cardboard, so as to make it possible to make an open box from what is left over. What will be the form of the cut-off pieces?

177. Draw a paper pattern for a conical lampshade whose base is an octagon, an n-gon, a circle. Invent various models whose shells can be made from a single piece of cardboard.

178. Given the plan for the exterior walls of a house, draw the horizontal projection of the roof, assuming that all parts of the roof make the same angle with the horizontal plane. How does the projection of the edge between

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two "faces" of the roof change, when the latter make different angles with the horizontal plane? What forms do the different faces have? (Plan of the house: a square; a rectangle; a rectangular figure with some protruding walls; a regular octagon).

179. An eaves-trough with semicircular cross section consists of two equal halves which are joined together under some angle. What is the form of their common edge?

180. A cube is cut at right angles to its space diagonal. What is made by the cut? How does it depend on the distance of the cut from a corner?

XIX. TOPOLOGY

181. Schematic drawing of a knot.

182. Composition of an ordinary string [twine] — of an electric cable (a great number of threads are wound around one another so as to give each of them the same exposure to the surface).

183. Chains which can be taken apart.

184. Drawing a lasso.

185. On paper, n points are pairwise connected by lines. Can this configuration be drawn in a single stroke without repetition? — when n=4, 5, 6, ...? — Can all edges of a tetrahedron, an octahedron, a cube be traced in the way described above? — Try to invent more such "graphs" for which this is possible or impossible, respectively.

186. Colour a disc [pizza] divided into 2, 4, 6 sectors in such a way that adjacent sectors are coloured differently. What is the minimal number of colours required to do this?

187. The same for the case of three, five, seven sectors.

188. The same for two concentric circular discs, when the outer ring (respectively: the inner disc) is divided into two, three, four, five parts by radial lines.

189. Divide a square into such regions that two, three, four, five, ... colours are needed in order to colour any pair of adjacent regions differently.

190. Colour a tetrahedron, a cube, an octahedron so as to give different colours to any two adjacent faces.

191. Attach coloured beads (or use wax, clay) to the corners of a tetrahedron, a cube, an octahedron in such a way that any two beads on the same edge have different colours. How many colours are needed?— Compare with the preceding problem.

192. Divide the surface of a torus [dough-nut] into five, six, seven parts so as to require five, six, seven colours, respectively, to distinguish them as above.

193. From a strip of paper, make a cylinder and paint each of its surfaces (inner and outer) with a different colour. From a similar strip, make a Möbius Band [4] and try to colour it the same way.

194. On the surface of the cylinder, apply a thin streak of paint to each of its two rims. Try the same on the Möbius Band.

195. Cut through the Möbius Band along a line parallel to the rim. Repeat the process on the result of the first cut. How many rims are created by each cut?

T. EHRENFEST-AFANASSJEWA

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[4] If A, B, C, D are the successive corners of a rectangular strip of paper, so that a cylinder is obtained by gluing A to B and D to C, a (ruled) Möbius Band is produced by gluing the side AB to CD in such a way that A coincides with C and D with B.

Hoechsmann

Problem-Based Learning in Mathematics

By Thomas C. O'Brien (posthumously)
with Chris Wallach and Carla Mash-Duncan

A teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations, he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for, and some independent means of, independent thinking.

George Polya, preface to the first edition, *How To Solve It*, **Princeton University Press, 1945.**

 For years problem-solving has been an aspect of the American school mathematics curriculum. But for most children— contacts with math educators around the country suggest 80 to 90 per cent of children—problem solving is limited to "word problems", i.e. computational exercises couched in words.

Word problems are a pretty narrow subset of the universe of problems. We can say with some authority that we have not solved a word problem outside a math classroom in many decades.

A more general definition of "problem" is a situation with a goal and the means to the goal is not known in advance. As the great mathematician George Polya said, [private conversation] "A problem is when you are hungry late at night and you go to the refrigerator and the refrigerator is empty. *Then* you have a problem."

In 2000 the National Council of Teachers of Mathematics' Principles and Standards for School Mathematics defined problem solving as follows:

> *Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on*

> *their knowledge, and through this process, they will often develop new mathematical understandings.* [Note 1]

Polya suggested two aims for elementary school mathematics. First are the "good and narrow aim of education."

> *… the good and narrow aim of the primary school: to teach the arithmetical skills — addition, subtraction, multiplication, division, perhaps a little more. Also to teach fractions, percentages, rates, and perhaps even a little more. … Arithmetical skills, some idea about fractions and percentages, some idea about lengths, areas, volumes, everybody must know this. This is a good and narrow aim of the primary schools, to transmit this knowledge, and we shouldn't forget it.*

And then there is a higher aim:

> *But I think there is one point which is even more important. Mathematics, you see, is not a spectator sport. To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. To solve certain problems of multiplication or addition, this belongs to the good and narrow aim. To the higher aims about which I am talking now, is some general tactics of problems. To have the right attitude to problems. To be able to be prepared to attack all kinds of problems — not only the very simple problems, which you can right away solve with the skills of the primary school, but more complicated problems, problems of engineering, physics and so on. This will be, of course, farther developed in the high school, still farther for those who take a technical profession at the university, but the foundations should be prepared already in the primary school. And so I think an essential point in the primary school is introduce the children into the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude, a general aptitude to the solution of problems. Well, so much about the general aim of the teaching of mathematics on the primary school level. [Note 2]*

Since Polya's death in 1985 there has been a burgeoning movement involving problem solving as a fundamental aspect of education which incorporates and goes beyond the development of problem solving tactics and attitudes.

Problem solving has become to be seen as a method of *causing learning to take place.*

At the heart of problem-based learning (PBL) is collaborative work among students in devising and solving problems involving conceptually complex material. [Note 3]

PBL, said to have been originally developed for the training of physicians at McMaster University in the late 1960s, has been incorporated into over sixty medical schools and other health-related programs such as nursing, dental and veterinary schools. Moreover, PBL is said to have been adopted by numerous disciplines including business, chemistry, biology, physics, mathematics, education, architecture, law, engineering, social work, history, English and literature, history, and political science. [Note 4]

The implementation of problem-based learning (PBL) entails not only the re-design of curriculum but also the development of effective facilitation-cum-coaching approaches. PBL curricula innovation typically involves a shift in three loci of educational preoccupation: from what content to cover to what real-world problems to present; from the role of lecturers to that of coaches; and, from the role of students as passive learners to that of active problem-solvers and self-directed learners. [Note 5]

What does this have to do with the mathematical education of children? PBL, it seems to us, is intimately related to the Piagetian notion that knowledge is a personal construction, not a set of fixed entities transmitted to be stored until text time. In classrooms, this means that interesting tasks, problems, and investigations should be actively engaged by learners.

The British mathematician Alfred North Whitehead hinted at PBL when he said, 90 years ago, "In training a child to activity of thought, above all things we must beware of what I will call 'inert ideas'—that is to say, ideas that are merely received into the mind without being utilized, or tested, or thrown into fresh combinations." [Note 6]

It is a complex task but teachers need to find out where the learner is in order to challenge the learner with problems and investigations which have a moderate mismatch with the learner's present status. Thus challenged, the child will revise or

extend or generalize his/her present fabric of ideas and relationships. This is what learning is all about, not the storage, rehearsal, and production-upon-command of inert facts.

Our task as educators is to come up with appropriate provocation, i.e., good problems and investigations to engage children's minds and imaginations.

That is to say, much of learning takes place by *provoked adaptation.* This is a message especially appropriate to mathematics education.

Recent Work with Children

During the past five years or so O'Brien has worked in with local teachers in elementary school math classrooms. The work was undertaken from a provoked adaptation point of view (which we now know is intimately related to PBL). That is, no teaching took place, problems were posted for children working collaboratively, and children were almost universally successful in their work.

Not the least, children's enthusiasm was such that we sometimes had to exert "crowd control" in the sense of giving children poker chips (two to each child) to be spent to in order to address the entire class, so energetic was their desire to share their findings.

In general, the tasks involved necessary inference—an utterly basic aspect of mathematical thinking—and in general the problems involved games devised by the author. By "inference" is meant the deriving of new information from old information.

(Suppose I hide a penny in one fist and don't tell you which fist, I show both fists, closed, to you. You choose one of the fists and find out that it is empty. You can't see it, but your mind can see that the penny is on the other fist.

(Or suppose I show you 8 pennies. You count them. Then I ask you to close your eyes and I cover some of the pennies with my hand. I ask you to open your eyes and tell me how many pennies are under my hand, You relate the three classes of chips—the original chips, the showing chips and the hidden chips—to *infer* the number of pennies under my hand, Interestingly, many teachers will predict that 5 and 6 year olds will subtract to get an answer. They don't.) [Note 7]

The results have been widely reported in the US and the UK. [Note 8]

The latest work was undertaken with first and fourth grade children in a private school in the midwestern USA.

First Grade

The activity is called Mystery Person. It was invented, so far as we know, by O'Brien.

A number of people are asked to sit in a circle and their initials are drawn on a large paper pad that everybody can see. In the diagram that follows, C is for Charles, etc.

```
    C
B       N
R       L
    T
```

The teacher secretly writes down the name of a Mystery Person. The players have to gather clues and infer who the Mystery Person is.

They ask the person who chose the Mystery Person about a particular person. If the person is the Mystery Person OR if it is next to the Mystery Person, the feedback from the teacher is "Hot." Otherwise, the feedback is "Cold."

The reader is asked to play this game with one or more adults. Then turn the tables; a different player hides the Mystery Person and the person who originally hid the Mystery Person has to gather clues and find the Mystery Person.

Once the reader has played the game several times, the challenge is to solve these problems with the list of people C-N-L-T-R-B as configured above. Mathematics is not a spectator sport.

In the circle above, C is cold. T is Hot.

Is there enough information to figure out who the Mystery Person is? If so, who is it? If not, what question would you ask next? The answers are given at the end of this article.

Up to now, you have played Mystery Person with one person hidden. We ask readers to go back to the C-N-L-T-R-B configuration above and challenge friends to a game.

We tried the Mystery Person game with first-graders at the beginning of the school year and were pleased to see that they succeeded. They enjoyed the games so much that Tom was accosted by a stranger in a super market. "You're working with my little Jamie with the Mystery Person game, yes? I want you to know that Jamie loves the games and insists that we play the games at home around the dinner table."

We stayed with the one-person game for two sessions, each about 25 minutes, and it was clear that the children were successful. There was rarely a wasted question and children knew when a conclusion had been reached. Children worked together enthusiastically and cooperatively. It was also clear to Chris that at their young age and at this early time in the school year, their attention span was such that that they needed a change of pace and so we took a break for other activities.

It was not until January that we got back to the Mystery Person games. We had done similar work with fifth graders in the past (Cite "What is Fifth Grade?) and we were keen to find out how children at this age would do with two Mystery Persons.

The group, as before, was Chris's math class, 14 children selected from three first grade classes in the school.

Chris asked 7 of the children to sit in a circle on the floor and she asked the rest of the children to sit in chairs in a circle surrounding them. She put the inner-circle children's names on the board.

```
    J
S       K
Ke      L
 E   C
```

First we played for one Mystery Person. Tom secretly wrote the name of one of the children in the inner circle and gave out data while Chris selected children from the entire group to ask questions and explain why they were asking about this or that person.

The only bit of "teaching" that took place, aside from reminding children of the rules of the game at the outset, was to ask children to note the consequences of the data they were given.

But Chris had an extra arrow for her bow. She commonly asked children to explain their thinking to her and to the class. And, unlike many American classrooms where the teacher moves on once a correct answer or a sensible explanation is given by one child, Chris asked several people to share their solutions and/or their thinking and often she chose a child whose solution was weak or incorrect. "What do you think of that?" was Chris's question to the class. Never did Chris say or imply that a child was incorrect.

Here is the way the game went. The consequences were placed in the pad by children taking turns.

A ring of fire meant hot. An ice cube meant cold.

A check meant that the person was a possible Mystery Person and an X meant that they had been ruled out as a Mystery Person.

Child	Chris	Consequences
1. Tell me about L?	L is cold	J

```
                    S      K̶
                   Ke    ┌─┐
                         │L̶│
                    E   ┌─┘
                        C̶
```

2. C?	2. Hot	

```
                      J̶
                 ╱ S      K̶
               K̶e    ┌─┐
                     │L̶│
                  E  └C̶┘
```
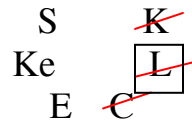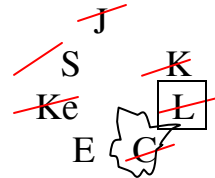
3. We're finished, It has to be E.

This was for warm-up. (Noteworthy, only this one game was needed.) We were pleased that, unlike much of the school curriculum, children were successful four months later.

Chris exchanged inner- and outer-circle children and the game went like this: (We won't provide a Consequences chart in order to provoke readers into following children's tactics.)

```
      H    C
 I         Je
 R         L
      N
```

Child	Chris
1. Tell me about L?	L is hot.
2. C?	Hot
3. Je?	Hot
4. R?`	Cold
5. H?	Hot
6. I?	Hot

7. We only need one more question and the game is finished,

What do you think about children's thinking? Did you infer both Mystery Persons? Try some of this with kids?

Fourth Grade

We worked with Carla's class of 14 fourth graders for 50-minute sessions for ten or so Thursdays staggered throughout the year by Tom's travel and school holidays and events.

For several of these sessions we worked with a game called Find It, also invented by Tom. [Find It is available for Palm PDAs from Handango: See http://www.professortobbs.com/software.htm]

As with the first graders, the sessions involved the whole group, with children encouraged to work out certain issues (such as "What's the best place to start? What are the consequences? What's a good next step?) in small groups.

Find It involves a 4 by 4 grid. Players can opt for 1-12 diversions to be placed randomly in the grid and the task is to infer where the diversions are. The player launches a probe from position 1 though 16 to look for the diversion.
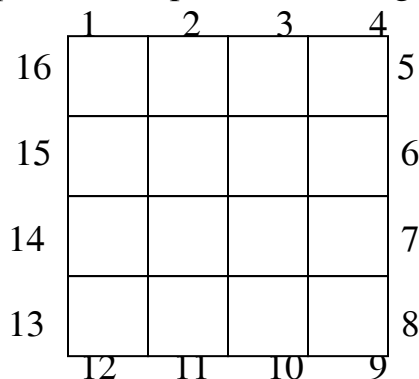
There are three games, Righties, Righties and Lefties, and Randoms. In the Rightie game, a probe makes a right turn when it hits a diversion. Righties and Lefties are a random mix of the two types of diversion and Randoms are randomly Righties or Lefties.

When a probe is launched, the destination and the number of diversions are reported. For example, suppose a player is playing Righties and has chosen that only one Rightie be hidden.

And suppose that the player shoots Probe 1 and finds that it exits at 12 with 0 hits.

This means that no Righties have been encountered. And thus four boxes in the grid can be eliminated.

```
            1    2    3    4

16      x                        5

15      x                        6

14      x                        7

13      x                        8
        12   11   10    9
```

Suppose the next shot is 16. And suppose the probe exits at 11 with 1 hit. You know with logical necessity that there is a Rightie in the 16-2 (or 16-11) box, The game is finished.

Here is a game involving 5 Righties. Can you locate the Righties? The answer is given below. This game took fourth graders 13 minutes to solve.

Start	9	10	11	12	16	1
Exit	7	3	6	9	1	15
Number of Hits	1	0	1	2	3	1

After two or three 20-minute sessions with Righties, the children were both efficient and confident. They had equilibrated. As is the case in most situations involving equilibration, they wanted to move higher.

Here are the data for 12 Righties. Fourth graders took 20 minutes. The answer is given below.

Start	1	2	3	4	6	7	8	11	10	12	14
Exit	16	1	2	5	4	6	9	10	8	11	14
Number of Hits	1	2	2	3	2	2	3	2	1	2	6

Summary

The results we report here are consistent with the previous five years' work. Children constructed important ideas in the face of a problem situation. They did so collaboratively, they respected one another's thinking, and their overall enthusiasm and eagerness to go further was at all times impressive.

The results here are consistent with the constructivist notion that moderate conflict (i.e., a problem which involves a moderate mismatch between the learner's original network of ideas and abilities and those needed to solve the problem) leading to provoked adaptation is at the heart of learning.

Perhaps most important, the activities here go somewhere. Polya said [private conversation], "First, a good problem must be difficult enough for the student, else it is an exercise and not a problem. Second, it should be interesting to the student. And most, important, it should *go somewhere.* Inference is an important "somewhere. " It is the glue that holds mathematics—and in fact, much of life—together.

The results here are consistent with the principles of problem-based learning.

Certainly problem-based learning is not entirely new to math teachers. Surely some teachers have used the principles of PBL in their classrooms from time to time, but no concerted and continuous thrust has been given PBL in American mathematics education in either research or practice.

The fact that PBL has been used widely and apparently successfully in a wide variety of fields is heartening. It is reasonable to suspect that leaders in a wide variety of disciplines, including medicine, do not adopt new polities and practices without good reason.

More important, the results are consistent with the Piagetian emphasis on equilibrium. Equilibrium and homeostasis are fundamental not only to the biological world but to the world of education.

Perhaps this is the time for American mathematics education to make some small starts away from the parrot-training that is so common and so fruitless.

Notes

1. National Council of Teachers of Mathematics (2000). Principles and standards for school mathematics, p.51, Reston, VA: NCTM

2. http://www.mathematicallysane.com/analysis/polya.asp

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http://www.udel.edu/ce/pbl2002/speaker_smith.html

4. http://www.uc.edu/pbl/intro_history.shtml

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http://www.udel.edu/ce/pbl2002/speaker_tan.html

6. The essay was first published in 1917. See Alfred North Whitehead, The Aims of Education and Other Essays, (New York: Free Press, 1957).

7. O'Brien, Thomas C., and Richard, June V. "Interviews to Assess Number Knowledge," The Arithmetic Teacher, May 1971.

8. Thomas C. O'Brien and Judy Barnett, "Fasten your seat belts," Phi Delta Kappan, 85(3), 201-6, November 2003.

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Answers

1. In the one-person Mystery Person game, the data are inconclusive.

2. Children's work (including answers) on the 5 Righties and 12 Righties tasks is shown below.

New perspectives on identification and fostering mathematically gifted students:

matching research and practice

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This special section of vol8,nos1&2 of *The Montana Mathematics Enthusiast* is a result of tremendous enthusiastic team work of many outstanding mathematics educators worldwide who are concerned with the issues related to mathematical giftedness and devoted to share with the international community their ideas, research results and best practices. The idea of the special issue on mathematical giftedness arose during the Topic Study Group 6 (TSG6) meeting at the 11th International Congress on Mathematics Education (ICME-11) in Monterrey, Mexico, in 2009 led by Viktor Freiman and Ali Rejali in collaboration with Mark Applebaum, Pablo Dartnell, and Arne Mogensen. More than 60 participants and 20 presentations resulted in invitations to scholars to share their findings in extended papers that meet the high standards of The Montana Mathematics Enthusiast. Each paper was rigorously reviewed by at least two renowned scholars. As a result of our work, we present 11 papers in this issue, 8 of which arise from the TSG6 work and 3 others are original papers written especially for this issue.

Dealing with the topic of mathematical giftedness is a very delicate and complex task because of the existence of multiple views, cultural perspectives, and pedagogical approaches to the subject. There is a growing interest of the mathematics education community in the field of giftedness and creativity that is supported by the intensive continuous work of the Topic Study Group on Activities and Programs for Mathematically Gifted at ICMI's Congresses, International Conferences on Mathematical Creativity and Education of Mathematically Gifted, as well as several recent publications

(Leikin, Berman, and Koichu, 2009; Sriraman, 2008; Sriraman, Freiman, and Lirette-Pitre, 2009; Sriraman and Lee, 2010).

Students we talk about can be identified by means of different terms (gifted, talented, promising, etc.) and different tools (psychological tests, standard assessments, school marks, teachers' observations, etc.) based on variety of criteria (problem solving behaviour, cognitive abilities, multiple intelligences, personal attitudes, etc.) and (or) their combination. Researchers and practitioners mostly agree that those students have special needs, deserve particular attention and require a different teaching approach.

At the same time, it seems that many mathematically gifted students may remain non-identified and non-nurtured in regular classrooms; they may even have difficulties complying with regular school routine and, under certain circumstances, become underachievers. Thus, their high potential may not be realized and get lost for the society which is in odds with modern trends of more inclusive school systems that care of all students helping them to become active, engaged and well-rounded citizens of the modern world. Although separating those so called gifted students from the school system may change their natural growth and diminish their ability to work with others, and also damage the level of other schools (Hatamzadeh and Rejali, 2008).

The work of the similar Topic Study Group at the ICME-10 has identified four main issues related to activities and programs for mathematically gifted:

1. Characteristics of giftedness and how such students can be identified.
2. Having identified the group of gifted students, it is now necessary to consider how such students should be met both inside and outside of the classroom.
3. Considering the materials that were presented to gifted students and discussed in particular, technology that might be of use.
4. Specific examples of problems and investigations.

By organizing our work at the ICME-11, we formulated following questions in order to pursue and extend our investigation:

a) What do we know from recent literature on the subject of mathematically gifted students?

b) Who is a mathematically talented student? What are her or his characteristics? What are the differences between the terms "mathematically gifted, mathematically promising, mathematically talented, mathematically able, and mathematical genius and others used by researchers and practitioners? How does it vary from one country to another?

(c) How can we identify them? What are the ways to search for mathematically gifted students at different ages and settings?

(d) How do we deal with students and kids who think they are (or their family think they are) mathematically gifted, but they are not according to identification criteria?

(e) What is the societal phenomenon of overreacting to mathematically gifted student and how it may affect the life and the future of these students?

(f) How do mathematically gifted students work with mathematics? What are their strengths and weaknesses on the subject? What are their attitudes and performances? How should we take all this into account in our teaching and assessment practices?

(g) What are special needs for mathematically gifted students (additional trainings, their school and everyday life experiences, their works at home, participation in extracurricular activities such as problem solving, mathematics clubs, mathematics houses, competitions, etc?)

(h) What should educational systems do in order to meet the needs of mathematically gifted? What are the (positive or negative) effects of curriculum as well as its implementation in practice inside or outside school on the development of mathematically gifted students?

(i) How should we teach mathematically gifted students (at different levels) and provide extra curriculum activities for them? How can we, as educators or teachers, help them to be more creative?

(j) How should we prepare teachers to work with mathematically gifted students?

(k) What are the challenges for gifted students and their mentors and how can these challenges be addressed?

(l) What is the future of mathematically gifted students and how can we help them realize their potential?

(m) What are the resources on the subject? What role may technology play in providing additional resources for mathematically gifted?

(n) What are other issues useful for further studies on the subject that are not mentioned in previous questions?

Neither the work of the particular group nor a special issue on mathematical giftedness could cover all aspects raised above questions. However, papers presented in this issue bring new perspectives in theoretical and methodological work, as well as their implementation in practice. Some other results have been presented at TSG6 in ICME-11 (http://tsg.icme11.org/tsg/show/7).

The eleven papers feature four themes: state of the research in the field and promising paths (Roza Leikin), *programs for gifted students* in different educational settings and cultures (Harvey B. Keynes and Jonathan Rogness; Arne Mogensen; Angela M. Smart; Mark Saul), *teacher education and professional development* (Mark Applebaum, Viktor Freiman, and Roza Leikin; Manon LeBlanc and Viktor Freiman), and *mathematical content, teaching approaches, and assessment* (Ed Barbeau, Paul Betts and Laura McMaster, Margo Kondratieva, Ildiko Pelczer and Fernando Gamboa Rodriguez).

We would like to thank the ICMI for inviting us to organize the Topic Study Group at the ICME-11, all participants of this group, all authors of papers for this issue

and reviewers who did a tremendous work of giving valuable comments and suggestions of how to improve the papers. We should acknowledge the help of Arne Mogensen and Mark Apelbaum as the associate editors, as well as many colleagues who acted as referees of the submitted papers for this issue: Agnis Andzans, Edward J. Barbeau, Scott A. Chamberlin, Lyn English, Sharade Gade, Gerald Goldin, Djordje Kadijevich, Alexander Karp, Petar Kenderov, Margo Kondratieva, Gregory Makrides, Peter Mitchell, Claus Michelsen, Sergey Pozdnyakov, Arthur B. Powell, Linda Sheffield, Peter J. Taylor, and Zalman Usiskin. All of them deserve special thanks.

Finally, we would like to thank Bharath Sriraman and the Editorial Board of the Journal for organising the special issue, the external reviews and hope the readers find the articles enjoyable and useful for future work on mathematical giftedness.

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Freiman & Rejali

The education of mathematically gifted students: Some complexities and questions

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Abstract: *In this paper I analyze some complexities in the education of mathematically gifted students. The list of issues presented in this paper is not inclusive; however, all of them seem to be typical on the international scope. Among these issues are: (1) the gap between research in mathematics education and the research in gifted education; (2) the role of creativity in the education of the gifted and the theoretical perspective on the relationship between creativity and giftedness, and (3) teaching the gifted and the teachers of gifted, including relationships between the equity principle in mathematics education and views on the education of gifted. In the paper I outline some actual research questions in the field of education of mathematically gifted.*

Key words: Educating the gifted, Mathematical creativity and giftedness, Research and practice

INTRODUCTION

Mathematics educators and researchers in mathematics education agree that any decisions made with respect to the education of mathematically talented children and adolescents should be based on research findings and on the deep understanding of mathematical thinking and learning. Following Schoenfeld (2000, 2002), who shed light on the two main purposes of research in mathematics education, I maintain that research in the field of mathematical giftedness and creativity must be carried out in two interrelated directions:

- First (theoretical) is to understand the nature of mathematical giftedness and mathematical creativity from the perspectives of thinking, teaching, and learning

- Second (applied) is to use such understanding to improve mathematics instruction in a way that helps realize mathematical giftedness and encourage mathematical creativity.

I demonstrate the shortage of systematic research in the education of mathematically gifted students, and outline some complexities in the education of gifted that require systematic research. I present some research questions that can be seen as a research agenda in the field of teaching mathematically gifted students.

1. RESEARCH IN MATHEMATICS EDUCATION AND RESEARCH IN GIFTED EDUCATION[1]

Educational literature related to the issues of high mathematical ability, mathematical talents, mathematical giftedness and mathematical creativity contains a variety of descriptive reports, instructional guidelines, and reference materials, but research reports in the field are less common. Analysis of the research literature in the fields of gifted education and mathematics education leads to the conclusion that the studies in these two fields moved in two tangential rather than intersecting directions. The following evidence clearly illustrates that mathematics education is underrepresented in the field of gifted education and, vice versa, the research on giftedness and gifted education is underrepresented in the field of mathematics education.

1.1 Publications in Research Journals Devoted to Giftedness

During the past decade seven key journals in the field of giftedness and intelligence (Gifted Child Quarterly, High Ability Studies, Journal for the Education of the Gifted, The Journal of Secondary Gifted Education, Creativity Research Journal, and the Journal for the Education of the Gifted) published only a few articles devoted directly to mathematical giftedness or creativity. The following twelve papers, from among more than 1,000 published in the past ten years, form an almost complete list: Chamberlin & Moon, 2005; Hodge & Kemp, 2006; Hong & Aqui, 2004; Koichu & Berman, 2005; Kwon, Park & Park, 2006; Mann, 2006; Nokelainen, Tirry & Merenti-Valimaki, 2007; Olszewski-Kubilius & Lee, 2004; Reed, 2004; Sriraman, 2003; 2005; Yim, Chong, Song & Kwon, 2008). Eight of the twelve studies are clearly connected with research in mathematics education.

Mann (2006) and Sriraman (2005) perform a theoretical analysis of the relationship between mathematical creativity and mathematical giftedness. Koichu & Berman (2005), Sriraman (2003) and Yim, et al. (2008) analyze problem-solving strategies used by mathematically gifted students. Chamberlin and Moon (2005) and Kwon, Park and Park (2006) suggest developing mathematical creativity based on earlier advances in mathematics education. Reed (2004) suggests and tests approaches to teaching the gifted at geometry lessons in heterogeneous classroom. Other studies consider good performance in mathematics as one of the several characteristics of general giftedness (Hodge & Kemp, 2006; Hong & Aqui, 2004), one of

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[1] If I overlooked any important publications in the course of the research, I apologize for the omissions.

the outcomes of attribution styles (Nokelainen, Tirry & Merenti-Valimaki, 2007) investigate the influence of attribution styles on the development of mathematical talent, and one of the subjects in extracurricular activities in and out of school (Olszewski-Kubilius & Lee, 2004).

1.2 Publications in Research Journals in Mathematics Education

A search of seven leading research journals in mathematics education (Journal for Research in Mathematics Education, Educational Studies in Mathematics, Journal of Mathematical Behavior, Focus on the Learning Problems in Mathematics, The International Journal on Mathematics Education – ZDM, Mathematical Thinking and Learning, and For the Learning of Mathematics) reveals that in the past decade only few publications were explicitly devoted to mathematical giftedness and creativity in these journals.

Only one publication in these journals is explicitly devoted to learning process of mathematically talented students, namely, Amit and Neria (2008) explore problem-solving strategies of talented pre-algebra students. About 10 publications in these journals directly address mathematical creativity: Presmeg (2003) and Ernest (2006) analyze and emphasize the importance of creativity in the development of mathematical meaning, and Lithner (2008) suggests a framework for analysis of mathematical activity and describes creative thinking in mathematics as opposed to imitative thinking. Liljedahl and Sriraman (2006) conduct a discussion about the meaning of mathematical creativity and its role in activities of professional mathematicians vs. mathematical activities of school children. This work provides a theoretical view on mathematical creativity, with connections to works by Polya, Hadamard, and Poincaré (for details about this theoretical perspective, see Liljedahl, 2009).

Sriraman (2009) argues that mathematical creativity is the main mechanism of the growth of mathematics as science. However he finds that the creativity "is a relatively unexplored area in mathematics and mathematics education." (p. 13). In his paper Sriraman provides a critical analysis of characteristics of mathematical creativity from different theoretical perspectives. Plucker and Zabelina (2009) stress the importance of defining creativity, admit the lack of literature that deals with the concept of creativity in mathematical education and provide their own definition of general creativity. Based on this definition they discuss domain-specific and domain-general creativity. Hoyles (2001) analyzes the role that a computer-based learning environment can play in navigation between skills and creativity in teaching mathematics. This analysis leads to observation that technology-based inquiry opens opportunities for the

advancement of students' mathematical creativity. Bibby (2002) provides a view from an elementary school mathematics classroom on the opportunities of simultaneous development of students' logical and creative reasoning. Shriki (2009) analyses pre-service mathematics teachers' views on mathematical creativity and demonstrates that their knowledge associated with mathematical creativity is insufficient for the discussion of the subject. Huckstep and Rowland (2000) review *Creative Mathematics*, a book by Upitis, Phillips & Higginson (1997), which provides insightful ideas for creative mathematical activities in school.

Note that numerous publications in Mathematics Education journals, in the past ten years, use the words "creative", "inventive," and "original" in their descriptions of mathematical activities suggested to students and of students' mathematical performance. Mathematics educators and researchers design, describe, and explore mathematical activities with a high potential for the development of mathematical creativity in school children. Works devoted to "doing mathematics" in classroom, to inquiry based classrooms and students' autonomy in such classrooms, to active construction of mathematical knowledge, and to students heuristics in problem solvingare implicitly related to mathematical creativity among students. However, in these works the words "creativity" and "inventiveness" are not part of terminology in the analysis of students' mathematical reasoning and the teachers' role in the classroom. Mathematics education must therefore pay more attention to research of different kinds of mathematical activities, with a clear focus on students' creative thinking and giftedness.

1.3 Other sources

A small number of publications in other journals focused on specific issues in the mathematical reasoning and problem solving of the gifted population. Among them are Gorodetsky & Klavir (2003); Livne, Livne & Milgram (1999); and Chiu (2009), who examine students' creativity in mathematical problem solving and suggest ways for analyzing students' creativity.

Several other research publications about students and adults with high mathematical abilities can be found in the Journal of Educational Psychology, Psychological Science Journal, and Journal of Applied Psychology. These studies, mostly by Lubinski, Benbow and their colleagues, are a part of larger longitudinal study that was precipitated by the study of Mathematically Precocious Youth (SMPY) at John Hopkins University which was initially spearheaded by Julian Stanley in earlier 1980s. For example, Lubinski, Webb, Morelock and Benbow (2001), on the basis of 10 years of observations, demonstrated that early identified

distinctions in intellectual strength predicted differences in the developmental trajectories and occupation pursuits of highly talented individuals. They also demonstrated the effectiveness of acceleration for individual cases in their 20-year follow-up study on 1975 mathematically gifted adolescents (top 1%). They demonstrate that earlier identified gender differences in mathematical reasoning of the participants predicted differential education and occupational outcome all of which were successful. Other publications by Lunibski and Benbow explore innovative evaluation tools for the identification of mathematical talents. For example, Lubinski & Benbow (2000) demonstrate that combination of theory of work adjustment concepts and psychometric methods facilitate positive development of talented youths. Another study (Webb, Lubinski & Benbow, 2007) demonstrates that spatial ability is significant for talent identification. Still, these studies focus mainly on general psychological characteristics of individuals and do not explore learning and thinking processes in mathematically gifted school students as associated with the contemporary theories of Mathematics Education (see elaboration and examples in Leikin, 2009a).

Lately there were several edited volumes devoted to these issues. Sriraman's (2008) monograph *Creativity, Giftedness, and Talent Development in Mathematics* includes contributions devoted to creativity and giftedness in mathematics, offers new perspectives for talent development in mathematics classroom and gives insights into the psychology of creativity and giftedness. However, the editor stressed the lack of systematic research of talent development in mathematics education. Leikin, Berman and Koichu (2009) edited a volume entitled *Creativity in Mathematics and the Education of Gifted Students*. As a result of a consolidated effort of a group of experts in the fields of mathematics education, psychology, educational research, mathematics and policy making the book puts in the foreground mathematical creativity and mathematical giftedness as important topics in educational research. The book includes several reports on the empirical studies related to mathematical creativity and giftedness along with theoretical framework for the analysis of mathematical creativity and giftedness. The editors stress the importance of empirical research in the field that must be performed in various spheres related to the education and identification of mathematically able students (see Leikin, 2009a).

1.4 International forums

At the international level one can see raising awareness of the importance of gifted education in mathematics. This awareness is reflected in a number of international forums that lately have focused their work on mathematical creativity and giftedness. ICME conferences twice included Topic Study Group (TSG) "Activities and programs for gifted students" (TSG-4 at ICME-10 in 2004 http://www.icme-organisers.dk/tsg04/; TSG-6 at ICME-11 in 2008 http://tsg.icme11.org/tsg/show/7). At ICME 11 Discussion group "Promoting creativity for all students in mathematics education"took place along with TSG-6 mentioned above (DG-9, http://dg.icme11.org/tsg/show/10). In summer 2008 ICMI Study-16 "Mathematical challenges in and beyond the classroom" discussed a variety of issues related to education of mathematically talented students. The results of the elaborated discussion by all the participants are expressed in the corresponding ICMI Study Volume (Barbeau & Taylor, 2009).

Since 1999 the main forum (founded by Meissner and Sheffield) that unites educational researchers, mathematicians and mathematics educators interested in education of mathematically gifted and development of mathematical creativity has been International Conference on Creativity in Mathematics and the Education of Gifted Students. Each of the 5 conferences (1999 – in Muenster, Germany; 2002—in Riga, Latvia; 2003—in Rousse, Bulgaria; 2006 – in Budejovice, Czech Republic; 2008 – in Haifa, Israel) issued proceedings including the conference papers. Eventually in Riga, Latvia in summer 2010 the participants of the conference established the International Group for Mathematical Creativity and Giftedness (MCG) (for the information about the group and the conferences see http://igmcg.org)

To sum it up, the discussion presented in this section of the paper underscores the need for advancement of the research-based perspectives on mathematical talent and mathematical creativity both in the direction of characterization of individuals with high mathematical ability (both analytical and creative) and the development of high mathematical abilities. Since Krutetskii's (1976) fundamental research on characterization of mathematical abilities in gifted students, there were performed several studies focusing very specifically on issues related to mathematics reasoning and problem solving of gifted students. Using the criteria suggested by Schoenfeld (2000, 2002) for theories and models in mathematics education, I argue that most of the existing works in the field must be further examined with respect to their explanatory and predictive power, scope, and replicability. The following sections in this paper describe several

complexities in the education of mathematically gifted student that can become focal points of the systematic research in the field of mathematics education.

2. MATHEMATICAL CREATIVITY AND MATHEMATICAL GIFTEDNESS

One of the research questions that requires special attention of the mathematics education community is the relationship between mathematical creativity and mathematical giftedness.

2.1 Creativity as property of professional mathematicians vs. creativity for all

One of the complexities related to the relationship between mathematical giftedness and mathematical creativity is rooted in the contrast between viewing mathematical creativity as a property of the mind of the professional mathematicians (Subotnik, Pillmeier & Jarvin, 2009; Sriraman, 2005; Liljedahl & Sriraman, 2006) and the opinion that mathematical creativity must and can be developed in all students (Sheffield, 2009; Yerushalmy, 2009; Hershkoivits, Peled & Littler, 2009).

According to Subotnik et al. (2009) creativity is fundamental to the work of a professional mathematician. In the course of their work, mathematicians find and solve problems that are substantive and challenging. Subotnik et al. (2009) describe the development of ability into competence, expertise, and finally scholarly productivity/artistry and argues that mathematicians need an array of psychosocial skills to be successful in such a highly competitive intellectual arena. Similarly Ervynck (1991) considers mathematical creativity as one of the characteristics of advanced mathematical thinking. Ervynck connected mathematical creativity with advanced mathematical thinking and considered it as the ability to formulate mathematical objectives and find inherent relationships among them.

Sriraman in his conversation with Liljedahl on the notion of mathematical creativity (Liljedahl & Sriraman, 2006) suggests that at the professional level mathematical creativity can be defined as "the ability to produce original work that significantly extends the body of knowledge (which could also include significant syntheses and extensions of known ideas)" or "opens up avenues of new questions for other mathematicians" (ibid. p. 18). Sriraman (2005) considers mathematical creativity as one of the characteristics of advanced research mathematicians. He defined seven levels of mathematical ability associated with mathematical creativity and giftedness. The abilities of professional mathematicians, according to this model, are at levels 5, 6, and 7, and he differentiated these levels with respect to the mathematicians'

measure of creativity: "Level 5" mathematicians are productive in mathematical research and have high levels of analytic and practical abilities, whereas creative mathematicians (levels 6 and 7) have higher levels of synthetic abilities, which allow them to "open up new research vistas for other mathematicians" (*ibid*., p. 30).

Sriraman (2005) stresses that creativity in school mathematics obviously differs from the creativity of professional mathematicians: "At the K–12 level, one normally does not expect works of extraordinary creativity; however, it is certainly feasible for students to offer new insights". Furthermore Silver (1997) and Sheffield (2009) address "creativity to all students" and consider solving problems and problem posing as main tools for the development of mathematical creativity in all the students. Along with this position Liljedahl and Sriraman (2006) argue that at school levels or even the undergraduate level "it is feasible for students to offer new insights/solutions" in mathematics. These insights/solutions are usually new with respect to mathematics the students have already learned and the problems they have already solved. Taking a developmental point of view, Sheffield (2009) suggests a continuum of mathematical proficiency through the development of creative ability in mathematics: innumeraters → doers → computers → consumers → problem solvers → problem posers → creators.

Viewing personal creativity as a characteristic that can be developed in schoolchildren requires distinction between *relative* and *absolute* creativity (Leikin, 2009). *Absolute creativity* is associated with "great historical works" (in terms of Vygotsky, 1930/1984), with discoveries at a global level. For example, examples of absolute creativity may be seen in discoveries of Fermat, Hilbert, Riemann and other prominent mathematicians (Sriraman, 2005). *Relative creativity* refers to discoveries of a specific person in a specific reference group. This type of creativity refers to the human imagination as it creates anything new (Vygotsky, 1930/1984).

2.2   The relationship between mathematical giftedness and mathematical creativity

While connecting between high mathematical abilities and mathematical creativity researchers express a diversity of views. Some researchers claim that creativity is a specific type of giftedness (e.g., Sternberg, 1999, 2005), others feel that creativity is an essential component of giftedness (Renzulli, 1978, 1986), while other researchers suggest that these are two independent characteristics of human beings (Milgram & Hong, 2009). Thus analysis of the relationships

between creativity and giftedness with specific focus on the fields of mathematics is important for better understanding of the nature of mathematical giftedness.

Creative thinking includes finding different solutions and interpretations, making various mathematical connections, applying different techniques, and thinking originally and unusually. In this sense creativity is a part of the problem solving process and one of the outcomes of learning mathematics. Another (overlooked) perspective on creativity we find in works of Vygotsky who stresses the role of creativity in the process of knowledge development, abstraction and generalization. Vygotsky (1930/1984) argued that creativity (imagination) is one of the basic mechanisms that allow development of new knowledge. A child activates imagination when connecting new and previously known concepts, when elaborating the known constructs, and when developing abstract notions. Thus imagination (or creativity) is a basic component of knowledge construction. Thus we deduce as follows about the complexity in the relationship between creativity and knowledge development: *to have knowledge is a necessary condition for a person to be creative while to have imagination is a necessary condition for knowledge construction.* These relationships are one of the central issues for investigation by mathematics education researchers.

Providing a precise and broadly accepted definition of mathematical creativity is an extremely difficult and probably unachievable task (Haylock, 1987; Liljedahl & Sriraman, 2006; Mann, 2006; Sriraman, 2005). Mann (2006) affirmed that analysis of the research attempting to define mathematical creativity revealed how the lack of an accepted definition for mathematical creativity hinders research effort. Following these observations, Leikin (2009a) suggested a model for the evaluation of creativity using Multiple Solution Tasks. The model includes operational definitions and a corresponding scoring scheme for the evaluation of creativity, which is based on three components: fluency, flexibility, and originality -- as suggested by Torrance (1974). For the evaluation of originality it utilizes Ervynk's (1991) insight-related levels of creativity in combination with conventionality of the solutions which comprises students' educational history in mathematics.

In several recent studies, that accepted developmental perspective on mathematical creativity, I and my colleagues implement the model for evaluation of mathematical creativity through Multiple Solution Tasks (Leikin, 2009b). In two of the studies (Levav-Waynberg & Leikin, 2009 and Guberman & Leikin, in preparation; Leikin, Levav-Waynberg & Guberman,

accepted) we examine development of mathematical creativity through mathematical instructions. Among other findings, we discovered that as the result of systematic implementation of Multiple Solution Tasks in mathematical instruction, students' flexibility and fluency significantly increased. Students' originality, however, decreased non-significantly, and resulting in a non-significant decrease in the creativity. Findings related to flexibility and fluency are naturally desirable.

Results related to originality have a reasonable explanation: when the students' flexibility increases, more students in the group produce more solutions and it becomes more difficult to produce a unique solution. Following these findings, we question the possibility of developing originality and hypothesize that in the fluency-flexibility-originality triad, fluency and flexibility are of a dynamic nature, whereas originality is a "gift".

Finally, our findings demonstrate that originality appeared to be the strongest component in determining creativity and the strength of the relationship between creativity and originality can be considered as validating our model, being consistent with the view of creativity as invention of new products or procedures. At the same time, our studies demonstrate that this view is true for both the absolute and the relative levels of creativity. We also assume that one of the ways of identification mathematically gifted students is by means of originality of their ideas and solutions.

Based on the above observations it is clear that systematic research should be performed to examine different ways of promotion of mathematical creativity in school students, identification of creative talents in school students, and between understanding of the relationship between mathematical creativity and mathematical giftedness.

3   TEACHING THE GIFTED AND TEACHERS OF THE GIFTED

3.1   Approaches and frameworks for teaching the gifted

Subbotnik et al. (2009) stressed that during the past 25 years multiple educational programs for talented youths have been proposed. Examples include Parnes's creative problem solving method (Parnes, Noller & Biondi, 1977), Renzulli's enrichment triad model (Renzulli, 1978; Renzulli & Reis, 1985), Johns Hopkins University acceleration program (Fox, 1974; Stanley, 1991), Tannenbaum's (1983) enrichment matrix, and many others. According to Nevo and Rachmel

(2009) programs for gifted education can be ranked by the intensity of the program, the most intensive being found in special schools for mathematically gifted students (Vogeli, 1997).

Usually characteristics of the effective learning environments for mathematically talented students follow specific characteristics of this population. These students tend to use self-regulatory learning strategies more often and more effectively than other students, and are better able to transfer them to novel tasks. In their review of research on the thinking process of highly able children, Shore and Kanevsky (1993) argued that if the gifted think more quickly and make fewer errors, and then we need to teach more quickly. Shore and Kanevsky stress that this is not entirely the case; adjustments have to be made in methods of learning and teaching, to take into account individual thinking differences Nisbet (1990) suggested several approaches to promote self-regulation in learning in science teaching that seem to be applicable to mathematics education:

- Talking aloud. According to this approach the teacher talks aloud while solving a problem so that the pupils can visualize work-out.
- Cognitive apprenticeship. This approach requires the teacher to demonstrate to students the processes that experts use to handle complex tasks, guiding the pupil via experiences.
- Discussion involves analysis of the processes of argument.
- Cooperative learning, which requires that pupils explain their reasoning to each other.
- Socratic questioning is based on careful questioning to force pupils to explain their thought processes and their arguments.

Nevo (2004) distinguished the methods of nurturing gifted children that exist around the world, and classified them according to three basic approaches relating to the capabilities of gifted students:

- *Acceleration* is usually defined as learning topics within the areas of studentsat accelerated pace. This can be expressed in early entrance into school, skipping grades, Advanced Placement, and/or earlier entrance to the university courses (Southern & Jones, 1991; Van Tassel-Baska, 2004a, b).
- *Broadening* is considered as studying a additional topics and subjects simultaneously with usual school mathematics. For example, studying extra-curricula topics in mathematical circles relates to the broadening approach. (e.g., Fomin, Genkin & Itenberg, 2000), learning belong to this approach. .

