The Mathematics Enthusiast

Volume 8
Number 1 *Numbers 1 & 2*

Article 16

1-2011

Gifted Students and Advanced Mathematics

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Abstract: The extension to a wide population of secondary education in many countries seems to have led to a weakening of the mathematics curriculum. In response, many students have been classified as "gifted" so that they can access a stronger program. Apart from the difficulties that might arise in actually determining which students are gifted (is it always clear what the term means?), there are dangers inherent in programs that might be devised even for those that are truly talented.

Sometimes students are moved ahead to more advanced mathematics. Elementary students might be taught algebra or even subjects like trigonometry and vectors and secondary students taught calculus, differential equations and linear algebra.

It is my experience over thirty-five years of contact with bright students that acceleration to higher level mathematics is often not a good idea. In this paper, I will articulate some of the factors that have led me to this opinion and suggest alternatives. At the same time, one needs to deal with truly exceptional students in an appropriate way.

Keywords: talented students, enrichment, acceleration

1. Beliefs and assumptions

The central question in mathematics education is, "Who owns the mathematics?" If the answer is not "the student", then our efforts within and without school are likely to be counterproductive. Traditional education has often led to a syllabus being imposed on students as passive recipients, so that whatever richness it possessed was not appreciated and thus not understood or retained.

If students are to enter into mathematics, it must be through an involvement that makes it intelligible, that ensures its applicability and that leads to an apprehension of its power. An overemphasis on covering material, whether in a traditional approach or in the enrichment of talented students, runs the risk of reducing the occasions for this involvement. The point was made by Nicolas Sarkozy, President of the French Republic, in an encyclical letter to educators on September 4, 2007 (I am indebted to the French Embassy in Washington for the translation):

Don't misunderstand me; my aim is not to increase the teaching hours still further; the timetable is already too heavy. It is not to add yet more new subjects to a list

The Montana Mathematics Enthusiast, **ISSN 1551-3440, Vol. 8, nos.1&2**, pp.319-328

> *which is already too long. On the contrary, to my mind,* ***the aim is to give back to our children time to live, breathe; assimilate what they have been taught****. [emphasis mine] We need to regain coherence in our educational system. · · · We need to restore coherence within each school subject and between these and society's expectations, once again find a lodestar for education, set for its principles, goals and simple criteria.*

It is this provision of room to breathe and sense of coherence and purpose behind what we present to students that will help them engage our discipline productively. The traditional curriculum scored quite well on coherence; elementary students got a solid exposure to arithmetic and secondary students might spend a whole year on subjects such as Euclidean geometry, analytic geometry, trigonometry, algebra and traditional applied mathematics, learning a range of results and techniques and doing exercises. It often lacked the opportunities for students to explore and experiment, to put their own stamp on the concepts and procedures they needed to master.

Teachers must not be put in the position of answering questions that students are not prepared to ask. If we are to proceed to more advanced mathematics, it is because the experiences of the students lead them to an apprehension of the need for it. It might be a more general approach that tidies up what might be otherwise unmanageable or of more powerful tools to handle situations that are difficult or impossible with the tools they possess. Arithmetic is a tool for convenient handling of quantitative information; algebra is an antidote to the over complexity of arithmetic solutions to word problems; the systematization of synthetic, analytic or transformation geometry allows the encompassing of an undisciplined slew of results.

Thus the pace of introducing new material should be sensitive to how well students have assimilated existing material, how flexibly they can negotiate it and their understanding of its uses and limitations.

2. Educational activity: past and present

The approach described has been implicit in many programs available to students over the years. The *Gelfand Correspondence Program* in Mathematics, first in the Soviet Union and latterly in the United States, has provided a curriculum for its adherents that is coherent and interactive (5). *Project SEED Mathematics*, originating in Berkeley, CA in the 1960s is another program, still continuing, that provides an in depth experience for students (7). The recent Volume for the 16th ICMI Study, *Challenging mathematics in and beyond the classroom* describes several initiatives, such as the *Creative problem solving in mathematics* (CPSM) course at Quincy Senior High School in Quincy, IL, that is based on a study of solid geometrical structures, and the *Maths `a modeler* research activities for students and teachers in France, one of which is focused on tiling (3, p. 189 seq).

