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The Montana Mathematics Enthusiast is an eclectic internationally circulated peer reviewed journal which focuses on mathematics content, mathematics education research, innovation, interdisciplinary issues and pedagogy. The journal is published by Information Age Publishing and the electronic version is hosted jointly by IAP and the Department of Mathematical Sciences- The University of Montana, on behalf of MCTM. Articles appearing in the journal address issues related to mathematical thinking, teaching and learning at all levels. The focus includes specific mathematics content and advances in that area accessible to readers, as well as political, social and cultural issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is interested in research based articles as well as historical, philosophical, political, cross-cultural and systems perspectives on mathematics content, its teaching and learning.

The journal also includes a monograph series on special topics of interest to the community of readers The journal is accessed from 110+ countries and its readers include students of mathematics, future and practicing teachers, mathematicians, cognitive psychologists, critical theorists, mathematics/science educators, historians and philosophers of mathematics and science as well as those who pursue mathematics recreationally. The 40 member editorial board reflects this diversity. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor in APA style. The typical time period from submission to publication is 8-11 months. Please visit the journal website at http://www.montanamath.org/TMME or http://www.math.umt.edu/TMME/

New Year Tidings

Bharath Sriraman, The University of Montana

About a year ago, I began an editorial on my way **from** Tromsø (Norway) to Montana, and strangely enough, and a year later, this editorial is penned as I make my way **to** Tromsø.

2009 proved to be a good year both for the journal and the monograph series, and I am hopeful that 2010 will result in positive tidings for the journal, its authors, editorial board members and the readers of *The Montana Mathematics Enthusiast*.

Some changes have occurred in the structure of the editorial board. Claus Michelsen (University of Southern Denmark) has served his term as an Associate Editor and is being replaced by Simon Goodchild of the University of Agder (Norway). I thank Claus for his support of TMME over the last 3 years. Simon brings tremendous editing experience as well as expertise in statistical techniques, methodological issues, as well as research in learning communities in mathematics teacher development. In addition I am pleased to welcome three new editorial board members, namely Raymond Bjuland and Reidar Mosvold both from the University of Stavanger (Norway), and Mehdi Alaeiyan from Iran University of Science and Technology. These three scholars add to the diversity and strength of the journal.

This issue of the journal includes 9 articles, representing a wide geographic range and topics. The opening article by Katz & Katz (Israel) looks at the important albeit forgotten place of non-Standard analysis in the teaching and learning of Calculus. The last article by Aztekin et al. examines a related topic in the context of researching notions of infinity among PhD students in Turkey. This article is based on Aztekin's PhD dissertation which made effective use of repertory grid methodology to get a nuanced view of different conceptions and misconceptions of infinity held by those with a fairly good academic background in mathematics.

Between the bookend pieces with non-Standard Analysis and infinity notions are seven articles reporting on issues related to the classroom, namely teacher development, implementation of mathematics content, mathematical learning in technological environments as well as larger societal issues. For instance Anjum Halai's article provides a description of a large scale project in rural Pakistan to understand implementation issues surrounding curricular changes in mathematics and science education aimed at poverty alleviation and gender equity. Her paper addresses policy and practice issues in impoverished areas.

This year there are two focus issues of the journal planned, one on mathematical giftedness and talent which is being compiled and edited by Viktor Freiman (Canada) and Ali Rejali (Iran). This focus issue is planned as vol7, no2 [June 2010]. A section of the vol7,no3 [October 2010] will focus on the regional work of women mathematics educators in the Northwestern region of North America [Montana, Idaho, Washington, Alberta]. This is being compiled by Ke Norman (Montana).

Finally two major monographs are in the works and on schedule for release in 2010. Monograph 10 is the massive *Sourcebook on Nordic Research in Mathematics Education*, which is a first of its kind 1000+ page tome on mathematics education research in Norway, Sweden, Iceland, Denmark and contributions from Finland. This is on schedule for release in Summer 2010. Monograph 11 focuses on *Interdisciplinarity and Creativity in the 21st Century*, slated for release in Fall 2010. These two monographs would not have been possible without the support and goodwill of over 100 authors scattered around the world.

Having conveyed the tidings for 2010, I will close by adapting a quote from Hermann Hesse, which I think is indicative of the spirit of the journal and could serve as its motto:

Not caring whether we are courted or cursed, we follow our true inner calling.

Happy New Year!

When is .999... less than 1?

Karin Usadi Katz and Mikhail G. Katz[0]

We examine alternative interpretations of the symbol described as *nought, point, nine recurring*. Is "an infinite number of 9s" merely a figure of speech? How are such alternative interpretations related to infinite cardinalities? How are they expressed in Lightstone's "semicolon" notation? Is it possible to choose a canonical alternative interpretation? Should unital evaluation of the symbol .999 . . . be inculcated in a pre-limit teaching environment? The problem of the unital evaluation is hereby examined from the pre-$\mathbb{R}$, pre-lim viewpoint of the *student*.

1. Introduction

Leading education researcher and mathematician D. Tall [63] comments that a mathematician "may think of the physical line as an *approximation* to the infinity of numbers, so that the line is a practical representation of numbers[, and] of the number line as a visual representation of a precise numerical system of decimals." Tall concludes that "this still does not alter the fact that there are connections in the minds of students based on experiences with the number line that differ from the formal theory of real numbers and cause them to feel confused."

One specific experience has proved particularly confusing to the students, namely their encounter with the evaluation of the symbol .999 . . . to the standard real value 1. Such an evaluation will be henceforth referred to as the *unital evaluation*.

We have argued [32] that the students are being needlessly confused by a premature emphasis on the unital evaluation, and that their persistent intuition that .999 . . . can fall short of 1, can be rigorously justified. Other interpretations (than the unital evaluation) of the symbol .999 . . . are possible, that are more in line with the students' naive initial intuition, persistently reported by teachers. From this viewpoint, attempts to inculcate the equality .999 . . . = 1 in a teaching environment prior to the introduction of limits (as proposed in [65]), appear to be premature.

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To be sure, certain student intuitions are clearly dysfunctional, such as a perception that $\forall\epsilon\exists\delta$ and $\exists\delta\forall\epsilon$ are basically "the same thing". Such intuitions need to be uprooted. However, a student who intuits $.999\ldots$ as a dynamic process (see R. Ely [23, 24]) that never quite reaches its final address, is grappling with a fruitful cognitive issue at the level of the world of proceptual symbolism, see Tall [63]. The student's functional intuition can be channeled toward mastering a higher level of abstraction at a later, limits/$\mathbb{R}$ stage.

In '72, A. Harold Lightstone published a text entitled *Infinitesimals* in the *American Mathematical Monthly* [38]. If $\epsilon > 0$ is infinitesimal (see Appendix A), then $1 - \epsilon$ is less than 1, and Lightstone's extended decimal expansion of $1 - \epsilon$ starts with more than any finite number of repeated 9s.[1] Such a phenomenon was briefly mentioned by Sad, Teixeira, and Baldino [51, p. 286].

The symbol $.999\ldots$ is often said to possess more than any finite number of 9s. However, describing the real decimal $.999\ldots$ as possessing an *infinite number of* $9s$ is only a figure of speech, as *infinity* is not a number in standard analysis.

The above comments have provoked a series of thoughtful questions from colleagues, as illustrated below in a question and answer format perfected by Imre Lakatos in [37].

2. Frequently asked questions: when is $.999\ldots$ less than 1?

Question 2.1. Aren't there many standard proofs that $0.999\ldots = 1$? Since we can't have that and also $0.999\ldots \neq 1$ at the same time, if mathematics is consistent, then isn't there necessarily a flaw in your proof?

Answer. The standard proofs[2] are of course correct, in the context of the standard real numbers. However, the annals of the teaching of decimal notation are full of evidence of student frustration with the unital evaluation of $.999\ldots$ This does not mean that we should tell the students an untruth. What this does mean is that it may be instructive to examine why exactly the students are frustrated.

Question 2.2. Why are the students frustrated?

Answer. The important observation here is that the students are not told about either of the following two items:

  (1) the real number system;
  (2) limits,

before they are exposed to the topic of decimal notation, as well as the problem of unital evaluation. What we point out is that so long as the number system has not been specified explicitly, the students' hunch that $.999\ldots$ falls infinitesimally short of 1 can be justified in a rigorous fashion, in the framework of Abraham Robinson's [47, 48] non-standard analysis.[1]

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[1] By the work of J. Avigad [3], the phenomenon can already be expressed in primitive recursive arithmetic, in the context of Skolem's non-standard models of arithmetic, see answer to Questions 6.1 and 6.2 below.

[2]The proof exploiting the long division of 1 by 3, is dealt with in the answer to Question 8.1.

Question 2.3. Isn't it a problem with the proof that the definitions aren't precise? You say that $0.999\ldots$ has an "unbounded number of repeated digits 9". That is not a meaningful mathematical statement; there is no such number as "unbounded". If it is to be precise, then you need to provide a formal definition of "unbounded", which you have not done.

Answer. The comment was not meant to be a precise definition. The precise definition of $.999\ldots$ as a real number is well known. The thrust of the argument is that *before* the number system has been explicitly specified, one can reasonably consider that the ellipsis "..." in the symbol $.999\ldots$ is in fact ambiguous. From this point of view, the notation $.999\ldots$ stands, not for a single number, but for a class of numbers,[3] all but one of which are less than 1.

Note that F. Richman [46] creates a natural semiring (in the context of decimal expansions), motivated by constructivist considerations, where certain cancellations are disallowed (as they involve infinite "carry-over"). The absence of certain cancellations (i.e. subtractions) leads to a system where a strict inequality $.999\ldots < 1$ is satisfied. The advantage of the hyperreal approach is that the number system remains a field, together with the extension principle and the transfer principle (see Appendix A).

Question 2.4. Doesn't decimal representation have the same meaning in standard analysis as non-standard analysis?

Answer. Yes and no. Lightstone [38] has developed an extended decimal notation[4] that gives more precise information about the hyperreal. In his notation, the standard real $.999\ldots$ would appear as

$$.999\ldots;\ldots 999\ldots$$

Question 2.5. Since non-standard analysis is a conservative extension of the standard reals, shouldn't all existing properties of the standard reals continue to hold?

Answer. Certainly, $.999\ldots;\ldots 999\ldots$ equals 1, on the nose, in the hyperreal number system, as well. An accessible account of the hyperreals can be found in chapter 6: *Ghosts of departed quantities* of Ian Stewart's popular book *From here to infinity* [55]. In his unique way, Stewart has captured the essense of the issue as follows in [56, p. 176]:

> The standard analysis answer is to take '$\ldots$' as indicating passage to a limit. But in non-standard analysis there are many different interpretations.

In particular, a *terminating* infinite decimal $.999\ldots;\ldots 999$ is less than 1.

Question 2.6. Your expression "terminating infinite decimals" sounds like gibberish. How many decimal places do they have exactly? How can infinity terminate?

Answer. If you are troubled by this, you are in good company. A remarkable passage by Leibniz is a testimony to the enduring appeal of the metaphor of infinity, even in its,

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[3]For a more specific choice of such a number, see the answer to Question 8.3.
[4]For details see Appendix A, item A.11 below.

paradoxically, *terminated* form. In a letter to Johann Bernoulli dating from june 1698 (as quoted in Jesseph [30, Section 5]), Leibniz speculated concerning

> lines [...] which are terminated at either end, but which nevertheless are to our ordinary lines, as an infinite to a finite.

He further speculates as to the possibility of

> a point in space which can not be reached in an assignable time by uniform motion. And it will similarly be required to conceive a time terminated on both sides, which nevertheless is infinite, and even that there can be given a certain kind of eternity [...] which is terminated.

Ultimately, Leibniz rejected any metaphysical reality of such quantities, and conceived of both infinitesimals and infinite quantities as ideal numbers, falling short of the reality of the familiar appreciable quantities.[5]

Question 2.7. You say that in Lightstone's notation, the nonstandard number represented by $.999\ldots;\ldots999\hat{9}$ is less 1. Wouldn't he consider this as something different from $.999\ldots$, since he uses a different notation, and that he would say

$$.999\ldots;\ldots999\hat{9} < .999\ldots = 1?$$

Answer. Certainly.

Question 2.8. Aren't you arbitrarily redefining $.999\ldots$ as equal to the non-standard number $.999\ldots;\ldots999\hat{9}$, which would contradict the standard definition?

Answer. No, the contention is that the ellipsis notation is ambiguous, particularly as perceived by pre-lim, pre-$\mathbb{R}$ students. The notation could reasonably be applied to a class of numbers[6] infinitely close to 1.

Question 2.9. You claim that "there is an unbounded number of 9s in $.999\ldots$, but saying that it has infinitely many 9s is only a figure of speech". Now there are several problems with such a claim. First, there is no such object as an "unbounded number". Second, "infinitely many 9s" not a figure of speech, but rather quite precise. Doesn't "infinite" in this context mean the countable cardinal number, $\aleph_0$ in Cantor's notation?

Answer. One can certainly choose to call the output of a series whatever one wishes. The terminology "infinite sum" is a useful and intuitive term, when it comes to understanding standard calculus. In other ways, it can be misleading. Thus, the term contains no hint of the fact that such an "$\aleph_0$-fold sum" is only a partial operation, unlike the inductively defined n-fold sums. Namely, a series can diverge, in which case the infinite sum is undefined (to be sure, this does not happen for decimal series representing real numbers).

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[5]Ely [24] presents a case study of a student who naturally developed an intuitive system of infinitesimals and infinitely large quantities, bearing a striking resemblance to Leibniz's system. Ely concludes: "By recognizing that some student conceptions that appear to be misconceptions are in fact nonstandard conceptions, we can see meaningful connections between cognitive structures and mathematical structures of the present and past that otherwise would have been overlooked."

[6]For a more specific choice of such a number, see the answer to Question 8.3.

Furthermore, the "$\aleph_0$-fold sum" intuition creates an impediment to understanding Lightstone extended decimals

$$.a_1 a_2 a_3 \ldots ; \ldots a_H \ldots$$

If one thinks of the standard real as an $\aleph_0$-fold sum of the countably many terms such as $a_1/10, a_2/100, a_3/1000$, etc., then it may appear as though Lightstone's extended decimals *add* additional positive (infinitesimal) terms to the real value one started with (which seems to be already "present" to the left of the semicolon). It then becomes difficult to understand how such an extended decimal can represent a number *less* than 1.

For this reason, it becomes necessary to analyze the *infinite sum* figure of speech, with an emphasis on the built-in *limit*.

Question 2.10. Are you trying to convince me that the expression *infinite sum*, routinely used in Calculus, is only a figure of speech?

Answer. The debate over whether or not an *infinite sum* is a figure of speech, is in a way a re-enactment of the foundational debates at the end of the 17th and the beginning of the 18th century, generally thought of as a Newton-Berkeley debate.[7] The founders of the calculus thought of

(1) the derivative as a ratio of a pair of infinitesimals, and of
(2) the integral as an infinite sum of terms $f(x)dx$.

Bishop Berkeley [8] most famously[8] criticized the former in terms of the familiar *ghosts of departed quantities* (see [55, Chapter 6]) as follows. The infinitesimal

$$dx$$

appearing in the denominator is expected, at the beginning of the calculation, to be nonzero (the *ghosts*), yet at the end of the calculation it is neglected as if it were zero (hence, *departed* quantities). The implied stripping away of an infinitesimal at the end of the calculation occurs in evaluating an integral, as well.

To summarize, the integral is not an infinite Riemann sum, but rather the standard part of the latter (see Section A, item A.12). From this viewpoint, calling it an infinite sum is merely a figure of speech, as the crucial, final step is left out.

A. Robinson solved the 300-year-old logical inconsistency of the infinitesimal definition of the integral, in terms of the standard part function.[9]

Question 2.11. Hasn't historian Bos criticized Robinson for being excessive in enlisting Leibniz for his cause?

Answer. In his essay on Leibniz, H. Bos [17, p. 13] acknowledged that Robinson's hyperreals provide

> [a] preliminary explanation of why the calculus could develop on the insecure foundation of the acceptance of infinitely small and infinitely large quantities.

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[7]See footnote 28 for a historical clarification.

[8]Similar criticisms were expressed by Rolle, thirty years earlier; see Schubring [52].

[9]See Appendix A, item A.3.

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F. Medvedev [41, 42] further points out that

> [n]onstandard analysis makes it possible to answer a delicate question bound up with earlier approaches to the history of classical analysis. If infinitely small and infinitely large magnitudes are regarded as inconsistent notions, how could they serve as a basis for the construction of so [magnificent] an edifice of one of the most important mathematical disciplines?

3. Is *infinite sum* a figure of speech?

Question 3.1. Perhaps the historical definition of an integral, as an infinite sum of infinitesimals, had been a figure of speech. But why is an infinite sum of a sequence of real numbers more of a figure of speech than a sum of two real numbers?

Answer. Foundationally speaking, the two issues (integral and infinite sum) are closely related. The series can be described cognitively as a proceptual encapsulation of a dynamic process suggested by a sequence of finite sums, see D. Tall [63]. Mathematically speaking, the convergence of a series relies on the completeness of the reals, a result whose difficulty is of an entirely different order compared to what is typically offered as arguments in favor of unital evaluation.

The rigorous justification of the notion of an integral is identical to the rigorous justification of the notion of a series. One can accomplish it finitistically with epsilontics, or one can accomplish it infinitesimally with standard part. In either case, one is dealing with an issue of an entirely different nature, as compared to finite n-fold sums.

Question 3.2. You have claimed that "saying that it has an infinite number of 9s is only a figure of speech". Of course "infinity" is not a number in standard analysis: this word refers to a number in the cardinal number system, i.e. the cardinality[10] of the number of digits; it does not refer to a number in the real number system.

Answer. One can certainly consider an infinite string of 9s labeled by the standard natural numbers. However, when challenged to write down a precise definition of $.999\ldots$, one invariably falls back upon the *limit* concept (and presumably the respectable epsilon, delta definition thereof). Thus, it turns out that $.999\ldots$ is really the limit of the sequence $.9, .99, .999$, etc.[11] Note that such a definition never uses an infinite string of 9s labeled by the standard natural numbers, but only finite fragments thereof.

Informally, when the students are confronted with the problem of the unital evaluation, they are told that the decimal in question is zero, point, followed by infinitely many 9s. Well, taken literally, this describes the hyperreal number

$$.999\ldots;\ldots999000$$

perfectly well: we have zero, point, followed by H-infinitely many 9s. Moreover, this statement in a way is truer than the one about the standard decimal, as explained above

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[10]See more on cardinals in answer to Question 6.1.

[11]The related sequence .3, .33, .333, etc. is discussed in the answer to Question 8.1.

