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Climbing and Angles: A Study of how two Teachers Internalise and Implement the Intentions of a Teaching Experiment

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Abstract: In this innovative teaching experiment, the context of climbing is used to induce the teaching and learning of angle concepts. This article reports on the outcomes of a three day teaching and climbing experiment and the impact of this experience on the teacher's understanding of meso and micro embodiments of mathematics, angle representations, as well as shifts in their attitudes about teaching/learning geometry.

Keywords: angles; embodied mathematics; geometry; guided un-earthing; mathematisation; mathematical archeology; mathematics and physical education; meso space; meso space representations; reflective research practice; teacher beliefs; teaching and learning

Introduction

Based on previous work in which a twelve-year-old girl discovered angles in her climbing experience, Fyhn (2006) posited that climbing discourse can be a possible resource in the teaching of angles in primary school. Consequently the girl's class was introduced to the physical activity of climbing, as an integrated part of the teaching of angles (Fyhn, 2008). The class and two teachers took part in a three-day teaching and climbing experiment (TCE). The first day was spent at a local climbing arena with a focus on angles and climbing, the second day was half day of follow-up work at school (ibid.), and the third day was a follow up climbing-and-angles day three months later.

Innovative research-based teaching is of little use unless teachers internalize and implement it. There is an entire body of research in teacher beliefs that supports the previous statement. The focus of this paper is a presentation of the TCE's intentions and how these intentions were internalized and implemented by the two participating teachers. The analyses aimed to search for regularities in each of the two different teachers' development. The main research question was: 'How do two different teachers internalise and implement the students' mathematising of climbing as an approach to the teaching of angles?' The term *mathematising* is used as by Freudenthal (1973), 'mathematising something' means learning to organise this 'something' into a structure that is accessible to mathematical refinements. The two teachers' development is compared to the researcher's own development towards this approach to teaching. Schoenfeld (1998) claims that teachers' knowledge, beliefs, and goals are critically important determinants of what teachers do and why they do it. So these three aspects are discussed as well: 'How are the teachers' beliefs related to their goals and knowledge?'

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Regularities in the most different cases indicate robustness and possibilities for generalisation (Andersen, 2003). A comparative case study approach was chosen to explore this question. The TCE was designed and performed by the researcher.

Theoretical Framework

According to Lakoff and Núñez (2000, p. 365) “Human mathematics is embodied; it is grounded in bodily experience in the world.” They further claimed that angles existed in the early geometry paradigm where space was just the naturally continuous space in which we lived our embodied lives. The naturally continuous space is unconscious and automatic (ibid.). This supports work on the angle concepts in primary school as an integrated part of the students’ physical activity. Berthelot and Salin (1998) claim that space could be conceptualised into three different categories:

microspace (corresponding to the usual prehension relations), mesospace (corresponding to the usual domestic spatial interactions), and macrospace (corresponding to unknown city, maritime or rural spaces...) In consequence, the space representation produced by the usual out-of-school experiences is not naturally homogenous, and is quite different from elementary geometry. (p. 72)

One goal of the TCE was to guide the students to build bridges between their embodied mesospace climbing experiences, and the part of school mathematics that concerns angles. The research focus was also whether and how the participating teachers attained this goal.

Inductive and Deductive Teaching

In Norway mathematics traditionally is taught deductively. Alseth, Breiteig and Brekke (2003) claimed that Norwegian mathematics lessons usually start with the teacher’s explanation on how to solve a particular task. Then the students work individually on solving similar tasks in their books.

The curriculum of 1987 (KUD, 1987) was interpreted to recommend deductive as well as inductive mathematics teaching even for the lowest grades: “The subject matter may be introduced at first by the pupils’ investigating and experimenting in well prepared learning environments, and/or by the teacher showing and explaining.” (p. 195, author’s translation)

In the following curriculum of 1997 (KUF, 1996a; KUF, 1996b) the paragraph ‘Approaches to the study of Mathematics’ focused on learning; the students’ experiences and practical work. In this curriculum “practical situations and pupils’ own experiences” played an important role throughout elementary school. Despite the claim that students construct their own concepts, this curriculum too could be interpreted to support a deductive approach to mathematics teaching. The 1997 curriculum was also very vague regarding inductive versus deductive approaches so that it can be interpreted that the curriculum makers were not aware of these two different kinds of approaches. The 2006 curriculum’s (KD, 2006) focus was that during each of the main stages the students should aim to achieve some specific competencies in the main mathematics areas. This curriculum’s intentions focused on students’ achievements and did not concern teaching. However, the TCE took place before this curriculum was implemented.

In the U.S.A, teachers meet different requirements than the Norwegian ones. The National Council of Teachers of Mathematics, *NCTM*, standard includes explicit demands regarding inductive and deductive geometry teaching; “in grades 6-8 all students should create and critique inductive and deductive arguments concerning geometric ideas and relationships” (NCTM, 2000). However the author is cognizant that the standards espoused by the NCTM do not constitute a national curriculum and are viewed merely as recommendations.

In 1973 Freudenthal (1973, p. 402) warned, “The deductive structure of traditional geometry has not just been a didactical success.” Many Norwegian mathematics teachers have no theoretical bases for designing inductive teaching and in addition they lack experience of teaching and learning geometry inductively. There are no requirements for inductive approaches to mathematics in the Norwegian Curriculum, opposed to for instance the NCTM standard. The TCE had a clear inductive approach to teaching, and therefore the Norwegian deductive teaching tradition had to be taken into account in the analyses of how the teachers internalized and implemented the TCE’s intentions.

Concretising and Mathematising

Freudenthal (1983) described Bruner’s triad *enactive, iconic, and symbolic*:

enactively the clover leaf knot is a thing that is knotted, iconically it is a picture to be looked at, and symbolically it is something represented by the word “knot” (p. 30)

He further claimed that this schema might be useful in work with concept attainment (ibid.).

Concretising is often used deductively in Norwegian mathematics lessons as a tool for explaining something to students who do not understand what is being taught. However, Freudenthal (1983) claimed not to teach abstractions by concretising them. He advised to use the converse approach, i.e., to start off by searching for a phenomenon that might lead the learner to constitute the understanding of angles (ibid.). He further pointed out that “angles, in contradiction to lengths are being introduced and made explicit in an already heavily mathematised context” (p. 360). In the TCE the students mathematised their climbing by identifying and describing different angles related to climbing. Furthermore they were asked to explain their climbing moves via the use of the described angles. So the teaching of angles did not take place in a heavily mathematised context.

