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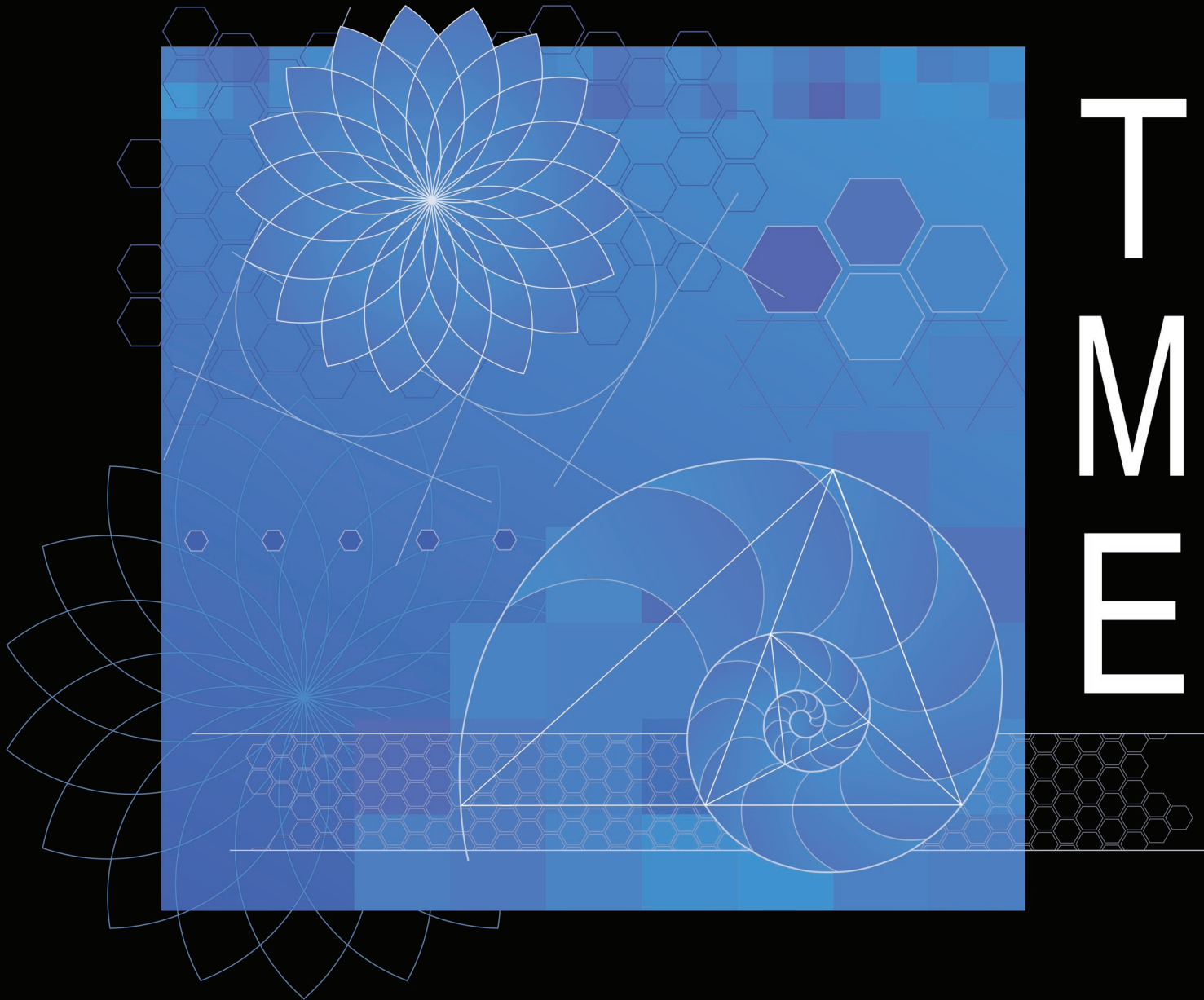
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THE MATHEMATICS ENTHUSIAST

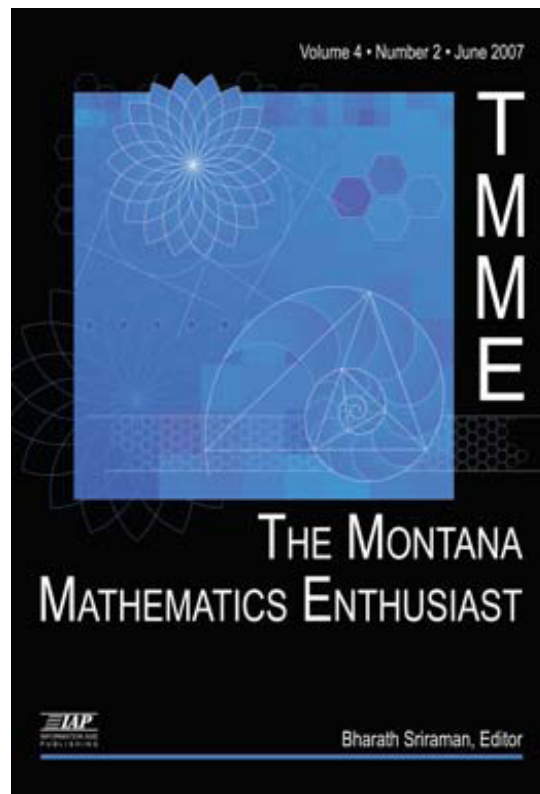
AN ECLECTIC RESEARCH JOURNAL AFFILIATED TO THE PSYCHOLOGY OF MATHEMATICS EDUCATION—
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The Mathematics Enthusiast is an eclectic internationally circulated peer reviewed journal which focuses on mathematics content, mathematics education research, innovation, interdisciplinary issues and pedagogy. The journal is published by Information Age Publishing and the electronic version is hosted jointly by IAP and the Department of Mathematical Sciences- The University of Montana, and affiliated to the professional association PMENA [Psychology of Mathematics Education- North America].

Articles appearing in the journal address issues related to mathematical thinking, teaching and learning at all levels. The focus includes specific mathematics content and advances in that area accessible to readers, as well as political, social and cultural issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is interested in research based articles as well as historical, philosophical, political, cross-cultural and systems perspectives on mathematics content, its teaching and learning. The journal also includes a monograph series on special topics of interest to the community of readers. The journal is accessed from 110+ countries and its readers include students of mathematics, future and practicing teachers, mathematicians, cognitive psychologists, critical theorists, mathematics/science educators, historians and philosophers of mathematics and science as well as those who pursue mathematics recreationally. The 40 member editorial board reflects this diversity. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor in APA style. The typical time period from submission to publication is 8-11 months. Please visit the journal website at <http://www.math.umt.edu/TMME/>.

The journal is published tri-annually and is a former affiliate of The Montana Council of Teachers of Mathematics, who have free access to the journal through the UM-Department of Mathematical Sciences.

Indexing Information

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"Dulce lumen et delectabile est oculis videre solem" -Qoheleth

AIMS, SCOPE AND EDITORIAL INFORMATION

0. Editorial – “Glocal”, “Glocavores”: Good Gadgetry?
Bharath Sriraman (Montana, USA).....pp.401-404

Special Issue: Optimizing Student Understanding in Mathematics (PMENA, 2010)

1. Research on Practical Rationality: Studying the justification of actions in
mathematics teaching
Patricio Herbst & Daniel Chazan (USA).....pp.405-462

2. Practical Rationality, the disciplinary obligation, and authentic mathematical
work: A Look at Geometry
Deborah Moore-Russo and Michael Weiss (USA).....pp.463-482

3. Complex Learning through cognitively demanding tasks
Lyn D. English (Australia).....pp.483-506

4. Conceptualizations and Issues related to Learning Progressions, Learning
Trajectories, and Levels of Sophistication
Michael T. Battista (USA).....pp.507-570

5. On the idea of Learning Trajectories: Promises and Pitfalls
Susan B. Empson (USA).....pp.571-596

6. Can Dual processing Theories of Thinking inform conceptual learning in
Mathematics?
Ron Tzur (USA).....pp.597-636

7. Table of Contents of Vol.9, nos 1& 2 [January 2012]



Editorial - "Glocal", "Glocavores": Good Gadgetry?

Bharath Sriraman
The University of Montana

In the movie "Up in the Air", *Glocal* is a neologism, a clever witticism conjured up by a naïve business school graduate at an agency hired by corporations to let go of employees that become redundant in corporate mergers/cuts. Her solution to the unpleasant nature of telling *real* human beings that they no longer had a job was to try and automate this inhumane process, i.e., have a scripted flowchart be read out at a safe distance via computers. In theory, a local call center could take care of firings done globally, at very little cost to the agency. Thus the neologism "glocal" suggests the gadgetry of global connectivity afforded through computer networks at our disposal for destructive and constructive purposes. The reader is wondering what the editorial alliteration has to do with the present issue of the journal. An explanation is in order.

First, the journal would not exist but for the support of the global community of scholars regularly contributing to it. Second, the title of the journal no longer has the local label "Montana" attached to it. This Spanish word meaning "land of mountains" has already been appropriated by a Danish furniture company that makes high end wooden furniture for homes, as well as characters from the film and cine media. It was high time for the journal to shed old skin and embrace the generic title "The Mathematics Enthusiast", which more accurately reflects the nature of the journal, and the directions in which it has grown.

In the last 6 years, approximately 5% of the submissions have come from Montana, and usually from my prodding locals to contribute to the journal. A perusal of the table of contents of the journal will reveal that a very large proportion of the articles come from the global community of scholars, and a smaller portion from those in the U.S. To this end the journal has begun to support *The Psychology of Mathematics Education- North America* (PME-NA). In October 2010, I was approached by several colleagues at the annual meeting in Columbus, Ohio with the suggestion that the journal be open to submissions from the community of scholars that form this professional organization. Given the sheer abundance of *unread* Conference proceedings conveniently available in pdf format, which can be mined via search engines, it seemed worthwhile to re-publish a selection of interesting papers in a special issue each year provided they passed an additional burden of peer review. This relationship with PME-

NA is meant to be **anti-symbiotic**, i.e., The Mathematics Enthusiast does not depend on PME-NA in any way- We do quite well on our own and do not need any professional organization to support or sustain us. In a similar vein PME-NA does not depend on The Mathematics Enthusiast either, since it publishes its own conference proceedings each year, and has been a tremendous professional organization for many mathematics education scholars in the U.S, myself included. The only reason the journal is supportive of PME-NA, is to give a possible journal outlet for colleagues at Institutions that do not recognize or value online conference proceedings. It is more or less a bibliometric fact that many Institutions do not give the same point value to a proceedings paper as opposed to a journal article unless the proceedings is listed in a recognized academic index (Sriraman, 2011). Vol8,no.3 of The Mathematics Enthusiast contains 6 extended contributions from the 2010 meeting of PME-NA. The theme of these papers is “optimizing student understanding in mathematics”.

The Mathematics Enthusiast is not a periodical like The Mathematics Teacher or The College Mathematics Journal. However, there are some elements of these two journals in articles addressing the teaching of mathematics content or simply mathematical content at the school and university levels respectively. The journal is also not a pure mathematics education research journal either, although it regularly features articles from the mathematics education research community. Our goal is to remain eclectic and open to the wider community of scholars besides mathematicians and mathematics educators. It is often the case that those looking into mathematics through a different disciplinary lens can offer perspectives that are surprisingly refreshing, and of interest to the community of readers.

Vol9, nos 1&2 [January 2012] of the journal will also be available in early August, in the online medium 6 months in advance. The print version of this issue will become available from Information Age Publishing in January 2012. Vol.9, no.3 [June 2012] will contain extended papers from the North Calotte Conference in Mathematics Education that took place in Tromso, Norway in 2010. The delay is due to being unable to locate appropriate reviewers for the submissions. The journal strives to find researchers who are capable of giving constructive reviews and familiar with the content of the article. Sometimes this becomes difficult, and the “objectivity” or the “black box” of blind review often results in reviews that are not helpful to the author in question, nor the journal. There is an analogy to the “firing” process at corporations mentioned in the first paragraph, and the “rejection” process of manuscripts in many journals. We are trying very hard to devise a completely open peer review system, where Latourian black boxes do not govern decisions that can affect authors (Sriraman, 2011).

The monograph series affiliated with the journal retains the “Montana” moniker and has 5 monographs in development for release in the next two years. One of these monographs is particularly ambitious because it attempts to cover the state of the art of mathematics education in China, Korea, Singapore, Japan, Malaysia, and India. This is slated for release late next year with a preliminary book of extended abstracts available free on the website for those interested.

On a parting note and in keeping with the neologism “glocal”, the community of 21st century readers of the journal can be thought of as glocavores (as opposed to locavores), since we readily consume ideas that spawn all around the world. In a more global sense, the Arab Spring is a testament to the fact the connectivity can be construed as a useful/constructive tool for instigating change- of the self, of ideas, and of the notion of “glocal” as a good thing, as opposed to the way it was conceived of by the female protagonist in “Up in the Air”.

Reference

Sriraman, B. (2011). Dogmatism and the Knowledge Industry: More Accurately Assessing the Work of Gifted Scholars. In D. Ambrose, R. Sternberg, B. Sriraman (Eds). *Confronting Dogmatism in Gifted Education*, Routledge, Taylor & Francis, in press.

Sriraman

Research on Practical Rationality: Studying the Justification of Actions in Mathematics Teaching

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&
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Abstract: *Building on our earlier work conceptualizing teaching as the management of instructional exchanges, we lay out a theory of the practical rationality of mathematics teaching – that is, a theory of the grounds upon which instructional actions specific to mathematics can be justified or rebuffed. We do that from a perspective informed by what experienced practitioners consider viable but also in ways that suggest operational avenues for the study of instructional improvement, in particular for improvements that enable students to do more authentic mathematical work. We show how different kinds of experiments can be used to engage in theory building and provide examples of initial work in building this theory.*

Keywords: Mathematics instruction; Practical Rationality; Theory of teaching; Teacher education

Introduction

In this paper we address the work of the mathematics teacher in instruction and the rationality behind this work. We first sketch out how the teacher's work could conceivably contribute to the creation of opportunities for students to do authentic mathematical work. In that sense we expect that the paper will add to

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our collective sense of what is conceivable and perhaps desirable to happen in classrooms. Most of the paper, however, is concentrated on elaborating on the grounds for possibility and justification of teachers' actions. In particular, what is the rationality that might (or might not) support teachers' management of authentic mathematical work by students?

In accounting for the rationality beneath teachers' actions and in regard to the possibility of enabling authentic mathematical work by students, we take some distance from two relatively commonplace ways of responding to a vision sketch. In one of these approaches, a vision of conceivable mathematical work in classrooms might be followed by an acknowledgment and analysis of the forces and structures that make the vision not viable. Such an approach would summon us to be like social critics of the current educational system, and to endorse a new educational system that would bring all our hopes to fruition. In the other approach, the vision sketch is followed by a busy shaping of persuasive rhetoric, design of efforts, and organization of resources, all of them aimed at making the vision happen against all odds. Such an approach would summon us to be like social engineers, relentlessly working to realize the vision, as if the only thing that separated the conceivable from the viable was the existence of the will to make the vision happen.

Without meaning to disrespect proponents of either of those approaches, we take a third approach, which combines the orientation to improvement of the second with the analytic disposition of the first but poses questions that call

neither for critique nor for engineering but rather for theory and research. We elaborate on the notion that the actions of teachers in classrooms are not mere expressions of their free will and personal resources; rather their actions also attest to adaptations to conditions and constraints in which they work. And yet that realization does not necessarily condemn us to accept the status quo; rather, it can suggest ways of working toward improvement in viable, incremental, and sustainable ways.

How can we think about the distinction, and the gap, between what is conceivable and what is viable in mathematics teaching? How can we find out how much of the vision can be realized within existing conditions and constraints? We argue below that what is required is first to understand and then to co-opt what we have been calling the practical rationality of mathematics teaching (Herbst & Chazan, 2003; Herbst, Nachlieli, & Chazan, 2011). We first recount how the story of practical rationality began and the big picture it serves.

How We Started Our Efforts to Explain Teaching

We started to work together back in 2000, following our common interest in understanding the teaching of mathematics at the secondary level and our shared sense of the importance of learning the wisdom of the practice (Shulman, 2004). But while the focus of our interest was convergent, our theoretical perspectives and our methods required some work. Chazan had been doing what Ball (2000) calls first-person research: He had been using his own practice

teaching Algebra I to investigate the dilemmas and dynamics that a teacher needs to manage (see Chazan, 2000; Chazan & Ball, 1999; also Lampert, 1985). Herbst had been using the more structuralist notions of didactical contract (Brousseau, 1997) and didactical transposition (Chevallard, 1985) to provide detached observer descriptions and explanations of the work of teaching and its effects on the classroom representation of knowledge (see Herbst, 1998; 1999; 2002a; 2002b). Our conversations at the time had found a good anchor concept in Bourdieu's (1998) notion of *disposition*: an element of practical reason that could be conceived as having two sides, like a coin. Dispositions could be seen by an observer as ordinances to which the individual is subject given the position in which they are, but dispositions could also be experienced as tendencies emanating from the individual and compelling them to act in particular ways (see Herbst & Chazan, 2003; cf. how Lampert, 1985, speaks of *commitments*). Early on the conversation was mostly theoretical, as we searched for ways to complement our perspectives; but then our conversation took a methodological turn.

At about the same time that we started talking about dispositions, the educational research community was dealing with a renewed interest in the use of experimental methods in education, which culminated with reports like Shavelson and Towne (2002) and the establishment of the What Works Clearinghouse by the US Department of Education (<http://ies.ed.gov/ncee/wwc/>). The notion was in the air that educational

research should aspire to the gold standard of using experimental design, randomly assigning participants to conditions; and our conversations started to include considerations of hypothesis testing in research on mathematics teaching. As we considered what experimental research in mathematics instruction could look like, it was odd to us that the image that first came to mind was that of research on whether the implementation of an instructional intervention might affect students' performance: Does curriculum X produce better gains than curriculum A on the scale N? To be clear, nothing is odd about thinking of curriculum or pedagogy implementation in terms of experimental research. What seemed odd to us was that those types of questions would appear as the prototypical examples of how our field might take on the challenge of experimental research.

Experimental research that gauged the achievement gains that could be caused by a particular treatment were clearly worthwhile questions, important for policy and practice, but they were also applied questions, not necessarily illuminating the fundamental phenomena of mathematics instruction. We wondered whether embracing an experimental paradigm would necessarily mean that research on mathematics instruction would be limited to asking questions of an applied nature, questions that took for granted that we knew the nature of mathematics instruction well and just had to design and test ways of improving it. Given our experience as classroom researchers, we knew that, at the time, mathematics education research (for a long time focused on learning

and the learner, and later on the individual teacher) still had some ways to go as far as understanding the nature of the activity of mathematics teaching. We thought there was a great need for basic (as opposed to applied) research on mathematics teaching, not just basic research on students or teachers. And so we wondered whether basic research on mathematics instruction had some use for an experimental paradigm.

Instructional Situations and their Norms:

A Focus for Basic Research on Mathematics Teaching

The fundamental idea, proposed by David K. Cohen among others (see Cohen, Raudenbush, and Ball, 2003; also Chevallard, 1985; Hawkins, 1974; Henderson, 1963), that instruction consists of the interactions among teacher, students, and content in environments was compelling to us and essential for defining an emerging field. We pondered what basic research on the nature of mathematics instruction could look like if it embraced an experimental paradigm: What kind of interventions could reveal aspects of the nature of *mathematics* instruction? And what aspects of mathematics instruction could we expect to find out about? These questions seemed important, on the one hand, in order to respond to the challenge of using an experimental paradigm. Those questions seemed important, on the other hand, in order to establish a foundation for basic (rather than applied) research on instructional practice in mathematics--research that asked questions distinct from the study of instruction writ large (which might assume that the subject does not matter or that it matters

the same regardless of the particular discipline from which it comes) as well as questions distinct from the study of people (teachers or students) which might perpetuate the reduction of mathematics education research to psychology.

One key idea presented itself as an aspect of mathematics instruction that we wanted to find out more about: If the subject matters in instruction, that is, if mathematics instruction in geometry is a practice distinct from instruction say in Calculus, American History, or Organic Chemistry, we would expect to see regularities of some sort across different cases of instruction in a specific domain. This was anchored by our mutual interest in justification and proof and our question of why, while those practices were current in geometry, they continued to be absent in algebra, in spite of calls for it in reports over the decades: How could it be that the same teacher with the same class, but perhaps at one year's remove, would talk and act so differently in regard to the source of mathematical truth simply due to a shift from geometry to algebra instruction? Additionally, if the regularities observed concerned mathematics instruction as an activity, we would expect to observe regularities that went beyond the knowledge being transacted to include similar ways in which teacher and students managed those knowledge transactions. The word "norm" used in the sociological sense as the normal or unmarked behavior that is tacitly expected in a setting, suggested itself as the name of the object of study. We hypothesized that instruction in specific courses of mathematical study (algebra, geometry, etc.) could be described as abiding by consistent sets of norms, much as other human practices like eating in

a formal dinner or getting a table in a restaurant abide by consistent sets of norms (Garfinkel & Sacks, 1970). And we thought that experimental research could be used to confirm that those norms exist.

Instructional Situations, their Norms, and the Notion of Breaching Experiment

While the observance of norms could be found at various layers of classroom activity (as we indicate below, in particular at the layer of the didactical contract and the layer of the mathematical task), we concentrated on studying norms at the layer that we've called the *instructional situation* (Herbst, 2006). Conceptually, an instructional situation is a type of encounter where an exchange can happen between (1) specific mathematical work done by students and their teacher in moment-to-moment interaction and (2) a claim on students' knowing of a specific item of knowledge at stake. Intuitively one could think of an instructional situation as including a mathematical task and the element of the curriculum that the completion of the task enables the teacher to lay claim on. We model instructional situations by spelling out norms that describe the knowledge and the work being exchanged, who is expected to do what, and when those different actions are supposed to happen (see Herbst & Miyakawa, 2008; Herbst, Chen, Weiss, & González, 2009).

Herbst's own research studying the work of the teacher managing the instructional situation of 'doing proofs' in high school geometry provided an example of a norm: students are expected to justify a statement in a proof with a reason before they move on to make the next statement. In proposing it as a

norm, we did not mean to endorse the norm as appropriate, but to describe what classroom participants – teacher and students – would consider appropriate. We were not willing to posit that those norms would necessarily be explicit for teachers or students: We expected that people might act as if they followed norms but not necessarily bring them up if and when they were asked to describe the activities they do. And we realized also that, unlike physical laws those norms of human activity could not be thought of as inevitable; they could in fact be broken – one could conceive of and actually find a teacher who had let a student make a new statement without having justified the previous one. While one would expect that a large number of observations of a similar instructional situation would reveal compliance with norms more often than non compliance, the notion that mathematics instruction is regulated by norms could not be validated solely through the observation of regularities in action. We needed empirical ways of attesting that even if a norm had actually been breached, people familiar with the practice would have expected it to be fulfilled.

The notion that basic research on mathematics instruction could consist of finding out about the norms of instruction in subject specific situations, along with the particular notion of a norm as a tacit, shared expectation for action, led us to an idea for how to pick up the challenge of doing experimental research. We were inspired by the ethnomethodological notion of breaching experiments (Garfinkel & Sacks, 1970; Mehan & Wood, 1975), which the first author was already adapting for use in classroom research (Herbst, 2003, 2006). We thought

this notion could be adapted to deliberately bring to the surface practitioners' sense of the norms of instruction. If we could represent to practitioners (for example, through a videotaped episode of instruction, but also possibly through an animation or through a virtual reality experience) action that purported to be of the same kind of what they would ordinarily do, but where a hypothesized norm of that action had been breached, we might be able to hear from practitioners whether they had expected the norm to hold. In that sense, a representation of teaching that included the breach of a norm could be expected to reproduce deliberately the phenomenon of interest, namely, that practitioners expected that norm to hold. The extent to which those procedures could be called *experiments* refers to Francis Bacon's notion of experiment in scientific inquiry: "there remains simple experience; which, if taken as it comes, is called accident," "if sought for, experiment" (cited in Durant, 1926, p. 146). That is, our earlier conception of doing experimental research only abided by the notion of experiment as the deliberate reproduction of a phenomenon. But one could also see at least as a possibility that the modern conception of experiment, which emphasizes reproduction of the phenomenon under controlled conditions by way of random assignment of participants to conditions, could be used to confirm that a norm holds: Imagine having two representations of teaching that differed only in that in one of them (the control condition) a hypothesized norm held while in the other (the treatment condition) the hypothesized norm has been breached. Imagine a sample of practitioners who have a comparable degree

of socialization in the practice where the norm is supposed to hold. Imagine randomly assigning those participants to one or another representation. Imagine having a way of gauging their satisfaction with the instruction experienced and comparing both groups in regard to that assessment. That gave us a skeleton of what basic experimental research on instruction could look like and some impetus for initial work on a project that we would later call Thought Experiments in Mathematics Teaching (ThEMaT).

Thought Experiments in Mathematics Teaching

The notions of instructional situation, norm, and breaching experiments led us first to gather video records from a geometry lesson on proofs where the teacher allowed a student at the board to omit the justification of a statement and to move on with the proof. We started by gathering focus groups of geometry teachers that looked at that video record and then examining the discourse of those focus groups for comments that might provide evidence that teachers in the focus groups had seen the actions of the videotaped teacher as breaching a norm (Herbst & Chazan, 2003; Nachlieli & Herbst, 2009; Weiss, Herbst, & Chen, 2009). At the same time that this work was being done we started exploring the use of animations to represent classroom scenarios and we wrote a grant proposal for Thought Experiments in Mathematics Teaching to the National Science Foundation, asking for support to create animations that helped us study what by then we had started calling the *practical rationality of mathematics teaching*.

Thought Experiments in Mathematics Teaching (ThEMaT) was funded in 2004 and, among other things, it enabled us to create seventeen families of animated classroom stories (the stories can be seen in LessonSketch, www.lessonsketch.org). The animations use simple cartoon characters and voice over to represent scenarios of classroom instruction. The use of animations allowed us to control the content of those scenarios, allowing us to design scenarios that breach a norm but comply with others. Animations also allowed us to produce breaches that had not been observed in actual classrooms (thus showing one important advantage over video records). And this media also allowed us to create stories that branched, thus depicting alternative scenarios that proceeded from a common trunk (thus our reference to families of stories, since many of them have several alternative stories; see Chazan & Herbst, 2012; Herbst, Chazan, Chen, Chieu, & Weiss, 2011; Herbst, Nachlieli, & Chazan, 2011). The generous support of the National Science Foundation has been crucial for us to maintain a research program that, in our view, has contributed to the field not only an important technique for data collection but also some useful theoretical and methodological ideas.

The goals of the research program are quite ambitious: To develop and test a theory of the rationality of instructional practices in mathematics. This theory of the rationality of instruction explains what instructional actions are justifiable by drawing on two elements (1) the norms that the practice of teaching a particular mathematics course imposes on whoever plays the role of teacher, and (2) the

obligations that the profession of mathematics teaching requires of anybody taking the position of mathematics teacher. Combined with the personal assets (including knowledge, skills, and beliefs) that an individual teacher brings with them to that position and that role, those norms and obligations can help explain teacher action and decision-making. The project is now on its second funding cycle in which we are designing and using an online interface (LessonSketch, www.lessonsketch.org) to deliver online multimedia experiences that include animations and other cartoon-based representations of teaching. The project designs multimedia experiences and questionnaires that confront individuals or groups of teachers with representations of teaching; the project will investigate how responses to those questionnaires correlate with measures of mathematical knowledge for teaching (MKT; Ball, Thames, & Phelps, 2008). Over the years, project ThEMaT has allowed us not only to probe and ground our ideas about norms and develop instruments but also to deepen the theory and make progress, though we have not yet used an experimental paradigm in quite the sense described above. Our interventions thus far are experiments in the sense that they reproduce predicted phenomena (evidence of the breach of a norm), but they have not yet reached the gold standard of controlled conditions by random assignment. These conditions may be fulfilled through our current efforts with LessonSketch: An authoring tool in the LessonSketch environment allows us to create online multimedia experiences that may be randomly assigned to participants (see Inglis & Mejía Ramos, 2009, for an example of a

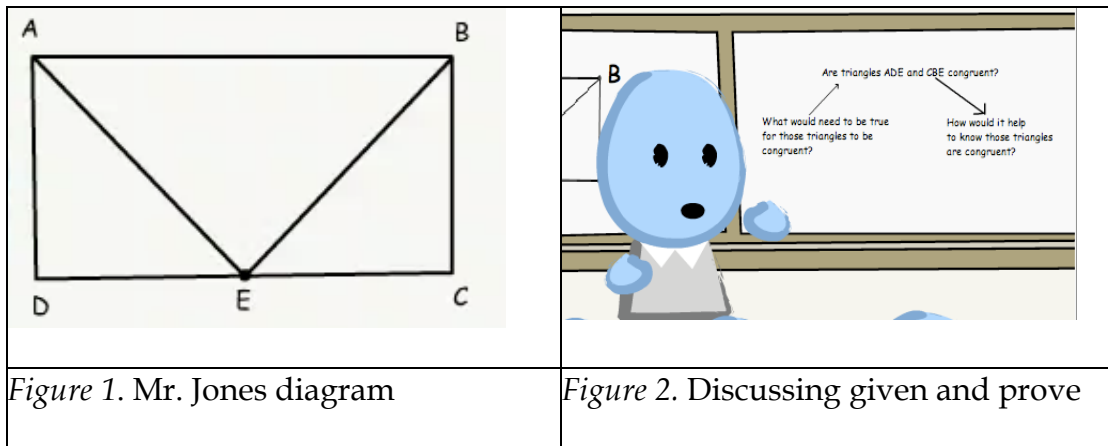
similar use of the internet in experimental research in mathematics education). While the foregoing describes the story of our work, we use the following sections to expand on the ideas and some of the methods.

Explicating Practical Rationality

A Classroom Scenario

Consider what we would call a thought experiment in mathematics teaching. The action happens in a high school geometry course in late November. The class has spent some time learning to use triangle congruence to prove statements and has begun the study of quadrilaterals. The teacher, Mr. Jones, has drawn a figure on the board (see Figure 1) and wants the class to prove a statement about the relationship between the sides of the rectangle $ABCD$. There is some hesitation. Somebody asks whether they could prove that \overline{AB} is longer than \overline{BC} while another student asks what they have to go on; the teacher lets those comments pass. A student asks whether triangles ADE and BCE are congruent. Mr. Jones writes this question on the board and draws two arrows from it. One arrow points toward a question he writes, "how would it help to know that those triangles are congruent?" The other arrow points toward another question he also writes, "what would you need to assume to be able to say that those triangles are congruent?" You can hear somebody say that it's obvious that they are congruent while another says that they could then say the triangles are isosceles. Another student says, "you'd need to know that AEB is a right angle;" Mr. Jones writes this on the board and asks the class what they have

to say about that (see Figure 2). Some students claim to not really know what the teacher means with that question but others raise their hands. One of these students says that she thinks it would be useful if the angle were right because then the angles at the top would be congruent with the small angles at E. Some kids perk up and one kid says, “and you could then say that AB is twice BC.” The teacher asks them to take a few minutes and see if they can prove that the ratio between the sides is 2 assuming as little as possible. You see a kid write, “Prove: The ratio is 2 ” while others have written “Given:” and are pensive.



For a few years now, in the context of the project Thought Experiments in Mathematics Teaching, we have been creating cartoon-based representations of teaching that illustrate conceivable scenarios of instruction. One of them is the story “A Proof about Rectangles,” a version of which we’ve just described. Now we want to use that episode to raise a few questions about mathematics

instruction in school classrooms and to elaborate on the ideas that this kind of material has helped us explore.