(the infinite string is never used in the actual standard definition). The hyperreal *is* an infinite sum, on the nose. It is not a limit of finite sums.

Question 3.3. Do limits have a role in the hyperreal approach?

Answer. Certainly. Let $u_1 = .9, u_2 = .99, u_3 = .999$, etc. Then the limit, from the hyperreal viewpoint, is the standard part of u_H for any infinite hyperinteger H. The standard part strips away the (negative) infinitesimal, resulting in the standard value 1, and the students are right almost everywhere.

Question 3.4. A mathematical notation is whatever it is defined to be, no more and no less. Isn't $.999\ldots$ defined to be equal to 1?

Answer. As far as teaching is concerned, it is not necessarily up to research mathematicians to decide what is good notation and what is not, but rather should be determined by the teaching profession and its needs, particularly when it comes to students who have not yet been introduced to $\mathbb{R}$ and lim.

Question 3.5. In its normal context, $.999\ldots$ is defined unambiguously, shouldn't it therefore be taught as a single mathematical object?

Answer. Indeed, in the context of ZFC standard reals and the appropriate notion of limit, the definition is unambiguous. The issue here is elsewhere: what does $.999\ldots$ *look like* to highschoolers when they are exposed to the problem of unital evaluation, *before* learning about $\mathbb{R}$ and lim.[12]

Question 3.6. Don't standard analysis texts provide a unique definition of $.999\ldots$ that is almost universally accepted, as a certain infinite sum that (independently) happens to evaluate to 1?

Answer. More precisely, it is a limit of finite sums, whereas "infinite sum" is a figurative way of describing the limit. Note that the hyperreal sum from 1 to H, where H is an infinite hyperinteger, can also be described as an "infinite sum", or more precisely H-infinite sum, for a choice of a hypernatural number H.

Question 3.7. There are certain operations that happen to work with "formal" manipulation, such as dividing each digit by 3 to result[13] in $0.333\ldots$ But shouldn't such manipulation be taught as merely a convenient shortcut that happens to work but needs to be verified independently with a rigorous argument before it is accepted?

Answer. Correct. The best rigorous argument, of course, is that the sequence

$$\langle .9, .99, .999, \ldots \rangle$$

gets closer and closer to 1 (and therefore 1 is the limit by definition). The students would most likely find the remark before the parenthesis, unobjectionable. Meanwhile, the parenthetical remark is unintelligible to them, unless they have already taken calculus.

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[12]A preferred choice of a hyperreal evaluation of the symbol ".999..." is described in the answers to Questions 8.2, 8.3, and 8.4.

[13]The long division of 1 by 3 and its implications for unital evaluation are discussed in detail in the answer to Question 8.1.

4. Meanings, standard and non-standard

Question 4.1. Isn't it very misleading to change the standard meaning of .999..., even though it may be convenient? This is in the context of standard analysis, since non-standard analysis is not taught very often because it has its own set of issues and complexities.

Answer. In the fall of '08, a course in calculus was taught using H. Jerome Keisler's textbook *Elementary Calculus* [33]. The course was taught to a group of 25 freshmen. The TA had to be trained as well, as the material was new to the TA. The students have never been so excited about learning calculus, according to repeated reports from the TA. Two of the students happened to be highschool teachers (they were somewhat exceptional in an otherwise teenage class). They said they were so excited about the new approach that they had already started using infinitesimals in teaching basic calculus to their 12th graders. After the class was over, the TA paid a special visit to the professor's office, so as to place a request that the following year, the course should be taught using the same approach. Furthermore, the TA volunteered to speak to the chairman personally, so as to impress upon him the advantages of non-standard calculus. The .999... issue was not emphasized in the class.[14]

Question 4.2. Non-standard calculus? Didn't Errett Bishop explain already that non-standard calculus constituted a debasement of meaning?

Answer. Bishop did refer to non-standard calculus as a *debasement of meaning* in his *Crisis* text [13] from '75. He clarified what it was exactly he had in mind when he used this expression, in his *Schizophrenia* text [15]. The latter text was distributed two years earlier, more precisely in '73, according to M. Rosenblatt [50, p. ix]. Bishop writes as follows [15, p. 1]:

> Brouwer's criticisms of **classical mathematics** [emphasis added–MK] were concerned with what I shall refer to as "the debasement of meaning".

In Bishop's own words, the *debasement of meaning* expression, employed in his *Crisis* text to refer to non-standard calculus, was initially launched as a criticism of *classical mathematics* as a whole. Thus his criticism of non-standard calculus was foundationally, not pedagogically, motivated.

In a way, Bishop is criticizing apples for not being oranges: the critic (Bishop) and the criticized (Robinson's non-standard analysis) do not share a common foundational framework. Bishop's preoccupation with the extirpation of the law of excluded middle (LEM)[15] led him to criticize classical mathematics as a whole in as vitriolic[16] a manner as his criticism of non-standard analysis.

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[14]Hyperreal pedagogy is analyzed in the answer to Question 6.10.

[15]A defining feature of both intuitionism and Bishop's constructivism is a rejection of LEM; see footnote 35 for a discussion of Bishop's foundational posture within the spectrum of Intuitionistic sensibilities.

[16]Historian of mathematics J. Dauben noted the vitriolic nature of Bishop's remarks, see [22, p. 139]; M. Artigue [2] described them as *virulent*; D. Tall [61], as *extreme*.

Question 4.3. Something here does not add up. If Bishop was opposed to the rest of classical mathematics, as well, why did he reserve special vitriol for his book review of Keisler's textbook on non-standard calculus?

Answer. Non-standard analysis presents a formidable philosophical challenge to Bishopian constructivism, which may, in fact, have been anticipated by Bishop himself in his foundational speculations, as we explain below.

While Bishop's constructive mathematics (unlike Brouwer's intuitionism[17]) is uniquely concerned with finite operations on the integers, Bishop himself has speculated that "the primacy of the integers is not absolute" [12, p. 53]:

> It is an **empirical fact** [emphasis added–MK] that all [finitely performable abstract calculations] reduce to operations with the integers. There is no reason mathematics should not concern itself with finitely performable abstract operations of other kinds, in the event that such are ever discovered [...]

Bishop hereby acknowledges that the *primacy of the integers* is merely an *empirical fact*, i.e. an empirical observation, with the implication that the observation could be contradicted by novel mathematical developments. Non-standard analysis, and particularly non-standard calculus, may have been one such development.

Question 4.4. How is a theory of infinitesimals such a novel development?

Answer. Perhaps Bishop sensed that a rigorous theory of infinitesimals is both
- not reducible to finite calculations on the integers, and yet
- accomodates a finite performance of abstract operations,

thereby satisfying his requirements for coherent mathematics. Having made a foundational commitment to the *primacy of the integers* (a state of mind known as *integr-ity* in Bishopian constructivism; see [15, p. 4]) through his own work and that of his disciples starting in the late sixties, Bishop may have found it quite impossible, in the seventies, to acknowledge the existence of "finitely performable abstract operations of other kinds".

Birkhoff reports that Bishop's talk at the workshop was not well-received.[18] The list of people who challenged him (on a number of points) in the question-and-answer session that followed the talk, looks like the who-is-who of 20th century mathematics.

Question 4.5. Why didn't all those luminaries challenge Bishop's *debasement* of non-standard analysis?

Answer. The reason is a startling one: there was, in fact, nothing to challenge him on. Bishop did not say a word about non-standard analysis in his oral presentation, according to a workshop participant [40] who attended his talk.[19] Bishop appears to

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[17]Bishop rejected both Kronecker's finitism and Brouwer's theory of the continuum.

[18]See Dauben [22, p. 133] in the name of Birkhoff [10, p. 505].

[19]The participant in question, historian of mathematics P. Manning, was expecting just this sort of critical comment about non-standard analysis from Bishop, but the comment never came. Manning wrote as follows on the subject of Bishop's statement on non-standard calculus published in the written

have added the *debasement* comment after the workshop, at the galley proof stage of publication. This helps explain the absence of any critical reaction to such *debasement* on the part of the audience in the discussion session, included at the end of the published version of Bishop's talk.

Question 4.6. On what grounds did Bishop criticize classical mathematics as deficient in numerical meaning?

Answer. The quest for greater numerical meaning is a compelling objective for many mathematicians. Thus, as an alternative to an indirect proof (relying on LEM) of the irrationality of $\sqrt{2}$, one may favor a direct proof of a concrete lower bound, such as $\frac{1}{3n^2}$ for the error $|\sqrt{2} - \frac{m}{n}|$ involved. Bishop discusses this example in [15, p. 18]. More generally, one can develop a methodology that seeks to enhance classical arguments by eliminating the reliance on LEM, with an attendant increase in numerical meaning. Such a methodology can be a useful *companion* to classical mathematics.

Question 4.7. Given such commendable goals, why haven't mainstream mathematicians adopted Bishop's constructivism?

Answer. The problem starts when LEM-extirpation is elevated to the status of the supreme good, regardless of whether it is to the benefit, or detriment, of numerical meaning. Such a radical, anti-LEM species of constructivism tends to be posited, not as a *companion*, but as an *alternative*, to classical mathematics. Philosopher of mathematics G. Hellman [28, p. 222] notes that "some of Bishop's remarks (1967) suggest that his position belongs in [the radical constructivist] category".

For instance, Bishop wrote [12, p. 54] that "[v]ery possibly classical mathematics will cease to exist as an independent discipline." He challenged his precursor Brouwer himself, by describing the latter's theory of the continuum as a "semimystical theory" [11, p. 10]. Bishop went as far as evoking the term "schizophrenia in contemporary mathematics", see [15].

Question 4.8. How can the elimination of the law of excluded middle be detrimental to numerical meaning?

Answer. In the context of a discussion of the differentiation procedures in Leibniz's infinitesimal calculus, D. Jesseph [30, Section 1] points out that

> [t]he algorithmic character of this procedure is especially important, for it makes the calculus applicable to a vast array of curves whose study had previously been undertaken in a piecemeal fashion, without an underlying unity of approach.

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version [13] of his talk: "I do not remember that any such statement was made at the workshop and doubt seriously that it was in fact made. I would have pursued the issue vigorously, since I had a particular point of view about the introduction of non-standard analysis into calculus. I had been considering that question somewhat in my attempts to understand various standards of rigor in mathematics. The statement would have fired me up."

The algorithmic, computational, numerical meaning of such computations persists after infinitesimals are made rigorous in Robinson's approach, relying as it does on classical logic, incorporating the law of excluded middle (LEM).

To cite an additional example, note that Euclid himself has recently been found lacking, constructively speaking, by M. Beeson [7]. The latter rewrote as many of Euclid's geometric constructions as he could while avoiding "test-for-equality" constructions (which rely on LEM). What is the status of those results of Euclid that resisted Beeson's reconstructivisation? Are we prepared to reject Euclid's constructions as lacking in meaning, or are we, rather, to conclude that their meaning is of a post-LEM kind?

Question 4.9. Are there examples of post-LEM numerical meaning from contemporary research?

Answer. In contemporary proof theory, the technique of proof mining is due to Kohlenbach, see [35]. A logical analysis of classical proofs (i.e., proofs relying on classical logic) by means of a proof-theoretic technique known as proof mining, yields explicit numerical bounds for rates of convergence, see also Avigad [4].

Question 4.10. But hasn't Bishop shown that meaningful mathematics is mathematics done constructively?

Answer. If he did, it was by a sleight-of-hand of a successive reduction of the meaning of "meaning". First, "meaning" in a lofty epistemological sense is reduced to "numerical meaning". Then "numerical meaning" is further reduced to the avoidance of LEM.

Question 4.11. Haven't Brouwer and Bishop criticized formalism for stripping mathematics of any meaning?

Answer. Thinking of formalism in such terms is a common misconception. The fallacy was carefully analyzed by Avigad and Reck [5].

From the cognitive point of view, the gist of the matter was summarized in an accessible fashion by D. Tall [63, chapter 12]:

> The aim of a formal approach is not the stripping away of all human intuition to give absolute proof, but the careful organisation of formal techniques to support human creativity and build ever more powerful systems of mathematical thinking.

Hilbert sought to provide a finitistic foundation for mathematical activity, at the meta-mathematical level. He was prompted to seek such a foundation as an alternative to set theory, due to the famous paradoxes of set theory, with "the ghost of Kronecker" (see [5]) a constant concern. Hilbert's finitism was, in part, a way of answering Kronecker's concerns (which, with hindsight, can be described as intuitionistic/constructive).

Hilbert's program does not entail any denial of meaning at the mathematical level. A striking example mentioned by S. Novikov [44] is Hilbert's Lagrangian for general relativity, a deep and *meaningful* contribution to both mathematics and physics. Unfortunately, excessive rhetoric in the heat of debate against Brouwer had given rise to

the famous quotes, which do not truly represent Hilbert's position, as argued in [5]. Hilbert's Lagrangian may in the end be Hilbert's most potent criticism of Brouwer, as variational principles in physics as yet have no intuitionistic framework, see [6, p. 22].

Question 4.12. Why would one want to complicate the students' lives by introducing infinitesimals? Aren't the real numbers complicated enough?

Answer. The traditional approach to calculus using Weierstrassian epsilontics (the epsilon-delta approach) is a formidable challenge to even the gifted students.[20] Infinitesimals provide a means of simplifying the technical aspect of calculus, so that more time can be devoted to conceptual issues.

5. HALMOS ON INFINITESIMAL SUBTLETIES

Question 5.1. Aren't you exaggerating the difficulty of Weierstrassian epsilontics, as you call it? If it is so hard, why hasn't the mathematical community discovered this until now?

Answer. Your assumption is incorrect. Some of our best and brightest have not only acknowledged the difficulty of teaching Weierstrassian epsilontics, but have gone as far as admitting their own difficulty in learning it! For example, Paul Halmos recalls in his autobiography [27, p. 47]:

> ... I was a student, sometimes pretty good and sometimes less good. Symbols didn't bother me. I could juggle them quite well ...[but] I was stumped by the infinitesimal subtleties of epsilonic analysis. I could read analytic proofs, remember them if I made an effort, and reproduce them, sort of, but I didn't really know what was going on.

(quoted in A. Sfard [53, p. 44]). The eventual resolution of such pangs in Halmos' case is documented by Albers and Alexanderson [1, p. 123]:

> ... one afternoon something happened ... suddenly I understood epsilon. I understood what limits were ... All of that stuff that previously had not made any sense became obvious ...

Is Halmos' liberating experience shared by a majority of the students of Weierstrassian epsilontics?

Question 5.2. I don't know, but how can one possibly present a construction of the hyperreals to the students?

Answer. You are surely aware of the fact that the construction of the reals (Cauchy sequences or Dedekind cuts) is not presented in a typical standard calculus class.[21] Rather, the instructor relies on intuitive descriptions, judging correctly that there is no reason to get bogged down in technicalities. There is no more reason to present a construction

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[20]including Paul Halmos; see [53], as well as the answer to Question 5.1 below.

[21]The issue of constructing number systems is discussed further in the answers to Questions 7.1 and 7.3.

of infinitesimals, either, so long as the students are given clear ideas as to how to perform arithmetic operations on infinitesimals, finite numbers, and infinite numbers. This replaces the rules for manipulating limits found in the standard approach.[22]

Question 5.3. Non-standard analysis? Didn't Halmos explain already that it is too special?

Answer. P. Halmos did describe non-standard analysis as *a special tool, too special* [27, p. 204]. In fact, his anxiousness to evaluate Robinson's theory may have involved a conflict of interests. In the early '60s, Bernstein and Robinson [9] developed a non-standard proof of an important case of the invariant subspace conjecture of Halmos', and sent him a preprint. In a race against time, Halmos produced a standard translation of the Bernstein-Robinson argument, in time for the translation to appear in the same issue of *Pacific Journal of Mathematics*, alongside the original. Halmos invested considerable emotional energy (and *sweat*, as he memorably puts it in his autobiography[23]) into his translation. Whether or not he was capable of subsequently maintaining enough of a detached distance in order to formulate an unbiased evaluation of non-standard analysis, his blunt unflattering comments appear to retroactively justify his translationist attempt to deflect the impact of one of the first spectacular applications of Robinson's theory.

Question 5.4. How would one express the number π in Lightstone's ".999...;...999" notation?

Answer. Certainly, as follows:

$$3.141\ldots;\ldots d_{H-1}d_H d_{H+1}\ldots$$

The digits of a standard real appearing after the semicolon are, to a considerable extent, determined by the digits before the semicolon. The following interesting fact might begin to clarify the situation. Let

$$d_{\min}$$

be the least digit occurring infinitely many times in the standard decimal expansion of π. Similarly, let

$$d_{\min}^{\infty}$$

be the least digit occurring in an infinite place of the extended decimal expansion of π. Then the following equality holds:

$$d_{\min} = d_{\min}^{\infty}.$$

This equality indicates that our scant knowledge of the infinite decimal places of π is not due entirely to the "non-constructive nature of the classical constructions using the axiom of choice", as has sometimes been claimed; but rather to our scant knowledge of the standard decimal expansion: no "naturally arising" irrationals are known to possess infinitely many occurrences of any specific digit.

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[22]See the answer to Question 7.3 for more details on the ultrapower construction.

[23]Halmos wrote [27, p. 204]: "The Bernstein-Robinson proof [of the invariant subspace conjecture of Halmos'] uses non-standard models of higher order predicate languages, and when [Robinson] sent me his reprint I really had to sweat to pinpoint and translate its mathematical insight."

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Question 5.5. What does the odd expression "H-infinitely many" mean exactly?

Answer. A typical application of an infinite hyperinteger H is the proof of the extreme value theorem.[24] Here one partitions the interval, say $[0,1]$, into H-infinitely many equal subintervals (each subinterval is of course infinitesimally short). Then we find the maximum x_{i_0} among the $H+1$ partition points x_i by the transfer principle,[25] and point out that by continuity, the standard part of the hyperreal x_{i_0} gives a maximum of the real function.

Question 5.6. I am still bothered by changing the meaning of the notation $.999\ldots$ as it can be misleading. I recall I was taught that it is preferable to use the y' or y_x notation until one is familiar with derivatives, since dy/dx can be very misleading even though it can be extremely convenient. Shouldn't it be avoided?

Answer. There may be a reason for what you were taught, already noted by Bishop Berkeley[26] [8] nearly 300 years ago! Namely, standard analysis has no way of justifying these manipulations rigorously. The introduction of the notation dy/dx is postponed in the standard approach, until the students are already comfortable with derivatives, as the implied *ratio* is thought of as misleading.

Meanwhile, mathematician and leading mathematics educator D. Tall writes as follows [63, chapter 11]:

> What is far more appropriate for beginning students is an approach build-
> ing from experience of dynamic embodiment and the familiar manipula-
> tion of symbols in which the idea of dy/dx as the ratio of the components
> of the tanget vector is fully meaningful.