Mathematical Archaeology

The term mathematisation (Freudenthal, 1973) is to a large extent the same as mathematical archaeology. But while mathematisation refers to building of knowledge and not to discovering anything, the word archaeology refers to something hidden that needs to be uncovered. Mathematics can be integrated into an activity to such a degree that it disappears for both the children and the teachers, and then there might be the need for making the mathematics explicit. A mathematical archaeology is an educational activity where mathematics is recognised and named. This involves being aware that some activities carried out in the classroom are in fact mathematics.

An aim of a mathematical archaeology is to make explicit the actual use of mathematics hidden in the social structures and routines. It is the process of digging mathematics out and drawing attention to how mathematics moves from being an explicit guide to becoming a grey eminence underlying, for instance, social and economic management. (Skovsmose, 1994, p. 95)

It is important to a project, which contains mathematics as an implicit element, to spend some time on mathematical archaeology. The reason is:

If it is important to draw attention to the fact that mathematics is part of our daily life, then it also becomes important to provide children with a means for identifying and expressing this phenomenon. (p. 95)

It makes a difference whether the teaching is built upon situations that contain possibilities for the application of mathematics or just for descriptive purposes. The mathematics in climbing is so implicit that it is invisible (Fyhn, 2006). One goal of the TCE was to provide students a means for identifying, describing and using angles as an integrated element in their climbing activity, and consequently mathematical archaeology was an important part of the project. The mathematics here was descriptive.

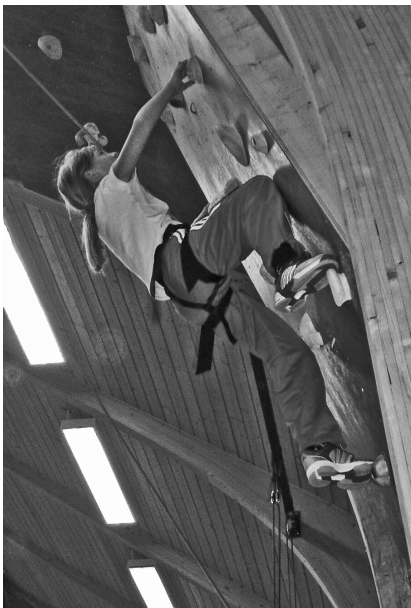


Figure 1. Teacher Frode: “They (the students) managed to ascend the climbing wall, and from different positions they named angles in their bodies. For instance our elbow can shape a right angle.” The idea is first to let the students identify angles in a climbing context. Second, they describe these angles, and third, they explain how the described angles influence their climbing. The teaching aims to guide the students to use ‘angle’, as a tool for improved climbing technique.

Torkildsen (2006) denoted performing mathematical archaeology on a subject as the *un-earthing* of mathematics in this subject. In the TCE, the researcher intended to guide the students through a kind of guided re-invention (Freudenthal, 1991), where the focus was on the un-earthing of descriptive mathematics. This teaching was denoted as *guided un-earthing*. The students’ mathematising of climbing with respect to angles, will be explained as the un-earthing of angles in climbing.

Different Approaches to Angles

Freudenthal (1983) recommended to “introduce angle concepts in the plural because there are indeed several ones; various phenomenological approaches lead to various concepts though they may be closely connected” (p. 323). He (ibid.) distinguished between angle as a static pair of sides, as an enclosed planar or spatial part and as the process of change of direction.

Mitchelmore and White (2000) found that the simplest angle concept was likely to be limited to situations where both the sides of the angle were visible; it is more difficult for children to identify angles in slopes, turns and other contexts where one or both sides of the angle are not visible.

Henderson and Taimina (2005) pointed out three different perspectives from which we can define angles: as a dynamic notion, as measure and as a geometric shape. Angle as shape referred to what the angle looks like; namely angle as a visual gestalt.

Krainer (1993) divided angles into four different categories: angle without an arc (angle as linked line/knee), angle with an arc, angle with an arrow (or oriented angle space) and angle with a rotation arrow (angle describing the rotation of a ray).

The TCE intended to let the teachers experience the guided un-earthing of angles in a climbing context. Figure 1 shows one of the climbing students with bent joints both in her knees, heels, hips, shoulders and elbows. The TCE referred to three different levels of understanding angles (Fyhn, 2008). At the first level students recognise angles as bent bodily shapes. These are mesospace angles with neither arcs nor arrows, and the students are not asked about these angles’ sizes in degrees.

At the second level the angles are described, either by what they look like (acute - right - obtuse), or by a drawing or by a rope demonstration. The right knee of the girl in Figure 1 shapes a right angle while her left heel shapes an obtuse angle. Angles can be

described by both meso- and microspace representations. At the third level angles are a tool for improved climbing technique; it is harder to ascend a climbing route if you cling to a handhold with your elbow in a right angle position, than if the angle is obtuse (ibid.).

Approaches to Angles in the Norwegian School

Johnsen (1996) warned that the most frequently used way of working with angles in Norwegian schools was measurement, and she further claimed that a large amount of Norwegian primary school students used the protractor incorrectly.

The Norwegian curriculum of 1997 (KUF, 1996a; 1996b) focused on students' experiences and their conceptual understanding. But regarding angles, 4th grade students were to gain experiences with 'important angle measurements' (KUF, 1996a, p. 162, author's translation). However, the English translation of the curriculum (KUF, 1996b) said 'important angles'. The curriculum text further continues "especially a whole turn as 360 degrees, a half as 180 degrees and a quarter as 90 degrees". This indicated a continuation of the measurement approach to angles in Norwegian primary school.

The curriculum of 2006 (KD, 2006) pointed out a clear measurement approach to angles: The word *angle* occurs only once and that is under the subject area 'measurement' for students at the fourth grade: "An aim for the teaching is that the student... is able to estimate and measure... angles" (ibid., p. 28, author's translation). According to Van den Heuvel-Panhuizen (2005) "Measurement and geometry are two domains, each with their own nature." (p. 13). In the curriculum of 2006 (KD, 2006) measurement and geometry occurred as two different sections, but angles are only treated in the section measurement.