Some of these questions concern the substance of this conceivable episode: What opportunities for students' mathematical work are made possible by how the teacher has been managing the instruction? Other questions are about theory: What kind of considerations about classroom instruction could help us describe and explain how teacher and students ordinarily transact mathematical ideas, in such a way that we could also account for possible avenues for improvement and foresee their consequences? Finally, other questions are about research methodology: What kind of data can help us ground those theoretical considerations? How to obtain it? These questions, though large, serve to explicate the program of research that we call the *practical rationality of mathematics teaching* (Herbst & Chazan, 2003; Herbst, Nachlieli, & Chazan, 2011).

Desirable and Customary Mathematical Work

What mathematical work are students doing in the episode described above? We could describe it as listing plausible statements about a figure and considering whether these plausible statements could be connected through logical necessity. The source of some of those statements seems to be perceptual—for example, the observation that angle AEB is right. Other statements seem to result from deduction—notably, the observation that if the angle AEB was right then one could conclude that side \overline{AB} would be twice as long as side \overline{BC} . But regardless of the origin of each of those statements, the

teacher is helping students connect all statements through abduction and deduction: Asking what assumptions would enable one to infer the plausible statement made and asking what inferences could be made if one took that plausible statement for granted. The assertion about the relative length of the sides of the rectangle eventually derives from the plausible truth of those earlier statements. The teacher is thus helping students reduce a question of truth (what could be true about an object) to a question of deducibility from possible statements about an object. They are using proof as a method to find things out.

Such use of proof as method in knowledge inquiry is essential to the discipline of mathematics (Lakatos, 1976). It is also behind the drive to mathematically model other fields of experience: The expectation that in those fields it will also be possible to reduce the problem of truth to a quest for deducibility, which can then warrant new, still unknown, possible truths is important in pure and applied science. Hanna & Jahnke (1996) have argued that, by using an empirical theory to predict empirical phenomena, scientists engage in modeling the world and deductively producing inferences based on assumptions, predictions that are eventually subject to confirmation by experimentation.

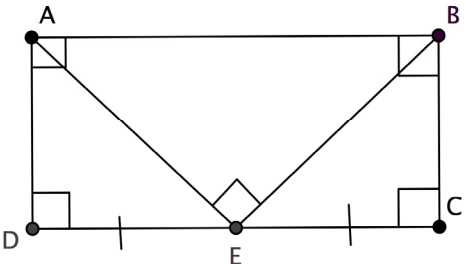
Being able to master such a form of inquiry can make a child resourceful in ways that can add to methodological resources they get from the study of other disciplines. Mathematical work of the kind depicted in the scenario is not only authentic mathematical work (Weiss, Herbst, & Chen, 2009) but also embodies

skills and processes that might empower students to contribute to knowledge production writ large. In that sense we would argue that Mr. Jones's questions to students about what could be deduced from a given statement, or what statement could entail what they think is true, are helpful ways of educating his students in the use of mathematical reasoning for making predictions about the world, in this case about the world of diagrams. A scenario where students could work on connecting plausible statements deductively is therefore conceivable and it could be represented using animations or comic strips with cartoon characters.

However, it is likely the case that few students encounter such opportunities to engage with proof in school mathematics in the way outlined by the foregoing scenario. The work they do during their school years rarely includes chances to acquire the skill or the appreciation of the methodological, model-making function of proof or even experiences doing work that could have had that exchange value.

It is more likely that the problem above would be presented to high school geometry students as shown in Figure 3. In particular, while students are ordinarily expected to prove propositions in high school geometry, it is ordinarily the teacher (or the book) who will state the givens and the conclusion of the propositions they prove. While efforts to change these norms have been made (e.g., the work with the Geometric Supposers reported in Schwartz, Yerushalmy, & Wilson, 1993), it rarely falls on the students to determine the

givens for a plausible conclusion, to deduce the conclusion from a set of givens, or to find both the givens and the conclusion for a theorem that relates to some plausible naïve conjecture.

	<p>•Given:</p> <p>$ABCD$ rectangle,</p> <p>E midpoint of \overline{DC},</p> <p>$\square AEB$ right angle</p> <p>Prove: $\frac{AB}{BC} = 2$</p>
<p>Figure 3. A more likely proof problem.</p>	

The Scenario as an Example of Norms and Instructional Situations

The expectation that, if students are to be held accountable for producing a proof, the teacher will have to provide for them the givens and the “prove” statement is an example of what we call a *norm* of the instructional situation “doing proofs.” It is a norm in the sense that an observer can describe teachers and students acting *as if* they expected that this would be the case. In consequence, if students and teacher were involved in an interaction about a problem for which the teacher did not provide the given and the “prove”, then it is likely that neither teachers nor students would describe those activities as doing proofs—they might describe them as something else (e.g., having a

discussion). The norm is that anytime the students are expected to produce a proof, teachers are expected to provide the givens and the conclusion to prove. Of course by “norm” we don’t mean ‘the correct thing to do’; it is certainly not “correct” from our perspective informed by our understanding of mathematical practice, though it may be experienced as correct or appropriate by teachers and students. We use “norm” and “normative” in two complementary senses: First, the sense in which ‘normative’ means ‘frequent’ or ‘usual;’ this could be corroborated empirically by observing, over a large number of high school geometry classrooms, the recurrence of this feature in proof activity. Second, the sense in which the participants in the situation act as if they expected such behaviors to be appropriate or correct.

Such norms are not just arbitrary belief systems, idiosyncratic and completely changeable; they are norms of interaction between teacher, students, and specific content and are thus ascribed not to individuals but to the specific instructional situation where that interaction happens. They have a particular purpose; they regulate the division of labor over time between student and teacher vis-à-vis a specific kind of instructional exchange. In this case, this norm regulates the exchange between the work students do when proving a proposition and the claim (that the teacher needs to substantiate in high school Geometry) that students know how to do proofs. In that sense, this norm is different in scope than the more general norms of the didactical contract, which are present across different instructional exchanges (e.g., the expectation that when the teacher asks

students a question, she already knows the answer). In trying to understand the practical rationality that underlies that norm and the possibilities to depart from it we are therefore asking not a question about instruction in general (e.g., Do teachers see it as possible, desirable, or appropriate to have students work on tasks where they determine the givens or the goal?) but rather a question about what counts as doing a proof in high school geometry: Do teachers see it as possible, desirable, or appropriate to hold students accountable for doing a proof and to do so in the context of tasks where students are in charge of providing the givens or the conclusion of the proof problem? To us it seems that such tasks would enable students to experience and learn about the methodological role of proof: Its instrumentality in finding new knowledge. But, such tasks are not common in classrooms.

“Doing proofs” in high school geometry illustrates what we mean by an instructional situation. These are frames for the encounter among teacher, students, and specific content: In these encounters an instructional exchange takes place—the exchange between the work that students do, for example, on a particular task, and the knowledge claim that such work enables the teacher to make by virtue of having done that work. Instructional situations can be modeled as systems of norms such as the one described above. Instructional situations are content-specific in two regards: They accommodate or make room for specific tasks, and they permit the exchange of work on those tasks for specific items of knowledge. The instructional situation “doing proofs” does not

customarily accommodate students' work in which students produce the givens or the 'prove' for a proof problem; rather, if they are ever involved in such work, their involvement does not count as knowledge of proof. Based on our understanding of the methodological role of proof in mathematics (Lakatos, 1976) we argue that such work (figuring out the givens or the conclusion) does not always precede but it is often part of the work of proving in mathematics.

Is it Feasible to Change Instructional Situations?

A motivation for our work has been to understand better whether the kind of mathematical work described above—the use of proof as a tool to know with—could feasibly be deployed in classrooms. One way of addressing that question focuses on the design of resources that can support that work. And some of our instructional experiments (e.g., Herbst, 2003, 2006) have included developing resources, including special lessons and units co-developed with teachers. In those, problems were designed to create contexts where proving could help students come up with an answer to the problem. Our focus on the feasibility of that work led us not only to investigate whether proof could play a role as a tool to know with (see Herbst, 2005) but also to investigate what kinds of disruptions of the work of teaching those tasks would cause (Herbst, 2003) and what sorts of negotiations a teacher needed to make to restore a sense of normalcy (Herbst, 2006).

Another way of addressing the feasibility question goes beyond investigating what is possible when teachers use different tasks to engage students in proving

and taps into the source of arguments that teachers could draw upon to justify or rebuff such tasks. Behind that version of the feasibility question is the fundamental hypothesis that classrooms are complex systems where actions are not merely a projection of the will or capacity of the actors or the richness of their resources. Rather, actions of individual actors contribute to the deployment of a joint activity system whose performance also feeds back, and thus gives shape, to the actions that the participants can take in that system. And at least tacitly and as a group, teachers of a given course know the demands of that system to the point that we should be well advised to canvass that knowledge if we intend to understand whether a particular improvement will be feasible or not. The question then is not simply how to design materials that enable desirable mathematical work or how to create in teachers the desire to promote that work. We also need to ask about the structure and function of the activity system where that work might be deployed and how this system might accommodate or resist attempts to deploy that work. In particular this requires thinking of mathematics instruction in school classrooms as a system of relationships that are deployed under various conditions and constraints. A conceptualization of this system could enable us to think in a more sophisticated and potentially accurate way about what teacher and students do and thus be able to foresee if given improvement efforts have a prospect of success.

An analogy with how mathematics educators have evolved in their thinking about students' errors can illuminate this conceptualization of instruction as a

system. There used to be a time when student errors were seen as indications of misfit, mishaps, or forgetfulness. Things changed when research on students' mathematical work started to be treated within a cognitive paradigm. For example, an international study led by Lauren Resnick, Pearla Nesher, and François Leonard (see Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989) on students' sorting of decimal fractions showed that students' errors had a conceptual basis: Their errors could be explained by the existence of conceptual, tacit controls such as the "fraction rule" or the "natural number rule." These were mathematical quasi-truths, or epistemological obstacles (Brousseau, 1997), true within a limited domain but false when that domain was extended. Students that made errors did so not out of the lack of knowledge but out of the possession of some knowledge. As a field, our stance toward students' errors thus changed from an early judgment stance to a later inquiry stance: Rather than judging students as irrational when they make errors, we now strive to understand what rationality leads them to make those errors.

We propose that we should think of the actions of teachers (and students) in the classroom by analogy with how we have come to think about error in children's mathematical thinking. The analogy we propose is that we could think of "error" in instruction—really teaching that deviates from what might be deemed desirable—not as an indication of misfit, ill will, or lack of knowledge, on the part of the practitioner. Rather, we should think of this "error" as an indication of the possible presence of some knowledge, knowledge of what to do,

which is subject to a practical rationality that justifies it. This is a rationality that we should try to understand better before judging teachers or attempting to legislate their practice. It is this rationality, rather than simple stubbornness, that explains why many reforms are not able to make their way into classrooms. Teachers and students act in classrooms in ways that attest to the existence of specialized knowledge of what to do; knowledge that outsiders to those classrooms are less likely to have even if they know the knowledge domain being taught and learned. For example, as it relates to the scenario narrated above, teachers and students of geometry would likely see it as strange for Mr. Jones to ask the students for the givens of the problem. We focus here on the rationality associated with the role of the teacher and how this might warrant or refute actions like that one.

Practical Rationality and the Role of the Teacher

The “teacher” of a specific course of mathematical studies, such as high school geometry, is an institutional role, not just a name to describe an aspect of an individual’s identity (Buchmann, 1986). There is a person who plays the role, for sure; that person comes to play the role with personal assets that are likely to matter in what he or she chooses to do. These assets are likely to include mathematical knowledge for teaching and skill at doing some tasks of teaching (Ball, Thames, and Phelps, 2008). It is widely believed that those assets make a difference; that teachers who have those assets may be able to figure out and do things that others may not be able to do. But while teachers’ causes and motives

to do things may have personal grounds, it is unlikely that their actions could be justified on personal grounds. One could imagine that Mr. Jones in the scenario above might have been bored with the prospect of giving his students another routine proof exercise or wanted to have a fun day teaching geometry. But we could not really expect him to use any of that as the warrant for doing what he did – his job is not to find activities that amuse him, but rather to teach geometry to his students. Even if the actual basis for his actions had been his own amusement, how could he justify having done that when talking with his peers? Those grounds for justification are what we call practical rationality.

The notion of practical rationality points to a container of dispositions that could have currency in a collective, for example, within the set of colleagues who teach geometry in similar settings. These are dispositions to abide by the *norms* of the specific instructional situation a teacher is engaged in (i.e., the norms of the situation of doing proofs in high school geometry) as well as dispositions to honor the *obligations* to the profession of mathematics teaching.

By dispositions we mean what Bourdieu (1998) describes as the categories of perception and appreciation that compel agents in a practice to act in specific ways. We interpret categories of perception to include the taken as shared ways in which practitioners perceive people, events, things, and ideas in the shared world of the classroom, as instantiated, for example in the language tokens they use to talk about the world of the classroom. We interpret categories of appreciation to include the principles and qualities on which practitioners rely to

establish an attitude toward people, events, things, or ideas. Dispositions tend to be tacit but they can be articulated to others when justifying to one's peers (or to other stakeholders) why one might or might not do something like what Mr. Jones did with that proof problem. The high school geometry course and the work of doing proofs, in particular, have been particularly fertile grounds for us to develop theory about instruction and the practical rationality of mathematics teaching.

Didactical Contract and the Role of the Teacher

To conceptualize the work of the teacher as the playing of a role, we start from the notion of the didactical contract (Brousseau, 1997): The hypothesis that student and teacher have some basic roles and responsibilities vis-à-vis a body of knowledge at stake. What does it mean that there is knowledge *at stake*? The relationship between teacher and students exists because of the assumption that there is knowledge that can be communicated from one to the other; this knowledge is at stake because such communication may or may not happen. The didactical contract is a tacit assignment of rights and responsibilities between teacher and student vis-à-vis the communication of that knowledge. These responsibilities include the expectation for the teacher to give students work to do that is supposed to create opportunities to learn elements of that body of knowledge, and the expectation for the student to engage in the work assigned, producing work that can be assessed as evidence of having (or not yet having) acquired that knowledge.

We use the word *norm* to designate each of those statements that an observer makes in an effort to articulate what regulates a practice: Actors act *as if* they held such statement as a norm, though they may be quite unaware of it. Each class has a didactical contract that can be modeled by listing its norms. From the perspective of the teacher, the didactical contract authorizes a basic exchange economy of knowledge that he or she has to manage: An exchange between work designed for, assigned to, and completed by students and elements of knowledge, prescribed by the curriculum, at stake in that work, and hopefully embodied in students' productions. The role of the teacher includes managing those exchanges between work and knowledge. This management includes, first, enabling and supporting mathematical work; and second, interpreting the results of this work, exchanging it for the knowledge at stake.

The hypothesis of a didactical contract only says that a contract exists that fulfills those goals; the hypothesis means to describe any mathematics teaching inside an educational institution. But it is also obvious that the teacher and student roles and responsibilities are under-described by that hypothesis: There are many ways in which the didactical contract could be enacted that would have at least those characteristics; contracts could be quite different from each other not the least because the mathematics at stake could be very different from course to course and thus require very different forms of work to be learned. Even for the same course of studies, say high school geometry, different contracts could further stipulate the roles and responsibilities of teacher and student

differently. In particular, it is conceivable that some contracts might include the expectation that every new task would require negotiation about how the general norms of the contract apply (e.g., What is it required of the teacher to get students to work on a particular task? What does it mean for students to work on that task?). It is also conceivable, and we argue more likely, that contracts rely on a manifold of instructional situations that forego the need for some of those negotiations much of the time. These instructional situations include mostly tacit but specific norms that specify how the didactical contract applies for a range of tasks and the specific items of knowledge to be exchanged for the students' work on those tasks.

While some research has endeavored to conceptualize, enact, and study the characteristics of alternative contracts (e.g., Chazan, 2000; Lampert, 1990, 2001; Yackel & Cobb, 1996), the first author has been interested in using a variety of approaches to study the usual high school geometry contract and the practical rationality behind the teachers' work managing the exchanges enabled by that contract. The reason for that is founded on the considerations about improvement made earlier. Sustainable improvement in instruction will not only need to provide new and better resources but also to be able to deal constructively with the inertia and possible reactions from established practice. Knowledge of how instruction usually works and what rationality underpins its usual operations is key for the design of reforms that are viable and sustainable. Furthermore, knowledge of how usual instruction works can encourage

piecemeal, incremental changes that don't throw the proverbial baby with the bathwater.

Instructional Situations and the Role of the Teacher

The situation of "doing proofs" has been a useful starting point in that research agenda. Historical analysis (Herbst, 2002b; González & Herbst, 2006) has showed how the general skill "how to do proofs" became an object of study in and of itself, leaving behind the important relationships between proofs and specific concepts, theorems, and theories. The work that students do has also evolved to the current state in which *what* a student can prove from available givens matters much less than *whether* and *how well* they carry out a proof. In exchange for a claim on that knowledge (to show that they know "how to do proofs") students are to show that they can connect a "given" with a "prove" by making a sequence of statements justified with prior knowledge (regardless of the strength or the importance of the proposition proved): In other words students are learning the logical form of proof at the expense of its methodological function. In describing such exchange as an instructional situation, we posit that this exchange is facilitated by a specialized set of norms that elaborate how the didactical contract applies.

From observing work in geometry classrooms we have noted that implicit expectations of who is to do what and when vary depending on the specifics of the object of study. In relation to diagrams, for example, the extent to which students can draw objects into a diagram or draw observations from a diagram

varies according to whether the work is framed as a construction, an exploration, or a proof (Herbst, 2004). While the didactical contract for a course may have some general norms that differentiate it from a contract for a different course, there is also differentiation between the more specific norms within a given course of studies, depending again on what is at stake. Much of those rules are cued in classroom interaction through the use of selected words such as *prove*, *construct*, or *conjecture*. These words frame classroom interaction by summoning special, mutual expectations, or *norms*, of who can do what and when. As noted above, we use the expression *instructional situation* to refer to each of those frames. Instructional situations are specialized, local versions of the didactical contract that frame particular exchanges of work for knowledge, obviating the need to negotiate how the contract applies for a specific chunk of work.

“Doing proofs” is an example of an instructional situation in high school geometry; “solving equations” is an example of an instructional situation in algebra I (Chazan & Lueke, 2009). We contend that these frames for classroom interaction, these instructional situations, are defaults for classroom interaction, tacit knowledge held by the classroom as an organization (Cook & Brown, 1999) that specifies what to do; knowledge perpetuated through socialization (and with the aid of textbooks and colleagues) that, in particular, provides cues for the teacher on what to do and what to expect the student to do. Instructional situations are sociotechnical units of analysis; they organize joint action with specific content.

Our perspective centers on the situation rather than the individual and has the power to explain why the same individual might happen to do quite different things in different situations by no fault of their own. To implement this focus on the situations thus far we have created models of those situations. A model is not a portrait of what is desirable but rather a simplified operational description of a reality, in this case a human activity. Our models consist of arrays of norms that describe each situation in terms of who has to do what and when (Herbst & Miyakawa, 2008). Those models facilitate research on the content of practical rationality.

Practical rationality is a container whose content includes the categories of perception and appreciation that are viable within the profession of mathematics teaching to warrant (or refute) courses of action in teaching. The notions of instructional situation, norm, and breach of a norm are the points of departure to study this rationality empirically. Based on the ethnomethodological notion of a *breaching experiment* (Mehan & Wood, 1975) we propose, as a methodological hypothesis, that if participants in an instructional situation are immersed in an instance of a situation where one of its norms has been breached, they will engage in repair strategies that not only confirm the existence of the norm but also elaborate on the role that the norm plays in the situation or on what might justify departing from the norm.

Our data collection technique relies on representations of breached instances of instructional situations – representations made in videos, slideshows, or comic

strips, sometimes using real teachers and students (e.g., Nachlieli & Herbst, 2009) or using cartoon characters (Herbst, Nachlieli, & Chazan, 2011). We confront usual participants in an instructional situation with a breached representation. For example, the classroom scenario narrated above is quite close in content to an animated classroom story, "A Proof about Rectangles," that we produced in order to study the rationality behind the tacit norm that the teacher is in charge of spelling out the givens and the prove. To find out about that rationality we attend to participants' reactions to the representation: Do they perceive the breach of the norm? Do they accept the situation in spite of the breach? What do they identify as being at risk because of the breach? What opportunities, if any, do they see being created or lost because of the breach?

Our aim is not to understand the participants themselves; our aim is to use the participants' experience with the situation to understand the *situation* better. In particular we want to discover the elements of the practical rationality of mathematics teaching that teachers consider viable justifications of breaches of situations that would arguably be desirable, say because they might create a more authentic kind of mathematical work (see Weiss, Herbst, & Chen, 2009). In the case of the story narrated above we would pose the following concrete question: On what account could a teacher justify (or rebuff) an action like the one Mr. Jones took? Clearly, researchers might be able to justify Mr. Jones' action and we have tried to articulate that from a mathematical perspective. But in spite of the fact that some of us have had experience teaching we don't know teaching

now in the way practitioners do. By virtue of the role that they play and the position from which they take on that role, teachers have to respond to specific obligations that shape their decisions.

Experimentation and Teachers' Responses to a Breach of a Norm

In the previous section we noted that our technique to study the practical rationality with which practitioners might justify abiding by or departing from a norm in an instructional situation consists in creating a representation of practice that instantiates the situation and where the norm in question has been breached, then listening to how teachers respond to that representation. When teachers respond to a breach in an instructional situation, they might reject the situation or might repair the situation. By *reject the situation* we mean that they would come across as saying "this class is not doing a proof;" key in such a categorization is (1) the recognition that someone might argue that the target situation (doing proofs) describes the scenario being enacted and (2) their denial of the validity of such a description. By *repair the situation* we mean a softer version of rejection: participants come across as describing the events using a different situation or as conforming to a contract different than the normative. For example, some teachers have said that Mr. Jones is leading students in an *exploration* rather than a proof.

“the only thing I could see him doing is that he was trying to get them the idea of making conjectures, okay? What, what can we assume about this picture” (ITH062806, 4, 81, Tina)²

“Maybe it's just like a -- kind of a like a blank canvas for just discussing without all of the restrictions tied on at this point just y'know lighter form of conversation y'know.” (ThEMaT082206, 10, 109, Lucille)

Key in categorizing those expressions as *repairs of the situation* are that (1) participants are describing the events in terms of the larger grain size of the teacher’s instructional goal and that (2) participants are using some conventional labels for recurrent classroom activity to describe what happened in ways that fail to recognize the situation as one of “doing proofs” (e.g., conversation, making conjectures).

A third alternative, also present in our data, can be described as participants’ *acceptance* of the situation, namely recognized it as a case of “doing proofs.” For the sake of coding data, whenever participants don’t reject or repair the situation we take that as an acceptance, even if this is tacit. In some of these cases their acceptance of the situation came with comments that indicated that something about the particular task in which “doing proofs” was embodied had not been done as it should have been done. For example, some of our participants said

² References to session data follow the convention (sessionid, interval, turn, speaker). All names are pseudonyms.

“So the fact that he's y'know not marking anything and asking them to kinda trust that drawing is kind of odd” (ThEMaT082206, 20,227, Edwin)

“we tell them not to assume anything that we draw.” (ThEMaT082206, 5,112,Tina)

Among those comments accepting the situation as doing proofs, some comments indicated a positive appraisal of what the teacher had done. For example:

“In the books we always go given-prove, right? So we don't really give them the option to even explore some of the nature of the figures.”

(ThEMaT082206, 10,116, Jillian)

We describe those responses as accepting the situation (the participant identifies or at least does not deny that the goal of the activity is to “do a proof”) but *repairing the task* (while the participant does not cast the situation as different than doing proofs, the participant recognizes some actions as deviating from the norm in that situation). A complete enumeration of contingencies includes, at least conceptually, the possibility that participants may accept the situation and accept the task: However, empirically one might observe those cases to be unmarked (e.g., the participant talks about something other than the breach). Incidentally, note that in this discussion we are proceeding rather globally and omitting considerations of the possible complexities of the unit of analysis for the sake of proposing how the experimental data could be aggregated: While the present considerations might be used to examine data gathered from individual practitioners providing a one-time response to a representation (for example,

responding to a multimedia questionnaire), data gathered from groups of practitioners in more extended conversations (such as those reported by Chazan & Herbst, 2012, or Herbst, Nachlieli, & Chazan, 2011) require more sophisticated considerations of the unit of analysis.

From those broad considerations about the way we might code data from practitioners' responses to a representation of an instructional situation we can anticipate a way of using this data to gauge the extent to which a hypothesized norm pertains to the situation under consideration—and in that way use experimentation to build basic knowledge about the practice of mathematics teaching. Consider first the case of practitioners responding to a representation of an instructional situation in which a hypothesized norm of that situation has been breached (e.g., the teacher asks students to provide the givens for a proof exercise). Consider further that the encounter between practitioners and representations is framed for them as a case of the situation (e.g., the instrument declares something to the effect of “we are going to see how a class works on a proof”) but no mention is made of the possibility that a norm might be breached nor is attention explicitly directed to the actions by which the breach is manifest. After the encounter, participants are asked to comment on how appropriately the teacher handled the situation (e.g., “what do you think of the way the teacher managed the class's engagement in proving”). The data is then coded in ways that permit the aggregation shown in the contingency table below (and drawing on the definitions of reject, repair, and accept given above).

	Accept Situation	Reject or Repair Situation
Reject or Repair Task	2	1
Accept Task	4	3

The hypothesis that the norm breached is a norm for the situation being represented would justify the expectation that data would aggregate in cells 2 and 3. Cell 2 represents responses of the kind ‘in this situation you’d rather do this other work instead’ (e.g., if you want students to do a proof, you give them the givens and the prove). Cell 3 represents responses of the kind ‘the kind of work you are doing there fits better in this other kind of situation’ (e.g., a question like that would be better off in a conversation than in a proof). Data that could be classified in any of those cells would provide evidence that adds credibility to the hypothesis that the norm applies. (Note that this evidence could but would not solely include repairs that specifically mention the norm breached—norms could stay tacit in spite of being breached and the evidence provided by participants might just reveal their sense that something has gone awry.) In contrast, cells 1 and 4 provide evidence that contradicts or at least provides no evidence in favor of the normative nature of the hypothesized norm.

Intuitively, under the hypotheses that the norm applies to the situation, that the representation breaches the norm, and that the participants are experienced

enactors of the situation, one would expect the aggregate of Cells 2 and 3 (repairs of situation or of task) to be higher than the aggregate of Cells 1 and 4. One could define a measure of the extent to which the representation elicits repairs (2 + 3) or percentage of teachers who repaired over those who provided comments.. More generally, given a representation (related to a norm N of a situation S) and a sample of practitioners, the representation could be classified a priori as breaching or non breaching N, and each practitioner could be classified as experienced or not experienced in S. The percentages of repairs could be used in particular, to test (this time using the modern sense of experiment) the extent to which experienced practitioners in a situation hold norm N.

Imagine a sample of experienced practitioners randomly assigned to one of the following two conditions. In the experimental condition the practitioners consider a breached representation, while in the control condition the practitioners consider a compliant representation. The responses from practitioners would then be summarized in corresponding repair ratios $r_{1,e}$ and $r_{0,e}$ as defined above and the difference between these proportions could be tested for significance. Similarly, one could pose the question of whether this norm is significantly more salient for teachers experienced in the situation of interest than for teachers who do not have such experience. This question could lead one to compare the ratios $r_{1,1}$ and $r_{1,0}$, that is, the repair ratios for experienced and non experienced practitioners confronting a breached representation. Finally, one could consider randomly assigning practitioners who

are either experienced or inexperienced in the situation to either a breached or a compliant representation, and analyzing the table of contingencies below. The Chi Square test could be used to examine whether acknowledgment of Norm N is specific to teachers experienced in Situation S.

	Experienced in S	Inexperienced in S
Breached Representation (of N _S)	r _{1,1}	r _{1,0}
Compliant Representation (of S)	r _{0,1}	r _{0,0}

Of course the preceding argument is only a sketch of what the research ahead requires. In addition to the problem of determining the unit of analysis noted above, there remains the problem of finding operational ways of determining repairs, rejections, and acceptances of task and situation. While we have made some important progress identifying norms of situations to be researched and creating representations that breach those norms, the work of developing measures of the repairs that practitioners produce in response to those representations is still incipient. Our current work in this area investigates the use of elements of systemic functional linguistics, particularly the notions of modality and appraisal (Halliday & Matthiessen, 2004; Martin & White, 2007), to anchor the notion of repair in linguistic performance. Furthermore, as far as the implementation of the technique, these considerations oversimplify the certainty with which one can say that a representation of a situation breaches a norm or

complies with all norms—it isn't only that the provisional nature of models challenges the extent to which one can ever say that a representation will be compliant, but the multidimensional and interactive nature of human activity makes it hard to represent breaches of a norm without other remarkable entailments needed for continuity's sake. Along those lines, and because of the extent to which an instance of a situation may instantiate more than the actions specific to a norm, a third challenge consists of being able to reproduce the phenomenon (participants' recognition of the norm) independently of the representation used: Would representations R and R' of different instances of the same situation S , each of which breaches the same norm N , produce similar responses from practitioners experienced in S ? Considering those methodological challenges, it is fitting to say that so far we have only been able to show how our theoretical agenda and basic research goals could use an experimental paradigm and within that to indicate more specific methodological goals.

The sketch above does indicate a path for using an experimental approach in basic research on mathematics teaching—specifically, research that identifies and confirms the existence of specific norms for specific instructional situations. But as noted above, practical rationality includes more than the norms of instructional situations; it includes the categories of perception and appreciation with which practitioners can relate to actual and possible actions in teaching. In particular, practical rationality includes the grounds on which a breach of a norm

might be recognized as a breach and yet appraised favorably. Notwithstanding the possible use of the experimental design sketched above to test hypotheses, it is probably just as important for theory and practice to deepen the descriptive research that can lead to more refined hypotheses, especially hypotheses that can account for the difference between justifiable and unjustifiable breaches of norms.

Practical Rationality and the Justifications for Breaches of Norms

The data that we collect from practitioners in response to breached representations usually contains more than repairs of those breaches. Practitioners not only recognize the presence of a norm when they repair its breach, quite often they do so using discourse that commits a stance toward such a breach. Those stances are not always negative; when these stances are positive, practitioners may engage in a rather visible practical argument to justify an action in spite of the norm against it. As part of the agenda to flesh out the content of practical rationality we are interested in inventorying and accounting for the dispositions used by practitioners to warrant actions that breach norms (as well as those actions that comply with norms).

Sometimes, teachers' responses to breaches of a norm may indict the teacher for breaching a norm and justify it with an argument that explicates why the norm exists. In the case presented above, the evidence we found suggests that the norm of providing the given and the "prove" may be justified on the grounds that it keeps students from making knowledge claims by relying on the

looks of a diagram. Indeed the line between, on the one hand, assuming something as given so as to start drawing necessary consequences from it and, on the other hand, assuming something else as true while one is drawing those consequences, may be blurry enough to justify keeping students from having to manage it. One could represent this argument for a norm by adapting Toulmin's (1969; see also Inglis, Mejía-Ramos, & Simpson, 2007) argument layout, as shown in Figure 4 (where instead of *data* and *claim* we use *circumstances* and *action* respectively).

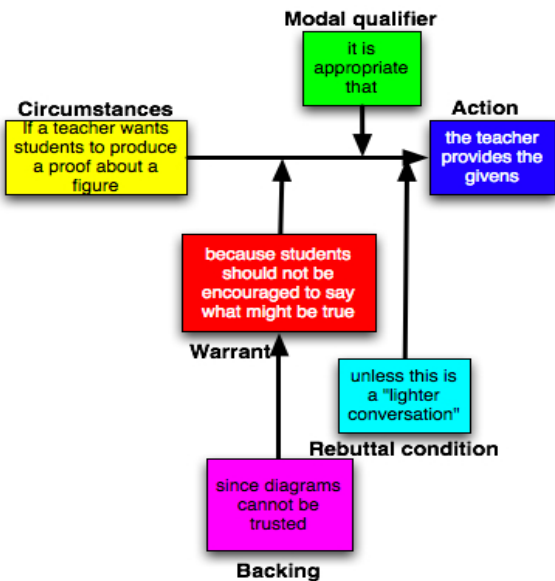


Figure 4. A practical argument using Toulmin's layout.