Tall writes that, following the adoption of the limit concept by mathematicians as the basic one,

> [s]tudents were given emotionally charged instructions to avoid thinking
> of dy/dx as a ratio, because it was now seen as a limit, even though the
> formulae of the calculus operated as if the expression were a ratio, and
> the limit concept was intrinsically problematic.

With the introduction of infinitesimals such as Δx, one defines the derivative $f'(x)$ as

$$f'(x) = st(\Delta y/\Delta x),$$

where "st" is the standard part function.[27] Then one sets $dx = \Delta x$, and defines $dy = f'(x)dx$. Then $f'(x)$ is truly the ratio of two infinitesimals: $f'(x) = dy/dx$, as envisioned by the founders of the calculus[28] and justified by Robinson.

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[24]See Appendix A, item A.9 for details.

[25]See Appendix A, item A.1.

[26]See footnote 8 and main text there.

[27]See Appendix A, item A.3 and item A.5.

[28]Schubring [52, p. 170, 173, 187] attributes the first *systematic* use of infinitesimals as a foundational concept, to Johann Bernoulli (rather than Newton or Leibniz).

6. A CARDINAL ISSUE

Question 6.1. How does one relate hyperreal infinities to cardinality? It still isn't clear to me what "*H*-infinitely many 9s" means. Is it $\aleph_0$, $\aleph_1$, the continuum, or something else?

Answer. Since there exist *countable* Skolem non-standard models of arithmetic [54], the *short* answer to your question is "$\aleph_0$". Every non-standard natural number in such a model will of course have only countably many numbers smaller than itself, and therefore every extended decimal will have only countably many digits.

Question 6.2. How do you go from Skolem to *point, nine recurring*?

Answer. Skolem [54] already constructed non-standard models of arithmetic a quarter century before A. Robinson. Following the work of J. Avigad [3], it is possible to capture a significant fragment of non-standard calculus, in a very weak logical language; namely, in the language of primitive recursive arithmetic (PRA), in the context of the *fraction field* of Skolem's non-standard model. Avigad gives an explicit syntactic translation of the nonstandard theory to the standard theory. In the fraction field of Skolem's non-standard model, equality, thought of as a two-place relation, is *interpreted* as the relation $=_*$ of being infinitely close. A "real number" can be thought of as an equivalence class relative to such a relation, though the actual construction of the quotient space ("the continuum") transcends the PRA framework.

The integer part (i.e. floor) function $[x]$ is primitive recursive due to the existence of the Euclidean algorithm of long division. Thus, we have $[m/n] = 0$ if $m < n$, and similarly $[m + n/n] = [m/n] + 1$. Furthermore, the digits of a decimal expansion are easily expressed in terms of the integer part. Hence the digits are primitive recursive functions. Thus the PRA framework is sufficient for dealing with the issue of *point, nine recurring*. Such an approach provides a common extended decimal "kernel" for most theories containing infinitesimals, not only Robinson's theory.

Question 6.3. What's the *long* answer on cardinalities?

Answer. On a deeper level, one needs to get away from the naive cardinals of Cantor's theory,[29] and focus instead on the distinction between a language and a model. A language (more precisely, a theory in a language, such as first order logic) is a collection of propositions. One then interprets such propositions with respect to a particular model.

A key notion here is that of an internal set.[30] Each set S of reals has a natural extension S^* over $\mathbb{R}^*$, but also atomic elements of $\mathbb{R}^*$ are considered internal, so the collection of internal sets is somewhat larger than just the natural extensions of real sets.

A key observation is that, when the language is being applied to the non-standard extension, the propositions are being interpreted as applying only to internal sets, rather than to all sets.

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[29]which cannot be used in any obvious way as individual numbers in an extended number system.
[30]See Appendix A, item A.2.

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In more detail, there is a certain set-theoretic construction[31] of the hyperreal theory $\mathbb{R}^*$, but the language will be interpreted as applying only to internal sets and not all set-theoretic subsets of $\mathbb{R}^*$.

Such an interpretation is what makes it possible for the transfer principle to hold, when applied to a theory in first order language.

Question 6.4. I still have no idea what the extended decimal expansion is.

Answer. In Robinson's theory, the set of standard natural numbers $\mathbb{N}$ is imbedded inside the collection of hyperreal natural numbers,[32] denoted $\mathbb{N}^*$. The elements of the difference $\mathbb{N}^* \setminus \mathbb{N}$ are sometimes called (positive) infinite hyperintegers, or non-standard integers.

The standard decimal expansion is thought of as a string of digits labeled by $\mathbb{N}$. Similarly, Lightstone's extended expansion can be thought of as a string labeled by $\mathbb{N}^*$. Thus an extended decimal expansion for a hyperreal in the unit interval will appear as

$$a = .a_1 a_2 a_3 \ldots ; \ldots a_{H-2} a_{H-1} a_H \ldots$$

The digits before the semicolon are the "standard" ones (i.e. the digits of $\mathrm{st}(a)$, see Appendix A). Given an infinite hyperinteger H, the string containing H-infinitely many 9s will be represented by

$$.999 \ldots ; \ldots 999$$

where the last digit 9 appears in position H. It falls short of 1 by the infinitesimal amount $1/10^H$.

Question 6.5. What happens if one decreases $.999 \ldots ; \ldots 999$ further, by the same infinitesimal amount $1/10^H$?

Answer. One obtains the hyperreal number

$$.999 \ldots ; \ldots 998,$$

with digit "8" appearing at infinite rank H.

Question 6.6. You mention that the students have not been taught about $\mathbb{R}$ and lim before being introduced to non-terminating decimals. Perhaps the best solution is to delay the introduction of non-terminating decimals? What point is there in seeking the "right" approach, if in any case the students will not know what you are talking about?

Answer. How would you propose to implement such a scheme? More specifically, just how much are we to divulge to the students about the result of the long division[33] of 1 by 3?

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[31]See discussion of the ultrapower construction in Section 7

[32]A non-standard model of arithmetic is in fact sufficient for our purposes; see the answer to Question 6.1

[33]This long division is analyzed in the answer to Question 8.1 below

Question 6.7. Just between the two of us, in the end, there is still no theoretical explanation for the strict inequality $.999\ldots < 1$, is there? You did not disprove the equality $.999\ldots = 1$. Are there any schoolchildren that could understand Lightstone's notation?

Answer. The point is not to teach Lightstone's notation to schoolchildren, but to broaden their horizons by mentioning the existence of arithmetic frameworks where their "hunch" that $.999\ldots$ falls short of 1, can be justified in a mathematically sound fashion, consistent with the idea of an "infinite string of 9s" they are already being told about. The underbrace notation

$$\underbrace{.999\ldots}_{H} = 1 - \frac{1}{10^H}$$

may be more self-explanatory than Lightstone's semicolon notation; to emphasize the infinite nature of the non-standard integer H, one could denote it by the traditional infinity symbol ∞, so as to obtain a strict inequality[34]

$$\underbrace{.999\ldots}_{\infty} < 1,$$

keeping in mind that the left-hand side is an infinite terminating extended decimal.

Question 6.8. The multitude of bad teachers will stumble and misrepresent whatever notation you come up with. For typesetting purposes, Lighthouse's notation is more suitable than your underbrace notation. Isn't an able mathematician committing a capital sin by promoting a pet viewpoint, as the cure-all solution to the problems of math education?

Answer. Your assessment is that the situation is bleak, and the teachers are weak. On the other hand, you seem to be making a hidden assumption that the status-quo cannot be changed in any way. Without curing all ills of mathematics education, one can ask what educators think of a specific proposal addressing a specific minor ill, namely student frustration with the problem of unital evaluation.

One solution would be to dodge the discussion of it altogether. In practice, this is not what is done, but rather the students are indeed presented with the claim of the evaluation of $.999\ldots$ to 1. This is done before they are taught $\mathbb{R}$ or lim. The facts on the ground are that such teaching is indeed going on, whether in 12th grade (or even earlier, see [65]) or at the freshman level.

Question 6.9. Are hyperreals conceptually easier than the common reals? Will modern children interpret sensibly "infinity minus one," say?

Answer. David Tall, a towering mathematics education figure, has published the results of an "interview" with a pre-teen, who quite naturally developed a number system where 1, 2, 3 can be added to "infinity" to obtain other, larger, "infinities". This indicates that the idea is not as counterintuitive as it may seem to us, through the lens of our standard education.

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[34]See answer to Question 8.3 for a more specific choice of H

Question 6.10. If the great Kronecker could not digest Cantor's infinities, how are modern children to interpret them?

Answer. No, schoolchildren should not be taught the arithmetic of the hyperreals, no more than Cantorian set theory. On the other hand, the study by K. Sullivan [57] in the Chicago area indicates that students following the non-standard calculus course were better able to interpret the sense of the mathematical formalism of calculus than a control group following a standard syllabus. Sullivan's conclusions were also noted by Artigue [2], Dauben [22], and Tall [58]. A more recent synthesis of teaching frameworks based on non-standard calculus was developed by Bernard Hodgson [29] in '92, and presented at the ICME-7 at Quebec.

Are these students greater than Kronecker? Certainly not. On the other hand, Kronecker's commitment to the ideology of finitism[35] was as powerful as most mathematicians' commitment to the standard reals is, today.

Mathematics education researcher J. Monaghan, based on field studies, has reached the following conclusion [43, p. 248]:

> [...] do infinite numbers of any form exist for young people without formal
> mathematical training in the properties of infinite numbers? The answer
> is a qualified 'yes'.

Question 6.11. Isn't the more sophisticated reader going to wonder why Lightstone stated in [38] that decimal representation is unique, while you are making a big fuss over the nonuniqueness of decimal notation and the strict inequality?

Answer. Lightstone was referring to the convention of replacing each terminating decimal, by a tail of 9s.

Beyond that, it is hard to get into Lightstone's head. Necessarily remaining in the domain of speculation, one could mention the following points. Mathematicians trained in standard decimal theory tend to react with bewilderment to any discussion of a strict inequality ".999..." < 1. Now Lightstone was interested in publishing his popular article on infinitesimals, following his advisor's (Robinson's) approach. There is more than one person involved in publishing an article. Namely, an editor also has a say, and one of his priorities is defining the level of controversy acceptable in his periodical.

Question 6.12. Why didn't Lighstone write down the strict inequality?

Answer. Lightstone could have made the point that all but one extended expansions starting with 999... give a hyperreal value strictly less than 1. Instead, he explicitly reproduces only the expansion equal to 1. In addition, he explicitly mentions an additional expansion–and explains why it does not exist! Perhaps he wanted to stay away from the

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[35]As a lightning introduction to Intuitionism, we note that Kronecker rejected actual (completed) infinity, as did Brouwer, who also rejected the law of excluded middle (which would probably have been rejected by Kronecker, had it been crystallized as an explicit concept by logicians in Kronecker's time). Brouwer developed a theory of the continuum in terms of his "choice sequences". E. Bishop's Constructivism rejects both Kronecker's finitism (Bishop accepts the actual infinity of $\mathbb{N}$) and Brouwer's theory of the continuum, described as "semimystical" by Bishop [11, p. 10].

strict inequality, and concentrate instead on getting a minimal amount of material on non-standard analysis published in a mainstream popular periodical. All this is in the domain of speculation.

As far as the reasons for elaborating a strict non-standard inequality, they are more specific. First, the manner in which the issue is currently handled by education professionals, tends to engender student frustration. Furthermore, the standard treatment conceals the power of non-standard analysis in this particular issue.

7. Circular reasoning, ultrafilters, and Platonism

Question 7.1. Since the construction of the hyperreal numbers depends on that of the real numbers, wouldn't it be extremely easy for people to attack this idea as being circular reasoning?

Answer. Actually, your assumption is incorrect. Just as the reals can be obtained from the rationals as the set of equivalence classes of suitable sequences of rational numbers (namely, the Cauchy ones), so also a version of the hyperreals[36] can be obtained from the rationals as the set of equivalence classes of sequences of rational numbers, modulo a suitable equivalence relation. Such a construction is due to Luxemburg [39]. The construction is referred to as the ultrapower construction, see Goldblatt [25].

Question 7.2. Non-standard analysis? You mean ultrafilters and all that?

Answer. The good news is that you don't need ultrafilters to do non-standard analysis: the axiom of choice is enough.[37]

Question 7.3. Good, because, otherwise, aren't you sweeping a lot under the rug when you teach non-standard analysis to first year students?

Answer. One sweeps no more under the rug than the equivalence classes of Cauchy sequences, which are similarly *not* taught in first year calculus. After all, the hyperreals are just equivalence classes of more general sequences (this is known as the ultrapower construction). What one does *not* sweep under the rug in the hyperreal approach is the notion of infinitesimal which historically was present at the inception of the theory, whether by Archimedes or Leibniz-Newton. Infinitesimals were routinely used in teaching until as late as 1912, the year of the last edition of the textbook by L. Kiepert [34]. This issue was discussed in more detail by P. Roquette [49].

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[36]One does not obtain all elements of $\mathbb{R}^*$ by starting with sequences of rational numbers, but the resulting non-Archimedean extension of $\mathbb{R}$ is sufficient for most purposes of the calculus, cf. Avigad [3].

[37]The comment is, of course, tongue in cheek, but many people seem not to have realized yet that the existence of a free (non-principal) ultrafilter is as much of a consequence of the axiom of choice, as the existence of a maximal ideal (a standard tool in algebra), or the Hahn-Banach theorem (a standard tool in functional analysis). This is as good a place as any to provide a brief unequal time to an opposing view [45]: *We are all Platonists, aren't we? In the trenches, I mean—when the chips are down. Yes, Virginia, there really are circles, triangles, numbers, continuous functions, and all the rest. Well, maybe not free ultrafilters. Is it important to believe in the existence of free ultrafilters? Surely that's not required of a Platonist. I can more easily imagine it as a test of sanity: 'He believes in free ultrafilters, but he seems harmless'.* Needless to say, the author of [45] is in favor of eliminating the axiom of choice–including the countable one.

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Question 7.4. I have a serious problem with Lightstone's notation. I can see it working for a specific infinite integer H, and even for nearby infinite integers of the form $H + n$, where n is a finite integer, positive or negative. However, I do not see how it represents two different integers, for instance H and H^2 on the same picture. For in this case, H^2 is greater than $H + n$ for any finite n. Thus it does not lie in the same infinite collection of decimal places

$$; \ldots 1 \ldots$$

so that one needs even more than a potentially infinite collection of sequences of digits

$$; \ldots ;$$

to cope with all hyperintegers.

Answer. Skolem [54] already constructed non-standard models for arithmetic, many years before Robinson.[38] Here you have a copy of the standard integers, and also many "galaxies". A galaxy, in the context of Robinson's hyperintegers,[39] is a collection of hyperintegers differing by a finite integer. At any rate, one does need infinitely many semicolons if one were to dot all the i's.

Lightstone is careful in his article to discuss this issue. Namely, what is going on to the right of his semicolon is not similar to the simple picture to the left. At any rate, the importance of his article is that he points out that there does exist the notion of an extended decimal representation, where the leftmost galaxy of digits of x are the usual finite digits of $\mathrm{st}(x)$.

8. HOW LONG A DIVISION?

Question 8.1. Long division of 1 by 3 gives $.333\ldots$ which is a very **obvious pattern**. Therefore multiplying back by 3 we get $.999\ldots = 1$. There is nothing else to discuss!

Answer. Let us be clear about one thing: long division of 1 by 3 does *not* produce the infinite decimal $.333\ldots$ contrary to popular belief. What it *does* produce is the sequence $\langle .3, .33, .333, \ldots \rangle$, where the dots indicate the *obvious pattern*.

Passing from a sequence to an infinite decimal is a major additional step. The existence of an infinite decimal expansion is a non-trivial matter that involves the construction of the real number system, and the notion of the limit.

Question 8.2. Doesn't the standard formula for converting every repeated decimal to a fraction show that $.333\ldots$ equals $\frac{1}{3}$ on the nose?

Answer. Converting decimals to fractions was indeed the approach of [65]. However, in a pre-$\mathbb{R}$ environment, one can argue that the formula only holds up to an infinitesimal error, and attempts to "prove" unital evaluation by an appeal to such a formula amount to replacing one article of faith, by another.

To elaborate, note that applying the iterative procedure of long division in the case of $\frac{1}{3}$, does not by itself produce any infinite decimal, no more than the iterative procedure

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[38]See the answer to Question 6.2 for more details.

[39]See Goldblatt [25] for more details

of adding 1 to the outcome of the previous step, produces any infinite integer. Rather, the long division produces the sequence $\langle .3, .33, .333, \ldots \rangle$. Transforming the sequence into an infinite decimal has nothing to do with long division, and requires, rather, an application of the limit concept, in the context of a complete number system.

Note that, if we consider the sequence

$$\langle .9, .99, .999, \ldots \rangle,$$

but instead of taking the limit, take its equivalence class

$$[.9, .99, .999, \ldots]$$

in the ultrapower construction of the hyperreals (see [39, 25]), then we obtain a value equal to $1 - 1/10^{[\mathbb{N}]}$ where $\langle \mathbb{N} \rangle$ is the "natural string" sequence enumerating all the *natural* numbers, whereas $[\mathbb{N}]$ is its equivalence class in the hyperreals. Thus the unital evaluation has a viable competitor, namely, the "natural string" evaluation.

Question 8.3. Isn't it odd that you seem to get a *canonical* representative for ".999..." which falls short of 1?

Answer. The hyperinteger defined by the equivalence class of the sequence $\langle \mathbb{N} \rangle = \langle 1, 2, 3, \ldots \rangle$ only makes sense in the context of the ultrapower construction, and depends on the choices made in the construction. The standard real decimal $(.999\ldots)_{\text{Lim}}$ is defined as the *limit* of the sequence $(.9, .99, .999\ldots)$, and the hyperreal $(.999\ldots)_{\text{Lux}}$ is defined as the class of the same sequence in the ultrapower construction. Then

$$(.999\ldots)_{\text{Lim}} = 1 \tag{8.1}$$

is the unital evaluation, interpreting the symbol $.999\ldots$ as a real number, while

$$\underbrace{.999\ldots}_{[\mathbb{N}]} = 1 - \frac{1}{10^{[\mathbb{N}]}}. \tag{8.2}$$

is the natural string evaluation.