In the Norwegian KIM project in geometry (Gjone and Nortvedt, 2001) more than one third of the participating sixth grade students were consistent in their reasoning about why a small angle with long sides is larger than a larger angle with shorter sides. This indicated a need for a new approach to the teaching of angles in Norwegian schools; neither Johnsen (1996) nor Gjone and Nortvedt (2001) could be interpreted to support the established measurement approach.

Teachers' beliefs

Törner, Rolka, Rösken and Sriraman (2010) paid attention to Schoenfeld's *Teaching-in-Context* theory, which pointed out interdependencies between the three fundamental variables knowledge, goals and beliefs, the *KGB* variables. "A teacher's spontaneous decision-making is characterized in terms of available knowledge, high priority goals and beliefs" (ibid., p. 403). Teaching here is understood as a continuous decision making process, and these three variables are considered as sufficient for understanding and explaining numerous teaching situations (ibid.). Lerman (2002) points out the cyclical relationship between changing beliefs and changing practices. Because one of the informants in the TCE is a trainee, it is of less value to research the two informants' change in practice. But the relations between their goals, beliefs and knowledge are visible to a large extent. So the analyses in this paper will relate Therese's and Frode's beliefs to their goals and previous knowledge.

Methodology

The TCE was closely related to design research (Gravemeijer & Cobb, 2006). In design research the designed teaching experiment undergoes iterative cycles of refining, while the TCE represented only one cycle. The TCE research focus was the process of teacher development.

The two participating teachers, Therese and Frode, were quite different people with different backgrounds; Therese was a trainee while Frode was an experienced mathematics teacher. When they joined the TCE both of them were acquainted with climbing and with mathematics teaching, even though their competencies and experiences differed to a large extent. Both Therese and Frode had had experienced inductive teaching, but these experiences mainly concerned science and other subjects. Frode had a couple of experiences with inductive mathematics teaching while Therese had none.

In addition to the teachers and the researcher, one more trainee and two other skilled grown-ups also took part in belaying¹ climbing students at day one, to make sure that as many students as possible could climb at the same time.

The Class

The entire class consisted of 18 students in the seventh grade. For different reasons some students were absent from different parts of the TCE. Nine girls and four boys in seventh grade participated the first day, the entire class participated the second day, while nine girls and six boys participated the third day.

One week before the TCE the students performed a pre-test with tasks that focused on angles and geometry. In this test more than half of the participating students failed in a KIM-test task where they should pick out the largest and the smallest among five given angles (Gjone & Nortvedt, 2001). This indicated that the students' conceptions of angles needed improvement (Fyhn, 2008).

The Researcher

The researcher designed the TCE and directed the completion of it, while the two teachers were assistant participants. The researcher's ability to let students mathematise their own climbing activities through performing mathematical archaeology was the result of a five-year long unguided process while she was teaching mathematics for trainee teachers.

The researcher's starting phase included three different aspects; firstly, an increased use of inductive teaching by use of artificial concretisations. Secondly, she performed meso space activities as basis for the teaching (Fyhn, 2002a), and finally, she performed some mathematical archaeology herself (Fyhn, 2001a; 2001b). The second aspect, meso space activities, turned out to make use of inductive approaches.

	Abstract symbols. Deductive approach	Artificial concretisation. Inductive approach
Micro space	A	B
Meso space	C	D

Table 1. Four different categories of geometry teaching. Category A shows the traditional Norwegian teaching, while the researcher's starting phase is presented in category D.

Table 1 presents four categories of approaches to teaching; the traditional deductive approach A and the researcher's inductive meso space approach D, where artificial concretisations were used. Table 2 presents four categories of mathematics, where E represents the mathematics that has been found in Norwegian curriculums until 2007, and H represents the mathematics that needs un-earthing in order to be described explicitly. The TCE focus on guided un-earthing of mathematics, was based upon the category H in table 2.

	visible	invisible
Construction by ruler and compass. Calculations. Proofs. Measurement. Manipulation of symbols.	E	F

Descriptive mathematics

G

H

Table 2. Four different categories of mathematics. Categories G and H represents the descriptive use of mathematics which is new in the 2007 curriculum (KD, 2006), which was implemented after this study took place. Invisible mathematics has never been explicitly focused in the Norwegian curriculum.

On one occasion the researcher succeeded in performing guided un-earthing of mathematics together with a fellow teacher educator (Mathisen & Fyhn, 2001). However, when a follow-up study was planned with two more colleagues, the result reflected several examples of what Skovsmose (1994) denotes as *artificial concretisations* of mathematics; category D in Table 1. In addition there was one single example of mathematical archaeology performed by the researcher; a focus on category H in table 2, and no more examples of guided un-earthing (Fyhn, 2002b). This is explained by the prematurity of the researcher's own ideas at that point in time. In 2004 the researcher finally succeeded in performing guided un-earthing of mathematics (Fyhn, 2006), and this time she was able to give a better description of what she did. On this occasion the participants were just the researcher and one single informant, and therefore the risk for interference from other people was minimised.

In planning the TCE, the researcher needed to avoid two possible events: a) the TCE could end up as a 'meso space artificial concretisation' project (category D in table 1) and b) a focus on category H where the teachers, and not the students, were performing the mathematical archaeology. Based on this, the researcher decided the teachers' roles in the project to be participant observers.

The teachers

By taking part in the project lead by the researcher, the teachers were provided with the experience of a *guided un-earthing* of angles in climbing. Even though both the teachers were familiar with mathematics and climbing, their competencies and experiences differed to a large extent. Both of them had studied one full year of university mathematics. There was a female professional climbing instructor who was a novice mathematics teacher, while Frode was a male experienced mathematics teacher who was a novice at climbing. They both had the qualifications for teaching mathematics at high school level in Norway. There's father was a scientist who lived in the Norwegian capital, while Frode's parents ran a small goat farm in the countryside in the northern part of the country.

Frode enjoyed taking part in physical leisure time activities and he was also partly responsible for his class' lessons in physical education. As for climbing, Frode was familiar with belaying climbers, and sometimes he went climbing himself. So he took active part in belaying the students.

There was an International Mountain Guide, and because of her climbing skills she was responsible for the security while the students were climbing. Being the researcher's trainee, she got some limited teaching tasks for each day of the project. There discussed these tasks with the researcher at the beginning and at the end of the TCE days.