- *Deepening* is usually associated with studying curricular topic at greater depth than prescribed by the curriculum or school textbooks. Deepening can include, for example, learning underlying rules for regular curricular topics.

Some of these approaches are highly appropriate for in-school framework as special classes for students with high abilities in mathematics can differ in the manner in which ability grouping is managed: through subject-based streaming, the provision of special classes, or the availability of special schools. Other activities such as math clubs, competitions, and student conferences can be found both in school and out of school. The integration of students in university courses, virtual courses, and personal mentoring are typical out-of-school solutions (Leikin, 2009a).

Despite the variety of frameworks for the education of mathematically gifted students, there is lack of empirical data about this field. It is necessary to conduct systematic empirical studies on various programs to gain better understanding of their effectiveness and suitability for the realization of the students' mathematical potential and the development of their creativity. We lack theoretical characterizations of effective courses and programs for mathematically talented students. Research should be directed at the theoretical characterizations of programs for students with high mathematical abilities

3.2 Equity principle and ability grouping

Some educational communities have provided special ability-grouping-based frameworks for treating mathematically gifted students. Among them special schools, as, for example, Kolmogorov's Schools in Russia (Kolmogorov, 1965; Kolmogorov, Vavilov & Tropin, 1981), or centers for gifted and talented youth, as, for example, CTY at John Hopkins University (http://cty.jhu.edu/about/index.html). These schools have shown to be effective and exciting frameworks for the education of gifted students (e.g. Karp, 2009; Vogeli, 1997). Nevertheless, some opponents of ability grouping argue that it contradicts the equity principle in mathematics education pronounced by the National Council of Teachers of Mathematics (NCTM, 1989). According to this principle "all students, regardless of their personal characteristics, backgrounds, or physical challenges must have opportunities to study – and the support to learn – mathematics". At the same time, special schools and classes for gifted may be seen as the expression of the equity principle because education must provide equal opportunities to all students to learn, realize their potential, which is comprised of intellectual abilities, personality and affective characteristics (NCTM, 1995; Sheffield, 1999; Leikin, 2009a). The central function

of the educational system is providing each and every student regardless of his/her social and economical status with learning opportunities that match their potential and promote it to the maximal extent.

Thus interpretation of the equity principle as associated with the education of mathematically gifted students is not trivial. In late 80th – earlier 90th the equity principle was (mis)interpreted as a recommendation to provide all students with identical instruction. The drive for social justice and the democratic view of education led to the cancellation of ability tracking in mathematics, and domination of heterogeneous mathematics education. Very often at a local level, school principles, mathematics coordinators or mathematics teachers echo this policy and held a mid-ability oriented position based on reasonable argument: If I will let high achievers learn "alone" then the average students will have nowhere to grow.

This conception also received a research base when in late 80s heterogeneous classroom was shown as an effective learning environment especially for students with middle level of abilities. Cahan, Linchevski and Igra (1995), Cahan and Linchevski (1996) and Linchevski and Kutscher (1998) demonstrated that mixed-ability grouping is more beneficial for mid-level student that grouping with low achieving students and that high achievers do not differ in their learning outcomes as either kind of ability grouping. The debate on the necessity of ability grouping is legitimate, and both proponents and opponents of heterogeneous mathematics education use valid arguments to justify their positions. NCTM (2000) re-conceptualized the equity principle and stressed that "Equity does not mean that every student should receive identical instructions; instead it demands that reasonable and appropriate accommodations be made to promote access and attainment for all students" (ibid., p. 12).

Ability grouping was shown as one of the ways of achieving the equity principle in the education of mathematically gifted students. Ability grouping may be essential for education of gifted both from cognitive and affective perspectives (Davis & Rimm, 2002), and it ought to supply special education to mathematically gifted students and prevent talent loss (Milgram & Hong, 2009). On the other hand ability grouping is still questionable both in light of the equity principle and of some research findings. For example, Shani-Zinovich and Zeidner (2009) report that gifted students in homogeneous (ability-level) classes demonstrated a higher degree of commitment than gifted students in heterogeneous classes. Homogeneous classes, however, can have a negative effect on students' self-evaluation, self-esteem, and emotional environment

In the light of the debate on ability grouping the following question demands careful and systematic investigation: What type of ability grouping is the most effective for mathematically gifted students?

3.3 The centrality of mathematical challenge for the realization of mathematical potential

A mathematical challenge is an interesting and motivating mathematical difficulty that a person can overcome (Leikin, 2007). Many authors recognize the centrality of mathematical challenge for the realization of mathematical promise and as a characteristic of the activities in which gifted mathematicians are involved. The importance of mathematical challenge, the approaches in teaching challenging mathematics, and the role of mathematical challenge in school curricula are analyzed from the international perspective in Barbeau & Taylor (2009). Taylor (2009) and Applebaum & Leikin (2007) analyzed types of mathematical challenges for school mathematics classrooms and stress the importance of teachers' mathematical, meta-mathematical and pedagogical knowledge associated with teaching challenging mathematical tasks. Movshovitz-Hadar and Kleiner (2009) consider mathematical challenge as one of the definitive conditions of mathematical courage that advances mathematics as science. They hypothesize that understanding of the underlying mechanisms of mathematical courage can shed light on the ways in which gifted students can be taught. Sheffield (2009) suggests ways in which mathematically promising students can be challenged, and stresses that challenges for students are differentiated according to their mathematical content knowledge, background, and interests.

Mathematical challenge is a necessary condition for realization of mathematical potential. It can appear in different forms in mathematics classrooms. There can be *proof tasks* in which solvers must find a proof, *defining tasks* in which learners are required to define concepts, *inquiry-based tasks, and multiple-solution tasks*. Mathematical challenge depends on the type and conceptual characteristics of the task, for example, conceptual density, mathematical connections, the building of logical relationships, or the balance between known and unknown elements. From the research perspective some questions can be interesting for the future investigation: What are the types of challenging tasks more appropriate for mathematically gifted students? What challenges better develop mathematical creativity? For example, what is the relationship between Olympiad tasks and students' mathematical creativity?

3.4   Teachers and teacher education in the education of mathematically talented students

The last and certainly not least important issue in the education of mathematically talented children and adolescents is the teacher's role in mathematics classroom, their ways of teaching and teacher preparation for the education of the gifted.

According to Brousseau's (1997) one of the teacher's central responsibilities is the devolution of good (challenging) tasks to learners. It is almost obvious that teachers ought to provide each and every student with learning opportunities that fit their abilities and motivate their learning. Sheffield (2009) maintains that teachers have to challenge students who are ready to move to a higher level, and provide hints to students who may be frustrated. Mathematical challenges directed at students' development usually entail scaffolding provided by a teacher. Consequently in Leikin (2009a) I recommend hanging the following motto on the door of all mathematics classrooms: *Exercises for homework – challenges for the classroom* (ibid. p. 405).

One way of helping teachers to use challenging mathematics in their classrooms is to provide them with appropriate learning material (e.g., a textbook) and make a large number of challenging tasks available to them (Barbeau & Taylor, 2009). However, merely providing teachers with ready-to-use challenging mathematics activities is not sufficient for the implementation of these activities. Teachers must be aware and convinced of the importance of mathematical challenges, and they should feel safe (mathematically and pedagogically) when dealing with this type of mathematics (Holton et al., 2008).

Furthermore, teachers must have autonomy in employing this type of mathematics in their classrooms (Krainer, 2001; Jaworski & Gellert, 2003). They should be able to choose mathematical tasks themselves, create these tasks, change them so that they become challenging and stimulating, and, of course, must be able to solve the problems.To fulfill these conditions, teachers' mathematical knowledge should allow them to cope with challenges presented to their students and their pedagogical knowledge and skills should support scaffolding that teachers provide to their students (Evered & Karp, 2000; Even et al., 2009). Moreover, teachers have to be committed to the purpose of talent development and believe that this purpose is valuable. Last but not least important, teachers have to be provided with multiple opportunities to advance their knowledge, to develop commitment and belief.

Many more questions, such as who can be a teacher of mathematically talented students and how these teachers should be educated are open for systematic research. The following questions need our attention: Should the teachers of gifted be gifted? Should the teachers be creative in order to develop students' creativity? How teachers' creativity can be characterized both from the mathematical and from the pedagogical points of view? What are the desirable qualities of teachers' knowledge, beliefs and personality that make them creative and gifted teachers?

CONCLUSION

Education of mathematically talented children and adolescents is an extremely complex field. People hold different views over the education of gifted which are strongly dependent on their personal experiences and histories related to the education of the gifted. This is true of school students, parents, teachers, teacher educators, educational researchers and educational leaders and managers. Learning opportunities are the most critical factor for the realization of human intellectual potential. Leikin (2009a) pointed out the components that are crucial in developing the students' mathematical potential:

- Parental support (not pressure) – both financial and intellectual;
- Availability of special settings and frameworks for highly capable students in schools and out of schools;
- The necessity of involving technological tools that promote mathematical creativity in students and support teachers' attempts to scaffold students mathematical inquiry;
- Mathematical challenges as a central characteristics of learning environment that develops creativity and promotes mathematical talent;
- Teachers' proficiency in choosing and managing mathematical challenges.

In this paper I argue that each of these components should be a subject for the systematic research in mathematics education.

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Historical perspectives on a program for mathematically talented students

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Abstract: The University of Minnesota Talented Youth Mathematics Program (UMTYMP) is a highly accelerated program for students who are extremely talented in mathematics. This paper describes our experiences running UMTYMP since its inception thirty years ago, the challenges in implementing such a program, and how changes in the student body have necessitated changes in the program over three decades.

Key words: accelerated, high school, mathematics program

1. Introduction

There is a wide variety of research into the many and multifaceted issues in providing opportunities for mathematically talented or gifted students, ranging from the identification of students to the best methods of instruction for the population. While these issues can separately be excellent sources for further discussion, the development and implementation of a large ongoing program involves addressing most of these concerns in a specific contextual and highly integrated fashion. This paper examines the evolution, success, and challenges of the 30-year old University of Minnesota Talented Youth Mathematics Program (UMTYMP), which continues to operate as the leading accelerated mathematics program in Minnesota. To our knowledge the program is unique in terms of number of students, scope of the curriculum, and the granting of college-level honors credit to students in middle or high school.

UMTYMP was started in order to provide Minnesota's most mathematically talented students with an alternative educational experience. Each year approximately 400 students in grades 6-12 take their mathematics courses through UMTYMP instead of their own schools. During their first two years of the program, students cover four years of standard high school curriculum: algebra I and II, geometry, and pre-calculus. The final three years of the program are comprised of honors-level collegiate courses in calculus, linear algebra, and vector analysis. Along the way, students must develop a strong work ethic and problem-solving

skills. Many continue on to upper-division and graduate level mathematics courses before finishing high school.

Our significant historical perspective allows us to identify and discuss practices and issues which have remained unchanged, and those which appear to be quite different than just ten to fifteen years ago. Because any discussion of our program requires knowledge of the context, Section 2 gives a brief overview of UMTYMP's design and goals; a more extensive description is given in [3]. In Section 3 we discuss the specific issues of student selection, retention, characteristics, and the evolution of the program over the past fifteen years. This is followed by a brief look in Section 4 at statistical data to see how well UMTYMP has achieved its goals. Section 5 describes how our internal assessments have resulted in changes to the UMTYMP structure, and some of the ways UMTYMP could be used in educational research. Finally, Section 6 discusses the inherent challenges in expanding or duplicating the program.

2. Program Description

2.1. Origins of UMTYMP. The idea of a mathematics program for talented students in Minnesota originated in the mid-1970s. Several faculty members, including one of the authors, had attended district-wide high schools for academically talented students during their childhoods and felt that their experiences were very positive influences on their mathematical success. In addition, a new faculty member in educational psychology who had actively participated in Julian Stanley's Study of Mathematically Precocious Youth (SMPY) at Johns Hopkins University was a strong proponent of accelerated courses for gifted students, and advocated for an accelerated mathematics program in Minnesota. (See [4] and [5] for summaries of the SMPY findings on acceleration for students similar to those in UMTYMP.)

While the formation of a new high school was not feasible for those faculty members, these ideas led the development in 1976 of a two-year program, located at Macalester College in St. Paul, Minnesota, which provided supplemental mathematics for talented students in the Minneapolis/St. Paul metropolitan area. It covered essentially the same material as the first two years of the current UMTYMP curriculum. While this program was quite successful, the

issue of providing additional advanced coursework in calculus for students entering ninth or tenth grade became evident.

The head of the mathematics department at the University of Minnesota at the time was very sympathetic to the idea of providing high level courses for young students, and the department agreed to develop a calculus course for the Macalester program graduates. Very shortly thereafter, the Macalester program lost its funding and the mathematics departments at the University of Minnesota agreed to fund and administer the entire sequence of courses. In the fall semester of 1980, the first UMTYMP high school and calculus level courses were offered at the University.

2.2. *Program Overview.* The overall goal of UMTYMP is to provide a challenging, stimulating and nurturing academic program for students who are exceptionally talented in mathematics.[1] In their home schools these students often face the stigma of being good at math; we provide them with a chance to immerse themselves in a culture of mathematics and meet other students with the same talents. We also emphasize how mathematics in general and UMTYMP in particular can increase their future opportunities. Family interest in college achievements of UMTYMP graduates seems to be a major factor in their support of the program; parents seem drawn to the fact that the schools most attended by our alumni include prestigious institutions such as MIT, Stanford, Harvard, the University of Chicago and Caltech. (See Table 2, Section 4.) Unfortunately, we sometimes have uninterested and poorly performing students who were pushed to participate in UMTYMP because their parents (incorrectly) see our program as a way to get their children in their "dream school," regardless of student performance.

Many details of the actual implementation of the program are dictated by logistic and administrative constraints. Classes meet for a two-hour session once or twice per week after the regular school day, totaling about 35 sessions from September through May. This highly compressed schedule makes every moment of class time valuable and has a profound impact on the curriculum, teaching styles, and even the screening process to get into the program; see Section 3 for details.

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[1] See [7] for an overview of effective learning environments for mathematically talented students.

The first two years of UMTYMP comprise the "high school component," and the content is aligned with Minnesota's state standards for high school mathematics.[1] In two years UMTYMP students cover four years' worth of high school curriculum in algebra, geometry and pre-calculus. A single hour of our class time corresponds to about one week's worth of material in a high school classroom. Our instructors are therefore forced to cover only the central ideas and techniques, leaving students to learn the computational details on their own while working on extensive homework assignments. This course structure led to initial concerns that high schools would not count UMTYMP courses towards their graduation requirements, but a state law passed in 1984 requires schools to grant high school credit on their transcripts for students who have completed our courses.

After completing pre-calculus, students move on to the three year "calculus component," which consists of honors level courses in single variable calculus, linear algebra, differential equations, multivariable calculus and vector analysis. The courses are more theoretical, and cover more topics, than the standard calculus sequence in our mathematics department. Students in this component receive honors level Institute of Technology credit for the courses on a University of Minnesota transcript; if they later choose to attend the University they will have already satisfied nearly half the requirements for a mathematics degree. If they enroll at a different undergraduate institution, the credits will either transfer or earn them placement into higher level courses there. Our intent is that no UMTYMP student would ever have to retake a course in the calculus sequences at any other institution and can proceed directly to post-calculus classes.

Based on our years of observations, we take the approach that understanding can be challenging and fun, but that learning computation skills, algorithms and how to use conceptual reasoning takes serious effort. Lecturing on the main ideas and then handing the students a textbook to learn the material on their own is not enough [6]. Thus a central feature of UMTYMP is the broad, deep support system, designed to enable virtually all interested and motivated students to be successful; see Section 3.2 for details. Significant emphasis is placed

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[1] This helps minimize difficulties for students at their own high schools. A Minnesota state law stipulates that if a student covers typical high school mathematics as part of an accelerated mathematics program at a college or university, the student's high school must recognize the courses as fulfilling the mathematics graduation requirements.

on developing effective work habits and individual problem solving skills, so that students learn that these abilities are as important as performance on classroom examinations. To stress the importance of clearly communicating mathematical ideas, each calculus homework assignment includes one problem whose solution should be written in a "professional" manner, roughly comparable to an example problem in a textbook. Students quickly learn that they must organize their work and write coherent explanations if they wish to earn full credit.

3. Specific Issues and Observations

This section describes our approaches to specific programmatic issues such as selection, recruitment and retention of students, and our teaching approach, all of which have remained largely unchanged during the last decade. We also discuss the defining characteristics (or lack thereof) of our mathematically talented students, and examine historical trends in our program.

3.1. *Student Selection and Recruitment.* Students who wish to enter either the high school or calculus components of UMTYMP must achieve satisfactory scores on entrance exams developed by our academic staff. In the past, we relied heavily on local schools to identify potential students. Before taking our entrance exam, a student would have to score in the 95th percentile or above on any national standardized mathematics examination and be recommended for their program by a teacher. In hindsight, however, this method of recruiting was too restrictive; in particular we discovered that teachers, whether through intentional or unintentional bias, tended to recommend male students over females of equal ability. In the interest of fairness our entrance exams are now open to any interested student in the appropriate grade levels.

The qualifying exams measure computational ability, but they stress critical thinking and speed. The qualifying exam for the high school component, for example, has 50 multiple choice questions to be answered in only 20 minutes. In each question students are given two quantities and asked to determine if one is larger than the other, if they are equal, or if there is not enough information to decide; see Figure 1 for examples. A high score indicates a solid

command of pre-algebra skills and the ability to process mathematics quickly. This has proven to be an effective way to find computationally strong students who can also handle the rapid pace of the courses. This process is far from perfect; it probably excludes students who are quite talented but work slower. However, budget and staffing issues require us to offer only accelerated courses, and hence UMTYMP is (unfortunately) not appropriate for those students.

While students who pass the entrance exams are invited to enroll in the program, we still ask parents and students to discuss the commitment with each other before accepting the offer.

Since all students entering UMTYMP have mathematical talent, the best predictors for student success are enthusiasm about mathematics in general (i.e. beyond algebraic computation), and the willingness to put in the effort to comprehend the ideas. These observations are consistent with recent studies on success in school and beyond [1]. Many of our underperforming students have time conflicts in their busy schedules or are simply not interesting in thinking deeply about mathematics.

We have found from earlier equity efforts that running mathematical enrichment programs throughout the academic year is a wonderful recruitment tool, since it introduces students to the type of mathematical thinking used in UMTYMP. Our Saturday morning classes are open to any students in grades 4-7 and cover subjects ranging from explorations of area and volume to spherical geometry and topology. Students who participate in the enrichment program are frequently eager to join UMTYMP, have better qualifying scores than general students, and are more successful in the program if they enroll.

As with many mathematics programs, we have difficulty consistently attracting females and students from traditionally under-represented minority groups. At times we have launched major initiatives to increase their numbers, such as the Bush Foundation Initiative described in [3], which succeeded in raising our female enrollment to over 40%. These gains are difficult to sustain once initiative funding ends. While the percentage of female students for UMTYMP still remains high for comparable programs, it has decreased to between 20% and 30%. In recent years we have started a new enrichment program which currently serves a diverse population of over 250 girls in grades 4-6, with the hope that a good number will eventually qualify for and enroll in UMTYMP

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(1) x and y are positive numbers and x < (x+y)/3.
    (a) x
    (b) y

(2) The sum of the remainders when each of these numbers is divided by 3:
    (a) 3, 10, 12, 19
    (b) 6, 11, 25, 27

Figure 1. Practice questions for the UMTYMP High School Component Entrance Exam. Students must determine the size relationship between the two quantities in each question.

3.2. *Retention.* A major challenge in UMTYMP is the retention of students once they enter the program, since the work and learning expectations both in the classroom and at home are typically very different from anything encountered in their K-12 education. For example, the conceptual approach and work expectations in UMTYMP Calculus significantly exceed those of Advanced Placement or International Baccalaureate courses. Moreover, the once-a-week format requires students to focus more intensely on the classroom lecture and activities. Note-taking skills are often nonexistent, causing difficulty when we cover concepts not in the textbook. Outside the classroom, students must learn to start their homework sufficiently early or risk turning in sloppy, half-finished assignments.

To counteract these problems we have created an extensive support system for students. Although class time is precious, we spend time in Calculus I discussing note-taking, study habits, and other tips for success in a college level course. Each semester we have ten optional study sessions to help students with their homework or to prepare for exams. We monitor exam and homework scores and quickly notify students (and their parents) when their work is below expectations, and frequently require them to attend study sessions or make other special arrangements. Because our students are not generally on the University campus, our instructors hold "virtual office hours" in which they answer questions from students via phone and email. As a general rule, nearly every student in UMTYMP who is interested in

mathematics and is motivated to work is able to complete the entire five year program at a satisfactory level.

Preventing UMTYMP students, especially females, from feeling isolated in class is another major retention concern. Our most successful strategy with females has been placing them in workshops which are 30%-50% female. When possible we also place them with female instructors or classroom assistants who are strong role models. This strategy has proven highly effective, and in recent years our retention rate with female students has been greater than or equal to the rate with male students.

3.3. *Student Characteristics*. Having dealt with thousands of mathematically talented students, we can make one fascinating observation: beyond a shared affinity for mathematics, there are no particular characteristics which set UMTYMP students apart from their peers.

In fact, our typical class is a microcosm of any American high school. We have musicians, athletes, and self-professed "computer geeks." Many of our students are introverted and awkward around other people, whereas others are extroverted, charismatic, and revel in being the center of attention. Some UMTYMP students are highly gifted and could be successful in any subject, while others have no particular interest or ability outside of mathematics. Each group has its own set of challenges. The gifted students are often involved in so many advanced courses that putting enough emphasis on UMTYMP to meet the heavy work demands can be an issue. Many of the students who are focused only on mathematics have not distinguished themselves academically, and it can be challenging for them to understand the high quality of writing and organization we expect in their work.

3.4. *Classroom Instruction*. UMTYMP can be a very challenging teaching assignment. In the high school component, for example, we cover a full year's worth of high school mathematics in about 30 hours, including 6 hours of testing. This would be impossible except for the fact that the students can learn routine topics–which comprise a large portion of the curriculum on their own without any formal instruction. Teachers must focus on the central ideas and most

significant types of problems and trust their students to develop computational skills on their own.

The range of students in our courses also has an effect on instructional practices. Sixty to seventy percent of the students in a typical algebra class are seventh graders, with the rest in sixth or eighth grade. Although they have an aptitude for mathematics, there can be a tremendous difference in learning styles and focus; anybody who has taught students at this level knows, for example, that an eighth grade female can be far more emotionally mature than a sixth grade male. Some of our students prefer to learn concepts through self-instruction. The end result is that the teacher does not always have the full attention of every student. Some may be working individually on a problem, or discussing it with their neighbor. Our instructors have to learn to deal with this apparent lack of focus, so long as it does not bother the other students in the classroom. The reader can find a full discussion of the challenges of teaching in UMTYMP in [3].

3.5. *Historical Stability and Changes*. The most notable stable feature of UMTYMP is the continuing high interest of extremely talented students and families to participate in a program of this scope and magnitude. This need has intensified in the current public school environment, in large part because of the recent emphasis on high-stakes standardized tests. Schools have been forced to focus their resources on the mainstream curriculum, with many fewer opportunities for mathematically talented students. As a result, students are drawn to UMTYMP but come with little mathematical exposure beyond routine (if excellent) calculation skills. Overall, the problem solving, study, and communication skills of incoming students are weaker than ten to fifteen years ago.

Student attitudes have also changed significantly in the last decade. Students nowadays are generally much more involved in extracurricular activities, and try to squeeze UMTYMP into a packed lifestyle. Many of the supportive features of our program go unused by students, not for a lack of interest or need, but rather a lack of time. Despite repeated warnings from the program to students and parents about this issue, the unfortunate consequence is that we occasionally lose good students who could succeed with more time and focus.

Although our students are busier, they are increasingly younger and younger. Through the year 2000, the median grade level of our Calculus I students was tenth grade. Now the

majority are in ninth grade, with a large contingent of eighth graders who started the high school UMTYMP program in sixth grade. This shift comes with a corresponding decrease in the overall maturity level of our students. Furthermore, our younger students are simply incapable of sitting still and staying focused for a long lecture. Whether this is a physiological fact or a byproduct of our cultural environment, we have had to adapt. Our class time has morphed from a rather traditional two hour lecture and workshop presentation to a more student-centered environment, with a blend of content presentation and student activities in small groups. The lectures focus on big ideas and central computations, while the workshops have group work specifically designed for UMTYMP students. This format remains effective even with the more mature students in their fourth or fifth year of the program.

One recent difficulty is the wish to include formal proofs and reasoning as part of the conceptual work. There is a dearth of textbooks that can meet our needs: rigorous treatment of a broad number of topics, but readable by a high school student. Overall we spend considerably more time searching for suitable textbooks now than in the past; currently all three years of the calculus program are undergoing a textbook search.

The high school program has some special issues. The time constraints and fast pace are more difficult for students in grades 6-8, and the difference in expectations between UMTYMP and their regular school work is more pronounced. The textbooks available at this level have also suffered more prominently in conceptual material and presentation. The so-called "college algebra" text used today is clearly less challenging than the high school text used twenty years ago. The geometry textbooks are even more problematic; although we strongly believe in group work and constructivist learning, we do not have enough class time for the exploration/conjecture model which is common in today's books. Our current text has evolved so far in this direction that it is now unsuitable for UMTYMP, and we are in the process of switching to a more traditional but well written book. However, this requires that the high-school instructors teach an UMTYMP course which will be significantly different than the courses and their own schools, causing an extra burden for them which was not present in 1995.

Field of Study	UMTYMP Alumni %	National %
Engineering	25.92%	5.80%
Mathematics	18.18%	1.05%
Physical Sciences	15.28%	1.50%
Computer Science	11.61%	2.75%
Biological	7.74%	5.35%

Table 1. Comparison of self-reported UMTYMP alumni degrees to national totals of all earned Bachelor's degrees (using an average of 1996-97 and 2001-02 data from [2].)

4. Outcomes

By their very nature, UMTYMP students are highly intelligent, so it comes as no surprise that many of them go on to be extraordinarily successful in their undergraduate studies and subsequent careers. This makes it difficult to measure the effect UMTYMP has had on our alumni and their success, particularly given the lack of any control group; it is not feasible for us to tell parents, "Your child has qualified for UMTYMP, but we would like to keep her out of the program and track her future progress."

We are initiating a large scale project to contact a thousand or more of our alumni in an effort to evaluate the long term effects UMTYMP has had on their undergraduate studies and subsequent careers.

In the meantime, although the absence of a control group makes it is difficult to make direct measurements of the program's impact on participants, it is possible to use our alumni database to make some broad observations which indicate a deep influence on students. Anecdotal data from surveys and other information from our alumni strongly support these observations.

One measure of the program's influence is which majors and careers are chosen by our alumni. A significantly higher percentage choosing paths in science, technology, engineering and mathematics (STEM) related fields may indicate that UMTYMP motivates and encourages students to use their mathematical talents in their careers. Table 1 compares the self-reported degrees earned by UMTYMP alumni to the national averages of all Bachelor's degrees. Even assuming our students' natural preference for mathematical and scientific subjects, they are earning degrees in STEM fields at a phenomenal rate. Given their aptitude

and the number of math credits earned in UMTYMP, it should come as no surprise that mathematics is the most common degree. However, the percentage is striking: 18 times the national average. Other areas are also impressive: physical sciences are 10 times the national average, engineering 5 times the national average, and computer sciences 4 times the national average. Nearly 39% of our alumni also go on to earn master's degrees, most commonly in mathematics, medicine, computer science and electrical engineering. About 19% of our alumni have earned doctorates in a wide variety of fields; including at least 18 Ph.D.'s in mathematics.

The remaining degrees not included in Table 1 are distributed between the humanities, social sciences, and various technical/professional fields. We do not view it as a programmatic failure when students finish UMTYMP and continue to a non-STEM field, since breadth and scope of education is the cornerstone of the liberal arts philosophy common at colleges and universities throughout the United States. Our anecdotal evidence also indicates that the intellectual demands and conceptually heavy content of UMTYMP encourage students to use these skills in future careers. Alumni pursuing careers in fields as varied as law to music performance state that, although they may never need to compute the value of a flux integral, the work habits, qualitative reasoning and problem solving skills developed in UMTYMP are invaluable to their future careers.

The program has had a profound impact at the University of Minnesota. A number of UMTYMP alumni have received graduate degrees in mathematics at the University of Minnesota, and two more are currently enrolled in the Ph.D. program. Our alumni permeate the department's faculty and staff as well, including: a highly respected Full Professor who was a graduate of the very first UMTYMP class; a member of the advisory committee for our Masters in Financial Mathematics program; and the director of our computer systems administration staff. At the undergraduate level, UMTYMP has been responsible for a large number of very high quality students enrolling at the University of Minnesota. The school has made attractive accommodations with credit, placement and scholarships in an effort to recruit our students. As Table 2 shows, we have been very successful at retaining these students who might have otherwise attended one of the other prestigious schools on the list.

Institution	Current	Historical
1.	University of Minnesota	University of Minnesota
2.	MIT	MIT
3.	Stanford	Stanford
4.	Berkeley	University of Wisconsin
5.	Harvard	Harvard
6.	Northwestern	Berkeley
7.	University of Wisconsin	Caltech
8.	Yale	University of Chicago
9.	Columbia	Northwestern
10.	Caltech	Carleton College
11.	Carleton College	Yale

Table 2. The fifteen most attended undergraduate institutions among our alumni, both current and historical.

5. Assessment and Research Perspectives

A key component in attracting students and their families to participate in UMTYMP is the presentation of current data on student admissions to colleges and universities and subsequent successful careers. This requires maintaining and regularly updating a robust alumni database. Our statistical database was originally created in the late 1980's to study the effect of certain programs aimed at increasing female participation in UMTYMP. It has since been updated to provide extended data about the undergraduate studies and career choices of our alumni.

Maintaining the database requires real effort, but the current and potential usage far outweighs the costs. The database has been an extremely valuable resource for grant proposal data as well as an impressive statistical history of UMTYMP students' achievements, including the data used in this paper. It has also provided evidence to help UMTYMP make effective programmatic decisions to better serve certain subgroups of our student populations. For example, careful analysis of female applicants who passed the high school entrance exam (see Section 3.1) showed that school teachers were doing a poor job of identifying quality female candidates for the program. This led us to develop new approaches to attract and retain female students, changes which have had a lasting effect on UMTYMP.

UMTYMP regularly provides parents and the University with data on alumni degrees, college admissions, schools attended, majors achieved, career directions and related data. In addition, several questionnaires have been collection from alumni concerning the usefulness of UMTYMP coursework in college majors, the role of UMTYMP's conceptual approach in college mathematics and science courses, and other similar questions. We are currently working to improve our data collection procedures and boost our response rate, so we can continue to perform detailed and accurate longitudinal analyses.

Current UMTYMP students are also generally quite willing to be involved in qualitative studies on various issues. The program has informally gathered information on the work and study expectations of UMTYMP compared to their regular school work, on social and scheduling issues to be involved and successful in UMTYMP, and on parental pressure. These informal studies could be made more formal and handled in more traditional ways.

In addition to these passive analyses, UMTYMP provides a relatively self-contained environment for researching pedagogical techniques or other educational issues. The courses, curriculum and examinations themselves allow interesting and longitudinal studies on understanding of important topics in single and multivariable calculus and linear algebra. For example, most exams are (covertly) broken into conceptual and computational components, and the sub-scores provide global pictures of student understanding as well as information to

help specific students improve performance. In a more formal approach, a current study of UMTYMP student understanding and misconceptions about series and sequences has provided some interesting initial results which could influence future instruction. This analysis will be continued for several years, and assess these concepts with the same students as they progress through the calculus program. More studies of this nature in the specific setting of UMTYMP students could be quite useful for other undergraduate issues in mathematics.

6. Issues Concerning Expansion and Duplication

The success of UMTYMP begs the obvious question: why has the program not been duplicated? To our knowledge nobody has ever tried to start a similar full-scale program at another location, although we have had modest success in expanding our own program to other sites throughout the state of Minnesota. This section describes those efforts and describes some of the challenges which would be faced by anybody interested in starting a similar program. Key aspects include long-term individual and institutional commitments along with the acceptance and support of local K-12 educational systems.

6.1. *Expansion within Minnesota*. Because UMTYMP receives crucial financial support from state government funds, UMTYMP has always been expected to make efforts to serve students throughout Minnesota. In the past, portions of UMTYMP have been offered at various "satellite" sites in cities throughout Minnesota such as Rochester, Saint Cloud and Duluth. Yet all of these cities have struggled with maintaining a full program. Several major issues appear to be common to all of these sites.

The demographics and geographic distribution of the population in Minnesota play a key role. About 65% (roughly 3.2 million) of Minnesotans live in the Minneapolis-Saint Paul (Twin Cities) Metropolitan Area. The satellite UMTYMP sites are all regional population centers, ranging from about 70,000 to 180,000 residents, which are surrounded by sparsely populated rural areas. The satellite programs have always begun with large classes – although still an order of magnitude smaller than the Twin Cities site – including some extremely talented students. As times passes, however, they inevitably experience lower and inconsistent enrollments. The smaller local populations force the sites to rely mostly on one major school district to provide the bulk of their students. When that district's interests change or administrative support for such a program wanes, there is a very significant effect on

UMTYMP participation. This issue is avoided in the Twin Cities site, which draws from dozens of large school districts. Decreased interest in one district is usually counterbalanced by increased interest in another.

The other major difficulty is finding high quality instructors who are both capable and willing to teach UMTYMP students. Because of the scope of the program and its curriculum, it requires both a dedicated high school teacher to handle the first component, and a college professor to teach the calculus courses.

While the high school component is important, the program is even more dependent on the availability of quality college faculty. This is clearly reflected at the Rochester site, where a highly technological and well-educated population base[3] ensures parent and student demand for the program; unfortunately, Rochester has no four-year college or university and hence no local mathematics faculty. The site has only been successful when University instructors travel from the Twin Cities to Rochester on a weekly basis to teach the UMTYMP courses. In contrast, St. Cloud and Duluth have large universities with faculty members who were once involved with UMTYMP in the Twin Cities and are enthusiastic about teaching the courses. However, the local economies and school districts in those cities are not producing enough students to sustain the sites indefinitely.

6.2. *Duplication outside of Minnesota.* While there are several outstanding summer programs[4] and a few programs which provide an accelerated academic year program of high school mathematics, UMTYMP is the only program we know of which systematically provides a five year program including honors level college courses. Anybody wishing to start a similar program would face all of the issues involved in expanding UMTYMP within Minnesota, magnified by the lack of the central office providing administrative and curricular support. A complete analysis of all the requirements for a successful program would be beyond the scope of this article, but some key components are:

• A long-term commitment of college mathematics faculty to create and teach the college-level courses. This also requires a commitment from the department chair and school administrators to support and professionally reward faculty for these efforts.

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[3] Rochester is home to both the internationally renowned Mayo Clinic and a major IBM facility.

[4] For example, see the programs supported by the American Mathematical Society's Epsilon Awards.

• College faculty knowledge of K-12 students and school curricula. This is essential in designing high school courses which simultaneously satisfy school criteria and provide suitable preparation for honors calculus courses.

• Prior experiences with K-12 schools and teachers in order to obtain their trust and confidence that the program will help bright students learn more mathematics, and not harm their own schools' mathematics program by "removing" their best students.

• An administrative office to handle complex issues such as qualifying exams, tuition and student fees, student transportation, and communication with students and parents. These issues cannot be effectively handled in an informal way, and can seriously undermine an otherwise intellectually exciting program.

• The enthusiastic support of the students who attend the program and their parents. Attending such a program is a deep and fairly expensive commitment at several levels. Being able to provide accurate and compelling data on the value of this effort is absolutely critical to maintaining the program.

There are a myriad of other issues which need to be addressed to run a successful program beyond the main items above. Yet UMTYMP has demonstrated that all of these issues can be successfully navigated and provide a unique experience for large numbers of mathematically talented students. The personal and academic pleasure of teaching students with these mathematical interests and capabilities is exceptional, with many instructors regarding these classes as highlights in their teaching careers. The sense of satisfaction of seeing these students grow mathematically and move onward to significant careers is comparable to watching one's undergraduate and graduate advisees succeed. These are among the best reasons for urging other mathematicians to become involved in similar programs. Anybody interested in developing a similar program is invited to contact us directly for more information.

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The proficiency challenge: An action research program on teaching of gifted math students in grades 1-9

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Abstract: *The paper describes design and outcome of a 3-year action research program on the teaching mathematics to gifted students in grades 1-9 in mixed ability classes in Denmark 2003-2006. The intention was to combine ideas and experience of many teachers with theories and suggestions of researchers to test and develop useful recommendations for future teaching.*

Key words: Action research; mathematically gifted; proficiency; differentiation.

Introduction

Different ability of students has been an accepted challenge to schools and debate on teaching for years. Recently the discussion in Denmark has been extended to challenging the extent and possible handling of differentiation to gifted students.

2003-06 the Municipal School Authority of Aarhus, Denmark in cooperation with VIA University College of Teacher Education initiated an action research program, where I was the researcher and also acted as the project manager. During this period we developed and tried out ideas on teaching of clever students in mathematics. Experience from this work and a sample of findings made in other countries was a platform to an extension from 5 teachers and 3 schools in the first year to 35 teachers at 13 schools in term 2004-05 and 18 teachers at 8 schools in term 2005-06. Almost all teachers and schools were changed every year.

Aim, target group and a proposed yearly schedule were sent with an invitation for taking part to all 52 primary & lower secondary schools in the municipal area. Almost every school has grades 1-9.

Aim of the program

The aim was to contribute to increased attention on the proficiency challenge in math teaching, and to develop and try out approaches, which first and foremost supports the

mathematically able. The assumption was that this can be done in an ordinary mixed ability classes and show profitable to all students.

The target group was schools with desire to optimize conditions to students with proficient qualifications – and teachers with a proficient background for math teaching (this was not meant to be a course on mathematics).

Yearly schedule and research design

The research-design involved close connection to actual teaching practice. Five mutual meetings during the school year were mainly informative to, from and among the teachers, and combined with my research between the meetings. The meetings thus provided information, collected findings and kept everyone informed on progress.

① August	*Start-up-meeting* with presentation of earlier results, appointments for try-outs and reporting.

The purpose of this first meeting was to ensure a common background to the collegial talks in the group. Second and third year of the project the teachers were shown two short Danish movies on gifted students and heard one of my taped interviews with a gifted student from the former year. A mathematical inlay was about the winning strategy in playing NIM. The outcomes were also these appointments and memos to participating teachers:

1. Prepare information to students and their parents on the developmental work.
2. Make appointment with coordinator, who will supervise 1-2 lectures. The purpose of my visit was to offer a concentrated collegial sparring on the routines or way teachers try to meet the mathematical challenging (gifted) students in their math teaching. Thus I visited all classrooms for at least one 45 minutes each and had a short talk afterwards with the every single math teacher on their strategies to the gifted students in their classes! Beforehand, the teachers were asked to point out the two (or some of the) most gifted students. I also suggested the teachers to be clarified on how to show the intended attention to these students when teaching them.
3. Read the report/book (on results from former year) before coordinator visits at schools.

② October	*Full-swing-meeting* with supplemental ideas and support.

At this meeting I presented the group to an overview on different routines noted during my visits to classes. The project teachers were asked to comment and justify these, e.g.:

Program of work of the lecture (day) on blackboard in class

Mental math routines

Connections to other subjects like P.E. and science

Explanations for only part of class

Teacher: ”I don't expect everybody to do all problems”, hard extra assignments to some

Mutual project with one number-able group among five (following the ideas of Howard Gardner)

Number-stories, focus on oral presentation

Guided discovery using concrete materials

Confidence on students organizing own investigation

IT as an extra possibility for differentiation

As *competitions* might be a suitable challenge to students with extra time and efforts, the teachers were also informed of some national and international possibilities. The Nordic KappAbel competition www.kappabel.comin all Nordic languages takes place every year and is meant for grade 8. The Kangaroo competition: http://www.mathkangaroo.org is not in Danish language, but suited for many more grade levels.

Every teacher was also asked to *prepare an answer* to one of these questions for the next meeting of the group:

1. **Thoughts on goal setting**

   How do you make gifted students aware and conscious on own goals?

2. **Thoughts on student's pre-understanding**

   How best to catch the special qualifications and experience of gifted students in a concrete area (eventually before a certain teaching sequence)?

3. **Thoughts on planning**

   How can the gifted students take part?

   How do I meet the expectations of these students?

   How do these students become co-responsible for planning?

4. **Thoughts on way of organization**

   Experience with gifted students in whole class teaching, group work and individual work?

   When does an organizational form work and when not?

5. **Thoughts on differentiation of teaching**

   What have you been changing and done differently to different students? Tasks, texts-formulations, materials, ...?

   Bring an example of something, you consider very successful and try to explain why?

6. **Thoughts on assessing with the students**

   How do you carry out a (mutual) evaluation, which also gives room to the gifted student?

   Give an example of a good method.

Every teacher was asked to arrange to *visit a colleague* (at another school), *and have a visit by a colleague* (preferably another). Appointments were made at this meeting.

③ December	*Mid-term-meeting* with evaluation so far and communication of new/more ideas.

The meeting was about:

➢ Impressions and considerations after the mutual sparring with colleague teacher

➢ Midway evaluation of the developmental work

➢ Best practice, ideas and strategies for mutual inspiration (every teacher was asked to bring at least one)

➢ Synopsis to the yearly report, year 3 this became a collection of recommended problems

➢ Separation in "writing-groups" – with responsibility for different grade levels

➢ Presentation of my interview-guide and actual appointments on interviews.

My interview-guide for interviews with 2 gifted students in every class was this:

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1. Are you good at mathematics? How do you know?
2. When do you feel, you learn the most in math lessons?
3. Give an example of a task, you find especially fruitful. Why do you find this task so good?
4. Are you working especially well with others in your class? Who for example?
5. How often do you talk with your math teacher about difficult tasks?
6. Do you think your math-teacher is demanding enough of you? Or too much?
7. Do you have a good advice to teachers with talented students in their class?
8. Eventually?

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④ March	*Almost-done-meeting* with mutual orientation and a frame for reporting. The participant's contribution to joint report on experiences and recommendations sent to coordinator for compilation.

I made interviews with 2 gifted students from every class in the project: 10 students in year 1, 69 students in year 2, 36 students year 3. All interviews were transcribed and a copy given to the teacher. At the March meeting I presented patterns and similarities from the student interviews:

> - The students are very different. All are gifted, but remarkably many are also good on quite different fields as sport, music, ...
> - Quite many have (also time consuming) other interests
> - Some students are rather special, but do get along well in classes. In any case nobody was interested in jumping past a grade in school, when I (jokingly) asked for that
> - Some, but far from everybody, are able to "explain" their interest in mathematics. Many consider it caused by parents (counting cars, some parents are even teachers themselves, etc.) and some by other reasons (a certain math teacher, a book-present including a calculator etc.)
> - Almost all gifted students were happy to be challenged more than most students in class! And some are not at all.

Following this presentation of findings we had a round in groups on coordinator findings. Every teacher had transcription of own two student's interview and was asked to select an essential statement (e.g. only 10 lines) from one of them. E.g. some statements about the teachers' handling of the proficiency challenge in mathematics teaching. The excerpts were shown and discussed in groups in order to find recommendations to the teacher or to the school(s).

Report / book

Before the yearly final meeting of the group I wrote the report/book on theoretical findings, contributions and recommendations from teachers, excerpts of student interviews and suggestions for new routines and strategies. Year 1 this was an internal report, year 2 this became a "real" book (more than 100 pages) and in year 3 the report became a problem book. The books were printed with support from the local authority (year 2) and the Ministry of Education (year 3), so they were sent for free to all 1.000 math teachers in the city of Aarhus. The rest can still be bought at printing cost (Mogensen, 2005).

⑤ April	*Final* meeting with publication of concluding report (and eventually a press release).

The final meeting presented some up-to-date resources, which teachers might want to draw on in their teaching, e.g. a digital math encyclopedia (in Danish) and interactive (electronic) blackboards. Speeches were held and everyone had the newly printed report/book.

In the following section I will present more of the overview and findings from this Danish action research project. These are also published 2008 in a report from the European Comenius 2.1 project: *Meeting in Mathematics* (Meeting, 2008).

What does it mean to be gifted?

All teachers in this project had students, they considered especially gifted or especially challenging. But how can a teacher know who they might be?

This decision was left up to the individual teacher. Some teachers based their choice on regular assessment through written tasks or tests. Some teachers had known the students for several years, some had just been appointed to the class. In each case the choice was not made until the action research program was three months underway.

Seen this way, the gifted students numbered two out of a typical total of 25 students in each class, or 8%. However, in intelligence research you will often meet the expression, "students with special qualifications". These students are approximately 2% of the total number by IQ-test, and might very well be among the gifted students mentioned above.

There was a large variation in teachers' perception of gifted students. The following characteristic may be a support for parents and teachers, who are in doubt. The table is provided by the Mensa organization (www.mensa.org). Although the two columns are not alternatives, Mensa members suggest the right column to present characteristics of the 2% most intelligent children.

Gifted student	Student with special qualifications
Is interested	Is extremely inquisitive
Has good ideas	Has wild crazy ideas
Ironical	Sarcastic
Answers questions	Poses questions to the answers
In the top of the class	Ahead of the class
Learns easily	Knows already
Popular among peers	Prefers adults
Remembers well	Makes informed guesses
Accepts information	Adapts information
Likes to go to school	Likes to learn
Fond of structured learning	Gets on with complexity
Has a talent	Has many talents
Becomes happy	Becomes ecstatic
Becomes angry	Becomes furious

Gifted students therefore do not necessarily constitute a homogeneous group, as they would fit in both columns of the table above. But they always challenge the teacher in matters regarding form and content in teaching. The challenge may not be noisy or obtrusive. Some of these students can be silent, pleased by a strong structure or "keeping their heads down", to be almost invisible in the classroom. Others may be seen as clumsy, anti-social or arrogant – and anyhow extremely visible in the classroom.