In more detail, in the ultrapower construction of $\mathbb{R}^*$, the hyperreal $[.9, .99, .999, \ldots]$, represented by the sequence $\langle .9, .99, .999, \ldots \rangle$, is an infinite terminating string of 9s, with the last nonzero digit occurring at a suitable infinite hyperinteger rank. The latter is represented by the string listing all the natural numbers $\langle 1, 2, 3, \ldots \rangle$, which we abbreviate by the symbol $\langle \mathbb{N} \rangle$. Then the equivalence class $[\mathbb{N}]$ is the corresponding "natural string" hyperinteger. We therefore obtain a hyperreal equal to $1 - \frac{1}{10^{[\mathbb{N}]}}$.

The *unital evaluation* of the symbol $.999\ldots$ has a viable competitor, namely the *natural string evaluation* of this symbol.

Question 8.4. What do you mean by $10^{\mathbb{N}}$? It looks to me like a typical sophomoric error.

Answer. The natural string evaluation yields a hyperreal with Lightstone [38] representation given by $.999\ldots;\ldots9$, with the last digit occurring at non-standard rank $[\mathbb{N}]$. Note that it would be *incorrect* to write

$$(.999\ldots)_{\text{Lux}} = 1 - \frac{1}{10^{\mathbb{N}}} \tag{8.3}$$

since the expression $10^{\mathbb{N}}$ is meaningless, $\mathbb{N}$ not being a number in *any* number system. Meanwhile, the sequence $\langle \mathbb{N} \rangle$ listing all the natural numbers in increasing order, represents an equivalence class $[\mathbb{N}]$ in the ultrapower construction of the hyperreals, so that $[\mathbb{N}]$ is indeed a quantity, more precisely a non-standard integer, or a hypernatural number [25].

Question 8.5. Do the subscripts in $(.999\ldots)_{\text{Lim}}$ and $(.999\ldots)_{\text{Lux}}$ stand for "limited" and "deluxe"?

Answer. No, the subscript "Lim" refers to the unital evaluation obtained by applying the *limit* to the sequence, whereas the subscript "Lux" refers to the natural string evaluation, in the context of Luxemburg's sequential construction of the hyperreals (the ultrapower construction).

Question 8.6. The absence of infinitesimals is certainly not some kind of a shortcoming of the real number system that one would need to apologize for. How can you imply otherwise?

Answer. The standard reals are at the foundation of the magnificent edifice of classical and modern analysis. Ever since their rigorous conception by Weierstrass, Dedekind, and Cantor, the standard reals have faithfully served the needs of generations of mathematicians of many different specialties. Yet the non-availability of infinitesimals has the following consequences:

(1) it distances mathematics from its applications in physics, enginering, and other fields (where nonrigorous infinitesimals are in routine use);
(2) it complicates the logical structure of calculus concepts (such as the limit) beyond the comprehension of a significant minority (if not a majority) of undergraduate students;
(3) it deprives us of a key tool in interpreting the work of such greats as Archimedes, Euler, and Cauchy.

In this sense, the absence of infinitesimals is a shortcoming of the standard number system.

Appendix A. A non-standard glossary

The present section can be retained or deleted at the discretion of the referee. In this section we present some illustrative terms and facts from non-standard calculus [33]. The relation of being infinitely close is denoted by the symbol $\approx$. Thus, $x \approx y$ if and only if $x - y$ is infinitesimal.

A.1. **Natural hyperreal extension f^*.** The *extension principle* of non-standard calculus states that every real function f has a hyperreal extension, denoted f^* and called the natural extension of f. The *transfer principle* of non-standard calculus asserts that every real statement true for f, is true also for f^* (for statements involving any relations). For example, if $f(x) > 0$ for every real x in its domain I, then $f^*(x) > 0$ for every hyperreal x in its domain I^*. Note that if the interval I is unbounded, then I^*

necessarily contains infinite hyperreals. We will sometimes drop the star * so as not to overburden the notation.

A.2. **Internal set.** Internal set is the key tool in formulating the transfer principle, which concerns the logical relation between the properties of the real numbers $\mathbb{R}$, and the properties of a larger field denoted

$$\mathbb{R}^*$$

called the *hyperreal line*. The field $\mathbb{R}^*$ includes, in particular, infinitesimal ("infinitely small") numbers, providing a rigorous mathematical realisation of a project initiated by Leibniz. Roughly speaking, the idea is to express analysis over $\mathbb{R}$ in a suitable language of mathematical logic, and then point out that this language applies equally well to $\mathbb{R}^*$. This turns out to be possible because at the set-theoretic level, the propositions in such a language are interpreted to apply only to internal sets rather than to all sets. Note that the term "language" is used in a loose sense in the above. A more precise term is *theory in first-order logic*. Internal sets include natural extension of standard sets.

A.3. **Standard part function.** The standard part function "st" is the key ingredient in A. Robinson's resolution of the paradox of Leibniz's definition of the derivative as the ratio of two infinitesimals

$$\frac{dy}{dx}.$$

The standard part function associates to a finite hyperreal number x, the standard real x_0 infinitely close to it, so that we can write

$$\mathrm{st}(x) = x_0.$$

In other words, "st" strips away the infinitesimal part to produce the standard real in the cluster. The standard part function "st" is not defined by an internal set (see item A.2 above) in Robinson's theory.

A.4. **Cluster.** Each standard real is accompanied by a cluster of hyperreals infinitely close to it. The standard part function collapses the entire cluster back to the standard real contained in it. The cluster of the real number 0 consists precisely of all the infinitesimals. Every infinite hyperreal decomposes as a triple sum

$$H + r + \epsilon,$$

where H is a hyperinteger, r is a real number in $[0, 1)$, and ϵ is infinitesimal. Varying ϵ over all infinitesimals, one obtains the cluster of $H + r$.

A.5. **Derivative.** To define the real derivative of a real function f in this approach, one no longer needs an infinite limiting process as in standard calculus. Instead, one sets

$$f'(x) = \mathrm{st}\left(\frac{f(x + \epsilon) - f(x)}{\epsilon}\right), \tag{A.1}$$

where ϵ is infinitesimal, yielding the standard real number in the cluster of the hyperreal argument of "st" (the derivative exists if and only if the value (A.1) is independent of the choice of the infinitesimal). The addition of "st" to the formula resolves the centuries-old

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paradox famously criticized by George Berkeley[40] [8] (in terms of the *Ghosts of departed quantities*, cf. [55, Chapter 6]), and provides a rigorous basis for infinitesimal calculus as envisioned by Leibniz.

A.6. **Continuity.** A function f is continuous at x if the following condition is satisfied: $y \approx x$ implies $f(y) \approx f(x)$.

A.7. **Uniform continuity.** A function f is uniformly continuous on I if the following condition is satisfied:

- standard: for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in I$ and for all $y \in I$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
- non-standard: for all $x \in I^*$, if $x \approx y$ then $f(x) \approx f(y)$.

A.8. **Hyperinteger.** A hyperreal number H equal to its own integer part

$$H = [H]$$

is called a hyperinteger (here the integer part function is the natural extension of the real one). The elements of the complement $\mathbb{Z}^* \setminus \mathbb{Z}$ are called infinite hyperintegers, or non-standard integers.

A.9. **Proof of extreme value theorem.** Let H be an infinite hyperinteger. The interval $[0,1]$ has a natural hyperreal extension. Consider its partition into H subintervals of equal length $\frac{1}{H}$, with partition points $x_i = i/H$ as i runs from 0 to H. Note that in the standard setting, with n in place of H, a point with the maximal value of f can always be chosen among the $n + 1$ partition points x_i, by induction. Hence, by the transfer principle, there is a hyperinteger i_0 such that $0 \leq i_0 \leq H$ and

$$f(x_{i_0}) \geq f(x_i) \quad \forall i = 0, \ldots, H. \tag{A.2}$$

Consider the real point

$$c = \mathrm{st}(x_{i_0}).$$

An arbitrary real point x lies in a suitable sub-interval of the partition, namely $x \in [x_{i-1}, x_i]$, so that $\mathrm{st}(x_i) = x$. Applying "st" to the inequality (A.2), we obtain by continuity of f that $f(c) \geq f(x)$, for all real x, proving c to be a maximum of f (see [33, p. 164]).

A.10. **Limit.** We have $\lim_{x \to a} f(x) = L$ if and only if whenever the difference $x - a \neq 0$ is infinitesimal, the difference $f(x) - L$ is infinitesimal, as well, or in formulas: if $\mathrm{st}(x) = a$ then $\mathrm{st}(f(x)) = L$.

Given a sequence of real numbers $\{x_n | n \in \mathbb{N}\}$, if $L \in \mathbb{R}$ we say L is the limit of the sequence and write $L = \lim_{n \to \infty} x_n$ if the following condition is satisfied:

$$\mathrm{st}(x_H) = L \quad \text{for all infinite } H \tag{A.3}$$

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[40]See footnote 8 for a historical clarification

(here the extension principle is used to define x_n for every infinite value of the index). This definition has no quantifier alternations. The standard (ϵ, δ)-definition of limit, on the other hand, does have quantifier alternations:

$$L = \lim_{n \to \infty} x_n \iff \forall \epsilon > 0 \, , \exists N \in \mathbb{N} \, , \forall n \in \mathbb{N} : n > N \implies d(x_n, L) < \epsilon. \qquad \text{(A.4)}$$

A.11. **Non-terminating decimals.** Given a real decimal

$$u = .d_1 d_2 d_3 \ldots,$$

consider the sequence $u_1 = .d_1, \; u_2 = .d_1 d_2, \; u_3 = .d_1 d_2 d_3$, etc. Then by definition,

$$u = \lim_{n \to \infty} u_n.$$

Meanwhile, $\lim_{n \to \infty} u_n = \mathrm{st}(u_H)$ for every infinite H. Now if u is a non-terminating decimal, then one obtains a strict inequality $u_H < u$ by transfer from $u_n < u$. In particular,

$$.999\ldots; \ldots\hat{9} = \underbrace{.999\ldots}_{H} = 1 - \tfrac{1}{10^H} < 1, \qquad \text{(A.5)}$$

where the hat $\hat{}$ indicates the H-th Lightstone decimal place. The standard interpretation of the symbol $.999\ldots$ as 1 is necessitated by notational uniformity: the symbol $.a_1 a_2 a_3 \ldots$ in every case corresponds to the limit of the sequence of terminating decimals $.a_1 \ldots a_n$. Alternatively, the ellipsis in $.999\ldots$ could be interpreted as alluding to an infinity of nonzero digits specified by a choice of an infinite hyperinteger $H \in \mathbb{N}^* \backslash \mathbb{N}$. The resulting H-infinite extended decimal string of 9s corresponds to an infinitesimally diminished hyperreal value (A.5). Such an interpretation is perhaps more in line with the naive initial intuition persistently reported by teachers.

A.12. **Integral.** The definite integral of f is the standard part of an infinite Riemann sum $\sum_{i=0}^{H} f(x)\Delta x$, the latter being defined by means of the transfer principle, once finite Riemann sums are in place, see [33] for details.

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High School Teachers' use of dynamic software to generate serendipitous mathematical relations

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Abstract: *In this study, we document and analyse problem-solving approaches that high school teachers exhibited as a result of using dynamic software (Cabri-Geometry) to construct and examine geometric configurations. What type of questions do teachers pose and pursue while representing and exploring mathematical tasks or objects dynamically? To what extent their initial problem solving strategies are enhanced with the use of the tools? Results indicate that the use of the tool offered the participants the opportunity of constructing geometric configuration (formed by simple mathematical objects) that led them to identify and explore key mathematical relations.*

Keywords: CAS; Dynamic geometry; mathematical relations; problem solving; teaching and learning of geometry; problem posing; conics

Introduction

The explosive development and availability of computational tools (Spreadsheets, Computer Algebra Systems –CAS- dynamic software (Cabri-Geometry, Geometry Inventor, Geometer's Sketchpad), and graphic calculators) have notably influenced not only the ways how mathematics is developed, but also how the discipline can be learned or constructed by teachers and students. In particular, tasks or problems' approaches based on the use of the tools offer teachers and students the opportunity of representing and examining the tasks from perspectives that involve visual, numeric, geometric, and algebraic reasoning. Thus, it is important to document the impact and types of

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transformations that the use of those tools brings into school mathematics. As Artigue (2005) mentioned "school, as is the case every time that it faces an evolution of scientific and/or social practices, can neither stand apart from this evolution, nor ignore the new needs it generates" (p. 232).

What type of mathematical competencies becomes relevant for teachers to promote during their instructional practices that enhance the use of computational tools? What hypothetical students' learning trajectories can be identified while examining the tasks through the use of computational tools? What features of mathematical thinking can be enhanced with the use of particular tools? These are important questions that need to be discussed in order to shed light on the relevance, for teachers, to systematically use diverse computational tools in their problem solving approaches. We recognize that different tools may offer various opportunities for teachers and students to identify, represent, and explore relationships embedded in mathematical problems.

> Regardless of the particular tools that are used, they are likely to shape the way we think. Mathematical activity requires the use of tools, and the tools we use influence the way we think about the activity…[Understanding] is made up of many connections or relationships. Some tools help students make certain connections; other tools encourage different connection (Hiebert, et al, 1997, p.10).

The use of dynamic software seems to offer teachers and students the possibility of constructing and analysing mathematical relationships in terms of loci that result from moving elements within the representation of the problem (Santos-Trigo, 2008). In general, the use of the tools can help teachers identify and explore potential instructional trajectories to frame the development of their lessons. In this perspective, it is relevant for teachers to use computational tools to document the type of mathematical thinking that can emerge in students' problem solving approaches.

Conceptual Framework

Fundamental principles associated with problem solving approaches and the use of computational tools were used to organize and structure the development of this study, namely:

(i) The recognition that teachers need to think of their mathematical instruction as a problem solving activity in which contents, problems or phenomena are seen as dilemmas that need to be examined, explained, and solved in terms of formulating and pursuing relevant questions or inquiry methods (Santos-Trigo, 2007). As Postman and Weingartner (1969) stated:

> Knowledge is produced in response to questions…Once you have learned how to ask questions –relevant and appropriate and substantial questions- you have learned how to learn and no one can keep you from learning whatever you want or need to know, (pp.23).

In similar vein, Romberg and Kaput (1999) recognize that teachers should provide proper learning conditions for their students to appreciate, value and develop a mathematical thinking consistent with the practice of the discipline. In particular, they need to participate in genuine mathematical inquiry.

> By genuine inquiry, we mean the process of raising and evaluating questions ground in experience, proposing and developing alternative explanations, marshaling evidence from various sources, representing and presenting that information to a larger community, and debating the persuasive power of that information with respect to various claims (p. 11).

Roschelle, Kaput, and Stroup (2000) recognize that the use of technology plays an important role in mediating the process of inquiry: "Inquiry allows incremental, continual growth of understanding from child's experience to the core subject matter concepts" (p. 50). That is, teachers need to problematize the content they teach by formulating and discussing questions that lead them to identify difficulties that might arise while their students use computational tools in their learning experiences. In this process, they

should constantly reflect on ways to articulate and structure their lesson and problem solving activities.

> Articulation requires reflection in that it involves lifting out the critical ideas of an activity so that the essence of the activity can be communicated. In the process, the activity becomes an object of thought. In other words, in order to articulate our ideas, we must reflect on them in order to identify and describe critical elements (Carpenter & Lehrer, 1999, p.22).

Jaworski (2006) discusses the relation between the notion of inquiry and cognitive and social perspectives.

> Inquiry, or investigative methods in mathematics teaching are seen to fit with constructivist view of knowledge and learning: they demand activity, offer challenges to stimulate mathematical thinking and create opportunities for critical reflection on mathematical understanding (Jaworski, 2006, p. 199).

Further, she also mentioned that:

> While inquiry tools might offer developmental possibilities for individuals within social settings, the prevalence of social norms and processes of social enculturation will be more powerful influences on learning than will cognitive stimulus central in constructivist theory (Jaworski, 2006, p. 200).

(ii) The importance for teachers and students to think of distinct ways to represent, explore or solve mathematics problems. Here, the use of technology might provide a means to examine problems representations from distinct perspectives. Guin and Trouche (2002) recognize the importance of the process, for people, to transform an artefact (a material object) into an instrument or problem-solving tool. This process involves aspects related to both the actual design of the artefact and the cognitive process shown by the user during the appropriation of the tool. Teachers' direct participation in designing mathematical tasks that involves the use of computational tools becomes important not only, for teachers, to recognize and discuss ways to employ the software in problem solving activities; but also to identify and analyse theoretical instructional trajectories that might help their students to transform the tool into a problem solving instrument.

 Students' engagement with, and ownership of, abstract mathematical ideas can be fostered through technology. Technology enriches the range and quality of investigations by providing a means of viewing mathematical ideas from multiple perspectives (The NCTM, 2000, p. 25).

(iii) Learning takes place within a community that promotes that active participation of all members. Thus, it is important to provide an environment in which each member values not only the need to express his/her own ideas; but also to listen and understand other's ideas. As Wells (2001, pp. 179-180) stated:

…the way in which an activity unfolds depends upon the specific participants involved, their potential contribution, and the extent to which the actualisation of this potential is enabled by the interpersonal relationships between participants and the mediating artifacts at hand…Knowledge is constructed and reconstructed between participants in specific situations, using the cultural resources at their disposal, as they work toward the collaborative achievement of goals that emerge in the course of the activity.

Thus, a reflective community promotes activities in which members have the opportunity of posing questions, making observations, using computational tools to identify, represent, explore mathematical relations, and communicate results.

A fundamental aspect of a community of learners is communication. Effective communication requires a foundation of respect and trust among individuals. The ability to engage in the presentation of evidence, reasoned argument, and explanation comes from practice (NRC, 1999, p.50).

The Research Design, the Participants, and General Procedures

Eight high school teachers, all volunteers, participated in three hour weekly problem-solving seminar during one semester. The aim of the sessions was to select, design and work on a set of problems that could eventually be used in regular instruction. In this process, all the participants had opportunities to reveal and discuss their mathematical ideas openly and use both dynamic software and hand-held calculators to solve the problems. The idea was that the participants became familiar with the use of the software

by representing themselves directly the mathematical objects dynamically. In addition, they also explore possible hypothetical learning trajectories that their students could follow during their instructional activities. While solving and discussing all the problems, themes related to curriculum, students' learning, and the evaluation of students' mathematical competences were also addressed. For example, the didactical sequence to study regular contents that appear in courses like analytic geometry was questioned since the use of dynamic software allows the study of the various conic sections dynamically at the same time. That is, one particular task could lead to the discussion of all conic section.

During the development of the sessions, the participants worked on the problems individually and in pairs. And later, they presented their work to the whole group. In general, the instructional activities were organized around a particular pedagogic approach in which the participants were encouraged to use an inquiry process to deal with the problems. As Jaworski (2006) indicates:

> It [the instructional approach] is a social process in the sense that a participant is a member of a community (e.g., of teachers, or of students learning mathematics) with its own practices and dynamics of practice which go through social metamorphoses as inquiry takes place. It is an individual process in that individuals are encouraged to look critically at their own practice and to modify these through their own learning practice (p. 2002).