The data also shows that teachers' responses sometimes acknowledge the breach, but rather than indicting the teacher for the breach they might justify it while relaying whatever reasons they might have for that justification. In this sense, the breaching experiments give access to other elements of the practical rationality of

mathematics teaching. In the data shown above, one of the comments appeared to justify the breach by elaborating on the grounds for exception noted above.

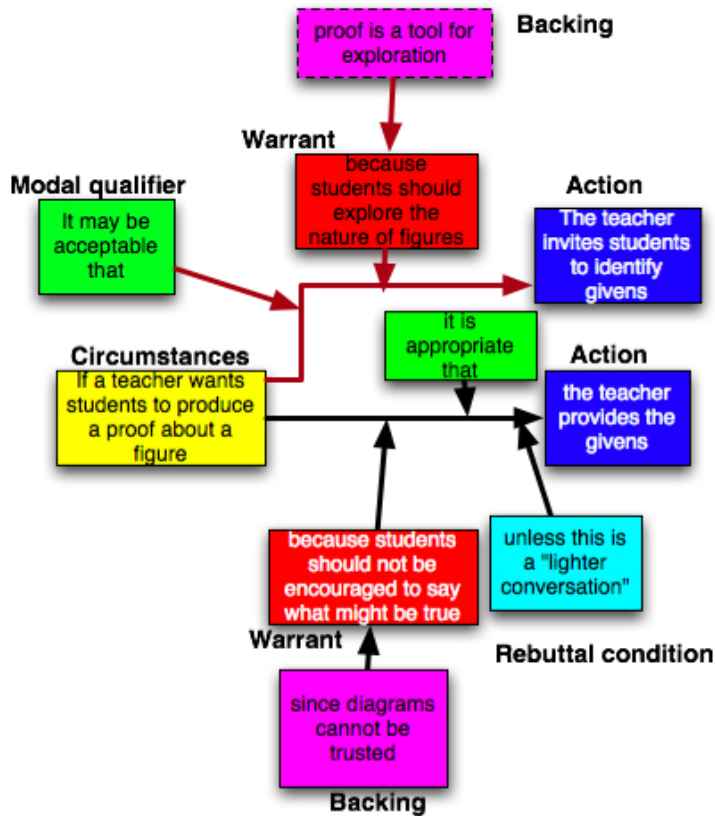


Figure 5. A practical argument for and against an action using Toulmin's layout.

The Norms and Obligations that Span Practical Rationality

From our work in the past five years, looking at the responses from teachers to animations that represent breaches of situations in geometry and algebra, we have built an initial model of this practical rationality. In this model, conceivable moves by a teacher are justified or rebuffed on the basis of principles or warrants that attest to the presence of two sets of regulatory elements. One of those sets of

regulatory elements describes the roles the teacher is called to play in the contract, the instructional situations, or in mathematical tasks. As noted above, we call all of those *norms*: Some are norms of the contract (they regulate work across the many objects of knowledge in a course of studies), while others are norms of the instructional situation (they regulate work that is specific to an object of knowledge). A third kind of norms, norms of the task (regulating how the teacher supports the milieu for the students' mathematical task) is also part of the model but is not discussed here (see Herbst, 2003; also Brousseau, 1997). The other set of regulations, which we explicate below, includes the professional obligations that tie an individual to the position of mathematics teacher, beyond the specific demands of a particular contract, situation, or task.

In general, the first set of regulations for actions in teaching come from the structure of the different 'games' the teacher and the student play with specific content. The various norms that justify teachers' actions respond to the requirements of the role the teacher is called to play in the contract for a course of studies, the situation that frames the different kinds of work that exchange for a particular object of knowledge, and a specific mathematical task. But these norms by themselves don't explain why practitioners see some breaches of norms as acceptable (see, for example, Nachlieli & Herbst, 2009; Herbst, Nachlieli, & Chazan, 2011). The data that we have gathered shows not only that the norm exists and what problems it would help solve, but also on what grounds it could be breached. As we analyze the data from study groups that considered the

many animations we created in ThEMaT, a more systematic way of accounting for those warrants has become useful to us.

Both the presence of norms and the breaches of norms can be accounted for by appeal to various professional *obligations* that we posit apply to the mathematics teacher (to some extent these obligations may also apply to the elementary teacher who teaches mathematics part of their time, but they likely need to be adapted). We propose that four professional obligations can organize the justifications (or refutations) that participants might give to actions that depart from a situational (or contractual) norm. We call these four obligations *disciplinary, individual, interpersonal, and institutional* (Herbst & Balacheff, 2009; see also Ball, 1993).

The *disciplinary obligation* says that the mathematics teacher is obligated to steward a valid representation of the discipline of mathematics. This may include the obligation to steward representations of mathematical knowledge, mathematical practices, and mathematical applications.

The *individual obligation* says that a teacher is obligated to attend to the well being of the individual student. This may include being obligated to attend to individual students' identities and to their behavioral, cognitive, emotional, or social needs.

The *interpersonal obligation* says that the teacher is obligated to share and steward their medium of interaction with other human beings in the classroom.

This may include attending to the needs and resources of shared discursive, physical, and social spaces within shared time.

And the *institutional (schooling) obligation* says that the teacher is obligated to observe various aspects of the schooling regime. These include attending to school policies, calendars, schedules, examinations, curriculum, extra curricular activities, and so on.

These obligations are not specific to a contract for a course of studies; they describe equally the teacher of AP Calculus and the teacher of informal geometry. They coalesce to justify contracts and their instructional situations; and they may combine with norms of contract, situation, or task in order to justify extraordinary actions. In general, combined with the norms of contracts, situations, and tasks these obligations span the practical rationality of mathematics teaching. The dispositions that compose practical rationality could be accounted for as combinations of norms and obligations. One can then say that the justifications for actions in teaching, either those actions that are usual or those that are unusual but viable, can be found by combining norms of the contract and situations that the teacher is enacting with obligations the teacher has to the profession of mathematics teaching.

Within that rationality one can see specific contracts (high school geometry, algebra I) and their instructional situations (doing proofs, solving equations) as sociohistorical constructions that have persisted over time by complying in some way with those obligations. To the extent that the obligations could contradict

each other, it is quite an accomplishment for teaching to have been able to develop stable contracts and situations over time (Herbst, 2002a).

Conclusion: Practical Rationality and Instructional Improvement

The theory of practical rationality is a way of accounting for existing, stable practices. To the extent that our interest in improving practice stresses the need for improvements to be responsible, incremental, and sustainable, it is appropriate for us to try to understand what justifies the norms of stable contracts and situations, even if we might want to modify or do away with some of them: Understanding stable systems of practices as well as understanding how those systems react to perturbations is fundamental for the design of new practices. Indeed, since improved practices will need to subject themselves to similar grounds for justification, practices that are close to those that are normal in existing instructional situations (as gauged by how many norms of a situation a practice breaches) may be easier to justify than others.

The theory also provides the means for the researcher to anticipate how instruction may respond to new practices: A novel task such as “what is something interesting that could be proved about the object in Figure 1” conjures up by resemblance one or more instructional situations (e.g., “doing proofs” and “exploration”) as possible frames for the work to be done. Models of those situations provide the researcher with a baseline of norms that could be breached as the work proceeds. Researchers can then use the obligations to anticipate what kinds of reactions teachers may have to the enactment those breaches. This

anticipation can be useful in examining the potential derailments in the implementation of new practices in classrooms. That anticipation may also be useful in the examination of teachers' responses to assessments or development, or their reactions to instructional interventions.

Thus the theory provides not only the basis for the design of probes for the rationality of teaching (Herbst & Miyakawa, 2008) but also a framework for an analysis of the reactions from participants. Combined with finer tools from discourse analysis (e.g., Halliday & Matthiessen, 2004) teachers' responses to representations of breaching (but arguably valuable) instances of an instructional situation can help us understand not only what justifies teaching as it exists today but also whether and how proposed new practices could be justified in ways that practitioners find compelling.

Along these lines, the theory also provides a framework for teacher development. This framework puts a premium on the teachers' noticing of actions in teaching, their consideration of alternative actions, and the consideration of justifications for those different actions. The various tools we have created, which include not only the animations and the cartoon characters but also software to create scenarios with them, software to annotate the scenarios individually or in forums, and software to author online sessions³ that

³ A dedicated software tool enables teacher educators to create an agenda for users to interact with representations of practice (e.g., videos, images), prompts and questions, and tools for the user to interact with the media (e.g., annotating, marking moments, etc.) and with each other.

use the materials, can be useful in implementing this development program.⁴ It is important to note that at the core of these developments there is a theory of teaching and its rationality that accounts for the teaching that is customarily seen in classrooms: At its base the theory attempts to be descriptive and explanatory rather than axiological or prescriptive. This is particularly visible in our identification of the obligations: We posit the institutional obligation in all its strength not necessarily out of advocacy for it but out of our recognition that practitioners are obligated to it regardless of anybody's feelings about it.

The theory does identify mechanisms for exploring empirically teaching that might be conceivable and desirable: The notions of situation, norm, breach, repair, and obligation can help examine a priori attempts to improve teaching and examine a posteriori the data from implementation. In that sense, the theory can support the piecemeal exploration of instructional improvement. The theory is a basic theory of mathematics instruction, a basic account of the activity of teaching mathematics in the school classroom—not an applied theory that reduces that phenomenon to the psychology of individual teachers. The psychology of mathematics teachers may still be useful to inform what enables and motivates individual teachers to do things, but the logic of action in mathematics teaching addressed by practical rationality may help us understand why some of those actions can be responsible, viable, and sustainable.

⁴ These tools and content, including examples of these learning experiences are available at www.lessonsketch.org

An important limitation of the theory in its current formulation is that it does not quite incorporate an explicit account of learning⁵ either by students or by teachers. Indeed the theory described above represents instruction as composed of stable patches of specific practices (contracts, situations, and tasks) and one might conclude that the theory describes only how knowledge is used by students and attested by teachers. Building on situated and socio-cultural accounts of learning and practice (e.g., Engestrom, 1992; Wenger, 1999) we contend that learning (by students and by the teacher) is accomplished in and through their practice in contracts, situations, and tasks. Additionally, the notion that contracts and situations can be breached by tasks that fall outside the norms of a situation or a contract is key in describing how the teacher might promote adaptive learning deliberately; and it has been foundational for Brousseau's (1997) theory of didactical situations. An explicit account of how this theory of instructional practice interfaces or complements accounts of student and teacher learning is needed and it remains a goal as we move ahead.

⁵ We appreciate Ron Tzur's comment to this effect in the occasion of the first author's plenary lecture at the 2010 PME-NA Conference.

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**Practical Rationality, the Disciplinary Obligation,
and Authentic Mathematical Work: A Look at Geometry**

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Grossman and McDonald (2008) recently argued that the research community needs to move its “attention beyond the cognitive demands of teaching ... to an expanded view of teaching that focuses on teaching as a practice (p. 185).” Building on the work of Bourdieu (Bourdieu and Wacquent, 1992; Bourdieu, 1985, 1998), Herbst and Chazan (2003, 2006) have written about mathematics teaching as a practice, just as law and medicine are considered practices, in an attempt to better understand the rationality that produces, regulates, and sustains mathematics instruction. This *practical rationality* is the commonly held system of dispositions or the “feel for the game” (Bourdieu, 1998, p. 25) that influences practitioners as to those actions that are appropriate in the classroom.

It is practical rationality that:

not only enables practices to reproduce themselves over time as the people who are the practitioners change, but also regulates how

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instances of the practice are produced and what makes them count as instances. (Herbst and Chazan, 2003, p. 2)

To better understand the practice of mathematics teaching, whether to improve it or communicate it to others, one must understand the practical rationality that guides it. However, practical rationality often “erases its own tracks” (Herbst and Chazan, 2003, p. 2) so that its practitioners come to view these practices as being natural. This rationality provides the regulatory framework that socializes its current and future practitioners into ways of thinking and acting that conform to expectations. For that reason, it is important to bring to the forefront a deliberate, conscious understanding of the rationality that drives the practice of mathematics teaching.

While practical rationality allows for a certain amount of diversity in its similarity, it is nevertheless given structure and cohesion by a complex system of norms. The word “norms” is used here not in the sense of a “standard” or something that is necessarily desirable, nor in the sense of an absolute requirement, but rather to denote that which is customary, typical, commonplace – behavior that passes without remark. Departures from a norm may occur, but when they do they are usually remarked upon and justified, thereby simultaneously confirming the norm and articulating the conditions under which it may be breached. These norms, and the grounds to which practitioners appeal to justify the norms and their breaches, provide the persistent continuity of the

practice.

Although norms are held in common among practitioners, they are usually not explicitly taught to novices. On the contrary, well before future teachers ever enroll in education courses, they already have firmly-established ideas about schools in general and mathematics instruction in particular (Ball, 1988a, 1988b). Through an apprenticeship of observation, they develop deep-seated ideas about mathematics and its teaching and learning (Lortie, 1975). These ideas often form the foundation on which they will eventually build their own practice of mathematics teaching (Millsaps, 2000; Skott, 2001).

A Look at Geometry

What do we know about the rationality that underpins geometry instruction? Herbst and Brach (2006) draw our attention to the practice of geometry instruction and provoke thought regarding the norms surrounding the teaching of proof,² but what about other key components of geometry courses? For example, definitions play a critical role in geometry. What norms exist for the teaching of definitions in geometry? Is the norm for students to be presented with finalized definitions? Under what conditions are students given opportunities to create, reflect on, and compare definitions (de Villiers, 1998)?

What is normative in regards to the introduction and use of the diagrammatic register (Weiss & Herbst, 2007) commonly encountered in geometry classes?

² Additional information on norms surround proving and proof is found at Herbst and Brach (2006).

What rationality guides teachers' and students' expectations in regard to the role of perception in the reading of geometric diagrams? What norms influence the teaching of subtle, yet key, concepts of geometry like existence and uniqueness? Are students given impossible problems³ as a means to discover existence? Are students allowed to explore situations that demonstrate uniqueness?⁴

Mathematics: Teachers' Beliefs and Practices

While many of the above questions are particular to geometry, others apply to the many branches of mathematics. Is it normative to encourage students to modify a problem (either to make it tractable, or to generate new avenues for exploration), or to introduce their own assumptions when solving problems? Do teachers commonly encourage students to pose their own problems? Do teachers model or introduce strategies like Brown and Walters' (2004) "what-if-not" strategy as a relatively simple means of generating new problems in their teaching practice?⁵

³ Questions of existence (or non-existence) arise in a wide range of problems, such as: Can one form a triangle with sides of lengths 2 cm, 3 cm and 10 cm? Can one locate a point in the interior of any polygon that is equidistant from all of its vertices? Under what conditions can a circle be constructed tangent to two intersecting lines at two specified points? This last problem is shown as a part of an instructional episode modeled in the ThEMaT (Thought Experiments in Mathematics Teaching) animations found at <http://grip.umich.edu/themat>.

⁴ Questions of uniqueness in geometry likewise arise in a range of problems, such as: Given two sides of a triangle and a non-included angle, how many different triangles can be constructed? Given any parallelogram, is there a uniquely determined quadrilateral whose midpoints are the vertices of the given parallelogram?

⁵ For example of a what-if-not application, consider how a compass and straightedge are used to construct a perpendicular bisector for a given line segment. Applying the "what-if-not" strategy could lead to the following questions. What if you wanted to construct a bisector that was not perpendicular to the line segment? How could you construct a perpendicular that did not bisect the segment?

Unfortunately, a large number of teachers view mathematics “as a discipline with *a priori* rules and procedures that ... students have to learn by rote” (Handal, 2003, p. 54). For many teachers in the U.S. “knowing” mathematics is taken to mean being efficient and skillful in performing rule-bound procedures and manipulating symbols (Thompson, 1992). Ball (1988b), in her doctoral study of preservice teachers’ ideas about the sources of mathematics and how mathematics is justified, found that many of them viewed mathematics as a mostly arbitrary collection of facts. While there are surely many factors that influence teachers’ practices, it would be naïve to assume that these and other beliefs teachers hold do not play a significant role. As a consequence, mathematics students often are “not expected to develop mathematical meanings and they are not expected to use meanings in their thinking” (Thompson, 2008, p. 45).

Targeting the Disciplinary Obligation

Herbst and Balacheff (2009) have suggested four obligations of teachers that frame their practical rationality. These obligations – which they refer to as the *disciplinary, individual, interpersonal, and institutional* obligations – may be invoked by teachers to justify normal instruction, but they also have the potential to organize a departure from normative practice.

Of the four, we focus here on the disciplinary obligation – the obligation of the teacher to faithfully represent the discipline of mathematics. We begin from

the premise that if teachers come to a more textured and authentic view of mathematics, this could lead to changes in what teachers deem as valid representations of mathematics, in the mathematical tasks they assign students, and in the ideas and attitudes they foster in students. Following Yackel and Cobb (1996) we note that what is taken as

mathematically normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants. At the same time these goals and largely implicit understandings are themselves influenced by what is legitimized as acceptable mathematical activity. (p. 460)

This focus on the disciplinary obligation brings into focus the question of what kind of work is “legitimized as acceptable mathematical activity” (in the words of Yackel and Cobb)? How does it correspond to the kind of work that mathematicians do?

Authentic Mathematical Practices

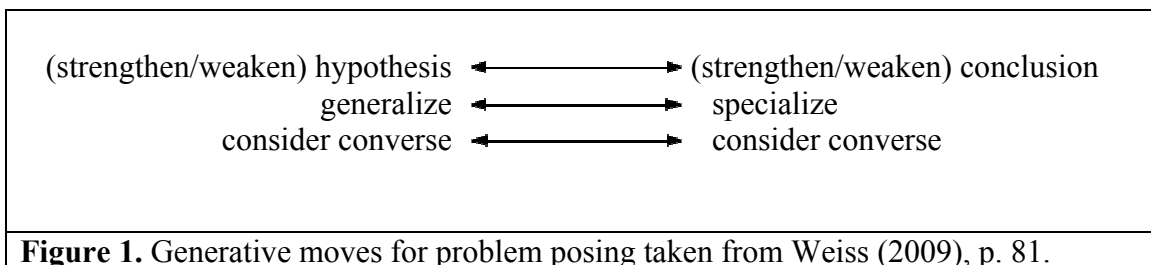
In Weiss, Herbst and Chen (2009) it was noted that, while the notion of “authentic mathematics” is frequently invoked in the literature, nevertheless “many of those who call for ‘authentic mathematics’ (or who use similar words or phrases, such as ‘genuine’ or ‘real’) in the classroom are actually talking about different things” (p. 276). In particular, Weiss, Herbst and Chen identify four distinct meanings of the slogan “authentic mathematics education”. Of particular

interest to us here is the one they refer to as AMP , i.e. the call for the cultivation of the *practices* that characterize the work of research mathematicians. Note, however, that in acknowledging the polysemy of the phrase “authentic mathematics” we allow for, and even anticipate, the possibility that these multiple kinds of “authenticity” may come into conflict with one another.

Mathematicians, those whose goals are to generate new and refine existing mathematical ideas and methods, are more than just proficient at mathematics. While they demonstrate exactly those qualities and competencies that have been identified by the National Research Council (2001) as goals of mathematics learning (namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition), mathematicians also demonstrate habits of “mathematical wondering” and an appreciation of mathematics that extends past their professional careers into their personal lives. They spend much of their time crafting new problems from existing ones, both out of pragmatism (some problems are more tractable than others at a given time) and out of curiosity.

In seeking to articulate the elements of the sensibility that characterizes mathematicians’ practices, Weiss (2009) analyzed a collection of narratives written by and about research mathematicians. This analysis reveals the fundamentally generative nature of mathematical practice, in which *problem posing* (asking fruitful and difficult questions of oneself and others) plays a role just as important as *problem solving*. The result of Weiss’ analysis is a partial

model of the mathematical sensibility, consisting of 15 mathematical dispositions, organized in 8 dialectical pairs (one disposition is its own dialectical counterpart). Weiss refers to the first five of those dispositions as *generative moves* by which a problem currently under consideration (whether solved or unsolved) can spawn a number of related problems. The five generative moves are shown in Fig. 1.



Authentic Mathematical Practice in the Work of Teachers

To what extent do the mathematical activities commonly seen in classrooms reflect authentic mathematical work? Do current norms in mathematics instruction promote either mathematical proficiency or curiosity? Does the rationality that drives mathematics teaching help encourage an appreciation of mathematics?

Herbst and Chazan (2011) has suggested that it is crucial that we recognize how instruction typically works, understanding the practical rationality that underpins teaching, if we are to design reforms that are viable and sustainable. It is through incremental changes, which recognize current practice, that permanent transformation is most likely to occur, but how might incremental

changes be introduced? What form might such changes take?

The key role of problem posing in mathematics instruction has long been recognized. Silver (1994) noted that problem posing is not only a prominent feature of mathematical activity; it also features heavily in “inquiry-oriented instruction” and can serve to create an environment in which students are more engaged.

Here we describe briefly how the five generative moves for problem posing (Fig. 1) could be relevant when describing the potential for secondary mathematics education to include instances of “authentic mathematical work”. Suppose a high school geometry class has been studying the properties of triangles, and has found (either through empirical exploration, deductive proof, or a combination of the two) that the three angle bisectors of any acute triangle always intersect in a single point. The following scenarios show how instructional interventions can change the direction of the task and have the potential to depart from normative geometry instruction.⁶

- One possibility is that the teacher might ask, “Does it really matter whether the triangle is acute or not?” Investigating this question could lead the class to the conclusion that, in fact, the initial restriction to the case of acute triangles was unnecessary, and that the conclusion obtains

⁶ The end goal is not for the instructor to make such interventions, but that all classroom participants, including students, begin to adopt this problem posing mindset.

for all triangles — a case of *weakening the hypothesis*, the first generative move in Fig. 1.

- Another possibility is that the teacher might encourage the class to seek to *strengthen the conclusion* of what has been proven, for example by providing additional properties that characterize the intersection point of the three angle bisectors of a triangle such as offering, “Not only do they intersect at a single point, but that point is the center of a circle that can be inscribed in the triangle.”
- A third possibility is that the class might seek to *generalize* their findings, for example by asking, “What happens if you construct the angle bisectors of *other* polygons? Do they meet at a point, and if not, what do you get?”
- A fourth possibility is that the class might seek to *specialize* their findings, for example by observing, “If you do this with an *equilateral triangle*, there seems to be more than can be said about the resulting figure — for example the point of intersection seems to be equidistant from the three *corners* of the triangle as well.”
- A class that has observed this last property might then *consider the converse* question: “If the angle bisectors of a particular triangle meet at a point that is equidistant from the three corners of the triangle, does that mean that the triangle in question *must* be equilateral?”

The examples above illustrate how the generative moves identified in Weiss (2009) can be used to describe and promote the practice of *wondering mathematically* about what is true, a core component of authentic mathematical practice. More examples could be generated *ad lib* by iterating and recombining these moves. For example, the generalization to the case of other polygons could lead to a subsequent specialization to the case of quadrilaterals (which in turn could be subsequently refined to the case of various “special quadrilaterals”). The many variations on this “angle bisector problem” have played a key role in the representations of mathematics teaching used by Herbst and his collaborators as probes of geometry teachers’ practical rationality (see Aaron, 2010; Herbst & Chazan, 2006; Weiss & Herbst, 2007; Weiss, 2009).

Authentic Mathematical Practice in Teacher Education

Many of the norms that characterize contemporary mathematics education are at a great distance from authentic mathematical practice. Herbst and Balacheff (2009) argue that an appeal to the disciplinary obligation can, in some cases, provide grounds for departing from those norms. This, however, requires that teachers hold a fuller and more nuanced view of authentic mathematical practice. In this section we address the role of teacher education in cultivating such a view.

Ball (1988b) identified a number of widespread views among preservice teachers, including “Mathematics is a mostly arbitrary collection of facts,”

“Doing mathematics means following set procedures,” and “Doing mathematics means using remembered knowledge and working step-by-step” (pp. 104-108). Her findings showed that preservice teachers predominantly view mathematics as a “closed” field, one in which there are no new questions left to ask. When asked to respond to the statements “Some problems in mathematics have no answers” and “There are unsolved problems in mathematics”, the preservice teachers in Ball’s study expressed confusion. For them, “wondering mathematically” simply does not exist as an activity.

The impact of these views of mathematical practice is significant. In a recent study, Cross (2009) showed that teachers who understand mathematics to be primarily about “formulas, procedures, and calculations” consistently defaulted to an initiate-respond-evaluate pattern in their interactions with students. In contrast, teachers who regard mathematics primarily as being about the “thought processes and mental actions of the individual” were more likely to engage their students in extended, continuous discourse (Cross, pp. 332-3). Cross concludes that teachers who do not hold beliefs consonant with supporting “learner-oriented classroom environments” should be engaged in programs intended to transform their beliefs.

The responsibility for cultivating an awareness of authentic mathematical practice in preservice teachers rests, by necessity, with teacher education. Mathematics teacher educators “have the dual responsibility of preparing teachers, both *mathematically* and *pedagogically* (Liljedahl, Chernoff, and Zazkis,

2007, p. 239).” Although many colleges and universities preserve an institutional separation between mathematics content courses and mathematics methods courses, undergraduate mathematics courses should not be the only opportunities for future teachers to develop a sense of and appreciation for authentic mathematical work. Learning to wonder mathematically can, and should, be a goal of teacher education courses. Experiences with mathematical discovery have been shown to have a profound, transformative effect on future teachers’ beliefs about the nature of mathematics and its teaching and learning (Liljedahl, 2005). Mathematics teacher education should make the processes and mechanisms of problem posing (including the generative moves of Table 1) explicit, and draw attention to how they can be used to navigate productively through open-ended problem spaces. Through engagement in, and explicit attention to, such mathematical activities, teachers might come to view mathematics differently. If they come to view mathematics differently, the disciplinary obligation that partly frames their instruction could lead to changes in what they deem valid representations of mathematics.

Besides implementing tasks that model authentic mathematical practice, mathematics education classes could provide future teachers with exposure to examples of the rich mathematical thinking that students are capable of and often bring to the classroom. Mathematics education classes should also help future teachers consider how to value and capitalize on students’ wondering as well as how to promote problem posing by and mathematical curiosity in their

students. Future teachers need exposure to and interaction with representations of classroom instruction (like case studies, videos, animations, etc.) that model authentic mathematical practice. Ideally teacher educators should be able to provide both actual and hypothetical episodes of instruction to show both what is currently possible and being done as well as foreshadowing what might be possible if current norms were questioned.

Mathematics educators could provide future (and also current) teachers opportunities to witness episodes of instruction that depart from normative practice but that exemplify authentic mathematical work. For teacher educators, a direct encounter with teachers' reactions to such breaches can help make visible the (usually tacit) norms that guide the rationality of teaching. These encounters have the potential to shape or transform teachers' views of the nature of mathematics and its teaching and learning.

Conclusions

The mathematics education community has a long history of efforts to improve teaching, and yet teaching remains largely resilient in the face of reform. One possible reason for this difficulty is that teacher education has struggled to instill a mathematical sensibility in preservice teachers, many of whom have little or no direct experience with authentic mathematical practice. A second possible reason for this difficulty is that reform efforts often fail to consider the norms that drive and sustain the practice of mathematics teaching as it exists currently. A

strong case can be made for the use of practical rationality as a lens for viewing both research and teacher education: if we are to design reforms that are viable and sustainable, it is crucial to understand the practical rationality that underpins teaching (Herbst & Chazan 2011).

It may be somewhat naïve to expect that, simply by providing preservice teachers with opportunities to experience authentic mathematical practice, we will somehow transform them into a different kind of teacher, one who creates opportunities for his or her own students to engage in such practices. On the other hand, it seems to us unassailable that such preservice teacher education is a *necessary*, even if not sufficient, condition for such an outcome. It is almost impossible to imagine teachers engaging students in the processes of wondering mathematically, when the teachers themselves have never experienced such activity. Cultivating a richer vision of mathematics as a discipline may make it possible (although by no means certain) that teachers can, in the future, appeal to the disciplinary obligation as grounds for change.

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Complex Learning through Cognitively Demanding Tasks

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Abstract: *The world's increasing complexity, competitiveness, interconnectivity, and dependence on technology generate new challenges for nations and individuals that cannot be met by "continuing education as usual" (The National Academies, 2009). With the proliferation of complex systems have come new technologies for communication, collaboration, and conceptualization. These technologies have led to significant changes in the forms of mathematical thinking that are required beyond the classroom. This paper argues for the need to incorporate future-oriented understandings and competencies within the mathematics curriculum, through intellectually stimulating activities that draw upon multidisciplinary content and contexts. The paper also argues for greater recognition of children's learning potential, as increasingly complex learners capable of dealing with cognitively demanding tasks.*

Keywords: complex systems; complex learning; models and modeling; 21st century technologies; teaching and learning

Although educational reformers have disagreed on many issues, there is a widely shared concern for enhancing opportunities for students to *learn mathematics with understanding* and thus a strong interest in promoting *teaching mathematics for understanding*. (Silver, Mesa, Morris, Star, & Benken, 2009, P.503).

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Introduction

In recent decades our global community has rapidly become a knowledge driven society, one that is increasingly dependent on the distribution and exchange of services and commodities (van Oers, 2009), and one that has become highly inventive where creativity, imagination, and innovation are key players. At the same time, the world has become governed by complex systems—financial corporations, the World Wide Web, education and health systems, traffic jams, and classrooms are just some of the complex systems we deal with on a regular basis. For all citizens, an appreciation and understanding of the world as interlocked complex systems is critical for making effective decisions about one's life as both an individual and as a community member (Bar-Yam, 2004; Jacobson & Wilensky, 2006; Lesh, 2006).

Complexity—the study of systems of interconnected components whose behavior cannot be explained solely by the properties of their parts but from the behavior that arises from their interconnectedness—is a field that has led to significant scientific methodological advances. With the proliferation of complex systems have come new technologies for communication, collaboration, and conceptualization. These technologies have led to significant changes in the forms of mathematical thinking that are needed beyond the classroom. For example, technology can ease the thinking needed in information storage, representation, retrieval, and transformation, but places increased demands on the complex thinking required for the interpretation of data and communication

of results. Computational skills alone are inadequate here—the ability to interpret, describe, and explain data and communicate results of data analyses is essential (Hamilton, 2007; Lesh, 2007a; Lesh, Middleton, Caylor & Gupta, 2008).

The rapid increase in complex systems cannot be ignored in mathematics education. Indeed, educational leaders from different walks of life are emphasizing the importance of developing students' abilities to deal with complex systems for success beyond school. Such abilities include: constructing, describing, explaining, manipulating, and predicting complex systems; working on multi-phase and multi-component component projects in which planning, monitoring, and communicating are critical for success; and adapting rapidly to ever-evolving conceptual tools (or complex artifacts) and resources (Gainsburg, 2006; Lesh & Doerr, 2003; Lesh & Zawojewski, 2007).

In this article I first consider future-oriented learning and then address some of the understandings and competencies needed for success beyond the classroom, which I argue need to be incorporated within the mathematics curriculum. A discussion on complex learners and complex learning, with mathematical modeling as an example, is presented in the remaining section.

Future-oriented learning

Every advanced industrial country knows that falling behind in science and mathematics means falling behind in commerce and property. (Brown, 2006).