In any case they are mathematically challenging to the teacher. And one should consider various approaches when meeting these students. Some teachers said: I don't think I have any really gifted students – although I have some who are smart. Perhaps you should see ability or giftedness as a wide spectrum and support the student differently.

Numerous attempts to uncover the competence of students have been made; this is reflected in many publications. The Russian psychologist Krutetskii (Krutetskii, 1976) suggested that mathematically gifted students were good at

- Reasoning quickly

- Generalizing

- Manipulating abstract concepts

- Recognizing and using mathematical structures seen before

- Remembering rules, patterns and solutions seen before

- Finding shortcuts, which means thinking "economical".

Krutetskii (1976) also mentions two significant norms of behavior of gifted students. Firstly working with mathematics does not tire them; they can keep on for hours. Secondly they have an ability to see cross-curricular problems through mathematical eyes.

In 1995 a report was published by the group: "Task Force on the Mathematically Promising" (NCTM, the American National Council of Teachers of Mathematics) prompted by the requirement to increase attention to talented math students in the USA. In the report Sheffield (Sheffield, 1999) describes mathematical promise *as a function of ability, motivation, belief and experience or opportunity. None of these variables are considered to be fixed, but rather are areas that need to be developed, so mathematical success might be maximized for an increasing number of promising students.*

The assumption that abilities **can** be enhanced and developed is supported by knowledge from brain research, where it is understood that experience results in changes in the brain.Together with the NCTM-report, this suggests that motivation should be affected and treated seriously when a school culture makes students keep low profiles to avoid being labeled as *nerds*. Self-confidence and good role models amongst classmates and teachers are decisive for students' attitude to the subject.

Sheffield suggests these characteristics of mathematically gifted students:

- Early and persistent attention, curiosity and good understanding of "quantitative" information.

- Ability to grasp, imagine and generalize patterns and connections.

- Ability of analytic, deductive and inductive reasoning.

- Ability to shift a chain of reasoning as well as the method.

- Ability of easy, flexible and creative handling of mathematical concepts.

- Energy and perseverance in problem solving.

- Ability to transform learning to a new situation.

- Tendency to formulate mathematical problems – not just solving them.

- Ability to organize and ponder information in many ways and sort out irrelevant data.

Please notice that this list does not include the ability of calculating fast and correctly! Of course many of them are capable of doing that – but Sheffield insists this it is neither a necessary nor a sufficient condition for being a mathematically gifted student. A lot of these students are impatient with details and reluctant to use time on computations.

Koshy offers the characteristics below partly based on work with British teachers (Koshy, 2001):
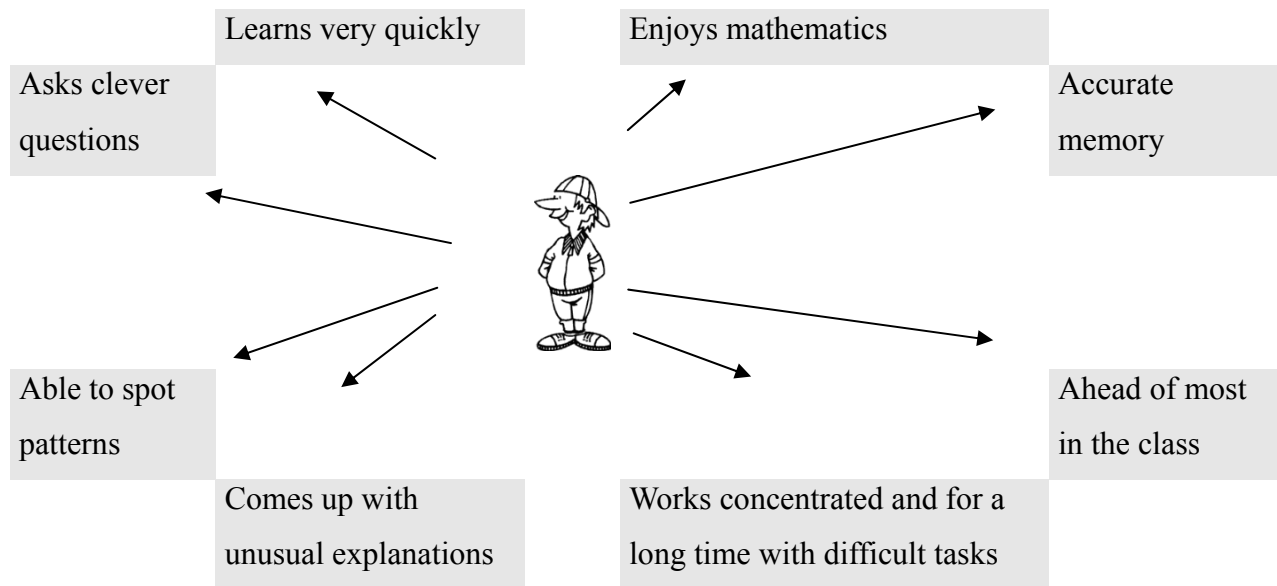
Risk

It is tempting to combine such suggested lists, so as to build a single checklist suited to estimate mathematical potential. However, there is a risk in using such a simplified list for the following reasons:

➤ Gifted students show their special talent only if there are stimulating opportunities for this.
➤ Some students play down their scope of abilities to avoid extra homework.
➤ Some students conceal their abilities in order not to be different – and be bullied.
➤ Multilingual students may have language problems.
➤ Some students have social problems or lack of self-confidence – e.g. no support from home.
➤ Other outside factors may also affect and provide ability, motivation, attitude and opportunities.

Of course teachers spot capable students more easily when there are challenging contexts of teaching and learning, i.e. these students get an opportunity to show their special abilities. This may take place in talks with classmates, elderly students or siblings, parents, teachers or school counselors. Observing how students approach and solve relevant tasks in and out of school may also help teachers to notice gifted students.

Parents' role

Some children show particular abilities before their start in school, and one could imagine a talk about this to take place with parents at the enrolment of kids in school.To make sure it happens, a line with focus on this should be included in the application form.

Parents' ambitions may also result in inquiries to the school about special consideration for their children. On the other hand there may be a total lack of support from home. Some countries are better than others at breaking the social heritage.

The role of parents regarding support and challenge was emphasized in interviews with some of the students and teachers in the action research project. Here is a typical statement by a Danish teacher in the action research project:

"The condition/A prerequisite to go further in teaching and learning than normally requested at a certain grade level is to explain at the first parents meeting how you intend to teach the students:

By keeping a focus on challenge also for the gifted students

By offering all students suitable and challenging opportunities

By assuring parents that nobody will be lost, the scope is to amass successes rather than defeats.

At a parent-teacher meeting, the teacher gives some examples of oral communication in teaching, e.g. the teacher could go through a teaching unit, and give the parents the same sort of tasks, which the teacher later would introduce to their children.

Ask the parents to reply, comment on the answers and tell them what teachers would expect, including creative remarks, add that these are welcome.

Concerning homework (or in periods the lack of same), it is likewise necessary to clarify that it is not volume, but quality that counts. The students must be able to explain their line of thought."

The statement suggests, that the role of the parents should be supportive, not demanding or a transfer of unfulfilled parents' ambitions.

Test

The qualifications or learning outcome of students can partly be assessed by a test. If written tests are used for all students, it is important to remember the limitations. Teachers may ask themselves:

➢ Will the test results tell something new about the individual student?

➢ Does the test contribute to planning of better teaching?

➢ Is the test also suited to the gifted students?

➢ Does the test method enable creative thinking?

➢ Is there a risk of losing surprising solutions or comments?

➢ Does the test fit the grade level and the curricular goals?

A test may be so easy that it either does not provide an optimal challenge or misleads some students to believeit to be more difficult than it actually is. The tests used by Krutetskii were not diagnostic but purely research tests. Each series reveals only one or few aspects and manifestations of the mathematical abilities being studied. And the 72 tests are of four basic categories, where three *"correspond to the three basic steps in solving a mathematical problem (gathering the information needed to solve the problem, processing this information while solving the problem and retaining in one's memory the results and consequences of the solution). The fourth category concerns the investigation of types of mathematical ability. (p.98)"*

This may be a reminder: Any cleverly designed test will map only some aspects of what might characterize mathematical giftedness.

Experience and strengths

How does a teacher use the experience and strengths of gifted students?

To make every teaching effective, you should start from recognizing the backgrounds of the students. But each of the strengths is accompanied by disadvantage, when teaching in a multilevel classroom. The following table makes use of some of the characteristics, Krutetskii, Sheffield and Koshy pointed out. Several tables like this one below appear in various publications (Baltzer, Kyed, Nissen & Voigt, 2006), and the description in the two columns is often found to explain the social challenge of some gifted students:

The strength	The disadvantage
Is curious	Poses questions, that may embarrass others
Thinking critical	Critical and intolerant towards others
Works alone	Seems superior and obstinate
Remembers earlier rules and solutions	Opposes exercises
Does abstract thinking	Rejects details, looks for simple solutions
Has high expectations	Perfectionist
Shows energy and patience in problem solving	Loses interest, when things do not develop as intended
Works goal-oriented	Is impatient with the slowness of others
Generalizes patterns and connections	Does not like routines, will easily be bored
Transfers learning to another situation	Formulates complicated rules and systems
Finds shortcuts	Gets frustrated by inactivity
Thinks "economically"	Interrupts and seems hyperactive

Goal

Are there especially good opportunities to make gifted and motivated students aware of and conscious about setting their own goals?

Yes, we can suppose so. And it may very well be a necessary step in order to meet the particular experience and strengths of these students. Well aware that cultures and settings may differ between schools and countries, I would like to mention that the following viewpoints are based on Danish experience.

When working with very capable students such common goals for a class may be too modest. The gifted student can aim higher than other students in the group. In the Danish action research scheme I interviewed 115 gifted students. Only very few felt too loaded by tasks and expectations from their mathematics teacher, who even had them in focus as especially gifted. On the contrary, to many students it was the other way around, i.e. most were eager to have at least a few more challenging tasks.

So three questions may be asked:

➢ Would it help to make goals more visible and involve the students in matters of organization and evaluation?

➢ How do teacher expectations affect the attitude and work of gifted students?

➢ Should teachers be ambitious on behalf of their students?

I will offer an answer to these questions below.

Planning

Can capable students co-operate in planning their math work? Yes, action research confirmed this. But it implies expectation, initiative and support by the math teacher. Learning a subject such as mathematics is an individual process, taking place in a social context. Co-operation is part of the learning process; in Denmark it is even included as an aim in the subject curriculum:

> *Danish Mathematics Curriculum grades 1-10 (Aims, section 2).*
>
> *Teaching shall be organized so that students build up mathematical knowledge and proficiency on the basis of their own prerequisites. Students shall, independently and together, experience that mathematics is both a tool for problem-solving and a creative subject. The teaching shall give students a vivid insight and further their imagination and curiosity.*

The curriculum is a common condition for all students, and it stipulates sharing responsibility in setting goals and choosing contents.However, the curriculum was not addressed to young students, i.e. it was not formulated in a language well-suited for young students, and it

is a majorchallenge for math teachers to pass it on and interpret the demands for the class. Nevertheless, teachers *ought* to do that.

As is the case in many countries, the Danish curriculum of mathematics is imbuedwith a constructivist view on learning, i.e. Knowledge and insight cannot just be fed from teacher to student, but have to be constructed by each student with the assistance of a teacher and in interplay with classmates. The learning process takes placed in a social setting where students can develop meta-cognitive abilities to monitor and direct their own learning and performance.

This means students share some responsibility in an active learning process. Here it is fundamental to success that the students practice self- and peer-assisted-evaluation. It is possibly the best argument for portfolios as tools of reflection and documentation in school.

It is certainly an important idea for the teacher to invite capable students to *think ahead*; having their own ideas, aiming further than the common goal in class, but still in correspondence with the math curriculum. In younger grades the teacher could encourage capable students to learn each their own tables way ahead of the rest of the class, or "tempt" them by mentioning prime numbers and square root. In lower secondary or middle school, capable students could be prompted to work with reduction or trigonometry at high school level. Teachers could encourage the capable students to go deeper or ahead.

Perhaps math teachers should take regular *developmental talks* with capable students individually or in groups – or might differentiation of goal and plan be handled in whole-class discussion? Many teachers in the action research project were considering advantages and disadvantages of various forms of organization. In every class students are different: they show different interests, intelligence and professional proficiency. Hence, when teacherswant to present the individual student with learning situations, which correspond to the student's background, they need to differentiate the teaching.

There are plenty of ways to differentiate:

1. *Short introduction to new content/tasks*

   You can make an arrangement with the class, setting students to work independently after a common introduction. The capable students are quick to catch the point and may on that account sooner than the rest continue their individually work. Students needing further

assistance can thereafter go through more examples. The capable students work individually or together with the tasks.

This form relies on teachers to discuss teaching organization with their students. One should not emphasize teaching of the able at the expense of weaker students. Through participation in meta-discussions, students will become conscious about learning in various ways, some are quick and pick up matters easily; while others are slow, having to struggle more with the issue at hand.

2. *Grouping by academic criteria*

This is when the capable students are put together in morepermanent groups, where they challenge each other.

In a group of academically capable students you could expect more independent work, but the group should continue to have the attention of the teacher. It must not become a suit-yourself group. When the students are grouped at levels, it is easier for the teacher to pose challenging questions and tasks and give further inspiration to the gifted as well as to the weaker ones. The grouping should be fixed for a period and made by the teacher based on joint decisions by teacher and students, possibly backed up by tests.

When a school has more classes at the same or close-age levels, the grouping could also be done by "setting". This means more teachers can cooperate to find and compose material suited to various levels and thus prepare a more goal-oriented teaching of the various groups.

3. *Amount of content/time*

Let students solve the same tasks at different levels – or differentiate in time. The more capable students can handle more tasks or the same tasks in shorter time. It is crucial that capable students are being challenged and develop a culture, which makes it attractive to get as far as they can. This means, you must have a stock of extra tasks, preferably different tasks. It may also imply that capable students must do more extensive work on tasks, for instance open-ended tasks, solvable at different levels.

4. *Different tasks*

Working within a content area, you may present tasks in various degrees of difficulty, which the student elects/gets handed. Likewise you could differentiate by materials, e.g. let capable

students use a 10-sided "dice" instead of a regular 6-sided one, use other basic arithmetical operations, etc.

Based on my experience and action research, I recommend the following variety of tools to teachers when it comes to differentiation:

Difference in demands	You do not have to be equally tolerant of the quality or the quantity of the individual work of the individual student. You should also be able to: - create interest around a topic - choose/produce good introductions - form teams or groups for collaboration - give the students sufficient time - promote the "mathematical discourse" - create rigorous discipline combined with a pleasant atmosphere.
Difference in time	The time, given to the individual students for one and the same task may differ. It is likewise important to make time to talk with a group or with individual students. On that account: - Fit out the classroom to enable students to be autonomous, e.g. in getting paper, scissors, glue, extra tasks, mathematical games, computer programs, calculators, etc. - Establish structure, e.g. giving your students a sense of propriety. - Arrange to have consecutive math lessons! Eventually this must be a collective decision at school.
Difference in assistance	- Prioritize your use of time for different students. - Make use of students helping each other.
Difference in topics	Give students frequent opportunities to work with different topics depending on need, interest, and inclination.

Difference in way of teaching	Vary your approach, of course adjusted to the different students. I recommend all these forms in a sensible balance: - Exposition by the teacher (of new content or homework). - Discussions between the teacher and the students and among students themselves. - Appropriate practical work. - Consolidation and practice of fundamental skills and routines - Problem solving, including the application of mathematics to everyday situations - Investigations and experiments.
Difference in educational resources	Textbooks are controlling! However, very few teachers will teach completely without textbooks. Apply also: - Supplementary written material. There is a lot: booklets, timetables, statistics, advertisements, news, etc. (Usually such material must undergo a certain adaptation). - Own introductory presentation (eventually with the assistance of colleagues) of activities of limited duration and specific goals or thematic work for longer time. - Student surroundings in a wide sense (TV, sport, preferences, opinions, experiences). - Observations of students and their work. - Calculators and computers are wonderful teaching tools also to increase variation in content and teaching style.

Difference in goals	Taking-off in continuous assessment the students will set for different goals. But the final goal of school and mathematics teaching must be the same to all! You may apply "untraditional" methods to obtain knowledge about the students' outcome of mathematics teaching, e.g.: • grade 6 students can tell all the class (and teacher) about the cost of a hobby • grade 7 students can write a report about quadrangles instead of a ordinary homework • grade 8 students can write in a log book once every other week about their mathematical findings.

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Designing and teaching an elementary school enrichment program: What the students were taught and what I learned

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Abstract: This article is a reflection on the experiences I had designing and teaching an elementary school enrichment program to gifted students in mathematics. In particular, I consider not just what I taught the students in the program but what I learned throughout the entire process. This article first focuses on a description of the program and my role within the program. I then describe in detail four of the lessons I designed and taught for the program. Central to the description of the lessons are my observations of the students' reactions to the lessons and my own growth as the instructor. The article concludes with a reflection on my pedagogic practices, the gifted students in the program, what I learned during the experience and what I learned after the experience.

Key words: mathematics enrichment, gifted students, elementary, constructivism

Introduction

In this article, I discuss my experience as a developer and instructor of a program for mathematically gifted elementary school students, entitled the Mathematics Enrichment Program. This program was intended to provide mathematically gifted students the opportunity to experience mathematics that goes beyond the regular curriculum. I begin with a brief description of the program, the school, and the students involved. I then describe my role in contributing to the design of the program and being the first instructor for the program. I outline four of the lessons I developed and taught for the program as well as some of my observations of the lessons. By providing rich details of the program, I offer information for others interested in developing a similar program. Lastly, this article includes a personal reflection on the development of my own mathematical knowledge and understanding as I worked with the program and afterwards as my own education provided more insight into the experience.

Program Description

The Mathematics Enrichment Program (MEP) took place at Roslyn Elementary School, a public elementary school located near the centre of Montreal. Approximately 530 students attend Roslyn from Kindergarten to Grade 6 (Roslyn School). Roslyn offers both an English stream as well as a French Immersion stream to its students, and is a member of an English school board.

The MEP was first piloted at Roslyn in autumn of 2007. Through a relationship with one of the local universities, Roslyn sought out a graduate student in Mathematics to work as a facilitator and instructor for the program. One of the local universities offers a graduate program in mathematics that focuses on mathematics education. Roslyn sought out a facilitator from this university program in hopes to hire someone with the expertise to teach within the MEP as well as someone who would have the availability part time, as this was not a full time position. I was the graduate student that was hired. During my first visit to the school, I met with the principal and vice principal to discuss the school's goals and intentions for the MEP. The school wanted to offer different and more creative mathematical opportunities, beyond the standard curriculum, for, as the school website states, students who showed "great talent in mathematics", or the mathematically gifted students (Roslyn School). The school decided who was considered to have great talent or was mathematically gifted under their own criteria. Specifically, the criteria for attending the MEP consisted of the classroom teacher's observations and assessment that the student was working two grade levels ahead in mathematics, that the student showed great talent and interest in mathematics, and parental permission. The school anticipated that the MEP, a program that was voluntary for these selected students to attend, would provide an opportunity for students gifted in mathematics to enhance their mathematical talents beyond the curriculum. The school also intended that while these students were attending the MEP, teachers would have the opportunity to focus more time on students in their classrooms who needed extra mathematical support.

It was planned that the MEP would take place during the regular school day. The students who attended were released from their regular classrooms during the time of the

program. The only expectations of these students were that they treat the program as though it was still normal class time and not a release time. The 25 students who attended the program were divided into three groups according to their current grade level: Grade 1 and 2 (five students), Grade 3 and 4 (11 students), and Grade 5 and 6 (nine students). The gender distribution was approximately equal. Each group of students separately attended hour-long lessons, which initially occurred once a week, and later up to twice a week once the program was fully organized. The students were only expected to attend the program and were never given any assignments or homework from the MEP. However, I did place great emphasis on encouraging the students to explore what they had learned from the activities on their own time at home.

There are a few questions raised about some of the above practices. In particular, the question of which students are gifted in mathematics is broached. According to the school, students working a two grade levels above are those who are gifted. Yet, according to research and literature on gifted students, this may be too suggestive a method of identification as those who are mathematically gifted may exhibit other features than just scholastic achievement (Bicknell, 2008; Clark, 2002; House, 1987; Rosario, 2008). Other questions that are brought forth in the literature, as well as in these situations, are: what are the needs of gifted students and how are they to be addressed? According to the school, the gifted students needed mathematical enrichment from a specialist, which was provided through special classes. Unfortunately, I did not collect any data other than my own observations so it is hard to judge the impact the program had on the individuals who took part. More research, potentially long term, is needed in this area if we are to be better able to answer whether educators are addressing the needs of gifted students appropriately.

My Role within the MEP

As aforementioned, I was hired as the first facilitator and instructor for the MEP. At the time, I was hired for two purposes, to work with the school to get the program started by taking care of some organizational aspects, and to develop and teach the lessons and activities for the program. The school officially categorized my position as a Math Enrichment Tutor, but it was mutually understood that I did much more than tutor.

My role within the MEP was also not limited to time spent within the school. The majority of the work I did for the MEP was outside the school, as I developed lessons and activities to meet the goals of the program. Once the program was organized to the point that students could start attending, my role within the school became that of strictly teaching the lessons and informally reporting the program's progress to the school administrators. Below I describe in more detail my roles in the MEP, both outside and inside the school.

Outside the school

Upon accepting the challenge to teach for MEP, I initially started looking for resources that could help me develop lesson plans. In particular I was searching for resources that described lessons or activities that I could use to meet the goals of the program. This proved to be a difficult task. Internet and literature searches provided a variety of interesting mathematical problems or games, but hardly anything that could be used as the basis for an entire lesson. For example, I found a lot of example of interesting mathematical number patterns or games that could be played with a deck of cards but I felt that the goals of the program were beyond this. As well, a number of the resources I located were on topics already covered in the curriculum, which was not what the school had in mind for the MEP. As such, I turned to the resource of my own experience to develop lesson plans.

I reflected on my own experiences in mathematics, from elementary school, where I was pulled out of class to attend a mathematics program for gifted students, to my undergraduate and graduate courses in pure mathematics, to generate some initial ideas. I created a list of the topics that stood out in my mind as having an impact on my own mathematical enrichment and organized this list into topics that could potentially be taught to elementary students. The biggest challenge was adapting topics to work within the constraint of the elementary students lacking extensive knowledge of algebra. This first list demonstrated my personal preference towards topics that a) encourage mathematical thinking that focused on purposes to mathematics, not just processes of mathematics, b) placed mathematics in realistic or geometric context situations, and c) demonstrate different representations of mathematics. Interestingly, my preferences align

with some similar recommendations, among many others, from the literature as areas to focus on to enhance mathematical skills (Davis & Maher, 1993; Freeman, 2003; House, 1987; Maccagnano, 2007; Nunes, 1993).

My preference for encouraging mathematical thinking, which focused on the purposes of mathematics, was evident as I developed lessons that required the students to reflect on their experiences, not just standard non-trivial problem solving processes. I wanted to avoid the teaching of mathematical procedures and instead focus on the purpose of the processes in problem solving. My preference for realistic geometric context situations was clearly an example of drawing on my own strengths in mathematics, as I prefer to treat mathematical problems with geometric models wherever possible. As such, a lot of my lesson plans employed realistic geometric context situations. I also wanted students to explore different representations of mathematical concepts and to establish links between these representations. By developing links between multiple representations, the students could potentially build a base for higher levels of abstraction within mathematics. Lastly, I included different cultural or social representations of mathematics, such as ancient alternative number systems, which became a feature of some of the lesson plans I developed for the MEP. Overall, the lesson plans that I designed were greatly influenced by my own experiences and beliefs about mathematics.

Inside the school

My role inside the school was that of a facilitator and the instructor. In the facilitator role, I ensured that the school was aware when I was coming, when I would be teaching each group, and what supplies I would need. The school provided me an empty classroom with a storeroom for supplies, which was essentially mine during the MEP. As the instructor my primary job was to conduct the lessons. I was very fortunate to be working with smaller groups of students than in most classrooms, which was advantageous as I was able to conduct lessons in a more informal round-table or seminar like scenario. I also provided the students with workbooks/journals to record their work, what they had learned, and make journal entries that reflected what they had learned and what they enjoyed. There has been some research that suggests that gifted students may

need extra emotional and social support from teachers (Clark, 2002). I aimed to address this dimension within my practice by being providing a classroom atmosphere that was very inclusive and positive. I encouraged the students from the beginning to talk about how they felt about the work, and whether they were comfortable with the subject matter and the classroom environment. After the first few weeks of the program I had one young boy ask if he could leave the group. Although he was doing very well with the subject matter, he stated that he was not interested in the program since all of his friends were still in the regular class. This aligns with what some of the literature says about gifted students and their self-concept image (Clark; Davis & Rimm, 1994).

Lessons

In the next section I describe some of the lessons I developed and taught for the MEP. The process of selecting topics for the lessons I developed for the MEP was made from a survey of my own mathematical experience and knowledge. The topics were then simplified to what I felt I could develop into interesting lessons that met the goals of the MEP and that aligned with the students' prior knowledge. Along with a portrayal of the lessons, I provided a brief account of my observations of the students' reactions to each lesson. As will be described, not all of the lessons I planned were responded to in a positive manner, and I speculate as to why this might have been. Although, these lessons were designed with the goals of the MEP in mind, and thus are beyond what the standard curriculum in this region required, I believe they could also be incorporated into a regular classroom setting for mathematical enrichment with some minor adjustments.

Cryptology

The cryptology lesson plan involved a) a description of what cryptology is and where it is used in our daily lives, b) an introduction to the concept of modular arithmetic, c) instructions on the different rules of a shift cipher, d) a demonstration of shift cipher using a Caesar Cipher, and e) an activity where the students encrypted and decrypted messages to each other. With only a few minor adjustments for the age groups, each group received relatively the same lesson. My purpose behind wanting to teach

cryptology was that it could be placed in a realistic context and allowed for an activity using the alternative representation of modular arithmetic.

I started by introducing the uses of cryptology within our daily lives, such as computer passwords, in order to demonstrate to the students a realistic context of mathematics. Teaching the students modular arithmetic took up the majority of the lesson and encompassed most of the mathematical concepts used. First, we discussed twelve-hour and twenty four-hour clocks and what is meant by modular arithmetic. We then moved onto some other modular bases and attempted a few practice samples of simple modular addition and subtraction problems, which were worked on in pairs until I felt comfortable that the students understood the concept. I then led from modular arithmetic into the idea of numbering the letters of the alphabet in order to represent them by numbers and eventually encrypt them. As a group, we numbered the alphabet from 0 to 25 and called this our plaintext code, recognizing that it was mod26. Once we had the basis of our plaintext and an understanding of modular arithmetic, I was able to demonstrate a simple Caesar shift cipher of key = 3, for the students. During the time remaining I encouraged the students to encrypt their own message using a key they had chosen and to switch with a friend and try to decrypt each other's messages.

For all three age groups, I introduced the idea of representing a number by a letter. I consciously refrained from using the word algebra when I introduced the symbols in the encryption formula. I had at first considered leaving blank spaces in the encryption formula. However, during the lesson I spontaneously drew a picture of a key in the formula to represent the number that was the key. The students did not voice any concern with this idea and so in an impromptu manner I wrote a P in the formula for the plaintext and C in the formula for the ciphertext (or the 'code', as we called it), leaving us with the formula $C = P + k \pmod{26}$ (for encryption), where k was the picture of a key. For example, if the key = 12 and the plaintext was 18 the students would have the formula $C = 18 + 12 \pmod{26}$ and assuming they did their modular arithmetic correctly, they would end up with $C = 4$. I do not recall any of the students struggling with the abstraction process of imagining P, C and k as numbers. Alternatively, they were able to rapidly abstract and accept the use of letters and pictures as representing different numbers.

In the months that followed the cryptology lessons, I would constantly get requests to do cryptology again. At Christmas time, we all wrote Christmas cards for our families in shift cipher codes. I heard reports from parents that the students were coming home from school and trying to teach the other members of their families how to encrypt messages. Cryptology turned out to be one of my most successful and talked about lessons.

Symmetry and the Art of Escher

The idea for a lesson on symmetry and the art of Escher came from a university geometry textbook entitled *Experiencing Geometry: Euclidean and Non-Euclidean with History* (Henderson & Taimina, 2005), where the authors of this text outline the seven different types of symmetry of line. The authors described symmetry using a definition of isometry, stating that, "an isometry is a transformation that preserves distance and angle measures" (Henderson & Taimina, p. 15).

For the lesson, I began by asking the students what they knew about symmetry and how they understood symmetry. I provided pictures and asked the students to tell me which were examples of symmetry. Through this discussion we started to agree as a group on what constituted symmetry and what did not. Initially, the students were limiting symmetry to only reflections. But as I offered more pictures and the students discussed the examples as a group, they were able to informally agree on a definition for symmetry that was similar, albeit simplified, to the definition of isometry offered by Henderson and Taimina (2005). In particular, the students agreed that they needed to look at the length and distances between the lines and the angles of the pictures. For the youngest group who had not been introduced formally to angles, we talked about paying attention to the corners of the pictures.

With this agreement on what to look for when searching for symmetry, I then demonstrated for the classes the seven different types of symmetry of a line on the overhead (Henderson & Taimina, 2005), using simple geometric shapes like triangles. Referring to the properties from the definition, we talked about each of the different symmetries, how they held these properties (with the exception of quasi-symmetry), and worked together to brainstorm other examples of these types of symmetry. Lastly, as a

class we went through examples of M.C. Escher's symmetry drawings. With these drawings, I asked the students to explore and identify the different types of symmetry they saw. Initially, I always asked the students to 'prove' to me that they had found some symmetry by showing me that the properties in the definition were there. After requesting this type of explanation a few times, the students started providing it without being asked and 'proving' or justifying solutions became a socio-mathematical norm in the MEP.

This lesson was the first time that I introduced the idea of formal definitions and properties to the students. The students were able to accept quite quickly the need to maintain properties. The few times that I provided contradicting examples to test the students' understanding, I was corrected and referred to the properties in the definition of isometry for clarification.

This lesson also provided me with my first, but not last, experience of being corrected by the students. I had chosen pictures from Escher that were bright and showed clear examples of symmetry to represent what I was introducing. For one picture I had not looked closely enough at all of the details and had decided that it was an example of reflection-symmetry, not half-turn symmetry that it actually was. More than one student noticed my mistake and referred me to the properties in the definition to demonstrate that they were right and I was wrong. This incident brought to my attention the confidence these students held in their own understanding. My experience as an instructor at university was in a different pedagogical setting where the teacher was perceived as 'all-knowing' and students were constantly looking for reassurance. This was never the case with the students in the MEP, which I feel is a reflection of the students' individual mathematic abilities as well as the opportunities that an exploratory mathematics atmosphere offers.

Roman Numeral Arithmetic

My goal when designing this lesson was to introduce the students to a representation of a number system different than the base-ten or Arabic numerals system. In the base-ten system we have ten symbols, 0-9, which can be used to represent any number. In particular, the base-ten system changes in symbolization with each increase of one unit. On the other hand, Roman numerals have symbols representing one and five

and any 10^n multiple of one or five up to n=3. As such, a change in symbols does not occur with each unit increase. I had hoped that the students would gain from this lesson an understanding of how the mathematics we use is socially constructed and how different societies have constructed alternative number system. I also wanted the students to start thinking flexibly about numbers as sums of their parts, which Roman numerals demonstrate quite nicely.

For the lesson, I introduced the Roman numerals to the students by displaying the Roman symbols and the corresponding base-ten numbers they represent. We spent a considerable amount of time talking about the rules for using Roman numerals and how to read Roman numerals. Once the symbols and rules were outlined, I explored briefly with the class some conversions of numbers back and forth from a base-ten system to Roman numerals.

The last activity the class investigated was addition and subtraction arithmetic with the Roman numerals. When the students first encountered the arithmetic problems in Roman numerals, they quickly converted then to base-ten numbers, conducted the arithmetic operation, and then converted the numbers back to Roman numerals. I took the time to point out to the students that the Romans did not convert their numbers to base-ten because they did not have base-ten. At this point, the students started exploring the arithmetic strictly within the Roman numeral system. For the youngest age group, I did not provide them arithmetic problems with sums larger than 20, but for the two older age groups, I utilized the entire range of Roman numeral symbols for the arithmetic problems.

The students quickly responded to the idea of using alternative symbols and rules to create numbers. No student questioned the logic of using Roman numerals. One student even mentioned that it reminded him of cryptology because he was just writing a new code for each number. As a follow up at the end of the lesson, I asked the students how many different types of number systems they thought we could have. After some discussion, the classes agreed that we could make as many number systems for which we could think of symbols and rules. Some students even mentioned that they might try making their own number system. Thus, for these young gifted students in mathematics, the idea of mathematics as being a social construction instead of absolute was a very easy

philosophy for them to accept. This was also an opportunity to introduce the students to other alternative number systems, such as base-two (binary) or base-three systems, which were explored in later sessions for the older two age groups.

Euclidean Straightedge and Compass Constructions

This lesson plan is the example of a lesson that did not work as I anticipated. Using a straightedge and compass, I had hoped to teach the students how to cut a line perfectly in half and how to draw an equilateral triangle, a square, and a hexagon. The goal of this lesson was to encourage the students to look at geometry figures in terms of their properties and particularly, the parts that make up the figures. I tried throughout the lesson to focus on the idea of the radius of the circle being the same distance from every point on the circle. This lesson was only attempted with the Grade 5/6 group, and after 20 minutes of little progress and much noise and confusion from the students, I decided to move onto a different lesson I had planned for the next MEP session. One of the reasons I speculate why this lesson did not work is because not all of the students arrived with a compass. I then suggested that everyone share with a partner and try the construction together. This also did not prove to be successful because as the students tried to share the compasses, they tended to not follow the instructions well.

I cannot predict whether this lesson would have worked if all of the students had brought compasses. It might have been that the topic was too advanced, or that my instructions were inappropriate to be incorporated into their prior knowledge. There could be other causes as well. One thing that the difficulty with this lesson did demonstrate to me is that at the time that I was working with the MEP, the program and myself as an instructor were both still in a developmental stage.

As was also mentioned previously, other lessons were also less than successful in how they were planned. In these situations, I found myself either having to adapt the original plan or in some cases, move onto a different lesson altogether. It was imperative that I be prepared for such circumstances inside the classroom. Since the lesson plans were all of my own design and not previously tested, situations where they needed adjustment or failed altogether were to be expected. Thus, while teaching I was also consciously and constantly evaluating the strengths and weaknesses of the lesson plan and adapting as I went along.

Reflection

My time as the instructor for the MEP lasted only six months, as I finished my graduate degree in Mathematics and moved to a different city. I am currently completing a graduate degree in Mathematics Education and I am able to reflect back on the MEP experience with some new perspectives based on focused studies on education. In particular, I have new theoretical and pedagogical perspectives, which cause me to rethink the teaching approaches I used in the MEP project. I also have a better understanding about the characteristics of gifted students and a familiarity with research on teaching mathematically gifted children of this age.

Reflection on my Teaching

Although I was not formally educated in educational theory at the time this program took place, I now see that there were instances and situations in my teaching that align with a constructivist view of education. According to Goldin (1990), a constructivist mathematics philosophy believes "mathematics [is] *invented* or *constructed* by human beings, rather than as an independent body of 'truths' or an abstract and necessary set of rules" (p. 31, emphasis in original). Some of the topics of my lesson plans aimed to demonstrate the constructed nature of mathematics. For example, in teaching Roman Numerals in comparison to the base ten number system my goal was to make obvious that mathematics has been socially and culturally constructed throughout history. Another example is when I facilitated the students developing, or constructing, a definition for symmetry on their own. The students also did activities like constructing their own ciphering systems. As well, I always encouraged the students to work in pairs or small groups.

Van de Walle and Folk (2007) provide six features that contribute to a constructivist teaching methods of mathematics. These features are a) children construct their own knowledge and understanding; we cannot transmit ideas to passive learners, b) knowledge and understanding are unique for each learner, c) reflective thinking is the single most important ingredient for effective learning, d) the socio-cultural environment

of a mathematical community of learners interacts with and enhances students' development of mathematics ideas, e) models for mathematical ideas help students explore and talk about mathematical ideas, and f) effective teaching is a student-centered activity (Van de Walle & Folk, p. 34). These features in no way make up an exhaustive list of what exactly a constructivist mathematics classroom should include, but they do provide a basis for features to look for.

On reflections, I did manage to include some of the features of a constructivist mathematics classroom in the MEP. For example, from the first class we used math journals to record any work and to reflect on the class, thus encouraging a reflection of the mathematics that was covered. For the youngest age group they might have drawn faces to describe how they felt about the lesson and were encouraged to write a few words about the class. The two older age groups responded to questions such as "what was math enrichment about today?" and "what did I learn?" After the first few weeks the students would start to answer these questions even before I instructed them to do so. I would read through the journal entries as a way to inform myself about their thinking. Further to this, I encouraged open discussions to allow students to listen to their peers and formulate their own understanding. I often felt it difficult to facilitate open class discussions and keep students on track and sometimes fell back to lecturing, but I also recognized that when the open class discussions were successful the level of understanding the students demonstrated was greatly increased.

Although, I now realize that there are many places where I did not honor a constructivist approach. The greatest example being that there were many instances of lecture style teaching where I was trying to transmit ideas to passive learners. In some cases, I did try to encourage some student discovery and always tried to activate the students' prior knowledge, but I was not consistent at this. I believe that my tendency to fall back on a lecture style teaching method was because of my current position at the time teaching introductory university mathematics courses, which were taught in this manner, as well as my own experience of participating in lecture style mathematics classrooms. Thus, I was working from the only example I had ever had.

The Gifted Students

I have also learned more about the characteristics of gifted students and approaches that prove to be beneficial. From numerous literature sources, characteristics of giftedness are described as including curiosity and understanding of qualitative features, thinking logically and symbolically about relationships, the ability to generalize patterns, see relationships, or make connections flexible mental processes, persistence in solving mathematical problems, rapid understanding of mathematical ideas, systematically and accurately working, confident in mathematical or quantitative situations, and creatively approaching problem solving, to name just a few (Applebaum, Freiman, & Leikin, 2008; Bicknell, 2008; House, 1987; Maccagnano, 2007; Pandelieva, 2008; Rosario, 2008).

As I reflect on my experience, I realize that I witnessed the students in the MEP exhibited similar traits. For example, as I mentioned earlier, the students in the MEP held no hesitation in correcting my mistakes, thus demonstrating some of their confidence in mathematics. Similarly, one very interesting observation about my experience in the classroom was that I hardly ever had to repeat instructions to the students. The students understood instructions on the first time or were very quick to work with a partner to ensure they understood the material, thus taking responsibility for their own understanding. I was also able to move through the lessons at a faster pace than I initially anticipated. I believe this is an example of the higher and rapid level of comprehension of the students in the MEP.

It was also the case that a number of times a student would draw conclusions about the mathematics we were working on that also showed a very strong level of comprehension, and an ability to generalize and see relationships. For example, while covering the ideas of modular arithmetic, the class had begun by looking at addition problems so that I could draw on their prior knowledge of clocks and time. While attempting a few addition modular arithmetic problems, one student took the opportunity to announce to the class that she had figured out the subtraction as well. Without being asked she went to the board and demonstrated it for the entire class. She thus exhibited her ability to rapidly comprehend the information and also to extend her understanding to cover alternative mathematical situations.

The Program

I now recognize that there are many resources available that offer suggestions of how programs like the MEP should be developed. For example, the NCTM emphasizes has a list of essential components of programs for the mathematically gifted, which include such features as teacher competence, high-order thinking skills, applications and problem solving, communication skills, encouragement of creativity, and integration of content (House, 1987). Another guide on developing programs for gifted students states that an enriched mathematics program should attempt such activities as using open-ended questions, avoid repeating the regular curriculum, do not grade, and ensuring topics are mathematically significant (Freeman, 2003). By reflecting on how I interpreted the goals of the program, I believe that I was able to attempt the majority of the NCTM essential components as well as Freeman's list of activities. Thus, the program did include a lot of features that the literature suggests it should.

Nonetheless, there are many areas were I can now say I could have improved the program. For example, although I constantly avoided repeated the regular curriculum, I am not sure if I could justify that all the lessons I planned demonstrated the significance of the mathematics involved. I also could have attempted to use more open-ended programs within the lessons. Similarly, offering more examples of where the content could be integrated with other curricular areas could have enhanced the program. I also would change my pedagogical approach to include more features of a constructivist teaching method to hopefully facilitate more creative activities and personal discovery. Overall, if I were to develop a similar program now, I would attempt to include these components.

Conclusions and Suggestions

Since I have left the MEP, other instructors have taken over. I had the opportunity to share some of my knowledge and experiences with the instructor that initially took my place. Other than that, I do not know what knowledge or wisdom has been passed on since I left. I do know that the program continued to run into a second school year and is planned to continue for a third. I also know that the school has expanded their Enrichment Program to also include literature, art, and engineering (Roslyn School).

From my experience, I have some suggestions for those who try a similar program in the future. First, it was difficult to find resources for lessons that matched the goals of the program. Although there is a lot of literature available on gifted mathematics students and alternative mathematics for the classroom, I could some find, but not a lot that could be incorporated into the lessons for the MEP. A lot of the material I found on non-curriculum mathematics was designed for larger lecture style classroom settings. Since I was aiming for more exploration and personal discovery with the MEP, these lessons were not appropriate. Thus, it would be very valuable for enrichment instructors of similar programs to have a place to share and exchange lesson ideas.

It is also important for an instructor to be very familiar with the material (s)he chooses to teach. As I demonstrated by my experience, not all lessons will be successful how they are planned. For an instructor to be able to flexibly adapt to the needs of the group, the instructor must have a deep conceptual understanding of the material. In some cases, it might even be most prudent to move on and perhaps return to a revised version of the lesson at a later date.

I also suggest that instructors only prepare the lessons to a certain point and then adjust and move with the pace of the class. For example, in the Roman numeral lesson, I had initially planned to take the opportunity to show the students how to read different Roman numeral dates that can be seen on the sides of old buildings. This was to help the students recognize a situation where we use Roman numerals. Right at the beginning of the lesson though, when I mentioned we would be doing Roman numerals, one student quickly stated that he knew how to read them already because he sees them on buildings around the city. Thus, I did not feel like I needed to include it in my lesson plan since the students spoke about it as a group without my initiation of the topic. Although,these suggestions could be relevant to any mathematics classroom.

Overall, I feel that the MEP, even in its infancy, was a very positive opportunity for the students who were deemed gifted in mathematics. The program took minimal effort for the school to run. All that was required was for the co-operation of the teachers to allow the MEP students to be pulled out of class and a room for the lessons to take place. The majority of the work was placed on the instructor, but I found it a very rewarding experience and was also compensated for my work. I would encourage other

schools to look into the possibility of providing a similar program for mathematical gifted students.

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An overview of the gifted education portfolio for the John Templeton Foundation

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Abstract: The John Templeton Foundation supported a philanthropic portfolio concerning the development of human genius. The work was contoured to some of the big questions of human activity: the nature/nurture question, the question of how cultures value and institutionalize support of exceptional students, and the 'continuum hypothesis' for gifted education. The first strikes at the heart of what makes us human while the second relates questions about high intelligence to the great social issues.

Key words: genius philanthropy exceptional cognitive ability

Note: Since the preparation of this article, the work of the Templeton Foundation has pursued other directions. This article reports on work completed with support from the Foundation.

The John Templeton Foundation is a large private philanthropic institution with an interest in, among other areas, the development of human genius. This report chronicles the start of a portfolio supporting individuals of exceptional cognitive ability.

This portfolio is assuming a shape contoured to some of the big questions of human activity: the nature/nurture question, the question of how cultures value and institutionalize support of exceptional students, and the 'continuum hypothesis' for gifted education. The first strikes at the heart of what makes us human while the second relates questions about high intelligence to the great social issues.

The 'continuum hypothesis' asserts that whatever constitutes genius, however we define it or choose to measure it, these qualities exists in a continuum throughout the human population. So, for example, Mozart was a genius. There have also been composers of lesser genius, but the difference, according this hypothesis, is quantitative, not qualitative. Likewise there are people who perform Mozart's music with genius,

others who perform it adequately, people who have a deep appreciation of the music although they cannot perform, people who have only a passing appreciation, and so on.

Of course, these questions need further refinement. Such refinements are part of the work of the investigators supported by the Foundation. And the answers to all such questions, of course, will not emerge from a single project, or a single series of projects, or even from a single generation's inquiry. Indeed, the individual investigator, within his or her field, may not see the work as guided by such a question. Often, it is only upon reflection from outside the work that we can put together the investigation of a small area of study with the resolution of a large question about human endeavors.

The following description of the 'genius portfolio' is an attempt to begin this process of reflection.

1. The Institute for Research and Policy on Acceleration (IRPA) at the University of Iowa

This institute continues the work started by the report "A Nation Deceived" (Colangelo, N., Assouline, S.G., Gross, M.U.M., 2004) about acceleration of gifted students, which Jack Templeton has called the 'signature product' of this portfolio of the JTF.

Housed at the Belin/Blank Center for Gifted Education and Talent Development, the Institute studies the implementation of acceleration for gifted students in the public schools, supports students and administrators in creating such programs, and catalyzes graduate and post-graduate research in the field of education and policy.
See: http://www.education.uiowa.edu/belinblank/acceleration/
   http://www.education.uiowa.edu/belinblank/bbc/default.asp

2. Templeton International Fellows at the Wallace Symposium

This two-year grant has catalyzed international engagement in the study of gifted students. Fifty-four fellows, from 40 countries were invited to Iowa to take part in the Wallace Symposium, a biennial gathering of researchers and educators working with gifted students. A special series of seminars was geared towards giving the Templeton

Fellows the tools to pursue research in and support for gifted education in their home countries.

Many countries have the resource of knowledgeable and energetic individuals supporting gifted students yet lack a coherent, institutional support program for their gifted students, informed by a concerted research effort (Gross, 1997). The Templeton Fellows learned about what exists in the US and other nations, and how similar efforts might be implemented in their own countries.