Data used to analyse the work shown by the participants come from electronic files that they handed in at the end of the sessions and videotapes of the pair work and plenary sessions. In addition, each participant, at the end of the semester, presented the collection of problems and results that they had worked throughout the sessions. Here, it is important to mention that, in this study, we are interested in identifying and discussing problem solving approaches that emerge while high school teachers use computational artefacts rather than analysing in detail individual or small group performances.

Presentation of Results and Discussion

To present the results, we focus on analyzing what the participants exhibited while working on one task that involves the use of Cabri-Geometry software. The participants relied on the use of the software directly to construct a geometric configuration that led them to identify and explore all conic sections. The participants were given two points A & B as the initial objects to construct a geometric configuration to identify mathematical relations. What can you do with two points, A & B situated on the plane? This was the initial question that led the participants to construct dynamic configurations that eventually helped them to identify and explore particular mathematical results.

We focus on presenting initially the work of a pair of teachers to show the type of mathematical activity that they engaged while working on this task. Later, they presented their work to the whole group and received comments and responded to some questions. At this stage, the initial pair's work became the group's work since all participants contributed to the search of both conjectures and arguments around the initial pair's report.

A pair, Martin and Peter, considered drawing line AB, the perpendicular bisector of segment AB and perpendicular lines to line AB (L_1 and L_2) passing by point A and B respectively.
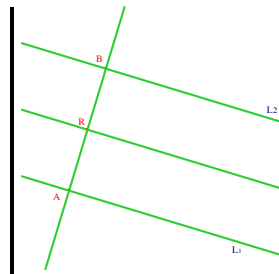
Figure 1: Drawing line AB, the perpendicular bisector of segment AB and perpendicular lines.

Martin and Peter were aware that Cabri Geometry software could be used to draw a conic that passes by five given points. Thus, their initial purpose was to situate five points in order to draw a conic passing by those points. Where should those points be situated? To this end, they drew two points P, Q on line L_2 and reflected those point with respect to the perpendicular bisector of AB to obtain points P' and Q' (figure 2). This arrangement was based on the consideration of the symmetry of an ellipse, for example (as possible candidate). Based on this information they drew (using the software) the conic that passes by the five points.
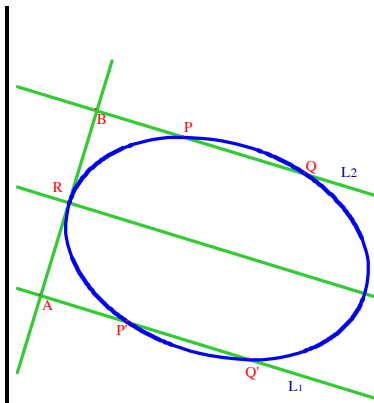
Figure 2: The figure that passes by points P, Q, R, P', and Q' seems to be an ellipse.

This pair recognized that when they decided to draw a conic (using the software command "conic") they thought of situating the five points on lines, so they can move them and observe the behavior of that conic. That is, they intended to construct a dynamic representation of a conic in which they could move certain points within the configuration and observe the behavior of the figure. Indeed, a crucial step to transform

the use of the software into a problem-solving tool is to think of mathematical objects in terms of dynamic representations.

From visual to formal arguments

It is important to mention that when this construction was presented (by the two participants) to the whole seminar, then all agreed that the figure generated by the software represented visually an ellipse (visual recognition). However, some participants stated that it was necessary to present an argument to justify that such figure held properties associated with the ellipse. How do we know that the figure satisfy the definition of the conic? Where are the foci located? Where is its center? These were important questions to answer in order to show that, for that figure, the sum of the distance from the foci to any point on the figure is a constant number (definition of ellipse). At this stage, all members of the seminar began to think of ways to argue that the figure was an ellipse.

During the development of the plenary session, it was recognized that a problem solving strategy to identify and examine properties of the figure was to use the Cartesian system. Thus, without losing generality, the participants decided to take the perpendicular bisector of segment AB as the x-axis. Similarly, they took the perpendicular bisector of segment AB as the ellipse's focal axis, R and R' the points where the focal axis crosses the ellipse, that is, the ellipse's vertices and point C the center of the ellipse as the midpoint of segment RR' (figure 3).
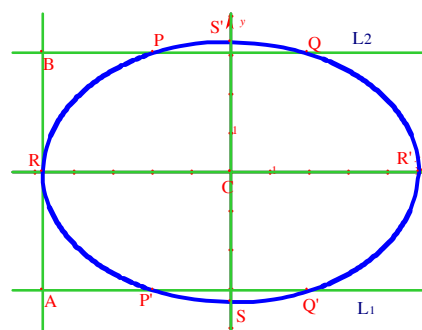
Figure 3: Identifying elements of the ellipse.

It is important to mention that the use of the software allows moving and changing the position of the objects easily. So, Figure 2 could easily be transformed into figure 3.

At this stage, the coordinator of the session (the researcher) asked: What type of equation represents an ellipse where its center is the origin of the Cartesian system? All agreed that the equation of the ellipse could be expressed as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } b^2 = a^2 - c^2$$

Based on this information, they argued that to identify the ellipse's foci it was sufficient to draw a circle with center at point S' and radius the length of segment CR', since the intersection points of that circle with the focal axis will determine the ellipse's foci (figure 4).
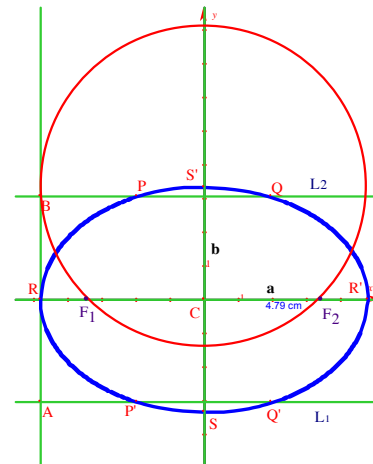
Figure 4: Drawing the ellipse's foci.

To verify that any point on the figure held the definition of the ellipse, Maria suggested to use the software to situate any point M on the figure and observed that by moving point M along the ellipse the sum of distances $F_1M + F_2M$ remains constant (figure 5).
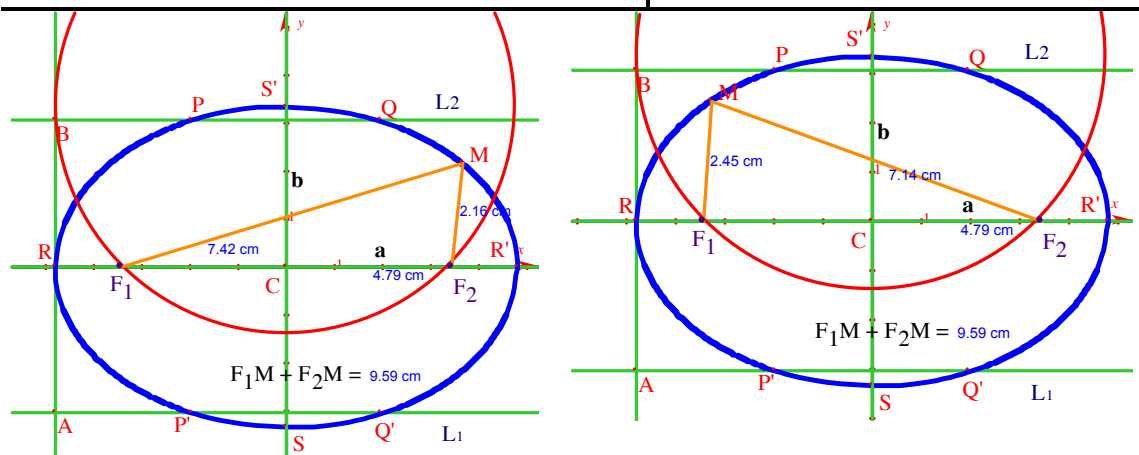
Figure 5: Verifying empirically the definition of the ellipse by moving point M along the ellipse.

Thus, they identified the equation of the above ellipse as $\dfrac{x^2}{(4.79)^2} + \dfrac{y^2}{(3.34)^2} = 1$ (Figure

5a).
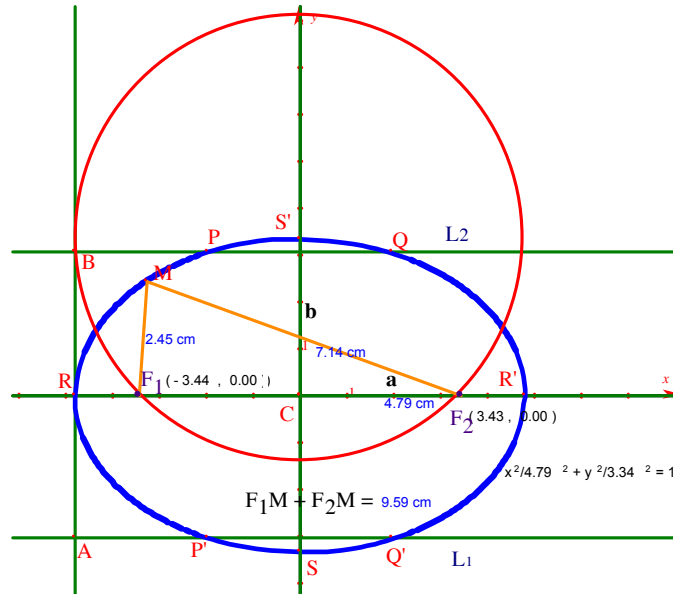
Figure 5a: Finding the equation of the ellipse.

Comment: The use of the tool helped the pair (Martin & Peter) situate five points on the plane to generate a conic section by using the five points software's command. To situate the points, Martin and & Peter relied on the idea of symmetry (P & P' and Q & Q' are symmetric with respect to the perpendicular to line AB that passes through point R) and a line on which points P & Q could be moved freely. Indeed, during the pair's presentation to the group other possibilities to situate the points on the plane were examined. For example, Robert proposed to situate P and Q on a circle with center on line L_2 (Figure 5b).
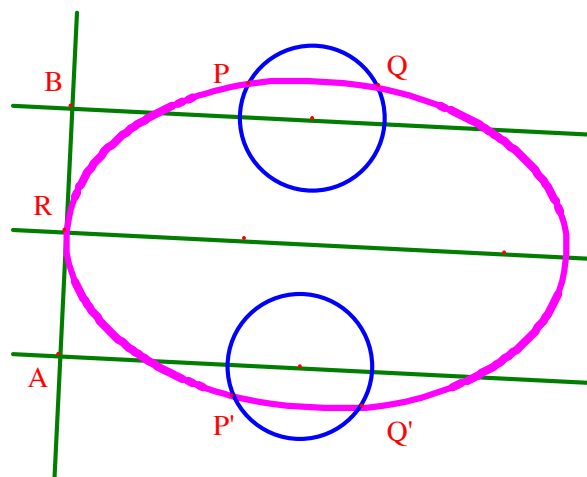
Figure 5b: Drawing a conic section by situating points P & Q on a circle.

It was observed that during the pair's presentation, the other members of the group not only proposed other ways to generate the conic section; but also explored properties attached to that figure. That is, the group became an inquiry community during the pairs' presentations and its members not only posed questions to examine properties of the figure; but also identified possible learning trajectories to frame their students' learning experiences.

Looking for other mathematical relations:
 At this stage, it was natural for the participants to start moving points within the representation to explore the behavior of the figure. Points P and Q are candidates to be moved since P' and Q' depend on the position of P and Q respectively.

What does it happen to the ellipse when points P is moved along line L_2? The software allowed the participants to explore this type of questions since geometric properties of the mathematical objects (perpendicular bisector, symmetric points, etc) are maintained within the construction. For example, when point P is moved along line L_2, they found that at certain position of P the ellipse became a hyperbola (figure 6).
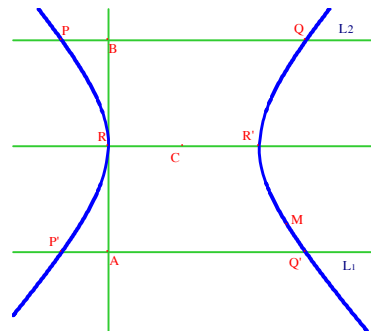
Figure 6: Moving P along line to generate a hyperbola.
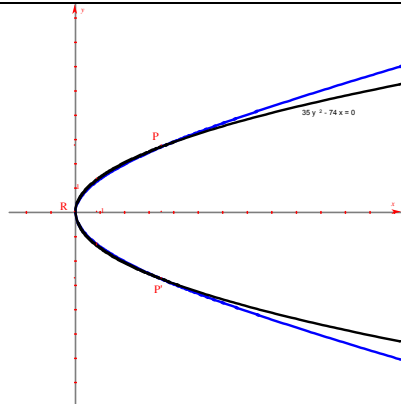
Figure 7: When point P is at infinity, a parabola will show.

Indeed, it could noticed that by moving either point P or Q along line other conics could be generated and their goal was to identify positions of P or Q in which a particular conic appears and also to verify the corresponding properties. In particular, they observed that when point Q goes to infinity a parabola is shown (figure 7).

Based on the idea of drawing symmetric points within a dynamic configuration, the participants proposed other isomorphic ways to draw some conics. For example, David's configuration involves drawing segment AB and lines L_1 and L_2 passing by point B and A respectively. These lines get intersected at point M. Line L_3 is the bisector of angle BMA. On line L_1, he situated points P and Q and reflect them with respect line L_3 to obtain P' and Q'. Using the software, he drew the conic that passes by point P, Q, P', Q' and R (intersection of segment AB and line L_3) to get figure 8. Again by moving point P along L_1 it was observed that at different positions of P other conics appear (figure 9).
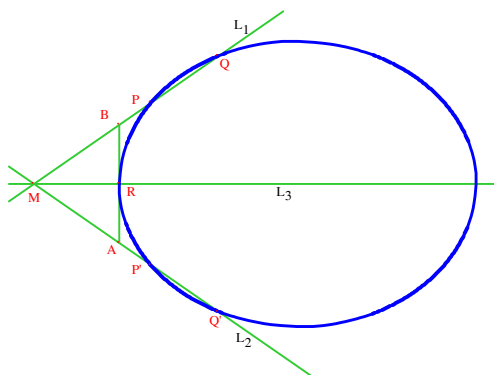
Figure 8: An isomorphic construction to generate all conics.
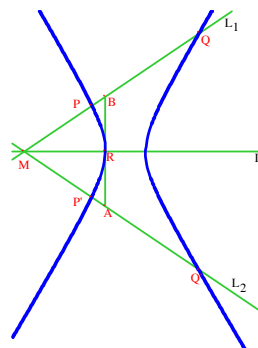
Figure 9: By moving point P along line L_1 other conic sections appear

Thus, the work presented by the Martin and Peter inspired the other participants to think of similar construction to generate the conic sections. That is, the community identified the essential elements associated with Martin & Peter's idea to develop a tool (geometric configuration) to identify and discuss properties associated with the conic sections. In addition, working on this type of tasks helped them recognize the importance of visualizing relations, analyzing particular cases, measuring attributes, and looking for analytical arguments. Thus, there is evidence that the participants have identified a set of strategies that are useful to represent and examine dynamic configurations. It is clear the use of the software facilitates they ways in which teachers and students can apply those problem-solving strategies. As the NCTM (2000) suggests "…no strategy is learned once and for all; strategies are learned over time, are applied in particular contexts, and become more refined, elaborate, and flexible as they are used in increasingly complex problem situations" (p. 54).

Reflections: The participants were surprised that the use of the software helped them identify and explore conic sections properties based on constructing a particular dynamic configuration. All recognized that the use of the software not only can offer their students

the possibility of exploring visually and empirically the behavior of mathematical objects, but it can also be useful to make sense of results expressed algebraically. In addition, the use of the software may help students participate in the process of reconstructing some mathematical results and help them transit from visual and empirical explanations to formal approaches to the problems. Here, all the participants were aware that the use of the software provides a different path to study the conics, compared with traditional approaches based on paper and pencil, and also to develop new results. They also recognized that the use of the software offers the potential of constructing dynamic configurations in which they can identify interesting mathematical relations. In particular, some teachers were surprised to observe that all the conics, that they teach in an entire analytic geometry course, could be generated by moving points within a representation that involves points, perpendicular lines and a perpendicular bisector. In addition, they observed that the software functions as a vehicle to explore relations among objects that are difficult to think of or identify using only paper and pencil approach. Furthermore, the facility to quantify lengths of segments allows the problem solver to identify and explore the plausibility of particular mathematical relations. Thus, the participants conceptualized learning as a process in which their ideas and approaches to tasks are refined as a result of examining openly not only what they think of the problem but also discussing and criticizing the ideas and approaches of other participants. In particular, the participants recognized that initial incoherent attempts to solve the problems could be transformed into robust approaches when the learning environment values and promotes the active participation of the learners. In general, they recognized that the use of the tool not only helped them visualize and explore mathematical relations but also to recognize potential learning trajectories that their students could follow.

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Gender and Mathematics Education in Pakistan: A situation analysis

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Abstract: This paper reports from a situation analysis of gender issues in mathematics education in Pakistan. It was undertaken at the initiation of a large scale project which aimed to understand how curriculum change in mathematics and science education may be implemented in ways that contribute to poverty alleviation and promote gender equity in disadvantaged rural settings. The paper posits that issues of access, achievement and quality of mathematics education are integrally linked with questions of equity, in this case gender equity. It identifies several questions and arenas for further research and makes recommendations for policy and practice in mathematics education.

Key words: gender equity, developing world, disadvantaged settings

INTRODUCTION TO THE CONTEXT AND BACKGROUND

Pakistan's Education system can be broadly divided into 'Basic Education' (primary, elementary and secondary levels) and Higher Education (post secondary and graduate levels). Both are governed by separate ministries with distinct management and financial systems. Mainstream or government schools offer primary education from class I – V (5 to 9) and then middle or elementary schooling, class VI-VIII (10-13) and finally secondary schooling, class IX – X (14 –

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Halai

15). In classes IX and X students take the secondary school matriculation examination which is conducted by the Boards of Secondary Education.

Mathematics is also taught as a compulsory subject in classes I – VIII. The curriculum content is organized mainly into five major strands, number and number operations, measurement, geometry, data handling and algebra. At the secondary level (classes IX & X) students can opt to take science group or general group. The former includes among other subjects physics, chemistry, biology and advanced mathematics (with a greater focus on algebra, functions, and trigonometry). The latter includes humanities and a course in general mathematics (with greater emphasis on arithmetic and less emphasis on algebra, functions and trigonometry). However, in 1995-96 the policy has changed according to which both groups take the same course in mathematics at the secondary level. In Pakistan policy making, including Education Policy and setting the strategic direction is the responsibility of the federal government[2]. Implementation is mostly carried out by the provincial governments and more recently from provincial to districts level. Curriculum development is the purview of the ministry of education in the federal government and is undertaken through a consultative process with the provincial governments through their respective education departments.