The students performed the pre-test and a similar post-test in Frode's lessons, and he marked the students' answer to these tasks. The researcher set up these tests and handled them to Frode.

The three Days

On day one, before the climbing started, the day's two focus-words, 'climbing' and 'angles', were written in bold letters on a flip-over and the students were reminded of this throughout the day. The climbing was top-roping; the rope goes from the climber's harness up to a carabiner in the ceiling and down again to another person. This person is belaying the

climber; the belayer keeps the rope to the climber tight at all time, to prevent the climber from reaching the floor if she or he falls.

After having been introduced to some examples, the students had to mathematise their own climbing by identifying and describing different angles shaped by their bodies, the ropes and the building. Assisted by the teachers, the researcher guided the students through this *mathematical archaeology* on climbing with respect to angles.

In the day's last lesson a meso space perpendicular bisection was constructed on the floor by use of a climbing rope, a chalk and the bodies of some students. The location was the climbing arena. The activity made use of a rope, which is an element in the local climbing context.

The aim of this day was to let the students transform their ideas from working with angles in meso space into working with angles in micro space; an approach towards abstraction both by use of words as symbols and through construction by ruler and compasses. The students shaped angles by using their own bodies, they studied and discussed how the rope passed through the belay device, and they made drawings from the climbing. The students were split into groups, and each group constructed the perpendicular bisection on the floor by use of rope and chalk before they constructed it with ruler and compasses in their books.

In the period between day two and day three, Frode tried to implement some mathematics into his physical education lessons. Therese and the other trainee presented their impressions from taking part in a researcher's field work, to the rest of their fellow trainees.

The aim of the third day was to provide the teachers and the students with time for reflection, and then to re-visit them after some months to see if any change had occurred. The students were divided into groups, and each group had to create one particular climbing route; they decided rules for which holds they were allowed to use. The groups got small pieces of coloured cloth to mark their holds; each group got their own colour. Afterwards the groups would describe how their routes were ascended, and these descriptions had to include something about angles.

The Data

Each of the three mornings the teachers wrote down their expectations, and in the end of each day they wrote down their impressions. In December, after the two first days, the teachers were interviewed on tape. In May, they got an e-mail asking their opinions about the use of climbing as basis for teaching about angles in primary school. In the end of June, each of them was visited by the researcher, in order to go through what was written about them so far. The intention was to make sure that their writings were interpreted as correctly as possible; to make sure that the English version of the collected and analysed data reflected their real opinions. However, maybe the teachers' experiences from the TCE made them change their minds, if so there could be some difficulties in validating the data.

In addition to the formal writings, some e-mail and sms correspondence took part when the researcher felt a need for contact, but this informal communication was not treated as data. The use of video in this study could have offered better possibilities to return to what exactly happened. Then there would have been more possibilities of analyses of the data and of restudying details too. But then the researcher's written material would have been an interpretation of what the teachers expressed in these videos. The focus of this research was whether and how the teachers internalized and implemented the intentions of the TCE, and video is considered not to be the best tool for getting valid data about this. Most of the data in this research was the teachers' own written material and that made the analyses close to the data. One aim of this paper was to focus on the teaching of angles and not on the teachers' beliefs. But the teachers' interviews and their e-mails showed that their beliefs and attitudes

were interwoven in their replies. Thus some attention is paid towards this aspect as well. The data in this study was:

- The teachers' writings about their expectations to and experiences from each of the three days
- The interviews with each of the teachers after day two
- Frode's e-mail about how he worked out an idea that he talked about in the interview
- Field notes from the presentation Therese and the other trainee held for their fellow trainees
- The teachers e-mail replies to the question "*Can climbing be used as basis for teaching about angles?*"
- The teachers' comments to the researcher's analyses of their writings
- The researcher's list of publications from the years 2001-2006

Analyses

The First Day

At the beginning of the day, Therese believed there was a great potential for angle teaching based on climbing. Frode expected both the students and the teachers to learn a lot about angles. Therese's expectations were categorised as uncovering invisible mathematics; categories F or H in table 2. Frode's expectations were more difficult to categorise.

In their writings at the end of the day, neither of them mentioned mathematising of climbing nor angles. But both of them filled about half of their lines with appreciation of the perpendicular bisection construction, which is interpreted as 'meso space artificial concretisation', category D in table 1. Therese lost the angle focus when she started focusing on belaying, while Frode did not mention the word angle in his text.

The perpendicular bisection activity far from fulfilled their expectations from the beginning of the day. This can be interpreted as that 'meso space artificial concretisations' corresponds with a view of mathematics that is found in category E in table 2. Maybe the teachers just claimed that they appreciated to experience some inductive meso space mathematics teaching; that could be interpreted to that they had reached what the researcher describes as her starting phase.

The Second Day

The aim of this day was to provide the students with follow-up work at school, and bridge the gap between their meso space experiences from the climbing wall and their micro space work with pen and paper. The students might mathematise their climbing with respect to angles through practical activities, climbing talk, drawings and oral discussions. Because the artificial concretisation of the perpendicular bisection showed to be a very popular activity, and belaying showed to be a very popular activity among the students, some extra attention was paid towards these two activities. The students' mathematising of the belay device's functioning with respect to angles would indeed fulfil the TCE intentions because it was descriptive use of apparently invisible mathematics; category H in table 2.

Therese put on a harness, attached herself to a rope by a belay device, and asked what to do if she was belaying someone who fell. She asked for angles shaped by the rope and the belay plate. But the students did not understand what she meant. Therese concluded that she should rather have let the students perform this activity themselves, and then more of them probably would have understood what she meant. She indicates the difference between whether the teacher or the students perform the *un-earthing* of angles; if the students had

performed the activity themselves, they could have identified and explained angles by trial and error, by repeatedly checking out how different ways to use the device worked out.

Frode was busy doing various other things so he did not write about his expectations and experiences this day. Unfortunately the researcher was not aware of this until afterwards. This is an example of the data's weaknesses, and these weaknesses need to be sorted out. Therefore there could not be pointed out any similarities between Frode's and Therese's writings from this day.

Most of Therese's expectations concerned mathematics. She was curious about how much of the students' understanding of angles there would be left from day one. She was curious about how the groups would succeed in the construction of the perpendicular bisection. She ended: "I believe I will learn about how to work with concepts in the classroom with angles as starting point." This can be an indication of some expectations about some further mathematics, beyond the results of the descriptive mathematical archaeology.