The project has already born fruit. A vibrant e-mail discussion has chronicled the work of Templeton Fellows in 30 of the 50 countries involved in the project.
See: http://itsnt710.iowa.uiowa.edu/fellows/

http://www.education.uiowa.edu/belinblank/events/researchsym/

3. Cogito

This grant to the Center for Talented Youth (CTY), at Johns Hopkins University, supports the development of a website for gifted students. Both a resource and a convener of community, the website serves these students as members, but also a larger population of 'surfers' who may not be included in the community of gifted students, but whose work holds promise (Olszewski-Kubilius, P., & Lee, S.Y., 2004).
See http://www.cogito.org

4. Genetics of high intelligence

A major project on this topic is led by Robert Plomin, a geneticist at Kings College, London, which will involve an international consortium of 12 outstanding geneticists on a series of studies of the genetic component of the phenomenon of high intelligence (Plomin, 1997).

A special issue of the Journal of Behavioral Genetics has been devoted to the work of this group. See http://www.springerlink.com/content/0001-8244.

5. Centers for Mathematical Talent: a developing concept.

In an effort to bring the mathematics research community into the support system for students of high ability, we are working to establish a series of regional centers, each

involving more than one university or research institution, which would coordinate efforts by mathematicians to work in this area.

This project, in its formative stages, may go far to bring coherence to the social institutions supporting intellectually gifted students.

6. Four Policy Studies: Thomas B. Fordham Foundation

The Fordham Foundation, an educational 'think tank', is studying, in four different ways, national and local policies that impact high-ability students:

a) A study of the effects of No Child Left Behind on gifted education;

b) A study of teachers' attitudes towards high-ability students;

c) An investigation into the effects of grouping by ability in the middle school;

d) A study of the Advanced Placement program, and the effects upon it of increased enrolment.

This project, viewed narrowly, is an investigation of government and local policies. But taken in context, it allows insight into how a large and loosely-organized educational structure (the American educational system) has reacted to the presence of students of high ability.

See: http://www.edexcellence.net/template/index.cfm

7. **David Lubinski** is a psychometrician at Vanderbilt University. Together with Camilla Benbow, they have been continuing one aspect of the work of Julian Stanley, a pioneer of gifted education.

This work involves an enormously longitudinal study of cohorts of students identified as being of high mathematical ability, following them through their careers (Lubinski, D., Webb, R.M., Morelock, M.J., & Benbow, C.P., 2001). Identification was through the usual SAT test, but given at ages 10-12. The first cohort is now in their mid-40s, and patterns of achievement are showing up which validate the identification process used in ways that have rarely been duplicated in educational research.

The importance of the work lies both in the validation of this method of identification of talent, and in the information we may glean about patterns of support for

gifted students, throughout their lives. Thus it addresses dead on the relationship between achievement and environment, one aspect of the nature/nurture question.

David Lubinski received the Templeton Award for Positive Psychology in 2000. See: http://www.vanderbilt.edu/Peabody/SMPY/david_lubinski.htm

http://www.exploration.vanderbilt.edu/news/news_wherearetheynow.htm

8**. In October 2007**, the Templeton Foundation sponsored a series of events at Princeton University marking the 100th anniversary of the death of John von Neumann. These included:

a) A panel discussion, *Budapest: the Golden Years- early 20th century mathematics in Budapest and lessons for today*. The panelists included:

Ron Graham:University of California, San Diego. Recipient of the Steele Prize for Lifetime Achievement.

Peter Lax, New York University, Recipient of the Wolf and Abel Prizes.

Laszlo Lovasz, EötvösLoránd University, recipient of the Wolf Prize

Marina von Neumann Whitman, University of Michigan, daughter of John von Neumann

Vera Sos, Alfred Renyi Institute, Hungarian Academy of Sciences

b) A workshop involving mathematicians and educators from the US, Hungary, Africa, and India, exploring ways to harness the power of the Hungarian system to other regions of the world. Some of the ideas generated include:

Making the journal *Komal*, which offers high-level mathematics and physics to high school students, internationally accessible in some form

Expanding the Hungarian summer programs to include international participants (the goal would be to offer the Hungarian programs as models for local programs in Africa and India)

Participation by a team from Senegal (in addition to the team from Benin) at the International Mathematical Olympiad. This project would be co-funded with the government of Senegal or other interested parties.

See: http://www.princeton.edu/piirs/von_neumann_event/

9. Building a presence in Africa

The Foundation is actively seeking new ways to support gifted individuals on the continent of Africa.

a) Dakar workshop on education

This was a workshop co-sponsored with the National Science Foundation, intended to bring together researchers in education from the United States and Africa. The Templeton support was for research on gifted education. The grant was administered by Quality Education for Minorities, in Washington, DC.

This workshop catalyzed several new partnerships, including some of the work described below.

b) The Pan-African Mathematical Olympiad (PAMO)

This program, run by the African Mathematical Union (AMU), is one of

the few serving high-ability students (in any content domain) on the continent. JTF has sponsored visiting scholars to their annual workshop for coaches, and also the attendance of a team from Benin to the International Mathematical Olympiad in 2009.

In addition, JTF sent an international 'committee of visitors' from Quality Education for Minorities to observe the program and suggest strengths and weaknesses. The Committee developed a report on the status of the PAMO and ways its work might be expanded.

c) International Mathematical Union (IMU) report on the status of mathematics in Africa. This project provides the philanthropic and scientific communities with a blueprint for work in this field in Africa
See:
http://www.mathunion.org/fileadmin/IMU/Report/Mathematics_in_Africa _Challenges___Opportunities.pdf

10. International Conference on Culture, Creativity and Mathematics Education in Haifa (Israel).

This conference took place in February 2008, and brought together 30 scholars from Israel, Europe and America, and 10 from predominantly Muslim countries, to discuss the role of culture as both a wellspring and a vehicle for creativity in mathematics.

Aside from the implications for questions about culture and intelligence, we hope this conference will stimulate continued thought and action in the Middle East. This region is now rich in natural resources, which will eventually run out. But human resources, properly developed, will never run out. The Templeton Foundation seeks to support development of the latter, putting the human resources of the region at the service of humanity, as the natural resources are now at its service.

A special issue of the Mediterranean Journal for Research in Mathematics Education is devoted to the proceedings of this conference. A book of essays and a volume of proceedings has also been published. See Leikin (2008) and Leikin (2009).

11. China: With Shing-Tung Yao, a world-famous mathematician, and several Chinese partners, the Templeton Foundation is developing a contest in research mathematics for high school students in China and abroad, on the model of the Westinghouse, Siemans, and Intel programs in the United States.

This nascent program is quickly growing. See http://www.yau-awards.org/introduction.php and http://www.yau-awards.org/overseas/

12. Publication series: To provide materials for gifted students, and to bring research mathematicians into the system, we are working with the American Mathematical Society (AMS) to start a series of publications. This will be a series of translations from foreign sources. Particularly in East Europe, there already exists a rich literature on this level, not available in English. Experience has found that material for this audience, when written well, finds secondary audiences in undergraduates, in teachers, even graduate students of adjacent fields.

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Saul

Prospective teachers' conceptions about teaching mathematically talented students: Comparative examples from Canada and Israel

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Abstract: In this paper we analyze prospective mathematics teachers' conceptions about teaching mathematically talented students. Forty-two Israeli participants learning at mathematics education courses for getting their teaching certificates, and fifty-four Canadian pre-service (K-8) teachers participating in mathematics didactics course were asked to solve a challenging mathematical task. We performed comparative analysis of problem-solving strategies, solution results and participants' success. Based on the discussion with 25 Israeli participants we composed an attitude questionnaire, in which prospective teachers were asked to express their degree of agreement with statements expressing different beliefs about education of mathematically talented students. The questionnaire was presented to 56 Canadian and 28 Israeli prospective elementary and middle school teachers. We describe similarities and differences between the attitudes of the two populations and suggest their possible explanations. Based on the results of this study we make several suggestions for teacher education programs.

Key words: Challenging task, teacher preparation, mathematically promising students

INTRODUCTION

Teacher preparation is a crucial factor in creating opportunities for mathematically promising students to realize their abilities by means of challenging mathematical tasks (Even et al., 2009, Sheffield, 1995). To what extent are teachers ready to work with mathematically promising students when they finish teacher education programs? We conducted an exploratory study in two different cultural contexts: in an Education College in the southern part of Israel and in French-language Canadian University in the south of New Brunswick. We asked prospective mathematics teachers enrolled in mathematics education courses to solve a challenging task and to answer a questionnaire that examined their beliefs about teaching mathematically promising students.

We start with review of literature related to the characteristics and educational needs of mathematically promising and mathematically talented students. We also discuss the role of teachers in the education of such students. We then describe the study structure, the results of the study and finish with some questions that remain open for future investigation.

MATHEMATICALLY PROMISING STUDENTS HAVE SPECIAL NEEDS

The NCTM Standards (2000) stressed that school mathematics has to provide all students, independently of their ability level, with equal opportunities in learning mathematics. Equal opportunities mean matching of the mathematics education to the mathematical potential of learners. NCTM (1995) set up a task force that defined the notion of mathematical promise as a function of four key factors: ability, motivation, belief, and experience. Wertheimer (1999) claimed that taking care of mathematically promising students is an essential educational issue because these students have the potential to become leaders and problem solvers in the future.

Sharing an inclusive view on the education of children with high ability in mathematics, we consider that both mathematically talented students and those that have potential to move beyond standard skills and are highly motivated are part of this group. Therefore, mathematically promising students may possess several characteristics known from the literature on mathematical giftedness such as excellent selective memory and faster progress in their learning (Ponamorev, 1986; Krutetskii, 1976). They also have strong motivation, increased concentration, intuition, originality, stability and flexibility (Goldin, 2009; Yurkevich, 1977; Ponamorev, 1986; Subotnik, Pillmeier & Jarvin, 2009). Krutetskii (1976) pointed at such high abilities in mathematics as formalization, abstraction, finding short solutions, inversion in thinking process and generalization. Mathematically talented students stand out for their ability to work systematically and quickly, getting an insight into the problem's mathematical structure (cf. Heintz, 2005). The ways they solve problems, usually differ from those of regular students (Krutetskii, 1976). Finally, many of these children are prominent in their higher ability to verbalize and explain symbolically their solutions (Freiman, 2006).

Several authors stress that mathematically promising students have to be provided with multiple opportunities that would foster their mathematical understanding,

creativity, curiosity, thoroughness and imagination (Ervynck, 1991; Piirto, 1999; Silver, 1997; Sheffield, 2003) and mathematical tasks for the mathematically promising students should be especially challenging (Applebaum&Leikin, 2007; Sheffield 2003; Freiman, 2006). Based on Polya (1973), Schoenfeld (1985), and Charles & Lester (1982), Leikin (2004) suggested that mathematically challenging task should (a) be motivating; (b) not include readily available procedures; (c) require an attempt; and (d) have several approaches to the solution. "Obviously, these criteria are relative and subjective with respect to a person's problem-solving expertise in a particular field, i.e. the task that is cognitively demanding for one person may be trivial (or vice versa) for another" (Leikin, 2004, p. 209).

Following Brousseau (1997) we acknowledge importance of teachers' role in "devolution of a good task" to any student and claim that this role is critical in creating suitable learning environment for mathematically promising students. In order to create such an environment a teacher should be mathematically educated, be able to assess students' potential and fit mathematical challenge to their abilities and needs. In this context, our exploratory study was aimed at analyzing (a) teachers' strategies when coping with challenging mathematical tasks (b) teachers conceptions about mathematically promising students and their education.

Teachers' knowledge associated with teaching mathematically promising students

Approaches implemented in each particular classroom and the mathematics employed depend on teachers' knowledge and beliefs. Research stresses the importance of teachers' knowledge (Shulman, 1986) and beliefs (Cooney, 2001, Thompson, 1992) for decision making in the process of teaching. Teachers' knowledge and beliefs are interrelated and have a very complex structure (see, for example, Leikin, 2006). In this study our focus is on the types of knowledge characterized by Shulman (1986) as composed of teachers' subject-matter knowledge i.e. knowledge of mathematics, and teachers' pedagogical content knowledge which includes the knowledge of how the students cope with mathematical tasks, and the knowledge of how to create an appropriate learning environment. We also differentiate between beliefs about the nature of mathematics and

beliefs about teaching mathematics with special attention to mathematics and mathematics teaching related to mathematically promising students.

The need of mathematically promising students' in especially challenging tasks may be negatively perceived by their teachers. The negative views depend on their previous experiences and the lack of mathematical and pedagogical readiness to deal with challenging tasks. They sometimes reflect teachers' skepticism about the possibility of increasing mathematical challenge in their classroom (Leikin, 2003). There is a lack of research evidence on how teachers deal with challenging investigative mathematical tasks intended for mathematically talented students and on their readiness to work with these students. Our paper is therefore focused on deepening our knowledge about the two above mentioned components: teachers' capacity to solve challenging tasks and their views on mathematics education of mathematically promising students.

Nowadays, mathematically talented students often study in heterogeneous classes and do not get special treatment, since teachers in these classes lack knowledge and skills to take care of them. Teachers often lack of instructional materials they may use with the students in the heterogeneous environment, and even when they have the appropriate material available, they do not know how to use it. Moreover, teachers are not always aware of the mathematical potential of their students, and consider as promising only those who get high grades and/or behave well. Besides when students do not follow all the prescriptions, choose their own ways of solving problems, perform their tasks quickly and misbehave when bored during the lesson, they are perceived by the teachers mainly as trouble-makers. Additionally, teachers themselves do not always understand students' original solutions and do not know why and how to encourage students' critical and independent thinking and creativity.

Considering specific learning needs of mathematically talented students we stress the special skills and knowledge the teachers need for organization of an appropriate teaching process. Is there a need for special preparation for teachers and if yes, what kind of preparation it should be? Different countries e.g., Australia, USA, Israel, Korea, Japan, Russia and others (Leikin, 2005) have different approaches in this matter.

Education of mathematically promising students, and their teachers in Israel and Canada

Research literature in the field of teacher education (Stigler & Hiebert, 1998) stresses that teaching is a culture-based activity. The authors of this paper have rich intercultural experience in mathematically promising students' education due to their personal histories. All three come from the former Soviet Union educational system, where school education of mathematically talented students was an important element. We studied in mathematical classes or special mathematical schools (e.g. Mathematical School #30- http://www.school30.spb.ru/), and attended mathematical summer camps. During our school years, we met a very special kind of teachers who were usually professional mathematicians who were themselves mathematically gifted and often graduated from similar special programs. Those teachers were very enthusiastic and committed to the concept of special educational programs for the gifted and talented (Evered & Karp, 2000; Freiman & Volkov, 2004; Karp, 2007). The later experience of the authors is based on the realities of Israeli and Canadian education. The present study has been conducted in two different countries, Canada and Israel.

In Canada, each province governs its own educational system. The issue of teaching mathematically talented students is viewed and resolved in different ways. In New Brunswick, there is strong emphasis on inclusive teaching and learning; all children should be involved in all activities. However, as result of recent study of inclusion in schools (MacKay, 2006), the government has started to develop and implement new policies that should better respond to the need of students with special needs. Gifted students are explicitly mentioned as part of this group (GNB, 2007).

Changes are already being made in many schools and some of them begin to take care of mathematically talented students (Freiman, 2008). At the Université de Moncton, prospective teachers work with challenging mathematical problems posted on the CASMI site (www.umoncton.ca/casmi, see the paper of LeBlanc & Freiman in this issue). The site allows prospective teachers to evaluate authentic students' solutions and may be used as resource in their future work. The problem that we use in this study was originally posted on this site and our preliminary analysis of submitted solutions allowed us to construct our investigation with Israeli university students. Working with challenging

tasks on the CASMI site, as well as some other projects we develop future teachers' awareness of the special needs of mathematically promising students. However, more is to be done in order to ensure their better preparation.

In Israel, during the past decade, awareness of the importance of promotion of high ability students has been growing. Education of talented children and adolescents is considered to be "the springboard for the development of democratic society strong in its scientific advancements, industry, high technologies, humanities, and arts". (Rachmel & Leikin, 2009, p. 6). A steering committee of the Division of Gifted Education in the Ministry of Education (Nevo, 2004) devised recommendations for the advancement of education of talented schoolchildren. Educational programs for students who are highly able in mathematics are coordinated by the Ministry of Education or by some non-profit organizations. Israeli Universities are also involved in promoting mathematics education of high ability students. Schools organize special mathematic classes, special mathematic groups (mainly starting in the 7th grade), mathematic circles, and competitions. Additionally, various out-of-school activities are developed for such students. Among those activities are mathematical clubs, Mathematical Olympiads, students' conferences and integration of school students in university courses (more details can be found in Rachmel, 2007; Rachmel & Leikin, 2009).

The Division of Gifted Education of Israeli Ministry of Education encourages teachers to get special education, though there is still a shortage of corresponding programs. In the last six years there were open five special teaching certification programs (in three teacher training colleges and two Universities) and the first M.A. program (in Haifa University) devoted to the education of gifted students. These programs are mainly interdisciplinary and are not focused on specific school subjects.

Mathematically promising students also get special treatment both through the efforts of the Ministry of Education (e.g., Epitomizing and Excellence in Mathematics Program, Zaslavsky & Linchevski, 2007) and those of different non-profit organizations (e.g., Excellence-2000 Association, MOFET Association, and more, e.g. Applebaum, Schneiderman & Leikin, 2006; Schneiderman, Applebaum & Leikin, 2006). Teachers of mathematics working in these programs have to participate in seminars devoted to

enrichment in secondary school mathematics. Unfortunately, there are still not enough courses specifically devoted to education of mathematically talented students.

As presented above the education of mathematically promising students and their teachers differs meaningfully in the two countries and thus we wondered whether the differences in the policy affected prospective teachers' conceptions associated with this issue. That is why, in our study, we ask participants from both countries about their beliefs regarding their own educational needs in preparation as professional teachers able to work with mathematically talented students.

THE STUDY

The purpose and the questions

The main purpose of the study presented in this paper was exploring prospective teachers' conceptions about teaching mathematically promising students. To examine teachers' mathematical knowledge we ask: How do teachers themselves cope with an investigation task intended for mathematically promising students? What problem-solving strategies do they use? In order to explore teachers' pedagogical conceptions associated with teaching mathematically talented students we ask: How do teachers define mathematically talented students? What do they think about mathematical tasks suitable for mathematically talented students? What are their views on the preparation of teachers for mathematically talented students? We compared the responses of participants from Israel and Canada.

Population and procedure

The study was conducted in two stages.

Stage A

Forty two Israeli college students enrolled in mathematics education courses as part of teaching certification program took part in solving a challenging task. Then 25 of these students participated in a follow-up discussion about the task they solved, the needs of mathematically promising students and the knowledge and skills of teachers of the gifted. The preliminary analysis of this stage of the study has been presented at the ICME-11 Congress (Applebaum, Freiman & Leikin, 2008). We performed qualitative analysis of the collected data: categorized problem solving strategies used by the prospective

teachers and performed content analysis of the discussion conducted by the first author of this paper. Categories derived from the analysis of the discussion were used in the attitude questionnaire at the second stage of the research.

Stage B

Fifty-four (New Brunswick) Canadian prospective elementary (K-8) teachers enrolled in mathematics didactics course and 28 Israeli prospective elementary school teachers enrolled in mathematics education course as part of their mathematics teacher training (Grades K-8) were asked to answer the questionnaire. All Canadian participants were asked additionally to solve the task that Israeli teachers had solved at Stage A of the study.

The tools: data collection and data analysis

The problem

The teachers were asked to solve the following problem:

Represent number 666 as a sum of consecutive natural numbers. Find as many different presentations as possible.

In accordance with above discussed theoretical views on needs of mathematically promising students and the role of challenging tasks in their education we proposed this problem to the participants of our study since: (a) this problem has more than one solution, (b) the problem allows different problem-solving strategies[1], (c) it is an inquiry-based problem and a solver can create his/her own strategy; (d) this problem does not demand any extracurricular knowledge. The challenging nature of this task for mathematically promising students was validated by the three authors.

The problem was solved individually by each participant during one 45-minute-long session. The analysis of teachers' problem-solving performance was done qualitatively. All teachers' problem solving strategies were described. We analyzed the effectiveness of

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[1] We differentiate between solution and solution strategy as follows: solution is the result obtained for the problem by the implementation of a solution strategy. The answer for a problem can contain a number of solutions since the problem considered herein is open-ended and has 5 different solutions on the set of natural numbers.

the strategies and the relationship between the strategies and the solutions. To summarize this analysis we quantified the results (presented later in Figure 1).

Discussion with the teachers

We supposed that teachers may have some knowledge about mathematically promising students despite the fact that their program did not include special courses devoted to this issue. The discussion allowed us to learn the participants' ideas about teaching such schoolchildren, their characteristics and needs. Twenty-five teachers participated in this whole group discussion. The discussion was recorded by an assistant. The content analysis of the discussion allowed us to reveal the main categories in teachers' responses. Later teachers' responses were used in the attitude questionnaire to compare beliefs of Israeli and Canadian participants.

Attitude questionnaire

According to research literature and the analysis of the discussion with 25 Israeli participants we composed an attitude questionnaire that allowed teachers to express their level of agreement with different beliefs related to the education of students with high abilities in mathematics. The questionnaire includes 3 main parts:

Part A: Characteristics of students that have high ability in mathematics,

Part B: Characteristics of tasks suitable for these students,

Part C: Preparation of teachers for teaching mathematically talented students.

Each part included statements that reflected Israeli teachers' beliefs expressed during the discussion. For each statement there were six ranks from which the teachers were asked to select the most appropriate level of agreement (form 1 - fully disagree to 6 - fully agree).

The validation of the questionnaire was performed both for the content validity and internal consistency of each questionnaire part. Content validity was examined in the course of the discussion of the three authors of this paper. All the clauses about which there was any kind of disagreement were changed. The reliability (internal consistency) of the questionnaire was checked for each of the three parts using Cronbach's alpha.

The reliability was found to be satisfactory to permit the use of this instrument: $\alpha = .91$ for Part A, $\alpha = .73$ for Part B, and $\alpha = .83$ for Part C of the questionnaire. We analyzed responses of teachers in Israel and Canada and compared them. T-test was applied to analyze whether the means of two groups are statistically different from each other for each part of the questionnaire and for each one of the questionnaire statements.

RESULTS

In the first part of this section, we discuss the strategies used by teachers when solving the problem as well as different resulting solutions. In the second part, we analyze several issues related to mathematically promising students raised during the follow-up discussion. In the third part we analyze the results of the attitude questionnaire.

Solving the problem: Strategies and solutions

Overall the teachers used five different strategies when solving this problem. Figure 1 presents the number of teachers who employed each strategy. As follows from the table, five different solutions were found by teachers and five different strategies were used. The table also shows which strategies led to particular solutions.

In the following section of this paper we provide in-depth analysis of the strategies and solutions. We describe different strategies used by the teachers and discuss the complexity of the solutions according to the level of mathematical knowledge and connectedness required in order to apply the strategy correctly and find as many solutions as possible. Then we analyse differences and similarities between Canadian and Israeli teachers in the use of different strategies.

No of solutions	221+222+223	165+166+167+168	70+71+…+77+78	50+51+…+60+61	1+2+…+35+36	Trial and error Israel	Trial and error Canada	Dividing 666 and surrounding a median number Israel	Dividing 666 and surrounding a median number Canada	Using properties of arithmetic sequence explicitly Israel	Using properties of arithmetic sequence explicitly Canada	Writing equations and using patterns Israel	Writing equations and using patterns Canada	'6' pattern Israel	'6' pattern Canada	Total No Israel	Total No Canada
Total No						16 38%	39 72%	15 36%	14 26%	2 5%	0	5 12%	1 2%	4 10%	0	**42**	**54**
One sol.	V					9	11	1						2		**12 29%**	**13 24%**
					V		2										
Two sol.	V	V				2	8	1	2			1				**9 21%**	**20 37%**
	V			V		1						1					
	V		V				2	3	2								
	V				V		4		2								
Three sol.	V	V	V			4	1	4	1					2		**14 33%**	**8 15%**
	V	V		V			1	1									
	V	V			V		2		2								
	V			V	V		1										
	V		V	V						1							
	V		V		V			2									
Four sol.	V	V		V	V		5		2							**3 7%**	**13 24%**
	V	V	V	V			1			1		2	1				
	V		V	V	V				1								
		V	V	V	V				2								
	V	V	V		V		1										
Five sol.	V	V	V	V	V			3				1				**4 10%**	**0**
No of solutions in which a result is attained by a particular strategy — 221+222+223						16	32	15	17	2		5	1	4		**42**	**50**
165+166+167+168						3	19	12	9	1		5	1	1		**22**	**29**
70+71+…+77+78						4	6	13	5	2		3	1	2		**24**	**12**
50+51+…+60+61						1	9	4	4	1		4	1			**10**	**14**
1+2+…+35+36							23	5				1				**6**	**23**

Figure 1:   Distribution of prospective teachers' solutions according to use of different solution strategies and different results

Trial and error strategy

Trial and error strategy was used by participants most frequently. Fifty-five of ninety-six (16 of 42 from Israel – 38% and 39 of 54 from Canada – 72%) teachers checked different combinations of numbers, some of which matched the problem conditions and some of which did not. Overall by using trial and error strategy solution $666=221+222+223$ was found by 48 teachers, solution $666=165+166+167+168$ was found by 22 teachers, solution $666=70+71+72+\ldots+78$ by 10 teachers, solution $666=50+51+52+\ldots+61$ was found by 10 teachers and solution $666=1+2+3+\ldots+36$ was found by 14 teachers (see Table 1).

All the teachers who used the "trial and error strategy" figured out that the solution can not contain only two addends. Twenty teachers (from both countries) found only 1 solution: $221+222+223=666$. Additionally, there were 2 teachers from Canada who found only 1 solution using this strategy: $1+2+3+\ldots+36=666$. They just did the addition starting with 1 and adding other numbers until they got 666. Seventeen teachers (3 from Israel and 14 from Canada) managed to find 2 solutions with this strategy. Nine teachers (4 from Israel and 5 from Canada found 3 solutions. Seven teachers (all of them from Canada) found 4 solutions.

None of the participants tried to analyse whether their solution includes the complete set of the solutions to the problem. Obviously, some teachers, when using trial and error strategy, could do it in more systematic way than others. Those who succeeded in finding more than one solution manifested higher level of flexibility; however they did not conduct an in-depth investigation of the problem applying more advanced mathematical methods (formulas, theorems, etc) as it was the case in other strategies we discuss below.

Dividing 666 and surrounding a median number

Twenty nine teachers (15 in Israel and 14 in Canada) divided 666 by different factors then putting the addends "symmetrically" and consequently around the quotient. Table 1 shows that almost all (3 of 4) teachers that found all five solutions used "dividing 666" strategy. There were a few solutions obtained with this strategy.

Some teachers divided 666 by 3. They found the following solution: $666 : 3 = 222$, and then they obtained three consecutive numbers by adding and subtracting 1: $222 - 1 = 221, 222 + 1 = 223$, therefore, $221 + 222 + 223 = 666$.

Other teachers divided 666 by 4 and received a non-integer number: 666:4=166.5, so they had to add and subtract 0.5 and 1.5 to obtain four natural addends: $166.5 - 0.5 = 166, 166.5 + 0.5 = 167, 166.5 - 1.5 = 165, 166.5 + 1.5 = 168$, so $165 + 166 + 167 + 168 = 666$.

The teachers that divided 666 by 9 found the following solution: $666 : 9 = 74$, then the sum of 9 addends was: $70+71+72+73+74+75+76+77+78=666$. Teachers that divided 666 to 12 found another solution: $666 : 12 = 55.5$, then the sum of 12 addends is $50+51+52+...+61=666$. Finally, the last solution was: $666 : 36 = 18.5$ leading to the discovery of the sum of first 36 natural numbers $1+2+3+...+36=666$ or $666 : 37 = 18$ and then the sum is $0+1+2+3+...+36=666$, that presents the same solution (if you decide that 0 can also be used).

Clearly, when implementing this strategy, teachers used the fact that consequent natural numbers form an arithmetic sequence. Dividing 666 by a particular number they searched for a median of a sequence which either belonged or did not belong to the sequence. Furthermore they used the property of an average of arithmetic sequence: The median member of an arithmetic sequence is a mean of all its terms. Thus the sum of all terms of an arithmetic sequence is: $x_1 + x_2 + ... + x_n = n \cdot median\{x\}$. This strategy also allows to prove that there exactly 5 solutions to this problem.

A. Thirty-six is the maximal number of addends: Since for 36 terms of the sum the minimal term is 1 then for bigger number of addends a sum must include addends smaller than 1, thus not natural. E.g., $666 : 37 = 18$, this leads to the following sum of consecutive numbers $0+1+2+3+...+36=666$ which includes 0 which is not natural.

B. In order to be able to form a set of natural numbers being symmetrically distributed around a median number; it (the median) must be a natural number or (natural+0.5). The median number of n consecutive numbers, when n is odd, belongs to the sequence (see sums of 3 and 9 terms earlier). All the numbers in the sequence are obtained by adding $\pm 1 \cdot k$ for natural k. The median number of n consecutive numbers, when n is even, does not belong to the sequence. For example $666 : 4 = 166.5$,

then 4 natural numbers around 166.5 are obtained by adding $\pm0.5, \pm1.5$ to 166.5. Similar results we obtain for the sums that include 12 and 36 terms. Since among numbers smaller than 37 only 3, and 9 are odd divisors of 666 and only 4, 12 and 36 are even numbers that divide 666 with reminder 0.5 there are no other solutions for the problem on the set of natural numbers.

Since none of the teachers provided these explanations explicitly we may claim that the teachers applied this strategy by intuitively using number sense and properties of arithmetic sequence.

Interestingly, this strategy was the most frequently used in finding solutions $165+166+167+168=666$ (used by 7 teachers out of 14 who found this solution), 70+71+72+...+78=666 (used by 5 out of 15 teachers), 1+2+3+...+36=666 (used by 4 out of 5 teachers who found this solution).

Using properties of arithmetic sequence explicitly

Two teachers (both from Israel) used formula of the sum of n first terms of arithmetic sequence. This led to the equation in two variables.

$$\left.\begin{array}{l} a_1 = m, \quad m \in N \\ d = 1 \\ N = n, \quad n \in N \\ S_n = 666 \end{array}\right\} \Rightarrow S_n = \frac{(2a_1 + d(n-1))n}{2} \Rightarrow 666 = \frac{(2m+1\cdot(n-1))n}{2} \Rightarrow 1332 = (2m+n-1)n$$

Then this equation was divided into a series of systems of two equations with two variables

$$\begin{cases} 2m+n-1=1332 \\ n=1 \end{cases}, \begin{cases} 2m+n-1=666 \\ n=2 \end{cases}, \begin{cases} 2m+n-1=444 \\ n=3 \end{cases}, ..., \begin{cases} 2m+n-1=1 \\ n=1332 \end{cases}.$$

One these teachers found 3 solutions for $n=3,4$ and 9:

$221+222+223=666$, 50+51+52+...+61=666 and 70+71+72+...+78=666.

The other teacher found 4 solutions: $221+222+223=666$, 50+51+52+...+61=666, 70+71+72+...+78=666 and $165+166+167+168=666$.

Using equations

Six teachers (five from Israel and one from Canada) used this strategy. Two teachers found 2 solutions by solving different equations that represented sums of consecutive numbers.

Three teachers found 4 solutions by constructing 11 equations and solving them:

$$x+(x+1)=666, \quad x+(x+1)+(x+2)=666, ..., x+(x+1)+(x+2)+...+(x+11)=666$$

One teacher found all 5 solutions by solving all the equations:

$$x+(x+1)=666, \quad x+(x+1)+(x+2)=666, ..., x+(x+1)+(x+2)+...+(x+35)=666$$

It seems that this strategy may arise from a routine procedure that students are used to applying in school.

Last number is 6

Four (all from Israel) teachers based their solution strategy on the fact that the last digit of the sum of consecutive numbers has to be 6.

Two teachers saw that $1+2+3=6$, and $(666-6):3=220$. They found only one solution: $221+222+223=666$

Two other teachers found 3 solutions by using sums of 3, 4 and 9 consecutive numbers whose sum ends in 6:

$221,222,223$ – as described above with $5+6+7+8=26$ and $(666-26):4=160$, thus the numbers are $165,166,167,168$.

Finally $0+1+2+3+4+5+6+7+8=36$ and $666-36:9=70$, thus the numbers are $70,71,72,73,74,75,76,77,78$.

Summary

From the analysis of the solutions produced by the participants from two different countries we learn about similarities and differences between these two groups of population.

Both in Israel and in Canada most of the participants used two main strategies: trial and error strategy and "dividing 666" strategy (98% of Canadian participants and 74% of Israeli participants). None of the teachers used more than 1 strategy when solving the problem. Israeli teachers used 5 different strategies while their Canadian colleagues only

3 strategies (see Table 1). Israeli participants also used properties of arithmetic sequence (5%) and using '6' pattern (10%).

More than one solution was found by 71% of the participants in Israel and by 76% of the participants in Canada. We found that 50% of Israeli participants and 39% of Canadian participants found 3, 4 or 5 different solutions. Only 4 (10%) Israeli teachers and none of Canadian participants found all five solutions.

Most of the participants in our study – prospective mathematics teachers in Israel and Canada - did not attempt to find the whole set of solutions for the given problem. This finding is disappointing. We assume that mathematics classroom for the mathematically promising students should be based on mathematics culture that encourages students to find the complete set of solutions for any problem. We consider that programs for mathematics teachers should include tasks of this kind and stress the importance of problems with multiple solution strategies and multiple results in education of mathematically promising students. Proving that a problem does not have an additional answer (besides those found) and examining the problem for additional results through implementation of different strategies should be a part of routine in teachers' courses as well as in school mathematics classrooms especially when dealing with the education of mathematically talented students.

TEACHERS' BELIEFS RELATED TO MATHEMATICALLY PROMISING STUDENTS AND THEIR EDUCATION

This section of the paper presents the results of the discussion with 25 Israeli participants about their conceptions about mathematically promising students and their education. The discussion was organized with focus on three main questions: (1) Who are they, the mathematically promising students and what are their needs? (2) What tasks are necessary to meet those needs, and (3) Do teachers feel ready to work with mathematically promising students in their classroom and what kind of education they need for that?

Who are they,-"Mathematically Promising Students"

When answering this question, the teachers addressed a wide range of characteristics of mathematically promising students. We organized the answers by the following

categories: mathematically promising students have advanced mathematical reasoning, they solve problems differently from other students, they work at a higher pace. Note that practically none of the teachers mentioned personal characteristics of the mathematically promising students such as motivation, beliefs, sensitivity. Only one of them (Tair) said that mathematically promising students are "thirsty for knowledge".

Mathematical reasoning

In this category we included teachers' replies that referred to students' mathematical reasoning. These teachers stressed that mathematically promising students may be characterized by different qualities of mathematical thinking, by advanced level of their logical reasoning and abstraction they perform. Michal and Inbal clearly expressed this opinion.

Michal: *The student who has developed **logical thinking**, **abstract thinking**, enjoys mathematics*

Inbal: ***Reasoning** is a very important component that proves that the student understood this material.*

Problem solving:

Some teachers stressed that mathematically promising students solve problems differently from other students, they find original problem-solving strategies, can solve unconventional problems, and can cope with many different tasks:

Yosuf: *The student who has high thinking skills and can solve problems from real life.*

Inbal: *The student who uses **original strategies** that he did not study at school and can apply them to new material. He can find **connections between different topics** in mathematics.*

Ruti: *The student who can **solve non-standard** and inquiry based problems.*

Hani: *I have 2 students in the 6-th grade who, from my point of view, are very promising, since they **solve all the tasks that I give them**. However, they cannot explain their solutions, but this is not a necessary characteristic of promising students.*

Pace of learning and thinking:

Many teachers believe that mathematically promising students are quicker than others when performing mathematical tasks:

Suad: *The student who can solve problem in **less time** than other students.*

Hani: *The student who understands the material **quickly** and could study with students of higher grade.*

Do Mathematically Promising Students have a different approach to learning Mathematics? If they do, what are those approaches?

When discussing this question, teachers referred to the main needs of the students: "deepening" (of their knowledge) (Yosuf, Hani, Ruti, Michal, and Aved), enrichment (Inbal and Aved) and acceleration (Inbal).

Different types of tasks

When reasoning about teaching approaches suitable for mathematically promising students many teachers focused on special mathematical tasks. Their ideas in this respect related to the abovementioned "deepening" or enrichment approaches.

Yosuf: I always prepare several special tasks aimed at mathematical thinking development for 3 students in my class who always complete their class work before other students.

Inbal: I bring **extra curriculum tasks** and the tasks for "deeper" learning to my class. The tasks can enrich students' comprehension in mathematics.

Hani: Such students usually complain when we are solving problems slowly, so I give them tasks on the **same topic but more complex**.

Ruti: At our school we have Mathematical Laboratory. Mathematically promising students attend it once a week and work there with **inquiry based problems**.

Michal: One of the ways is to ask the mathematically promising student what s/he prefers. What kind of tasks does s/he want to solve? He should be able to

choose ***between inquiry based, open–ended or other types of complex problems.***

Aved: Challenging the mathematically promising students ***with the tasks from various Mathematical Olympiads***. The tasks that contain ***"deepening"*** and ***enrichment***.

Social interactions of different kinds

Some teachers mentioned that mathematically promising students require different learning environments with regard to the social interactions in which they are involved. Yosuf thought that these students should help others which may be useful for themselves. Ayad expressed an opinion that learning in homogeneous classes may better suit the needs of talented students.

Yosuf: I ask the students who have completed their work to help other students. This helps them to organize their own thinking. And I discovered that students usually understand the explanations of the mathematically promising classmates better than mine.

Ayad: Promoting mathematically promising students will become more effective if they study in ***homogenous groups***.

Acceleration

Some teachers think that mathematically promising students should be taught at a higher pace in order to realize their mathematical potential

Inbal: Another way is to ***move the student to a different class where s/he can learn with students of the same mental level.***

Is the task you solved suitable for mathematically promising students?

Despite the complexity of the task the participants appreciated the importance of the incorporation of such tasks in teacher education programs. During the whole group discussion 25 Israeli participants agreed that this problem was challenging and suitable for mathematically promising students. Their arguments were: the problem has different solutions, there is more than one answer, it was inquiry based problem and so on.

Inbal: *The task **does not demand extensive knowle**dge, but rather higher order thinking skills. So this task can be a **challenge for the students** of various grades from the 3rd up to the 12th .*

Michal: *[The task that requires] not only search of solutions but **hypothesizing or developing some theory** may be very challenging for students.*

Inbal: *The **beauty of this question** is that unless you found the correct approach **you never know if you have all the solutions**. So you are in some conflict with yourself.*

Tair: *This task **has different solutions**   unlike almost all the usual tasks in a primary school. More than that, there are different ways for finding these solutions…*

Do teachers need special preparation for teaching "mathematically promising students"?

When this question was discussed, all teachers expressed their disappointment about not having at least one course in their Teachers' Training Program that focuses on special approaches to teaching mathematically promising students and their needs.

Michal: *One of the courses has to cover the topic: "**various needs of students**".*

Ruti: *In teaching mathematics the problem with the mathematically promising students is very complicated. **In addition to a special course, teachers need to get experience**.*

Yosuf: *I feel that during all my years in the Pedagogical College I learned nothing about work with the mathematically promising students.*

Raya: *I think that we **need the course** that will instruct how to choose problems, what kind of problems are preferable for each age, what are the materials and ways for teaching mathematically promising students and so on…*

Tair: *In my opinion, there must be a special separate course during teachers' training that has to touch upon the problems we have talked about.*

Based on opinions expressed by this group of participants, we developed an attitude questionnaire presented below.

Attitude questionnaire

As we described in the methodological section, the attitude questionnaire was based on the analysis of the beliefs expressed by the Israeli prospective mathematics teachers that participated in the whole group discussion at stage A of our study. At stage B, the questionnaire was given to one group of Israeli prospective teachers (N=28) and two groups of Canadian prospective teachers (N=56). As presented in the methodology section the three parts of the questionnaire were composed by combining teachers' statements during the discussion and the beliefs described in the literature

There were three parts in the questionnaire:

Part A: Characteristics of mathematically promising students,

Part B: Types of mathematical tasks suitable for advancement of high ability students,

Part C: Education of mathematics teachers for teaching talented students.

As presented in the methodology section, all three parts of the questionnaire had high internal consistency that allowed quantitative analysis of the data. We compared the responses provided by the Israeli participants to the responses of Canadian participants (using T-test). Figures 2A, 2B and 2C show the results of the analysis of the three parts of the attitude questionnaire and compares results received for Israeli and Canadian participants. Figures 3A, 3B and 3C present percentage of the teachers who agreed or strongly agreed with the statements in the questionnaire.

Characteristics of mathematically promising students

In general Israeli teachers' agreement with statements about special characteristics of mathematically talented students was stronger than that of Canadian participants (Figure 2A). The average score for Part A of the questionnaire in the Canadian group of teachers was neutral (M=3.89 with SD=0.85; between 3 – slightly disagree and 4 – slightly agree), whereas for Israeli teachers it was positive (M=4.56 with SD=0.46; between 4 – slightly agree and 5 – agree). Both in Israel and in Canada none of the items in Part A of the questionnaire received a score higher than 5. Mean scores between 4 (slightly agree) and 5 (agree) were found for 11 of 13 statements (all except 1.4 and 1.7 in Fig 2A) for the

Israeli group of teachers and for 5 of 13 statements (1.2, 1.3, 1.6, 1.8, 1.9 in Fig 2A) for the Canadian group of participants.

The three highest mean scores for Canadian group of prospective teachers were obtained for the following categories: 1.9 – Mathematically talented students can solve new problems – those that were not solved in the classroom previously (M=4.376 SD=1.03); 1.6 – Mathematically talented students enjoy solving mathematical problems (M=4.30, SD=1.10), 1.3 – Mathematically talented students can understand more abstract mathematics than usual students (M=4.16, SD = 1.21). The Israeli group of teachers chose as most correct the following statements: 1.3 – Mathematically talented students can understand more abstract mathematics than usual students (M=4.96, SD=0.88), 1.9 – Mathematically talented students can solve new problems – those that were not solved in the classroom previously (M= 4.93, SD=0.72), 1.1 – Mathematically talented students solve mathematical tasks quicker than other students (M=4.89, SD = 0.96).

In spite of the fact that both Canadian and Israeli teachers chose statements 1.3 and 1.9 among the three most acceptable there were significant differences in their responses. As mentioned earlier, Israeli teachers provided higher agreement scores to almost all the statements in the questionnaire. Thus the highest mean score in the Canadian group (M=4.37; SD=1.03) is smaller than ninth mean score in the Israeli group (M=4.57, SD=0.69). This observation possibly explains significant differences that we found between the attitudes of Israeli and Canadian teachers to 10 of 13 items in Part A of the questionnaire.

The most significant differences between the attitudes of the two groups are observed for statements 1.1 (Mathematically talented students solve mathematical tasks quicker than other students) and 1.5 (Mathematically talented students participate in mathematics lessons more enthusiastically than other students). Whereas Israeli teachers agreed with these statements ($M_{1.1}$=4.89, $SD_{1.1}$=0.96; $M_{1.5}$=4.79, $SD_{1.5}$=0.88), Canadian teachers' mean agreement score for the same statements was lower than "slightly agree" ($M_{1.1}$=3.72, $SD_{1.1}$=1.62; $M_{1.5}$=3.82, $SD_{1.5}$=1.01). Interestingly these differences relate to cognitive (1.1) and affective (1.5) characteristics of mathematically talented students.

The characteristics that got the lowest agreement score in both countries were 1.4 (Mathematically talented students like helping other students), 1.7 (Mathematically

	M (SD)		t
	Canada N=56	Israel N=28	
Part 1: Characteristics of mathematically talented students	**3.89** **(0.85)**	**4.56** **(0.46)**	**4.69*****
1.1 Mathematically talented students solve mathematical tasks quicker than other students	3.72 (1.62)	4.89 (0.96)	4.19***
1.2 Mathematically talented students prefer learning with students who are good in mathematics	4.02 (1.21)	4.32 (1.31)	1.04
1.3 Mathematically talented students can understand more abstract mathematics than usual students	4.16 (1.21)	4.96 (0.88)	3.51**
1.4 Mathematically talented students like helping other students	3.63 (1.04)	3.96 (1.35)	1.17
1.5 Mathematically talented students participate in mathematics lessons more enthusiastically than other students	3.82 (1.10)	4.79 (0.88)	4.36***
1.6 Mathematically talented students enjoy solving mathematical problems	4.30 (1.10)	4.75 (0.80)	2.15*
1.7 Mathematically talented students like to work in small groups with students of different levels of knowledge in mathematics	3.38 (0.97)	3.75 (1.14)	1.46
1.8 Mathematically talented students can solve problems in original ways	4.12 (1.14)	4.79 (0.88)	2.97**
1.9 Mathematically talented students can solve new problems – those that were not solved in the classroom previously	4.37 (1.03)	4.93 (0.72)	2.92**
1.10 Mathematically talented students know many facts in mathematics	3.89 (1.06)	4.57 (0.69)	3.53**
1.11 Mathematically talented students remember any mathematical statement they ever learned	3.53 (1.04)	4.29 (0.85)	3.58**
1.12 Mathematically talented students ask many questions unpredicted by the teacher	3.91 (0.98)	4.57 (1.14)	2.63*
1.13 Mathematically talented students like to participate in mathematical competitions	3.95 (1.14)	4.68 (0.91)	3.2**
*p<0.05; **p<0.01; ***p<0.001			

Figure 2A: Attitudes towards characteristics of mathematically talented students.
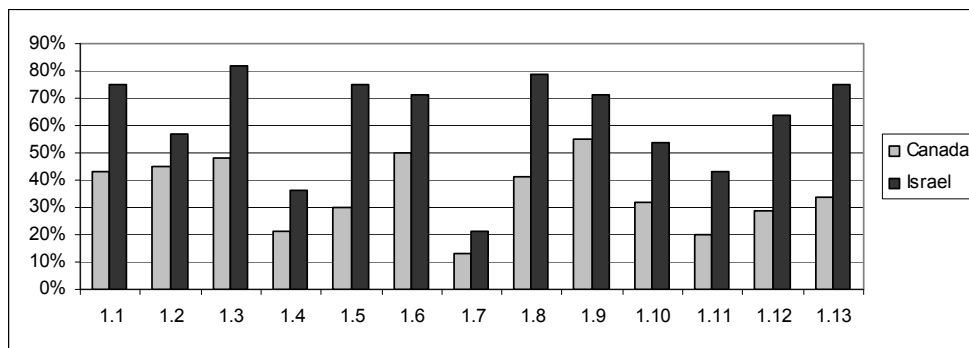

Figure 3A: Percentage of strongly positive attitudes (6–completely agree, 5–agree) to characteristics of mathematically talented students

talented students like to work in small groups with students of different levels of knowledge in mathematics), 1.11 (Mathematically talented students remember any mathematical statement the students ever learned). In both groups mean agreement scores for these characteristics were close to the middle of the scale.