The National Curriculum 2006 is organized in five standards which have been kept broad for flexible interpretations. These standards are: I) Numbers and Operations, II) Algebra, III) Measurements and geometry, IV, Information handling, V) Reasoning and logical thinking. This last is a significant new addition because as the documents states it would enable a focus on standards and bench marks for development of mathematical thinking. In the national curriculum for mathematics the teachers' role has been rerouted from dispensing information to planning investigative tasks, managing cooperative learning environment, and supporting students'

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[2] Pakistan is a federation with four provinces i.e. Punjab, Sindh, Baluchistan and North West Frontier Province (NWFP); federally administered areas, Azad Jammu and Kashmir, and the federal capital Islamabad. Sindh and Baluchistan include some of the most poverty stricken regions in the country.

creativity in developing rational understanding of the concepts of mathematics (MoE 2006, National Curriculum for Mathematics grades I – XII, p. 2-3).

Besides the government schools in the mainstream there is burgeoning private sector in education. According to certain reports the private sector provision at primary levels is as large as 30 -33% (Andrabi 2008). However, this report is mainly concerned with the government schools.

The situation analysis reported in this paper was undertaken from 1st June 2006 – 31 May 07 as part of initiating a large action research project to study the process of implementation of curriculum reform to generate knowledge that would improve education quality for the poorest in the world and raise gender equity. The research team included university academics and school teachers as primary participants. Secondary participants included district education officers, head teachers, parents and students. While the research took place in several countries this paper focuses on Pakistan. (for details see www.edqual.org).

The situation analysis involved a literature review, interviews with key education stakeholders on their perceptions and understanding of gender issues in mathematics classrooms and anlaysis of textbooks from a gender perspective. The education stakeholders included teachers, district education officers, head teachers and teacher educators from a poverty stricken district in Sind in southern Pakistan. Selection of the district was made on the basis of a District Education Index (DEI). The DEI investigates the dispersion in the educational status of districts irrespective of their economic status and measures the average shortfall from a perfect score of 100 percent. The closer the value of the DEI is to 100, the better endowed it is with respect to education variables (SPDC, 2002-03). District Thatta was found to be at the bottom quintile amongst all districts in Sindh on the basis of the DEI. From the nine Talukas (sub-district unit) of Thatta, Mirpur Sakro was selected as it is among the most poverty-ridden talukas in the district. Schools for inclusion in the action research project were selected from this taluka. Of a total twelve high schools in taluka Mirpur Sakro four were included in the action research. Hence detailed school profiles were developed of these four schools including information on teacher qualifications. It was found that there was great disparity in teacher qualification in boys and girls schools. Women teachers in the participating schools did not have mathematics as their major in the

undergraduate or graduate studies. The male teachers had studied mathematics at least up to their first degree. The disparity in teacher quality in girls' school and boys' school reflects the general pattern in teacher quality in Pakistan (Warick & Reimers, 1995; SPDC, 2003). It raises other gender issues which are discussed later in this paper.

What follows is a brief review of literature in the area of gender and mathematics education in Pakistan, followed by key issues from the field about access, achievement, teacher and teaching materials in mathematics classrooms.

GENDER AND MATHEMATICS EDUCATION IN PAKISTAN

The relationship between gender and performance in mathematics has received considerable attention in education. In particular the last few decades have seen an increase in research on issues related to boys and girls' access, performance and achievement, and participation in mathematics (Gallaher & Kaufmann, 2005). Providing an overview of the history of schooling (Govinda, 2002) maintains that culturally and traditionally gender roles and expectations have defined males as bread earners and providers and females as care givers to the family at home. In line with these traditional roles and expectations access to schooling and education has been in favour of boys. While most industrially developed nations in the north and west have bridged the disparity in schooling participation rates, many countries in South Asia continue to have significant gaps, with the proportion of girls not attending schools being much higher than that of boys (Jha & Kelleher, 2006).

Pakistani society is segregated strongly along the lines of gender and this segregation is also reflected in the education system. Pakistan takes gender into account when setting up and administering government schools. Boys' schools are those with male students and male teachers, and girls' schools are with female students and female teachers. This is especially the case at elementary and secondary schools levels. Parents prefer to send their girls to a single sex school. In case where secondary schools for girls are not available, parents opt not to send their girls to a co-education secondary school. In case some private schools offer co-education at secondary levels, boys and girls usually sit in separate sections of the same class.

A national survey of schools (grades 4 and 5) in Pakistan studied if this use of gender made any difference to the achievement of male and female students in mathematics and science (Warick

& Reimers, 1995; Warick & Jatoi, 1994). The researchers selected 500 government schools, 1000 teachers, and 11,000 grad 4 and 5 students and 300 supervisors all over the country through standard methods of probability sampling. The sample included 47 percent male, 28 percent female and 25 percent coeducational schools. The researchers gave curriculum based achievement test in mathematics and science and a brief questionnaire to the students of class 4 and 5 of the selected school. They conducted interviews of teachers, head teachers and supervisors of those schools.

It was found that students of male teachers had significantly higher achievement scores in mathematics than students of female teachers in the same grades. However, the study goes on to examine and provide explanations for this finding. In contrast to student gender it was found that teacher gender explained ten times more regarding student differences in their mathematical achievement. However, it was not possible to look at the independent effect of school gender on students' achievement in mathematics as the former overlapped with both student and teacher gender. As such, school gender is likely to be a proxy indicator for student and teacher gender rather than an independent influence on mathematics achievement. The study concluded that rural elementary schools are the main source of gender gap in mathematics achievement. Their most critical deficiency is in the inability of rural schools for female students to retain women teachers with adequate training in mathematics. About 75% of the women teachers come from cities. The study goes on to elaborate the reasons and issues that women teachers from cities face when they are posted in rural areas. Also, the study reports that the gender gap in achievement in mathematics could be caused by higher average level of education for male teachers than for female teachers (Warwick & Reimers, 1995, p.70-72).

The study also provides evidence that the gender gap favouring male teachers is highly significant in rural schools, particularly teachers responsible of more than one grade. It disappears or is reversed in urban setting. For teachers who have university degrees, the gender gap favours women. Female teachers, who cover more curriculum areas, have students with significantly higher achievement scores. It implies that students' achievement depends not only on education of teachers, location of schools but also how the teachers actually teach mathematics. The study highlights the deficiency of rural elementary schools as the main source

of gender gap. The female schools in rural areas generally failed to retain women teachers who are adequately trained. Well qualified women from urban areas do not prefer to teach in rural areas because of the challenging living conditions and also they do not receive any incentive for being transferred in to rural areas.

More than a decade later, the findings of this study are still relevant because gender disparities in access, provision and quality of education still persist. While there is improvement in bringing gender parity to access and quality of education the progress is slow and deep inequities against girls persist (MoE, 2006). For example a recent country gender assessment report by World Bank highlights that only 46% of the sample villages in Sind and Punjab had had an elementary girls' school inside the village. In contrast 87% had a boys' elementary school within the village (World Bank, 2005). Likewise, UNESCO report holds that, access to and participation at primary school level has greatly increased in the region and total enrolment rose by 19%. But a fifth of children of official primary school age remain out of school. Two-thirds of out-of-school children are girls (21 million) and the region has the greatest gender disparities in primary education. Very large disparities are found in Pakistan with a female/male enrolment ratio of 0.74. (UNESCO, 2003-04).

Besides access to basic schooling (and therefore to mathematics), girls' under performance and underachievement in mathematics has been a source of much concern the world over with a widespread belief that males outperform girls in mathematics. (Chipman, 2005). However, internationally since 1980's trends in students' achievement in United States, UK and other technologically developed countries in Europe show that girls are either closing the gap in mathematics achievement or doing better than boys. This trend is also evident in reports from the results of large international studies on student achievement in mathematics and science such as TIMMS[3] 2003 and PISA, 2005 as noted in (MA, 2008). Pakistan is not a participant in TIMMS

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[3] Trends in International Mathematics and Science Study (TIMSS), conducted by the International Association for the Evaluation of Educational Achievement (IEA) which surveys student achievement in mathematics and science. The OECD Programme for International Student Assessment (PISA) which surveys reading, mathematical and scientific literacy levels.

or PISA. However, results of the NEAS[4] a large scale national assessment in mathematics taken at grade four (8-9 yrs) and again at grade eight (12-13 yrs) shows that girls' mean score is higher than boys at grade four level but is lower than boys at grade eight levels. Student achievement in the matric examination results nationwide show that girls are doing better generally than boys in terms of securing the top positions. A closer examination of mathematics results disaggregated for gender is not possible because the Boards of Secondary Education in the public sector do not maintain the data sets for results disaggregated for gender and mathematics. However, the newly launched Aga Khan Examination Board[5] (AKU-EB) in the private sector does maintain results which made it possible to do an analysis of student achievement in mathematics which was separated for boys and girls.

From the two-year examination result data for mathematics, in the AKU-EB has showed that male enrollment is more than the female in both the years; but the performance of females in the examination is significantly better the males. The pass percentage of the females is relatively more than that of the males, and also the high grades are more achieved by females than males. This advantage in favor of girls is also carried through in mathematics (see appendix tables 2 & 3). While the AKU-EB results cannot be taken to represent national trends because it is a private examination board with mostly private schools subscribing to it, it does confirm that in a given sample of students in spite of being proportionally less in number of enrolment the results are better for girls.

The results from AKU-EB and NEAS are confounding and must be probed and considered carefully because it masks several other issues and inequities in mathematics education specifically and secondary education generally and which have been described in some detail in the preceding section. One explanation for girls performing better in matric examination could be

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[4] National Education Assessment System, Ministry of Education Government of Pakistan

[5] Aga Khan University Examination Board (AKU-EB) is a Federal Board of Intermediate and Secondary Education established by Aga Khan University (AKU) in response to demand from schools for more appropriate school examinations. AKU-EB was founded in August 2003 in accordance with Ordinance CXIV of the Government of Pakistan to offer examination services for both Secondary School Certificate (SSC) and Higher Secondary School Certificate (HSSC) throughout Pakistan and abroad.

that girls who do eventually make it to the secondary schools are from a relatively advantaged backgrounds in terms of socio-economic strata, in terms of opportunities available in urban settings or and are therefore doing better than boys. Alternatively, it could be explained that girls are known to do better at items which require recall of facts and information and the matric examination is mainly based on such items therefore girls are doing better in spite of the high odds stacked against them in the context of education in Pakistan. Further research is required to undertake a comparative analysis of achievement in mathematics of boys from similar backgrounds

Moreover as (Khan, 2007) points out that enrolment numbers at primary levels are high but progressively decrease at secondary college and tertiary levels. It is estimated that less than 3% of the 17-23 age group make it to higher education and that the gender ratio in public sector universities is at 30-35%, with female enrolment in university at 63, 110 and male enrolment at 131 861. More significantly for the purpose of this paper, Khan confirms that the women who do pursue higher education, very few go on to study mathematics and natural sciences.

Besides access and achievement an issue in mathematics classrooms is that teachers often perpetuate or reinforce the patterns of gender stereo typing and gender bias with implications for boys and girls as learners of mathematics. For example, research has shown that teachers of all grade levels tend to call more often on boys than girls, ask them more complex questions and provide them more academic feedback and attribute their success to ability. More teachers believe that girls succeed in mathematics because of their hard work (Gilbert & Gilbert, 2002). In Pakistan classrooms also teachers reflected similar views about girls and boys learning mathematics. For example, (Halai, 2006) reports from a survey of a cohort of in-service teachers comprising 80% women and 20% men, that 86% of the cohort agreed with the statement provided in the questionnaire "boys are better mathematicians, do you agree? Provide reasons for your answers". Their main reason was that boys are inherently better and therefore do better at mathematics while girls are well behaved and hardworking and so they succeed. Additionally, teachers maintained that boys were better mathematicians because "boys want to apply their learning to different contexts as they participate in activities like shopping for groceries, etc". while girls "want to rote learn and follow rules because they are expected to stay at home and be obedient" (Halai, 2006, p.118-119).

Curriculum content especially textbooks also reinforce gender stereo types in mathematics. Shah and Ashraf (2006) and Halai (2006) conducted systematic textbook analysis from classes I to V of the Sindh Textbook Board from a gender perspective. Their analysis reveals the following representations for boys/men and girls/women which reflect the gendered dimensions of the roles and responsibilities of women, and the implicit gender bias in the mathematics textbooks:

- Frequency of fe/male characters in illustrations they found that the number of female illustrations depicted was greater (i.e. 89 females compared to 61 males). In contrast, there were more frequently cited male characters in texts (123 males in comparison to only 49 females).

- Additionally, men have been portrayed as more powerful in terms of:

- Variety of professions that they are able to take. For example, females have been depicted in 4 professions as compared to males who represent as many as 12 different professions;

- Variety of attire (Males are seen to wear a variety of attires that suit the jobs that they perform. On the other hand, females are shown to be wearing the local dress shalwar kameez on each and every occasion irrespective of the nature of their jobs)

- Males were shown to be using 'technology' (e.g. operating a computer; riding a motorbike) on six occasions whilst females were depicted in a similar role merely once. Thus, access and variety of technologies are represented to be more characteristic of males.

- Females are shown to participate in games less often (n=2) while males are portrayed more frequently (n=12) in similar activities.

- Males are depicted slightly more frequently (i.e. 26 times) in different locations as compared to females who are shown 19 times. This 'locus of activities' has some important gender implications. It suggests that males have more access and mobility to different places as compared to females who are generally home-bound and have limited access. Hence, males are seen to have more opportunities than females to socialise.

- Male names have been used more frequently (n=111; 73%) as compared to female names (n=41; 27%). Similarly, the names of males have been capitalised on more occasions (n=128; 69%) in comparison to the names of females (n=55; 31%). Such bias is likely to suggest a greater involvement of males in mathematics.

- Furthermore, the different types of male and female possessions are also brought out. Boys have been depicted alongside tractors, cars, robots, etc. that imply training for their future role as men. In contrast, girls have not been projected in such a futuristic role preparation as women. (Shah & Ashraf, 2006)

In sum, there is compelling historical evidence that access and achievement to schooling and therefore to mathematics in the case of Pakistan is strongly in favour of boys. There is progress in improving access but deep rooted cultural and social biases about gender roles and expectations permeate textbooks and teacher thinking.

ACCESS AND EQUITY IN MATHEMATICS CLASSROOMS: CHANGING TRENDS

The discussion so far illustrates that gender segregation permeates the social and cultural norms and practices in Pakistan and this is also true in the context of education. However, a change in societal trends towards girls' education could be seen at the grassroots level. For example, in the course of developing school profiles in the course of the situation analysis it was found that one of the participating schools was documented as a "Boys School" in the records of the district education office. Initial conversation with teachers and principal were also under the assumption that the research team was entering into a boys' school. However as data collection started on the school profile it was discovered that in secondary classes (IX & X) there were 8 girls each sitting with the boys. The head teacher stated that there was no secondary school for girls in the village and the parents wanted the girls to continue with their education. Hence they had been accommodated in the boys' school. On inquiring if the district education office was aware that this was the situation. He stated that community members in particular the seniors in the community were aware of it. He then started to provide an apologetic explanation that he had requested to the district office to upgrade the elementary girls school to the status of secondary school so that the girls did not face this inconvenience. However, the bureaucracy moved very slowly and the application is still pending. This incident of sending girls to co-education schools and of creating spaces for girls' participation in schooling particularly at secondary levels is a major breakthrough in stereotypical attitudes towards gender segregation. Likewise change in societal position about girls and boys participation in schooling can be seen from the influx of private co-education schools in Pakistan. By some accounts the prevalence of private not for

profit schooling is as high as 30% (Andrabi, 2008). While opening basic education to market forces raises other questions, for the purpose here is the fact that almost all private primary and secondary schools are co-education, suggests that the community is ready to send the girls to school irrespective of the school gender.

However welcome it is, the changing trend of providing access to schooling and therefore to mathematic has to be seen in conjunction with quality and equity. Perspective of teachers and head teachers was that with the "same textbooks and syllabus" boys and girls had available the same quality of mathematics teaching once they were in the classroom. However, it was found that girls were being taught by women teachers most of whom had done very little mathematics beyond high school and also had very low self concept as mathematicians. For example, in the interview, one teacher picked up the textbook and pointing to the word problems said "Miss we ourselves do not know how to solve these problems involving volumes of cylindrical shapes. How can we teach problem solving to our students". On the contrary the male teachers maintained that they did not need support in mathematics content knowledge and expressed satisfaction with their teaching because they said that their students all passed in the annual school and matric examinations. While the knowledge gap in teachers could be addressed through intensive in-service provision or through better pre-service preparation and selection, these observations raise other issues. For example, Pajares (2005) maintains that self efficacy beliefs are a significant contributor in decision about whether or not the students would pursue higher education in say mathematics and that "verbal persuasion and vicarious experiences nourished the self efficacy beliefs of girls and women as they set out to meet the challenges to succeed in male dominated academic domains" (p.308). However, in the case of girls in middle and secondary school classrooms in Pakistan are more likely to find teachers who have a low sense of self efficacy in mathematics with concomitant implications for their confidence building and enjoyment of learning mathematics.

Classroom observations showed that the teaching in boys' school was more or less the same as that in the girls' school. It was characterized by teacher explaining to the class mathematics rules and procedures and the students following those rules procedures and memorizing them for recall. This comparable quality was not surprising because in both cases teachers aimed at

preparing students for the matric examination which did not necessarily promote problems solving or other higher order mathematical thinking skills.

However, the variable teacher qualification did have implications for the action research project when it came to introducing the problem solving framework in the classroom. As is reported elsewhere the male teachers' uptake was stronger and more confident. It was evident that teacher in the girls' school lacked confidence and adequate knowledge so that their efforts at introducing problem solving had to be provided a scaffold through co-teaching with the university researchers (Halai, forthcoming).

ACCESS, QUALITY, EQUITY: CONCLUDING REFLECTIONS AND RECOMMENDATIONS

This paper reports on the basis of literature review and field work with teachers, district education officers and other stake holders in mathematics education in a highly gender segregated environment of Pakistan. The findings and discussion so far raises several issues and implications for policy and practice for mathematics education.