Many nations are highlighting the need for a renaissance in the mathematical sciences as essential to the well-being of all citizens (e.g., Australian Academy of Science, 2006; Pearce, Flavell, & Dao-Cheng, 2010; The National Academies, 2009). Indeed, the first recommendation of The National Academies' *Rising above the Gathering Storm* (2007) was to vastly improve K-12 science and mathematics education. Likewise the Australian Academy of Science has indicated the need to address the "critical nature" of the mathematical sciences in schools and universities, especially given the unprecedented, worldwide demand for new mathematical solutions to complex problems. In addressing such demands, the Australian Academy emphasizes the importance of interdisciplinary research, given that the mathematical sciences underpin many areas of society including financial services, the arts, humanities, and social sciences.

The interdisciplinary nature of the mathematical sciences is further evident in the rapid changes in the nature of the problem solving and reasoning needed beyond the school years (Lesh, 2007b). Indeed, numerous researchers and employer groups have expressed concerns that schools are not giving adequate attention to the understandings and abilities that are needed for success beyond school. For example, potential employees most in demand in the mathematical sciences are those that can (a) interpret and work effectively with complex systems, (b) function efficiently and communicate meaningfully within diverse teams of specialists, (c) plan, monitor, and assess progress within complex, multi-stage projects, and (d) adapt quickly to continually developing technologies

(Lesh, 2008). Research indicates that such employees draw effectively on interdisciplinary knowledge in solving problems and communicating their findings. Furthermore, although such employees draw upon their school learning, they do so in a flexible and creative manner, often generating or reconstructing mathematical knowledge to suit the problem situation (unlike the way in which they experienced mathematics in school; Gainsburg 2006; Hamilton 2007; Zawojewski, Hjalmarson, Bowman, & Lesh, 2008). Indeed, such employees might not even recognize the relationship between their school mathematics and the mathematics they apply in solving problems in their daily work activities. We thus need to rethink the nature of the mathematical learning experiences we provide students, especially those experiences we classify as “problem solving;” we also need to recognize the increased capabilities of students in today’s era.

In his preface to the book, *Foundations for the Future in Mathematics Education*, Lesh (2007b) pointed out that the kinds of mathematical understandings and competencies that are targeted in textbooks and tests tend to “represent only a shallow, narrow, and often non-central subset of those that are needed for success when the relevant ideas should be useful in ‘real life’ situations” (p. viii). Lesh’s argument raises a number of issues, including:

What kinds of understandings and competencies should be emphasized to reduce the gap between the mathematics addressed in the classroom (and in standardized testing), and the mathematics needed for success beyond the

English

classroom?

How might we address the increasing complexity of learning and learners to advance their mathematical understanding within and beyond the classroom?

Understandings and competencies for success beyond the classroom

The advent of digital technologies changes the world of work for our students. As Clayton (1999) and others (e.g., Hoyles, Noss, Kent, & Bakker, 2010; Jenkins, Clinton, Purushotma, Robinson & Weigel, 2006; Lombardi & Lombardi, 2007; Roschelle, Kaput, & Stroup, 2000) have stressed, the availability of increasingly sophisticated technology has led to changes in the way mathematics is being used in work place settings; these technological changes have led to both the addition of new mathematical competencies and the elimination of existing mathematical skills that were once part of the worker's toolkit.

Studies of the nature and role of mathematics used in the workplace and other everyday settings (e.g., nursing, engineering, grocery shopping, dieting, architecture, fish hatcheries) are important in helping us identify some of the key understandings and competencies for the 21st century (e.g., de Abreu, 2008; Gainsburg, 2006; Hoyles et al., 2010; Roth, 2005). A major finding of the 2002 report on workplace mathematics by Hoyles, Wolf, Molyneux-Hodgson and Kent was that basic numeracy is being displaced as the minimum required mathematical competence by an ability to apply a much wider range of

mathematical concepts in using technological tools as part of working practice. Although we cannot simply list a number of mathematical competencies and assume these can be automatically applied to the workplace setting, there are several that employers generally consider to be essential to productive outcomes (e.g., Doerr & English, 2003; English, 2008; Gainsburg, 2006; Lesh & Zawojewski, 2007). In particular, the following are some of the core competencies that have been identified as key elements of productive and innovative work place practices (English, Jones, Bartolini Bussi, Lesh, Tirosh, & Sriraman, 2008; Hoyles et al., 2010). I believe these competencies need to be embedded within our mathematics curricula:

- Problem solving, including working collaboratively on complex problems where planning, overseeing, moderating, and communicating are essential elements for success;
- Applying numerical and algebraic reasoning in an efficient, flexible, and creative manner;
- Generating, analyzing, operating on, and transforming complex data sets;
- Applying an understanding of core ideas from ratio and proportion, probability, rate, change, accumulation, continuity, and limit;
- Constructing, describing, explaining, manipulating, and predicting complex systems;
- Thinking critically and being able to make sound judgments, including being able to distinguish reliable from unreliable information sources;

English

- Synthesizing, where an extended argument is followed across multiple modalities;
- Engaging in research activity involving the investigation, discovery, and dissemination of pertinent information in a credible manner;
- Flexibility in working across disciplines to generate innovative and effective solutions.
- Techno-mathematical literacy (a “techno-mathematical literacy, where the mathematics is expressed through technological artefacts.” Hoyles et al., 2010, p. 14).

Although a good deal of research has been conducted on the relationship between the learning and application of mathematics in and out of the classroom (e.g., de Abreu 2008; Nunes & Bryant 1996; Saxe 1991), we still know comparatively little about students’ mathematical capabilities, especially problem solving, beyond the classroom. We need further knowledge on why students have difficulties in applying the mathematical concepts and abilities (that they presumably have learned in school) outside of school—or in classes in other disciplines.

A prevailing explanation for these difficulties is the context-specific nature of learning and problem solving, that is, competencies that are learned in one situation take on features of that situation; transferring them to a new problem situation in a new context poses challenges (Lobato 2003). This suggests we need

to reassess the nature of the typical mathematical problem-solving experiences we give our students, with respect to the nature of the content and how it is presented, the problem contexts and the extent of their real-world links, the reasoning processes likely to be fostered, and the problem-solving tools that are available to the learner (English & Sriraman, 2010). This reassessment is especially needed, given that “problems themselves change as rapidly as the professions and social structures in which they are embedded change” (Hamilton, 2007, p. 2). The nature of learners and learning changes likewise. With the increasing availability of technology and exposure to a range of complex systems, children are different types of learners today, with a potential for learning that cannot be underestimated.

Complex learners, complex learning

Winn (2006) warned of the “dangers of simplification” when researching the complexity of learning, noting that learning is naturally confronted by three forms of complexity—the complexity of the learner, the complexity of the learning material, and the complexity of the learning environment (p. 237). We cannot underestimate these complexities. In particular, we need to give greater recognition to the complex learning that children are capable of—they have greater learning potential than they are often given credit for by their teachers and families (English, 2004; Lee & Ginsburg, 2007; Perry & Dockett, 2008; *Curious Minds*, 2008). They have access to a range of powerful ideas and processes and

English

can use these effectively to solve many of the mathematical problems they meet in daily life. Yet their mathematical curiosity and talent appear to wane as they progress through school, with current educational practice missing the goal of cultivating students' capacities (National Research Council, 2005; *Curious Minds*, 2008). The words of Johan van Benthem and Robert Dijkgraaf, the initiators of *Curious Minds* (2008), are worth quoting here:

What people say about children is: "They can't do this yet."

We turn it around and say: "Look, they can already do this."

And maybe it should be: "They can still do this now."

As Perry and Dockett (2008) noted, one of our main challenges here is to find ways to utilize the powerful mathematical competencies developed in the early years as a springboard for further mathematical power as students progress through the grade levels. I offer three interrelated suggestions for addressing this challenge:

1. Recognize that learning is based within contexts and environments that we, as educators shape, rather than within children's maturation (Lehrer & Schauble, 2007).
2. Promote active processing rather than just static knowledge (*Curious Minds*, 2008).
3. Create learning activities that are of a high cognitive demand (Silver et al., 2009).

In the remainder of this paper I give brief consideration to these suggestions. In doing so, I argue for fostering complex learning through activities that encourage knowledge generation and active processing. While complex learning can take many forms and involve numerous factors, there are four features that I consider especially important in advancing students' mathematical learning. These appear in figure 1.

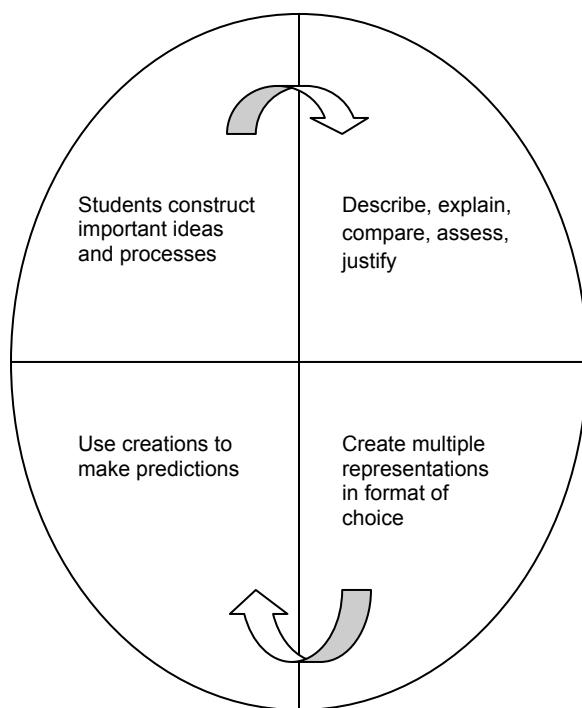


Figure 1. Key Features of Complex Learning

Research in the elementary and middle school indicates that, with carefully designed and implemented learning experiences, we can capitalize on children's conceptual resources and bootstrap them towards advanced forms of reasoning

not typically observed in the regular classroom (e.g., English & Watters, 2005; Ginsburg, Cannon, Eisenband, & Pappas, 2006; Lehrer & Schauble, 2007). Most research on young students' mathematical learning has been restricted to an analysis of their actual developmental level, which has failed to illuminate their potential for learning under stimulating conditions that challenge their thinking—"Research on children's current knowledge is not sufficient" (Ginsburg et al., 2006, p.224). We need to redress this situation by exploring effective ways of fashioning learning environments and experiences that challenge and advance students' mathematical reasoning and optimize their mathematical understanding.

Recent research has argued for students to be exposed to learning situations in which they are *not* given all of the required mathematical tools, but rather, are required to create their own versions of the tools as they determine what is needed (e.g., English & Sriraman, 2010; Hamilton, 2007; Lesh, Hamilton, & Kaput, 2007). For example, long-standing perspectives on classroom problem solving have treated it as an isolated topic, with problem-solving abilities assumed to develop through the initial learning of basic concepts and procedures that are then practised in solving word ("story") problems. In solving such word problems, students generally engage in a one- or two-step process of mapping problem information onto arithmetic quantities and operations. These traditional word problems restrict problem-solving contexts to those that often artificially house and highlight the relevant concept (Hamilton, 2007). These problems thus

preclude students from creating their own mathematical constructs. More opportunities are needed for students to generate important concepts and processes in their own mathematical learning as they solve thought-provoking, authentic problems. Unfortunately, such opportunities appear scarce in many classrooms, despite repeated calls over the years for engaging students in tasks that promote high-level mathematical thinking and reasoning (e.g., Henningsen & Stein, 1997; Silver et al., 2009; Stein & Lane, 1996).

Silver et al.'s recent research (2009) analyzing portfolios of "showcase" mathematics lessons submitted by teachers seeking certification of highly accomplished teaching, showed that activities were not consistently intellectually challenging across topics. About half of the teachers in the sample (N=32) failed to include a single activity that was cognitively demanding, such as those that call for reasoning about ideas, linking ideas, solving complex problems, and explaining and justifying solutions. Furthermore, the teachers were more likely to use cognitively demanding tasks for assessment purposes than for teaching to develop student understanding. While Silver et al.'s research revealed positive features of the teachers' lessons, it also indicated that the use of cognitively demanding tasks in promoting mathematical understanding needs systematic attention.

Modeling Activities

One approach to promoting complex learning through intellectually challenging tasks is mathematical modeling. Mathematical models and modeling

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have been interpreted variously in the literature (e.g., Romberg, Carpenter, & Kwako, 2005; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; English & Sriraman, 2010; Greer, 1997; Lesh & Doerr, 2003). It is beyond the scope of this paper to address these various interpretations, however, but the perspective of Lesh and Doerr (e.g., Doerr & English, 2003; Lesh & Doerr, 2003) is frequently adopted, that is, models are “systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system” (Doerr & English, 2003, p.112). From this perspective, modeling problems are realistically complex situations where the problem solver engages in mathematical thinking beyond the usual school experience and where the products to be generated often include complex artifacts or conceptual tools that are needed for some purpose, or to accomplish some goal (Lesh & Zawojewski, 2007).

In one such activity, the *Water Shortage Problem*, two classes of 11-year-old students in Cyprus were presented with an interdisciplinary modeling activity that was set within an engineering context (English & Mousoulides, in press). In the *Water Shortage Problem*, constructed according to a number of design principles, students are given background information on the water shortage in Cyprus and are sent a letter from a client, the Ministry of Transportation, who needs a means of (model for) selecting a country that can supply Cyprus with water during the coming summer period. The letter asks students to develop such a model using the data given, as well as the Web. The quantitative and

qualitative data provided for each country include water supply per week, water price, tanker capacity, and ports' facilities. Students can also obtain data from the Web about distance between countries, major ports in each country, and tanker oil consumption. After students have developed their model, they write a letter to the client detailing how their model selects the best country for supplying water. An extension of this problem gives students the opportunity to review their model and apply it to an expanded set of data. That is, students receive a second letter from the client including data for two more countries and are asked to test their model on the expanded data and improve their model, if needed.

Modeling problems of this nature provide students with opportunities to repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to create a structurally significant product—structural in the sense of generating powerful mathematical (and scientific) constructs. The problems are designed so that multiple solutions of varying mathematical and scientific sophistication are possible and students with a range of personal experiences and knowledge can participate. The products students create are documented, shareable, reusable, and modifiable models that provide teachers with a window into their students' conceptual understanding. Furthermore, these modeling problems build communication (oral and written) and teamwork skills, both of which are essential to success beyond the classroom.

Concluding Points

The world's increasing complexity, competitiveness, interconnectivity, and dependence on technology generate new challenges for nations and individuals that cannot be met by "continuing education as usual" (The National Academies, 2009). In this paper I have emphasized the need to incorporate future-oriented understandings and competencies within the mathematics curriculum, through intellectually stimulating activities that draw upon multidisciplinary content and contexts. I have also argued for greater recognition of children's learning capabilities, as increasingly complex learners able to deal with cognitively demanding tasks.

The need for more intellectually stimulating and challenging activities within the mathematics curriculum has also been highlighted. It is worth citing the words of Greer and Mukhopadhyay (2003) here, who commented that "the most salient features of most documents that lay out a K-12 program for mathematics education is that they make an intellectually exciting program boring," a feature they refer to as "intellectual child abuse" (p. 4). Clearly, we need to make the mathematical experiences we include for our students more challenging, authentic, and meaningful. Developing students' abilities to work creatively with and generate mathematical knowledge, as distinct from working creatively on tasks that provide the required knowledge (Bereiter & Scardamalia, 2006) is especially important in preparing our students for success in a knowledge-based economy. Furthermore, establishing collaborative, knowledge-building

communities in the mathematics classroom is a significant and challenging goal for the advancement of students' mathematical learning (Scardamalia, 2002).

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Conceptualizations and Issues Related to Learning Progressions, Learning Trajectories, and Levels of Sophistication¹

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Abstract: In this paper the nature of learning progressions and related concepts are discussed. The notions of learning progressions and learning trajectories are conceptualized and their usage is illustrated with the help of examples. In particular the nuances of instructional interventions utilizing these concepts are also discussed with implications for the teaching and learning of mathematics.

Keywords: Learning progressions; Learning trajectories; teaching; Instructional interventions

Learning progressions (LP) are playing an increasingly important role in mathematics and science education (NRC, 2001, 2007; Smith, Wiser, Anderson, & Krajcik, 2006). They are strongly suggested for use in assessment, standards, and teaching. In this article, I discuss the nature of learning progressions and related concepts in mathematics education, and I illustrate issues in their construction and use. I emphasize the different ways that LP and related constructs *represent* learning for teaching. Finally, I illustrate the need that teachers have for LP.

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Definitions and Constructs

According to the National Research Council, "Learning progressions are descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (2007, p. 214). A similar description of learning progressions is given by Smith et al. who define a learning progression "as a sequence of successively more complex ways of thinking about an idea that might reasonably follow one another in a student's learning" (2006, pp. 5-6). Unlike Piaget's stages, but similar to van Hiele's levels³, it is assumed that progress through learning progressions is "not developmentally inevitable" but depends on instruction (Smith et al., 2006).

Common Characteristics of the LP Construct

In the research literature, descriptions of the LP construct possess both differences and similarities. The characteristics that seem most common to different views of learning progressions are as follows:

- LP "are based on research syntheses and conceptual analyses" (Smith et al., 2006, p. 1); "Learning progressions should make systematic use of current research on children's learning " (NRC, 2007, p. 219).
- LP "are anchored on one end by what is known about the concepts and reasoning of students. ... At the other end, learning progressions are

³ Because many of my examples refer to the van Hiele levels, I have included a very brief synopsis of the levels in Appendix 1.

anchored by societal expectations. ... [LP also] propose the *intermediate* understandings between these anchor points that ... contribute to building a more mature understanding" (NRC, 2007, p. 220).

- LP focus on core ideas, conceptual knowledge, and connected procedural knowledge, not just skills. LP organize "conceptual knowledge around core ideas" (NRC, 2007, p. 220). LP "Suggest how well-grounded conceptual understanding can develop" (NRC, 2007, p. 219).
- LP "recognize that all students will follow not one general sequence, but multiple (often interacting) sequences" (NRC, 2007, p. 220).

Differences in LP Construct

There are several differences in how the learning progressions construct is used in the literature.

- LP differ in the time spans they describe. Some progressions describe the development of students' thinking over a span of years; others describe the progression of thinking through a particular topic or instructional unit.
- LP differ in the grain size of their descriptions. Some are appropriate for describing minute-to-minute changes in students' development of thought, while others better describe more global progressions through school curricula.

- LP differ in the audience for which they are written. Some LP are written for researchers, some for standards writers, some for assessment developers (formative and summative), and some for teachers.
- LP differ in the research foundation on which they are built. Some LP are syntheses of extant research; some synthesize extant research then perform additional research that elaborates the syntheses (the additional research may be cross-sectional or longitudinal).
- LP differ in how they describe student learning. Some focus on numerically "measuring" student progress, while others focus on describing the nature or categories of students' cognitive structures and reasoning.

Learning Trajectories

Another important construct that is similar to, different from, and importantly related to, learning progressions is that of a "learning trajectory⁴." I define a learning trajectory as a detailed description of the sequence of thoughts, ways of reasoning, and strategies that a student employs while involved in learning a topic, including specification of how the student deals with all instructional tasks and social interactions during this sequence. There are two types of learning trajectories, hypothetical and actual. Simon (1995) proposed that a "*hypothetical learning trajectory* is made up of three components: the

⁴ Although some people use the terms "learning progression" and "learning trajectory" similarly, I think it is extremely useful to carefully distinguish learning progressions and learning trajectories.

learning goal..., the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136). In contrast, descriptions of *actual learning trajectories* can be specified only during and after a student has progressed through such a learning path. Simon states that an "actual learning trajectory is not knowable in advance" (p. 135). Steffe described an actual learning trajectory as "a model of [children's] initial concepts and operations, an account of the observable changes in those concepts and operations as a result of the children's interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes. Such a learning trajectory of children is constructed during and after the experience in intensively interacting with children" (2004, p. 131).

Clements and Sarama's (2004) "conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain and a related, *conjectured* route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (2004, p. 83). In their *hypothetical learning trajectories*, they specify instructional tasks that promote (and assess) progression through their levels of thinking.

One critical difference between my definition of learning progressions and my definition of learning trajectories is that *trajectories include descriptions of instruction, progressions do not*. One of the most difficult issues facing researchers who are constructing learning trajectories for curriculum development is determining how instructional variation affects trajectories. That is, how specific is the trajectory to the instructional sequence in which it is embedded? If the sequence has been tested for one curriculum, how well does it apply to other curricula? Also, how do actual trajectories for individual students vary from the hypothetical trajectory for a curriculum? That is, a learning trajectory for a curriculum is in some sense an "average" of actual trajectories for a sample of individual students – and, as an average, it is a prediction for a target population, and thus it is necessarily hypothetical. And the "standard deviation" of the distribution of actual trajectories may be as relevant as the mean.

Pedagogical Uses of LP

Beyond the scientific value of LP/LT descriptions of students' mathematics learning, these descriptions are powerful tools for teaching. LP/LT can be used for formative and summative assessment, and to guide instructional decisions made in curriculum development and moment-to-moment teaching. Indeed, Simon states, "I choose to use 'hypothetical learning trajectory' ... to emphasize aspects of teacher thinking that are grounded in a constructivist perspective and that are common to *both advanced planning and spontaneous decision making*" (1995, p. 135). Such a hypothesized trajectory (or LP) helps

teachers make instructional decisions based on their "best guess of how learning might proceed" (Simon, 1995, p. 135). Thus, from the constructivist perspective, LP and LT should ideally help teachers not only plan instruction, but understand students' learning on a moment-to-moment basis and appropriately and continuously adjust instruction to meet students' evolving learning needs.

Another difference between learning progressions and learning trajectories derives from their intended use and consequent development. If one is designing and testing a curriculum, one is more likely to develop a learning trajectory based on the fixed sequence of learning tasks in that curriculum. If, in contrast, one is focusing on a formative assessment system that applies to many curricula, one is more likely to develop a learning progression based on many assessment tasks, not those in a fixed sequence. A general learning progression describes students' various ways of reasoning about a topic, irrespective of curriculum; it focuses on understanding and reacting to students' current cognitive structures. A curriculum-based learning trajectory describes students' ways of reasoning within a fixed curriculum; it focuses on understanding and reacting to students' cognitive structures, relative to the curriculum sequence. The advantage of learning progressions is that they are widely applicable and focus tightly on general student cognition. The advantage of learning trajectories is their specificity in tracing students' movement through a fixed curriculum.

LP as Cognitive Terrain

It is useful to think of learning progressions as describing the terrain on a mental mountain slope that students must ascend to learn and become fluent with particular mathematical topic. From a curriculum-development, instructional planning perspective, we try to determine the most efficacious ascent path (the one for which most students are most likely to succeed), as depicted by the fixed path in Figure 1a (the hypothetical prototypical learning trajectory). But to meet individual students' learning needs, often we must zoom in on individual deviations from the path to more precisely determine the next steps that students can make successfully. Critical to aiding a student's moment-to-moment climb is flexibly and reactively choosing tasks that provide them with successful hand- and foot-holds in this cognitive terrain (Figure 1b).



Figure 1a



Figure 1b

Theoretical Frameworks for Learning Progressions

Another way to understand differences between learning progressions is to examine their postulated learning mechanisms. For instance, the original van Hiele theory relates progression through the levels of geometric thinking to phases of instruction. In contrast, Battista uses constructivist constructs such as

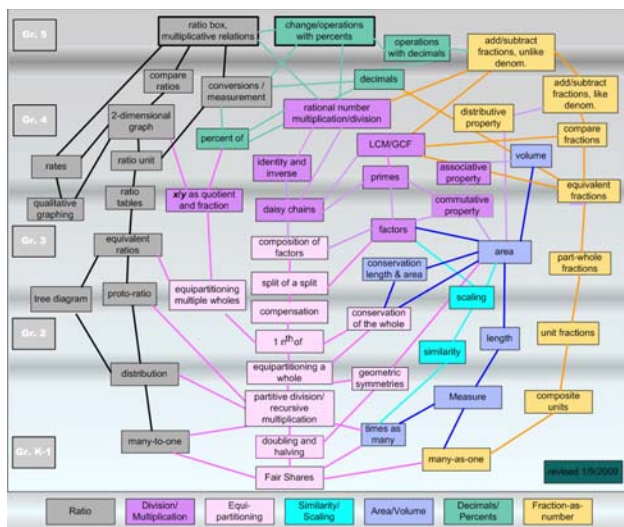
levels of abstraction to describe students' progression through the van Hiele levels (see also the theories of abstraction of Simon, et al. (2004) and Mitchelmore & White (2000), as well as Pegg & Davey's analysis of geometric learning, 1998).

We might also contrast a constructivist approach to teaching to the approach taken in Gagne's "programmed learning" hierarchies⁵, which seem much more fixed, logical, prescribed, and less interactive.

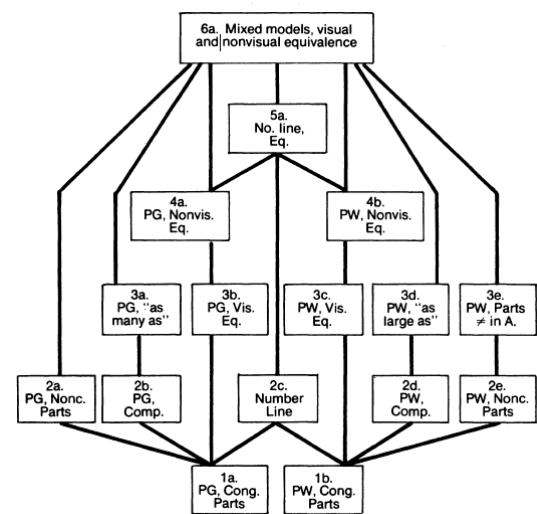
Beginning with the final task, the question, is asked, What kind of capability would an individual have to possess if he were able to perform this task successfully, were we to give him only instructions? ... Having done this, it was natural to think next of repeating the procedure with this newly defined entity (task). What would the individual have to know in order to be capable of doing *this* task without undertaking any learning, but given only some instructions? ... Continuing to follow this procedure, we found that what we were defining was a *hierarchy* of subordinate knowledges [sic], growing increasingly "simple" ... Our hypothesis was that (a) no individual could perform the final task without having these subordinate capabilities ... and (b) that any superordinate task in the hierarchy could be performed by an individual provided suitable instructions were given, and provided the relevant subordinate knowledges could be recalled by him (Gagne, 1962, p. 356).

⁵ A hierarchy was empirically validated by examining student success rates on various items in the hierarchy (similar to examining item difficulties in current quantitative approaches). So it was not intended that hierarchies be developed strictly logically.

It is interesting that, on the surface at least, representations of learning progressions from different theoretical frameworks can look similar. For instance, compare the overall appearance of the learning progression of Confrey et al. (from a more constructivist perspective) to the Gagne-like hierarchy described by Novillis (see Figure 2). It would be revealing to analyze how these progressions differ at a micro- versus macro-level.



Confrey et al., 2009, p. 1-4
 Learning Trajectories Map for Rational Number Reasoning.
 Figure 2. Trajectory versus Hierarchy



Novillis, 1976, p. 132
 A Hierarchy of selected subconcepts of the fraction concept

The Nature of Levels

A critical component of learning progressions is the notion of "levels." Because the concept of level is not straightforward, and because how one defines level determines how one views (and measures) level attainment, I examine this concept in more detail, using the van Hiele levels as an example. Indeed, the issues discussed below for the van Hiele levels are critical because any attempt to develop, assess, and use levels in learning progressions must address these issues in some way.

Levels, Stages, and Hierarchies

Clements and Battista (1992) described the difference between researchers' use of the terms *stage* and *level* as follows. A *stage* is a substantive period of time in which a particular type of cognition occurs across a variety of domains (as with Piagetian stages of cognitive development). In contrast, a *level* is a period of time in which a distinct type of cognition occurs for a specific domain (but the size of the domain may be an issue). Battista defines a third construct—a *level of sophistication* in student reasoning as a qualitatively distinct type of cognition that occurs within a hierarchy of cognition levels for a specific domain.

Example: The van Hiele Levels

In discussing the van Hiele levels, Clements and Battista (1992) suggested several characteristics that might apply to levels.

- "Learning is a discontinuous process. That is, there are 'jumps' in the learning curve which reveal the presence of discrete, qualitatively different levels of thinking.
- The levels are sequential and hierarchical. For students to function adequately at one of the advanced levels in the van Hiele hierarchy, they must have mastered large portions of the lower levels. ... Progress from one level to the next is more dependent upon instruction than on age or biological maturation. ... Students cannot bypass levels and achieve understanding (memorization is not an important feature of any level). The latter requires working through certain "phases" of instruction" (1992, pp. 426-7).

Types of Levels-Hierarchies

When considering *hierarchies* of levels in learning progressions, it is helpful to distinguish two types. A "weak" levels-hierarchy refers to a set of levels that are ranked in order of sophistication, one above another, with no class inclusion relationship between the levels necessary. A "strong" levels-hierarchy refers to a set of levels ranked in order of sophistication, one above another, with class inclusion relationships between the levels required. That is, in a "strong" levels-hierarchy, students who are reasoning at level n are assumed to have progressed through reasoning at levels 1, 2, ... ($n-1$). The van Hiele levels were originally hypothesized to form a strong levels-hierarchy (which is generally supported by the research—but there are issues), while Battista's levels of sophistication in reasoning about length to be discussed below form a weak levels-hierarchy. (I will return to this idea when I discuss quantitative methods for examining learning progressions.)

Being "At" a Level

What, precisely, does it mean to be "at" a level? Battista (2007) argued that students are *at* a van Hiele level when their overall cognitive structures and processing causes them to be disposed to and capable of thinking about a topic in a particular way. So students are "at" van Hiele Level 1 when their overall cognitive organization and processing disposes them to think about geometric shapes in terms of visual wholes; they are at Level 2 when their overall cognitive organization disposes and enables them to think about shapes in terms of their

properties. Also in this view, when students move from familiar content to unfamiliar content, their level of thinking might decrease temporarily; but because students are disposed to operate at the higher level, they look to use that level on the new material, and quickly become capable of using that level (Battista, 2007). So, for instance, in moving from studying quadrilaterals to studying triangles, students who are at Level 2 for quadrilaterals might initially process triangles as visual wholes, but right from the start they look for, and fairly quickly discover and use, triangle properties.

A Different Approach: Vectors and Overlapping Waves

Some studies indicate that people exhibit behaviors indicative of different van Hiele levels on different subtopics of geometry, or even on different kinds of tasks (Clements & Battista, 2001). So an alternate view of the development of geometric reasoning is that students develop several van Hiele levels simultaneously. To represent this view, Gutiérrez et al. (1991) used a vector with four components to indicate the degrees of acquisition of each of van Hiele levels 1 through 4. For example, a student's degree of acquisition vector might be: 96.67% for Level 1, 82.50% for Level 2, 50.00% for Level 3, and 3.75% for Level 4. Using this vector approach, Gutiérrez et al. described six profiles of level-configurations in students' reasoning about 3d geometry. To illustrate, Profile 2 was characterized by complete acquisition of Levels 1 and 2, high acquisition of Level 3, and low acquisition of Level 4. However, even though level acquisition was described in terms of the vector model, the profiles could easily be re-

interpreted in terms of levels only. For instance, Profile 2 could be thought of as Level 2 or transition to Level 3.

Similar to the vector approach to the van Hiele levels, several researchers have posited that different types of reasoning characteristic of the van Hiele levels develop simultaneously at different rates, and that at different periods of development, different types of reasoning are dominant, depending on the relative competence students exhibit with each type of reasoning (Clements & Battista, 2001; Lehrer et al., 1998; see Figure 3). The "waves" depicted in Figure 3 are the competence growth curves for the different types of reasoning.