Statements 1.4 and 1.7 belong to the group of 3 of 13 statements in Part A for which no significant difference between the responses of Israeli and Canadian participants was found. Additional statement on which no significant difference was obtained is statement 1.2 – Mathematically talented students prefer learning with students who are good in mathematics. All three statements belong to the group of social characteristics of mathematically talented students and prove that teachers characterize them as learners who prefer working with students of the same ability level (if at all).

Additional evidence of the significance of differences between the attitudes of Canadian and Israeli teachers can be seen in Figure 3a that shows percentage of the participants in each country who marked middle and high level of agreement for different statements. Figure 3a demonstrates that whereas more than 50% of Israeli participants agreed or strongly agreed with 10 of 13 the statements in Part A of the questionnaire (except 1.4, 1.7 and 1.11), less than 50% of Canadian participants chose these levels of agreement for 12 of 13 statements.

Tasks suitable for the mathematically talented students

In this section we present comparative analysis of the attitudes of Israeli and Canadian prospective mathematics teachers demonstrated in Part B of the questionnaire.

Part B of the questionnaire reveals additional differences between beliefs of Canadian and Israeli teachers about mathematical tasks suitable for mathematically talented students. The level of agreement with the statements in Part B of the questionnaire expressed by Israeli participants is higher than that expressed by Canadian participants (Figure 2B) for five of six statements. The average score for Part B both groups was between "slightly agree" and "agree" levels (For Canadian group: M=4.10, SD=0.69; for Israeli group: M=4.61, SD=0.58).

Unlike Part A of the questionnaire where in both groups no statement received a score higher than 5, in Part B in the Israeli group the mean score of at least 5 was obtained for 2 of 6 statements: 2.5 – Problem form mathematical Olympiads are suitable for

mathematically talented students (M=5.00, SD=0.77) and 2.6 – New problems – those that were not solved in the classroom previously – are suitable for mathematically talented students (M=5.25, SD=0.80). We found significant differences between the attitudes of the two groups to these two statements. The most serious difference related to suitability of Olympiad problems for students with high abilities in mathematics.

		M (SD)		t
		Canada N=56	Israel N=28	
Part 2:	**Mathematical tasks suitable for mathematically talented students**	**4.10 (0.69)**	**4.61 (0.58)**	**3.52****
2.1	Difficult problems that regular students cannot solve are suitable for mathematically talented students	3.73 (1.04)	4.11 (1.23)	1.39
2.2	Problems from extra-curricular topics are suitable for mathematically talented students	4.32 (0.99)	4.18 (1.12)	-0.57
2.3	Regular problems that all students solve are suitable for mathematically talented students	3.80 (1.20)	4.29 (1.05)	1.90
2.4	Investigation problems that require discovery of new facts and their proof or refutation as suitable for mathematically talented students	4.38 (0.95)	4.82 (1.02)	1.90
2.5	Problems from mathematical Olympiads are suitable for mathematically talented students	3.86 (1.07)	5.00 (0.77)	5.61***
2.6	New problems -– those that were not solved in the classroom previously -– are suitable for mathematically talented students	4.53 (0.94)	5.25 (0.80)	3.67**
*p<0.05; **p<0.01; ***p<0.001				

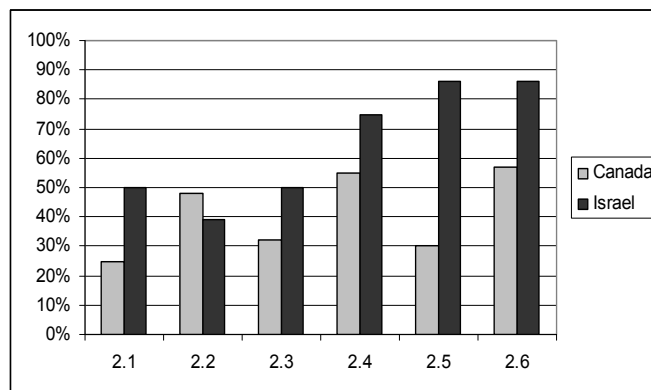Figure 2B: Attitudes towards the types of tasks suitable for mathematically talented students

Figure 3B: Percentage of strongly positive attitudes (6–completely agree, 5–agree) towards the types of tasks suitable for mathematically talented students

Both Israeli and Canadian participants chose "New problems – those that were not solved in the classroom previously" as most suitable for the education of mathematically talented students, though these scores were significantly different (see Figure 2B).

The mean scores showing attitudes of Israeli teachers to four remaining statements in Part B of the questionnaire were all between 4 (slightly agree) and 5 (agree). This way Israeli teachers demonstrated positive attitudes to all the statements included in this part. In contrast attitudes of the teachers in Canadian group were neutral, from 3 (slightly disagree) to 4 (slightly agree) for 3 of 6 statements. The two highest mean scores for Canadian teachers were obtained for statements: 2.4 – Investigation problems that require discovery of new facts and their proof or refutation are suitable for mathematically talented students (M=4.38 SD=0.95) and 2.6 – New problems – those that were not solved in the classroom previously – are suitable for mathematically talented students (M=4.53, SD=0.94).

Statement 2.2 – Problems from extra–curricular topics are suitable for mathematically talented students – was scored by Canadian teachers (M=4.32, SD=0.99) slightly higher than by Israeli teachers (M=4.18, SD=1.12), though the difference was not significant. Statement 2.1 – Difficult problems that regular students cannot solve are suitable for mathematically talented students received the lowest agreement score in both countries (see Figure 2B).

Additional evidence of the differences between the attitudes of Canadian and Israeli teachers towards tasks suitable for mathematically talented students can be seen in Figure 3.B that shows percentage of the participants in each country who marked neutral and high level of agreement for different statements. Figure 3b demonstrates that *more than* 70% of Israeli participants agreed or strongly agreed with 3 of 6 the statements in Part B of the questionnaire (2.4, 2.5 and 2.6), and additionally about 50% of participants agreed or strongly agreed with 2 other statements (2.1 and 2.3). More *than* 50% of Canadian participants chose these levels of agreement only for 2 of these 3 statements. Statement 2.5 received an even lower level of agreement – only 30%.

Education of teachers of mathematically promising students

	M (SD)		t
	Canada N=56	Israel N=28	
Part 3: Education of mathematics teachers for work with talented students	**4.01 (0.90)**	**4.85 (0.57)**	**5.23*****
3.1 To teach talented students teachers have to learn more mathematics than other teachers	3.50 (1.21)	4.79 (1.03)	5.08***
3.2 To teach talented students teachers have to study special classroom settings	4.20 (1.29)	5.25 (0.65)	5.00***
3.3 To teach talented students teachers have to learn ways for identification of high abilities students	4.41 (1.17)	5.25 (0.65)	4.23***
3.4 To teach talented students teachers have to learn how to solve investigation problems	4.50 (1.18)	5.54 (0.51)	5.63***
3.5 To teach talented students teachers have to know their special psychological characteristics	4.70 (1.11)	4.64 (1.42)	-0.19
3.6 To teach talented students teachers have to be gifted in mathematics	2.73 (1.24)	3.64 (1.34)	3.00**
*p<0.05; **p<0.01; ***p<0.001			

Figure 2C: Attitudes towards characteristics of the education of teachers of mathematically talented students.
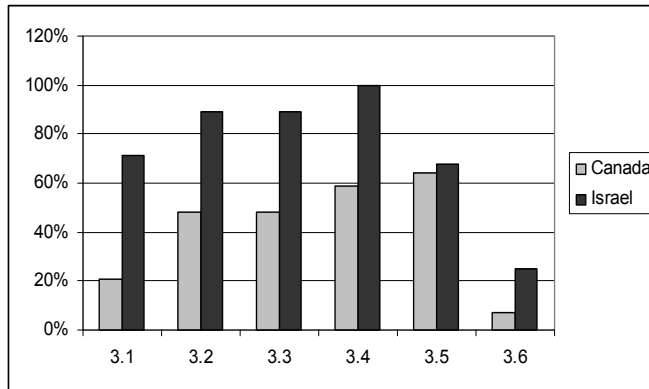

Figure 3C: Percentage of strongly positive attitudes (6–completely agree, 5–agree) towards the education of teachers of mathematically talented students.

Results from Part C of the questionnaire reveal highly significant differences between Canadian and Israeli teachers' attitudes. Similar to parts A and B, the level of agreement of Israeli participants with the statements in Part C is significantly higher than that of

Canadian participants (Figure 2C). The average score for Part C in the Canadian group of teachers was slightly positive (M=4.01, SD=0.90), whereas for Israeli teachers it was close to the middle of the scale (M=4.85, SD=0.51). Similar to Parts A and B, mean scores for all the statements in Part C for Canadian participants were below 5 (agree). Looking at the attitudes of Israeli participants, we learn that 3 of 6 scores were above 5 (agree), whereas those of Canadian group were between 4 (slightly agree) and 5 (agree) for 4 of 6 statements.

The two highest mean scores for Canadian group of students were obtained for the following categories: 3.5 To teach talented students teachers have to know their special psychological characteristics (M=4.70, SD=1.11) and 3.4 To teach talented students teachers have to learn how to solve investigation problems (M=4.50, SD=1.18).

The teachers from Israeli group demonstrated the highest agreement (between "agree" and "completely agree" levels) with the following three statements: 3.4 Teachers have to learn how to solve investigation problems (M=5.54, SD=0.51), 3.2 Teachers have to study special classroom settings (M=5.25, SD=0.65), and 3.3 Teachers have to learn ways for identification of high abilities (M=5.25, SD=0.65).

The statement that got the lowest score in both countries was statement 3.6 To teach talented students teachers have to be gifted in mathematics (see Figure 2C). The mean agreement score of Israeli teachers was almost neutral (M= 3.64; CD=1.34), and Canadian teachers' attitudes were even negative (M=2.73, CD=1.24). From this observation, it becomes clear that our participants do not think that being gifted is a necessary condition for teachers working with gifted students.

Significant differences between the attitudes of Canadian and Israeli teachers can be seen in Figure 3C that shows percentage of the participants in each country who marked middle and high level of agreement for different statements. Figure 3C demonstrates, for example, that whereas 100% of Israeli participants agreed or strongly agreed that teachers need to learn how to solve investigation problems, only 60% of their Canadian colleagues seem to share this point of view at the same level of agreement.

The most striking difference can be observed in the statement affirming that talented students' teachers have to learn more mathematics than other teachers (more than 70% of Israeli teachers vs. 20% of Canadian peers agreed or strongly agreed with this

statement). Similar percentages of participants from both countries seem to agree equally only with the statement that teachers need to learn more about psychological characteristics of gifted and talented students.

Additional comparison

To finish this report we provide additional information in Figure 4 that demonstrates the percentage of teachers who express positive attitude (at slightly agree, agree, and strongly agree levels) to the statements in the questionnaire.

Figure 4 demonstrates that Israeli participants were more positive in all the three parts of the questionnaire. More than 80% of Israeli teachers agree (at different levels) with 11 of 13 statements in Part A of the questionnaire and with 5 of 6 statements in Parts B and C. Among Canadian participants less than 80% agreed with 10 of 13 statements in Part A, and with 5 of 6 statements in Parts B and C.

Figure 4 demonstrates that both Israeli and Canadian participants were the most positive with respect to statements 1.6, 1.9, 2.4, 2.6, 3.4, and 3.5. At the same time the difference in the attitudes of Israeli and Canadian participants is very clear in case of statements 1.1, 1.11, 2.3, 2.5, 3.1, 3.2 and 3.6. The least popular statements among the participants from both countries were 1.7, and 3.6.
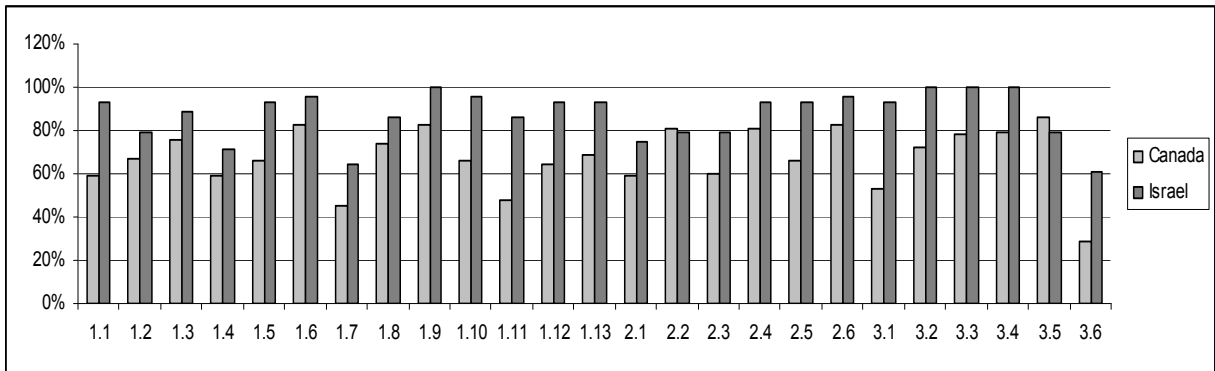

Figure 4: Percentage of positive attitudes of teachers to the beliefs statements in the questionnaire.

In conclusion, dissimilarities in the views of the representatives of two countries speak of interesting and meaningful differences in the education of prospective mathematics teachers as related to the issue of mathematically talented students. These differences were reflected in the participants' attitudes revealed in our questionnaire. We suppose that

a detailed qualitative investigation of teacher training in different countries can explain many of the findings of this study.

CONCLUDING REMARKS

The results of our data analysis, can lead to several conclusions about teaching mathematically promising students.

First, our findings show that teachers cope with challenging tasks with varying levels of success. The majority of teachers used 'non-systematic' strategies, without analysis of the efficiency of the strategies. Indeed, these results suggest that teachers need better mathematical preparation in terms of solving open-ended challenging tasks that would enable them not to limit the problem solving process with finding of one suitable solution. On the contrary, teachers should be encouraged to perform in-depth investigation, assess of strategies' efficiency, search for different ways to solve problems, and for possible generalizations in terms of developing mathematical theories. Acquiring such cognitive and meta-cognitive skills will help teachers in guiding their students on the way to deeper and more meaningful mathematical knowledge.

Comparing solutions and strategies of Israeli and Canadian participants we were not able to draw far going conclusions. However, we can state that Israeli teachers used both non-systematic strategies and systematic ones (that they have previously learned in a different context), whereas most Canadian prospective teachers used only non-systematic strategies. Comparative analysis of school mathematical curricula and of the teacher educational programs in the two countries may shed more light on the findings of this study. We assume that continuation of the study that will employ different types of challenging tasks (Applebaum & Leikin, 2007) also can contribute to our understanding of the discovered phenomena.

Our findings from the discussion with Israeli participants suggest that they were aware of the qualities of mathematically promising students in mathematics classrooms. While the list of characteristics of these students given collectively lacks some important features, teachers recognize special learning needs of mathematically promising students and value investigation and challenging tasks as important for the mathematical development of these students. Namely, the task they were asked to deal with was

characterized by teachers as potentially rich in terms of higher order thinking, theory building, and leading to the development of appropriate strategies.

According to the teachers participating in the discussion, special needs of mathematically promising students can be met with particularly challenging, open-ended and investigative tasks of higher difficulty level and increasing complexity. However, the teachers saw such tasks as rather exceptional for today's mathematics classroom and rarely used by teachers This confirms the need for a more challenging curriculum for mathematically promising students already mentioned by several researchers working with the mathematically promising students (e.g., Sheffield, 2003, Freiman, 2006). In spite of the afore-mentioned opinion expressed by the teachers in the discussion they themselves did not feel prepared for dealing with such tasks in their classroom. Their feeling was consistent with the data obtained in the first part of our analysis that shows that only few teachers were able to find (almost) all solutions to the problem. Their mathematical background should be, therefore, reinforced by mathematically challenging tasks and investigations.

Regarding the social aspects of teaching mathematically promising students, the teachers' opinions vary meaningfully. Some teachers speak about the benefits of homogeneous learning environment, while others consider that mathematically promising students will benefit more while helping less capable students in heterogeneous classes.

The results of the questionnaire analysis can deepen our knowledge of beliefs of prospective teachers regarding the definition of mathematically promising students, their particular educational needs and teachers' readiness to meet those needs in the process of education.

Why did Israeli prospective teachers agree with questionnaire statements (23 of 25) more than their Canadian colleagues? One plausible explanation can be that the statements were built according to the results of a discussion in which only Israeli teachers took part. Discussion with Canadian prospective teachers could reveal other statements and lead to a different distribution of answers. However, using our data, we can investigate further whether the level of mathematical preparation can be a factor reinforcing Israeli teachers' perception of the necessity of stronger mathematical background for work with mathematically promising students. Finally, our data from

questionnaires suggest that there was a wider variety of opinion about social than about cognitive issues related to mathematically promising students. The fact that Israeli participants agreed less about psychological aspects needs further investigations.

Our study is an exploratory small-scale study. It would be interesting to use our instruments with larger and culturally more diverse prospective teachers' populations. There is also a need for more rigorous study of the preparation of mathematics teachers for the education of mathematically promising students. While more rigorous studies would be needed to get into the situation details, some recommendations can be made regarding teachers' training and professional development associated with teaching mathematically talented students. Teacher education programs should:

- Expose teachers to the complexities of teaching mathematically promising students.

- Develop in teachers stronger higher-order thinking skills and their abilities to investigate challenging tasks by proposing such tasks during their training.

- Amplify teachers' didactical inventory of teaching strategies to allow identification and fostering of mathematically promising students' abilities using inquiry- based, challenging and investigative tasks.

At the next stage we intend to investigate how mathematically promising students deal with the mathematical problem used in this study, what are their own views on their needs, and compare teachers' beliefs and expectations with the real situation in their classes.

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Mathematical and Didactical Enrichment for Pre-service Teachers: Mentoring Online Problem Solving in the CASMI project

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Abstract: In order to teach successfully, future teachers should not only be educated about students' conceptions, but also about different forms of knowledge and classroom culture. In our research, we examined whether the participation in the Internet-based challenging problem solving community CASMI contributes to the development of the aforementioned awareness and understanding in order to meet the needs of all students including the gifted ones. The results obtained enabled us to note that the pre-service teachers' perceptions of the project as a source of enrichment are mainly positive. However, analyzing schoolchildren's strategies, the participants preferred to use pre-determined criteria instead of writing personal formative comments adapted to the mathematical reasoning presented in the solution. Research shows that such comments could enrich the feedback by better reflecting the diversity of the learners' styles, thus helping them to reach their full potential. We suggest more attention needs to be given to the analyses of this diversity in pre-service teacher training and professional development in order to enable teachers to differentiate their teaching.

Key words: Online Problem Solving, Pre-Service Teacher Training, Diversity of Schoolchildren's Strategies, Asynchronous Assessment, Mathematical Enrichment

RATIONALE OF THE STUDY

What should future teachers know to teach successfully in a mathematical classroom that becomes more and more diverse (in terms of children's background and abilities) and at the same time be inclusive? Setting up an early 21st century research agenda for teacher's professional development and teacher education, Even & Tirosh (2002) base their recommendations on an important body of refereed literature that focuses on the development of mathematical awareness

and understanding of student mathematics learning and thinking. According to them, this should be coordinated by three major axes: educating about student conceptions, educating about different forms of knowledge, and educating about classroom culture. A complex approach to teacher education is thus needed in order to eventually help meet educational needs of children struggling with mathematics and those of gifted ones who may get lost while not being challenged enough (Diezmann, Thornton & Watters, 2003; Diezmann & Watters, 2005; Freiman, 2006; Freiman, Manuel & Lirette-Pitre, 2007; Johnson, 2000b; Kettler & Curliss, 2005; Sheffield, 2003).

In our paper, we will examine whether participation in the Internet-based challenging problem solving community CASMI contributes to the development of the aforementioned awareness and understanding in order to meet the needs of all students including the gifted ones. During the semester, pre-service teachers enrolled in mathematics education courses in two Canadian universities were involved in the analysis of K-12 children's solutions by giving them an asynchronous feedback.

Working with a vision of the diverse and inclusive classroom, we keep in mind that gifted students, independently of how we define and identify them, may need additional resources that are not directly available in a regular classroom. Therefore, we believe that the Internet may provide teachers and their students with appropriate activities for every child. Several studies show that rich, contextual, and open-ended mathematical problems posted on a website can challenge all children and give them an opportunity to produce new mathematical knowledge in a situation when the answer is not obvious and the strategy is to be chosen or constructed by using different ways of reasoning and communicating. This situation may be potentially fertile for mathematically gifted learners, meeting their special needs for more challenge (Applebaum & Leikin, 2007; Barbeau & Taylor, 2009; Diezmann *et al*., 2003; Diezmann & Watters, 2005; Freiman, 2006; Freiman & Lirette-Pitre, 2008; Freiman *et al*., 2007; Johnson et al., 2007; Johnson, 2000b; Kettler & Curliss, 2005; Leikin, Levav-Waynberg & Applebaum, 2008; Sheffield, 2003). While the analysis of children's mathematical production by pre-service teachers has become an important part of mathematics education courses, little is known about the impact of participation of pre-service teachers in online activities with schoolchildren and even less about their capacity to guide young learner by means of asynchronous feedback.

In our mathematics education classes, with pre-service teachers, we explore a variety of solutions to mathematical problems submitted electronically by schoolchildren. We aim to help pre-service teachers appreciate the diversity of such solutions and learn how to guide schoolchildren in a personalized and caring manner, nurturing their curiosity, interest and perseverance, which are very important for all children and especially for the gifted ones.

In our previous publications, we discussed some data about pre-service teachers' perceptions of the CASMI project (Freiman, Vézina & Gandaho, 2005). In this paper, we will report on our exploratory research in which we combined the information gathered from questionnaires regarding pre-service teachers' perceptions of the project with their feedbacks on schoolchildren's solutions. More precisely, two particular goals have been set for our enquiry:

a) to look at how pre-service teachers perceive their participation in the project regarding online challenging problem solving as a source of enrichment.

b) to examine if, being faced with a multitude of problem solving strategies, pre-service teachers are able to evaluate the correctness of students' mathematical reasoning and to provide them with an adequate feedback.

We found that very few research data are available on these questions. Therefore, our study aims to contribute to a better understanding of teacher – student retroactive communication on problem solving and to identify promising paths of improvement in pre-service teachers' mathematics education, in order to enable future teachers to provide students with richer learning opportunities.

THEORETICAL PERSPECTIVE

In order to understand the value of mathematical enrichment activities supported by the virtual CASMI environment, we looked at the literature that analyzes the role of challenging problems in today's mathematics classroom and their importance for meeting the needs of gifted students. We also searched for different studies on virtual problem solving environments and formative feedback. In the next three subsections, we will briefly review the most pertinent findings and recommendations from the studies that guided us in our data collection and data analysis.

Problem solving in today's school mathematics and the needs of gifted students

In today's mathematics classrooms, problem solving is seen as an important vehicle for the enrichment of mathematical culture because it puts strong emphasis on the development of abilities to communicate and to reason mathematically (OECD Program for International Student Assessment, 2003). In Canada, more precisely, new approaches in teaching problem solving in mathematics are following common trends set up by the NCTM Standards (2000). These trends explicitly define the central role of problems in learning mathematics and the importance to use mathematics as problem solving tools in real life interdisciplinary contexts, therefore facilitating knowledge transfer (Tardif, 1999).

Whether it is in connection with problem solving or with the learning of mathematics in general, it has been established that gifted students learn differently than their peers. The scale defined by PISA (OECD Program for International Student Assessment, 2003) assesses several levels of mathematical literacy. The highest level described by this scale features many characteristics of mathematically gifted students. Among others, these students show insight in the solution of problems, develop abilities in mathematical interpretation of problems in real-world contexts (also see Krutetski's (1976) notion of mathematical cast of mind), identify relevant mathematical tools or methods in order to find solutions to problems set in unfamiliar contexts, solve problems involving several steps, reflect on results and generalize findings and use reasoning and mathematical argument to explain solutions and communicate outcomes. Moreover, they usually are quicker at grasping concepts and the depth of their understanding surpasses the one of other students (Johnson, 2000a). It is thus important to ask ourselves what can be done to differentiate instruction for gifted students. Among others, Johnson (2000a) makes these different suggestions:

- Students should be allowed to explain their reasoning (orally and in writing).
- Resources used in the classroom should be numerous and varied.
- Open-ended problems should be privileged.
- Students should be asked "why" and "what if" questions.
- Problems and activities should extend beyond the curriculum.

Furthermore, studies conducted in the past decades, including studies of mathematical giftedness, state the need for more challenging tasks for all students but also reveal a lack of opportunities of solving such problems for students in the regular classroom (Barbeau & Taylor, 2009). However, a new approach to problem solving provided by virtual environments has the potential to increase learning opportunities for students. Indeed, a growth in Internet-based learning opportunities in mathematics can be observed. The technology itself is developing towards socially friendly, flexible and dynamic environments in which many schoolchildren can access virtual resources from school or from home. They can now get an instant interactive access to more challenging mathematics, solve problems and submit their solutions using virtual tools. Moreover, these new learning environments provide learners with a variety of contents and tools, giving them the choice between multitudes of activities adapted to their particular pace and needs. "Technology can provide a tool, an inspiration, or an independent learning environment for any student, but for the gifted it is often a means to reach the appropriate depth and breadth of curriculum and advanced product opportunities" (Johnson, 2000a, p. 5). One of the elements that become important in such environments is the kind of feedbacks students receive. Indeed, within the socio-constructivist teaching and learning paradigm, teachers need to make valid references about children's strategies (Willson & Kenney, 2003). This can be done, among others, by giving high quality feedbacks about children's solutions. In our paper, we will focus on pre-service teachers involved in a mentoring task based on the analysis of schoolchildren's solutions to challenging mathematical problem solving online activities.

Virtual opportunities of challenging problem solving: assessing diversity

When students solve open-ended problems, they mobilize a multitude of resources (Schoenfeld, 1989). This mobilization of resources is recognized as the use of a set of skills (mathematical or not) by the Program for International Student Assessment (OECD Program for International Student Assessment, 2003). It is through this mobilization of a set of resources and a metacognitive reflection that students are able to elaborate not only divergent strategies for solving problems but also several different solutions (Poirier Proulx, 1999).

Open-ended and challenging problem solving is therefore seen as a process where students should be evaluated on the bases of their own ways of reasoning and communicating. According to Lesh & Doerr (2000), the challenge for teachers is to maintain and nurture the

diversity of students' approaches, encouraging them to verbalize their thinking and explain their strategies. One of the possible solutions is to make teachers familiar with a "Problems of the week" model which proves to be an effective way to develop students into more independent learners (Webb, 2003).

This type of model is found in the CASMI, an Internet-based learning environment. Researchers argue that the use of such environments allows more schoolchildren to participate in mathematically rich contextual problem solving activities. Pre-service teachers can thus learn from students' solutions by analyzing their reasoning and communication abilities (Charbonneau, 2000; Renninger & Shumar, 2002) in didactic contexts that are more practice oriented (Bednarz, 2004). In such contexts, teachers play the role of a mentor by guiding students in their learning.

Guiding students with an effective formative feedback

Formative feedback is defined as "information communicated to the learner that is intended to modify the learner's thinking or behavior for the purpose of improving learning" (Shute, 2007, p. 1). Thus, the main goal of formative feedback is to help students understand their errors and further their reasoning. But is all feedback good feedback? It has been recently argued by Hattie & Temperley (2007) that feedback is "most effective when it aids in building cues and information regarding erroneous hypothesis and ideas and then leads to the development of more effective and efficient strategies for processing and understanding the material" (p. 102). According to Shute (2007), formative feedback serving as a corrective function should, at the least, indicate the correctness of students' answers and provide information about the correct answer. However, she specifies that a certain number of researchers agree that feedback, to be more effective, needs to give information pertaining to the improvement of the answer (instead of simply indicating the correctness of the work). Indeed, unspecific feedback can be considered useless or frustrating by students.

Galluzzo, Leali, and Loomis (2000) identified key elements linked to an effective feedback by resuming the works of Brophy. Among others, the authors insist that the teacher must:

- give a feedback which is specific to students;
- not strictly put his focus on the students' errors but also state the accomplishments;
- be specific in his comments (rather than global).

The authors also underline the importance of the knowledge of the discipline taught. Indeed, one cannot give a specific feedback if he or she is not comfortable with the discipline.

Shute (2007) did a review of the formative feedback literature and came up with these nine guidelines to enhance learning (p. 30):

- Focus feedback on the task, not the learner.

- Provide elaborated feedback to enhance learning.

- Present elaborated feedback in manageable units.

- Be specific and clear with feedback messages.

- Keep feedback as simple as possible but no simpler (based on learner needs and instructional constraints).

- Reduce uncertainty between performance and goals.

- Give unbiased, objective feedback, written or via computer.

- Promote a learning goal orientation via feedback.

- Provide feedback after learners have attempted a solution.

    She also specifies three guidelines for high-achieving learners (p. 33):

- Consider using delayed feedback, especially for complex tasks.

- Use facilitative feedback, which aims to guide students by giving them comments and suggestions in link with the problem that needs to be solved. Telling students what to do is considered directive feedback rather than facilitative feedback.

- Verification feedback, which gives information pertaining to the correctness of the answer, may be sufficient. On the other hand, elaboration feedback gives more information to students, allowing them to correct their work.

Summarizing and projecting our literature review on our research questions, we claim that the combination of challenging problem solving in an online environment and the opportunity to analyze genuine schoolchildren's solutions and to produce a formative feedback provides us with an insight into pre-service teachers' ability to evaluate and to guide students based on the diversity of their strategies and solutions. In the next sections, we describe in more details how we proceeded with data collection and data analysis.

METHODOLOGY

In our exploratory study, we analyzed quantitatively pre-service teachers' experiences with the assessment of open-ended challenging problems in the online environment. According to our two goals, we wanted to learn about pre-service teachers' perceptions on the importance of such experiences and their impact on future classroom practices regarding the use of the enrichment activities with their students. We were also interested in the evaluation of the quality of the feedbacks given by pre-service teachers. We thus studied their abilities to understand children's strategies and communication styles. In this section, we will describe the virtual environment CASMI (*Communauté d'Apprentissages Scientifiques et Mathématiques,* www.umoncton.ca/casmi)[1], the mentoring activities in which the pre-service teachers were involved and how these activities have been evaluated. We will also present the samples and data collection tools.

Virtual environment

In the CASMI environment, schoolchildren are invited to solve challenging mathematical problems and submit their solutions electronically (Freiman & Lirette-Pitre, 2008). Pre-service teachers then analyze every solution and write a personal feedback. The problems of the week are grouped in four categories according to their level of difficulty and posted online. These problems present a variety of contexts to which schoolchildren are supposed to apply mathematical concepts from all domains of school mathematics (arithmetic, algebra, geometry, statistics).

Figure 1 (p. 11) presents one of the problems students had to solve in the CASMI. In this problem, "The Valentine's Day card", students had to find the original width and length of a piece of paper that had been folded. The problem contains a context familiar to French Canadian schoolchildren and is attractive. A variety of answers can be produced, since the only constraint is that the sum of the width and the length of the original piece of paper must be equal to 50 centimeters. Children with different abilities may extract different mathematical relationships

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[1] Although the research project took place when the website was called CAMI (Chantier d'Apprentissages Mathématiques Interactifs), the abbreviation CASMI will be used throughout this article in order to facilitate its reading.

representing and exploring them in many different ways. One solution submitted by a grade 6 student is presented in figure 2 (p. 11) and an extract from the personal feedback given to her by a pre-service teacher is presented in figure 3 (p. 12).

> The Valentine's Day card
>
> Valentine's day is coming and Reuben decides to make a Valentine's day card for Sophie.
>
> As you probably did before, Reuben takes a piece of red construction paper and folds it vertically in two. He then folds the piece horizontally and finally draws hearts and flowers while writing beautiful words of friendship everywhere.
>
> The perimeter of the folded card is 50 centimeters. Find the length and the width of the original piece of paper (before it was folded). Clearly explain your reasoning.

Figure 1. Mathematical problem presented in the CASMI

> The Valentine's day card
>
> If we unfold it, it's going to be twice as big, and if we unfold it again, it's going to be twice as big again.
>
> $50 \times 2 = 100$
> $100 \times 2 = 200$
>
> Answer: 200 centimeters

Figure 2. Solution submitted electronically in the CASMI

Extract from the feedback

I believe that you tried to find the perimeter of the original paper (before it was folded). However, the problem was to find the length and the width of this piece of paper. I invite you to verify your answer. I am sure that you can solve this problem!!!

Thank you for participating. Bravo for your efforts! I wish to receive other solutions from you in the next few weeks.

Figure 3. Extract from the personal feedback written by a pre-service teacher

The first paragraph of the extract from the feedback contains various aspects mentioned in our theoretical framework. First of all, the fact that the student didn't seem to understand exactly was she was looking for is underlined and an "appropriate interpretation" of the question is given. Moreover, the student is invited to review her work. Finally, a strong belief in the child's capacity to correctly solve the problem is visible. The second paragraph, written in the last section of the feedback, values the student's participation and efforts and aims to encourage her to solve more problems in the CASMI in the near future.

While all children are asked explicitly to explain their reasoning, not all of them show their work and sometimes, it is not obvious to see mathematical reasoning beyond the explanations. All this may represent important challenges to pre-service teachers who are not used to solving problems in different ways, analyzing reasoning and giving critical comments back to students. Therefore, working within the CASMI environment, they get this genuine opportunity to look at this variety of mathematics created by children.

Participants

During the 2006 winter semester, a total of 70 pre-service teachers participated in our research. Thirty-two were enrolled into the Middle School (5-8) Teacher Preparation Program and 18 were enrolled in the High School (9-12) Teacher Preparation Program at Université de Moncton. Twenty more were enrolled in the Secondary Mathematics Teacher Preparation Program (7-11) at Université Laval. The collaboration between the two researchers never aimed to make any comparisons between the two groups. There was no specific interaction between the two groups. According to our theoretical perspective, we focused on each participant's perception using a survey and we assessed the quality of randomly selected feedbacks. In this case, we can consider these two groups as one combined population (one group) rather than as two different populations.

Instruments

During the semester, feedbacks were written to schoolchildren using an electronic form built into the CASMI site (figure 4, p. 14)[2]. All pre-service teachers had to log-in individually to assess solutions randomly assigned to them. Our form was divided in three sections. The first section, *Greeting*, was situated at the beginning of the form and allowed pre-service teachers to make a first contact with students by writing comments pertaining to their participation or the efforts that were made, as well as general comments with regard to the submitted solution. The second section of our form, the rubric, contained six different components used by Math Forum to score solutions: interpretation, strategy, exactness in calculations, completeness, clarity, and quality of reflection. We developed our own pre-built set of criteria according to the specific features of each component. These criteria were presented as multiple choice items. Thus, in their formative feedback, pre-service teachers could choose one of these pre-determined criteria for each component. The chosen criterion could also be accompanied (or replaced) optionally by an open comment, which permitted the personalization of the feedback. Finally, in the last section of the electronic form called *Signature*, pre-services teachers could summarize their thoughts about the student's production and invite them to visit the CASMI again in order to

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[2] An English version of this electronic form is presented in appendix 1 (p. 29).

solve more problems. So, in every section of the electronic form, pre-service teachers were capable of writing comments and thus of personalizing the feedback given to schoolchildren.

Figure 4. Electronic form in the CASMI site

At the end of the semester, a questionnaire including open-ended questions as well as multiple choice questions was distributed. The questionnaire was divided into ten sections, pertaining to different aspects of the project: 1) General information on the participants; 2) CASMI project and the didactics course; 3) CASMI project and the student doing mathematics; 4) CASMI project and teachers; 5) Appreciation of the CASMI website; 6) Accessibility of the problems; 7) Problems' content; 8) Functioning of the CASMI website; 9) Continuation of the CASMI project; and 10) Use of the site with the preservice teachers' future students. The answers to the questions as well as the comments gathered in the questionnaire permitted us to

collect qualitative and quantitative data concerning pre-service teachers' perceptions pertaining to the CASMI project and teacher training as well as to the CASMI project and teaching and learning mathematics.   The multiple choice questions employed a four-point Likert scale: 1 = Completely in agreement, 4 = Completely in disagreement.

Procedures

Université de Moncton.

At the Université de Moncton, pre-service teachers enrolled in the Elementary (K-8) Teacher Preparation Program must take two courses in mathematics education.   Within each course, they conduct a project related to CASMI. Most of the pre-service teachers participating in our project were enrolled in their second math education course and were already familiar with the resource. While during the first course they are required to do reflective analyses of their experience and are guided by the course instructor in their assessment process, the second course requires more autonomous work and better quality of feedback. Fifty students evaluated up to ten solutions each.   During the math education classes, each problem as well as different ways of solving it and communicating related strategies were discussed. Pre-service teachers thus understood the problems before having to assess schoolchildren's work.

Université Laval.

The participants at Université Laval were all enrolled into the Secondary Mathematics Teacher Preparation Program. In this program, pre-service teachers have to take three courses in mathematics education.   Within the framework of our research project, twenty pre-service teachers enrolled into their third and final math education course received a brief presentation of the CASMI, which they were not familiar with. A document explaining the evaluation rubric and presenting examples of feedbacks was also given to them. In a four weeks period, each pre-service teacher evaluated a total of twelve productions submitted by students.

At the beginning of each week, before they received students' productions, pre-service teachers had to solve the four "problems of the week" presented in the CASMI. These problems were then revised in class. This revision made it possible to avoid any confusion that could be allotted to the various problems. Moreover, pre-service teachers were asked to present different strategies used when solving these problems. Therefore, they were made aware of different ways

to solve one problem. It is important to note that the pre-service teachers' feedbacks were strictly evaluated on formative bases. Following each week, comments pertaining to pre-service teachers' feedbacks were emitted by the professor. These comments made it possible for the pre-service teachers to adjust their formative evaluations week after week.

Data Analysis

A total of 65 pre-service (47 from Université de Moncton and 18 from Université Laval) answered the questionnaire. A theme analyses of the qualitative data collected in the questionnaire was realized. Frequency distributions were calculated to analyze the multiple choice items.

In addition to data from the questionnaire, we analyzed formative feedbacks written by pre-service teachers. Out of a total of 924 schoolchildren's solutions submitted to ten problems posted during the semester, we randomly selected 200. We developed and validated an evaluation grid containing 53 variables[3]. These variables reflected elements reported in our theoretical framework and were divided into nine categories. The first category was *General* and it permitted us to determine the correctness of students' answers and then check if pre-service teachers had identified that answer as being correct or incorrect. The same variables were repeated for the next two categories, *Greeting* and *Signature*. We were interested to see if pre-service teachers added personalizing elements to their message (i.e. smiley, humor, etc.) and if they congratulated students on their work or thanked them for participating. Elements of feedbacks more directly in link with the mathematical aspect of the student's solution also interested us. For each of our six components, we evaluated if pre-service teachers had chosen the appropriate criterion in the pre-built set of multiple choice items specific to these components. Ideas present in the feedback examined were analyzed. For each idea, we checked, among others, if pre-service teachers underlined the correctness of the answer, the correctness of the reasoning and if they identified students' errors. Elements more linked with the quality of feedbacks, like specificity or reference to students' work, were also evaluated. In addition to that, we checked if pre-service teachers gave facilitative, verification or elaboration feedback.

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[3]Some of these variables were repeated for every criterion or for every different idea present in a comment. The evaluation grid thus contains a total of 271 variables.

This analysis enabled us to gather information about the quality of submitted solutions by the students as well as the quality of the feedback provided by the pre-service teachers.

Resume and Analyses of the Most Important Findings

a) How do pre-service teachers perceive their participation in the project regarding their future work on challenging problem solving in the mathematics classroom?

The participation in the CASMI project allowed pre-service teachers to analyze concrete solutions of real schoolchildren. The first part of our analyses concerned pre-service teachers' perceptions of the CASMI project and according to the previously described elements of our questionnaire, we found that 84.6 % of participants agreed or strongly agreed that their feedbacks were important for schoolchildren. Eighty percent found that the project helped them better understand schoolchildren's reasoning and 67.7 % found that it helped them better understand the problem-solving process in mathematics. Moreover, 83.1 % affirmed that they had learned more about formative feedbacks, 66.1 % say that the project gave them the chance to review mathematical concepts, and 78.5 % of pre-service teachers said that the project gave them ideas for teaching. Finally, 81.5 % of them agree or strongly agree that the CASMI project not only enables teachers to differentiate their teaching but also enriches the mathematics curriculum. The complete results on pre-service teachers' perceptions of the CASMI project are presented in table 1 (next page).

Table 1
Pre-service teachers' perceptions of the CASMI project

	Completely in agreement		In agreement		In disagreement		Completely in disagreement		I don't know	
	Frequency	%	Frequency	%	Frequency	%	Frequency	%	Frequency	%
Your feedback is important for the student	40	61.5	15	23.1	2	3.1	1	1.5	6	9.2
The content of the problems enriches the math curriculum	26	40.0	27	41.5	7	10.8	3	4.6	0	0.0
The project…										
helped me to understand the student's reasoning	7	10.8	45	69.2	10	15.4	3	4.6	0	0.0
helped me better understand the problem-solving process in math	10	15.4	34	52.3	15	23.1	5	7.7	1	1.5
allowed me to perfect my techniques in formative evaluation in math	18	27.7	36	55.4	6	9.2	3	4.6	2	3.1
allowed me to review math concepts	19	29.2	24	36.9	15	23.1	5	7.7	1	1.5
gives teachers ideas for math courses	28	43.1	23	35.4	11	16.9	1	1.5	2	3.1
allows teachers to differentiate their teaching	19	29.2	34	52.3	7	10.8	1	1.5	4	6.2

The results obtained from the analyses enable us to note that the pre-service teachers' perceptions of the CASMI project as a source of enrichment are mainly positive in all aspects of the questionnaire. Those results are consistent with our previous data (Freiman *et al.*, 2005). However, in this study, we decided to conduct an in depth analyses of the quality of the feedbacks given by pre-service teachers in order to track their abilities to assess students' solutions and to guide them, in the process, towards better problem solving strategies. The second part of our analyses, concerning the types of feedbacks given by pre-service teachers, is presented in the next section.

b) Being faced with a multitude of problem solving strategies, are pre-service teachers able to evaluate the correctness of students' work and to provide students with an adequate feedback in order to guide them and to help them improve their problem solving skills?

Our methodological framework defined certain aspects that are important when giving a feedback. Among those aspects, pre-service teachers need to be able to assess if the solution submitted by a student is correct. It is also important for schoolchildren to be guided and to get feedback which is directly linked with the work they have done. Keeping this in mind, we analyzed 200 feedbacks given by pre-service teachers in order to study their ability to evaluate students' work and to give a quality feedback. We found that in 78.5 % of cases, pre-service teachers were able to correctly identify if students' answers were correct. They made an incorrect evaluation 10.5 % of the time (i.e. indicating to a student that his answer was correct when it wasn't and vice versa) (table 2, p. 21). Moreover, for each component of the evaluation rubric, pre-service teachers were invited to choose a criterion specific to the component and linked with the student's work (table 3, p. 21). They chose the appropriate criterion 70.0 % of the time for the component Interpretation, 72.5 % of the time for the component Strategy and 72.0 % of the time for the component Clarity. This percentage goes up to 79.0 % in the case of the component Correctness, 77.5 % for the component Completeness, and 80.0 % for the component Quality of reflection.

Table 2
Choice of the criterion in order to identify if the student's answer was correct

	Frequency	Percent
No criterion selected	3	1.5
Incorrect choice of criterion	21	10.5
Partially correct choice of criterion	19	9.5
Correct choice of criterion	157	78.5
Total	200	100.0

Table 3
Choice of the criterion specific to the component

Component	No criterion selected		Incorrect choice of criterion		Partially correct choice of criterion		Correct choice of criterion	
	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent
Interpretation	7	3.5	11	5.5	42	21.0	140	70.0
Strategy	9	4.5	29	14.5	17	8.5	145	72.5
Clarity	19	9.5	13	6.5	22	11.0	144	72.0
Exactness in calculations	10	5.0	20	10.0	12	6.0	158	79.0
Completeness	14	7.0	13	6.5	16	8.0	155	77.5
Quality of reflection	14	7.0	14	7.0	10	5.0	160	80.0

Since every feedback could be personalized by writing a comment, we then asked ourselves which kind of analysis and recommendations were present in the individual comments that were written. The analyses of the 200 feedbacks given by pre-service teachers shows that 70.5 % of these feedbacks place little or no importance on the successes of students and tend to strictly focus on their errors or on challenges for them to overtake (table 4, p. 22). Moreover, although the majority of comments do refer implicitly to schoolchildren's work, 60.5 % of them are general and lack in precision (table 5, p. 22).