At the end of the day she wrote that she was satisfied and pleased about how much the students had absorbed about angles:

A physical approach to angles leads some misconceptions to surface. The students are not sure which angle we refer to. Many of them thought that the angle disappeared when the rope was straightened. And that is correct in a way. But I believe they absorbed that the straight rope represents a 180° angle. The students really differ in how fast they understand this. But with this approach I believe that we reached all the students at some level, and that all of them have got something from this.

Therese here nicely described how the students' conceptions of angles were extended because their intuitive ideas of angles were challenged while they tried to understand how to belay a fellow climber. Here Therese experienced a *guided un-earthing* of angles; the students' un-earthing of apparently invisible mathematics for a descriptive purpose. These students' mathematising of climbing with respect to angles caused extension to their conceptions of angles.

The Interviews

The 'meso space artificial concretisation' (category D) experience of the perpendicular bisection construction fulfilled the expectations that the teachers presented in the interviews. Frode had experienced practical mathematics teaching that took place outside the classroom; categories C or D, while Therese had experienced mathematics teaching that differed from the traditional deductive teaching she was used to; categories B or D.

Therese appreciated observing the students' growing consciousness about angles in their bodies:

The students said that, well... there are no angles in our bodies... and then the consciousness-raising that happens throughout such a day... On the second day, when they were asked to perform an acute angle and a right angle by their bodies, then we could see all these different ways to stand and move. That was nice.

Her description here was interpreted as students' un-earthing of angles; a move from category H to category G. When she was asked if she thought that the students would think about angles related to climbing in the future, she answered that they would have to put their 'angle glasses' on. This statement was interpreted that she thinks the students can use their climbing bodies as models for angles. Furthermore she said:

Then the natural activity can take its own course, but the mathematics is still there. I like that. If the subject is all about mathematics I believe there will be some impatience, because you do not get the natural flow that we had on that particular day. I really appreciate the balance we got that day, to get the mathematics in while they performed activities ...and talked about it ...and related it and associated it to mathematics, yes ... that is more natural.

According to Therese the students' basic knowledge on the second day differed from their basic knowledge on the first day; angles seemed to concern them in a way.

Further on in the interview she pointed out that the first day's focus was on climbing while the second day's focus was on mathematics, and she appreciated "the natural progression to get more focus on the subject." Therese's utterances could be interpreted as a reference to the *guided un-earthing* of angles in climbing. Frode's utterances did not indicate a similar focus.

According to Frode the students' attitude towards mathematics was very negative when he started teaching them this autumn. He was not sure if he has managed to change any of this, but many of the students had started claiming that they enjoyed mathematics.

When Frode was asked if he believed that the students would be relating angles to climbing in the future, he answered: "We have worked with angles related to climbing. I believe that in future talk about angles the word climbing will show up, and consequently they will think about what we did." This way of connecting angles to climbing is interpreted as the students' use of climbing bodies as model for angles. Therese made a similar claim about models, and they both are interpreted to believe that the students' will remember the move from category H to category G in Table 2.

The Teachers' Attitudes

According to Lerman (2002) "It is in the recognition of conflict between what one wishes to do, or believes oneself to be doing, and the perceived reality of one's teaching that can bring about change"(p. 234). Törner, Rolka, Rösken and Sriraman (2010) support this by pointing at the relationship between goals and beliefs. The interviews showed that the teachers recognised such conflicts, and consequently they had a positive attitude towards participating in the TCE. The focus in this paper was on the teaching of angles and not on the teachers' beliefs, but it turned out that some beliefs came to surface.

Frode's school focused on 'outdoor schooling'; schooling outside the ordinary school building. His school even offered guiding in how to use outdoor schooling in theory and in practice. Frode wanted to be loyal to his school's aims. However, if he asked his colleagues for the mathematics content in their outdoor schooling, they answered that you could take the children to the shore and count stones and pebbles.

I think you can do that with students at the lower grades... You have to look ahead, try something... that is where I feel I really need something more. How to use outdoor schooling in mathematics teaching for students at 7th and 8th grade?

Frode had a conflict between his personal goals and what he experienced in his classroom, and consequently he was ready for a change in teaching practice (ibid.). He probably had two reasons for wanting to improve his teaching. Firstly, with respect to the practical work that ran like a connecting thread throughout the curriculum. Secondly, with respect to 'outdoor schooling' in the way it was focused on at his school.

Therese's statements described a conflict between her personal goals and her experiences from the mathematics classrooms:

I became a mathematics trainee teacher just to get the paper that proves I am a teacher. Mathematics is a subject that I had left far behind; actually I am really fed up with it. Now that I have started teaching myself, I find myself in the worst case scenario regarding mathematics. I experience my own teaching as dreadful and boring... This goes deep into my soul... I really do not want to force this upon other people because I think this is not all right.

Therese was interpreted to have two reasons for her positive attitude towards the TCE. Firstly, she wanted to experience mathematics teaching that was based on students' mastery experiences. Secondly, she wanted to experience mathematics teaching that differed from the deductive approach of which she disapproved.

Both of the teachers' attitudes were interpreted as that they wanted something that differed from deductive micro space teaching (category A in Table 1).

The Trainees' Presentation

A few weeks later, Therese and the other trainee did a presentation to their fellow trainees. They were free to choose among everything that happened on day one. They could have chosen to focus on the climbing approach to angles, by for instance letting their fellow trainees mathematise some belay devices. However, they chose to demonstrate the meso space perpendicular bisection.

There could be several reasons for their choice; maybe the other trainee's view of mathematics teaching to a great extent corresponded with a 'meso space artificial concretisation' (category D) approach to some visible (category E) mathematics, and therefore she did not see the point in the *guided un-earthing* of invisible (category H) mathematics. Maybe Therese was misled by listening to her well-meaning fellow trainee, just as the researcher was by listening to her well-meaning fellow teacher educators, on her way towards an idea of how to perform *guided un-earthing* of descriptive mathematics.

Another possibility was that the other trainee just believed in climbing as a social and exciting activity, and that the researcher's enthusiasm for mathematising of climbing had influenced and overwhelmed both of them. Maybe the other trainee was just as convinced about the climbing approach to angles, as the audience who was able to see "The Emperor's new clothes". Then the perpendicular bisection would represent an acceptable mathematics alibi.