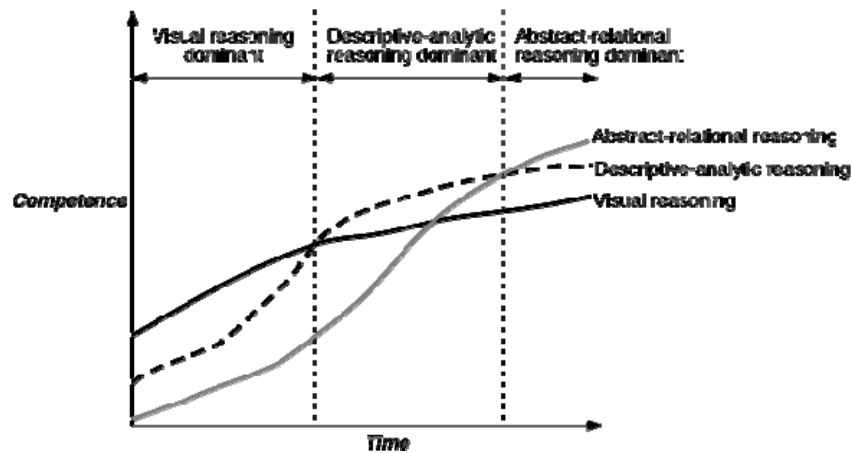


Figure 3. Waves of acquisition of van Hiele levels

Lehrer et al. (1998) argued that ... geometric development should be characterized "by which 'waves' or forms of reasoning are most dominant at any single period of time" (p. 163). Clements and Battista (2001) also proposed the view that the van Hiele levels (seen as types of reasoning) develop simultaneously but at different rates. Visual-holistic knowledge, descriptive verbal knowledge, and, to a lesser extent initially, abstract symbolic knowledge grow simultaneously, as do interconnections between levels. However, although

these different types of reasoning grow in tandem, one level tends to become ascendant or privileged in a child's orientation toward geometric problems. Which level is privileged is influenced by age, experience, intentions, tasks, and skill in use of the various types of reasoning.

Although the vector and wave models for the van Hiele theory have merit, embedded within both is a difficult issue—distinguishing *type of reasoning* from *level of reasoning*. That is, sometimes the term *visual-holistic* is used to refer to that type of reasoning that is strictly visual in nature, and sometimes it is used to refer to a period of development of geometric thinking when an individual's thinking is dominated and characterized by visual-holistic thinking. For instance, Gutiérrez et al. (1991) used vectors to indicate students' "capacity to use each one of the van Hiele *levels*" (p. 238). This statement makes sense only if van Hiele levels are taken as types of reasoning, not periods of development characterized by qualitatively different kinds of thought. Similarly, Clements and Battista (2001), along with Lehrer et al. (1998), talked about "waves of acquisition" of *levels* of reasoning defined by van Hiele. Thus, broadly speaking, researchers have intermingled and not yet completely sorted out (a) van Hiele levels as types of reasoning, and (b) van Hiele levels as periods of development of geometric reasoning. The waves theory described above is similar, but not identical, to Siegler's overlapping waves theory (2005). Indeed, the vertical axis in Siegler's theory is "relative frequency" of use, not competence, as shown in the van Hiele interpretation above. Frequency of use may be connected to

Battista

competence, but also to other factors such as personal preference, social pressure, and so on.

In summary, given the variability in strategy use and reasoning that seems to accompany learning, even if we develop an adequate definition for what it means for a student to be "at" a level, the periods of time when students meet the strict requirement for being at levels may be short, with students spending most of the time "in transition."

Level Determination

Empirical determination of levels of reasoning is a major issue in the van Hiele theory, and LP/LT levels in general, because it operationalizes researchers' conceptions of the qualitatively different types of reasoning that occur in the LP/LT. For instance, consider some of the different ways that researchers have determined van Hiele levels. Some studies (Carroll 1998; Usiskin, 1982) used paper and pencil tests, judging that a level was achieved if a given number of items designed to assess that level were *answered* correctly. In other studies (Fuys et al., 1988; Clements & Battista, 1992; Battista, in prep) students' reasoning (as recorded in interviews or open response written tasks) was coded by matching students' reasoning to characteristics of the van Hiele levels. Beyond the answers versus reasoning dichotomy, there have been additional differences in level determination. For instance, in the Usiskin van Hiele test, three of the tasks used to assess property-based (Level 2) reasoning about quadrilaterals involved diagonals, but the Battista and Clements and Battista studies focused on

visually salient "defining" properties of shapes. Thus, the properties assessed by Usiskin's test were more likely to be unfamiliar to students than those assessed by Battista and Clements and Battista.

A totally different approach to assessing van Hiele levels was devised by another group of researchers (Battista, 2007). In a collaborative effort to find ways to assess elementary students' acquisition of the van Hiele levels in interview situations, Battista, Clements, and Lehrer developed a triad sorting task, that, with variations, both Clements and Battista (2001), Lehrer et al. (1998), and Battista (in progress) used in separate research efforts. In this task, students were presented with three polygons, such as those shown in Figure 4, and were asked, "Which two are most alike? Why?" Choosing B and C and saying that they "look the same, except that B is bent in" was taken as a Level 1 response. Choosing A and B and saying either that they both have two pairs of congruent sides or that they both have four sides was taken as a Level 2 response. The purpose of this task was to determine the type of reasoning used on a task that students had not seen before (so it was unlikely to elicit instructionally programmed responses).

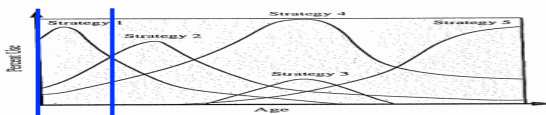


Figure 4. Triad polygon sorting task.

One difficulty with this analysis is that giving the number of sides of a polygon is a "low-level" use of properties. That is, there are different types of

geometric properties. The simplest property involves describing the number of components in a shape. For instance, a quadrilateral has four sides; a triangle has three angles. A second, more sophisticated type of property describes spatial relationships that are particularly salient in identifying shapes (e.g., opposite sides of a rectangle are congruent and all angles are right angles). In some sense, these properties are the "psychological defining characteristics" of shapes for Level 2 students. The third type of property describes other interesting but less salient relationships (e.g., the diagonals of a rectangle are congruent and bisect each other⁶). These properties are likely to be derived once students understand the meaning of shape classification—so they are more likely to occur in Level 3.

The distinction in properties described above suggests that students' use of number of sides of a polygon may not be a very good indicator of Level 2 thinking, which should focus on relational properties. Thus some jumps in levels on triad tasks observed by Lehrer et al. (1998) may have been caused by coding students' use of number of sides as Level 2. Because a critical factor used in distinguishing van Hiele levels is how students deal with geometric properties, clarifying the meaning of *properties*, as it relates to the van Hiele levels, is important.

Another factor that should be considered with the triad task is that saying Shape B is more like Shape C is not necessarily a less sophisticated response than focusing on number of sides. That is, Shape B is actually more like Shape C if we

⁶ Of course, it is true that some "interesting" properties *logically* can be used to define shapes.

consider how much movement it takes to transform B into C, compared to B into A. In fact, one could imagine a metric that quantifies the amount of movement required. Thus, the “morphing” response described by Lehrer et al. (1998), and also observed by Clements and Battista (2001), may be an intuitive version of a notion whose mathematization is far beyond the reach of elementary students.

Another issue with the triad-task approach is pointed out by differences in the ways the researchers used the triads. Lehrer et al. (1998) construed *each* triad task as an indicator of type of reasoning. So students’ use of different types/levels of reasoning on different triads was taken as evidence of differences in levels of response. In contrast, Clements and Battista (2001) used a *set* of 9 triad items as an indicator of level of students. To be classified at a given level, a student had to give at least 5 responses at that level. If a student gave 5 responses at one level and at least 3 at a higher level, the student was considered to be in transition to the next higher level. Of course, because it aggregates responses, this approach obscures intertask differences and variability in reasoning. It focuses on determining the *predominant* level of reasoning that a student used on the triad tasks.

Another difference between the researchers’ approaches is also important. In analyzing students’ reasoning on the triad tasks, Lehrer et al. (1998) classified student responses solely on the basis of the type of reasoning that students employed. In contrast, in determining students’ van Hiele levels, Clements and Battista (2001) attempted to also account for the “quality” of students’

reasoning—each reason for choosing a pair in a triad was assessed to see if it correctly discriminated the pair that was chosen from the third item in the triad. In this scheme, the van Hiele levels for students were determined based on a complicated algorithm that accounted for both type of reasoning and discrimination score^{7,8}.

Cognition Based Assessment (CBA): Levels, Progressions, Trajectories, and Profiles

I now describe my work on the Cognition Based Assessment project to illustrate the relationship between learning progressions and learning trajectories as *representations of learning for teaching*⁹. The description of CBA also illustrates that to be useful for teachers, learning progressions must be embedded within an interconnected system of *LP-based* formative assessments, interpretations of students' reasoning, and instruction.

The CBA View of Learning and Instruction

According to the "psychological constructivist" view of how students learn mathematics *with understanding*, the way students construct, interpret, think about, and make sense of mathematical ideas is determined by the elements and

⁷ Additional discussions of van Hiele levels measurement issues can be seen in articles by Wilson (1990), and Usiskin and Senk (1990).

⁸ It is worth noting that quantitative methods for determining levels face the same issues described here for qualitative methods. For instance, using the Saltus method can still leave us with many students who cannot be clearly placed in a level (e.g., Draney & Wilson, 2007).

⁹ CBA development was partially supported by the National Science Foundation under Grant Nos. ESI 0099047 and 0352898. Opinions, findings, conclusions, or recommendations, however, are those of the author and do not necessarily reflect the views of the National Science Foundation.

organization of the relevant mental structures that the students are currently using to process their mathematical worlds (e.g., Battista, 2004). To construct new knowledge and make sense of novel situations, students build on and revise their current mental structures through the processes of action, reflection, and abstraction. A major component of psychological constructivist research on mathematics learning and teaching is its attention to students' construction of meaning for specific mathematical topics. For numerous mathematical topics, researchers have found that students' development of conceptualizations and reasoning can be characterized in terms of "levels of sophistication" (e.g. Battista & Clements, 1996; Battista et al., 1998; Cobb & Wheatley, 1988; Steffe, 1992; van Hiele, 1986). These levels lie at the heart of the CBA conceptual framework for understanding and building upon students' learning progress. Selecting/creating instructional tasks, adapting instruction to students' needs, and assessing students' learning progress require detailed, cognition-based knowledge of how students construct meanings for the specific mathematical topics targeted by instruction.

CBA Assessment and Instruction

To implement mathematics instruction that genuinely and effectively supports students' construction of mathematical meaning and competence, teachers must not only understand cognition-based research on students' learning of particular topics, they must be able to use that knowledge to determine, monitor, and guide the development of their own students'

reasoning. Cognition-Based Assessment supports these activities by including the following five critical components.

1. Descriptions of core mathematical ideas and reasoning processes that form the foundation for students' sense making and understanding of elementary school mathematics.
2. For each core idea, research-based descriptions of levels of sophistication in the development of students' understanding of and reasoning about the idea (these are CBA LP).
3. For each core idea, coherent sets of assessment tasks that enable teachers to investigate their students' mathematical thinking and precisely locate students' positions in the cognitive terrain for learning that idea.
4. For each assessment task, a description of what each level of reasoning might look like for the task.
5. For each core idea, descriptions of instructional activities specifically targeted for students at various levels to help them move to the next higher level.

These five components are critical for an assessment "system" that focuses on understanding and guiding the development of students' mathematical reasoning.

Learning Progressions and Trajectories for Length

The CBA levels of sophistication, or learning progressions, for a topic (a) start with the informal, pre-instructional reasoning typically possessed by

students; (b) end with the formal mathematical concepts targeted by instruction; and (c) indicate cognitive plateaus reached by students in moving from (a) to (b). As an example, Figure 5 outlines the CBA levels of sophistication for the concept of length.

Non-Measurement Reasoning	Measurement Reasoning
<p>N0: Student Compares Objects' Lengths in Vague Visual Ways</p> <p>N1: Student Correctly Compares Whole Objects' Lengths Directly or Indirectly</p> <p>N2: Student Compares Objects' Lengths by Systematically Manipulating or Matching Their Parts</p> <p>N2.1. <i>Rearranging Parts to Directly Compare Whole Shapes</i></p> <p>N2.2. <i>One-to-One Matching of Parts</i></p> <p>N3: Student Compares Objects' Lengths Using Geometric Properties</p>	<p>M0: Student Uses Numbers in Ways Unconnected to Iteration of Unit-Lengths</p> <p>M1: Student Iterates Units Incorrectly</p> <p>M1.1: <i>Iterates Non-Length Units (e.g., Squares, Cubes, Dots) and Gets Incorrect Count of Unit-Lengths</i></p> <p>M1.2: <i>Iterates Unit-Lengths but Gets Incorrect Count</i></p> <p>M2: Student Correctly Iterates ALL Unit-Lengths One-by-One</p> <p>M2.1: <i>Iterates Non-Length Units (e.g., Squares, Cubes) and Gets Correct Count of Unit-Lengths for Straight Paths</i></p> <p>M2.2: <i>Iterates Non-Length Units (e.g., Squares, Cubes) To Correctly Count Unit-Lengths for Non-Straight Paths</i></p> <p>M2.3: <i>Explicitly Iterates Unit-Lengths and Gets Correct Counts for Straight and Non-Straight Paths</i></p> <p>M3: Student Correctly Operates on Composites of Visible Unit-Lengths</p> <p>M4: Student Correctly and Meaningfully Determines Length Using only Numbers—No Visible Units or Iteration</p> <p>M5: Student Understands and Uses Procedures/Formulas for Perimeter Formulas for Non-Rectangular Shapes</p>

Figure 5. CBA Levels for Students' Reasoning about Length (Battista, accepted)

The set of CBA levels of sophistication for the topic of length are graphically depicted in Figure 6. Also shown, are an ideal hypothetical learning trajectory (in red) and a typical actual learning trajectory for students (in green). The CBA levels represent the "cognitive terrain" that students must ascend during an actual learning trajectory.

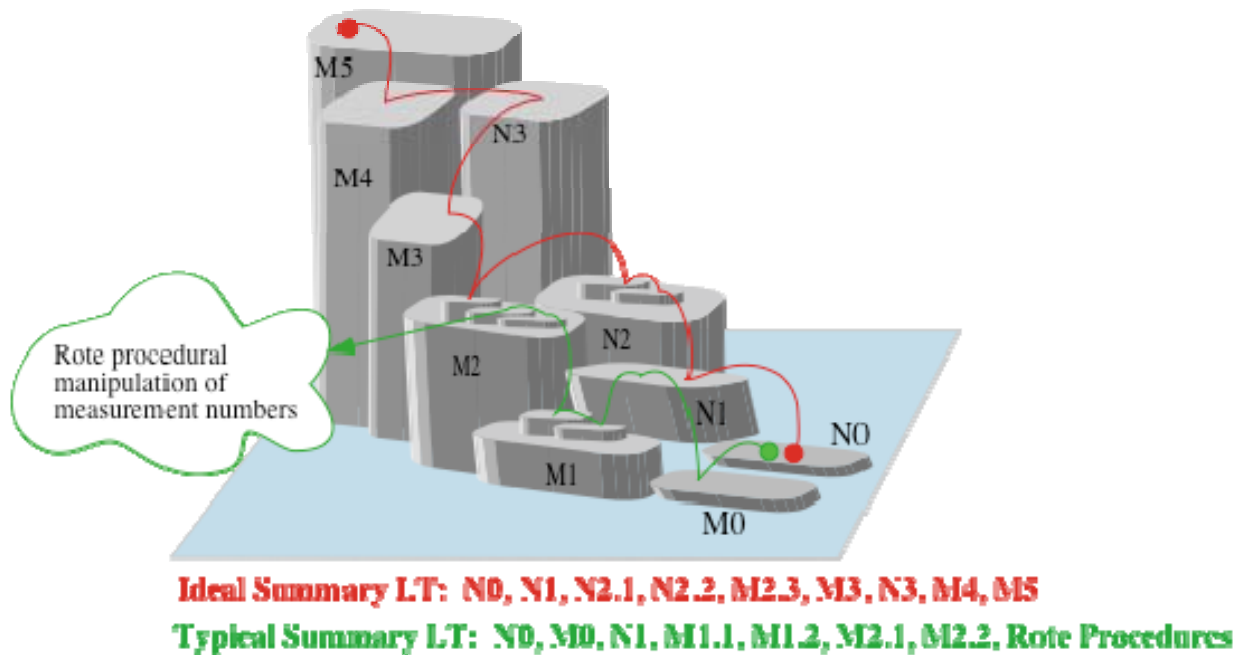


Figure 6. Levels of Sophistication Plateaus and Learning Trajectories for CBA Length

A CBA levels-model for a topic describes not only cognitive plateaus, but what students can and cannot do, students' conceptualizations and reasoning, cognitive obstacles that obstruct learning progress, and mental processes needed both for functioning at a level and for progressing to higher levels. The levels are derived from analysis of both the mathematics to be learned and empirical research on students' developing conceptualizations of the topic. The jumps in the ascending plateau structure of a CBA levels-model represent cognitive restructurings evidenced by observable increases in sophistication in students' reasoning about a topic. Furthermore, an ideal CBA levels-of-sophistication model for a topic provides indications of jumps in sophistication that are small enough to fall within students "zones of construction." That is, a student should be able to accomplish the jump from conceptualizing and reasoning at Level N to

conceptualizing and reasoning at Level N+1 by making a significant abstraction, in a particular context, while working to solve an appropriate problem or set of problems¹⁰. For instance (See Figure 7), in Situation A the student has to make a cognitive jump that is too great. In Situation B, the student can progress from Level 1 to Level 2 by making cognitive jumps to successive sublevels.

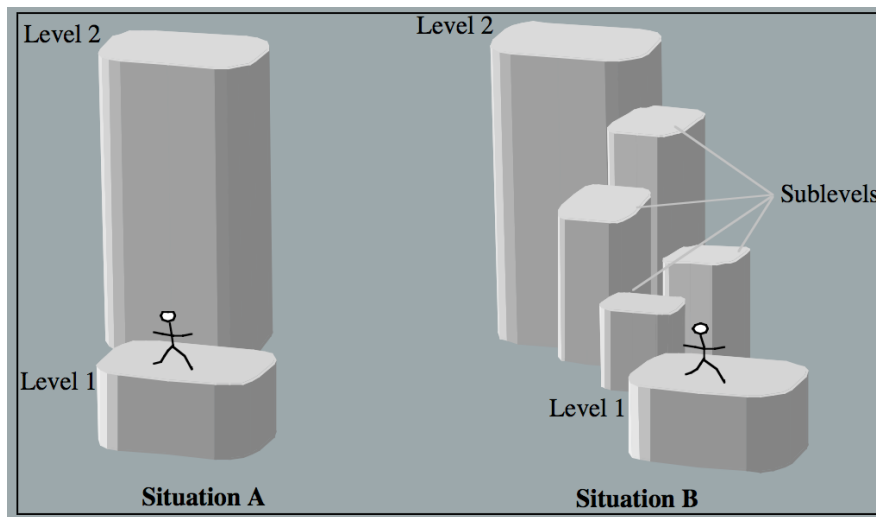


Figure 7

However, because the levels are compilations of empirical observations of the thinking of many students, and because students' learning backgrounds and mental processing differ, a particular student might not pass through every level for a topic; he or she might skip some levels or pass through them so quickly that the passage is difficult to detect. Even with this variability, however, the levels still describe the plateaus that students achieve in their development of reasoning about a topic. They indicate major landmarks that research has shown students often pass through in "constructive itineraries" or learning trajectories

¹⁰ The jump in reasoning may apply to restricted contexts, not to all contexts connected with the mathematical topic. That is, the jump may be tightly situated rather than global.

for these topics. Thus, such levels provide an excellent conceptual framework for understanding the paths students travel to achieve meaningful learning of a topic.

Digging Deeper into the LP/LT Representations

As hypothetical or average learning trajectories, the trajectories depicted in Figure 6 are still simplifications of actual learning trajectories traversed by individual students. To illustrate, I describe one portion of the actual learning trajectory of a fifth grader, RC, who was having particular difficulty with the concept of length (the trajectories of most other students were much simpler). Figure 8 shows RC's learning trajectory for 34 consecutive length items (start with the green point, end with the red point). This actual learning trajectory is extremely complex because it contains so much back-and-forth movement between levels. Note that RC's performance is consistent with the variability in strategy choice described by Siegler (2007, var).

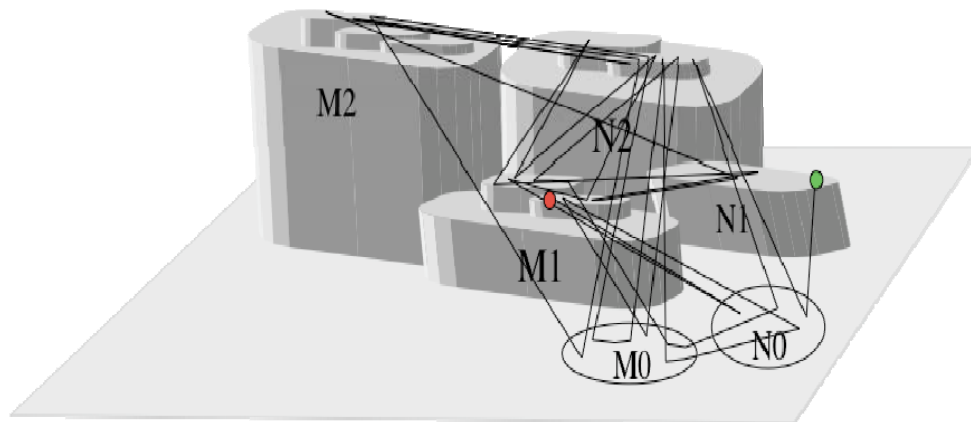


Figure 8 RC's learning trajectory for 34 consecutive length items

Figure 9 provides a better representation of this complicated portion of RC's learning trajectory. This figure starts with RC's levels on initial assessment items, moves to his responses during an instructional intervention, and ends with his reasoning on reassessment items.

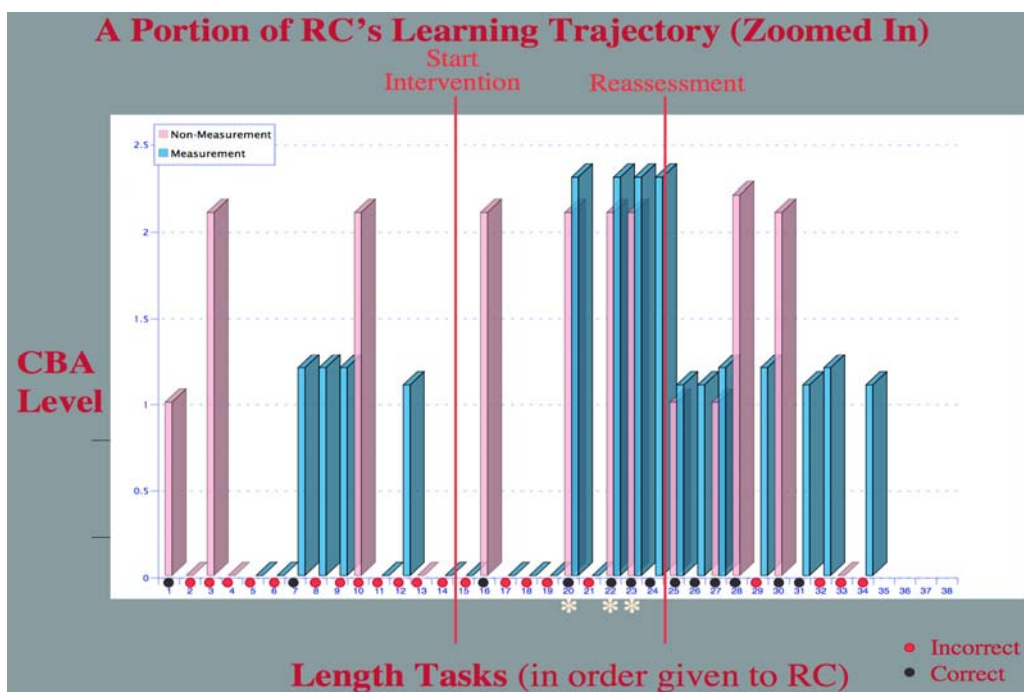


Figure 9. Another representation of RC's learning progress

But even Figure 9 does not represent RC's learning trajectory with enough detail to be maximally useful for instruction. We need a *narrative description* of (a) what tasks he was attempting, and (b) his level of reasoning on each task. Below, this information is provided for the critical period of instructional intervention in which RC made progress (see the three starred items in Figure 9).

During the instructional intervention, RC was given items of the following type.

Item 23 (see Figure 10). Suppose I pull the wires so they are straight. Which wire would be longer, or would they be the same? How do you know? Predict an answer, then check with inch rods (the black/gray sections on the student sheet were each 1 inch in length). [Items 20-22 were similar.]

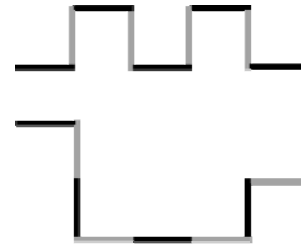


Figure 10

On Item 23, RC counted unit lengths as shown in Figure 11 and concluded that the top wire was longer. He checked his answer by placing inch rods on both wires then straightening each set of rods to compare the lengths directly.

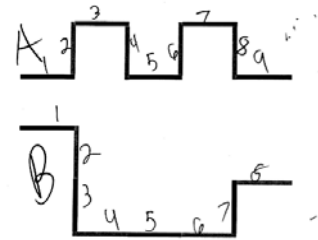


Figure 11

Importantly, on Item 23 and several other problems, RC used *both* M2.3 and N2.1 reasoning. On the last problem of this type (Item 24), RC did not check his answer by straightening—he seemed sure of his prediction, having empirically abstracted that comparing counts of unit lengths predicted the results of comparing straightened wires.

In the reassessment period, RC's thinking regressed when he attempted problems that were different from the ones he successfully used M2.3 reasoning on. For instance, on Item 27, at first, RC counted unequal segments, then dots (Measurement Reasoning), then imagined straightening paths (Non-Measurement reasoning), forming contradictory conclusions.

Item 27. Which path is shorter, or are they the same? How do you know?

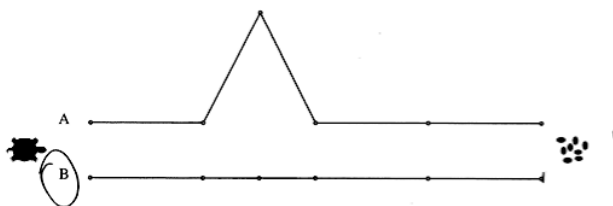


Figure 12

- RC: [Counts gaps between dots on the bottom path, then on the top path] 1, 2, 3, 4, 5. 1, 2, 3, 4, 5. Hmm. Which one do I think is shorter... [Counts dots on the top path, then on the bottom path] 1, 2, 3, 4, 5, 6. 1, 2, 3, 4, 5, 6. This one's [pointing to the bottom path] shorter. ...
- I: Okay. Now you got 6 both times? But you still think this one [pointing to path B] is shorter?
- RC: Yep.
- I: Why is it shorter?
- RC: Because if you pull this one out [pinching the endpoints of A with his fingers]...it'll be like right there [moving his fingers horizontally outwards to just past the endpoints of A]. You can't pull this [pinching the endpoints of B] out anymore.

So in the face of a seeming conflict between measurement and non-measurement reasoning, RC correctly relies on his non-measurement reasoning.

- I: Okay. Could using these rods help you think about this problem [placing inch rods on the paper]?
- RC: Yes. [Counts the segments on path A.] 1, 2, 3, 4, 5. ... So here and right up here [draws marks at the two ends of the straight line of 5 rods he places above path A]. [RC counts gaps between dots on path B, then counts 5 inch rods that he places at the top of the sheet, between the two marks that he previously made when rearranging path A] 1, 2, 3, 4, 5. 1, 2, 3, 4, 5.
- I: Okay, now how did you get 5? Is that for the bottom path? For B?
- RC: Yeah.
- I: How did you get B? Show me.
- RC: Because [counting gaps between dots on path B, then on path A] 1, 2, 3, 4, 5. 1, 2, 3, 4, 5.
- I: ... Do 5 of these [rods] fit? Like if you put 1 here [placing a rod on the leftmost segment of the bottom path], and one here [places a second rod over the second and third gaps between dots from the left end of the bottom path].
- RC: Well, I can make it like a string.
- I: Do you want to use this [hands RC a line of cylindrical inch rods strung on a wire]?
- RC: [Places the first 4 rods over path B and makes a mark at the right end with his pen.]
- I: So what are you thinking?
- RC: This one [pointing at line A] is longer. This one [pointing at path B] is shorter.
- I: Okay. And how did you figure that out?
- RC: I lined these up [pointing at the string of rods]. And there was some more right there [pointing to the 'hill' on the top path]. ...

In the above episode, the interviewer attempted to get RC to use inch rods and measurement M2 reasoning. However, RC used the inch rods mainly to correctly implement the non-measurement N2.1 strategy.

In the episode below, the interviewer was even more directive in encouraging RC to develop correct *measurement* reasoning.

Item 32

- I: [See Figure 13a] If these are wires and I pull them so they are straight, which will be longer, or will they be the same? Is there any way that counting can help you solve this problem?

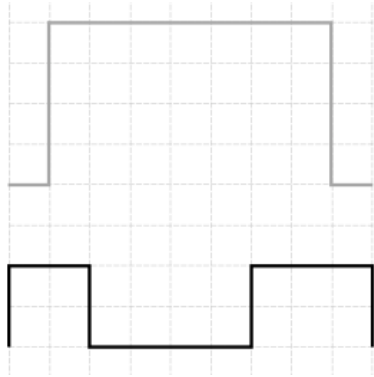


Figure 13a

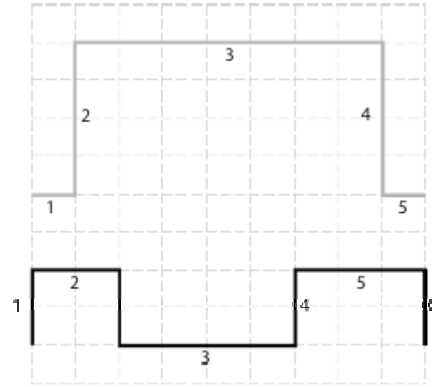


Figure 13b

RC: Um, yes.

I: So what would you do?

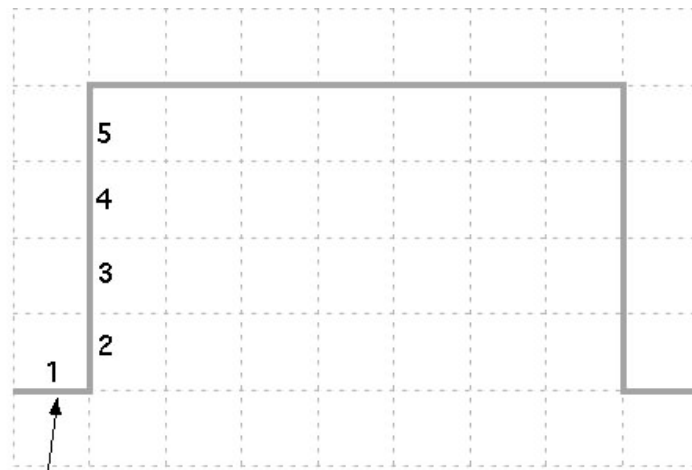
RC: Count these [counts the unequal straight portions of the bottom wire, but skips the second vertical segment from the left—see Figure 13b] 1, 2, 3, 4, 5, 6. [Counts the unequal straight portions on the top wire] 1, 2, 3, 4, 5.

RC: But this one [pointing to the top wire] would actually be longer. Because if you pull it out it'll come right there [pulling his hands out from the endpoints of the top wire to several inches past the endpoints]. And if you pull it out, it'll come right there [pulling his hands out from the endpoints of the bottom wire to a few inches past the margins].

So RC uses incorrect measurement and non-measurement reasoning on this task.

Item 34

I: Okay, could counting rods like this [tracing the unit segment at the left end of the top wire] help at all?