Table 4
Feedback in the form of positive feedback or focusing on the student's errors

	Frequency	Percent
Positive feedback	161	29
Focusing on student's errors or on challenges	391	70.5
Total	552	99.5

Table 5
General or specific comment

	Frequency	Percent
General comment	336	60.5
Specific comment	219	39.5
Total	555	100.0

Thus, the problem may not reside as much in the criteria-based assessment of students' answers as in the (informal) feedback they give (or do not give). Among the 200 solutions that were analyzed, 100 contained some incorrect reasoning or calculation mistakes (table 6, p. 23). Our findings show that for 81.0 % of these solutions, at least one comment, directly linked to one of the components in the rubric, was made by pre-

service teachers (table 7, p. 23). However, even though 81.0 % of the incorrect solutions were commented on at least once, in several cases (i.e. for several components), pre-service teachers seemed to be satisfied by choosing one of the pre-determined criteria and didn't write any comments in order to enrich their feedback. We do not know why they did not take the time to write more comments. In our future work, we will need to conduct interviews with the participants in order to learn more about their reasons for choosing a particular criterion over another.

Table 6
Correctness of the student's answer

	Frequency	Percent
No answer	1	0.5
Incorrect answer	100	50.0
Partially correct answer	36	18.0
Correct answer	63	31.5
Total	200	100.0

Table 7
Feedbacks given to students whose answers contained some incorrect reasoning or calculation mistakes

	Frequency	Percent
No feedback	7	7.0
Feedback directly linked to one of the six components of the rubric	81	81.0
Feedback given in the sections *Greeting* or *Signature*	12	12.0
Total	100	100.0

Moreover, they do not seem to fully appreciate the diversity of students' approaches which, according to Lesh & Doerr (2000), is a challenge for teachers. It is important for them to maintain and nurture that diversity. The pre-service teachers that participated in our study were not in a guiding mode and did not encourage

schoolchildren to further their reflection. Indeed, instead of being built on students' work, comments that were written strove students' thinking towards pre-determined answers which is contrary to current tendencies in mathematics education (Astolfi, 2006).

CONCLUSION

This study draws its originality from the fact that it focuses not only on pre-service teachers' perceptions but on the link existing between these perceptions and the quality of their formative asynchronous feedback. Linking pre-service teachers' perceptions of what an Internet resource on problem solving can bring to improve mathematics teaching and learning to their ability to analyze children's thinking, we aimed to develop practical recommendations on how to build more solid assessment competences in pre-service teachers.

Participation in the online project allowed pre-service teachers to experience new mathematical problem solving approaches which stress the use of a multitude of strategies and communication means by schoolchildren. They perceived their experience as valuable since it permitted them to better understand the problem solving process and, in particular, children's ways of communicating their reasoning. They observed that some problems allow different data interpretation, different solving strategies and sometimes different answers.

Some strategies may be plausible, even ingenious. Others may contain misinterpretations, misconceptions, or alternative views. In order to be able to guide children through their learning, pre-service teachers have to become competent not only in mathematics but also in feedback pedagogy, which sometimes work in the counter direction of the traditional didactical contract (Brousseau, 1986, 1988, 1998; Poirier, 2001). When communicating with schoolchildren about problem solving, our pre-service teachers get the chance to work on contextual open-ended problems revising their own views of problem solving and its role in mathematics learning. They also reinforce their own conceptual understanding of mathematics and develop a better understanding of how children think and explain their thinking. While writing feedbacks, pre-service teachers put in practice their ability to understand the problem itself and to guide children towards better problem solving strategies (Freiman *et al.*, 2005; Metallidou, 2009).

It's not easy to understand a child's reasoning when it is expressed out loud. Asynchronous assessment is even more challenging because there is no opportunity to give feedbacks in another way than written comments. But our data shows a lack of such personal comments. The comments' general character may be a result of the pre-service teachers' lack of mathematical background as well as lack of time. If the first two issues can be address by better teacher training strategies, the last one may raise a concern. Indeed, when schoolchildren are allowed to use a variety of strategies and communication means, teachers must give feedback to every one of them. If pre-service teachers don't have the time to do it with 10 students, how will they find the time to do it with 30 students, and possibly 30 different strategies? Are changes necessary to the school system or to the working ethics of pre-service teachers?

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Appendix 1. English version of the electronic form in the CASMI site

Analysis of the solution

Section Greeting:

Greeting text:

General section:

Section - Data:

○ You correctly identified the important data of the problem and you wrote them down.

○ You partially identified the important data of the problem.

○ I would have liked for you to write down the data of the problem. This stage is very important in problem solving.

Feedback

Section - Interpretation:

○ The goal of the problem was well understood and mastered. Bravo!

◉ The goal of the problem was partially understood and you are on the right track to complete the problem.

○ The goal of the problem was partially understood. Here is some advice which will help you solve the problem.

○ The goal of the problem does not seem to have been understood. Here is some advice which will help you solve the problem.

Gifted Students and Advanced Mathematics

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Abstract: The extension to a wide population of secondary education in many countries seems to have led to a weakening of the mathematics curriculum. In response, many students have been classified as "gifted" so that they can access a stronger program. Apart from the difficulties that might arise in actually determining which students are gifted (is it always clear what the term means?), there are dangers inherent in programs that might be devised even for those that are truly talented.

Sometimes students are moved ahead to more advanced mathematics. Elementary students might be taught algebra or even subjects like trigonometry and vectors and secondary students taught calculus, differential equations and linear algebra.

It is my experience over thirty-five years of contact with bright students that acceleration to higher level mathematics is often not a good idea. In this paper, I will articulate some of the factors that have led me to this opinion and suggest alternatives. At the same time, one needs to deal with truly exceptional students in an appropriate way.

Keywords: talented students, enrichment, acceleration

1. Beliefs and assumptions

The central question in mathematics education is, "Who owns the mathematics?" If the answer is not "the student", then our efforts within and without school are likely to be counterproductive. Traditional education has often led to a syllabus being imposed on students as passive recipients, so that whatever richness it possessed was not appreciated and thus not understood or retained.

If students are to enter into mathematics, it must be through an involvement that makes it intelligible, that ensures its applicability and that leads to an apprehension of its power. An overemphasis on covering material, whether in a traditional approach or in the enrichment of talented students, runs the risk of reducing the occasions for this involvement. The point was made by Nicolas Sarkozy, President of the French Republic, in an encyclical letter to educators on September 4, 2007 (I am indebted to the French Embassy in Washington for the translation):

> *Don't misunderstand me; my aim is not to increase the teaching hours still further; the timetable is already too heavy. It is not to add yet more new subjects to a list*

Barbeau

*which is already too long. On the contrary, to my mind, **the aim is to give back to our children time to live, breathe; assimilate what they have been taught**. [emphasis mine] We need to regain coherence in our educational system.* · · · *We need to restore coherence within each school subject and between these and society's expectations, once again find a lodestar for education, set for its principles, goals and simple criteria.*

It is this provision of room to breathe and sense of coherence and purpose behind what we present to students that will help them engage our discipline productively. The traditional curriculum scored quite well on coherence; elementary students got a solid exposure to arithmetic and secondary students might spend a whole year on subjects such as Euclidean geometry, analytic geometry, trigonometry, algebra and traditional applied mathematics, learning a range of results and techniques and doing exercises. It often lacked the opportunities for students to explore and experiment, to put their own stamp on the concepts and procedures they needed to master.

Teachers must not be put in the position of answering questions that students are not prepared to ask. If we are to proceed to more advanced mathematics, it is because the experiences of the students lead them to an apprehension of the need for it. It might be a more general approach that tidies up what might be otherwise unmanageable or of more powerful tools to handle situations that are difficult or impossible with the tools they possess. Arithmetic is a tool for convenient handling of quantitative information; algebra is an antidote to the over complexity of arithmetic solutions to word problems; the systematization of synthetic, analytic or transformation geometry allows the encompassing of an undisciplined slew of results.

Thus the pace of introducing new material should be sensitive to how well students have assimilated existing material, how flexibly they can negotiate it and their understanding of its uses and limitations.

2. Educational activity: past and present

The approach described has been implicit in many programs available to students over the years. The *Gelfand Correspondence Program* in Mathematics, first in the Soviet Union and latterly in the United States, has provided a curriculum for its adherents that is coherent and interactive (5). *Project SEED Mathematics*, originating in Berkeley, CA in the 1960s is another program, still continuing, that provides an in depth experience for students (7). The recent Volume for the 16th ICMI Study, *Challenging mathematics in and beyond the classroom* describes several initiatives, such as the *Creative problem solving in mathematics* (CPSM) course at Quincy Senior High School in Quincy, IL, that is based on a study of solid geometrical structures, and the *Maths `a modeler* research activities for students and teachers in France, one of which is focused on tiling (3, p. 189 seq).

Even though, as the Study Volume indicates, many educational researchers are studying and creating programs for gifted students, it is hard to avoid the impression that evidence of their efficacy is largely anecdotal. Success seems to be dependent to a large degree on the expertise and passion of their proponents and on the readiness of students to embrace them. In many cases, the student participants are either self-selected or identified by adults as being suitable. I am not aware of longitudinal studies that any particular regime leads to greater mathematical awareness and prowess, either among those amateurs of mathematics that melt into the general public or those who proceed to higher study. Nor am I aware of systematic programs that have been adopted over a large jurisdiction to bring along those that are especially interested or capable in mathematics.

3. Algebra and calculus

Algebra and calculus both have the characteristic of being general methods, capable of treating a wide range of problems and situations. In so being, they tend to suppress particularities and to see problems as belonging to broader categories. The situation is mediated through a specially created formalism that is efficient and sophisticated, so that a first-hand feeling for the situation may be lost in the application of a standard procedure. Both algebra and calculus are systems of great mathematical power, but this is often traded off against transparency and intelligibility. An inexperienced student might see these as machines, to be used indiscriminately.

Students should be exposed to these advanced areas only when they can appreciate their significance and understand their use. Indeed, it might be said that the most important thing that a young student needs to know about either algebra or calculus is when not to use it.

Consider algebra. Its utility for most middle school and early secondary students is in the reformulating and solving of word problems. Some such problems can often be more conveniently handled by arithmetic or proportional techniques. Consider the following example:

Example 1. Two old ladies, Olga and Tamara live in separate towns some distance apart that are joined by a single road. One morning at sunrise, the two ladies set out simultaneously to walk to the town of the other, each walking at her own constant speed. The two passed on the road at noon. Olga reached her destination at 4 pm, while Tamara did not arrive at hers until 9 pm. What time was sunrise that day?

When this problem is given to students who have had some algebra, their first impulse it to set up some equations and try to solve them. Invariably, they find this a tough task and often do not succeed. Part of the difficulty is the introduction of superfluous detail, such as the actual distance between the towns or the speeds of the two ladies, which serve to obfuscate the situation. What gets lost is the significance of the proportionality inherent in the situation: when a person walks at a constant speed, the distance travelled is proportional

to the time taken. Suppose the time taken by both ladies before noon is T hours. The distance walked by Tamara in the morning is the same as that walked by Olga in the afternoon, and *vice versa*. Appealing to the proportionality quickly leads to T: 4 = 9: T and the answer T = 6. Thus, the sun rose at 6 am.

Accordingly, gifted students should be presented with arithmetic and proportionality problems of varying difficulty, and challenged to solve them through basic reasoning. Some might be encouraged to use the sort of diagrammatic methods espoused by Singapore texts. (For an example, see (3, p. 290-291).) However, they will find some problems tough when only arithmetic methods are available, but routine when algebra can be used.

Example 2. A man is 6 years older than his wife. He noticed 4 years ago that he had been married to her exactly half of his life. How old will he be on their 50th wedding anniversary if in 10 years she will have spent two-thirds of her life married to him? [International Mathematical Talent Search, Round 17. Consult (6).]

The student who tries to meet this on arithmetical terms has a real challenge, and will appreciate how the definition of variables and the use of algebra will clarify the situation. However, the application of algebra is not completely automatic; the rapidity of achieving success on this problem will depend on how astutely the variables are defined and the equations are set up.

Algebra should presented in a context where the student can be expected to decide on where and how to use it; sometimes it is better avoided; other times, it is essential. Gifted students need to learn algebra, which after all is the language of mathematics, but it should be presented in a measured way so that its power is made manifest and the student can absorb and dwell naturally in its higher level of abstraction. Once algebra is embarked upon, its use as a tool in all sorts of problems should be explored, not only in the setting up and solving of equations, but in problems of maximization, analytic geometry, trigonometry, combinatorics.

For students at the secondary level, calculus is in an analogous position. When it is introduced prematurely, students seem inclined to address it only on operational terms. Since the only functions secondary students are going to have to deal with to any degree are polynomials and the standard transcendental functions, they are not sensitive to the issue that differential calculus applies to functions that are smooth and may see it as applicable to anything in sight (such as the absolute value function). By not being aware of the more subtle issues and the range of validity of calculus techniques, their long term growth as mathematicians may be stunted. Two case studies will illustrate the point.

This should not be construed as an argument against giving calculus to minors, but only that it is given to students with a well-rounded experience in algebra and geometry. As any fan of the Putnam competition knows, calculus can be the arena of its own clever and elegant challenges.

Case Study 1. *Functional equations*. In recent years, it has been common to include among competition questions, particularly at the Olympiad level, functional equations that have to be solved. Frequently, there are no conditions on the function apart from the equation, so the solution sought is completely general. However, students immersed in algebra and calculus, will often, probably unconsciously, assume that the function in question is a polynomial, that it is continuous or that it possesses a derivative. Acting on this often leads them into formidable computational territory; their extra knowledge often prevents them from addressing the problem at its most basic and natural level.

Example 3. Problem 2 on the 2008 Canadian Mathematical Olympiad (4) sought the solution of the functional equation

$$f(2f(x)+f(y)) = 2x + y,$$

where f is defined on the rationales and takes rational values.

The algorithmic-bound student who either assumes that *f(x)* is a polynomial or differentiates immediately introduces unwarranted complications and restricts the scope of the problem. The solver of this needs to take to heart that stark but comforting fact that everything available is stated in the problem and has to try to squeeze out of this meagre store the maximum possible information.

This might involve guessing a solution, to see where one might be headed. Or it might be trying some basic substitutions (setting variables equal to zero or to each other are reasonable options) to get simplifications or more workable conditions. In this case, one can find that

$$f(0) = 0, f(2x + y) = 2f(x) + f(y) \text{ and } f(2x) = 2f(x) \text{ for all } x \text{ and } y.$$

The experience that students might obtain from such problems might well lead to an easier embrace of axiomatic systems in their later studies, where they need the discipline of assuming exactly what is given and putting aside extraneous details.

Example 4. Solve the functional equation

$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$$

for real x and y,

Noting the role of $x \pm y$, one can try the substitution $u = x + y, v = x - y$ to obtain

$$vf(u) - uf(v) = (u^2 - v^2)uv$$

for real u, v. One can "separate the variables" to obtain

$$\frac{f(u)}{u} - u^2 = \frac{f(v)}{v} - v^2$$

from which it is deduced that $f(x) = cx + x^3$ for some constant c. It can be checked that this works.

Notice in this example how little one has to rely on technical results and processes, and how much depends on intuition and ability to draw out the significance of the equation resulting from separating the variables. Students will often have initial difficulties with these sorts of problems, but given enough time and experience, they will develop the experience and divergent thinking that will more reliably lead to success.

Case Study 2. Inequalities and optimization. If students are exposed to calculus while their algebraic background is sparse, they are inclined to see every inequality and optimization problem as an occasion for taking the derivative. It is useful to defer calculus until students have learned various algebraic techniques for dealing with inequalities. These often involve techniques such as completing the square, expansion, rearranging and factoring of expressions to expose clearly the sought inequality, and so provide the student with practice in reading algebraic expressions and extracting information from them. This could be combined with the derivation of and experience in dealing with standard inequalities such as the arithmetic-geometric means inequality, power mean inequalities and the Cauchy-Schwarz inequality. Indeed, an examination of Olympiad inequalities suggests that probably three quarters of them can be handled with an astute application of the arithmetic-geometric means inequalities.

The student who resorts to calculus to solve inequality problems runs three risks. The first is that, in missing the salient features of an inequality or optimization problem, she complicates the situation. The second is that the solution may not be complete; having found the condition for the vanishing of a derivative, the student may neglect giving an argument to justify the nature of the optimum. The latter danger is particularly pronounced if the student is equipped with the howitzer of Lagrange Multipliers; this is a neat technique, but often the classification of the optimum can be tricky. The third is that she might not develop the valuable ability to "read" algebraic expressions and develop an instinct for performing the most appropriate and effective manipulations.

Another pitfall that occurs in this area is that students forget that the essence of solving a problem is to reduce it to something more elementary and straightforward. There is an unlimited supply of inequalities of ever increasing sophistication and power, and many students lack the maturity to adjust the strengths and generality of the tool to the situation at hand.

Example 5. A good example appeared as Problem 3 on the 2008 Canadian Mathematical Olympiad (4). Candidates were asked to show, for positive reals *a, b, c* satisfying *a + b + c* = 1, that

$$\frac{a-bc}{a+bc}+\frac{b-ac}{b+ac}+\frac{c-ab}{c+ab}\le\frac{3}{2}.$$

This drew more solutions than expected, that ranged from very straightforward to extremely complicated; a few appealed to the very general Muirhead majorization inequalities (for which I had to access *Google* for enlightenment) (8). However, elementary algebraic manipulation leads to the equivalent $ab+bc+ca\ge 9abc$ or

$$9\le\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c)$$

which is a consequence of the Cauchy-Schwarz inequality.

Example 6. For x, y, z > 0, show that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\le 10$$

Since combining terms on the left side or using calculus to maximize it is particularly nonappetizing, it is best to look for elementary methods and insights.

The basic $\left(x-\sqrt{yz}\right)^2\ge 0$ leads to $(x+y)(x+z)\ge x\left(\sqrt{y}+\sqrt{z}\right)^2$ and a quick evolution to the solution.

While neither example is obvious, both underscore the utility of learning how to read the structure and seeking insight rather than charging ahead with a standard approach.

In summary, the mathematical growth of students in the exercise of judgment should be kept commensurate with the exploration of new and higher level material.

4. Subjects suitable for gifted students

In selecting a program for gifted students at the precollege level, the emphasis should be on broadening the experience of the regular syllabus rather than on acceleration. One has the opportunity of covering topics that are attractive, yet not likely to figure in main stream mathematics education. I will mention some of these:

Geometry. School geometry tends to be sparse, and in many jurisdictions, there is a tendency towards empirical geometry using technology. The use of resources such as *Geometer's Sketchpad* is a welcome addition to the syllabus, but it may displace other important aspects of the mathematical experience. Unless it is part of an enriched program, Euclidean geometry is unlikely to figure as part of a student's mathematical education.

Elementary geometry is all about circles and triangles, figures that admit an unlimited supply of properties and relationships. It betokens the fecundity of mathematics; after 2500 years, new results are still being found and old ones reestablished in ever more elegant ways. It links mathematicians across time and culture, is shared by amateurs and professionals, hones analytic and logical skills, fosters competency in exposition, sharpens the aesthetic sense and provides an ample stage for investigation, ingenuity and achievement. It provides a

handsome supply of tools – traditional Euclidean derivation, transformations, vectors, analytical geometry, complex numbers – for the solutions of problems.

The ability to use transformation arguments, in particular, is particularly exciting for the novice, as such arguments rely on exposing and exploiting the basic structural aspects of the situation and give an insight into why the result holds that Euclidean or analytic methods often fail to do.

Number theory. Elementary number theory is another attractive area for young students. Not only should they learn the basics of prime factorization, common multiples and divisors, but they should master the use of modular arithmetic (something is probably easier picked up by the young than by many students later at college age). The solution of Diophantine equations provides an excellent challenge for students at the secondary level, as they are required to assimilate and select the right algebraic and numerical facts and techniques. A particular equation that is ideal for the young is Pell's equation. It is easily motivated and readily understandable, and provides the occasion for a great deal of empirical investigation. Yet the methods for treating this equation provide a natural home for surds (a topic given at most a cursory treatment in the standard curriculum) and provides direct experience in issues that will be taken up in more detail in the study of computation and number theory, modern algebra. An indication of what is possible is provided by my book *Pell's equation* (2).

However, again care needs to be taken when more advanced work is undertaken or referred to that students do not lose a sense of appropriateness and judgment. They need to realize that the Dirichlet result about the infinitude of primes in certain arithmetic progressions is deep, and not to be thrown into a solution when simpler resources are available.

Polynomials. For about nine years, I presented a course on polynomials to secondary students that terminated in an optional final examination. This was an ideal topic for gifted students, as it combined practicality with an concrete gentle introduction to important areas of advanced mathematics, including complex analysis, inequalities, number theory, modern algebra, approximation theory, dynamical systems, combinatorics and, yes, calculus. This eventually resulted in a book (1). As there were many topics that would be useful for students to know, but that might likely not meet in a college course, it could be regarded as an amplification of high school work in which student derived practical experience with examples of higher level theory later encountered at college.

Functional equations. An area that was almost non-existent two decades ago, functional equations now occur regularly on competitions. This is an excellent realm of challenge for secondary students, who often require only basic reasoning and elementary facts, but need to collect the evidence about the unknown function carefully and cogently.

Combinatorics. Although combinatorics has increasingly become part of the undergraduate mathematics curriculum, there is an elementary dimension to this division of mathematics that makes it eminently suitable for the young. The Pigeonhole Principle and Inclusion-Exclusion Principle are two techniques that are at once powerful and accessible. The use of generating functions provides exercise in algebraic techniques along with an indication of how one area of mathematics can enrich another. As with geometry, a high premium is put on careful argumentation, so that the skills of the student in organization and exposition can be enhanced.

Recursions and Dynamical systems. Elementary finite differences, in particular the solving of recursions, is an elementary topic that can be part of the arsenal of gifted students. Linear recursions share many structural properties with linear systems of algebraic or differential equations, and so provide a larger context for linear algebra that will be studied later. Dynamical systems, particularly the study of the logistic recursion, requires only basic algebraic and calculus background, and serves as an occasion for computer investigation and a study of approximation.

Trigonometry. This branch of mathematics has become considerably emaciated in the standard syllabus in North America. This is unfortunate, as trigonometry is an elegant formulation for dealing with situations that at root involve similar triangles in a powerful way. It stands at the crossroads of pure and applied mathematics, and provides a firm foundation for studies in either of these directions. Combining ideas of algebra and geometry, it is a platform to encourage facility and insight in both areas, one should that be part of the educational experience of any gifted student in mathematics. It also provides a home for complex numbers, which lives only as an orphan in the standard school syllabus; many trigonometric manipulations can be handily done using complex techniques.

Cardinality. Many students are confounded at college by a failure to understand the nature of the continuum. For gifted students, this can be circumvented by embarking on an early and leisurely examination of the real number system to get a feel for its complexity. This includes understanding countability and uncountability, and realizing that by this criterion, the sets of rationals and nonrationals are essentially different. The study of infinity is often quite difficult even at the college level, but an argument can be made for dealing with it early before students have had a chance to form prejudices.

History. Young students can usefully be introduced to some aspects of the history of mathematics. There is value in seeing how our predecessors tackled problems before modern mathematical structures were in place and to gain some understanding of how these structures were conceived and formulated. Apart from Euclidean geometry, students can study with

profit the solution by Euler of the Konigsberg bridge problem, precalculus determination of areas and tangents (the cycloid gives some beautiful case studies), the beginnings of number theory at the hands of such masters as Fermat (see (2) for a treatment of Pell's equation), attempts to solve exactly or approximately polynomial equations and the analysis of algebraic curves. The Mathematical Association of America and the American Mathematical Society both produce books that can be read by secondary students.

§4. **Conclusion**.

In dealing with gifted students, the guiding principle should be to broaden the experience of the students at each level, and not to proceed to more advanced work unless it is carefully prepared for. Advanced mathematics involves more abstraction and generality, and so is inclined to increase the intuitive distance between the student and the mathematics, unless the intuition itself is enriched. There is a trade-off between the intelligibility of particular situations presented at a lower level and their capacity for inclusion in a broader sphere at a higher level. To appreciate the power and elegance of higher mathematics, and to exploit it judiciously, students need time and experience to develop comfort and facility with sophisticated matter.

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Disrupting gifted teenager's mathematical identity with epistemological messiness

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Abstract. Mathematics is widely perceived as a universal and uncontested discipline, contrary to the philosophy of mathematics literature. Other researchers have considered the potential role of philosophy in school, but there is little work with gifted students engaged with issues concerning the nature of mathematics. We developed a philosophy of mathematics unit intended to enlarge gifted students' perceptions of the nature of mathematics by exposing the uncritical and tidy rendering of mathematics within school math. Using a narrative methodology, we attended to gifted student's students' stories of relationship with mathematics, based on the premise that a person's relationship with mathematics is inextricably woven together with their identity. In this paper, we will focus on the experiences of three gifted teenagers during our philosophy of mathematics unit. We found that these students were disrupted and compartmentalized their school math and philosophy of mathematics experiences and beliefs. We conclude that substantive experiences with the nature of mathematics should be a regular component of school math.

Key words: philosophy of mathematics, gifted high school students, mathematical identity, narrative

INTRODUCTION

> For me, yeah I don't like it – grayness in math. I think of math as right or wrong (Dorothy, a high school student in the IB program).

It is well known that mathematics is perceived as a universal and unquestioned body of knowledge. This positioning of the nature of mathematics in wider society is reflected in curricular documents and in the teaching of mathematics. Curriculum recommendations do not, to our knowledge, ever refer to possible philosophically-based goals or learning outcomes (see, for example, National Council of Teachers of Mathematics, 2000; Western and Northern Canadian Protocol, 2006; Manitoba Education, 2008). Teachers, en masse, believe that mathematics is absolute (at a superficial level), and reproduce these beliefs among their students (Philipp, 2007).

Contrary to the popular positioning of mathematics in school math and the wider society, philosophers have debated the epistemological status of mathematics at least as far back as Socrates. These debates revolve around the questioning of mathematics as an absolute body of knowledge and are far from resolved. Various fallibilist positions have been developed by mathematicians (e.g., Davis & Hersh, 1981), mathematics educators (e.g., Ernest, 1998), philosophers (e.g., Lakatos, 1976), and cognitive scientists (e.g., George Lakoff & Nunez, 2000).

Given the premise that school math is presented as a neat, tidy and undisputable collection of facts, we developed a "messy" conception of the nature of mathematics, and then developed activities intended to explore this messiness with gifted teenagers. Our goal was to use messiness to expand gifted high school students' conceptions of the nature of mathematics. Not surprisingly, given many years of exposure to a narrow and tidy vision of mathematics in school, the gifted students we worked with struggled to make sense of mathematics as messy.

Philosophy based programs of study for children and young adults are not a new idea. For example, the Philosophy for Children (P4C) program was initiated in the seventies by Lipman, premised by the idea that children and young adults can think philosophically, and so philosophy should not be relegated to college-level study (Lipman, Sharp, & Oscanyan, 1980). These programs also tend to share the following qualities: (1) pedagogy is rooted in open dialogue, where a context, such as a story (e.g., Lipman, 1988;), or a story beginning (e.g., Matthews, 1984) is used to trigger a teacher-facilitated discussion of a philosophical issue; and (2) content is usually focused on general philosophical issues such morals, ethics, truth, and rarely considers discipline-based issues. The effects of these programs have been well documented. In general, these programs improve the thinking (e.g., Naji & Ghazinezhad, 2008) and other curriculum-based skills of students (e.g., Trickey & Topping, 2004).

Specific issues arise when considering exposing children and young adults to ideas from the philosophy of mathematics. Given that mathematics is perceived as (superficially) absolute within school math and by the wider society, what is there to discuss philosophically? Daniel et.al. used the P4C model to develop a philosophy of mathematics program (called P4CM) for children and young adults (Daniel, Lafortune,

Pallascio, & Schleifer, 1999). Stories with mathematical content were used to trigger open dialogue concerning philosophy of mathematics. They found various kinds of evidence that participation in P4CM is beneficial; for example, negative attitudes toward mathematics are reduced (Lafortune, Daniel, Pallascio, & Schleifer, 1999). Others have successfully implemented variations on P4CM. For example, while working with junior high students, Martin (2008) used a story that raises the issue of making a perfect cube to trigger a conversation about whether a perfect cube actually exists; the ideas within the conversation of these students were consistent with ontological views of Aristotle and Plato.

For all of these philosophical programs, questions remain concerning the process of students' development of enhanced thinking. In particular, while participating in open philosophical dialogue, students would be individually making sense of philosophical issues. There would likely be changes in their informal, implicit personal philosophies. What sorts of changes might occur and how do they occur? These questions apply equally to gifted and the general student population. In particular, it is not clear in what ways gifted students would respond to a unit of activities focusing on issues concerning the philosophy of mathematics. For example, we found various positioning of gifted students toward a philosophy of mathematics unit, including confusion, resistance and engagement (McMaster & Betts, 2007).

We wondered if the gifted students we work with would be more or less open to entertaining alternative visions of mathematics. Would they be responsive? Would their gifted abilities contribute to or hinder their responsiveness? We considered this research to be a first foray into these questions, and therefore deliberately decided to take an exploratory approach. Because we hoped to expand students' perspectives of mathematics, because we believed gifted students would be able to handle ideas that would appear foreign to their past experiences with mathematics, and because our activities come near the end of a course on philosophy, we did not expect to be disruptive of their relationship with mathematics. Our efforts disrupted student relationships with mathematics, but it is unclear whether their perspectives were enlarged in a stable way. In this paper, we will look closely at how three gifted students adapted to the disruptions triggered by a "messy" rendering of the nature of mathematics. We will suggest that

these students navigated the disruption by compartmentalizing their experiences, which likely allowed them to protect their identities in relation to mathematics.

RESEARCH METHODS

In this research project, our goal was to expand our IB student's appreciation of mathematics. To detect this goal, we used a narrative methodology (Clandinin & Connelly, 2000) – we sought to detect student's stories of identity in relation to mathematics. Narrative assumes that we use story to make sense of experience and that experience is storied (Clandinin & Connelly, 2000). Hence, we looked for student stories that suggested how they were positioning themselves in relation to the ideas presented during our philosophy of mathematics unit.

A narrative approach is suitable to the nature of our research questions. This paper represents results from an initial project concerning gifted student's positioning of self and others that are triggered by experiences with the nature of mathematics. We are less concerned with what students know or learned about mathematics or the nature of mathematics. Rather, we sought to understand the role of their identities (as learners and gifted students) as they struggled with novel ideas concerning the nature of mathematics. Our focus is on experience and identity; hence the uses of a narrative approach. In what follows, we describe the participants and their context for this study, methods of data collection and analysis, and our philosophy of mathematics unit based on epistemological messiness.

Participants and context

The students we worked with were enrolled full time in the International Baccalaureate (IB) program, and in their final year of high school. We consider these students to be gifted because they are high performing academically and highly motivated to be successful. The IB program is a "demanding two-year curriculum that meets the needs of highly motivated students" (International Baccalaureate Organization, 2005-2009c), and so it is considered an advanced placement program of study, attracting students with the highest grades in regular studies. At the very least, all the students were

academically precocious, based on grades. The three students we focus on in this paper are among the best in the IB program at this school. Dorothy is a multi-sport athlete with high marks in all subjects, and scored in the top 15% in English among all IB programs in the world. Mary consistently receives the highest grades in all subjects among all students in the IB program in her school. John scores high marks in all courses, and is considered brilliant in math and science by his teachers.

The IB program is implemented by high schools around the world, all following an academically advanced and standardized curriculum (International Baccalaureate Organization, 2005-2009b). The IB curriculum includes a course called Theory of Knowledge (ToK), which focuses on ways of knowing, including epistemological issues specific to major disciplines (International Baccalaureate Organization, 2005-2009a). The philosophy of math unit within ToK is an ideal location to introduce the notion of epistemological messiness. All students who participated in this project were enrolled in the Theory of Knowledge course, as well as other IB courses that would be considered advanced versions of standard high school courses, such as Math, Science, and English. The philosophy of mathematics unit came near the end of the ToK course, so general ideas (e.g., Plato's Forms, aesthetics) where available to apply to the particular case of the discipline of mathematics.

Data collection and analysis

Stories of student identity in relation to mathematics were constructed from data collected before, during and after the philosophy of mathematics unit. Before beginning the unit, we interviewed each participant, seeking to establish their appreciation of and attitudes about mathematics. These interviews revealed what we expected: mathematics is absolute and so why would there be a need to consider philosophical aspects of mathematics. Thus, we knew at the start of the unit that students would tend to story the experience with narratives such as "math is inaccessible" and "math is black and white." We speculated that these stories would be intimately tied to how they made sense of ideas. For example, Plato's Forms may prop up a student's identity of "yes, math is inaccessible." We also suspected that students would need to negotiate tensions between ideas developed during the unit and their life of experiences with mathematics dominated

by just one vision of mathematics, namely, math is a perfect and uncontested body of knowledge.

During the unit, we asked students to write a reflective journal at the end of each class. These served a pedagogical purpose: they provided us with insight on student thinking for assessment purposes, allowed us to provide feedback to students for the purpose of encouraging further elaboration of their ideas, and were used to showcase student ideas in subsequent classes. The journals were also used for research purposes. They became a source of data for detecting student's stories of identity in relation to mathematics. We also kept field notes of interesting conversations that occurred during the classes, which also served as a source of data.

After the unit was complete, we selected ten students to participate in an in-depth interview. We used two criteria to select candidates. First, we sought candidates that seemed to display exceptional giftedness. Although this tended to correlate with grades, we looked for students who displayed exceptional thinking during classes, such as an ability to develop an idea or the soundness of their ideas. Second, based on the journals and field notes, we tried to select candidates with differing reactions toward the unit. For example, John aggressively accepted Formalism throughout although began to consider Embodiment at the end of the unit, whereas Mary quietly embraced Platonism throughout, whereas Dorothy seemed undecided throughout but tentatively considered Proofs and Refutations (see next subsection for descriptions of these philosophical positions). The final interview lasted about an hour, was open ended, and focused on encouraging and challenging students to describe and develop their views concerning the nature of mathematics.

We decided to describe the stories of three students, Dorothy, Mary and John. We believe that each of these students are quite different, and taken as a whole are reflective of the diversity of the class. They are also among the most gifted of the students who participated. And yet, despite the diversity suggestive of our small sample, we found a common theme in their stories, namely, a navigation of disruption of their identity in relation to mathematics. In the next sections, we will try to illustrate these stories of navigation of disruption.

Data analysis proceeded in two phases. First, we focused solely on developing the story of each of the three students (and the other students who participated in the post-interview), without comparison. We each developed a story and then, through a process of dialogue and reexamination of the data, we came to an agreement on each story. Our differing perspectives as regular-teacher-of-these-students and researcher-from-outside-the school were complementary, and, we believe, adds to the trustworthiness of our interpretations. From these stories, we selected three for further analysis. We then looked for themes across the stories of the three case participants. It was during this second phase that we came to agree on the theme of disruption, and when we began orienting their stories as one of navigating disruption.

<u>An epistemologically messy philosophy of mathematics unit</u>

Numerous mathematicians have described the work they do using journey or process metaphors, which belie the neat and tidy presentations of mathematics found in most expository texts, including school math text books. For example,

> *When asked what it was like to set about proving something, the mathematician likened proving a theorem to seeing the peak of a mountain and trying to climb to the top. One establishes a base camp and begins scaling the mountain's sheer face, encountering obstacles at every turn, often retracing one's steps and struggling every foot of the journey. Finally when the top is reached, one stands examining the peak, taking in the view of the surrounding countryside and then noting the automobile road up the other side! (Kleinhenz, 2007)*

What is described in math textbooks and taught in school math classes is the "automobile road up the other side," which clearly hides most of what it means to do math. And yet, if students are to appreciate math (a goal found in all curriculum documents we are aware of!), then they should experience the doing of mathematics. The quote above begins to question a tidy rendering of doing mathematics – there are frustrations, false starts, back tracking, and numerous other accomplishments and setbacks along the way. At the very least, problem solving is more than a linear sequence of steps, and there is always more to do even after a problem is solved. This is a starting point for recognizing that school math experiences hide philosophical issues. It is with this rejection of the tidiness in the representations of school math that we use as a starting point for the nature of

mathematics as messy.

We wish to be critical of a tidy vision of the nature of mathematics that seems to be universally propagated by school math. Mathematics as a perfect and uncontested body of knowledge is a tidy position – there is no uncertainty and hence no messiness. But numerous philosophers have questioned the certainty of mathematics. For example, Davis and Hersh (1981), who are mathematicians, suggest that mathematics is a human endeavor, and hence subject to the same fallibilism as any human endeavor. Ernest (1998) developed a fallibilist position by drawing on a social constructivist perspective. Although school math does not explicitly present a philosophy of mathematics, its tidy enactment commonly engenders superficial absolutist positions among student's personal working philosophies.

We developed a philosophy of math unit based on exploring four distinct philosophical positions concerning the discipline of mathematics. Each of these positions were given credence as viable philosophies, where deeper explorations of each were intended to invite students to attend to and critique the tidy renditions of school math common to their mathematical experiences. First, we broadly distinguish between Absolutism (math is universal and infallible) and Humanism (math is fallible). We then developed two example positions for each broad category: Platonism and Formalism for Absolutism, and Proofs and Refutations and Embodiment for Humanism.

We gradually developed each of these positions through a series of activities, each activity usually built from a specific high school math context but examined from a philosophical perspective and with minimal attention to teaching the mathematics involved (we ensured that the mathematical concepts explored were familiar to students). A messy rendition for the nature of mathematics emerges in two ways: each philosophical position by itself carries numerous opportunities for critique of the neatness of school math, and the availability of multiple positions for the nature of mathematics is an opportunity to perceive the philosophy of mathematics as a contested body of knowledge. In what follows, we provide a brief description of each position, and of one example activity, to illustrate the content of our philosophy of mathematics unit [see Betts (2007) and McMaster & Betts (2007) for more detailed descriptions of our philosophy of mathematics unit].

Platonism is an absolutist position based on Plato's "allegory of the cave" and Plato's "Forms" (Govier, 1997). Forms are the ideal – a universal representation of the particulars accessible to humans. The cave allegory suggests that humans perceive only shadows of perfection – being the Forms – and are chained down, unable to be free of the cave to experience perfection. So, in particular, mathematical concepts, such as the fraction ½ and a drawing of a line, are but imperfect representations of the Form for a concept. School math pretends to present ideas as if they are ideal. A line is drawn as if it is a "perfect" line, rather than as a representation of a line that is good enough for the purposes of the current mathematical argument. Platonism can trigger critique of the neatness of school math because humans cannot access the ideal – the Forms – and hence must account for the imperfection of a human representation of a mathematical idea. Erdos, one of the most prolific mathematicians ever, was a proponent of Platonism (Hersh, 1997).

Formalism is also an Absolutist position, and is based on the premise that error enters into mathematics when its ideas are operationalized in human contexts (Hersh, 1997). For example, Russell's paradox arises because it is represented using language, and so is subject to the fallibility of language. Mathematicians such as Hilbert set out to formalize mathematics as a symbolic system independent of language (Mancosu, 1998). In essence, mathematics is a set of symbols and rules for manipulating these symbols, which have no meaning in the real world. According to the mathematician Hardy, only pure mathematics, mathematics that is unconcerned with application in the real world, is real mathematics (Hardy, 1992). Students can engage with the idea that mathematics does not come in a perfect package; rather, mathematicians have worked hard to remove error from mathematics. Hardy would argue that school math is not real mathematics, which can lead students to question the tidy renditions of school math as misleading.

One of the Humanist positions is based solely on the ideas developed by Lakatos in his book Proofs and Refutations. Lakatos used the historical development of Euler's formula to describe various iterations of the following process: conjecture, proof of conjecture, refutation of conjecture (e.g., by counter example; critique of proof, definitions, and/or axioms), leading to a new conjecture (by modifying definitions, axioms, and/or the actual conjecture) (Lakatos, 1976). This position is messy in two

ways. First, a result is sanctioned by the mathematical community not just by proof, but by a process of error detection and adjustment to account for the error. Mathematical decisions can be based on criteria other than logic, such as aesthetics. Second, a theorem can always come under scrutiny, even if it has been sanctioned as true by the mathematics community – in other words, we can never be 100% sure that a conjecture and its proof is true because another refutation may arise in the future.

The other humanist position, which we call Embodiment, is based on the ideas of Lakoff and Nunez. The reader should consult other writers for more detailed descriptions of embodied cognition in general (e.g., G. Lakoff & Johnson, 1999) and as it relates to mathematics (e.g., George Lakoff & Nunez, 2000). A key principle is that mathematical ideas start from our experience as humans and are built up via a series of metaphorical mappings. For example, the notion of continuity of the real number line comes from our embodied experience of motion. The real numbers is a discrete and infinite collection of numbers, but is also represented as a continuous line. We can manage these realizations of real numbers because we can experience continuous motion between two points, which is also a travelling of an infinite number of discrete points (e.g., the halfway point). The embodied experience of motion from A to B is metaphorically mapped onto the notion of an interval of the real number line, such as all real numbers from 0 to 1. An embodied vision of mathematics is messy because the idea that mathematics is universal and independent of humanity is completely rejected.

One of the activities we used near the beginning of the unit involved the circle. We asked the students to come up with more than one answer to the following question, and to be able to justify their answers: How many sides does a circle have? We know of 5 distinct and mathematically viable answers to this question, of which, we will describe three: (1) no sides because sides are straight and a circle is curved; (2) one side, which is the edge going all the way around the circle; and (3) infinitely many, because a circle is the limiting case of a regular n-sided shape as n approaches infinity (in the limit, there is an infinite number of sides, each of length 0). After generating a list of answers that seemed mathematically correct and trying to justify which answer could/should be the correct answer, we asked students to reflect and discuss what this situation means for the nature of mathematics.

This activity allowed us to develop several philosophical issues, based on the ideas of students. If, for example, we pick one answer, how do we know for certain it is correct, which allows us to point to a broad distinction between Absolutism and Humanism. The distinction arises because of the potential for opting for an answer that later turns out to be rejected – does this mean that mathematicians can eventually remove all error with careful analysis, or is mathematics a human endeavor so that it must be a fallible body of knowledge. Another issue arises concerning the inaccuracy that must arise in drawing a circle, which leads to the idea of a perfect circle and Plato's Forms. Finally, in the debate about which answers to accept, the issue of agreeing on the definition of a side arises, leading to a discussion of Lakatos' heuristic, where we consider the refutation of an idea through the contesting of a definition.

The circle activity is not immediately used to illustrate Embodiment. The Embodied position is difficult to develop because it is based on attending to subtle and taken-for-granted aspects of human experience. After an initial encounter with Embodiment that is not grounded in a mathematical context, we revisit previous examples for evidence of this position. For the circle example above, we wonder how to imagine how n-sided shapes of increasing n approach a circle but the circle doesn't disappear in the limiting case. How do we do this? We can't draw a circle as an infinite number of sides of length zero. But we can experience a circle as a continuous curving line that loops back onto itself, which is metaphorically mapped onto the limiting case definition of circle.

The example above also illustrates the pedagogical principles used to implement our philosophy of mathematics unit. We followed teaching ideas used in the P4C model. In particular, we sought to establish an environment where open dialogue concerning the nature of mathematics was facilitated. We encouraged students to state and defend philosophical positions. We resisted the urge to tell students about the philosophical positions of others or to suggest a "best" position. We considered it tantamount that students not consider us, as teachers, to be the final arbiters of a correct philosophical position or argument. Rather, we sought opportunities to validate student thinking by labeling their ideas as following a specific philosophical position. So, for example, when

a student argued that the circle example above suggests that mathematical results change over time, we suggested that their position was similar to that of Lakatos.

Students were encouraged to and did begin to develop their own personal philosophical positions concerning the nature of mathematics, rather than merely reproducing ideas from us. Our pedagogical emphasis on dialogue and refusal to sanction one philosophical position as correct led students to think deeply about the philosophical implications of the mathematical contexts we explored and about their own experiences with mathematics. Students had no difficulty applying to the case of mathematics ideas previously developed in their philosophy course. We noticed students suggesting that aesthetics is an important consideration, leading to an interrogation of proof as the only arbiter of mathematical truth. Students also were able to critique mathematical aspects of the contexts we presented. The idea that we must decide on the meaning of a side during the circle activity above was brought forward by the students without prompting from us. With some scaffolding we were able to help students notice superficial uses of Humanism and Absolutism. They, for example, began to recognize that falliblism is not the same as solipsism. Most significantly was the critical thinking inherent in the questioning of students: How do we gain knowledge of the Forms if we can only access the imperfect – if we are trapped in the cave? Why isn't it possible for mathematical results to be certain even though they emerge from human experience? These questions respectively represent a critique of Platonism and a synthesis of Absolutism and Embodiment. The students, in general, did engage with philosophical ideas, critique the neatness of school math, and begin to appreciate the philosophy of mathematics as a contested body of knowledge [see McMaster & Betts (2007) for further details].

RESULTS - DESCRIBING EACH STUDENT'S STORY OF NAVIGATING DISRUPTION

Our focus in this paper is not what students learned. We saw significant evidence that the students engaged with philosophical ideas, and take it as a given that they learned about philosophies of mathematics. Our focus on student identity leads us to notice how their thinking about the nature of mathematics was intricately woven together with their relationship with mathematics and their identity in general. Philosophical ideas

concerning the nature of mathematics are not evident within these students' prior experience, but their identities do matter as they take-in and work with these ideas, independent of their common experiences of math as a neat collection of facts and rules to follow. We found that many student's identities in relation to mathematics were disrupted. In previous work, we developed a general description and characterization of the disruption for all students who participated in our philosophy of mathematics unit (see McMaster & Betts, 2007). In this paper, we focus on a deeper description of how three students navigated the disruption of their identity in relation to mathematics. Each student started the unit believing mathematics was tidy and uncontested, and this belief was implicitly challenged by the activities during the unit. For each case, we try to establish a chronology for each story, based on their identity before the philosophy of math unit (as per pre-interview), during the unit (journals and in-class observations), and after the unit (as per post-interview).