Even though, as the Study Volume indicates, many educational researchers are studying and creating programs for gifted students, it is hard to avoid the impression that evidence of their efficacy is largely anecdotal. Success seems to be dependent to a large degree on the expertise and passion of their proponents and on the readiness of students to embrace them. In many cases, the student participants are either self-selected or identified by adults as being suitable. I am not aware of longitudinal studies that any particular regime leads to greater mathematical awareness and prowess, either among those amateurs of mathematics that melt into the general public or those who proceed to higher study. Nor am I aware of systematic programs that have been adopted over a large jurisdiction to bring along those that are especially interested or capable in mathematics.

3. Algebra and calculus

Algebra and calculus both have the characteristic of being general methods, capable of treating a wide range of problems and situations. In so being, they tend to suppress particularities and to see problems as belonging to broader categories. The situation is mediated through a specially created formalism that is efficient and sophisticated, so that a first-hand feeling for the situation may be lost in the application of a standard procedure. Both algebra and calculus are systems of great mathematical power, but this is often traded off against transparency and intelligibility. An inexperienced student might see these as machines, to be used indiscriminately.

Students should be exposed to these advanced areas only when they can appreciate their significance and understand their use. Indeed, it might be said that the most important thing that a young student needs to know about either algebra or calculus is when not to use it.

Consider algebra. Its utility for most middle school and early secondary students is in the reformulating and solving of word problems. Some such problems can often be more conveniently handled by arithmetic or proportional techniques. Consider the following example:

Example 1. Two old ladies, Olga and Tamara live in separate towns some distance apart that are joined by a single road. One morning at sunrise, the two ladies set out simultaneously to walk to the town of the other, each walking at her own constant speed. The two passed on the road at noon. Olga reached her destination at 4 pm, while Tamara did not arrive at hers until 9 pm. What time was sunrise that day?

When this problem is given to students who have had some algebra, their first impulse it to set up some equations and try to solve them. Invariably, they find this a tough task and often do not succeed. Part of the difficulty is the introduction of superfluous detail, such as the actual distance between the towns or the speeds of the two ladies, which serve to obfuscate the situation. What gets lost is the significance of the proportionality inherent in the situation: when a person walks at a constant speed, the distance travelled is proportional to the time taken. Suppose the time taken by both ladies before noon is T hours. The distance walked by Tamara in the morning is the same as that walked by Olga in the afternoon, and *vice versa*. Appealing to the proportionality quickly leads to T: 4 = 9: T and the answer T = 6. Thus, the sun rose at 6 am.

Accordingly, gifted students should be presented with arithmetic and proportionality problems of varying difficulty, and challenged to solve them through basic reasoning. Some might be encouraged to use the sort of diagrammatic methods espoused by Singapore texts. (For an example, see (3, p. 290-291).) However, they will find some problems tough when only arithmetic methods are available, but routine when algebra can be used.

Example 2. A man is 6 years older than his wife. He noticed 4 years ago that he had been married to her exactly half of his life. How old will he be on their 50th wedding anniversary if in 10 years she will have spent two-thirds of her life married to him? [International Mathematical Talent Search, Round 17. Consult (6).]

The student who tries to meet this on arithmetical terms has a real challenge, and will appreciate how the definition of variables and the use of algebra will clarify the situation. However, the application of algebra is not completely automatic; the rapidity of achieving success on this problem will depend on how astutely the variables are defined and the equations are set up.

Algebra should presented in a context where the student can be expected to decide on where and how to use it; sometimes it is better avoided; other times, it is essential. Gifted students need to learn algebra, which after all is the language of mathematics, but it should be presented in a measured way so that its power is made manifest and the student can absorb and dwell naturally in its higher level of abstraction. Once algebra is embarked upon, its use as a tool in all sorts of problems should be explored, not only in the setting up and solving of equations, but in problems of maximization, analytic geometry, trigonometry, combinatorics.