Rods like this

Figure 14

RC: I already counted that. [Pointing to each straight portion of the top wire again] 1, 2, 3, 4, 5.

I: Oh, but I was wondering if, could you count like [counting a few unit segments on the top wire, moving from left to right] 1, 2, 3, 4, 5 like that [see Figure 14]? Would that help?...

RC: I think so. [Counting squares along the top wire; see Figure 15] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. [Counting squares along the bottom wire] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.

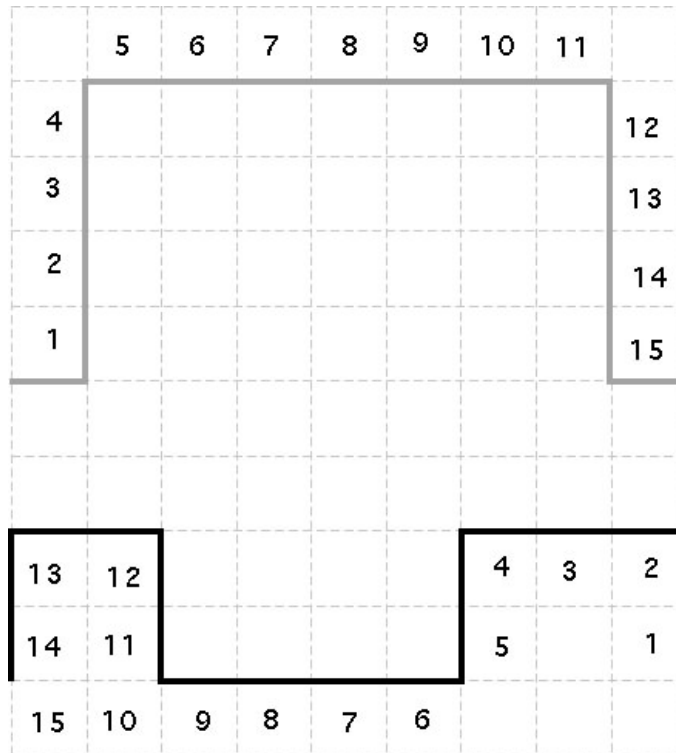


Figure 15

I: So what do you think?

RC: Probably the same length.

Given this narrative data on RC's reasoning, how should we represent his current knowledge structure with respect to length in a way that is most helpful for instruction? Rather than using an actual learning trajectory, the CBA approach is to construct a "profile" of RC's reasoning, using CBA LP levels of sophistication as the conceptual framework. To see what this profile looks like, note that in the context of problems like Item 23, in which the "wires" could be straightened using actual inch rods, RC had seen *empirically* that counting unit lengths could predict which was longer. So, for the last of these problems, he adopted the scheme of comparing wires by counting unit lengths in them. At first, he checked his answers by physically straightening a set of inch rods for each wire; but he curtailed this physical check on the last problem. We can

conclude that in this context, RC had abstracted a particular reasoning scheme. However, for problems in different contexts, where dots or squares were salient, RC did not apply his new scheme (but he also did not apply his original M0 scheme). Furthermore, throughout the sessions, RC kept returning to the non-measurement scheme of straightening the paths (N2.1). So, the profile of RC's reasoning in terms of the CBA LP for length is: (a) he still relies heavily on non-measurement N2.1 reasoning; (b) he has started to see that measurement reasoning M1.2 (counting rods) can help him determine which path is longer; but (c) he does not yet understand the critical properties of unit length iteration (no gaps, overlaps; uniform lengths – M2.3).

So, future instruction must help RC (a) connect his iteration of inch rods (M2) to straightening paths (N2.1), (b) develop understanding of the properties of unit length iteration (M2.3), and (c) generalize a correct unit iteration scheme to new contexts (M2.3). For instance, in problems like Item 32, we would encourage RC to use inch rods (matching square size) to check his answers by *counting and straightening*. In response to this type of instruction/intervention, many students constructed generally applicable schemes, overcoming the fixation on the visually salient squares. Additionally, we also need to give RC tasks that highlight the importance of unit length iteration properties. For instance, we need to give RC problems in which he can determine by straightening that counting unequal segments gives incorrect comparisons.

It is the condensed, synthesized narrative profile of RC's reasoning, described in terms of CBA tasks and levels, that enables us to appropriately characterize and diagnose RC's reasoning in a way that is most useful for designing instruction that best matches his learning needs. Knowing the average CBA level for these tasks, or having a numerically valued vector or table of CBA level numbers, is insufficient for proper diagnosis and remediation.

Qualitative versus Quantitative Approaches to Developing LP

Both qualitative and quantitative methods have been used to develop learning progressions in mathematics (and science). *Both approaches are equally careful and scientific.* Generally, both approaches involve (a) synthesizing, integrating, and extending previous research to develop conceptual models of the development of student reasoning about a topic (hypothesized learning progressions); (b) developing and iteratively testing assessment tasks; (c) conducting several rounds of student interviews in support of steps (a) and (b); and (d) iteratively refining LP levels. In qualitative approaches, the cycle of iteration, testing, and revising eventually "stabilizes" into final levels, as determined by current level descriptions being used to reliably code all data. In contrast, quantitative methods compare the data to statistical model predictions (which often are derived using mathematical iteration), and, if needed, make adjustments to assessment item sets and levels.

Rash Rush to Rasch? Issues with Quantitative Methods

There have been numerous recommendations (sometimes demands) to use quantitative techniques to develop learning progressions (e.g., NRC, 2001), with a hint that using non-quantitative techniques is less "scientific." For example, Stacey & Steinle state that there have been "repeated suggestions made by colleagues over the years, which implied that we had been remiss in not using this Rasch analysis with our data" (2006, p. 89). However, using Rasch and other IRT approaches raises often-ignored serious issues that I now highlight.

First, Rasch/IRT models are "measurement" models. For instance, Masters and Mislevy state that "The probabilistic partial credit model ... enables *measures* of achievement to be constructed" [italics added] (1991, p. 16). Or, Wilson, who describes the Saltus model as an example of "psychometric¹¹ models suitable for the analysis of data from assessments of cognitive development" (Wilson, 1989, p. 276). However, the whole enterprise of "measuring" in psychological research has been criticized, with less than compelling rebuttals (Michell, 2008).

Second, many of the assumptions of numerical models do not seem to fit our understanding of the process of learning and reasoning in mathematics. For instance, the Saltus model "assumes that each member of group h applies the strategies typical of that level consistently across all items" (Wilson, 1989, p. 278). Or, "The saltus model assumes that all persons in class c answer all items in a

¹¹ Of course psychometrics is "the measurement of mental capacity, thought processes, aspects of personality, etc., esp. by mathematical or statistical analysis of quantitative data" (OED online).

manner consistent with membership in that class.... In a Piagetian context, this means that a child in, say, the concrete operational stage is always in that stage, and answers all items accordingly. The child does not show formal operational development for some items and concrete operational development for others" (Draney & Wilson, 2007, p. 121). But, as has been discussed earlier, the levels in learning progressions are not necessarily stages, and often do not form a strong levels-hierarchy, making quantitative models problematic:

"From this research, one can only conclude that there are situations in which students appear to reason systematically...When these situations arise, evidence about student understanding can be summarized by [numerical] learning progression level diagnoses, and educators can draw valid inferences about students' current states of understanding. Unfortunately, inconsistent responding across problem contexts poses challenges to locating students at a single learning progression level and makes it unclear how to interpret students' diagnostic scores. For example, how should one interpret a score of 2.6? A student with this score could be reasoning with a mixture of ideas from levels 2 and 3, but the student could also be reasoning with a mixture of ideas from levels 1, 2, 3, and 4. Such challenges prompt additional studies to support the valid interpretation of learning progression diagnoses" (Steedle & Shavelson, 2009, p. 704).

Thus, use of Rasch-like models to examine cognitive development, such as Wilson's Saltus model or latent class analysis, assumes that students are "at a

level" (Briggs...; Draney & Wilson), which returns us to the problem discussed earlier about a student being at a level. Research on learning suggests that quite often, the state of student learning is not neatly characterized as "being at a specified level," which causes problems for interpretation of model results: "The results from this study suggest that students cannot always be located at a single level of the learning progression studied here. Consequently, learning progression level diagnoses resulting from item response patterns cannot always be interpreted validly" (Steedle & Shavelson, 2009, p. 713). See also my previous discussion of overlapping waves.

Third, Rasch/IRT models are based on item difficulty, which does not capture critical aspects of the nature of student reasoning, as Stacey and Steinle argue:

Being correct on an item for the wrong reason characterises DCT2 [their decimal knowledge assessment]. It is one of the reasons why the DCT2 data do not fit the Rasch model, because these items break with the normal assumption that correctness on an item indicates an advance in knowledge (or ability) that will not be 'lost' as the student further advances. ... A student's total score on this test might increase or decrease depending on the particular misconception and the mix of items in the test. This does not fit the property of Rasch scaling stated in Swaminathan (1999), that 'the number right score contains all the information regarding an examinee's proficiency level, that is, two

examinees who have the same number correct score have the same proficiency level' (p. 49). Neither the total score ... nor Rasch measurement estimates provides a felicitous summary of student performance on the decimal comparison items of the DCT2 test" (2006, pp. 87-88).

Indeed, Stacey and Steinle further state that, "Conceptual learning may not always be able to be measured on a scale, which is an essential feature of the Rasch approach. Instead, students move between categories of interpretations, which do not necessarily provide more correct answers even when they are based on an improved understanding of fundamental principles" (2006, p. 77). Even more, how to place rote performance on items becomes extremely problematic in such models. For instance, in Noelting's hierarchy for proportional reasoning, the highest level is the formal operational stage in which the "child learns to deal formally with fractions, ratios, and percentages" (Draney & Wilson, 123). But using a formal procedure rotely is not a valid indication of formal operational reasoning. Stacey and Steinle concluded that there is nothing to gain in using the Rasch approach to the case of decimals that they studied and many other contexts. "Learning as revealed by answers to test items is not always of the type that is best regarded as 'measurable', but instead learning may be better mapped across a landscape of conceptions and misconceptions" (2006, p. 89).

Methods for Determining Levels in Learning Progressions

The most accurate way to determine levels in learning progressions (once the framework has been developed) is administering individual interviews, which are then coded by experts, using the LP levels framework. The difficulty with this approach is that it is time consuming. However, many teachers can learn to make such determinations, both with individual interviews and during class discussions. Another way to gather such data is using open-ended questions. Again, students' written responses must be coded, and many students do not write enough for proper coding. However, if teachers help students learn how to accurately describe their reasoning in writing, written responses can be a valuable means for gathering strategy information.

An alternate, less time-consuming, way to gather data is through multiple choice items that have distracters that are generated from interviews and that correspond to specific levels (Briggs et al., 2006, have labeled format "Ordered Multiple-Choice"). CBA has also experimented with teacher coding sheets—students describe their reasoning but the teacher or a classroom volunteer chooses the options in a multiple-choice-like coding sheet. However, beyond convenience, there are several issues that one must consider when using these alternate formats (e.g., Alonzo & Steedle, 2008; Briggs & Alonzo, 2009; Steedle &

Shavelson, 2009)¹²? For instance, students may not recognize which multiple-choice description matches the strategy they used to solve the problem.

When assessments are used summatively, however, taking a numerical approach can be both practical and useful. However, if one stores the data as numerical levels codes, in order to use the data for individual diagnoses, teachers must consult the theoretical model on learning that underlies the levels framework.

In Summary

When using quantitative methods to develop levels in learning progressions, the validity and usefulness of interpretations of results depends on (a) the adequacy of the underlying conceptual model of learning, (b) the fit between the statistical/mathematical model (including its assumptions) and the conceptual model of learning, and (c) the fit between the data and the statistical/mathematical model's predictions. Unfortunately, use of quantitative methods often ignores factor (b). For example, adopting the Saltus model might cause one to neglect explicit consideration of the critical issue of what it means to be at a level. Also, although many users of quantitative approaches argue that implementing such approaches enables them to test their models, too often, these tests are restricted to factor (c). Researchers in mathematics education need to resist external pressures to apply quantitative techniques without deeply

¹² Also at issue is whether Rasch techniques are the appropriate model when Ordered Multiple-Choice format tasks are employed (Briggs & Alonzo, 2009).

questioning their validity, because such adoptions result in the techniques being applied in ways that we would call in other contexts instrumental or rote procedural. Instead, researchers must investigate much more carefully the conceptual foundations of these techniques (a daunting task, given the statistical/mathematical complexity underlying the procedures)¹³.

Learning Progressions and Curriculum/Assessment Standards

In the current era of "high standards," testing, and accountability, it seems reasonable to base both the content and grade-level locations of standards on research-based learning progressions. Indeed, the CCSSM state, "the development of these Standards began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (CCSSM, 2010, p. 4). However, there are aspects of the CCSSM, in particular for geometry, that seem to contradict this claim. As an example, consider the consistency of the CCSSM with the van Hiele levels. Although modern researchers have expressed several misgivings about the nature of the levels, recent reviews agree that "research generally supports that the van Hiele levels are useful in describing students' geometric concept development" (Clements, 2003, p. 153; Battista, 2007).

A major landmark in the van Hiele levels is when students develop

¹³ One way to investigate the conceptual foundations of the approaches is to apply both to the same sets of data.

property-based reasoning about geometric shapes. For instance, at van Hiele Level 2, a student conceptualizes a rectangle, not as a visual gestalt, but, say, as a figure that has the properties:¹⁴ "4 right angles," and "opposite sides parallel and equal." The CCSSM rightly recognize the critical importance of Level 2 reasoning. However, they specify that the development of this reasoning occurs at grades 4 and 5, which ignores van Hiele-based research that strongly suggests that, for most students, this reasoning is very difficult to achieve before ninth grade (Battista, manuscript in preparation). Indeed, the percent of students at or above Level 2, before and after high school geometry, has been reported as 31% before and 72% after by Usiskin (1982), and 51% before and 76% after by Frykholm (1994). Even after high quality instruction specifically targeting increasing students' van Hiele levels, research shows that the highest percent of students in grades 5-7 that achieved Level 2 reasoning or above was about 58%. So existing research casts serious doubt on the achievability of the CCSSM geometry standards *for most students*.

It should be noted, however, that this research often uses different kinds of level indicators. For instance, in the Usiskin assessment of van Hiele levels (which was also used by Frykholm), property assessment tasks involved diagonals of quadrilaterals, which may have been studied less as opposed to basic defining, and more familiar, properties of classes of quadrilaterals.

¹⁴ At Level 2, students do not understand minimal definitions. Instead, definitions tend to be lists of all the visually salient properties that students know (stated in terms of formal geometric concepts).

Battista

Furthermore, in Battista's study of fifth grade students working in his *Shape Makers* curriculum, if Level 2 was assessed by the triad tasks described above (which should be considered "transfer" tasks), 58% achieved Level 2 or higher on the posttest. But if Level 2 was assessed by students' knowledge of properties of shapes that had been explicitly explored in the curriculum, 83% were judged as achieving Level 2 or higher. However, Battista's research also suggests that, *in general*, junior high students' level of reasoning on these same familiar quadrilaterals is quite low (only 22% achieving Level 2).

This example illustrates several issues:

1. Standards too often are not sufficiently based on research. For example, given the research cited above, expecting ALL fourth or fifth graders to achieve Level 2 reasoning seems unreasonable.

2. Integrating various research studies into coherent learning progressions can be difficult because of variability in methods and assessments. For instance, assessments of van Hiele Level 2 have variably focused on knowledge of properties of familiar shapes, use of properties in transfer tasks, and knowledge of derived/secondary, as opposed to defining, properties (Battista, in preparation).

3. Although it is sometimes possible for students to make great progress in LP when using LP-based curricula and being taught by excellent teachers, this situation is not the norm. Basing standards on what happens in the best situations seems unwise.

4. For learning progressions to be useful in standards setting, the goals of the standards must closely match the knowledge acquisition described in the progressions. For instance, exactly which properties are targeted by CCSSM—familiar defining properties, or unfamiliar derived properties?

5. Should standards set benchmarks that all or most (say 80%) students can achieve, or should they target benchmarks that only, say, 50% (or 30%) of students might reasonably be expected to achieve? This is a critically important issue that may inadvertently place equity concerns in opposition to concerns about ensuring that sufficient numbers of students enter advanced mathematics and science careers in the US.

Teachers' Use of and Need for Learning Progressions

Professional recommendations and research advocate that mathematics teachers possess extensive knowledge of students' mathematical thinking (An, Kulm, Wu, 2004; Carpenter & Fennema, 1991; Clarke & Clarke, 2004; Fennema & Franke, 1992; Saxe et al., 2001; Schifter, 1998; Tirosh, 2000). Teachers must "have an understanding of the general stages that students pass through in acquiring the concepts and procedures in the domain, the processes that are used to solve different problems at each stage, and the nature of the knowledge that underlies these processes" (Carpenter & Fennema, 1991, p. 11). Research shows that such knowledge can improve students' learning (Fennema & Franke, 1992; Fennema et al., 1996). Indeed, "There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a

learning task, use this knowledge as a starting point for new instruction, and monitor students' changing conceptions as instruction proceeds" (Bransford et al., 1999, p. 11). Thus, there is a great need to study teachers' learning, understanding, and use of learning progressions in mathematics.

Related to the study of teachers' use of learning progressions, there is much research investigating the nature of the knowledge teachers have and need to teach mathematics, with the scope of this work described by the "egg" domain-map of Hill, Ball, and Schilling (2008, p. 377) (see Figure 16). Battista's Cognition Based Assessment, Phase 2 (CBA2) research project is focusing on one component in this domain, "Knowledge of Content and Students" (KCS), which Hill et al. define as, "Content knowledge intertwined with knowledge of how students think about, know, or learn that content" (p. 378).

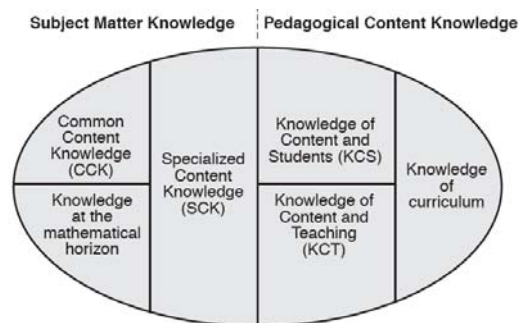


Figure 16

The Hill/Ball/Schilling framework puts mathematical knowledge at the forefront in describing mathematics-related teacher knowledge. Consistent with this content-primary perspective, Park and Oliver state, "it is transformation of subject matter knowledge for the purpose of teaching that is at the heart of the definition of PCK" (2008, p. 264).

In contrast, the CBA2 approach to studying KCS focuses on teachers' "cognitive/psychological knowledge" of students' mathematical thinking, and a major component of this research is connected to teachers' understanding and use of learning progressions. Although cognitive/psychological knowledge and mathematical knowledge are distinct, they are intertwined with each other and with knowledge of teaching and curricula. See Figure 17.

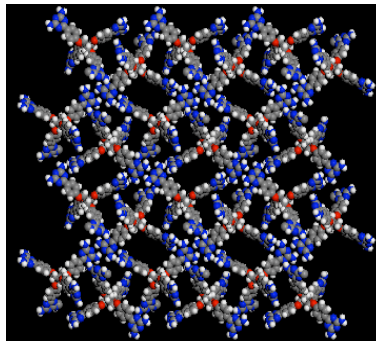


Figure 17. Intertwined Teacher Knowledge [Mathematics gray, Psychological white, Teaching blue, Curricula red]

In the CBA2 project, we are conducting case studies that qualitatively describe (a) the nature of teachers' conceptualizations of students' mathematical thinking, (b) the processes by which teachers come to understand research-based knowledge on the development of students' mathematical thinking (as represented in CBA LP), and (c) how teachers use this knowledge (including CBA assessments and instructional guidance) in assessment and teaching.

One Teacher's Use of CBA

Before describing several issues in teachers' understanding and use of learning progressions, it is worthwhile to note the power that many teachers obtain with CBA's linked LP, assessments, and instructional guidance. So I

quickly summarize a case study of one teacher in the CBA project who used several extremely detailed CBA learning progressions in his teaching and assessment. As Teacher 19 learned and used CBA ideas and materials, he made major progress in:

- understanding students' learning progressions
- understanding assessment tasks
- deciding what's most important in the curriculum
- diagnosing and remediating students' learning difficulties
- deciding on the effectiveness of instruction— are there problems in the teaching, or are students not quite ready to learn a particular concept
- improving informal assessments by helping to him ask better questions and more quickly understanding what students say
- understanding and building on students' reasoning and procedures as they occurred in frequent class discussion
- helping parents understand their children's mathematics program and progress through it.

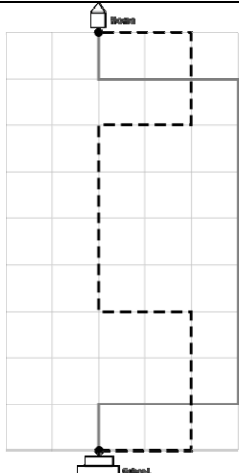
In much of T19's discussion of CBA, he described how important it was for him to be able to say to himself, "Well, they're here and this is where I need to take them," a major affordance of CBA LP. This is practical, decision-making information needed for everyday mathematics teaching. Finally, T19 was impressed by the great progress in learning his students made (especially those who were struggling), which he attributed to his learning about and use of CBA

materials. As an especially important example for him, he described how one of his struggling students started the school year at a kindergarten level in mathematics and by mid-year was functioning at a third grade level.

Teachers' Understanding of Students' Reasoning about Length: The Need for LP

To illustrate why research-based LP are so important for teachers, I describe one example of teachers' understanding and misunderstanding of students' reasoning about length measurement, a topic that almost all elementary students have difficulty with. Examination of this example illustrates the kind of content that is needed in LP written for teachers.

Teachers were shown the work of Student X and asked to analyze it (see Figure 18).

<p><i>Student Problem</i> <i>Which sidewalk from home to school is longer, the dotted one, the gray one, or are they the same?</i></p>	
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Teacher Task
 Consider Student X who used the strategy below on the Student Problem (above).

Student X: [Counts squares along the gray path 1-14, then along the dotted path 1-15.] The gray path is shorter because it has less squares.

(a) Is Student X's reasoning correct or incorrect? If it is incorrect, what is wrong with it?

(b) What would you do instructionally [to help Student X]?

Figure 18. Student problem and teacher task

To illustrate the difficulties that teachers had with analyzing Student X's reasoning, I describe two examples of how teachers conceptualized (a) X's reasoning, and (b) subsequent instruction for X.

Teacher1: [X's reasoning] is incorrect because ... she is counting the boxes instead of the side length for the unit. Like on this first box [in the gray path; see Figure 19] she is just counting it as one unit even though there are two sides there that should be measured.

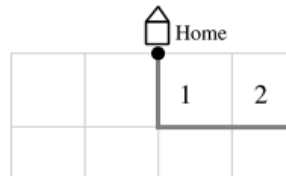


Figure 19.

Teacher3: [X's reasoning] is incorrect. She is not recognizing that she is counting two segments as one [pointing to the first turn in the gray path] because she is looking at area. So she is looking at the area of the squares, not counting the sides or segments.

Although both teachers understood that X's reasoning is incorrect, several features of the teachers' conceptualizations of X's reasoning are problematic. First, there was no evidence in any of X's work that she was mistaking area for length. Instead, X implemented the procedure of "placing" squares along a path, without properly relating this procedure to unit-length iteration. X did not understand the concept of unit-length iteration or the procedure for

implementing it. Conceptualizing X's error as looking at area mis-conceptualizes X's reasoning *psychologically*. One of the key features of LP is that they provide psychologically sound, and pedagogically useful, interpretations of students' reasoning.

The second important feature is the statement by both teachers that X is counting 2 length units instead of 1. Thus, both teachers misinterpret X's conceptualization and error. *X is iterating squares, not different-sized linear units*. Both teachers focus on the mathematical consequences of X's errant strategy, rather than its psychological root.

To further examine this misinterpretation and its consequences, we look at how the teachers' conceptualizations of X's reasoning affects their view of the instruction X needs.

Int: What would you do instructionally to move X to this next type of reasoning [correct iteration]?

T1: Well I think she needs to understand what the unit is, and that the units have to be ... consistent as she is measuring. So she would need to see that this unit that she labeled as one [draws Figure 20A] is more than this unit [draws Figure 20B].



Figure 20.

So like you could show her that this unit and this unit are not the same cause if you straighten it out this would be two units, and this would just be the one unit.

T3: We used inch rods cut out of straws ... and physically put those along [the paths] ... And that helped them to recognize that they weren't counting the sides when they were using squares. They were missing something.

T1 has a valid *long-term* instructional goal—X must learn to iterate a constant unit-length. However, because T1 misinterprets X's conceptualization, she chooses an inappropriate *short-term/immediate* instructional goal. Telling X that

Battista

she counted 2 units instead of 1 would confuse X. Students who are conceptualizing length measurement as iterating squares along a path must first see that a totally different kind of unit—linear—must be iterated. This is surprisingly difficult for many elementary school students. Understanding the properties of unit-length iteration—equal-length units, no gaps/overlaps—comes after understanding the nature of the iterated unit. LP provide not only long-term instructional goals but the kind short-term/immediate instructional goals that are critical for guiding and supporting students' moment-to-moment learning.

In summary, T1 understands X's reasoning *mathematically* but not *psychologically*. *It seems that focusing on the mathematical consequences of counting squares, while critical to determining the validity of X's reasoning, caused T1 to incorrectly conceptualize the nature of X's reasoning.* Consequently, although T1's instructional goal was worthwhile, her plan does not adequately build on X's current reasoning. Interestingly, although T3's conceptualization of X's reasoning was also problematic, probably because she had previously been interactively guided in the appropriate use of length activities by CBA staff, she was still able to appropriately choose which CBA instructional activity was appropriate for X. Nevertheless, to appropriately build on X's current reasoning, teachers must fully and psychologically understand the nature of her reasoning. It is insufficient to merely know that X is getting incorrect counts of units, or even where the incorrect counts occur. Teachers must understand that X is using

the wrong kind of measurement unit. And it is the CBA LP on length that provides the appropriate framework for this understanding.

To fully understand and respond to misconceptions like those of X and other students, teachers need research-based learning progressions that describe the range of conceptualizations that students possess about length and length measurement. Knowledge of such progressions not only helps teachers understand students' thinking psychologically, it expands a teacher's focus beyond mathematical, to pedagogically critical psychological, interpretations of students' mathematical thinking. And for LP to be maximally useful for teachers in instruction, LP must be linked to (a) appropriate assessment tasks that reveal students' reasoning, and (b) instructional tasks specifically designed to address students' learning needs at various locations in the LP.

Balance: How Much Detail Is Needed in LP for Teachers?

In discussing the use of learning progressions for formative assessment by teachers, Popham states, "It's important to stress that there must always be a balance between (1) the level of analytic sophistication that goes into a learning progression and (2) the likelihood of the learning progression being used by teachers and students" (2008, p. 29). So a central issue in describing learning progressions written for teachers is how much detail teachers can handle in the progression descriptions.

Although space does not permit me to provide a full analysis of this issue in the CBA2 project, it is true that almost all of the teachers who participated in the CBA2 project for at least a year did learn to use the great amount of detail in CBA LP. However, a comment made by many teachers who participated in the CBA2 project is that most/many teachers would have difficulty learning the great amount of detail in the CBA materials. Consequently, some of these teachers suggested giving teachers simplified versions of the CBA materials. The following episode illustrates that this approach, if it oversimplifies a LP, can lead to difficulties.

Misinterpretation of "Simplified" Level Descriptions

One idea that we experimented with in the CBA2 project is providing "simplified" descriptions of CBA levels to teachers. As an example, in the regular CBA materials, Level M1 for length was described and numerous examples of student work were provided. In contrast, some teachers were given the very abbreviated description of Level M1 below. Notice that in this abbreviated description, the terms "gaps" and "overlaps" were not elaborated or illustrated. It was assumed that teachers would understand the terms, given the context.

Abbreviated Version

"CBA Length Level M1. Incorrect Unit Iteration

Students do not fully understand the process of unit-length iteration; their iterations contain gaps, overlaps, or different length units, and are incorrect."

How several teachers misinterpreted the terms "gaps" and "overlaps" is revealing. T17¹⁵ made the following comment in deciding which CBA level Student X evidenced on the home-to-school problem.

T17: Well what do you mean gap? An opening that is not counted. ... [X] didn't count it [pointing at the "2" on the dotted path, See Figure 21a]. ... So it has to be a gap.

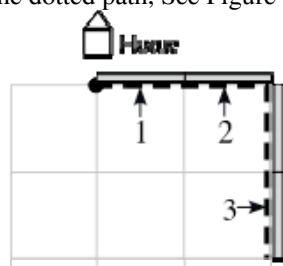


Figure 21a



Figure 21b




In this case, T17 interpreted the term "gap" as a mismatch in the correspondence between the number sequence "1, 2, 3" and the sequence of 4 unit-lengths that should have been iterated along the portion of the dotted path shown above. If X were counting unit lengths, she should have counted "1, 2, 3, 4" for this portion of the path. But she omitted the count for the third segment; so there was a "gap" in her counting sequence (see Figure 21b). T17's interpretation of *gap* was very different from the meaning of *gap* that the CBA author intended (see CBA document excerpt below). And, like T3 and T1 above, T17's interpretation of *gap* seemed to contribute to her mis-interpretation of X's conceptual difficulty.

"M1.2: Iterates Unit-Lengths but Gets Incorrect Count

Students iterate unit-lengths rather than shapes. So when iterating unit-lengths, they draw line segments, not squares, rectangles, or rods. However, because they do not understand the properties of

¹⁵ T1 and T3 had read the full CBA document on length; T17 had not read any CBA length material other than the abbreviated descriptions like that shown above.

unit-length iteration, their iterations contain gaps, overlaps, or different length units (see below)"
(Battista, in press).

<i>gaps</i>	
<i>overlaps</i>	
<i>different length units</i>	

The issue of determining how teachers can use learning progressions in their teaching and formative assessment, and how learning progressions should be described to facilitate this use, is central to supporting mathematics teaching that develops deep conceptual knowledge and problem-solving proficiency in students.

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Appendix 1: The van Hiele Levels

Below I describe the van Hiele levels in a way that is consistent with Clements' and Battista's (1992) analysis and synthesis of research on the levels. My recent elaborations and extensions of the levels are described in Battista (2007, 2009).

Level 0 Pre-recognition

At the pre-recognition level¹⁶, children perceive geometric shapes, but perhaps because of a deficiency in perceptual activity, may attend to only a subset of a shape's visual characteristics. They are unable to identify many common shapes. They may distinguish between figures that are curvilinear and those that are rectilinear but not among figures in the same class. That is, they may differentiate between a square and a circle, but not between a square and a triangle.

Level 1 Visual

Students identify and operate on geometric shapes according to their appearance. They recognize figures as visual gestalts. In identifying figures, they often use visual prototypes, saying that a given figure is a rectangle, for instance, because "it looks like a door." They do not, however, attend to geometric properties or traits that are characteristic of the class of figures represented. That is, although figures are determined by their properties, students at this level are not conscious of the properties. For example, they might distinguish one figure from another without being able to name a single property of either figure, or they might judge that two figures are congruent because they look the same; "There is no why, one just sees it" (van Hiele, 1986, p. 83). By the statement "This figure is a rhombus," the student means "This figure has the shape I have learned to call 'rhombus'" (van Hiele, 1986, p. 109).

Level 2 Descriptive/analytic

Students recognize and can characterize shapes by their *properties*. For instance, a student might think of a rhombus as a figure that has four equal sides; so the term "rhombus" refers to a collection of "properties that he has learned to call 'rhombus'" (van Hiele, 1986, p. 109). Students see figures as wholes, but now as collections of properties rather than as visual gestalts; the image begins to fall into the background. The objects about which students reason are classes of figures, thought about in terms of the sets of properties that the students associate with those figures. Students experientially discover that some combinations of properties signal a class of figures and some do not. Students at this level do not see relationships between classes of figures (e.g., a student might contend that figure is not a rectangle because it is a square).