Some months later Therese was asked why she let their fellow trainees do just the perpendicular bisection. Without hesitating she answered, that this was a simple and practical task that was easy to perform in the actual room, and that they had had some discussion before they decided what to do. She continued, "By activating the other students we could describe how to do mathematics in a practical way, and we illustrated how you wanted to work with mathematics." This claim can be interpreted to be that she thought that the researcher's aim was practical work just like what the curriculum points out. Therese was interpreted to be in a phase similar to the researcher's first phase.

Frode's Teaching Practice

During the interview, Frode suggested to integrate some mathematics into his physical education lessons. A couple of weeks later, he was e-mailed and asked how this had worked out. Frode had tried to keep mathematics in the back of his mind throughout two physical education lessons with the class. He had used the words 'line', 'velocity', 'angle' and 'direction' in his instructions, "many of the concepts are mathematical, but many were everyday concepts which we regularly use in everyday language." Regarding physical education as additional subject he wrote:

We do many coordinating exercises where especially the angle concepts are used; usually the students do somersaults in many different ways. I see that we can use the concept of rotation here. I give instructions to the students like that "*in your next jump you shall rotate horizontally 360°*"

Frode's descriptions of these lessons show that he is performing mathematical archaeology on his physical education lessons to a large extent, he has a focus on un-earthing the category H mathematics. Furthermore, he chose a deductive meso space (category C) approach in teaching his recently acquired descriptive view of mathematics (category G) as part of his physical education teaching. This could be interpreted to mean that Frode was in a similar phase as the researcher was, when she was performing mathematical archaeology on her students' work (Fyhn, 2001a; 2001b).

Frode wrote, “he he, you have opened my eyes a bit here”. This was a strong statement, which was interpreted as that he had internalized and implemented some of the intentions of the TCE. “It is all about possibilities, not limitations”, he wrote. Frode was becoming acquainted with *un-earthing* mathematics from his students’ physical experiences, but there was no sign of guided un-earthing.

The Third Day

In the morning, Therese and the students started out by writing about their memories from the first two days, and their expectations to this day. Unfortunately Frode was prevented from taking part in this writing session. This lack of data made Therese’s writings less useful because they could not be compared to Frode’s writing. The last time Therese thoroughly learned that demonstrations are useful simply to create an image of something to copy; she claimed that to learn something the students need to have something in their own hands, and try it themselves afterwards. In addition she had learned new ways of thinking about angles. Her text is interpreted to be that she disapproves of deductive micro space teaching (category A).

Therese had no expectations concerning mathematics this day. She was curious about the day, and expected a nice day with possibilities for her to give some climbing advices to the students. Afterwards Therese enjoyed watching the students making routes and discussing what holds that were natural to use related to their movements. And “It seems as if the students’ conceptions of angle are more profound now than the last time.” Her writing can be interpreted as that the students’ improved conceptions of angles was caused by building of bridges between visible and invisible angles (categories H and G).

Furthermore Therese pointed out a misconception caused by language; the Norwegian word *rett* means both straight and right. “A straight leg with a 180° angle can quite easily be called a right angle in Norwegian. And that is not unnatural because that is what we connect with the word right.” This writing could be interpreted as a description of how she experienced that the *guided un-earthing* of angles in climbing helped the students to get over an expected misconception. Frode, however, did not make any claim about how to prevent possible misconceptions.

Most of Frode’s writing concerned mathematics and mathematising. He even claimed that the students were aware of un-earthed angles:

At the end of the day, during the presentations, I observed that the students had learned to put angles down in words. They managed to ascend the climbing wall, and from different positions they named angles in their bodies. For instance our elbow can shape a right angle.

His writing could be interpreted as students’ improved conceptions of angles was related to mathematical archaeology on climbing; “Mathematics has to be recognised and named” (Skovsmose, 1994). Freudenthal (1991, p. 64) claimed similarly, “Name-giving is a first step towards consciousness.” However, Frode’s words “...the students had learned...” did not indicate whether he believed that they learned it through guided un-earthing or because of meso space teaching (category C).

Furthermore Frode appreciated that the students had found out how to use a table as a tool for organising and structuring their information; the students were able to analyse their information after having put it into a table. This activity was the students own mathematising of climbing, but not with respect to angles this time; the students turned mathematics from invisible (category F) into visible (category E) by un-earthing it. This was the students’ own idea, and no guiding took place. What Frode described here was how the focus on mathematical archaeology seemed to generate original mathematical reasoning from the students. Therese did not mention this event.

Beliefs, goals and knowledge

The *KGB* variables (Törner, Rolka, Rösken & Sriraman, 2010) will enlighten relations between the teachers' beliefs, and their knowledge and goals. Therese, who is a trainee, had no previous knowledge about mathematics teaching except for her experiences from being a student in mathematics classrooms. But she had expressed two clear goals with her participation in the TCE: i) to experience mathematics teaching that was based on the students' mastery experiences, and ii) to experience mathematics teaching that differed from the deductive boring approach she was used to. She had reached the goals, but her limited knowledge about mathematics teaching indicates that it probably is unrealistic that she would be capable to have implemented the TCE's intentions. But on the other hand, her claims were interpreted that she believed the TCE was a nice approach to teaching. That is not necessarily the same as that she was able to carry out a similar teaching herself.

Some of Frode's knowledge regarding mathematics teaching came to surface during the TCE. He was an experienced mathematics teacher with a solid background as for the subject mathematics. He was aware that teachers, who were less competent for teaching the higher grades, influenced the mathematics teaching at his school. His goal was to experience practical mathematics teaching outside the classroom. And this teaching had to correspond to the higher grades' syllabus. According to his writings, he had reached this goal. His students had been aware of un-earthed angles and that they had found out on their own how to use a table as tool for organizing their information. Both Therese and Frode may be interpreted that they believe that climbing might function as a basis for the teaching of angles.

A Couple of Months later

The data could be interpreted to indicate that the teachers to some extent had internalized and implemented the intentions of the TCE. However, maybe the teachers did not want to disappoint the researcher, and consequently wrote what they believed she expected from them. So, the data needed careful validating, and the teachers were e-mailed some months later and asked to reply in two to ten lines: "May climbing be used as basis for teaching about angles?"