DOROTHY

Before the philosophy of mathematics unit began, Dorothy expressed a joy for learning in general and math in particular. She expressed a real satisfaction in obtaining the right answer in math, which is a feeling that she has valued since early elementary school. She prefers certain branches of math, such as algebra and trigonometry, over others such as probability, because she doesn't like having what she calls "options" in probability. She also likes the process of working through a precise sequence of steps, where that process is clear and linear, a process she describes as "exactly how things fall into place." She wants to know how it works, but only wants it to work in one way. One right answer is what she wants. At the conclusion of this pre-interview, she states, though not rudely, that she just wants to "stop talking about math." Dorothy is interested only in the business of doing math that involves arriving at the right answer, and finds it uncomfortable and disconcerting to delve into the philosophical issues that accompany it. Dorothy carries a tidy rendering of the nature of mathematics: results are either right or wrong, there is one method for solving each problem, and each method is essentially an algorithm. Dorothy's implicit personal philosophy of mathematics is a superficial form of absolutism, propped up by her success with doing school math.

During the philosophy of mathematics unit, Dorothy freely expresses feelings of confusion. She is clearly not comfortable with this feeling, and tends to seek simplistic resolutions to the issues presented in class. For example, in an early journal she wrote: "It is just simpler to accept what we are told than to dispute it." She does not want to enter this debate at all, but since she is required to, she advocates for math that is simple and useful. In a later journal, she agrees with the Humanist position because it is simple, not just from a mathematical point of view, but from a human point of view. She wrote: "We must make adjustments to mathematical concepts for sheer simplicity in life." She objects to discussing the issues. This discussion frustrates her because she does not see its purpose. We believe this is because she has been indoctrinated to be very goal oriented, rather than to see the value of the messy discussion that we undertook in class.

After the philosophy of math unit, Dorothy was still reluctant to talk about the nature of math, and seeks to keep issues of messiness in math from threatening her prior experiences with a right/wrong dichotomy approach to math that she has been trained to value. She is willing to discuss various elements of math as long as those discussions do not threaten what she sees as math's fundamental operations, such as how formulas work, or whether they work. She is so comfortable with the right/wrong approach to math that she is only willing to discuss the issues underpinning math if she can consider them like separate issues that don't threaten what she feels she actually does in math class - different philosophical positions can call different issues into question but that doesn't mean that there is more than one answer to problems she is asked to solve in math class. Dorothy seems to be able to compartmentalize math to make these discussions feel safe to her; that is, we can talk about the philosophy of math as long as it doesn't prevent her from being able to seek the right answer to a math problem.

Throughout her post-interview, the interviewer challenged Dorothy to consider her position more critically, especially her tendency to agree with both Absolutism and Humanism as acceptable philosophies of mathematics. For example, she wants math to always produce one right answer (Absolutism), but she also wants math to be personal, under the control of the person performing mathematics, and describes the evolution of mathematical knowledge in humanist terms. She becomes aware that her position is

untenable, but this is not enough for her to change her position. We believe this illustrates just how deeply seated her ideas about math are.

When challenged further to defend her position, Dorothy's sense of security, sourced in following rules, procedures, using formulas, and getting to the right answer, is threatened. Throughout the interview, she repeatedly changes the topic, laughs, flirts, indicates she doesn't care about the issues raised, and tries to brush off the interviewer. For example:

> Interviewer: So there is no interpretation or opinion in mathematics?
>
> Dorothy: No [laughs].
>
> Interviewer: And yet you did talk about grayness coming into our philosophy of math?
>
> Dorothy: Ahhh okay [laughs].
>
> Interviewer: Your turn.
>
> Dorothy: Noooo [laughs] it shouldn't be my turn!

Given the frequency of these exchanges, we don't believe these comments are random – Dorothy is profoundly uncomfortable. At one point, she even makes a borderline inappropriate comment (i.e., "Men!") which targets the interviewer. We see this as additional evidence of her attempts to get out of the tight spot in which she finds herself. In addition, she sees the discussion itself as combative, even saying at one point to the interviewer "You win". The student is uncomfortable, defensive, and almost rude. A process that she has found satisfying and which has fed her self-concept regarding mathematics, the process of targeting and then obtaining a right answer, is being seriously challenged, and she is seeking ways to bail out. Her desire to avoid the issues altogether is closely connected to Dorothy's need for control. If the philosophy of math is integrated into her mathematical experience, she feels a loss of control, and literally doesn't know what to do. A huge source of her feelings of academic success becomes threatened, and her self-concept along with it. She tries to avoid issues from the philosophy of mathematics to keep it separate from her experience of school math.

MARY

Based on the initial interview, Mary believes that she does well in math because she works hard. She doesn't believe she is good at mathematics, although she does enjoy doing mathematics and that enjoyment seems to be strongly tied to her experiences of success with math due to hard work. Mary does what she is supposed to do in math class (and in all courses). She accepts the knowledge of the instructor at face value and without question – the math that is taught in school was developed by mathematicians in the past and is true without question. There is no need to question the results of mathematics. Mary is happy with this state of affairs because it is easy to figure out what responses are correct, so that she can be successful in terms of grades and feel good about her hard work. Mary perceives mathematical results as either right or wrong, which are sanctioned by teachers as the communicators of the work of mathematicians. This is a tidy rendering of the nature of math because of the simple relationship between teachers and mathematicians and the unquestioned acceptance of the ultimate and universal truth of the mathematics learned in school. For Mary, these beliefs about school math extend to all of mathematics.

Mary's conception of mathematics was challenged during the philosophy of mathematics unit. We presented the idea that mathematics might not be absolute and that humans might be inextricably implicated with what is considered true in mathematics. Now, Mary must face the possibility that the canons of mathematics, which she is so successful at reproducing on math tests, might not be so certain. She faces the possibility that the nature of mathematics involves uncertainty, which causes a problem for her desire to detect and reproduce right answers.

Mary adapts to the discomfort caused by epistemological messiness in two ways. First, she keeps mathematics at arm's length. For example, during the final interview, she said:

> I also agree with the fact that math has always existed and is not created by human beings or anyone else. When we, as humans, find out some new mathematical concept, we are really just discovering something that was always there.

This quote is representative of Mary's position in two ways. She rarely used "I" to state her position, and when she did, she reverted to "we" (which was much more common), as if to distance herself from the position. Further, the quote represents Mary's belief that math is ubiquitous – it is "everywhere." Mary deliberately places mathematics outside her personal experience, and the only reason she experiences math is because "we" cannot help bumping into it – it is needed for "us" to "survive." Keeping mathematics at arm's length is comfortable for Mary. It allows her to keep mathematics as objective and separate from us, which protects her comfortable acceptance of the absolutism of mathematics.

The second way Mary adapts to her discomfort during the unit is to be slow to commit to an answer or to sit on the fence. For example, in the first journal she wrote:

> It can be argued whether math is independent and can act alone or if it needs language to exist.

In the last journal, when asked to pick one of the four camps, she wrote:

> My philosophy of math is Platonism, as it is the philosophy of math that makes the most sense to me. I feel that there is not really one philosophy of math that is completely right.

In the first quote, she states a contentious issue, but will not take a position. In the second quote, she selects a position but makes a qualification. Throughout the final interview, she was slow to answer, tried to give short and non-committal answers, and would qualify with phrases such as "I'm not sure." The only idea that Mary would commit to was that "we" can never be "sure." She uses uncertainty in general to protect her belief that mathematics can be certain. She qualifies or doesn't commit because she is looking for the school sanctioned right answer to reproduce. The Theory of Knowledge course reinforced the idea that knowledge is never certain. Mary is doing what school has taught her to do, namely, to reproduce the right answer.

When Mary does commit to a position it is because there is a strong emotional connection to her zone of comfort with mathematics. In the final journal, when reflecting on whether school math has influenced her beliefs, she wrote:

> Although high school math has been a major influence on my beliefs of Platonism, I think my personal traits and the way in which I think also contribute

to my Platonist views. I like things that are black and white that give me definite answers. I do not want to be caught in a no man's land, as I will not know what to do because I will not know what the right thing to do is. Platonism tells me that concepts have definite answers. This is what makes me happy because I will know what I am doing, and can tell if I am doing the right thing.

A Platonist view of mathematics is a security blanket for Mary. During the final interview, when pressed on this issue, she admitted as much. Formalism is rejected because math loses its real life ubiquitous nature (she is perhaps worried that formal math is so abstract that she will no longer be able to understand it) – this is the safety of keeping math at arm's length. Embodiment is rejected because math is not separate from humans, and so she cannot maintain an impersonal relationship with mathematics. Proofs and Refutations is hedged by the possibility of finding absolute answers or the surety that "we" can never be sure. These are strongly emotional positions, in the sense that she feels strongly about keeping mathematics impersonal and separate. She selects Platonism because she feels strongly about wanting to feel happy about knowing there are right answers that she can correctly reproduce.

JOHN

John is extremely good at achieving 100% on math tests and exams. He was also one of the few participants who expressed a genuine love of mathematics. During his pre-interview, he stated, "I like the fact that in math you can derive an answer and be certain of it…" He admires the work of mathematicians, and feels a sense of pleasure when his ability to be the only one in a math class who can solve a challenging problem positions him as the "mathematician" of the class. A key word used in his pre-interview is "comfortable." He likes math because it makes him feel comfortable. He knows what to do, he's good at doing it, and he experiences satisfaction at the achievement of the one, unique answer. Math is at the center of his self-concept; he in fact claims it to be at the center of "everything." John has simplified the nature of mathematics by conflating what he does in school math with the work of a mathematician. He sees himself as a problem solver, and his success on math tests props up this perception of mathematics – he is comfortable with his perceptions. His comfort with school math generates a blind spot in

recognizing the potential differences between how math is rendered in school and how mathematicians experience mathematics.

During the philosophy of math unit, John's discomfort was minimal at first but slowly increased. After the first class he wrote: "I think about math today in another light, one in which I am not used to thinking." John is just a little bit worried because the first class has triggered the thought that the math he is comfortable with might not be the math of mathematicians. In subsequent journals, we find evidence of an increase in his concerns about mathematics. Later in the unit he wrote:

> I learned about the idea of embodiment today. I, however, don't buy it. I believe that we, as humans, despite our given restrictions within the reality by which we live, are capable of extrapolating our knowledge into areas and dimensions unprovable by our current capabilities. I still believe firmly in absolutism.

The words "however" in sentence two and "firmly" in the last sentence are not needed by John to express his ideas. Their presence suggests how important John felt it was to emphasize his position, and hence his increasing concern with the ideas presented.

Part of his discomfort is rooted in his respect for mathematics and mathematicians. During class discussions, we learned that John read about mathematicians and mathematics out of interest (not as required school reading). In one of his journals he wrote: "We <u>have</u> a problem. And a mathematician <u>must</u> [emphasis not added] be able to accept it." He admires mathematicians, but is discovering an element of being a mathematician that is outside his comfort zone. His discomfort with the ideas presented during class activities has increased. We believe this is because he has available to him increasing evidence that the mathematics he is comfortable with is not so neat and tidy. The one right answer he is certain exists for every problem and takes pleasure in finding has been challenged.

John also values critical thinking – he is curious about ideas but is also skeptical. We found this evident in his questioning and challenging disposition during class discussions. We also found this evident in journal entries. For example, from two of his journals:

> I feel that this is an extremely deep topic, in need of further explanation, and look forward to further exploring it.

And:

> There are always abstract exceptions to mathematics. It is indeed fascinating to wonder about it and analyze, realizing we might never truly achieve an answer. It is rather the thought process that makes it all worthwhile.

John values thinking for the sake of thinking. John believes that an idea must stand up to critique before it is accepted, and he wants to engage in such critical thought processes. He finds pleasure in engaging with ideas. Thus, when he found his ideas about mathematics to be challenged, he took this challenge seriously because he values critical thinking.

He must find a way to navigate the disruption in his comfort with the math he has experienced in school. He does this, with pleasure, through critical thinking. Although he adamantly agreed with Absolutism, through his skeptical challenging and questioning, he eventually found problems with both Platonism and Formalism. His initial reactions to both Proofs/Refutations and Embodiment was rejection because they represented a rejection of Absolutism. Now his skeptical disposition was to question and critique in order to find reasons to also reject these positions. But Embodiment, in his perception, was difficult to reject. We spoke several times after class about Embodiment – it was clear that his valuing of pure thinking was the essence of his curiosity and questioning. He wanted to make sure he understood in order to make sure the ideas could withstand critical evaluation. So, although he rejected Embodiment in his second last journal, his last journal started to describe a philosophy of math that he labeled "Embodied Absolutism." John is finding a way to protect his absolute vision of and experience with mathematics through his pure joy with pure thinking.

During the final interview, John's explication of his ideas continued, predicated on his joy of engaging with ideas. He sees Embodied Absolutism as a philosophical project – a thought experiment – in which the problem is deciding what is absolute and what is embodied. For example, John argued that although error may arise due to perception, and this is because of our embodiment, the concept that is perceived is still absolute. When challenged on this idea, he acknowledged that he might be wrong about "where Absolutism stops and embodiment starts." His post interview is singularly

focused on his recognition that there is still thinking to be done on his philosophy, and he is willing and happy to do that thinking.

John's comfort/pleasure with math is inextricably tied up with his comfort/pleasure with thinking, where the thinking he values is oriented by both curiosity about pure thought and by skepticism of all ideas. But absolutism is the one idea, at least at the ontological level, which cannot be challenged – there must be some absolutes. For example, he noted that the "…fact that we are embodied…is absolute." At the beginning of the post-interview, he described the ideas as fascinating (a word used several times in his journal as well). The philosophy of math has been a cerebral game, but he loves playing this kind of thinking game, and so his comfort with thinking about ideas "for the fun of it" protects him from the disruption of his comfort with school math. School math becomes compartmentalized – his experiences with school math remain separate and protected from his thinking about the nature of mathematics.

CONCLUSIONS

In summary, Dorothy starts from her comfort with the absolutism of mathematics, and then the messiness of philosophy of mathematics disrupts this comfort. To deal with this discomfort, she seeks simplistic answers. For example, she tries to simultaneously agree and disagree with a Humanist position. She tries to avoid philosophy of mathematics issues altogether. Epistemological messiness thwarts what she really likes about math, what she describes as its lack of "greyness." Dorothy experiences profound discomfort, and to protect her sense of identity in relation to mathematics, she compartmentalizes philosophy of mathematics to keep it separate from her experiences with school math.

Mary wants to maintain her identity with mathematics as an objective and separate body of knowledge with which she need not think or feel personally about. When the ubiquitous, objective and absolute mathematics that she is happy with (because she can successfully reproduce it for her teachers) is challenged, she feels discomfort. She does not want to face the prospect that a nature of mathematics, which she is happy with, might not be representative of mathematics. She protects her sense of identity by being non-committal, qualifying her answers, or keeping ideas at arm's length from her personal beliefs. This allows her to maintain a sense of success – if she doesn't commit,

she doesn't need to face being wrong. When she does commit, it is to maintain and protect a strong emotional connection to mathematics – that is, that she is happy with math and that would allow her to maintain an impersonal relationship with math.

John genuinely loves mathematics and pure thinking and these are intricately tied together. But it is skepticism that ultimately protects his identity in relation to mathematics. Although his comfort with school math is disrupted, he is critical of all ideas except the idea that there must be some absolutes. That there must be some absolutes and his joy of pure thinking leads him to synthesize absolutism with embodiment. This allows him to ultimately protect his identity in relation to school math because, in the end, it is pure thought that matters and is valued. Mathematics is based on pure thought. Descartes would be proud.

We would like to highlight several features of these three stories of navigating of disruption. First, a story of identity in relation to mathematics is intrinsically and fundamentally bound up with a story of identity in general. Dorothy's outgoing nature was the story of her sense of indecision in terms of the philosophies of mathematics that we presented. Mary's quietness is her way of seeking the answer that she will quietly embrace and, given the opportunity, reproduce on tests if her answer is the curriculum-sanctioned correct answer.   John's skepticism is fundamental to both his continued rejection of ideas but also his eventual acknowledgement of the skepticism of the Embodiment position.

Second, we believe that these students compartmentalize their disruption. The Theory of Knowledge (ToK) course is a mental game. For Dorothy, the game doesn't really matter. For Mary, she quietly plays the game by looking for the sanctioned right answer, which is absolute in math class and "there is no right answer" in ToK. John enjoys playing the mental game of debating ideas, but when in math class, he understands the procedure presented and gets mad at himself when he makes a "stupid" mistake on a test – the skepticism of TofK does not carry over into math class. We believe this compartmentalizing is important for maintaining a sense of coherent identity in relation to mathematics for these students. If they did not compartmentalize their experience of our messy philosophy of mathematics unit, their experience outside the unit would also

be disrupted, which could potentially undermine their status as the "smart" (math) students as sanctioned by their teachers

From the results of this project, we make several recommendations. It may seem that our philosophy of mathematics unit failed to enrich these gifted student's beliefs concerning the nature of mathematics – their beliefs were only disrupted leading to a compartmentalization of school math and philosophy of mathematics. But our philosophy of mathematics unit was only a two-week intervention compared to 12 years of enculturation into a narrow and tidy vision of math. These gifted students are focused on maintaining their success (read grades) in the IB program. In particular, their IB mathematics teacher was resistant to the ideas explored in our unit, so the "geography" of the school math course and ToK course may have contributed to the compartmentalization we observed. Given the social milieu of our project, perhaps mathematics as messy is too foreign for these gifted students to occasion change in their relationship with or perceptions of mathematics. Other renderings of mathematics could be used to enrich our messy framing of the nature of mathematics, such as by Byers (2007), who uses concepts such as mystery and ambiguity to describe key processes in the development of mathematical ideas. It may be that activities can be created based on mystery/ambiguity that resonate, rather than disrupt, while still occasioning richer conceptions of mathematics among these gifted students.

We also believe that the disruption and compartmentalization experienced by these gifted students is a curricular issue. All mathematics curricula, to our knowledge, state a major goal is for students to appreciate the products and processes of mathematics. And yet, a richer exploration of the nature of mathematics with gifted high school students is disruptive of their personal identity in relation to mathematics. This is because mathematics curriculum, as enacted in math classrooms, is singularly narrow in its tidy vision of mathematics. Most curricula try to point to the richness of mathematics through a list of mathematical processes (e.g., problem solving, reasoning) that should be infused throughout the teaching of all skills and concepts. But this list is easily framed by a narrow vision of mathematics. We believe that curriculum documents should endorse a "critical engagement" mathematical process, which signals teachers concerning some of the messiness of mathematics. The goal would be opportunities for students to

experience some of the messiness of mathematics as regularly as problem solving and throughout their K-12 school math program. A critical engagement mathematical process could be enacted throughout K-12, so that children/teachers are not enculturated/enculturating a narrow and tidy vision of mathematics.

The curricular recommendations above have significant implications for teacher professional learning. The expansion of mathematics curriculum to include messiness places considerable demands on all K-12 teachers, especially given the common belief among teachers and wider society that the nature of mathematics is uncontested and uncontestable. The professional learning of teachers is a significant concern because it effects all kinds and levels of teacher education, at a time when it is not clear how to effectively invite teachers into the current agenda (e.g., National Council of Teachers of Mathematics, 2000) to reform mathematics education; and because current calls for reform are effectively silent concerning philosophically-based goals or learning outcomes, so are insufficient based on our curricular recommendations. We consider these implications as a call for collective and action-based research that raises the status of philosophy of mathematics among all educational stakeholders.

Finally, this project raises questions for further research. Our research questions for this initial project were exploratory in nature and focused on gifted students. Subsequent research could consider more closely how student identity is related to student relationships with mathematics. For example, what is the relationship between expanded or disrupted perceptions of the nature of mathematics and success in school math? Would gifted abilities contribute to or hinder a student's responsiveness to issues concerning the nature of mathematics? More precise research questions are needed to expand the literature, and subsequent research could consider the general student population as compared to gifted students.

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The promise of interconnecting problems for enriching students' experiences in mathematics

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Abstract: The *interconnecting problem approach* suggests that often one and the same mathematical problem can be used to teach various mathematical topics at different grade levels. How is this approach useful for the development of mathematical ability and the enrichment of mathematical experiences of all students including the gifted ones? What are the benefits for teachers' and what would teachers need to implement this approach? What directions would further research on these issues take? The paper discusses these and closely related questions.

I propose that a long-term study of a progression of mathematical ideas revolved around one interconnecting problem is useful for developing a perception of mathematics as a connected subject for all learners. Having a natural appreciation for linking learned material, mathematically-able students exposed to this approach could develop more comprehensive thinking, applicable in many other problem solving situations, such as multiple-solution tasks. Because the problem's solutions vary in levels of difficulty, as well as conceptual richness, the approach allows teachers to form a strategic vision through a systematic review of various mathematical topics in connection with one problem.

General pedagogical ideas outlined in this paper are supported by discussions of concrete mathematical examples and classroom applications. While individual successful practices of using this approach are known to be taking place, the need for more data collection and interpretation is highlighted.
Key words: multiple-solution problems, connectedness of mathematics, constructions in geometry, teaching support of mathematically inclined students.

1. Interconnecting problems and giftedness in mathematics

Mathematically gifted learners differ from average learners in their ability to perceive and retain mathematical information (Krutetskii, 1976). Apparently, they possess a well-organized interconnected web of mathematical knowledge (Noss&Hoyles, 1996) which manifests itself in flexibility of handling data, originality of interpretations, ability to transfer and generalize mathematical ideas (Greenes, 1981), and creativity of approaches taken when problem solving. According to Polya (1973), besides extracting relevant information from the memory, "in solving a mathematical problem we have to construct an argument connecting the material recollected to a well-adapted whole" (Polya, p.157). This ability to logically organize and process mathematical information is yet another distinguishing characteristic of mathematical talent (Krutetskii, 1976).

A learner could be a *good exercise doer* but still be incapable of adjusting standard techniques for answering unfamiliar questions (see e.g. discussion in Greenes, 1981). In teachers' words, "some of them [students] who solve standard problems quickly and easily meet an impasse when solving problems requiring independent thoughts" (Krutetskii, p. 176). This observation implies that the goal of the teacher consists of helping a dedicated learner go beyond *instrumental* understanding secured by knowing mathematical procedures, and achieve *relational* understanding (Skemp, 1987) between different mathematical topics, which assumes connections of various mathematical ideas. "An ability to establish and use a wide range of connections offers students alternative paths to the solution. … with a formulation of each new connection … the likelihood of discovering a solution in enhanced" (Hodgson, 1995, p.19). The emphasis on making connections is important not only for the teaching of mathematically gifted learners but is becoming one of the core didactical principles of the modern mathematical curricula (NCTM, 2000).

Researchers distinguish several ways of manifesting students' higher ability: in quality of the product, in characteristics of the process, and as a subjective experience. There also exists a variety of possibilities to describe and study the phenomenon of creativity (see e.g. Sriraman (2004a) for a review of this topic). As for the driving force of mathematical creativity, interaction of ideas in the mind of the thinker is considered as one of the most important factors in this process (Ervynck, 1991). Consequently, some

authors proposed to measure flexibility of thinking and creativity in mathematics by the number of produced solutions to a given problem as well as the ability of the solver to switch between different representations of the problem (Krutetskii, 1976; Laycock, 1970, Silver, 1997). From this perspective, problems which allow multiple solutions present a promising tool for nurturing of giftedness and enhancement of the quality of teaching in general (Stigler & Hiebert, 1999; Fennema & Romberg, 1999). Leikin and her collaborators extensively studied *multiple-solution connecting tasks* which they define as "tasks that contain an explicit requirement for solving the problem in multiple ways" (Leikin & Levav-Waynberg, 2008, p.234). They view these tasks as a valuable tool for the examination of mathematical creativity (Leikin & Lev, 2007).

The approach considered in this paper also focuses on problems with multiple solutions but those problems are used with a different pedagogical emphasis. The idea is not to solve the problem in many different ways at once. Instead, one problem is used throughout a learner's development over a long period of time. Each problem's solution is considered from different perspectives as the learner builds his mathematical confidence over several years of schooling. In particular, problems connecting elementary and advanced solutions as well as various methods and techniques are valuable for this purpose. The intuition developed through elementary approaches to the problem may be used by the learner for a better understanding of more advanced methods and at the same time for making connections between the various approaches. While learners at different stages of their growth "may be able to solve a particular problem, the manner of solution and the consequences of long-term development of learning can be very different, moving from rigid use of a single procedure through increasing flexibility to symbolic operations on thinkable concepts" (Tall, 2006, p.200). Multiple-solution problems used to specifically support the progression of the learner are the subject of this paper.

I call a problem *interconnecting* if it possesses the following characteristics:

(1) allows simple formulation (without specialized mathematical terms and notions);

(2) allows various solutions at both elementary and advanced levels;

(3) may be solved by various mathematical tools from distinct mathematical branches, which leads to finding multiple solutions, and

(4) is used in different grades and courses and can be understood in various contexts.

Due to the wide range of difficulty levels of its solutions, the same interconnecting problem may appear at the elementary school level, and then in progressive grades until the advanced level. The students, familiar with the problem from their prior hands-on experience, will use their intuition to support the more elaborated techniques presented symbolically in the upper grades. This would allow students to see their old problem in a new light and interpret new methods in terms of an old and familiar example, and thus linking the new concept with the existing schemata. Rephrasing Watson and Mason's description of reference examples, an interconnecting problem is "the one that becomes extremely familiar and is used to test out conjectures, to illustrate the meaning of theorems" (Watson & Mason, 2005, p.7).

From a learner's standpoint, a problem is *interconnecting* if its solution has been understood by the learner from several conceptual perspectives after working on the problem over an extended period of time. This definition of interconnectedness does not only characterize a problem but also demands a continuous engagement and certain cognitive effort from a learner, suggesting that same problem can be interconnecting for one student but not yet for another. Thus, the possibility of identifying and developing mathematically gifted students is embedded in the definition of interconnecting problems. Once understood, an interconnecting problem may be used by the solver as a model of flexible thinking in another problem context. The possibility for creative solutions arises from the learner's familiarity with other interconnecting problems because this familiarity allows the learner to have a comprehensive grasp of the new problem. In the next section I discuss interconnecting problems in comparison with various types of other mathematical activities and teaching approaches.

2. The place of interconnecting problems among other teaching approaches

There are various types of mathematical activities students face during their lessons. Different activities have different learning objectives. For instance, mathematical exercises help students to develop proficiency with various standard techniques and rules. In contrast, recreational problems appeal to students' common sense and intuition. There are also problems which combine some features of both the exercise and recreational types. These problems, on the one hand, are very intuitive and on the other hand incorporate special knowledge in a natural fashion. Their elementary solutions may not be immediately apparent but when found they demonstrate how several basic facts can be useful in a non-routine situation. They help to activate and connect basic knowledge and allow the student to discover new relations and properties. According to Polya (1945) and Schoenfeld (1985), this type of problem plays a very important role in the development of a strong mathematical background of a learner.

Careful and meaningful construction of appropriate learning environments for gifted students is a difficult pedagogical issue. First, according to Diezmann & Watters (2002) in order to have a cognitive value for a learner, the mathematical task must have a level of difficulty appropriate for the learner, that is, it must be at the psychological edge between his/her comfort and risk-taking zones (Vygotski, 1978).In addition, if suitable learning-stimulating tasks are not given "at the right moment, then some intellectual abilities may not have the chance to develop"(Sierpinska, 1994, p.140). Students need to be challenged during all years of education because "when the student comes to study mathematics at the university level, the propitious moment [in his/her development] would have passed, and it may be too late for the teaching intervention to have any effect" (Sierpinska, 1994, p.140).

Tasks which require finding multiple solutions present a challenge not only for students but also for their teachers. Besides a general direction to employ different representations of the same mathematical concept (NCTM, 2000), teachers are insufficiently advised how to incorporate multiple-solution tasks in their lessons and how to assess their students' progress in solving them (Leikin&Levav-Waynberg, 2007). I suggest that familiarity of students with interconnecting problems during their entire educational process creates a culture of mathematical thinking that makes solving

multiple-solution tasks more accessible. Through interconnecting problem, students may acquire the habit of analyzing a given problem in multiple ways as a systematic approach to problem solving and learning mathematics.

In a way, the interconnecting problem approach complements the *strand of problems* approach (Weber et al, 2006; Powell et al, 2009). The strand of problems approach uses *isomorphic* problems (English, 1993; Hung, 2000; Maher & Martino, 1996; Sriraman, 2004b), which appear to be different but employ the same underlying mathematical structure, and allows students to develop "problem-solving schemas within a specific mathematical domain" (Powell et al, p.139). Both approaches employ Bruner's proposal of spiral curriculum, the view that curriculum should revisit basic topics and ideas learned over an extended period of time. This proposal correlates with the phenomenon of the spacing effect found in studies of memory: learning of fewer items in a longer period of time is more effective than repeated studies in a short period of time (Crowder, 1976). Thus reinforcement and revisiting is necessary in order to achieve fluency in understanding and comprehension of some material. But the revisiting can happen in different ways. In the *strand of problems* approach, the learner returns to the same mathematical idea or technique by solving a number of different problems. Here the challenge is to recognize that different problems have the same mathematical structure and thus the same method can be employed to solve all of them.

In contrast, in the *interconnecting problem* approach the learner always deals with the same problem but employs different mathematical ideas and consequently, methods to solve it. This leads to establishing links between different topics learned in mathematics curriculum. In sum, the two complementary approaches are based on different paradigms: *one problem linked with multiple ideas (or concepts)* and *many problems linked with one idea (or concept)*, which allows building a network of knowledge, especially if the approaches are used in a combination. This view is schematically presented in Figure 1.
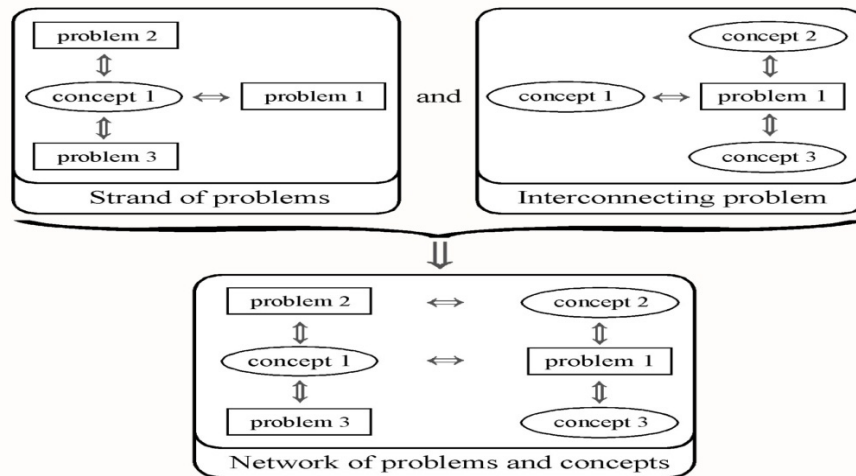
Figure 1: Strand of problems and interconnecting problems generate a network of concepts and problems.

In this respect, the interconnecting problem approach becomes an integral part of a teaching strategy aimed at creating a learning environment fostering mathematical intellectual growth and giftedness in particular.In the next section I give an example of interconnecting problem and examine its potential for learner's development.

3. An example of an interconnecting problem

As many other good mathematical questions, this problem arose from practical needs in an engineering design project. It was conveyed to me in a conversation with my friend, who also mentioned that the majority of his colleagues, former university graduates, could not find a reasonable solution to it. I took it as a challenge to illustrate that the problem can be solved at different levels of grade school education and thus serve as an interconnecting problem for a learner of mathematics.

Problem: *Start with an arbitrary angle ABC and point E inside the angle. The problem is to draw a circle tangent to the sides of the angle and passing through the point E (that is we need to construct the center and the radius of the circle).*

In this section I will consider four possible approaches to this problem that can be applicable at different stages of learner's cognitive development and related to different mathematical tools and representations of the question. The first approach is very

intuitive and can be demonstrated with manipulatives. This corresponds to *enactive* stage of problem representation (Bruner, 1966). Two other approaches, similarity-based and parabola-based, are geometrical approaches. They can be classified in Bruner's terminology as *iconic* because they involve reasoning based on the properties of the drawn objects. The third method develops further the idea of parabola-based approach by moving it towards algebraic formalization and rigorous description of the solutions in terms of their coordinates. The local network of knowledge build around this problem over time can be schematically shown in the following figure.
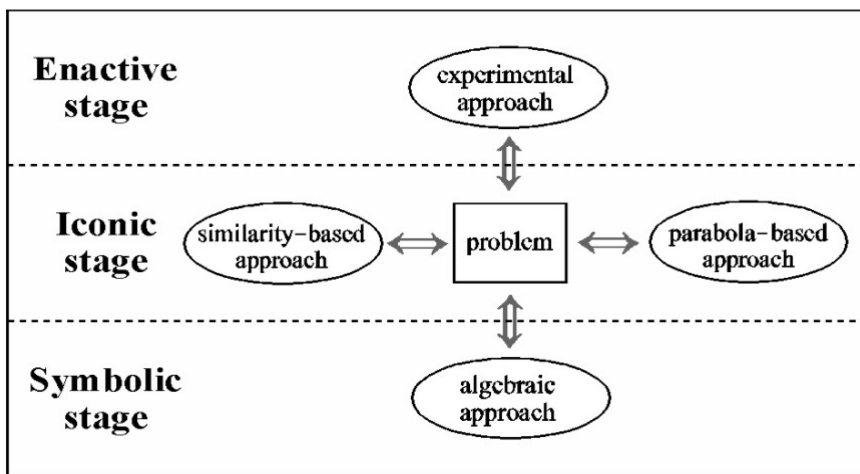
Figure 2: Approaches to the problem appropriate during several developmental stages.

Below I present mathematical details pertinent to each of the approaches. In this section I give a more algorithmic, step-by-step description of each method. The next section discusses ideas and concepts underlying these methods.

A. Experimental approach:

We bring into play a 3D model to help students understand that the solution to the problem exists. Consider a conical basket and imagine putting your finger on a point located inside the basket. Keeping the basket and the finger in the static position, ask if it is possible to find a ball or spherical balloon such that when it is placed in the basket the finger will touch the surface of the balloon. It is clear that if the balloon is too small, then

the finger will be far from its surface, while if the balloon is too big, the finger will deform or break the surface. Is it possible to get a balloon of the right size? The solution then is very intuitive: we place a small balloon and inflate it until it touches the finger. This experiment can convince students that the problem has a solution no matter what the size of the cone is and where the finger points. It does not define the radius and position of the center yet, but shows that it can be determined mechanically, doing the experiment with real manipulatives. Note that our original problem is a plane section of this 3D model.

The next two approaches are purely geometrical. They can be discussed with a child who starts to notice and understand properties of drawn objects such as circles, triangles, tangent lines, perpendicular segments, etc.

B. Similarity-based approach:

For this approach I refer to Figure 3.
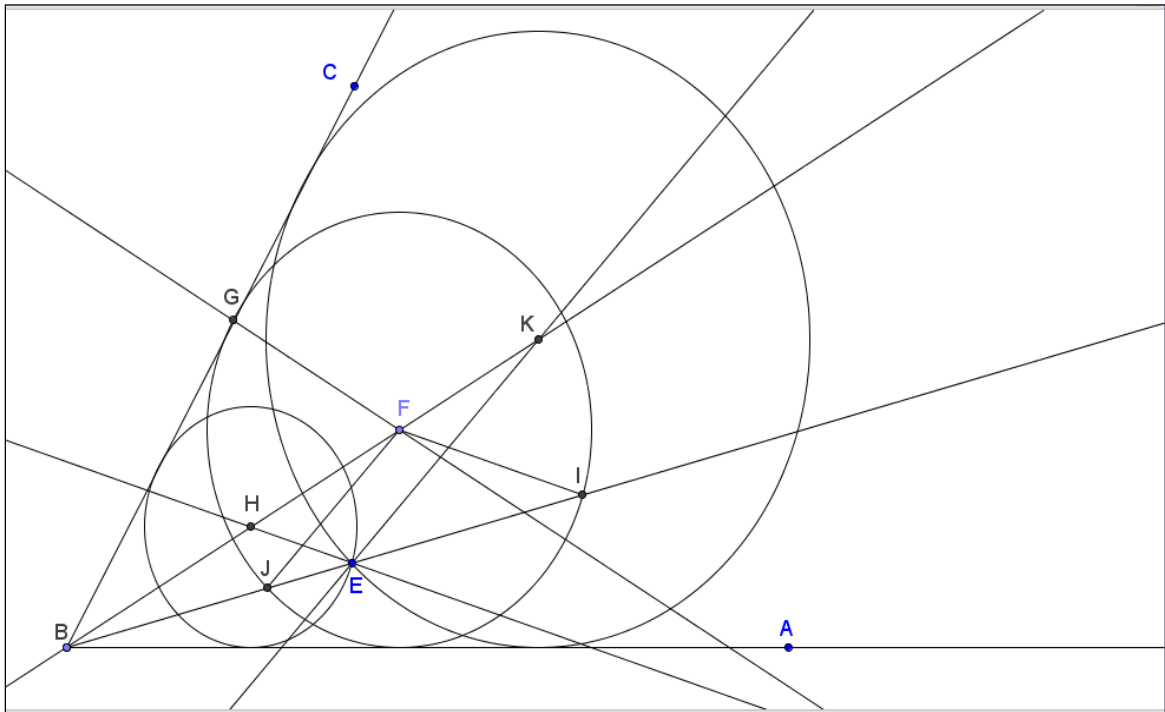
Figure 3: Pure geometrical similarity-based approach.

I. First we draw an arbitrary *auxiliary circle* tangent to the sides of the angle but not passing through the point E. We do it by the following steps:

1. Draw an angular bisector of ABC; we know that all circles tangent to the sides of the angle have their centers on this bisector.
2. We pick an arbitrary point F on the bisector as the center of the auxiliary circle.
3. We drop a perpendicular from the point F to one of the sides of the angle, BC.
4. The intersection point of the perpendicular and the side is called by G, and FG is the radius of the auxiliary circle.

II. Our second step is to connect the vertex B of the angle and the given point E by a ray BE. Since point E lies inside the angle, the ray BE intersects our auxiliary circle in two points, called J and I. The segments FJ and FI are radii of the auxiliary circle.

III. Our last step is to draw two lines through point E: one line is parallel to segment FJ and another is parallel to segment FI. These two lines intersect with the angular bisector BF at points K and H respectively.

We claim that points K and H are the centers of the required circles; their radii are segments KE and HE respectively.

This method is not applicable if E lies on the bisector BF or on one of the sides of the angle. The latter case is discussed in (Jones, 1998) along with an analysis of students' approaches to solve the problem. In the special case when E lies on the bisector BF we follow another approach, which is in fact easier (see Figure 3a).
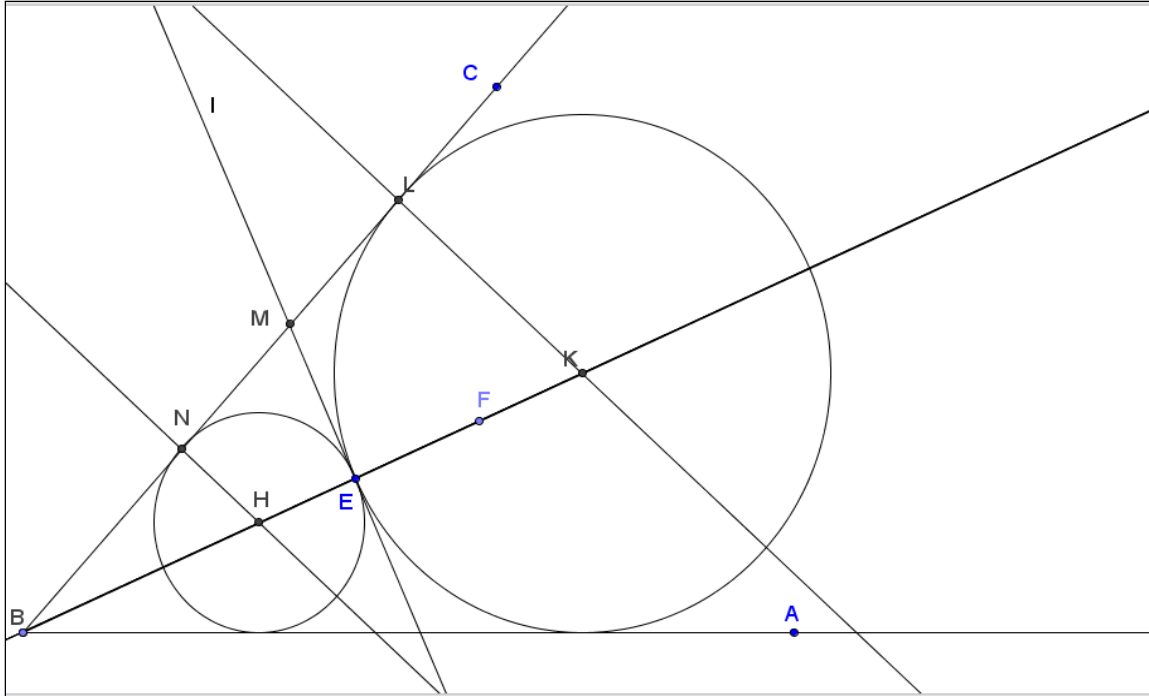
Figure 3a. Special case: point E lies on the angle bisector.

First, we draw a line perpendicular to BF passing through point E. This new line intersects the side BC at point M. We put points L and N on side BC such that LM=ME=MN. Two lines perpendicular to the side BC and passing through points L and N intersect the angular bisector at points K and H respectively. These are the centers of the required circles. Similarly, if E lies on one of the angle's sides, say, AB, we find the center of the circle as an intersection of the angular bisector BF and the line perpendicular to the side AB and passing through E.

C. Parabola-based approach:

    I.   We first draw the angular bisector of ABC.

    II.   Our second step is to draw a parabola with focus at given point E and the directrix being one of the angle's sides, say AB. Recall that *parabolais the set of points which are equidistant from given point* (called *focus*) *and a given line* (called *directrix*). Thus we draw it in the following way (Figure 4):
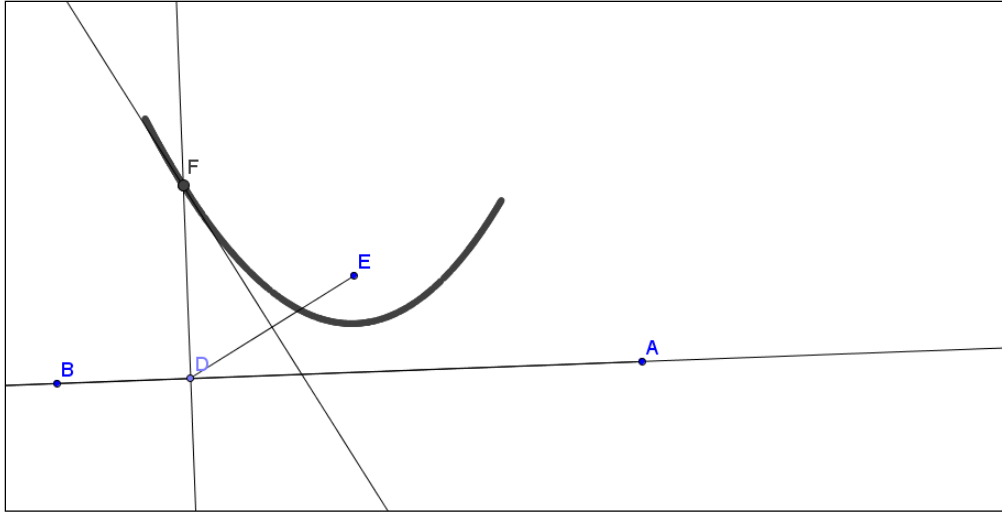
Figure 4: Drawing a parabola with focus at E and directrix AB. Here EF=FD

1. Take an arbitrary point D on side AB.

2. Draw a perpendicular to the side AB through point D.

3. Draw a perpendicular bisector to the segment ED.

4. These two lines intersect at a point F which lies on the parabola.

5. As D moves along the line AB, the intersection points form the parabola.

The parabola is a locus of centers of all circles which pass through point E and are tangent to the side AB. This parabola intersects with the angular bisector at two points, call them H and G (Figure 5). We claim that these two points are the centers of the circles we need to construct. Note that the second step, the drawing of a parabola with given focus and directrix, can alternatively be performed with a help of special mechanisms (linkages) known to ancient Greeks and widely used in the Middle Ages (see e.g. Henderson and Taimina, 2005, p.300). Modern geometry software such as *GeoGebra* has this tool as a built in option.
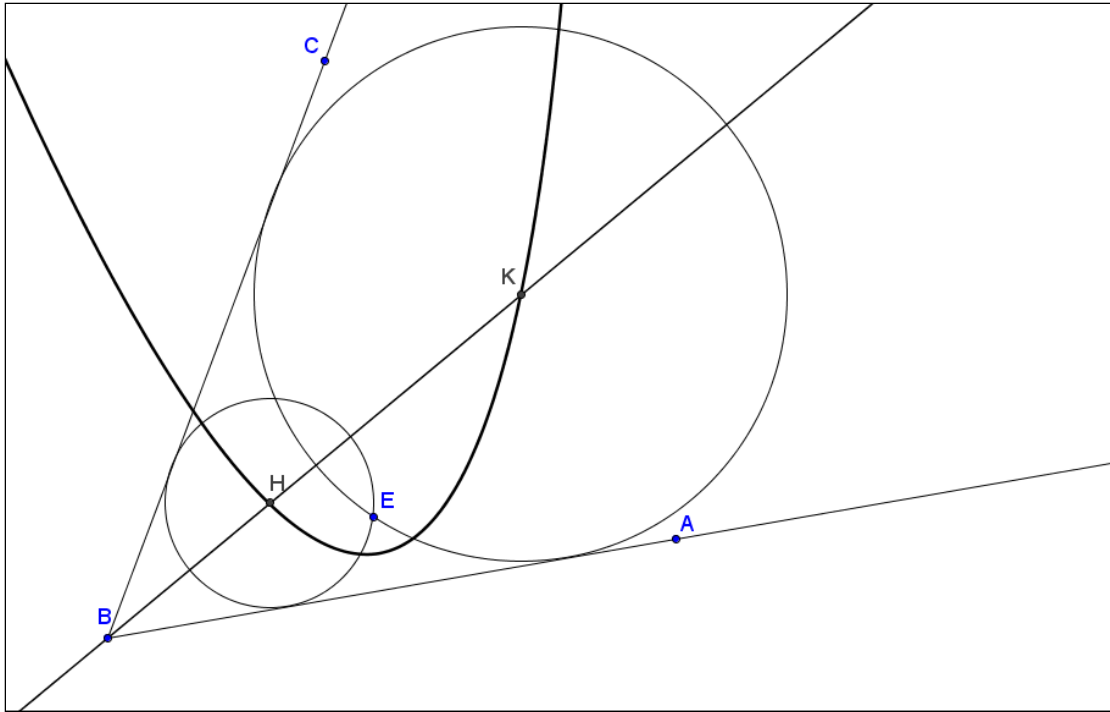
Figure 5: Approach involving geometrical definition of parabola.