¹⁶ Not described by van Hiele, but argued for by Clements and Battista (1992).

Level 3 Abstract/relational

Students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand and sometimes even provide logical arguments in the geometric domain. They can classify figures hierarchically (by ordering their properties) and give informal arguments to justify their classifications (e.g., a square is identified as a rhombus because it can be thought of as a "rhombus with some extra properties"). Thus, for instance, the "properties are ordered, and the person will know that the figure is a rhombus if it satisfies the definition of quadrangle with four equal sides" [van Hiele, 1986], p. 109).

As students discover properties of various shapes, they feel a need to organize the properties. One property can signal other properties, so definitions can be seen not merely as descriptions but as a way of logically organizing properties. It becomes clear why, for example, a square is a rectangle. The students still, however, do not grasp that logical deduction is the method for establishing geometric truths.

Level 4 Formal deduction

Students establish theorems within an axiomatic system. They recognize the difference among undefined terms, definitions, axioms, and theorems. They are capable of constructing original proofs. That is, they can produce a sequence of statements that logically justifies a conclusion as a consequence of the "givens."

Level 5 Rigor/metamathematical

Students reason formally about mathematical systems. They can analyze and compare axiom sets.

Battista

On the Idea of Learning Trajectories: Promises and Pitfalls¹

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Learning mathematics is a complex and multidimensional if not an inherently indeterminate process. A necessary goal of research on learning is to simplify this complexity without sacrificing the ability of research to inform teaching. This goal has been addressed in part by researchers focusing on how to represent research on learning for teachers and on how to support teachers to use and generate models of students' learning (e.g., Franke, Carpenter, Levi, & Fennema, et al., 2001; Hammer & Schifter, 2001; Simon & Tzur, 2004; Steffe, 2004). Recently, the idea of *learning trajectories* has gained attention as a way to focus research on learning in service of instruction and assessment. It is influencing curriculum standards, assessment design, and funding priorities. In this paper – which grew out of my response to Michael Battista's keynote address on learning trajectories at the last annual meeting of the North American chapter of Psychology in Mathematics Education (Battista, 2010) – I examine the idea of learning trajectories and speculate on its usefulness in mathematics education.

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The National Research Council (2007) described learning progressions as “successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic” (p. 214). The recently released *Common Core Standards in Mathematics* (CCSM) (2010), noted that the “development of these Standards began with research-based learning progressions detailing what is known today about how students’ mathematical knowledge, skill, and understanding develop over time” (p. 4). The idea of learning trajectories has a great deal of intuitive appeal and may offer a way to bring coherence to how we think about learning and the curriculum. As research on learning trajectories proliferates and is brought to bear on some of the most vexing problems in teaching and learning mathematics, however, it is worth considering what it foregrounds and what it may obscure.

In this paper, I briefly describe the origins of *learning trajectories* in mathematics education and then consider three points for us to keep in mind as we study learning and apply our findings to serve the purposes of understanding and addressing the problems of practice.

- 1) The idea that learning progresses is not especially new. What do we know about learning mathematics and how does it fit with the idea of a trajectory?
- 2) Learning trajectories focus on specific domains of conceptual development and may be limited in characterizing other valued aspects of the mathematics curriculum.

- 3) Learning in school is function of teaching. Too tight a focus on learning trajectories may lead us to oversimplify or ignore critical drivers of learning associated with teaching.

My goal in making these points is not to state the obvious but to foreground the question of what the idea of learning trajectories affords us education researchers and practitioners, and what it might obscure.

Origins of Learning Trajectories

The term *learning trajectory* appears to have been first used in mathematics education in Marty Simon's oft-cited 1995 paper, "Reconstructing Mathematics Pedagogy from a Constructivist Perspective." As I reread this paper, the most important things I noticed – besides the fact that that the actual words "learning trajectory" did not appear until 21 pages into the article – were that a) a learning trajectory did not exist for Simon in the absence of an agent and a purpose and b) it was introduced in the context of a theory of teaching. According to Simon, a *hypothetical learning trajectory* is a teaching construct – something a teacher conjectures as a way to make sense of where students are and where the teacher might take them. It is hypothetical because an "actual learning trajectory is not knowable in advance" (p. 135). Teachers are agents who hypothesize learning trajectories for the purposes of planning tasks that connect students' current thinking activity with possible future thinking activity. A teacher might ask, "What does this student understand? What could this student learn next and

how could they learn it?" and create a hypothetical learning trajectory as a way to prospectively grapple with these questions.

The idea of *learning progressions* appears to have emerged first in the context of science education and is now virtually synonymous with learning trajectory. In a special issue of the *Canadian Journal of Science, Mathematics, and Technology Education* devoted to the topic of "long-term studies" of learning in science education, Shapiro (2004) traced the notion of learning progression in part to Rosalind Driver in her 1989 article, "Students' Conceptions and the Learning of Science." In it, Driver drew attention to the increasing number of studies of the development of children's thinking in specific science domains that documented patterns in what she called *conceptual progressions* and sequences of conceptual progressions, which she termed *conceptual trajectories* (Shapiro, 2004, p. 3). In contrast to Simon, the focus in that special issue of CJSMT was on describing children's learning as it had actually occurred under a given set of conditions, rather than on a thought experiment about how it could occur. Neither of these senses of *learning trajectory* – as a teacher-conjectured possible progression or a researcher-documented progression of actual learners– predominates in current conceptions of the notion.

Since 2004, there has been a groundswell of research that explicitly identifies itself as concerned with learning trajectories or progressions, as reflected in conferences and special journal issues (Clements & Sarama, 2004; Duncan & Hmelo-Silver, 2009), reports (Catley, Lehrer, & Reiser, 2005; Cocoran, Mosher, &

Rogat, 2009; Daro, Mosher, & Cocoran, 2011), and books (Clements & Sarama, 2009). A report by the Center for Continuous Improvement in Instruction (Daro, et al., 2011) treats learning trajectories as interchangeable with learning progressions, reflecting the general trend.

Because the metaphor of trajectory implies a sequenced path, researchers who focus explicitly on learning trajectories have taken pains to draw attention to their multidimensional character. For example, Clements and Sarama (2004) defined learning trajectories as complex constructions that include “the simultaneous consideration of mathematics goals, models of children’s thinking, teachers’ and researchers’ models of children’s thinking, sequences of instructional tasks, and the interaction of these at a detailed level of analysis of processes” (p. 87). Confrey and colleagues (2009) defined them as “researcher-conjectured, empirically-supported description[s] of the ordered network of experiences a student encounters through instruction ... in order to move from informal ideas ... towards increasingly complex concepts over time” (p. 2).

Three Points to Keep in Mind

Learning Trajectories are Not Really New – So What does the Metaphor Buy Us?

The idea that students’ learning progresses in some way as a result of instruction is at the very heart of the enterprise of mathematics education. Researchers have been studying students’ mathematics thinking and what it could mean for that thinking to progress in identifiable ways since long before the term *learning trajectories* was introduced. Chains of inquiry focused on

children's mathematics learning – we could call these research trajectories – have stretched over decades. For example, Glenadine Gibbs's (1956) study of students' thinking about subtraction word problems helped to pave the way for later researchers such as Carpenter and Moser (1984) to create frameworks portraying the development of children's thinking about addition and subtraction, and for Carpenter, Fennema, and Peterson to study how teachers used this information about children's thinking to teach for understanding (Carpenter, et al., 1989; Carpenter, et al., 1999). Les Steffe and John Olive's recent (2010) book on *Children's Fractional Knowledge* detailing the evolution of children's conceptual schemes for operating on fractions synthesized two decades' worth of prior research, as did Karen Fuson's findings on the development children's multidigit operations (1992). None this work mentioned learning trajectories as such, but each focused on elucidating the development of children's understanding and identifying major conceptual advances.

Why then talk about learning trajectories now? The metaphor emphasizes the orderly development of children's thinking and draws our attention to learning targets and possible milestones along the way.

To what extent is this kind of assumption about learning warranted? That is, in what sense does children's mathematics learning follow predictable trajectories? Some domains appear to readily lend themselves to analysis in terms of a pathway, such as the development of young children's counting skills (Gelman & Gallistel, 1986). The progression of children's strategies for addition

and subtraction story problems from direct modeling, to counting, to the use of derived and recalled facts also has been well established (Carpenter et al., 1999; Carpenter, 1985; Fuson, 1992). Yet even given such a robust progression in a basic content domain, how and when – and sometimes whether – children come to understand and use these strategies depends on a variety of factors differing from classroom to classroom and from child to child. Trying to represent research on learning in terms of trajectories quickly gets complicated, even for as fundamental a concept as rational number (e.g., Figure 2 “Learning Trajectories Map for Rational Number Reasoning,” in Confrey, Maloney, Nguyen, et al., 2009) or measurement (Figure 1).

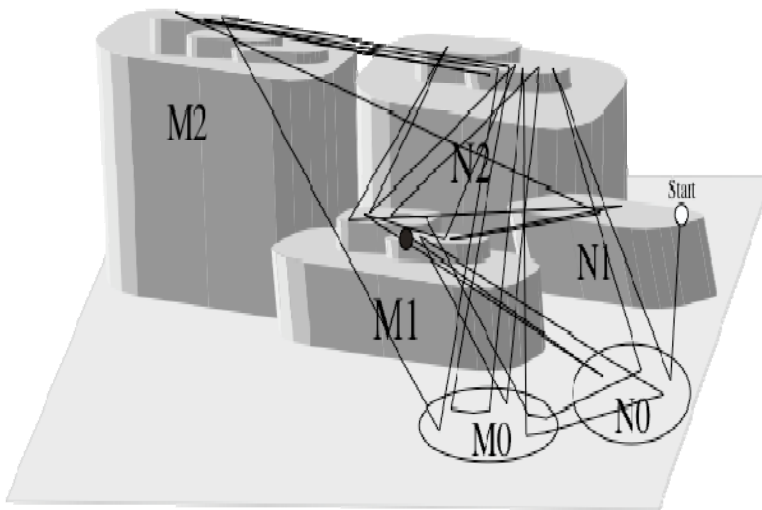


Figure 1. Battista’s (2010) representation of one student’s actual learning path in measurement

Other research suggests that the development of much of children’s thinking is more piece-meal and context-dependent than representations of learning trajectories might lead us believe (DiSessa, 2000; Greeno & MMAP, 1998). For

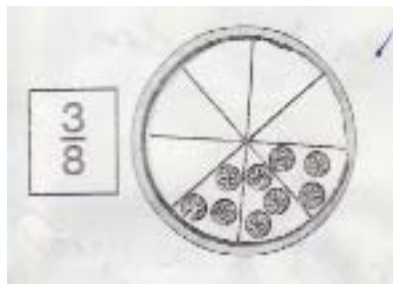
example, in a cross-sectional, cross-cultural study, Liu and Tang (2004) found differences in progressions of students' conceptions of energy in Canada and China over several years of schooling, which they attributed to differences in curriculum and instruction in each country. The topic of rational numbers in mathematics has an ample research base that illustrates, in some cases meticulously, how children's thinking about fractions could progress (Behr, Harel, Post, & Lesh, 1992; Davydov & Tsvetkovich, 1991; Empson & Levi, 2011; Hackenberg, 2010; Steffe & Olive, 2010; Streefland, 1991; Tzur, 1999). Taken collectively this research does not appear to converge on a single trajectory of learning.

Why might this be? In practice, learning cannot be separated from tasks and the instructional context; the "selection of learning tasks and the hypotheses about the process of student learning are interdependent" (Simon & Tzur, 2004, p. 93). What children learn is sensitive to the context in which they learn it - a context that is constituted by many factors, including most immediately the types of instructional tasks and how teachers organize students' engagement with these tasks.

For example, in classrooms where part-whole tasks (Fig. 2a) dominate instruction on fractions, children learn to think about fractions in terms of counting parts rather than as magnitudes (Thompson & Saldanha, 2003). Students are likely to think about $5/8$ as "5 out of 8" and of $8/5$ as an impossible fraction. In classrooms where teachers have students solve and discuss equal

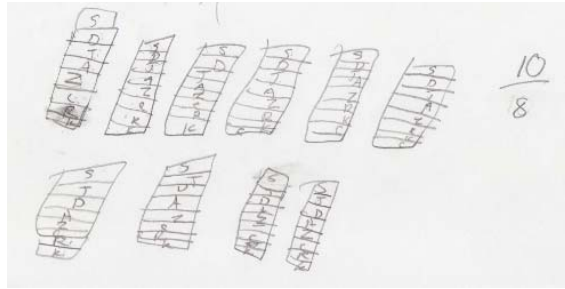
sharing tasks (Fig. 2b), children learn to think about fractions in terms of relationships between quantities and later in terms of a multiplicative relationship between numerator and denominator (Empson, Junk, Dominguez, & Turner, 2005; Empson & Levi, 2011). They are more likely to think of $5/8$, for example, as 5 groups each of size $1/8$, instead of “5 out of 8.” In classrooms where teachers engage students in reasoning about multiplicative comparisons of measures (Fig. 2c), students learn to think about fractions as a ratio of measures (Brousseau, Brousseau, & Warfield, 2004; Davydov & Tsvetkovich, 1991; Steffe & Olive, 2010). Children learn to interpret $5/8$ as a multiplicative comparison between 5 and 8. Both of these latter types of tasks – equal sharing and measuring – coupled with norms for engaging in tasks that put a premium on intellectual effort and agency (Hiebert & Grouws, 2007) – appear to constitute productive approaches to learning fractions.

How much pizza is left on the plate?



(a)

8 children want to share 10 candy bars so that each one gets the same amount. How many candy bars can each child have?



(b)

How long is B compared to A? A compared to B?



(c)

Figure 2. Examples of types of tasks to teach fractions: a) part-whole, b) equal sharing (with sample solution), and c) measurement

Within the context of documenting regularities and patterns in the development of children's thinking, however, it's important to recognize individual children's ways of reasoning and the significant contributions this reasoning could make to a group's learning. To return to my research on equal sharing, for example, we found that students frequently produced strategies for solving problems that were, from the perspective of a trajectory, "out of sequence" and presented rich learning opportunities for other students (e.g., Turner et al., in press). There was a progression in what students learned but "deviations" were consistent and numerous, and, I am suggesting, fruitful - not anomalies to be ignored but significant occurrences that teachers could use to advance everyone's learning.

Consider first a simple progression of strategies for equal sharing (Empson &

Levi, 2011). To figure out how much one person got if 8 people were sharing 6 burritos equally, a child using a basic strategy might draw all 6 burritos, decide to split each burrito into 8 pieces, and give each person 1 piece from each burrito for a total of 6 pieces. A more sophisticated strategy would involve imagining that each burrito could be split into 8 pieces and mentally combining those pieces to conclude that one person's share consisted of 6 groups of $1/8$ burrito or $6/8$ burrito. Ultimately, children come to the understanding that the problem can be represented by $6 \div 8$, which is the same as $6/8$.

Within this simplified progression, there are several other ways to solve the problem that do not fall into a sequence and do not appear as an inevitable consequence of development. These other strategies were a function of specific quantities in a problem as well as what tools children were using and children's prior knowledge. For example, a fifth grader solved the problem by reducing it to an equivalent ratio involving $1 \frac{1}{2}$ burritos and 2 children, which she easily solved by finding half of 1 and half of $1/2$ and combining the amounts (Fig. 3). Another fifth grader used a similar strategy, but used cubes to represent each quantity (8 total cubes for sharers, 6 total cubes for burritos), specifically highlighting the ratio character of the strategy. These strategies were appropriated by several children who saw them as more efficient and they provided an opportunity for the teachers to address concepts of fraction and ratio equivalence.

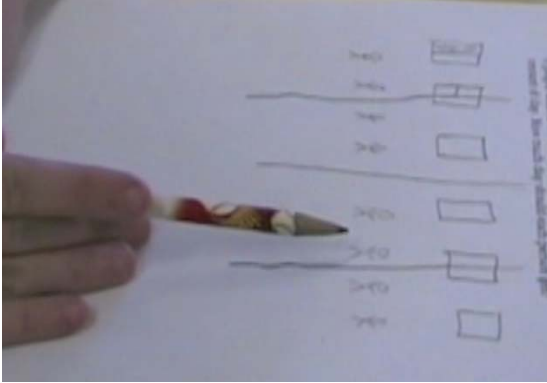


Figure 3. A solution for $6 \div 8$ involving the equivalent ratios 3 for 4 and $1 \frac{1}{2}$ for 2

As students' understanding develops and diversifies, they become more likely to see and make connections between their ways of thinking and different ways of thinking expressed by their fellow students. Making these connections enriches learners' understanding and cultivates their ability to recognize and pursue new avenues of reasoning independently of the teacher's direction and to monitor their thinking. The balance in instruction between supporting students' agentic initiative and aiming to instill specific conceptions can be difficult to manage. Indeed, some researchers have cautioned that representing learning as progressive sequences of content understanding could lead teachers to direct students through the sequences at the expense of allowing students to "express, test, and revise their own ways of thinking" (Lesh & Yoon, 2004, p. 206; Sikorski & Hammer, 2010). At the same time, other research suggests that, at the right level of abstraction, representations of the progressive development of students' understanding can enhance teachers' ability to respond to students' thinking in ways that open up or are generative of new possibilities (e.g., Franke et al. 2001).

In either case, it's important to recognize that research on learning in specific mathematics domains has a long history that, while concerned with progress, may not fit easily into the idea of a single trajectory.

Learning Trajectories Involve Specific Domains of Conceptual Development – So Their Reach May be Limited

Researchers have made the study of mathematics learning more tractable by focusing in particular on conceptual development in specific content domains, represented by sets of well-defined, interrelated tasks. Steffe and Olive's (2010) research on the development of fraction concepts and Clements and Sarama's (2009) research on children's understanding of measurement are examples of such an approach. This work, like a great deal of the research in mathematics education including my own, is informed by a Piagetian-like view of learning, if not in its emphasis on levels, then certainly in its emphasis on a conceptual trajectory, in which less sophisticated concepts give way to more sophisticated concepts. Because this work is based on children's thinking about specific types of tasks, its power lies in its capacity to inform teachers' use and interpretation of these tasks to foster students' conceptual development in a coherent unit of study (e.g., Fennema et al, 1996; Simon & Tzur, 2004).

However powerful, these kinds of portrayals of learning necessarily represent only one dimension or a small set of what we value as a field about mathematics and wish for students to learn. Learning is a multidimensional process, comprised of a variety of intertwined cognitive and social processes. In

particular, since the publication of *Everyone Counts* (National Research Council, 1989) and the *Curriculum and Evaluation Standards* (National Council of Teachers of Mathematics, 1988), mathematics educators have increasingly focused on teaching students to engage in practices such as posing and solving problems (Hiebert et al., 1996), constructing models (Lesh & Doerr, 2003), and making convincing arguments (Lehrer & Schauble, 2007) – that is, to do mathematics. Doing mathematics involves a complex and integrated set of content understanding *and* disciplinary practices (Bass, 2011; Kilpatrick, Swafford, & Findell, 2001) as well as the ability to monitor the interplay between these things (Schoenfeld, 1992).

The ability to engage in mathematical practices such as the ones above is as critical as content knowledge to a well-developed capacity to think mathematically, but it is less amenable to analysis in terms of sequences of development. For example, students engaged in mathematical modeling or problem solving may draw on multiple content domains and work collaboratively on tasks that have many possible resolutions such that the solutions they produce appear to follow no predictable trajectory over time. Examples of such tasks include creating simulations of disease spread (Stroup, Ares, & Hurford, 2005), optimizing the occupancy of a hotel during tourist season (Aliprantis & Carmona, 2003), and designing a template to generate a quilt pattern (Lesh & Doerr, 2003). These kinds of tasks and thinking practices pose considerable challenge for researchers seeking to codify and systematically

represent learning in terms of a trajectory, because of the variety of understanding and practices that students bring to bear in their solutions.

Learning trajectories may be limited in what they can and cannot specify in terms of learning mathematics over time; and in particular, they may not be applicable to certain critical aspects of the mathematics curriculum. Catley, Lehrer, and Reiser (2005) recognized this potential limitation when they argued that “scientific concepts are never developed without participation in specialized forms of practice” and “concepts are contingent on these practices” (p. 4) – such as the ones listed in the Common Core Standards in Mathematics (2010). Among others, these practices include making sense of problems and persevering in solving them; using appropriate tools strategically; attending to precision; and looking for and making use of structure (CCSM, 2010, pp. 6-8). Most, if not all, current characterizations of learning trajectories do not address the practices that engender the development of concepts – although it’s worth thinking about alternative ways to characterize curriculum standards and learning trajectories that draw teachers’ attention to specific aspects of students’ mathematical practices as well as the content that might be the aim of that practice.

What is a reasonable unit of students’ mathematical activity for teachers to notice? If a unit is too small or requires a great deal of inference (e.g., a mental operation), then teachers in their moment-to-moment decision-making may not be able to detect it and respond to it; likewise if a unit is too broad or stretches over too long a period of time (e.g., “critical thinking”), teachers may not

Empson

recognize it when they are seeing it. The most productive kinds of units of mathematical activity would allow teachers to see and respond to clearly defined instances of student's thinking during instruction and to gather information about students' progress relative to instructional goals. For example, in research in elementary mathematics, *strategies* and *types of reasoning* are productive units because we know that teachers can learn to differentiate students' strategies and use what they learn about students' thinking to successfully guide instruction (e.g., Fennema, et al., 1996). Catley and colleagues (2005) proposed "learning performances" as a way to represent the "cognitive processes and associated practices linked to particular standards" (p. 5). Formative assessments that include a variety of points of access and possible solutions and that require students to engage in various mathematics practices could also yield rich information about students' understanding of and engagement in mathematics (cf., Aliprantis & Carmona, 2003; Lesh & Doerr, 2003). The important thing is to take into account the interplay of practices and content in students' learning over time.

Teaching is Integral to Learning and Learning Trajectories

Learning school mathematics depends on teaching. To support learning, teachers need be able to "understand, plan, and react instructionally, on a moment-to-moment basis, to students' developing reasoning" and coordinate these interactions with learning goals (Battista, 2010). Similarly, Daro and colleagues (2011) concluded that:

Teachers are going to have to find ways to attend more closely and regularly to each of their students during instruction to determine where they are in their progress toward meeting the standards, and the kinds of problems they might be having along the way. Then teachers must use that information to decide what to do to help each student continue to progress, to provide students with feedback, and help them overcome their particular problems to get back on a path to success. (Daro et al., 2011, p. 15)

We know very little about how teachers do these things, in contrast to what we know about children's learning, whether it falls under the rubric of learning trajectory research or not. As teachers interact with students and decide how to proceed, there are many types of decisions to be made - how to gather information about children's thinking, how to respond to it appropriately in the moment, how to design tasks that extend it, and even what to pay attention to. With the right tools, teachers have access to the most up-to-date information about each student, what they understand and are able to do, their disposition, their history, and so on, and can make decisions based on their own informed understanding of these things and their relationships. Good tools, such as formative assessment frameworks in particular, enhance this knowledge and support teachers to engage in the active, contingent process of creating instructional trajectories informed by knowledge of actual children's learning.

Further, learning mathematics in school takes work and depends fundamentally on interpersonal relationships of trust and respect, which cannot be designed into a tool or a list of learning goals. Teaching is a relational act and the relationship between the teacher and the student is at the center of students' learning in school (Gergen, 2009; Grossman & McDonald, 2008). These

relationships can have a profound effect on what students learn and how they come to see themselves.

In the face of what can seem like a tidal wave of top-down mandates, I suggest that we mathematics educators keep sight of the fact that teaching is driven essentially by interpersonal relationships and happens from the bottom up, beginning with the teacher and the student relating to each other and the content. We need to be sure that teachers are equipped with knowledge of the domain and its learning milestones without forgetting that both teachers and students are active agents in learning.

Closing Thoughts

“Clearly ... the trajectories followed by those who learn will be extremely diverse and may not be predictable” (Lave & Wenger, 1991)

In choosing to focus on learning trajectories, we embrace a metaphor that, for all its appeal, implies that learning unfolds following a predictable, sequenced path. Everyone knows it is not that simple; researchers and educators alike acknowledge the complexity of learning. As Simon (1995) emphasized, learning trajectories are essentially provisional. We can think of them as the provisional creation of teachers who are deliberating about how to support students’ learning and we can think of them as the provisional creation of researchers attempting to understand students’ learning and to represent it in a way that is useful for teachers, curriculum designers, and test makers.

I firmly believe that a critical part of our mission as researchers is to produce something that is of use to the field and serves as a resource for teachers and curriculum designers to optimize student learning. No doubt this includes creating, testing, and refining empirically based representations of students' learning for teachers to use in professional decision-making and, further, investigating ways to support teachers' decision-making without stripping teachers of the agency needed to hypothesize learning trajectories for individual children as they teach. This focus would add a layer of complexity to our research on learning and invite us to think seriously about how to support teachers to incorporate knowledge of children's learning into their purposeful decision-making about instruction. Further, I suggest we consider, in the end, "Whose responsibility is it to construct learning trajectories?" (Steffe, 2004, p. 130). If we researchers can figure out how to supply teachers with knowledge frameworks and formative assessment tools to facilitate their work, teachers will be able to exercise this responsibility with increasing skill, professionalism, and effectiveness.

Because of the growing popularity of *learning trajectories* in education circles, it is worth thinking hard about the role of learning trajectory representations in teaching, and in particular, whether a learning trajectory can exist meaningfully apart from the relationship between a teacher and a student at a specific time and place. Simon's (1995) perspective on teaching and learning suggests not. As the field moves forward with research on learning trajectories and strive for

Empson

coherence in learning across the grades, I would like to remain mindful of both the affordances and constraints this particular type of representation offers for teachers and students alike.

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Can Dual Processing Theories of Thinking Inform Conceptual Learning in Mathematics?

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*Abstract: Concurring with Uri Leron's (2010) cross-disciplinary approach to two distinct modes of mathematical thinking, intuitive and analytic, I discuss his elaboration and adaptation to mathematics education of the cognitive psychology dual-processing theory (DPT) in terms of (a) the problem significance and (b) features of the theory he adapts. Then, I discuss DPT in light of a constructivist stance on the inseparability between thinking and learning. In particular, I propose a brain-based account of conceptual learning – the Reflection on Activity-Effect Relationship (Ref*AER) framework – as a plausible alternative to DPT. I discuss advantages of the Ref*AER framework over DPT for mathematics education.*

Key Words: constructivism, reflection, anticipation, activity-effect, dual processing, heuristic-and-bias, intuitive, analytic, brain.

This theoretical paper extends an article (Tzur, 2010b) in which I discussed Uri Leron's (2010) plenary address during the last annual meeting of PME-NA. Being invited to discuss his paper re-acquainted me with the inspiring empirical and theoretical work that he and his colleagues were conducting in the last two decades (Leron & Hazzan, 2006, 2009). It also provided me with an important window into literature outside mathematics education (e.g., cognitive

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Tzur

psychology), which I consider as both thought provoking and relevant to our field. Last but not least, after reading his paper(s) I realized how naturally his approach linked with recent efforts in which I have been participating – to relate mathematics education research with cognitive neuroscience (brain studies). I concur with Leron's belief that bridging between intuition and analytical thinking can contribute to optimizing student mathematical understandings and am delighted to provide my reflections on this endeavor.

In itself, the main thesis that human thinking and judgment (or rationality) consist of two qualitatively distinct modes is not new to mathematics education. Skemp's (1979) seminal work has already articulated and linked both modes, which he termed *intuitive* and *reflective* intelligences. To the best of my knowledge, Skemp's constructivist theory evolved independently of the commencement of the 'heuristic and bias' approach (Kahneman, Slovic, & Tversky, 1982; Kahneman & Tversky, 1973; Tversky & Kahneman, 1973, 1983). Moreover, I believe that, in mathematics education, this distinction can be traced back to Dewey's (1933) notion of *reflective* thought (contrasted with *unconscious* mental processes), and to Vygotsky's (1986) notion of ZPD and his related distinction between *spontaneous* and *scientific* concepts.

However, two novelties in Leron's contribution seemed very useful for mathematics education. First, his review of cognitive psychology literature pointed out to empirical studies in which a dual view of thinking processes has been robustly elaborated on (Evans, 2006; Kahneman & Frederick, 2002;

Stanovich, 2008) and ‘mapped’ onto corresponding, differentiated brain regions (Lieberman, 2003, 2008). Thus, a similarly important and timely direction, of linking mathematics education with brain studies (Medina, 2008), is supported by relevant findings from cognitive psychology (see Section 2). Second, he reported on studies (Leron & Hazzan, 2006, 2009) informed by DPT that demonstrated its applicability to our field, including articulation of instructional goals and design criteria. Next, I further discuss both contributions.

1. SIGNIFICANT QUESTIONS! USEFUL THEORY?

1.1 Significance of DPT

Like many teachers of mathematics and mathematics educators, Leron and his colleagues noticed a phenomenon that seemed to equally puzzle researchers in other fields. Quite often, researchers observing people’s solutions to various problems framed them as recurring faulty judgments (reasoning processes and conclusions). Examples of such solutions abound in the aforementioned papers; I will present three of my own below. Interestingly, studies of such examples in the ‘80s and ‘90s fueled a debate about human rationality that quite tightly conjoined epistemology and psychology (Goldman, 1994; Kim, 1994; Nisbett & Ross, 1994; Quine, 1994). For example, alluding to computational complexity, Cherniak (1994) considered ‘ideal’ (normative) rationality as intractable. Instead, using the example of mathematicians working on unfeasibly long proofs he proposed ‘minimal’ rationality, owing much of its

functionality to 'quick-and-dirty' heuristics that evade practical (mental) paralysis.

As I see it, addressing this puzzling phenomenon and significant problem by mathematics educators is more pressing and weighty than by cognitive psychologists and/or economists. As challenging as it might be to solidly explain why/how the human mind produces erroneous judgments, in those other fields it may suffice. The works of Leron (Leron, 2010; Leron & Hazzan, 2006, 2009) and others (Katz & Katz, 2010; Viholainen, 2008) indicate, however, that in our field such an explanation is but a start. In this sense, Leron made two key contributions: (a) *clarifying a goal* for student and teacher learning—closing the rather prevalent gap between intuitive and analytic reasoning, and (b) *explicating mathematics educators' duty* to figure out ways of thinking about, designing, and implementing teaching that can foster student development of and disposition toward analytic reasoning. To these ends, Leron identified four vital questions for mathematics educators:

- i) What differentiates among those who solve problems correctly and incorrectly, that is, why do the latter fail to use analytic reasoning whereas the former do so?
- ii) Using the above as a basis—how can we explain observations about the '*cueing impact*' of changes in a problem format or context have on correctly solving a problem, and what does this entail for instructional design?

- iii) When using puzzling problem situations in our teaching (e.g., earth circumference), what strategies can be used to effectively capitalize on students' "Aha" moments that follow those puzzlements?
- iv) How may we design instruction to promote (a) students' (and teachers') *awareness* of the potential use of improper intuitive reasoning and (b) disposition toward constant activation of analytic reasoning to override the faulty intuitions (i.e., resist and critique the intuitive)?

1.2 Dual Processing Theory (DPT): Is It Useful for Mathematics Education?

To articulate what purposes DPT can serve in mathematics education, I first briefly present its key features by alluding to one of Leron's examples and three of mine (to keep it short, language does not precisely replicate the original problems).

- A. Adults with college education were asked: Two items cost \$1.10; the difference in price is \$1. How much does each item cost? (Over 50% submit to impulse and respond: \$1 & \$0.10)
- B. In the elevator, the 7th floor button is already lit. A person who also wanted that floor gets on the elevator and, though seeing the lit button, pressed it again.
- C. Grade 3 students were asked to reason which side will a next (fair) coin flip show, 'Head or Tail', after it showed 4 'Heads' in a row. Roughly 50% said 'Head', because it's always been the case; the rest said 'Tail',

because it could not always be 'Head'. Virtually no one reasoned 50-50, and that previous flips were irrelevant.