For students at the secondary level, calculus is in an analogous position. When it is introduced prematurely, students seem inclined to address it only on operational terms. Since the only functions secondary students are going to have to deal with to any degree are polynomials and the standard transcendental functions, they are not sensitive to the issue that differential calculus applies to functions that are smooth and may see it as applicable to anything in sight (such as the absolute value function). By not being aware of the more subtle issues and the range of validity of calculus techniques, their long term growth as mathematicians may be stunted. Two case studies will illustrate the point.

This should not be construed as an argument against giving calculus to minors, but only that it is given to students with a well-rounded experience in algebra and geometry. As any fan of the Putnam competition knows, calculus can be the arena of its own clever and elegant challenges.

Case Study 1. *Functional equations*. In recent years, it has been common to include among competition questions, particularly at the Olympiad level, functional equations that have to be solved. Frequently, there are no conditions on the function apart from the equation, so the solution sought is completely general. However, students immersed in algebra and calculus, will often, probably unconsciously, assume that the function in question is a polynomial, that it is continuous or that it possesses a derivative. Acting on this often leads them into formidable computational territory; their extra knowledge often prevents them from addressing the problem at its most basic and natural level.

Example 3. Problem 2 on the 2008 Canadian Mathematical Olympiad (4) sought the solution of the functional equation

$$f(2f(x)+f(y)) = 2x+y,$$

where *f* is defined on the rationales and takes rational values.

The algorithmic-bound student who either assumes that *f(x)* is a polynomial or differentiates immediately introduces unwarranted complications and restricts the scope of the problem. The solver of this needs to take to heart that stark but comforting fact that everything available is stated in the problem and has to try to squeeze out of this meagre store the maximum possible information.

This might involve guessing a solution, to see where one might be headed. Or it might be trying some basic substitutions (setting variables equal to zero or to each other are reasonable options) to get simplifications or more workable conditions. In this case, one can find that

$$f(0)=0, f(2x+y)=2f(x)+f(y) \text{ and } f(2x)=2f(x) \text{ for all } x \text{ and } y.$$

The experience that students might obtain from such problems might well lead to an easier embrace of axiomatic systems in their later studies, where they need the discipline of assuming exactly what is given and putting aside extraneous details.

Example 4. Solve the functional equation

$$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$$

for real x and y,

Noting the role of $x \pm y$, one can try the substitution $u=x+y, v=x-y$ to obtain

$$vf(u)-uf(v)=(u^2-v^2)uv$$

for real u, v. One can "separate the variables" to obtain

$$\frac{f(u)}{u}-u^2 = \frac{f(v)}{v}-v^2$$

from which it is deduced that $f(x) = cx + x^3$ for some constant c. It can be checked that this works.

Notice in this example how little one has to rely on technical results and processes, and how much depends on intuition and ability to draw out the significance of the equation resulting from separating the variables. Students will often have initial difficulties with these sorts of problems, but given enough time and experience, they will develop the experience and divergent thinking that will more reliably lead to success.

Case Study 2. Inequalities and optimization. If students are exposed to calculus while their algebraic background is sparse, they are inclined to see every inequality and optimization problem as an occasion for taking the derivative. It is useful to defer calculus until students have learned various algebraic techniques for dealing with inequalities. These often involve techniques such as completing the square, expansion, rearranging and factoring of expressions to expose clearly the sought inequality, and so provide the student with practice in reading algebraic expressions and extracting information from them. This could be combined with the derivation of and experience in dealing with standard inequalities such as the arithmetic-geometric means inequality, power mean inequalities and the Cauchy-Schwarz inequality. Indeed, an examination of Olympiad inequalities suggests that probably three quarters of them can be handled with an astute application of the arithmetic-geometric means inequalities.

The student who resorts to calculus to solve inequality problems runs three risks. The first is that, in missing the salient features of an inequality or optimization problem, she complicates the situation. The second is that the solution may not be complete; having found the condition for the vanishing of a derivative, the student may neglect giving an argument to justify the nature of the optimum. The latter danger is particularly pronounced if the student is equipped with the howitzer of Lagrange Multipliers; this is a neat technique, but often the classification of the optimum can be tricky. The third is that she might not develop the valuable ability to "read" algebraic expressions and develop an instinct for performing the most appropriate and effective manipulations.