Therese's reply arrived less than two hours later. She started out claiming that there are lots of angles both in the climbing bodies and in the climbing gear for belaying. She argued that she found the adjusting of angles in arms and legs to be an element in the climbing moves. Her writing was interpreted to mean that the students were able to mathematise their climbing; that they were able to *un-earth* angles in their climbing experiences. Consequently Therese was interpreted to have internalized the TCE's intentions:

The climber, who is conscious about this, can feel it in her own climbing, and make active use of it as an element in the climbing technique. Good climbing technique is based on the least possible use of force. This is active thinking about angles.

Therese's text was interpreted to mean that mathematising of climbing with respect to angles is easy, because the climbing context is pervaded with static and dynamic angles both in the ropes, in the wall and in the climbers' bodily joints. However, she could not be interpreted to have implemented the intentions before she had tried to implement guided un-earthing in her own teaching. Maybe then she would end up like Frode, who seemed to be satisfied with performing the mathematical archaeology himself. However, she had got as far as possible for her at that given moment. In addition she pointed to the students' positive attitude toward this activity: "Most people experience climbing to be exciting and fun".

Therese did not clearly point out anything about the teacher's role regarding the *guided un-earthing* of angles in climbing. Her beliefs and goals here concern angles in climbing and not an inductive approach to teaching. As previously pointed out, Therese is a trainee, and consequently she knows almost nothing about inductive mathematics teaching.

But as a professional climbing instructor, she knows a lot about climbing, and her goals and beliefs regarding angles as an element in the mathematics teaching is coloured by this knowledge.

Frode's reply arrived three days later. He wrote 15 lines concerning his opinion about physical activity in school in general, "... Children enjoy physical activities, and so do adults. Physically active children are happy children!" followed by 11 lines where he focused on the question. His answer was yes, but 'experience' was the only reason he gave. There was a great risk in interpreting Frode's e-mail to be that he was not convinced about anything related to climbing and angles. Because the TCE was a comparative case study with only two informants, it was natural to make one more inquiry to investigate if this really was his answer.

But there was a risk that Therese's and Frode's e-mails did not reflect what they really meant about climbing as basis for teaching mathematics. Maybe their e-mails just revealed what they thought the researcher expected them to write or what they felt persuaded to write. They were asked to reply to a question which concerned the intentions of the TCE, but there was no guarantee of how they would interpret the question.

A final Visit

In the end of June, Therese verified most of the writings about her. Frode immediately pointed out that his last e-mail was meant as a start of some longer writing, but that this longer writing never was continued. So his last e-mail did not reflect what he actually meant.

Frode explained that when his students worked with time and velocity during this spring's mathematics lessons, they started with performing a 'running experiment'. They finished with making a written report that explained what they had done, and how they could find the average velocity. What he says here can be interpreted as the students' un-earthing of mathematics from their meso space activity. Frode did not claim whether his approach to teaching here was inductive or deductive, but he had guided the students to build a bridge between their embodied meso space experiences and school mathematics. He immediately made a new version of his reply to the question. At first he wrote that climbing was a great fundament for the teaching of angles,

Children use their bodies to shape different angles. This gives them a closer relationship to angles. The students in my class enjoy climbing, and after the climbing days some of the students said: '*Angles are fun!*' I believe the students will remember '*angles*' in their future climbing.

In addition Frode was interpreted to claim that angles would concern his students' future climbing activity. His belief here makes sense when related to his goal; to experience practical mathematics teaching outside the classroom. His knowledge and experience about mathematics and mathematics teaching was the background for his goal.

Frode also showed that his own teaching practice was changing. He and his students had un-earthed mathematics from their running this spring, and they had even made a written report about this. This is what Lerman (2002) claimed; a change in teaching practice is related to a change in belief. Frode was interpreted to have implemented the intention of performing mathematical archaeology on the students' physical activities. Both Frode and Therese were interpreted to claim that they have internalised and implemented some of the intentions of the TCE. However, the difference is that Therese sticks with the guided un-earthing, while Frode tends to perform the un-earthing himself and present the un-earthed mathematics to his students in a deductive way. This might be due to the teachers' knowledge and goals. Therese had no knowledge about inductive mathematics teaching but a lot of knowledge about climbing, while her goals focused on inductive teaching and mastery experiences. In the end she seemed to believe in guided un-earthing of the students mastery experiences from climbing. But she had not had any possibility of performing such teaching. Frode's goal

concerned mathematics teaching outside the classroom, and his change in practice is a strong indication to that his new belief concern un-earthing of mathematics from physical activities. Both of them show a development that has much in common with the researcher's development. The difference is that Frode is interpreted to be loyal towards a deductive teaching.

Discussion

Wood and Berry (2003) underlined the importance of creating a shared knowledge base for teaching. They claimed that research on the process of development extends the idea of a 'product'; "the process involved can become the product that is sought" (p. 197). Regarding teacher development, they ask for reports of research that study the process-into-product models. The TCE intends to be such a report.

Leatham (2006) warned researchers against assuming that teachers easily can articulate their beliefs. He also pointed the simplistic thinking of there being a one-to-one correspondence between what teachers state and what researchers think those statements mean. Leatham's (2006) warning against the dangers of simplistic one-to-one correspondence between what teachers state and what teachers mean matches the TCE's analysis; as shown in Frode's first e-mail reply to the question about climbing as basis for teaching about angles. The main data source of the TCE was the teachers' written statements. The analyses of the teachers' written statements were presented to the teachers, in order to have the analyses as close as possible to what they really meant.

According to Brekke, Kobberstad, Lie and Turmo (1998) it had been problematic for Norwegian students to grasp that 180° is an angle. A strengthened rope represents a 180° angle where both of the sides are visible. At the end of day two, Therese wrote, "many of the students thought that the angle disappeared when the rope was straightened.... But I believe they absorbed that the straight rope represents a 180° angle." This writing indicated that the angles shaped by climbing ropes can represent a useful contribution to the teaching of angles; that students' mathematizing of the belaying of climbers could prove to be useful to extend the students conceptions of angles.