D. As a Sudoku enthusiast, I made two careless errors while solving a 'black-belt' puzzle (see Figure 1). In the puzzle on the left (1a), I considered and *almost* wrote '4' in the bottom-middle square while transposing the digits to a different cell and ignoring the vertical 'conflict'. Two minutes later, while solving the puzzle on the right (1b), I actually committed a similar error (considering only vertical '9' and writing the small '9' digits where the top one conflicts with a horizontal, given '9').

6	5			9		8	3	
2 ⁷	9	1		8				
4	7					2		
			4		5		1	
				6				
7		4		9				
	3	7	4	4			←	
	4		7		8	1		
1	6			4		3	7	2

	9		8		5		
8	5	8	5	9	9	3	4
	2		5		8	9	
2	1		5			3	6
8	8		6	3			
6				1	5	7	8
→ 9	4	2		9		8	5
5	6	1					
9	6	8			5	4	

Figure 1a. Processing error not committed; almost placing '4' in mid-lower left cell (transpose row, ignore vertical)

Figure 1b. Same error repeated & committed; '9' in left-lower cell (checked for vertical only)

The key insight about human thinking, which led to different variants of DPT, is that responses to vastly diverse problems, faulty or correct, may all share

a common root. As implied by its name, the basic tenet of DPT is that two different modes of brain processing are at work (Evans, 2003, 2006; Stanovich, 2008; Stanovich & West, 2000). The first mode, '*intuitive reasoning*' (or '*heuristic*'), is considered evolutionary more ancient and shared with animals. It is characterized by automatic (reflexive, sub-conscious), rapid, and parallel in nature processing, with only its final product available to consciousness. The second mode, '*analytic reasoning*', is evolutionary recent and considered unique to humans. It is intentional (reflective, conscious), relatively slow, and sequential in nature. The principal roles attributed by DPT to the second mode are monitoring, critiquing, and correcting judgments produced by the first mode. Said differently, the second mode of processing suppresses/inhibits default responses; it serves as a failure-prevention-and-correction mental device. As Leron (2010) pointed out, some cognitive psychologists refer to the intuitive mode as System-1 (S1) and to the analytic mode as System-2 (S2). They further emphasize that, quite often, both systems work in tandem, which basically means that S1 produced a proper judgment that S2 did not need to correct.

A second tenet of DPT is that, in essence, faulty responses given by problem solvers reflect failure of their analytic processes to prevent-and-correct output from their intuitive processes. A key, corresponding assumption that seems to be taken-as-shared by most proponents of DPT and to underlie the notion of '*rational judgment/actions*', is that at any given problem situation a person intends to accomplish a correct solution that serves her or his own

purposes (e.g., economic benefit, academic success, etc.). In the four examples above, a person would like to properly solve the problems but, as DPT explains, the fast-reacting insuppressible S1 tends to “hijack” the subject’s attention and thus yields a non-normative answer (Leron, 2010). Thus, in Example A, S1 ‘falls prey’ to the cost of one item (\$1) being equal to the difference. In Example B, S1 brings forth and directs execution of the planned action (get on elevator, identify-and-press 7th floor button) before S2 could re-evaluate necessity in the circumstances. In Example D (Figure 1b), S1 directed my actions to place the digits with only partial checking before S2 detected that partiality. This occurred *soon after* I actually thought of placing the ‘4’ where it is shown in Figure 1a, but then consciously (S2 override) avoided this error. Example C (predicting results of a coin flip) was selected to highlight a few hurdles with DPT, particularly the impact of problem solvers’ cognitive abilities on their solutions (Stanovich & West, 2000). Clearly, what to an observer would appear as *non-normative* responses (e.g., it’s most likely to be ‘Head’) was the proper response within the children’s cognitive system—a case of S1 and S2 working in tandem for the reasoner, though erroneously for an observer.

Before turning to hindrances I find in DPT, a few more comments seem noteworthy. Evans (2006) highlighted a key distinction to keep in mind—between dual *processes* and dual *systems*. This is important for mathematics education particularly because, as he asserted, dual system theories are too broad. Thus, he asserted the need to elaborate specific dual-reasoning accounts at

an intermediate level that explains solutions to *particular tasks*. To me, his goal (particular task) seems primary whereas the means (dual accounts, or singular, or triple) seems secondary.

This leads to my second comment—the need to pay particular attention to solution processes—and kinds of problem situations—in which analytic/reflective processes successfully monitor and correct S1's 'run' *before* reaching and submitting to the latter's judgment. For example, when I first read Example A in Leron's paper, I immediately identified the task as 'inviting' the faulty conclusion. I also immediately noticed my conscious, pro-active 'flagging' of this tendency and, consequently, selected an analytic process instead. This mental adjustment happened before I even calculated the faulty difference (90 cents), precisely the desired state of affairs indicated in question #iv above. My case indicates the need for precisely analyzing the way intuitive and analytic processes *interact*. Initial forms of DPT assumed sequential operation, where outcomes of intuitive processes (or S1) serve as input for analytic processes only when/if S2 identified S1's output as a faulty response. Recently, the possibility for parallel processing of both modes was postulated, including the idea that they often compete for the immediate or final judgment in a given problem situation (Evans, 2006). To further theorize such interaction, Evans suggested 3 principles: (a) *singularity*—epistemic mental models are generated and judged one-at-a-time, (b) *relevance*—intuitive (heuristic) processes contextualize problems to maximize relevance to the person's current goals, and (c)

Tzur

satisficing—analytic processes tend to accept intuitive judgments unless there is a good reason to reject and override them. While essential, it seems that these principles fall short of accounting for how I solved Example A.

My last comment refers to factors that were found to make a difference in ways groups of people, or even an individual, solve particular problems. Stanovich and colleagues (2008; Stanovich & West, 2000) provided a good review of those. Here, I refer to a critical factor for mathematics education that was highlighted in Leron's (2010) address, namely, the impact of problem format ('packaging') on suppression of intuitive judgments. A substantial portion of Leron's work, which I see as a major contribution to our field, focused on the design of bridging tasks that are more likely to trigger what he considered solvers' available analytic processes. These tasks, in turn, enabled student solutions of the mathematically congruent tasks that were difficult to unpack without such bridging. This indirect allusion to assimilatory conceptions of those for whom bridging is required points to a hindrance.

From a constructivist perspective, a major theoretical and practical hindrance I find in DPT is the unproblematic application of an observer's frame of reference—considered as 'normative'—to the evaluation of people's responses—considered as 'rational' (or not, or partial). In essence, if the 'same' task is solved differently by people of different cognitive abilities (the observing researchers included), and if many who failed on a structurally identical task can solve a bridging task (and later also the failed one), then what a problem solver

brings to the task must be explicitly distinguished from the observer's cognitive toolbox. Simply put, the presence of two cognitive frames of reference is glossed over by DPT's equating of normative with rational (for more about this, see Nisbett & Ross, 1994).

Theoretically, and crucial for mathematics education, what this lack of distinction fails to acknowledge is both the different interpretation(s) of a task and different mental activities available to the observed person for solving it. That is, it fails to acknowledge the core construct of *assimilation* (Piaget, 1980, 1985; von Glasersfeld, 1995). Recent research in cognitive psychology did point out to possible differences between observer and observed interpretations (Stanovich & West, 2000), but the key theoretical implication of those findings—simultaneously addressing two frames of reference—did not seem to follow. In my view, distinguishing the observer (Roth & Bautista, 2011; Steffe, 1995; von Glasersfeld, 1991) and using assimilation as a starting point is necessary in our field in order to move beyond cognitive psychology's focus on thinking and reasoning into accounts of learning as a conceptual advance that can be observed, and fostered, in other people's minds. And, as Skemp (1979) so eloquently asserted, for a mathematics education theory of teaching to be useful—at its core one must articulate learning as a process of cognitive change in what the learner already knows.

Practically, overlooking learners' available conceptions when analyzing their solutions, correct or faulty, precludes the powerful design of bridging tasks

demonstrated in Leron's (2010) paper. Indirectly, both the specific features of those tasks (e.g., the need to cue for a nested sub-set, or steps to 'see' the invariant length of string-around-earth when different shapes increase) and the rationale and criteria he provided for introducing those features (e.g., make the problem *accessible* to the *solver's intuition*), draw on conjectured inferences about how a person may interpret and solve the alternative tasks. That is, such tasks require inferences into students' existing (assimilatory) conceptions. This leads to the discussion of DPT's core hindrance.

2. A CONSTRUCTIVIST LENS ON DPT: 'BRAINY' MATHEMATICS EDUCATION

2.1 Taking Issue with DPT

As a constructivist, I adhere to the core premise common to Piaget's (1970, 1971, 1985), Dewey's (Dewey, 1902; 1949), and Vygotsky's (1978, 1986) grand theories, that knowing (thinking, reasoning) cannot be understood apart from the 'historical process' in which one's knowing evolved. This premise entails my twofold thesis about hurdles in adopting and adapting DPT to mathematics education. First, a sole focus on normative and faulty modes of thinking/reasoning in mathematics or other domains (aka cognitive psychology), falls short of the theoretical accounts needed to intentionally foster optimal student (and teacher) understandings. Second, although DPT can inform our work, mathematics education already has frameworks that interweave

articulated accounts of knowing, coming to know (learning), and teaching (Dreyfus, 2002; Dubinsky & Lewin, 1986; Hershkowitz, Schwarz, & Dreyfus, 2001; Pirie & Kieren, 1992, 1994; Sfard, 1991, 2000; Steffe, 1990, 2010; Tall & Vinner, 1981; Thompson, 2002, 2010; Thompson, Carlson, & Silverman, 2007). As I shall discuss below, one framework that my colleagues and I have been developing—reflection on activity-effect relationship (*Ref^{*}AER*)—seems to (a) singularly resolve issues of faulty/normative *reasoning* and of *conceptual learning* (with or without teaching) and (b) explain different modes of thinking without alluding to 2 systems (or distinct processes). Moreover, the *Ref^{*}AER* framework is supported by and gives support to cognitive neuroscience models of the brain. Due to space limitations, the brief exposition below makes wide use of references to comprehensive versions. I begin by listing seven critical questions for mathematics education that Leron's work and accounts of DPT raised, and a framework such as *Ref^{*}AER* needs to address:

1. Why does the mental system of some people make an error (e.g., selects \$1 and 10 cents in the price example A) whereas other people focus also on the difference? Unless one considers solvers' assimilatory conceptions, this question (and 2-4 below) cannot be resolved by DPT assumptions that S2 has no direct access to the perceived information or that S2 selects accessible instead of relevant information.
2. When a person's response is non-normative, is it a case of (a) having the required conceptions but failing to trigger them (e.g., Sudoku and

elevator examples), (b) having a rudimentary form of those conceptions that require explicit prompting (e.g., sub-set in Leron's (2010) bridging task; renegotiating the difference aspect in the price problem and/or making the numbers more 'difficult'), or (c) not having a conception for monitoring S1 (e.g., my next coin-flip example and the original medical base-rate example in Leron's paper)? And how can we distinguish among these three cases?

3. How does S2, which failed to monitor S1 in a specific task, become capable of doing so? Is the process of learning different for each of the three cases above?
4. How do new monitoring capacities learned by S2 'migrate' to S1 (become automatic)?
5. What is the source of learners' surprise (e.g., string-around-earth example), how may it be linked to learning, and how might teaching capitalize on this?
6. What role do specific examples play in learning (by S2 and/or S1)?
7. Can we explain why particular bridging tasks promote some learning in some students but not others, and provide explicit ideas for changing them in the latter case?

2.2 A Brain-Based Model of Knowing and Learning

In recent years, a few cross-disciplinary meetings among cognitive neuroscientists and mathematics educators took place. One of those (Vanderbilt, 2006) focused on the design of tasks that (a) reveal difficult milestones in mathematics and (b) can be examined at the brain level (e.g., fMRI). Using the *Ref*AER* framework of knowing and learning (Simon & Tzur, 2004; Simon, Tzur, Heinz, & Kinzel, 2004; Tzur, 2007; Tzur & Simon, 2004; Tzur, Xin, Si, Woodward, & Jin, 2009), I presented fractional tasks to the group. This presentation, and the fertile dialogue with brain researchers that ensued, led to an elaborated, brain-based *Ref*AER* account (Tzur, accepted for publication) that seems highly consistent with DPT studies of the brain (Lieberman, 2003, 2008).

Briefly, *Ref*AER* depicts knowing (having a conception) as *anticipating* and *justifying* an *invariant relationship* between a single (goal-directed) activity-sequence the mental system executes at any given moment (Evans' *Singularity* principle; see also Medina, 2008), potentially or actually, and the effect it must bring forth. Learning is explained as transformation in such anticipation via two basic types of reflection. Reflection *Type-I* consists of ongoing, automatic comparison the mental system executes continually *between the goal* it sets for the activity-sequence and *subsequent effects* produced and noticed. As Piaget (1985) asserted, the internal global goal (anticipated effect) serves as a regulator of the execution for both interim effects and the final one (Evans' *Relevance* principle) (see also Stich, 1994). The effects either match the anticipation or not (Evans' *Satisficing* principle). By default, the mental system runs an activity-sequence to

Tzur

its completion as determined by the goal (e.g., the elevator example). Yet, the execution may stop earlier if (a) the goal detects unanticipated sub-effects (e.g., Sudoku example in Figure 1a) or (b) a different goal became the regulator, including possibly a sub-goal within the activity-sequence overriding the global goal. Reflection *Type-II* consists of comparison *across (mental) records of experiences*, each containing a linked, re-presented bit of a 'run' of the activity and its effect (*AER*), sorted as match or no-match. Critically, Type-II reflection does not happen automatically—the brain may or may not execute it. The recurring, invariant *AER* across those experiences are linked with the situation(s) in which they were found anticipatory of the proper goal and registered as a new conception.

Accordingly, *Ref*AER* postulates that the construction of a new conception proceeds through two stages. The first, *participatory*, necessitates reflection Type-I and is marked by an anticipation that a problem solver can access *only when and if somehow prompted* for the novel, provisional *AER* (Tzur & Lambert, in press, linked this stage with the Zone of Proximal Development—ZPD). The second, *anticipatory* stage necessitates reflection Type-II and is marked by independent, spontaneous bringing forth, running, and possibly justifying the novel anticipation. It should be noted that although developed independently, *Ref*AER* is consistent with Skemp's (1979) theory; the reflection types and stage distinctions extend his work.

To link the *Ref*AER* framework with brain studies, I separated and ‘distributed’ von Glasersfeld’s (1995) tripartite notion of scheme–situation, activity, and result—across three major neuronal systems in which they are postulated to be processed. The assumption regarding both knowing and learning is that the fundamental unit of analysis in the brain is not a single synaptic connection or a neuron (Hebb, 1949, cited in Baars & Gage, 2007; Crick & Koch, 2003; Fuster, 1997, 2003). To stress neuronal ‘firing’ in the brain and the life-long growth, change, and decay of neuronal networks (Medina, 2008), I use the term Synapse Inhibition/Excitation Constellation (*SIEC*)—any-size aggregate of synapses of connected neurons that, once ‘firing’ and updating, forms a stable pattern of activity (Baars, 2007b). The roles and functions of *SIECs* are described in terms of the three neuronal networks where they may be activated (Baars, 2007a): a ‘Recognition System’ (RecSys), which includes the sensory input/buffer and various long-term memories; a ‘Strategic System’ (StrSys), which includes the Central Executive; and an ‘Engagement-Emotive System’ (EngSys). Within these networks, solving a problem, as well as learning through problem solving, is postulated as follows (indices in the diagram correspond to those in the text below):

1. Solving a problem begins with assimilating it via one’s sensory modalities into the *Situation* part of an extant scheme in the RecSys. This *SIEC* is firing and updating until reaching its activity pattern

(recognizing state), and activates firing and updating of a Goal *SIEC* in the StrSys.

2. A Goal *SIEC* is set in the StrSys as a desired inhibition-excitation state that *regulates* the *execution* and *termination* of an activity sequence. The goal *SIEC* also triggers:
 - a. Corresponding *SIECs* in the EngSys that set the desirability of the experience and the sense of control the learner has over the activity (McGaugh, 2002; Medina, 2008; Tzur, 1996; Zull, 2002). These were found linked to activity in the anterior cingulate cortex (Bush, Luu, & Posner, 2000; Lieberman, 2003, 2008).
 - b. A temporary auxiliary *SIEC* checks if an activity has already been partly executed and can thus be resumed. If its output is 'Yes', it re-triggers the *AER's* execution in the StrSys from the stopping point (go to #4); if 'No,' it triggers the Goal *SIEC* to trigger #3 below.

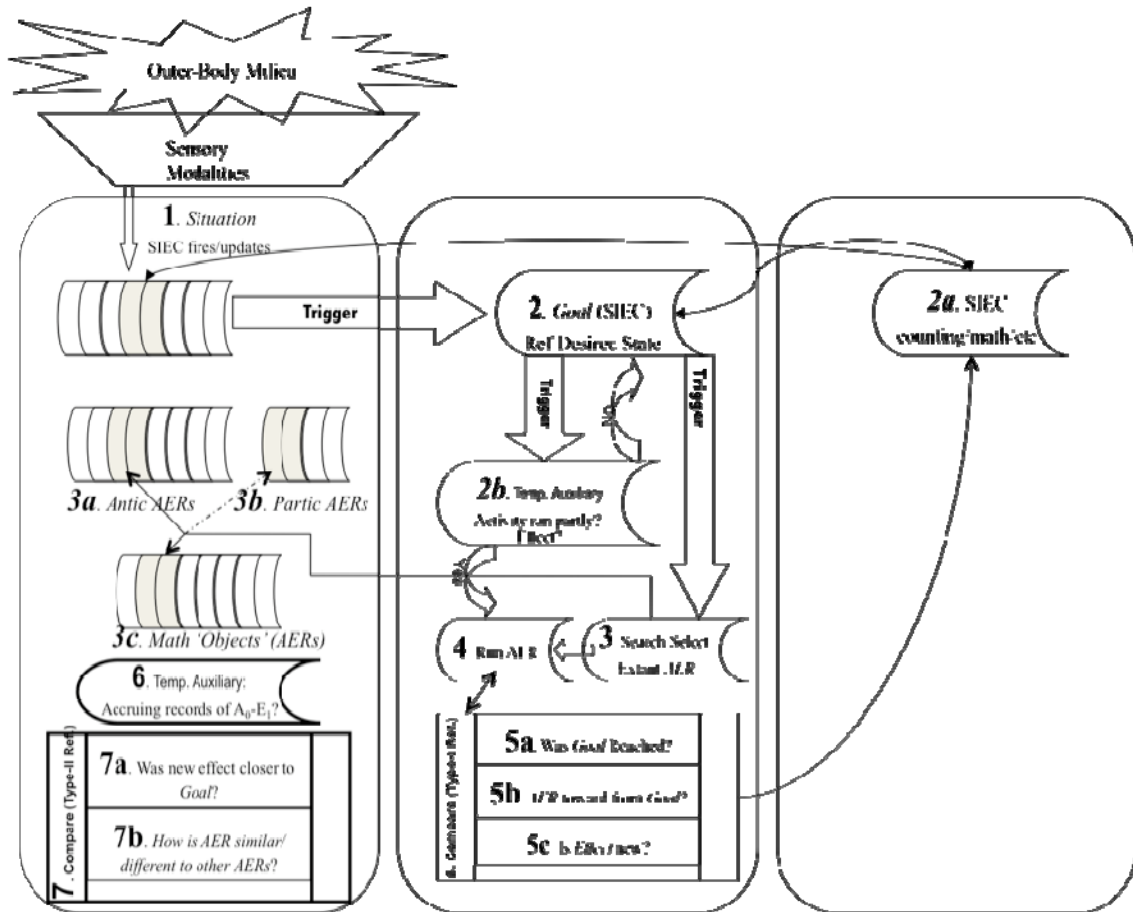


Figure 2. Brain problem solving and learning processes

3. A SIEC responsible for searching-and-selecting an available AER is triggered by the Goal SIEC. The search operates on three different long-term memory 'storages' of SIECs (3a, 3b, 3c below). Using a metaphor of 'road-map', Skemp (1979) explained that, within every universe of discourse (e.g., math, economy), the 'path' from a present state to a goal state may consist of multiple activity-sequences, among which one that is eventually executed is selected (see also multiple-

trace theory in Nadel, Samsonovich, Ryan, & Moscovitch, 2000).

Searched and selected *AERs* include:

- a. Anticipatory *AERs* – a mental operation carried out and its anticipated effect;
 - b. Participatory *AERs* that the learner is currently forming and can thus be called up only if prompted, as indicated by the dotted arrow;
 - c. Mental (e.g., mathematical) ‘objects,’ which are essentially anticipatory *AERs* established and encapsulated previously (e.g., ‘number’ is the anticipated effect of a counting operation).
4. Once an operation and an ‘object’ *AERs* were selected, the brain executes them while monitoring progress to the goal via a meta-cognitive *SIEC* in the StrSys responsible for Type-I reflections. Skemp’s (1979, see ch. 11) model articulates this component in great details, including how it can be carried out automatically (intuitive) and/or reflectively (analytic). This goal-based monitoring component seems compatible with Norman and Shallice’s (2000) model of schema activation, Corbetta and Sulman’s (2002) notion of ‘circuit breaker’, and Kalbfleisch, Van Meter, and Zeffiro’s (2006) identification of brain internal evaluation of response correctness. Mathematical operations are mainly activated in the Intraparietal Sulcus (IPS, see Nieder, 2005).

5. The execution of the selected *AER* is constantly monitored by Type-I reflection to determine 3 features:
 - a. Was the learner's goal, as set in *SIEC* 2a, met?
 - b. Is the *AER* execution moving toward or away from the goal (see McGovern, 2007 for relevant emotions)?
 - c. Is the final effect of the executed portion of the *AER* different from the anticipated, set goal? Goldberg and Bougakov (2007) suggested that this is a function of prefrontal cortex (PFC).

Each feature (5a, 5b, 5c) can stop the currently executed *AER* (e.g., seeing the lit elevator button halts the process leading to pressing it again). If the output of 5c is 'No', that 'run' of the *AER* is registered as another record of experience of the existing scheme (see Zull, 2002). Symbolically, such no-novelty can be written: Situation₀-Goal₀-*AER*₀ (Tzur & Simon, 2004). If the output is 'Yes', symbolized as Situation₀-Goal₀-*AER*₁, a new conceptualization may commence (see next). This perturbing state of the mental system (von Glasersfeld, 1995), seems related to anticorrelations of brain networks (Fox, et al., 2005).

6. Type-II reflective comparisons *may* then operate on the output records of Type-I reflection. Whenever the output of Type-I question 5c is 'Yes,' the brain *updates a new SIEC* for that recently run *AER* and stores it in a temporary auxiliary in the *RecSys* (symbolized A₀-E₁, or *AER*₁).

Each repetition of the solution process for which the output of 5c is 'Yes' adds another such record to the temporary auxiliary.

7. The accruing records of temporary AER_1 (novel) compounds are continually monitored by Type-II reflective comparison $SIEC$ in terms of two features:
 - a. Is the effect of the new AER (E_1) closer to or further away from the Goal?
 - b. How is the new AER_1 similar to or different from the extant anticipatory and/or participatory $AERs$ in the *RecSys*? This aspect of Type-II reflection seems supported by Moscovitch et al.'s (2007) articulation of the constant interchanges between MTL and PFC.

The output of recurring Type-II reflective comparisons is a new $SIEC$ (AER_1). The anticipatory-participatory stage distinction implies that a new $SIEC$ can initially be accessed by the Search-an-Select $SIEC$ (#3) only if the learner is prompted for the activity (A_0), which generates the noticed effect (E_1) and thus 'opens' the neuronal path to using AER_1 in response to the triggering situation ($Situation_0$). Over time, Type-II comparisons of the repeated use of AER_1 for $Situation_0$ produces a new neuronal pathway from the $Situation_0$ $SIEC$ to the newly formed AER_1 , that is, to the construction of a new, directly retrievable, anticipatory $SIEC$ (scheme symbolized as $Situation_1$ - $Goal_1$ - AER_1). This construction of an anticipatory AER seems to explain how repeatedly correct analytic judgments may become intuitive (automatic).

3. DISCUSSION: BRAIN-BASED REF*AER VS. DPT

I contend that *Ref*AER*, with its brain-based elaboration, simultaneously resolves not only the reasoning puzzlement addressed by DPT, but also central problems of mathematics learning and teaching. Concerning what an observer considers *normative* solutions, *Ref*AER* explains and predicts their production as the outcome of either an anticipatory conception, which can run automatically and/or reflectively, or a compatible participatory conception that was made accessible by a prompt—self/internal (e.g., Sudoku-1a) or external (e.g., Leron's bridging task, apple falling on Newton's head). Accordingly, *faulty* solutions may be the outcome of (a) partial, inefficient, and/or flawed execution of a suitable anticipatory conception (e.g., Sudoku-1, elevator), (b) prompt-dependent inability to access a suitable participatory conception (e.g., solving the \$1.10 incorrectly when difference=\$1 and correctly with other amounts), and, quite often, (c) lack of a suitable conception for correctly solving the given problem (e.g., 3rd graders facing the next coin-flip problem, Leron's students who could not solve the bridging task).

I further contend that, for mathematics education purposes, and possibly also cognitive psychology, *Ref*AER* resolves DPT problems better. Instead of postulating two systems (or processes), it explains how the brain gives rise to a multi-part *single thought process* by which a problem solver may reach a normative or a faulty *answer*. Furthermore, it stresses that a '*solution*' must encompass not only the answer, but also the crucial (inferred) *solver's* reasoning

processes used for producing it. A good demonstration of such analysis, and the vitality of intuitive solutions (e.g., for finding limits of sequences), were provided by Hersh (2011). *Ref*AER* accomplishes such inferences via analyzing the solver's: (i) goal and sub-goals (see Stanovich & West, 2000, for differing researcher/subject goals), (ii) entire or partial activity-sequence selected and executed (see Kahneman & Frederick, 2002, for the notion of Attribute Substitution), (iii) suitability of objects operated on (see Leron's, 2010, specific explication of objects, such as length *gap* in the string-around-earth task and the *nested sub-set* in his RMP task), (iv) sub- and final effects noticed, and (v) successful/failed reflections (both types).

Most importantly, *Ref*AER* analyses are rooted in an explicit distinction between two frames of reference operating in the evaluation of solvers' judgments—the observer's advanced, well-justified frame and the observed's evolving and sensible frame in terms of his or her extant conceptions (Roth & Bautista, 2011; Steffe, 1995). Thus, consistent with Stich's (1994) assertion that cognitive systems serve one's goals and not absolute truths, *Ref*AER* evades the pitfalls of equating normative with rational. Instead, it clarifies that upon a solver's assimilation of a problem situation and setting her/his goal(s), one path among multiple extant activity-sequences (spontaneously known or prompted) is selected, executed and being monitored by the goal. By default, the brain runs the sequence to its completion, which is signaled via Type-I comparison (goal *SIEC*), and can thus be portrayed by an observer as intuitive/automatic.

However, at any given moment during the activity-sequence execution or after its completion, the system's regulator (goal *SIEC*) may notice effects that require interruption and/or correction to the run and/or even to the goal itself (portrayed as analytic/reflective). In paraphrasing Gigerenzer's (2005) "I think, therefore I err", we shall say: "I learn to think, therefore I may adjust (initially) erroneous anticipations."

Consequently, *Ref*AER* seems to provide a basis for resolving two problems that, while not addressed by DPT, are vital for mathematics education, namely, explaining (a) how *learning* to reason—both intuitively and analytically—may occur and (b) how can *teaching* capitalize on it and foster (optimize) students' mathematical progress. The former has been articulated above in a way that seems to address each of the 7 questions presented in Section 2. The latter (implications for teaching) exceeds the scope of this paper; it was articulated elsewhere (Tzur, 2008a, 2008b, 2010a) as a 7-step cycle that proceeds from analysis of students' extant conceptions. To briefly convey the potential of this *Ref*AER*-based 7-step cycle, I return to Leron's example of a bridging (RMP) task.

In designing that task, Leron made explicit the two-phase activity-sequence of considering base-rate ($1/1000$) and diagnostic information (5% false positive) as necessarily linked sub-goals. What's more, the 'objects' on which his alternative sequence would operate were replaced, from multiplicatively related quantities (fractions, percents) to frequencies of whole numbers considered

Tzur

additively up to the final multiplicative calculation. In terms of *Ref*^AAER*, these alterations explain why some of the students who incorrectly solved the DMP problem could correctly solve the RMP problem. The alteration was more likely to orient solvers to (a) explicitly coordinated sub-goals (*specifying* each of the nested sub-sets) of the task's global goal and (b) selection of and operation on *accessible quantities* – anticipatory *AER* ('objects') – in place of quantities that are notoriously prompt-dependent (or lacking) in youngsters and adults and thus, not surprisingly, '*neglected*'. Accordingly, these insightfully designed task alterations explain the educative power of a bridging task. It seemed to bring forth an anticipatory *AER* that, I conjecture, could have served Leron's students as an internal prompt for correctly selecting-and-executing the entire activity-sequence for operating similarly on the more difficult-to-grasp multiplicative quantities and relationships.

Leron's design of bridging task not only fits well within the *Ref*^AAER*-based, 7-step teaching cycle, but also with a teaching practice we recently found in China (Gu, Huang, & Marton, 2006; Jin & Tzur, 2011). Our study was based on Xianyan Jin's dissertation, which provided a penetrating inspection of how bridging ('*xianjie*') tasks are consistently fitted within a 4-component lesson structure in Chinese mathematics teaching. She further 'mapped' the 7-step cycle onto the Chinese lesson structure, while highlighting the role that bridging tasks, like those designed by Leron et al. (Leron, 2010; Leron & Hazzan, 2009), can play in the cycle's critical first step – activating students' extant (assimilatory)

conceptions. Alluding to Leron's (2010) closing slogan, I believe that, without positing thinking dualities, mathematics teaching informed by the brain-based Ref*AER framework, and designed to bridge between available (assimilatory) and intended mathematical ideas, can nurture the power of natural (intuitive) thinking, address the challenge of stretching it, and inform the beauty of overcoming it (via anticipatory analytic processes).

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AIMS, SCOPE AND EDITORIAL INFORMATION

1. Visualizations and intuitive reasoning in mathematics
Kajsa Bråting (Sweden)
2. If not, then what if as well- Unexpected Trigonometric Insights.
Stanley Barkan (Israel)
3. Mathematical Competitions in Hungary: Promoting a Tradition of Excellence & Creativity
Julianna Connelly Stockton (USA)
4. From conic intersections to toric intersections: The case of the isoptic curves of an ellipse
Thierry Dana-Picard, Nurit Zehavi & Giora Mann (Israel)
5. Increasing the Use of Practical Activities through Changed Practice
Frode Olav Haara & Kari Smith (Norway)
6. Student Enrolment in Classes with Frequent Mathematical Discussion and Its Longitudinal Effect on Mathematics Achievement.
Karl W. Kosko (USA)
7. Translating and Adapting the Mathematical Knowledge for Teaching (MKT) Measures: The Cases of Indonesia and Norway
Dicky Ng (USA), Reidar Mosvold & Janne Fauskanger (Norway)
8. Reflections on Teaching with a Standards-Based Curriculum: A Conversation Among Mathematics Educators
Jill Newton, Ricki Geller, Lindsay Umbeck & Lisa Kasmer (USA)
9. Number Theory and the Queen of Mathematics
Megan Wagner (Montana)
10. Was Pythagoras Chinese- Revisiting an Old Debate
Ross Gustafson (Montana)