First, access to, and entry into mathematics classrooms was largely seen as equity in mathematics education. For example, teachers maintained that both boys and girls did the same mathematics, followed the same textbook and syllabus. Hence there was no inequity in mathematics education once access was provided to girls along with the boys. Second, improving the quality of mathematics curriculum cannot be devoid of concerns of gender equity. For example, it was seen that a pattern persists of female teachers being less mathematically qualified and then their male counterparts. This state had serious negative connotations for the gender segregated classrooms in Pakistan with the implication that girls would be taught mathematics by teachers who were not necessarily academically qualified in mathematics and who had most likely a low self concept as mathematicians. Third, access, quality and equity at the grassroots level in mathematics classrooms is linked to gender equity in the policies and provision of teachers, teacher development and textbook preparation. Unless these two levels of mathematics education are addressed access to mathematics classrooms will not translate into equity in mathematics classrooms. For example, provision of teachers with comparable qualification must be ensured

with appropriate implementation of policies and provision of support for teachers to upgrade their knowledge of mathematics. Likewise, officially prescribed curriculum materials must be developed along guidelines to ensure gender equity. Fourth, improving gender parity in mathematics classrooms is a sociological process rooted in deep seated beliefs about gender roles and expectations. Teachers would need to question their deep rooted assumptions about gender roles in society and its implications for positive or weak role models for mathematics learners. A recommendation for teacher education courses is to build on teachers' lives and experiences and avoid false dichotomies in personal and professional as these dichotomies impede a change in teachers' belief and perceptions about gender and mathematics teaching and learning.

To conclude access, achievement and quality of mathematics education in Pakistan shows persistent gender bias in favour of boys. However, there are signs of change in trends towards girls' access to education and to mathematics. An important element would be to see this trend in conjunction with equity in quality of mathematics education for boys and girls in Pakistan.

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Appendix

Table 1

	Year							
	2007				**2008**			
	Gender				**Gender**			
	F		**M**		**F**		**M**	
Overall status	**Count**	**Column N %**	**Count**	**Column N %**	**Count**	**Column N %**	**Count**	**Column N %**
Absent	12	1.9%	45	4.1%	7	.7%	25	2.0%
Fail	105	16.2%	255	23.3%	124	12.4%	323	25.7%
Pass	516	79.8%	774	70.7%	872	86.9%	907	72.2%
Withheld	14	2.2%	20	1.8%	0	.0%	2	.2%
Total	**647**	**100.0%**	**1094**	**100.0%**	**1003**	**100.0%**	**1257**	**100.0%**

Source: http://www.aku.edu/AKUEB/onlineresults.shtml retrieved October 2008

Table 2

Subject Status	Year							
	2007				2008			
	Gender				Gender			
	F		M		F		M	
	Count	Column N %	Count	Column N %	Count	Column N %	Count	Column N %
Absent	7	1.1%	33	3.0%	5	.5%	15	1.2%
Fail	106	16.4%	187	17.1%	112	11.2%	286	22.8%
Pass	534	82.5%	873	79.8%	886	88.3%	956	76.1%
Withheld	0	.0%	1	.1%	0	.0%	0	.0%
Total	**647**	**100.0%**	**1094**	**100.0%**	**1003**	**100.0%**	**1257**	**100.0%**

Source: http://www.aku.edu/AKUEB/onlineresults.shtml retrieved October 2008

Early Intervention in College Mathematics Courses: A Component of the STEM RRG Program Funded by the US Department of Education

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Abstract: *Student preparedness in entry level college mathematics courses has decreased in the past. In order for students to be successful in these courses, early intervention is required. Texas A&M International University (TAMIU) has implemented an early intervention program, the Mathematics Enrichment Program (MEP), which will target freshman students entering College Algebra courses in Spring 2009. This project is part of the STEM Recruitment, Retention, and Graduation program, recently funded by the US Department of Education. This paper will highlight the components of the project, most importantly the extent of the intervention, how the project is planned, and preliminary results.*

Keywords: early intervention; collegiate mathematics education; retention; STEM RRG programs; Assessment and evaluation

Introduction

There has been a question for academics and institutions of higher education: why are students dropping out and performing poorly in entry-level college mathematics courses? From a recent poll, 76% responded saying it is because school is boring and 42% say it is because they are not learning enough (Education Trust West, 2004). In addition, it was also learned, as concluded by Hodges's study (Hodges, 1998) that a critical factor in the success of a student was an adequate and effective prerequisite for the class. Students who take

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rigorous mathematics courses are much more likely to attend college (U.S Department of Education, 1997). Accordingly, if the prerequisite necessary for college algebra does not prepare students to be successful in college algebra, then the system needs to be revisited, remedial actions be generated, and early intervention must be implemented.

Early intervention programs can provide college preparation for more minority students so institutions of higher educations can address the lack of representation of minorities in certain career fields, including mathematics and science. Most minority students (83.7%) are enrolled in lower-cost and public institutions (Wilds & Wilson, 1998).

This paper reports on a project that offers a pre-freshman camp for incoming Hispanic and other low-income minority students to help them make connections between mathematics and science, to encourage success in freshman courses, to develop more sophisticated understanding of their own study skills and strengths, and to create and develop a learning communities of peers.

This workshop has been designed to expose students to adequate mathematics foundations and emphasizes basic skills, critical thinking and problem solving capabilities required to be successful in college courses. Sessions are conducted by experienced TAMIU mathematics faculty. Each workshop participant is promised a weekly stipend of up to $300.00, at the completion of the sessions. The goal is that after these sessions, participants will be able to succeed in freshman-level mathematics courses at TAMIU or elsewhere. Participants also gained experience by working in groups.

Program Synopsis and Format

Six hours of activities were planned for each day of the week from January 5 to 9, 2009, before the beginning of the Spring semester at TAMIU. This exposes students for up to 30 hours of mathematics instruction and activities, equivalent to a three semester credit course at the college level. There were two sessions in the morning, each 1 hour and 25 minutes long separated by a 10 minutes break. Students were dismissed for an hour lunch, and would come back to work for three hours in ALEKS. ALEKS is a web-based system that

diagnosis the knowledge of a student. ALEKS does this by asking questions of various difficulty levels, under the assumption that if a student can answer correctly a question of a certain difficulty, then the student can answer correctly an easier question in the same subject. Through questions of different levels of difficulty ALEKS can accurately diagnose the level of understanding and knowledge of a student in a specific subject. Once the current level of knowledge and understanding have been determined, ALEKS determines what the student is ready to learn, and offers a "learning path", through topics that a student is ready to learn, based on earlier assessments. In order to work in ALEKS students needed Internet access, and so all sessions were held in the University's Technology & Enrichment in Mathematics (TEMA) Computer Lab equipped with 28 personal computers with Internet access. It is widely known the technology makes a difference in college mathematics teaching (Adams, 1997) and that the introduction of software tools and other hybrid structures have resulted in some partial success (Kinney, 2001).
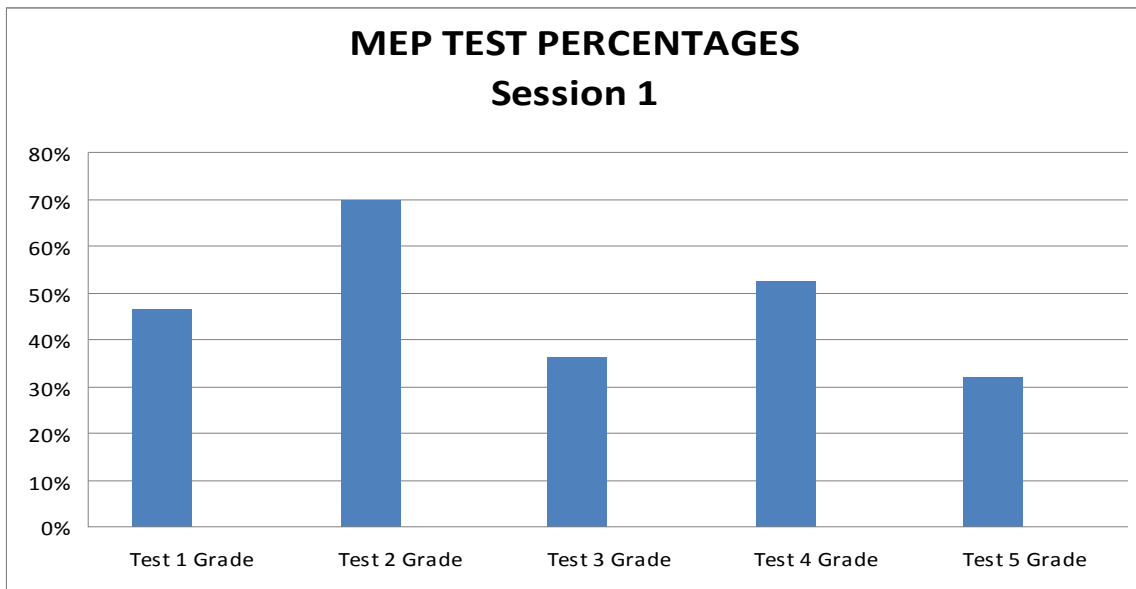
Figure 1: *Average scores by tests*

The first morning session was conducted in a mixed setting. This included an instructional session followed by a list of problems to be completed by the participants. This list of problems determined the type of lessons for the next day, and was determined also by the performance of students in ALEKS in previous sessions. It is anticipated that the students' use of software will contribute to a higher performance in the sessions. Figure 1 shows that

students are progressing well in some selected areas, as also still need to learn from the assigned topics as the workshop progresses. A list of problems based on a review chapter of a standard college algebra textbook similar, to the review chapter of the College Algebra book by Barnett, Ziegler, and Byleen, 2008, was divided equally for five days. Students attempted to solve these problems at the end of the first morning session. Figure 1 reflects the average scores received by the participants in each assignment given in the five morning sessions of the workshop. It is evident that there is a lot to learn after each day of activity.

The second morning session was in a lecture format. Participants were involved in the topics throughout discussion and analysis. Students were encouraged to think about basic concepts and procedures that they took for granted from their earlier education, discuss why a method would work, or how could geometrical objects be modeled by mathematical equations. Before the workshop, students had studied mathematics mostly from an algorithmic point of view. In this setting, students discussed why these algorithms worked. This would prepare them to solve more complex equations in their future courses in mathematics.

During the afternoon session, students worked in ALEKS. On the first meeting, students were explained what ALEKS was, registered into ALEKS and completed an initial assessment. In subsequent sessions, students worked in ALEKS in learning mode. A group of two participants was assigned to a single ALEKS license. Their task was to complete the ALEKS pie consisting of 197 items.

Students were told to work through ALEKS in groups, in the way that best suited their styles. Even though most people would feel that a license for ALEKS should be personal (and not a group activity), we believe through this experience that there is good value in working ALEKS in groups. On the one hand, students were working cooperatively and so the strengths of one member were passed to the weaker student. On the other hand, it gave students a reason to come back to ALEKS when one of the members needed a break from the strenuous amount of work that ALEKS demands from each student. Group load was shared equally among group members, and we feel that this system served them well.

Workshop facilitators and two students supervised their activities and provided help as needed. The two group members worked on a problem completely and collectively before one of them entered the answer into ALEKS. If students had a problem they could not solve, they could either raise their hands to have a tutor explain the problem to them or they could click on the "Explain" button, which would provide a complete explanation of the problem the student was presented to solve. After having read an explanation in ALEKS, a student would press the "Practice" button to solve a problem of similar difficulty to the one the student had missed.

Once a group of students had spent some considerable time in learning mode, ALEKS would switch to assessment mode and check that the student had retained what he they had been practicing mode. Students worked together during learning mode and assessments. What ALEKS believes that a student has mastered is shown in a way of a bar, and students, as well as instructors, can check the growth of that bar. It is refreshing for both the student and the faculty member in charge of preparing the student to see the bar grow and provides an extra incentive to continue. In the experience that we got in this workshop, the average group bar grew by at least 25%. This is a very impressive accomplishment of the students in the workshop. We expect that the students that participated in this workshop will use ALEKS more effectively in College Algebra than those who did not participate in the workshop and that this will be shown in their final grades at the end of the Spring 2009 semester.

An interesting characteristic of the group was that most of them had the same strengths and weaknesses. While students showed that they understood basic properties of the real numbers, they also showed that they had problems with expression containing radicals or with rational expressions as revealed on the first day of sessions. Students were encouraged to work in the topics that they were the weakest in, given that they had available at that time all the help that they needed in order to overcome their problems.

Finally, at the end of last day, all participants shared the lessons learned and appreciation for the workshop activities and organizers before they received a certificate of completion reflecting the number of hours they had spent in the workshop.

Curriculum

The curriculum is mainly confined to pre-algebra lessons for college students. The number of ALEKS Preparation for College Algebra licenses was adequate to provide this background. The ALEKS software was the main focus for applications, tutorials, and problem solving. Tables 1, 2, and 3 provide the list of items covered in these sessions based on the review chapter of the College Algebra book by Barnett, Ziegler, and Byleen (2008).

Table 1: *Topics for Algebra I: Lesson/Activity (9:00 am – 10:25 am)*

Basic Algebraic Operations

Day 1: Algebra and Real Numbers

Day 2: Exponents and Radicals

Day 3: Polynomials: Basic Operations

Day 4: Polynomials: Factoring

Day 5: Rational Expressions: Basic Operations

Table 2: *Topics for Algebra II: Lesson/Activity (10:35 am – 12:00 noon)*

Equations, Inequalities and Graphs

Day 1: Linear Equations and Applications

Day 2: Linear Inequalities

Day 3: Quadratic Equations and Applications

Day 4: Cartesian Systems and Equation of a Line

Day 5: Distance, Mid Point and the Equation of a Circumference

Table 3: *Topics for ALEKS Mathematics Tutorials (1:00 pm – 4:00 pm)*

Learning with ALEKS

Day 1: Registration, Introduction, Initial Assessment and Learning Mode

Day 2-4: Learning Mode and Assessment

Day 5: Final Assessment

Methodology and Evaluation of the Workshop

Recruitment of participants was done by using e-mail contacts. E-mail addresses of all registered college algebra students enrolled for Spring 2009 as of December 15, 2009 were obtained from the Office of the University Registrar. Students were sent an e-mail inviting them to participate and apply. A reminder was sent a week later. Participants need to submit an essay describing their career plans and goals for future in order for them to be accepted for the workshop.

Among a series of week-long workshops to be held in the next two years, this first cohort of the participants consisted of 23 students who planned to take College Algebra in the Spring 2009 semester. Of which, there were 13 female participants and 96% are Hispanics. The rest is represented by other minority students. Almost of all of them attended workshop regularly. Holidays hampered some of the recruitment efforts that had been previously planned, but since we had originally planned to conduct this workshop for 20 students, the total of 23 students that attended this workshop exceeded our expectations.
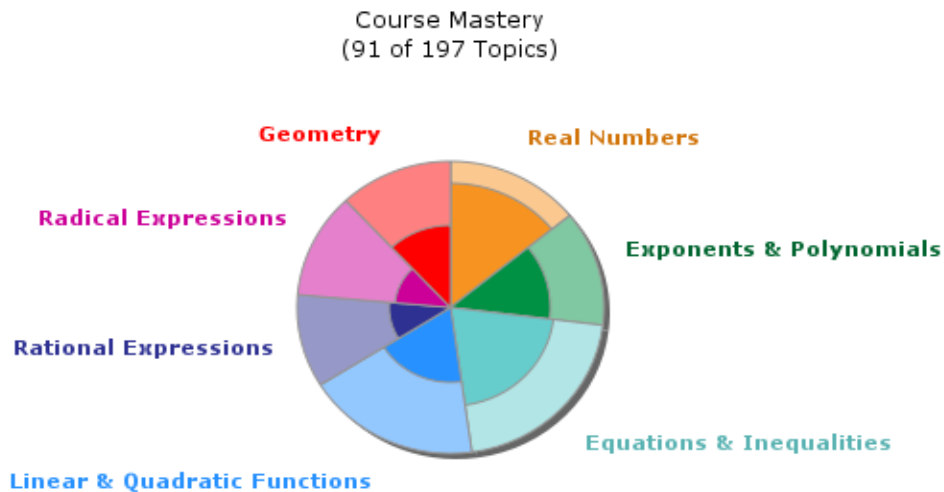
Figure 2: *Average composition of a pie after the initial assessment*

Figure 2 outlines the typical composition based on participants' knowledge as they enter the workshop. Their course mastery is 91 items from a total of 197 topics (46%) as depicted in Figure 2.
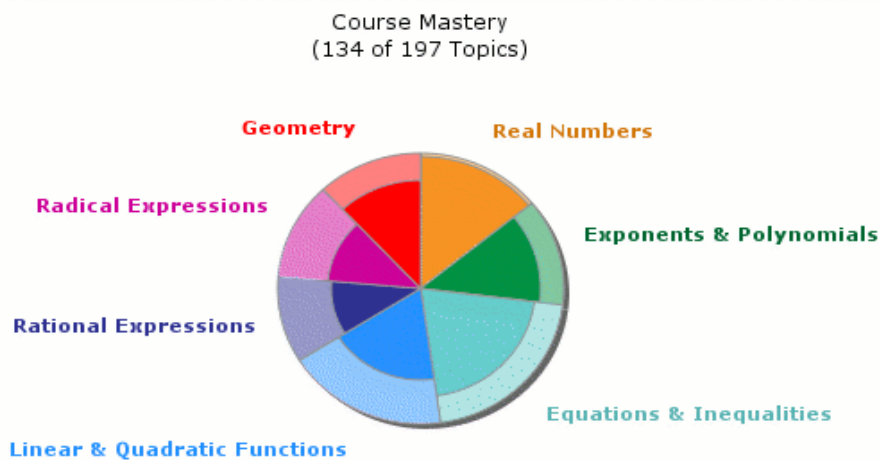
Figure 3: *ALEKS pie average items completed by the end of the last day*

Figure 3 shows a dramatic improvement in their knowledge base including to some extent recovery of their knowledge from their early years after successful completion of the workshop sessions. It was evident that 68% from the 197 topics assigned to participants throughout the workshop for ALEKS tutorial effectively provided this knowledge transfer.

Lessons Learned

In order to determine the success of this workshop undertaking and the extent of the adjustments to be made for future workshops, a feedback form was distributed among the 23 participants to solicit their input and comments. The feedback forms were evaluated using the scale: 1 - strongly agree, 2 - agree, 3 - neutral, 4 - disagree, 5 - strongly disagree. A summary of the findings is summarized in Figure 4. One item requires a response of the types, yes or no.