According to Gravemeijer and Cobb (2006) the Dutch *RME* (Realistic Mathematics Education) had emerged

in resistance to instructional and design approaches that treated mathematics as a ready-made product... A process of guided reinvention then...requires the instructional starting points to be experimentally real for the students, which means that one has to present the students problem situations in which they can reason and act in a personally meaningful manner.(p.15)

In the TCE, the students' conceptions of angle were treated as something the students created as an integrated part in the development of their climbing talk. None of the climbers asked why they had to climb, or what they needed these experiences for; this is interpreted to be that the students found the activity to be meaningful to them. According to van den Heuvel-Panhuizen (2003)

Models are attributed the role of bridging the gap between the informal understanding connected to the 'real' and imagined reality on the one side, and the understanding of formal systems on the other. (p. 13)

This corresponds to one intention of the TCE; to guide the students to build a bridge between their (embodied meso space) experiences and school mathematics. This matches Frode's claim, that he believes the students will remember angles in their future climbing.

Findings and Conclusions

The main focus of this research was whether and how teachers internalised and implemented guided un-earthing of angles in climbing as an approach to the teaching of angles. The data concerning the teachers were compared to the development of the researcher's publications in order to search for common developmental features. Five years passed from the first time the researcher performed guided un-earthing (Mathisen & Fyhn, 2001) and until she managed to work it out the second time (Fyhn, 2006). The first time she hardly was able to describe the un-earthing, but the second time she had developed a tool for describing it explicitly. Through these five years, the researcher was easily misled into what Skovsmose (1994) denotes as *artificial concretisation* by listening to well-meaning fellow teacher educators. For a period she was even satisfied with performing the un-earthing of mathematics by herself, instead of guiding her students to perform it.

The findings indicate some regularity in the two teachers' development, and their processes of development are to a large extent similar to the researcher's development towards guided un-earthing. The TCE was a three day descriptive work with mathematics, with no explicit focus on problem solving or task solving. This was unclear to the researcher and therefore the informants were not informed about it. Many teachers do not treat descriptive work with mathematics as real mathematics (Skovsmose, 1994). The teachers' lack of knowledge about descriptive use of mathematics might have influenced their goals and beliefs about the TCE. According to Schoenfeld (1998) the questions what a teacher will do next, and why, can be illuminated by describing interactions between her knowledge, goals and beliefs.

There were strong indicators of relations between the teachers' knowledge, goals and beliefs. The trainee Therese's beliefs were restricted because of her limited knowledge about inductive mathematics teaching; her goals were to experience mathematics teaching that differed from the deductive micro space approach that she was familiar with. Frode's beliefs and goals were related to his knowledge about how his teacher colleges taught mathematics, and his beliefs were related to his knowledge about gymnastics teaching.

Before the TCE, none of the teachers were familiar with inductive approaches to teaching mathematics, but they were familiar with inductive approaches to teaching physics. At the end of day one, none of them mentioned the climbing approach to angles. However, both of them appreciated artificial inductive meso space teaching (category D in Table 1). They are interpreted to have entered a phase similar to the researcher's phase when she was trying to grasp mathematical archaeology and mathematising.

At the end of the second day, Therese nicely described how the students' intuitive ideas of angles were challenged, while they tried to understand how to belay a fellow climber. She is also interpreted to have experienced and appreciated a situation where *guided un-earthing* of angles caused extension to the students' conceptions of angles. She can still be interpreted to be in a phase where her implementation of the guided un-earthing of angles is premature or diffuse. Together with her fellow trainee, she chose to present their fellow trainees with some artificial inductive meso space teaching (category D in Table 1). They could as well have chosen to guide their fellow students in un-earthing of angles in the belay device's functioning. This interpretation indicated that Therese's development followed similar pattern as the researcher's development.

The interviews indicated that Therese and Frode both to some extent believed in the students' un-earthing of angles from climbing as an appropriate approach to the teaching of angles. Both the teachers were interpreted to claim similar utterances: The students had grasped that climbing bodies shaped angles, and that different bodily moves would shape different angles. But maybe their claims in the interview reflected just what they believed the researcher wanted them to say. According to Lerman (2002) it is a methodological weakness

to assume that interviews and questionnaires can reveal beliefs, which is the main determinant of a teacher's action in the classroom. This yields particularly Therese, who is a trainee and has no class on her own. At the end of day three the teachers' writings were interpreted to present the TCE's intentions to some extent; they wrote several lines about the students' un-earthing of mathematics from climbing. However, some months after the project, none of them wrote anything that could be interpreted as *guided* un-earthing of angles in climbing; none of them mentioned anything regarding neither the teacher's role nor how the learning should take place.

During the period between day two and day three, Frode performed mathematical archaeology on the activities in his physical education lessons. But this led to a teaching that was interpreted as deductive meso space teaching, category C in Table 1. During the school year Frode's teaching practice changed: he and his students un-earthed mathematics from their running, and his students had even made a written report about this. This indicates that he had internalized and implemented some of the intentions of the TCE. However, there is no sign of inductive approaches in Frode's teaching. There is no claim regarding inductive work in the Norwegian mathematics curriculum (KD, 2006). This is opposed to the NCTM (2000) geometry standard which points out explicitly that the grade 6-8 students should create and critique inductive and deductive arguments.

The TCE findings lead to the hypothesis that teachers can be guided to re-invent a climbing approach to angles the following way: At first, a phase where the teacher experiences different approaches to teaching: deductive versus inductive, and meso space versus micro space. These constitute the four categories in table 1. One important point here may be a discussion about how to bridge the gap between the different categories. A short instructional DVD has been made as a basis for such a discussion (Fyhn, 2007). In addition the teachers need to discover the power of a mathematical archaeology approach by performing it themselves. After having experienced this phase, the teacher is ready for trying to perform a *guided un-earthing* of angles in climbing. One more instructional video has been made for this purpose (Fyhn, 2008). However, neither these videos nor these ideas have been researched.

Six months after the last climbing day, Therese was working with outdoor education at a non-degree granting college, and there she paid no attention to mathematics education. Frode was approaching the subject of geometry in his teaching schedule, and without explaining he underlined that his way of teaching differed from the researcher's. He claimed: "You must let the students perform activities that they enjoy. The challenge is to find the mathematics in these activities." He still was interpreted to mean that he was performing the mathematical archaeology on his students' activities, and then he explains the un-earthed mathematics via a deductive approach. The findings from the TCE indicate that future research and instructional design of this kind carefully need to give the teachers time for gathering experiences with and reflecting upon guided un-earthing opposed to other approaches to teaching.

Notes

¹ To belay means to secure the climber with a breaking device connected to the rope in case the climber falls. The climber will then be hanging by the rope.

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