Another pitfall that occurs in this area is that students forget that the essence of solving a problem is to reduce it to something more elementary and straightforward. There is an unlimited supply of inequalities of ever increasing sophistication and power, and many students lack the maturity to adjust the strengths and generality of the tool to the situation at hand.

Example 5. A good example appeared as Problem 3 on the 2008 Canadian Mathematical Olympiad (4). Candidates were asked to show, for positive reals *a, b, c* satisfying $a + b + c = 1$, that

$$\frac{a-bc}{a+bc}+\frac{b-ac}{b+ac}+\frac{c-ab}{c+ab}\le\frac{3}{2}.$$

This drew more solutions than expected, that ranged from very straightforward to extremely complicated; a few appealed to the very general Muirhead majorization inequalities (for which I had to access *Google* for enlightenment) (8). However, elementary algebraic manipulation leads to the equivalent $ab+bc+ca\ge 9abc$ or

$$9\le\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c)$$

which is a consequence of the Cauchy-Schwarz inequality.

Example 6. For x, y, z > 0, show that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\le 10$$

Since combining terms on the left side or using calculus to maximize it is particularly nonappetizing, it is best to look for elementary methods and insights.

The basic $\left(x-\sqrt{yz}\right)^2\ge 0$ leads to $(x+y)(x+z)\ge x\left(\sqrt{y}+\sqrt{z}\right)^2$ and a quick evolution to the solution.

While neither example is obvious, both underscore the utility of learning how to read the structure and seeking insight rather than charging ahead with a standard approach.

In summary, the mathematical growth of students in the exercise of judgment should be kept commensurate with the exploration of new and higher level material.

4. Subjects suitable for gifted students

In selecting a program for gifted students at the precollege level, the emphasis should be on broadening the experience of the regular syllabus rather than on acceleration. One has the opportunity of covering topics that are attractive, yet not likely to figure in main stream mathematics education. I will mention some of these:

Geometry. School geometry tends to be sparse, and in many jurisdictions, there is a tendency towards empirical geometry using technology. The use of resources such as *Geometer's Sketchpad* is a welcome addition to the syllabus, but it may displace other important aspects of the mathematical experience. Unless it is part of an enriched program, Euclidean geometry is unlikely to figure as part of a student's mathematical education.

Elementary geometry is all about circles and triangles, figures that admit an unlimited supply of properties and relationships. It betokens the fecundity of mathematics; after 2500 years, new results are still being found and old ones reestablished in ever more elegant ways. It links mathematicians across time and culture, is shared by amateurs and professionals, hones analytic and logical skills, fosters competency in exposition, sharpens the aesthetic sense and provides an ample stage for investigation, ingenuity and achievement. It provides a handsome supply of tools – traditional Euclidean derivation, transformations, vectors, analytical geometry, complex numbers – for the solutions of problems.

The ability to use transformation arguments, in particular, is particularly exciting for the novice, as such arguments rely on exposing and exploiting the basic structural aspects of the situation and give an insight into why the result holds that Euclidean or analytic methods often fail to do.

Number theory. Elementary number theory is another attractive area for young students. Not only should they learn the basics of prime factorization, common multiples and divisors, but they should master the use of modular arithmetic (something is probably easier picked up by the young than by many students later at college age). The solution of Diophantine equations provides an excellent challenge for students at the secondary level, as they are required to assimilate and select the right algebraic and numerical facts and techniques. A particular equation that is ideal for the young is Pell's equation. It is easily motivated and readily understandable, and provides the occasion for a great deal of empirical investigation. Yet the methods for treating this equation provide a natural home for surds (a topic given at most a cursory treatment in the standard curriculum) and provides direct experience in issues that will be taken up in more detail in the study of computation and number theory, modern algebra. An indication of what is possible is provided by my book *Pell's equation* (2).