The idea of the parabola-based approach could be converted into an algebraic method by a learner who knows how to describe geometrical objects such as lines and circles analytically, to reformulate the question in terms of related algebraic equations and solve those equations. We outline this approach in the following subsection.

D. Algebraic approach:

Let the angle measurement be α, where $0 < \alpha < \pi$. Consider a coordinate system in which the angle is formed by the ray AB with equation $y = 0, x \geq 0$ and ray BC with equation $y = x \tan(\alpha)$ in the first quadrant or second quadrant (Figure 5a). Let a given point E lie inside the angles and have coordinates (x_0, y_0). We are looking for the coordinates (x, y) of the center of a circle which passes through E and is inscribed in the angle. As we previously observed, the center lies on the angular bisector, and thus we have one relation $y = kx$, where $k = \tan(\alpha / 2)$. The ray representing the angular bisector

lies in the first quadrant. Another relation comes from the observation that the distance between the center and point E must be equal to the ordinate of the center. Squaring both values, we obtain $(x - x_0)^2 + (y - y_0)^2 = y^2$. We note that since both values, the distance and the ordinate, are nonnegative, squaring does not affect the roots of the equation.

Now, the system of two equations leads to one equation with respect to the abscissa of the unknown center, $(x - x_0)^2 + (kx - y_0)^2 = k^2 x^2$. After a simplification it becomes a quadratic equation $x^2 - 2x(x_0 + ky_0) + x_0^2 + y_0^2 = 0$, and thus we find two possible solutions $x = x_0 + ky_0 \pm \sqrt{2kx_0 y_0 + y_0^2(k^2 - 1)}$, which correspond to the abscissas x_1 and x_2 of the centers H and K of the two circles. Consequently, the ordinates y_1 and y_2 of the centers are $y = kx = k(x_0 + ky_0 \pm \sqrt{2kx_0 y_0 + y_0^2(k^2 - 1)})$. By construction we have y_1 =EH and y_2 =EK. An analysis of these formulas reveals the cases when there is only one solution possible: when point E lies on the side of the angle, that is either $y_0 = 0$ or $y_0 = x_0 \tan(\alpha)$. In the first case, the center has coordinates (x_0, kx_0), and in the second we get $(x_0(1 + k^2)/(1 - k^2), kx_0(1 + k^2)/(1 - k^2))$.

Also, note that the formula simplifies when point E lies on the angular bisector, i.e. $y_0 = kx_o$. Then we obtain $x = x_0(1 + k^2 \pm k\sqrt{1 + k^2})$, $y = kx_0(1 + k^2 \pm k\sqrt{1 + k^2})$.

This approach is essentially an algebraic realization of the second geometrical approach, C, based on the intersection of a ray with a parabola.
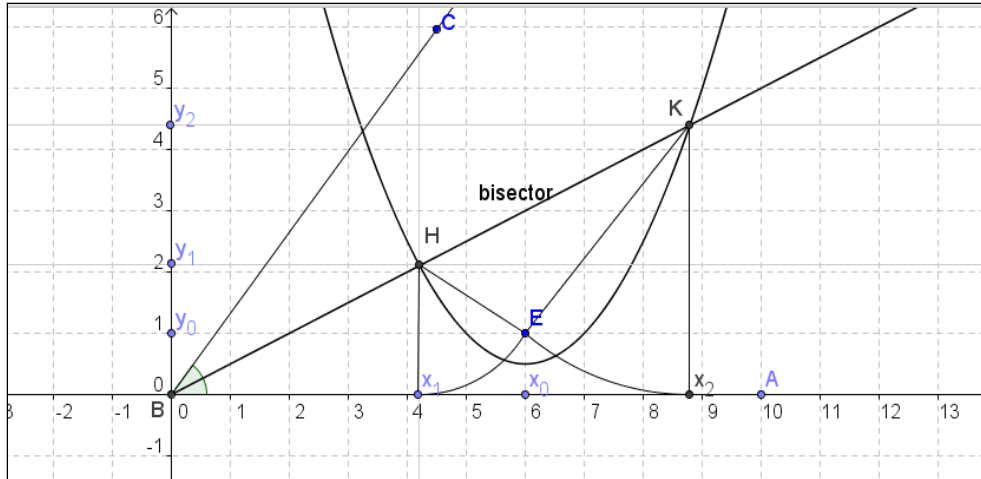
Figure 5a: Algebraic approach: Graphs in the coordinate plane.

The parabola, which consists of centers of all circles passing through E (x_0, y_0) and tangent to the ray $y = 0$, $x \geq 0$ has equation $y = (x - x_0)^2 / (2y_0) + y_0 / 2$ because its focus lies at E and the x-axis is its directrix. Together with the equation of the ray $y = kx$, this yields exactly the same quadratic equation as we have analyzed above in approach D.

4. Discussion of the key ideas of each of the four approaches.

Gifted students often grasp the formal structure of the problem and produce their solutions from exploration of certain key ideas associated with this perceived structure (Krutetskii, 1976). Polya (1973) distinguishes between the stages of designing a plan in problem solving and implementing the plan. The design is based on the conceptual grasp of the problem situation, whereas its implementation requires more of instrumental knowledge. Since identification of concepts and ideas relevant to a given problem is essential for the solvers' success, training of able students must include a deep analysis of each solution accompanied by the explicit identification of its main ideas. Observe that approaches B, C, and D, if presented to a student as such, will indeed guide him/her to the right answer. Yet, without an appropriate reflection by the learner, without identification and understanding of the reason for each step of the construction, the solutions remain useless for learning to solve problems in general. In this section I listsome ideas and concepts associated with more algorithmic step-by-step solutions presented in the previous section.

The approach A based on the experiment with an inflating balloon is not quite a solution of the problem but it plays an important role in the exploration, visualization and internalization of the situation. It shows that a solution exists and can be found as a result of a continuous process. Embedding this problem in 3D, we allow for a physical realization of the question. Similarly, using modern dynamic geometry (or engineering) software one can easily perform the task approximately just by a trial and error method in the interactive 2D environment. The size and position of the circle can be continuously adjusted in order to obey the requirements of the problem. Most of students (and engineers!) would employ this approach sufficient for a particular configuration. Thus it may take some effort to convince them to find a solution for a general configuration based on mathematical concepts and ideas. Some of them are as follows.

Each of the other three mathematically more advanced approaches B, C, and D uses the fact that the *center of the circle inscribed in an angle lies on the angular bisector*. This observation is essentially based on one's *embodied knowledge* because it refers to the axial symmetry of the geometrical figure and may be demonstrated to a child by folding the picture along the angular bisector. In addition, every approach has its key mathematical ideas, which I outline below.

The fact that *similarity results from dilatation (or uniform scaling)* is the key idea of the first geometrical solution (approach B). Figure 6 shows two circles inscribed in an angle. An inner ray started at the vertex of the angle intersects each of the circles in two points, I, J and K, L respectively. Triangles IJD and KLF, formed by the points of intersection with the ray and the centers D and F of the circles, are similar. Again, one can appeal to the embodied cognition, the natural sense of geometrical perspective, to view the second circle as a magnified copy of the first. This view implies that the sides of the triangles are parallel, which forms the basis for the construction employed by approach B.
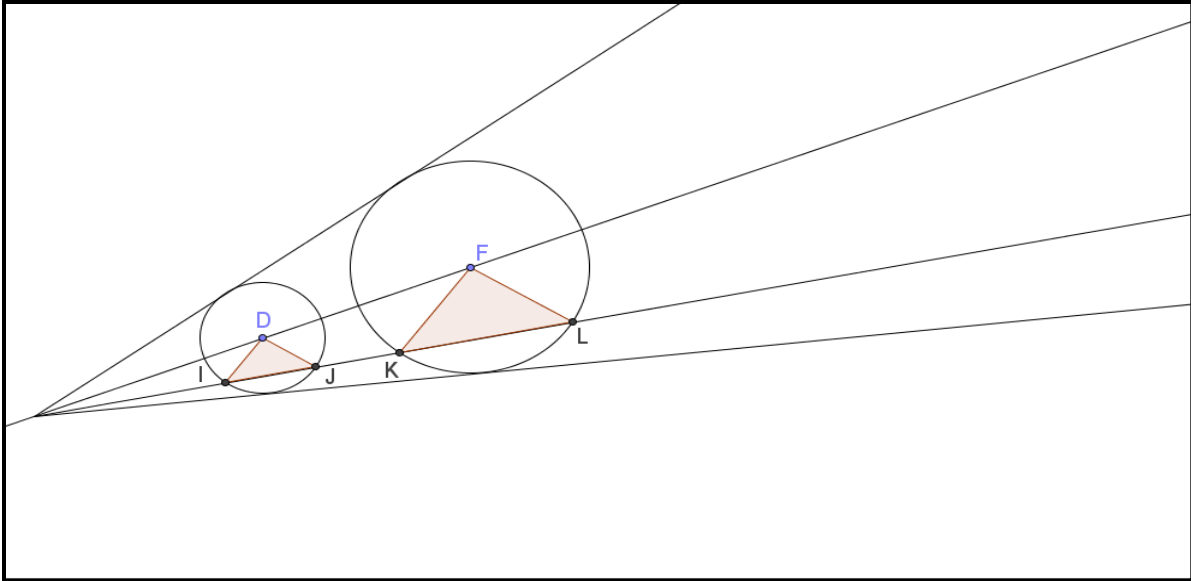
Figure 6: Two similar triangles IJD and KLF viewed as a result of dilatation.

   The following key ideas form a foundation for the solution with a parabola (approach C): the set of all circles inscribed in an angle form a family; their centers lie on the ray which is the angle bisector. Similarly, the set of circles passing through E and tangent to one side of an angle form another family; their centers lie on a parabola with focus at E and the directrix being the side of the angle. The center of the required circle is at the same distance from the angle's sides as it is from the given point E, thus the *elements common to both families give the required circles*.

   The algebraic solution (approach D) is based on the following key ideas: In an appropriate system of coordinates, an equation of the angular bisector involves a homogeneous linear function with slope expressed via the value of given angle. The distance between two points given by their coordinates is calculated by the Pythagorean Theorem. This leads to the equation of a circle, which is a set of points equidistant from one given point, its center. *In order to find intersection points of two curves, one needs to solve a system of equations describing the curves.*

   Note that in this paper I only listed elementary solutions accessible for students in grade school. One may also identify some approaches from university mathematics curriculum, e.g. methods of complex analysis, relevant to the problem. But even if solved by elementary methods, we see that the problem offers a range of mathematical ideas to be explored. These ideas become connected as learners discover them one by one in a

course of continuous engagement with the problem. Furthermore, this long-term commitment to the same problem helps to develop students' "capacity for work on one interesting problem for a long period of time", which was found to be one of the characteristics of "creative-productive giftedness in mathematics" (Velikova et all , 2004). If we want our students to make sense of mathematics "we cannot expect any brief program on problem solving to do the job. Instead we must seek the kind of long term engagement in mathematical thinking" (Resnik, 1988, p.58), and this thinking can be organized around an interconnecting problem, its possible solutions and their interplay.

I conclude this section with an illustration of the effect of such an interplay or interconnectivity of ideas employed in different solutions. The following geometrical fact emerges from a comparison of approaches B and C.

Theorem. *Consider an arbitrary circle and parabola drawn in such a way that the same line is tangent to the circle and is the directrix of the parabola, and both the circle and the parabola lie on the same side from the line (see Figure 7). Pick arbitrary point A on this line. Let O denote the center of the circle and F the focus of the parabola. Assuming that line passing through point A and O intersects the parabola in two points, call points of the intersection D and E. Assuming that the line passing through point A and F intersects the circle, call points of the intersection B and C. Then segments FD and CO are parallel and so are segments FE and BO.*
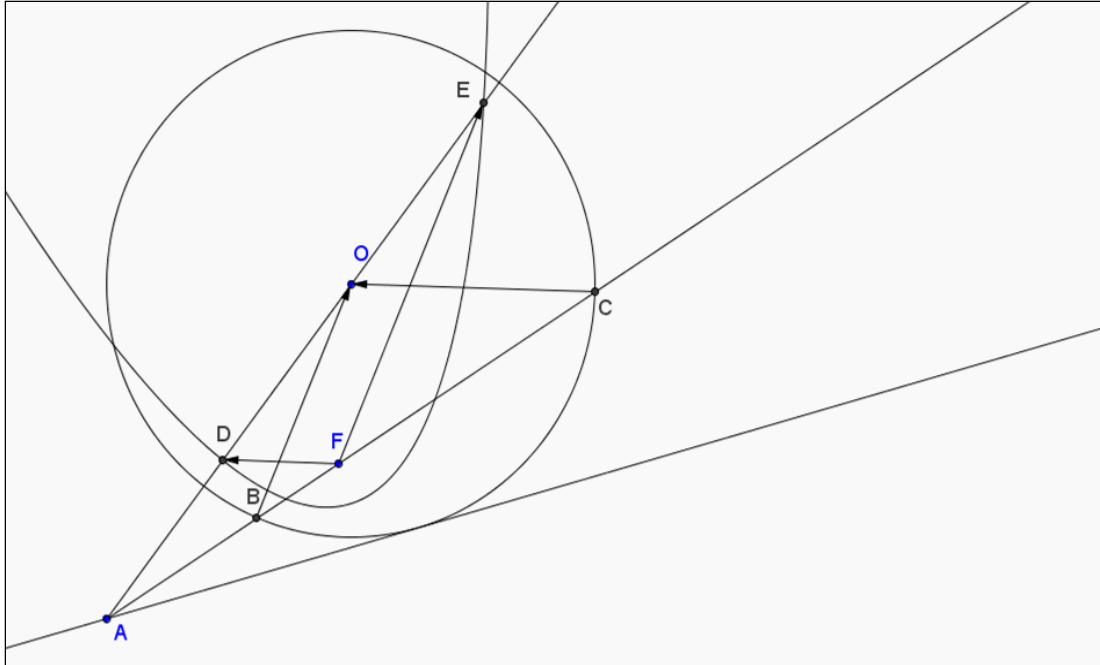
Figure 7: New theorem emerged from approaches B and C to the initial problem.

Proving this statement would be a challenging task for a majority of secondary school students. It would constitute a good question in a mathematical contest and thus can be used for identifying and fostering mathematical giftedness. Note however, that the statement becomes obvious if one identifies points D and E in Figures 7 and 6 with points H and K in Figures 3 and 5, or in other words, if one connects the ideas learned in two approaches to our initial problem. We leave it for the reader to reproduce the proof in full details. While doing this, the reader is advised to focus on his/her own experience and observe how familiarity with an interconnecting problem may lead to understanding of new mathematical facts in the process of rewiring various mathematical ideas.

5. Teaching issues related to interconnecting problems

Mathematics' teachers can play a pivotal role in helping students make connections. Teachers' commitment to this role is reflected in how they select curriculum materials, express personal interest in solving problems, explore and learn new connections in mathematics, negotiate meaning, and search for adequate pedagogical approaches (Koshy, 2001, p.123). The success of the interconnecting problems approach implementation depends on mathematics teachers' readiness to implement it in general, and as a method of nurturing mathematical talent, in particular.

Today's teachers have access to many problems and mathematical activities through books, Internet, journals, conferences, and other channels. Thus, it is unreasonable to say that the teachers are in need of more problems. But precisely because the number of available problems is large, teachers necessitate a systematic approach which would help them select problems appropriate for creating a coherent and connected representation of mathematical ideas for their students. By making this choice teachers would need to deal with such issues as ensuring that problems make mathematical sense, are clear and non-ambiguous. But the real challenge the teachers face is not just to pick a good problem and discuss it with the students, but also let the students experience usefulness of previously learned methods as well as develop an understanding of needs and possibilities of more advances approaches. Interconnecting problems also allow teachers to form a strategic vision and use it in their choice of tasks and actions in a classroom.

However, to be able to successfully implement the interconnecting problem approach, and especially if teaching a gifted group, teachers would benefit from (Barbeau et al., 2010):

- Having personal experiences of problem-solving (in particular, having experience with multiple-solution connected tasks and ability to identify the place of each solution within mathematical curriculum) and investigations to draw upon. This would also help teachers to distinguish the markers of giftedness from just getting good marks in standard assessments or memorizing and following procedures diligently.

- The ability to accept that some of the pupils they encounter will indeed be quicker and more intelligent than they are, but also that they have a role in nurturing whatever talent they find; put more emphasis on modeling the process of problem solving by their own example of thinking out loud rather that just providing student with information and techniques;

- Becoming familiar with the resources so that they can orchestrate a program that will benefit their pupils, and having peers outside the school available for advice, assistance and mentoring. All of these presuppose a level of self-confidence that many teachers lack;

- Having administrative support for working with the same group of students for a longer period of time. It is possible that a proper assessment of giftedness requires contact over a long time, as the teacher needs to understand how a given student thinks. Instead of having a new teacher each year at school, perhaps pupils need fewer teachers, each for several years. This allows a dynamic to be created between the teacher and the class and allows the teacher to get to know the student in a way not possible over a single year.

In relation to this new approach, it would be helpful to find out what teachers' views are on good mathematical problems, what they value, how they select questions for their students; what their beliefs about useful learning recourses are and how close are teachers' descriptions of good problems to the idea I am developing in this paper. In short, the following two questions are essential for the successful use of the approach: (1) Would practicing teachers identify interconnecting problems as good problems? (2) Would teachers be able to see good problems as interconnecting ones? A discussion of teachers' perspective on interconnecting problems goes beyond the scope of this paper. Further investigation of teachers' readiness to implement the approach and their related understandings, knowledge, perspectives and experiences will provide some empirical evidence of benefits of proposed approach and guide its effective implementation in practice.

Conclusion

Being an instructor of mathematics, I often find myself leading a classroom discussion around problems illuminating the essence of a mathematical method. Some of the problems I bring into play appear to be universally useful in a variety of courses. Students attending my classes enjoy recognizing them and comparing how different ideas and techniques can be applied to address the same mathematical question. My observations suggested identification of problems useful for systematical use in various university level courses. Similar practices are discussed in literature. For example, Mingus (2002) refers to "calculation of *n*-th roots of unity" as a problem which "encourages students to see connections between geometry, vectors, group theory, algebra

and long division". By means of investigation of this problem in different courses "students were able to review concepts from previous courses and improve their understanding of the old and new concepts" (Mingus, 2002, p.32). Further discussion reveals that "proving identities involving the Fibonacci numbers provide a solid connection between linear algebra, discrete mathematics, number theory and abstract algebra". In my view, these are examples of interconnecting problems. The practice of using such problems effectively responds to the proposal that students' achievements at university level courses are greatly influenced by the degree of interconnectedness of their basic mathematical knowledge, in particular, by connectedness between mathematical terminology, images, and the properties of the objects represented by these terms (Kondratieva & Radu, 2009). My own experiences resonated with like-minded instructors' practices led me to the formulation of the approach described in this paper, which I propose to apply to the whole mathematics curriculum with particular consideration of the needs of gifted students.

Modern curriculum is moving from a formal approach towards more exploration-based and inquiry-based study of mathematics. While making connections and multiple representations of mathematical ideas are recognized as primary goals in teaching and learning mathematics, it is not always clear how teachers can implement this agenda. House & Coxford (1995) argued that presenting mathematics as a "woven fabric rather than a patchwork of discrete topics" is one of the most important outcomes of mathematics education. However, there is also a need for practical teaching strategies "for engaging students in exploring the connectedness of mathematics" (House & Coxford, 1995, p. vii).

The interconnecting problem approach is one of such strategies. I hope that this article shows the potential of interconnecting problems and provides some practical ideas for teachers who pursue this direction in mathematics education.

I suggest that the use of the interconnecting problem approach at different stages of students' cognitive growth can foster the intellectual ability of the best students, identify mathematically-able students and engage them in analysis of connections between various ideas and methods. In addition, the application of different methods to the same mathematical problem throughout the years of schooling can:

- save classroom time devoted for exploration in high school by having necessary investigations and hands on experiences in earlier grades;

- foster earlier transitions to the study of algebraic methods by means of reference to pictorial or other previously employed representations of the problem;

- motivate students through freedom of exploration and experimental observations;

- improve students' logical skills by letting them reason in familiar terms;

- improve retention of basic facts by using them in the context of the problem and connect to other basic facts used in the same problem earlier;

- develop students' visualization skills and rely on their hand-on experience with geometrical objects when a more advances mathematical method is employed.

- help with producing multi-step solutions by building connections between various topics.

One may point at the obstacles the use of interconnecting problems may face because by the time students are in high school they may forget what they have done in previous years. Therefore, I emphasize the importance of very careful planning through the years of school curriculum for using of this approach. Elementary and secondary level teachers may need to collaborate in order to identify useful interconnecting problems and outline the direction of emphasis through elementary grades required for the secondary level studies appealing to the same problem. Teachers need to ensure that the experience with interconnecting problems obtained in earlier years of education is memorable. For that, each investigation needs to be concluded with a concise summary of the key ideas and perhaps illustrated by special schematic images which students will associate with the problem in the future. The purpose of such images is to allow the students quickly evoke previous experiences associated with the problem and thus prepare them for learning new skill related to the old ones. As an example one may consider the notion of "procept" viewed as an amalgam of processes, an object emerged from them and the symbol which both represents and evokes it (Gray & Tall, 1994). Another example is the Shatalov's "support signals" also helpful for "to reward successes—however small—and thus build up the child's natural enthusiasm for learning and confidence to be creative (Johnson, 1992, p. 59).

To summarize, I am not claiming that the interconnecting problem approach is easy to implement but it is worth trying because students equipped with a comprehensive view of one interconnecting mathematical problem will likely exhibit more confidence, mathematical insight, and elegancy in problem solving than those who have studied an equivalent number of disconnected and arbitrarily contextualized mathematical facts. Teachers who care about coherent picture of mathematics they teach may observe more signs of giftedness in their classrooms.

Acknowledgment

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Kondratieva

Creativity assessment in school settings through problem posing tasks

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Abstract: Research in math education on mathematical creativity relies on the idea that creativity is potentially within all students and it can be fostered by properly structured activities. The tasks most commonly used for its assessment are problem solving and problem posing. In our approach we use problem posing tasks to get insight into students' creativity. Based on a qualitative analysis of the participants' answers to the questionnaire that followed the task, we define *algorithmic, combined* and *innovative creativity* as constructs that can be put in correspondence with the types and level of knowledge involved in the problem posing task. We propose criteria to identify these types of creativity and discuss aspects related to the quality of the resulting problems. A second set of criteria is defined in order to assess the novelty of the posed problems.

Keywords: assessment criteria for creativity, mathematical knowledge.

Introduction

The first accounts of mathematical creativity emerged in the context of the work of professional mathematicians (Poincare, 1948; Hadamard, 1954). These accounts were subjective and often associated with a "genius" view of creativity (Weisberg, 1988). However, over the past decades, the approach to creativity in the mathematics education research community has shifted and now creativity is seen as an ability that can be enhanced in students by properly selected mathematical activities. In this view, creativity is closely connected to deep knowledge of a domain; it is associated with long periods of work and reflection and might be influenced by previous experience and instruction (Holyoak & Thagard, 1995; Sternberg, 1988; Silver, 1997). Since creativity became a subject of research in mathematics education, several research issues and paths have

emerged. One of which,, its assessment, will be discussed in this article. Two main approaches to assessment of creativity can be identified. The first one relies on an interpretation of the main components of creativity as they were defined by Torrence (1974). Fluency, flexibility and novelty are interpreted as number of identifiable changes in the approach to the problem, number of generated solutions and the level of their conventionality (Silver, 1997; Ervynck, 1991; Leikin, 2007). Another approach is represented by researchers who look at the relation between the traits, abilities and certain behaviours during task resolution and creativity. Balka (1974) in his article synthesized a set of criteria for measuring mathematical creative ability based on the works of Guilford; Torrance; and Meeker. He listed both convergent thinking, characterized by finding patterns and breaking from established frames of mind, and divergent thought defined as formulating mathematical hypotheses, evaluating unusual mathematical ideas, and splitting general problems into specific sub-problems. Haylock (1997) mentions two of these as being key-aspects for creativity: the ability to overcome fixations in mathematical problem-solving (like, for example, breaking away from stereotyped solutions), and the ability for divergent production within mathematical situations. Meanwhile, the two approaches are not independent; they focus on different aspects. In the first one, we have quantitative measures that allow the comparison between students performing the same task; the second approach gives us ways for fostering creative behaviour in problem solving.

As settings for the assessment, there are two major approaches: problem solving and problem posing tasks. Both have been recognized as being appropriate for this purpose. Namely, Ervynck (1991), Silver (1997) and much earlier, Polya (1973) among others, argued that solving problems in multiple ways is an expression of creative thought. In fact, Silver (1997) in his article stressed that inquiry-oriented mathematics instruction which includes problem-solving and problem-posing tasks and activities can assist students to develop more creative approaches to mathematics. Jensen (1973) said that for students to be creative in mathematics, they should be able to pose mathematical questions that allow exploration of the original problem as well as solve the problems in multiple ways.

We focus on the relation between problem posing and mathematical creativity; in particular, on the issue of defining criteria for creativity assessment through problem posing tasks in classroom settings. Our interest is to connect between mathematical knowledge and creativity. From this point of view, our approach is more related to the one of Haylock (1997), in terms that this would eventually lead to insight on stages of creative behavior and suggest ways for fostering students' mathematical creativity.

We shall start by describing the adopted working definitions for mathematical creativity, on one hand, and for classroom problem posing, on the other. Next, we present our methodology. In section three, we present arguments for the potentially creative nature of the problem posing process. In the next two sections we describe and give examples for the criteria derived from the experiments. We finish with conclusions and an outline of future research paths.

Definitions

Mathematical creativity

In the literature, we can find many definitions of mathematical creativity, but none is a commonly accepted one (Mann, 2006). Treffinger, Young, Selby and Shepardson (2002) identified over 100 contemporary definitions. Runco (1993) defines creativity as a construct involving both "divergent and convergent thinking, problem finding and problem solving, self-expression, intrinsic motivation, a questioning attitude, and self-confidence" (p. ix). Krutetskii (1976) characterized mathematical creativity in the context of problem formation (problem finding), invention, independence, and originality. Ervynck (1991) defines creativity in a framework of mathematical knowledge: "mathematical creativity is the ability to solve problems or to develop thinking in structures, taking into account of the peculiar logical-deductive nature of the discipline, and of the fitness of the generated concepts to integrate into the core of what is important in mathematics." (p. 47)

At the same time, researchers stressed the need to have workable definitions that can be applied at classroom level (Pehkonen, 1997; Freiman & Sriraman, 2007). A good, commonly agreed definition would help, on one hand, to identify students with creative mathematical thinking and, on the other hand, design meaningful tasks for them. In our

paper, we shall adopt the definition given by Sriraman (2005) and also accepted by other authors (Liljedahl & Sriraman, 2006; Freiman & Sriraman, 2007). Mathematical creativity at classroom settings is defined as a) the process that results in novel and / or insightful solutions and b) the formulation of new questions and/or possibilities that allow an old problem to be regarded from a new point of view.

Problem posing in classroom setting

For mathematicians, problem posing refers to the process by which they formulate a problem that has not been solved by anyone before. In most empirical studies, though, problem posing means the formulation of novel problems with the solution unknown at least for its creator (Van den Heuval-Panhuizen et al. 1995). In other contexts it is understood as reformulation of an existing problem (Cohen & Stover, 1981), mostly ill-defined one. Silver's (1994) synthesizes these aspects in his definition in accordance with which „problem posing refers to both the generation of new problems and the re-formulation, of given problems". (p. 19)

We shall adopt the definition given by Van den Heuval-Panhuizen et al. (1995). Therefore, in this study we define problem posing in a specific topic as the process of formulating questions about 1) the existence of a mathematical object; 2) the relation between different mathematical objects; 3) new properties of a given object deduced or related to a set of specified properties. The classroom setting means that the problem posing happens "inside of a class", in a context shaped by school curriculum. We mean the "inside of a class" as opposite to the work of mathematicians; therefore all students and teachers are included here, no matter their experience in mathematics. This also implies that the problem posing process is initiated by teachers as purposeful, goal-oriented learning activity performed with the students in the mind (even if physically not present during the experiment). Teachers want to illustrate, through these problems, mathematical methods or concepts, rather than considering problem posing an end itself.

Methodology

Subjects

In the experiments participated high school and first year university students from Romania along with secondary / high school teachers, all from Romania, and Olympiad participants from Mexico. University students were of 18-20 years old and entered to university after completing an admission exam. None of the students has been involved in training on problem posing. High school students were 16-17 years old and they just have studied sequences as part of the school curriculum. Olympiad participants were 15-18 years old and had no previous contact with sequences (as topic in introductory calculus). The teachers who participated in the experiment had varied experience in teaching and, at the time of the experiments, they were participating at an in-teacher education program. Overall, in the experiments participated 44 high school students; 25 university students; 22 Olympiad participants; 41 middle school teachers and 22 high school teachers.

Task

Participants received the following instructions: *Consider three consecutive elements of a sequence, a_{n-1}, a_n and a_{n+1}, and the usual algebraic operations (inequality included). With these elements pose three problems such that to have an easy, one of average difficulty and a difficult problem. At the end, you need to handle in the drafts of your work.* At the moment of handling their problems in, they received a questionnaire about the following aspects of the problem posing process: the existence of an initial idea (for each problem of different difficulty), change of the idea during generation, problem types from which to start the generation process, a theorem or generalization as from where to trigger the problem posing process and difficulty criteria they used.

A remark needs to be made: no further clarifications were made about the difficulty of problems. Each participant could establish his own criteria for difficulty based on his experience. We plan to analyze our data from this point of view in the future.

Data analysis

In the first step of our analysis we looked at the problem posing process from the point of view of overall dynamics. The purpose was is identify recurrent actions that could, eventually, be grouped and considered as phases of the posing process. Further, by

analyzing these phases we were hoping to have elements that would situate classroom problem posing as an instance of mathematical creativity.

What are the processes or phases that characterize creativity? Silver (1997, p.76) asserts "…It is in this interplay of formulating; attempting to solve, and eventually solving a problem that one sees creative activity. Both the process and the products of this activity can be evaluated in order to determine the extent to which creativity is evident."

In the analysis of our data, we hypothesized that, during the problem posing process, the knowledge available to the student is under a continuous reordering as the relevance of a piece of knowledge is under change. The aspects proposed by Silver (1997), like shifts in direction or reformulations or explored paths, relate to the change between on-focus and off-focus state of a particular mathematical object and property of the object. In one phase, there is a broad field which is briefly explored such to focus, immediately after, on a particular aspect of the mathematical object or property. Such a "reordering" allows cognitive change to occur and it is the base for the "shift from association-based to causation-based thinking, which facilitates the fine-tuning and manifestation of the creative work" (Gabora, 2002). Therefore, in the analysis of our data we paid special attention to the cases when participants reported changes in their approach or when that change was identifiable from the scratch work (even if not reported in the questionnaire). We conclude that problem posing is creative because involves the same mechanisms that are present during a creative endeavor. We shall give two examples to illustrate these ideas. The first one is presented in figure 1 and was given by a teacher. The teacher reported in the questionnaire: *The idea was to combine the theorems* (Fig1a.) *and I tried several expressions* (Fig. 1b) *until I get to the final form.*(Fig. 1c)

Figure 1. a. Theorems reported by the teacher

b. The expressions tried by the teacher

c. final expression of the problem

In the reformulation of an expression one draws on experience, searches for analogies or for new associations, meanwhile during the evaluation of the newly formed expression needs to search for causal relations, assess general characteristics of the problem and the aptness of the problem with the initial constraints or goals.

As second example we give the answer of an Olympiad participant to the question *Did you have from the very first moment what sort of problem will you generate at each level of difficulty? If you answer Yes, please specify it.* The answer was: *Neither yes, neither a no. I had an initial sketch of what I wanted to do, but the final product was not what I thought of initially.* The next question of the questionnaire referred to the change in ideas: *If you answered yes to the previous question, did the idea change during the process? If case of a yes answer, please specify.* The student's answer reveals that one switches from broad to focused look and continuously monitors the problem in formation: *Probably because the original ideas were not in concordance with the level of difficulty of the problem I was just creating and, also, because I was trying to create something new (especially for the difficult problem).*

We see creativity in the problem posing task due to this cyclic alternation between the two types of thinking: an association based one during which ideas flow and a causal

type of thinking that allows assessing the creation done so far and setting a new context for the next cycle.

Once we identify problem posing as a creative act, we concentrate in the next section on defining criteria to assess it. The assessment can be seen at process and results level. We shall focus on mathematical knowledge as a key factor in such an enterprise.

Criteria for creativity assessment through problem posing

In this section we define criteria for assessing creativity at process and result level. As we shall illustrate, a separation into these two aspects was necessary given that, in repeated situations, the quality of the generation process was not matching the quality of final results. First, we shall focus on the process of problem posing and, then, on the resulted problems.

Assessment of the problem posing process

In order to formulate the criteria, we analyzed the drafts handled in by the participants. The details from the drafts were interpreted, where possible, in terms of actions (steps taken towards the accomplishment of the task). At the same time, the actions (like for example, replacement of a constant with a variable) rely on knowledge and, therefore, we considered that the definition of criteria should relate to knowledge. A problem posing task always has a context given by the topic of the posed problems. As such, and especially at classroom level, we can identify a cluster of knowledge that typically is used in problems related to that topic. We shall refer to this as domain or topic specific knowledge.

In school mathematics, we consider the concepts, theorems, corollaries presented at a topic as the domain specific knowledge of that topic or domain. The clustering of knowledge based on its relevance to a particular task is a common practice between researchers. For example, Leikin (2007) introduces, between others, the concept of expert spaces as being the space of solutions to a problem given by an expert in the domain. After we delimited the domain and not-domain specific knowledge for the topic / domain of sequences by analyzing several textbooks, we categorized the steps, actions, taken by the participants during the posing process based on the belonging of the involved

knowledge to domain specific or not clusters. After a systematical classification of the processes seen in the drafts, we identified three main ways for posing a sequence problem. Our criteria for the creativity of the process are based on these three categories.

We define as first level of creativity (*algorithmic)* one that it is characterized solely by the employment of domain-specific algorithm. Typical examples are the cases where problem generation is based on a rule, on problem types or specific techniques. For a rule based generation, consider the example presented in figure 2. The elements of the rule would be instantiated by known cases that are known from class or individual study. The problem was posed by a high-school student and refers to the rule: the sequence obtained as product between a sequence having the limit zero and one that is bounded, is convergent to zero. As it can be seen in the figure, the student defines a bounded sequence (a typical example for bounded sequences in Romanian textbooks) and one that converges to zero, and then asks for the limit of the product.

Figure 2. Problem posing by using a domain specific rule

However, it has to be said that the rule is not always profoundly understood (the relations between elements); situation that often leads to erroneous problems. Since at this moment we look at the process itself, it has to be underlined that high school students rely mostly on memory when trying to instantiate the rule elements. Knowledge is too rigid and not interconnected, leading to many unsolvable problems or, when solvable, they lack interest. The expertise in instantiating elements of a rule or of some technique

will impact the quality of the result. We give three examples for the use of the same technique as base of the generation process with different results.

The first one is the one presented in Figure 1. Two others, one made by teacher and one posed by a high school student are shown in figure 3a and 3b.
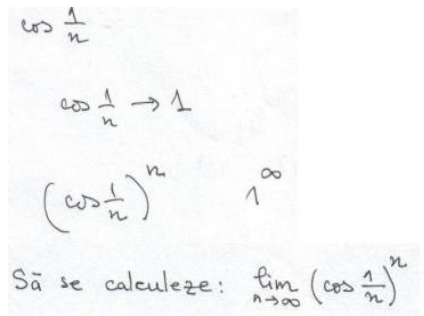
a. Example of a teacher´s problem
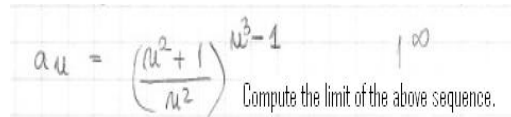
b. Example of a high school student´s problem for algorithmic creativity

Figure 3. The use of a known limit as start point in the generation process

It can be seen, in an attempt to solve, that problem at point b, quickly leads to infinite as result, since the exponents of n are chosen so. In comparison with this, the problems posed by the teachers, need to be worked until the end in order to have a result and, also, require having knowledge about trigonometrically functions. In conclusion, even if the approach to problem posing is fundamentally the same, the quality of the resulting problems can vary significantly.

A second level of creativity is defined as the application of some domain-specific rule along with some other type of knowledge. We shall use the term *combined creativity* for this case. The "other knowledge" would be from another domain and its application not straightforward for the most. However, this not-topic specific knowledge plays a central role in defining the problem; the problem is structured around this knowledge and connected to the topic through the formulation of the problem. Example:

$$\text{Consider } (a_n)_{n \in N} \text{ such that } a_0 = 1 \text{ and } a_n = \frac{na_{n-1}}{a_{n-2}}. \text{ Prove that } \frac{(2n+1)!}{a_n a_{n+1}} \geq \frac{2^{2n+1}}{2n+1}.$$

It is interesting to see the procedure followed by this Olympiad participant to generate the problem. His answer in the questionnaire was: *Getting to n! is trivial and then I tried to "out inside" the combinatorial identity*. We give in Figure 4, the fragment with the most important step (from creativity point of view).

Thinking that

$$a_n \cdot a_{n+1} = n!\,(n+1)!$$

We complete such to obtain some kind of combination

$$\frac{2n+1!}{a_n \cdot a_{n+1}} \approx \frac{(2n+1)!}{n!\,(n+1)!} = \binom{2n+1}{n}$$

$$\binom{2n+1}{0} + \binom{n+1}{1} + \dots + \binom{2n+1}{2n+1} = 2^{(2n+1)}$$

We look for an inequality, from the identity -->

Then, we can observe that

$$\binom{2n+1}{n} \geqslant \frac{2^{(2n+1)}}{(2n+1)}$$

$$\therefore \quad \frac{(2n+1)!}{a_n \cdot a_{n+1}} \geqslant \frac{2^{(2n+1)}}{2n+1}$$

Figure 4. Example of problem generation

In this case, the question of the problem is not one typical for sequences. Indeed, the problem is about combinatorial, but it is formulated as one of sequences. The combination of knowledge from different topics can lead to a situation that is considered as worth for exploration. A second example comes from a first year University student and it was posed for the average difficulty problem:

Consider a sequence $a_{n+1} = (a_n + a_{n-1}) \bmod 100$ with $a_0 = 0, a_1 = 1$. Prove that the sequence is periodical.

In the questionnaire, the student reported that he wanted to build a periodical sequence, so he thought of the pigeonhole principle and then tried to define something to fit this idea. Once again, the problem is structured around this not-topic specific knowledge that also becomes essential for solving the problem. As a remark on the "quality" of the problem, it has to be said that under the current formulation the problem is straightforward, an aspect that seems to be ignored by the student (since he specifies the problem as average difficulty). Small changes in the initial values, and maybe other question could have turned the problem into a challenging one. In conclusion, the quality of the problem is not always in direct relation with the creativity shown during the generation process.

A third level of creativity was tagged as *innovative creativity* and it is defined as the process of using solely knowledge from outside of the topic for which the problem is

generated. Example: *Consider the following sequence:* $a_1 = 3$, $a_{n+1} = a_n^2 - a_n$. *Decide whether 396,138,794,300,000 is term of the sequence.*

In the above case, the rule of divisibility with 3 was applied (as the start point) and generated a question. The result is an extremely simple problem, yet unusual at first. The main point we highlight is that they used knowledge and techniques from a completely different domain and, then, reformulated the problem in terms of the requested domain.

A second example comes from a University student: *Consider* $f : (0, \infty) \to R, f(x) = \dfrac{x+1}{2x+3}$. *Note with* $f_n(x) = \underbrace{(f \circ f \circ ... \circ f)(x)}_{n}$. *Prove that*

$f_n(x) = \dfrac{a_n x + b_n}{c_n x + d_n}$ *where* $a_n, b_n, c_n, d_n \in N^*$.

The problem is built around function composition and uses no knowledge from sequences (as seen in introductory calculus). With regard to the problem, we observe that it is not a difficult one to solve, however – as a homographic function - it leads to an interesting exploration and far-reaching results.

As we underlined, it is not necessary that certain creativity in the process to lead to interesting or challenging problems and vice versa. Therefore, it is important that when judging the creativity of a student we pay attention also to the process by which he arrived to the results and not only to the final problem.

Assessment of the result

Plucker and Beghetto (2004), in their review on creativity, stressed that there are two key elements of creativity, specifically novelty and usefulness. We observe that this definition allows evaluating the results of the creative process, especially as usefulness is concerned. At this point we focus on the novelty of the posed problems, considering that their usefulness is given by the fact that we situate ourselves in a classroom setting, therefore the problems are useful because they carry a potential pedagogical value.

The novelty of a problem is judged in comparison with already existing problems, therefore we need to define the elements of the problems that should be compared at this phase. In the particular case of classroom setting, the core set of problems supposed to be

known are those from the textbooks and some problem books. Generally speaking, we treat a problem as having a given part, requested part, form of the question, restrictions (when asking to apply some particular method, for example) and solutions. In some particular domain (sequences, for example), we can speak about problem types as determined by the expressions involved in the given part. Based on these specifications we define the following levels of novelty.

At the lowest level we define the *algebraic novelty* which consists of differences in the expressions in the given or requested part, meanwhile all the rest remains unchanged (the problem structure, type, and possible solution method). A very common way is to change the values of a constant thus obtaining something new (in terms of the expression involved in the problem), but in the same time having the same problem from structural point of view. High school students tend to generate problems with algebraic novelty only, especially if they experience a failure before.

The second level of novelty consists of a significant change in the given or requested or "form of the question" part, but the structure remains identical to the initially known problem. Such change it is reflected at the level of the nature of the used expression, therefore we shall use the term of *conceptual novelty*. The simplest example consists of *parameterization*, the process by which a constant is changed into a parameter. The new problem, though structurally identical with the initial one, is novel since it opens up a space for discussion based on the parameters values.   This interpretation of novelty refers to comparing an initial (retrieved) problem and a new one, but can be easily extended to define the novelty of a problem in relation with a set of problems.

A third level of novelty is the *methodological* one. Let's analyze the following example given by a secondary teacher:

Consider the sequence 1,2, 2, 3, 3, 3,4,4,4, 4, … Answer the followings:

1. Which are the next three terms of the sequence?

2. Is the sequence monotone?

3. Prove that the last digit of the index of the last elements from the part of equal numbers is not divisible with 4.

The question 3 can be considered as one bringing a methodological novelty in play, since the question that can't be answered by the same method as the previous ones. In this case, we have a modification that turns the problem into a new one, and this novelty can be identified at the level of the applicable solution methods. It might seem that novelty is easy to be achieved, but often even a small change in the value of a constant can turn a problem previously easy to solve into a very difficult one. Therefore, one needs a good understanding of the problem's structure in order to maintain the problem solvable and well defined.

Conclusions

In our paper, we defined criteria for the assessment of mathematical creativity in classroom settings through problem posing tasks. The criteria were identified as result of a qualitative analysis of a series of problem posing experiments ran with high school, university students, teachers and Olympiad participants. The structuring element of the analysis was the topic-specificity of the knowledge involved in the process. Based on this, we introduced and illustrated the constructs of *algorithmic*, *combined* and *innovative* creativity. In each case, we outlined the possible links between the quality of the result and the creativity involved in the problem posing process. In the last part, we introduced criteria for the assessment of the results' novelty. Three constructs were given and exemplified: *algebraic*, *conceptual* and *methodological* novelty.

As future line of research, we envision the study of the co-growth of the body of mathematical knowledge and understanding along the creativity exhibited during a problem posing task. A second line of research concerns the development of activities that could foster creativity of students. As a third line, we want to study the constraints teacher consider during the posing process whether those are tacit or not by nature.

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Forthcoming TMME vol8,no.3
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&
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Bharath Sriraman, Editor

The next issue of the journal includes several articles that have already been accepted for publication as well as two sections devoted to mathematics education meetings in 2010. The first section includes extended papers from the *North Calotte Conference in Mathematics Education held in Tromsø, in February 2010*, and the second section includes extended versions of the plenary papers from the 2010 Annual Conference of the Psychology of Mathematics Education: North American Chapter, held in Columbus, Ohio, in October 2010.

As the reader will note, this double issue is nearly 400 pages (!) in its length, and space restrictions for the print version of the journal does not allow us to include more articles that are already in the pipeline. We appreciate the patience of authors that have to wait until the next issue to see their articles in print. Another feature of this particular issue is to allow an eclectic style of fonts and formatting for journal articles, particularly those that require the use of special math fonts. In the past, the quest for "font" uniformity has sometimes resulted in many math symbols disappearing or changing into unintelligible icons when converted to portable document format. However the universality of the intended "meaning" of operators given the particular contexts in which they disappeared in some articles in the past, did not take away from the understanding of the person reading these papers. Another reason to appreciate the language of mathematics!

On a concluding note, I wish to thank each and every author contributing to this issue for their work and adding to the prestige, eclecticism and scope of the journal. I hope 2011 bodes well for everyone.