However, again care needs to be taken when more advanced work is undertaken or referred to that students do not lose a sense of appropriateness and judgment. They need to realize that the Dirichlet result about the infinitude of primes in certain arithmetic progressions is deep, and not to be thrown into a solution when simpler resources are available.

Polynomials. For about nine years, I presented a course on polynomials to secondary students that terminated in an optional final examination. This was an ideal topic for gifted students, as it combined practicality with an concrete gentle introduction to important areas of advanced mathematics, including complex analysis, inequalities, number theory, modern algebra, approximation theory, dynamical systems, combinatorics and, yes, calculus. This eventually resulted in a book (1). As there were many topics that would be useful for students to know, but that might likely not meet in a college course, it could be regarded as an amplification of high school work in which student derived practical experience with examples of higher level theory later encountered at college.

Functional equations. An area that was almost non-existent two decades ago, functional equations now occur regularly on competitions. This is an excellent realm of challenge for secondary students, who often require only basic reasoning and elementary facts, but need to collect the evidence about the unknown function carefully and cogently.

Combinatorics. Although combinatorics has increasingly become part of the undergraduate mathematics curriculum, there is an elementary dimension to this division of mathematics that makes it eminently suitable for the young. The Pigeonhole Principle and Inclusion-Exclusion Principle are two techniques that are at once powerful and accessible. The use of generating functions provides exercise in algebraic techniques along with an indication of how one area of mathematics can enrich another. As with geometry, a high premium is put on careful argumentation, so that the skills of the student in organization and exposition can be enhanced.

Recursions and Dynamical systems. Elementary finite differences, in particular the solving of recursions, is an elementary topic that can be part of the arsenal of gifted students. Linear recursions share many structural properties with linear systems of algebraic or differential equations, and so provide a larger context for linear algebra that will be studied later. Dynamical systems, particularly the study of the logistic recursion, requires only basic algebraic and calculus background, and serves as an occasion for computer investigation and a study of approximation.

Trigonometry. This branch of mathematics has become considerably emaciated in the standard syllabus in North America. This is unfortunate, as trigonometry is an elegant formulation for dealing with situations that at root involve similar triangles in a powerful way. It stands at the crossroads of pure and applied mathematics, and provides a firm foundation for studies in either of these directions. Combining ideas of algebra and geometry, it is a platform to encourage facility and insight in both areas, one should that be part of the educational experience of any gifted student in mathematics. It also provides a home for complex numbers, which lives only as an orphan in the standard school syllabus; many trigonometric manipulations can be handily done using complex techniques.

Cardinality. Many students are confounded at college by a failure to understand the nature of the continuum. For gifted students, this can be circumvented by embarking on an early and leisurely examination of the real number system to get a feel for its complexity. This includes understanding countability and uncountability, and realizing that by this criterion, the sets of rationals and nonrationals are essentially different. The study of infinity is often quite difficult even at the college level, but an argument can be made for dealing with it early before students have had a chance to form prejudices.

History. Young students can usefully be introduced to some aspects of the history of mathematics. There is value in seeing how our predecessors tackled problems before modern mathematical structures were in place and to gain some understanding of how these structures were conceived and formulated. Apart from Euclidean geometry, students can study with profit the solution by Euler of the Konigsberg bridge problem, precalculus determination of areas and tangents (the cycloid gives some beautiful case studies), the beginnings of number theory at the hands of such masters as Fermat (see (2) for a treatment of Pell's equation), attempts to solve exactly or approximately polynomial equations and the analysis of algebraic curves. The Mathematical Association of America and the American Mathematical Society both produce books that can be read by secondary students.

§4. **Conclusion**.

In dealing with gifted students, the guiding principle should be to broaden the experience of the students at each level, and not to proceed to more advanced work unless it is carefully prepared for. Advanced mathematics involves more abstraction and generality, and so is inclined to increase the intuitive distance between the student and the mathematics, unless the intuition itself is enriched. There is a trade-off between the intelligibility of particular situations presented at a lower level and their capacity for inclusion in a broader sphere at a higher level. To appreciate the power and elegance of higher mathematics, and to exploit it judiciously, students need time and experience to develop comfort and facility with sophisticated matter.

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