Information and Communication before the workshop was timely and accurate

3 (Neutral) 30%

1 (Strongly Agree) 44%

2 (Agree) 26%

The workshop was scheduled at a suitable time

2 (Agree) 30%

1 (Strongly Agree) 70%

The workshop facilities and location were appropriate and satisfactory

3 (Neutral) 9%

2 (Agree) 13%

1 (Strongly Agree) 78%

Handouts (if provided) were clear and useful

3 (Neutral) 9%

1 (Strongly Agree) 52%

2 (Agree) 39%

The staff responded to questions in an informative, appropriate and satisfactory manner

2 (Agree) 26%

1 (Strongly Agree) 74%

The workshop material was presented in a clear and organized manner

3 (Neutral) 9%

1 (Strongly Agree) 56%

2 (Agree) 35%

Figure 4: *A summary of feedback received from the workshop participants*

The workshop participants were highly appreciative about the way the workshop was conducted as is evident from Figure 4.

Conclusions

The organizers were quite impressed by the quality of the students that they had in this workshop. Students took a very long time to do the Initial Assessment in ALEKS (that is, they really did it consciously!), they participated enthusiastically in all sessions, and were eager to learn and see new points of view on matters they already knew. The organizers were very glad to have worked with them in this workshop. The initial assessment showed that they were very weak in handling expressions that involve radicals or exponents, or rational expressions, so they had to spend some time working with them in this type of problems. Students also worked in ALEKS on these topics, and they showed a clear improvement at the end of the workshop. Even though the final result was not that students completed the pie (a pie chart consists of various lesson plans), nor that they mastered the areas in which they were the weakest at (due to lack of time), it is felt by both the organizers and participants that the result was extremely positive, because it provided a venue for them to learn about their weaknesses, as well as it provided a way for them to overcome them. Students made substantial improvement in their weaknesses and we are confident that they will continue to do so. Overall, we are very satisfied with the workshop and the performance of the participants. The organizers also concluded that the number of participants in the workshop and the duration of the workshop are appropriate.

Among the comments made by students was the fact that the workshop allowed them to realize their weaknesses, and through ALEKS made a conscious effort to overcome them. Students acknowledged the value of the new points of view that had been presented during both morning sessions, as it helped them understand better material they already thought to have mastered.

Future Work

The authors plan to conduct the same set of analyses for the second workshop to be held in early August 2009. Plans are underway to recruit participants for the next workshop. The participants of this workshop are currently enrolled in Math 1314, College Algebra at Texas A&M International University. Their final semester grades will be compared with rest of the students in the class to see as to what extent they have been benefited from these types of intervention. Also, their standardized exam results will be collected to make a final determination of the effect of this workshop on them.

Acknowledgements

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http://www.aleks.com/

http://www.tamiu.edu/~rbachnak/STEMRRG/index3.html

"What Was *Really* Accomplished Today?"

Mathematics Content Specialists Observe a Class for Prospective

K–8 Teachers

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Abstract: *One of the important activities mathematics teacher educators engage in is the development of teachers at both the in-service and pre-service levels. Also of importance is the professional development of these professional developers. In the summer of 2004, a summer institute was held that allowed mathematics teacher educators watch the teaching of a mathematics content course for prospective K–8 teachers. This paper examines the manner in which a specific group of mathematics content specialists experienced this professional development.*
Key words: pre-service teacher education; mathematics content for teachers; middle school mathematics; teacher professional development

Introduction

        One of the important activities mathematics teacher educators engage in is the development of teachers at both the inservice and preservice levels. Teacher educators in the field of mathematics education work in a variety of settings, including university departments of

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mathematics, schools or colleges of education, local school districts or state departments of education, or private organizations. These teacher educators may engage mathematics teachers in mathematical content, pedagogical strategies and techniques, or some combination of content and pedagogy. They design experiences to help teachers improve their understanding of mathematics and develop their pedagogical practices. Learning to teach is an ongoing process and teachers at both the preservice and inservice stages are expected to engage in professional development activities to further refine and improve their practice.

Teacher educators however, are also first and foremost teachers. So if learning to teach is a continual process, it makes sense to speak of the ongoing development not only for classroom teachers, but for teacher educators, as well. This article discusses a professional development opportunity designed to allow mathematics teacher educators to examine, explore and discuss the development of preservice teachers in the context of mathematical knowledge for teaching (MKT) (Hill, Rowan, & Ball, 2005). The idea of formal professional development for teacher educators is a relatively recent phenomenon in the field of mathematics education and little is known about how teacher educators might engage in improving their own practice or how one might design a professional development experience for participants from these varied backgrounds and work settings. Thus, as we examine the manner in which these teacher educators experienced the professional development, it is critical to ask who these teacher educators are, what they think and believe about the practice of teaching and learning, and how they might examine and improve their practice.

Institute Description

During the summer of 2004, The Center for Proficiency in Teaching Mathematics (CPTM), a NSF-funded research effort at the University of Georgia and the University of Michigan, held an eight day summer institute entitled "Developing Teachers' Mathematical Knowledge for Teaching". The internal language of CPTM planners often described the purpose of the institute as *the professional development of professional developers*. The institute had 65 participants, along with numerous special guests, doctoral students as participant-researchers or staff, outside observers, and part-time visitors. Participants included mathematicians; university-based mathematics educators from departments of mathematics or schools or colleges of education; school-, district-, and state-level professional developers; and representatives of independent professional development organizations.

The institute centered on a mathematics content course for prospective K–8 teachers. The specific content of the course was the conceptual understanding of fractions. This setting was used as a context to study the learning of mathematics teacher educators. Some of the overarching questions to be answered by CPTM included "What do teacher developers know and believe? What do they need to learn? What is challenging about their work that many do not learn simply from experience? What content knowledge do they need? What do they need to know about their own learners and about how to relate to them effectively?"(Sztajn, Ball, & McMahon, 2006).

The mathematics content course met in a large banquet room, which allowed participants to observe the lessons. Participants were not allowed to interact with members of the class or to interrupt the proceedings of the class. Before class each day, the entire group of participants met for forty-five minutes with the instructor or with one of the institute planners to examine the lesson plan for the day. In these *Preparation* sessions*,* participants were encouraged to question the instructor as to her goals and plans and to make specific suggestions for how each class might be conducted. Participants also met with the instructor or with institute planners for forty-five minutes immediately after each class to discuss their observations. For these *Lab Analysis and Discussion* sessions, participants were divided into three, approximately equal, focus groups*.* For three days of the institute, each focus group was given a specific assignment during the lab class, either to focus on the mathematics of the class, the teaching of the class, or student learning in the class, so that over the course of the three days, each focus group attended to each of the three foci.

Data Sources and Research Questions

Each of the lab classes was videotaped from multiple perspectives, as were the Preparation sessions and the Lab Analysis and Discussion sessions with participants. Participants were each given a notebook in which to record their observation notes and comments on the lab classes. At various times during the institute, they were also given prompts for reflection or for their reaction to specific events relative to the planning or implementation of the lab class. These notebooks were collected, scanned, and returned to the participants. Field notes on individual participants as well as whole group field notes during the Preparation and the Lab Analysis and Discussion sessions were taken.

The overall goal of studying the Summer Institute is to better understand the interactions between the participants and the learning opportunities they experience in this professional development initiative. To this regard, we will carefully examine who the institute participants are, as well as the planning and development of experiences in which they will engage this summer. (Ball, Sztajn, & McMahon, CPTM internal document, June 2004)

During the institute, participants were asked to sign a consent form in which the research goal was again stated as "to better understand the interactions between the participants and the learning opportunities they experience in this professional development initiative." Specific research questions of the institute related to the goals and evolving design of the institute's curriculum, the ways in which participants with differing characteristics interacted with the curriculum and how those interactions might change during the institute, and how participants viewed the mathematical knowledge and work of elementary preservice teachers and how those views might change during the institute. Interest in participants' backgrounds is expressed in the goal statement above and is implicit in each research question stated above.

We chose to focus on the question "How do participants with differing characteristics view the mathematical knowledge and work of preservice elementary teachers?" In this paper, we describe the identification of a particular subgroup from within the 65 participants and analyze selected data to address the question. We present evidence of how seven *mathematics content specialists* looked at and responded to a laboratory class for prospective K–8 mathematics teachers during a one-week professional development institute for teacher educators.

Participant Selection

We decided to focus on a group of mathematics educators that we called *mathematics content specialists*. This subgroup of participants were selected based on the following criteria:

1. Work in a department of mathematics
2. Do not teach mathematics pedagogy courses for K–8 preservice teachers
3. Teach mathematics courses, though not necessarily for K–8 preservice teachers.

We found 21 such participants in the institute. These 21 mathematics content specialists were equally divided among three subgroups. We selected one of the subgroups for our study. Our choice of one particular subgroup was based on our observations that several of the participants in this group were particularly vocal in group discussions during the institute. They seemed open

to sharing ideas and opinions on each of the three foci of the institute: student learning, teaching, and mathematics. Descriptive information on these seven people, taken from their applications to attend the institute, is shown in Table 1.

Table 1:

Seven content specialists from Focus Group 1 and selected characteristics.

Pseudonym	# of math courses taught to K–8 PSTs	# of math education courses taught to K–8 PSTs	# of other courses taught to K–8 PSTs	# of other math courses taught	# of other math education courses taught	# of other courses taught	# of math courses you plan to teach to K–8 PSTs	# of math education courses you plan to teach to K–8 PSTs	# of other courses you plan to teach to K–8 PSTs	K–8 teaching experience (in years)
Donna	10	0	0	5	3	0	4	0	0	0
Sharona	4	0	0	8	0	0	4	0	0	0
Lona	3	0	0	8	0	1	2	0	0	0
John	3	0	0	11	0	0	4	1	0	0
Emily	1	0	0	6	0	0	2	0	0	0
Darryl	0	0	0	9	0	0	1	0	0	0
James	0	0	0	5	0	0	1	0	0	0

Each of these participants reported that he or she encountered K–8 preservice teachers only in mathematics content courses, if at all. None had K–8 teaching experience or taught mathematics education courses for K–8 preservice teachers, and only one planned to teach an education course for K–8 preservice teachers in the future. Hereafter, when we use the term *participants*, we refer only to the seven mathematical content specialists chosen for the analysis in this paper.

Analysis and Discussion

In order to examine how the participants viewed the mathematical knowledge and work of the preservice teachers we used researcher field notes, written transcriptions of discussions that were held after Lab Analysis and Discussion sessions as well as participant's individual notebook entries. For most of the participants these notebook entries do not contain a reflective or critical component. They are, rather, lists of events and direct quotations of statements made

by students in the class, much like the notes one might take during a lecture without comment or reflection. They consist of a simple record of what happened and what was said. There are exceptions in that some of the seven participants wrote comments beside their observations or wrote summary reflections after the lab class.

The Cookie Jar Problem

In order to analyze participants' views of the mathematical knowledge and work of the preservice teachers, we examined a particular event within the lab class and the participants' reactions to that event. Even though this event occurred on a single day of the course, discussion of the event continued throughout the three days in which our participants attended to teaching, learning, and mathematics. Students in the lab class were presented with the following problem on the second day of the class:

> There was a jar of cookies on the table. Kim was hungry because she hadn't had breakfast, so she ate half the cookies. Then Stan came along and noticed the cookies. He thought they looked good, so he ate a third of what was left in the jar. Nita came by and decided to take a fourth of the remaining cookies with her to her next class. Then Karen came dashing up and took a cookie to munch on. When Patty looked at the cookie jar, she saw that there were two cookies left. "How many cookies were in the jar to begin with?" she asked Kim (lesson plans).

Students worked in pairs and individuals presented solutions for class discussion. The event on which we wish to focus is a presentation by one of the students, Tessa; however, in order to place her presentation in context, we first briefly describe the preceding presentations.

Stan presented a solution to the cookie jar problem in the form of a sketch, as shown in Figure 1. Stan solved the problem by working backward through the given information. That is, he first drew the two squares in the upper left box to represent the two cookies Patty saw in the end. He then drew the lower left square to represent the one cookie taken by Karen. The middle square on the bottom row represents the cookie taken by Nita, or one-fourth of the cookies remaining. He then drew the two squares in the middle to represent one-third of the remaining cookies taken by Stan. His final step was to draw the six squares on the right to represent the one-half of the cookies first taken from the initial state of the cookie jar. His answer to the question then became a simple matter of counting the squares he had drawn to represent the cookies to arrive at an answer of 12.
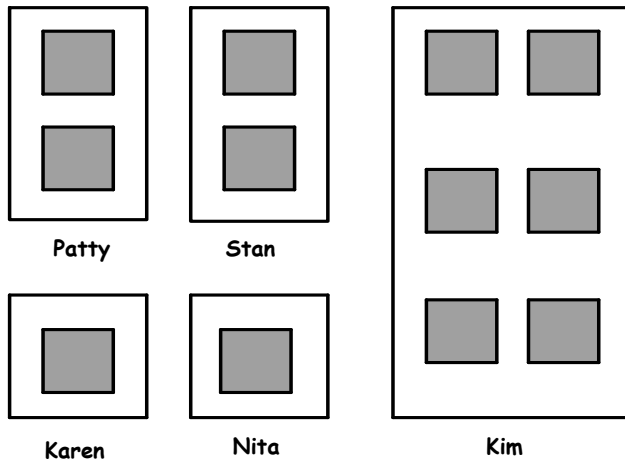
Figure 1: Stan's solution to the cookie jar problem by working backward.

Nita, as shown in Figure 2, presented an algebraic solution. Since our intent in this paper is merely to use these earlier solutions to illustrate the context for Tessa's later work, we do not discuss Nita's algebraic solution in detail. However, we should comment that Nita's incorrect answer was the result of a mistake of taking a fraction of the cookies *remaining* versus taking a fraction of the cookies already taken. Students seemed to be enamored of Nita's approach and discussion of this algebraic solution consumed a considerable portion of class time. The cookie jar problem was revisited at the beginning of the next class. Students again worked in pairs to produce alternate solutions. Sharon worked with Nita to produce a solution that Shelly presented to the class, as shown in Figure 3.

$$\left(\frac{1}{2}x - \frac{1}{3}\left(\frac{1}{2}x\right) - \frac{1}{4}\left(\frac{1}{3}\right)\left(\frac{1}{2}x\right) - \frac{1}{4}\left(\frac{1}{3}\right)\left(\frac{1}{2}x\right)\right) - 1 = 2$$

$$\left(\frac{1}{2}x - \frac{1}{3}\left(\frac{1}{2}x\right) - \frac{1}{4}\left(\frac{2}{3}\right)\left(\frac{1}{2}x\right)\right) - 1 = 2$$

...

$$x - 3 = \frac{1}{2}x + \frac{1}{6}x + \frac{1}{24}x$$

$$x - 3 = \frac{12}{24}x + \frac{4}{24}x + \frac{1}{24}x$$

$$x = \frac{17}{24}x + \frac{72}{24}$$

$$x\left(\frac{24}{17}\right) = \frac{72}{24}$$

$$x \approx 2.12$$

Figure 2: Nita's algebraic solution.

		1/4 eaten by Nita	
1/2 eaten by Kim			
		1/3 eaten by Stan	

Figure 3: Shelly's solution to the cookie jar problem

Unlike Stan's solution, Shelly worked forward through the problem statement by first drawing the large outer rectangle to represent the cookie jar in its original state. She drew a vertical segment to divide the rectangle in half, and labeled the left half as "1/2 eaten by Kim." She divided the right half into thirds horizontally and labeled the bottom third as "1/3 eaten by Stan." She then divided the upper square on the right side into fourths and labeled the unshaded square as "1/4 eaten by Nita." She reasoned that, since there were three small squares remaining, these must represent the one cookie eaten by Karen and the two cookies remaining in the jar. She concluded that each small square must represent a single cookie, and was thus able to divide the larger rectangle into 12 small squares to arrive at an answer of 12 cookies originally in the jar. Shelly's solution might be characterized as an area model for the solution to the cookie jar problem, and it is in this context that Tessa offered her solution.

In Figure 4, we see Tessa's solution to the cookie jar problem. As she produced this sketch, she mapped the words of the problem onto her emerging representation. First, she drew a circle to represent all the cookies in the cookie jar at the beginning of the problem. She then drew
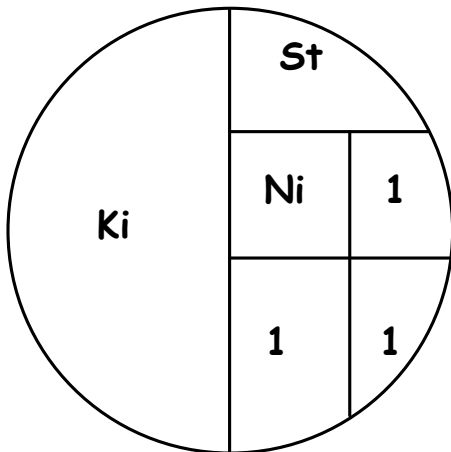
Figure 4: Tessa's solution

a vertical segment and labeled the left part as "Ki" to represent the one-half of the cookies taken by Kim. She then drew the two horizontal segments in order to "divide this into three parts" and labeled the top portion as "St," to represent the one-third of the remaining cookies eaten by Stan. Tessa then divided the remaining, unlabeled, portion of the circle into four parts by drawing a vertical segment. She experienced some confusion about how to label these four new parts, and initially placed the numeral 1 in the lower right portion before erasing it to rethink her solution. Her confusion seemed to remain until the instructor of the course said, "Who took one-fourth of the remaining cookies?" With this question, Tessa labeled one of the remaining parts as "Ni," for the one-fourth of the remaining cookies eaten by Nita. She then quickly placed a "1" in each of the remaining parts to represent the one cookie taken by Karen and the two cookies remaining. She then had no trouble arriving at an answer of 12 cookies for the number of cookies originally in the cookie jar.

One student in the lab class commented that when Tessa drew the horizontal segments in the right portion of her circle, she did not really create equal parts. Tessa's response was, "I don't do very many math problems. Although I've seen the pie diagrams, it just didn't occur to me. When I drew this [the thirds], I knew it didn't look like they were equal portions, but you can certainly take a third of a half. I decided not to worry about them not being proportional. But it did bug me and I was hoping I wouldn't have to deal with any algebra …." The instructor interrupted to ask, "Are you saying your drawing is or is not representing equal parts, or are you just not worrying about it, or …." Tessa replied, "It works for *me.*" There was a short discussion in which some students saw Tessa and Shelly's representations as equivalent and others

expressed a personal, but not mathematical, preference for Shelly's rectangular representation. The discussion concluded when one student commented that this way made sense to Tessa and that was all that mattered. We will use participants' comments on Tessa's partition of the semicircular region into three parts, as well as the cookie jar problem in general, as a context to examine, the participants' views of the mathematical knowledge and work of the preservice teachers.

The Cookie Jar Problem and Mathematical Knowledge

In general, the participants focused on correctness of mathematics and clarity of explanations. Each of the participants noted that one or another of the students in the lab class *got it* or *didn't get it*, or *understood* or *didn't understand*. We can only infer that the participants made these dichotomous judgments against some absolute standard of mathematical correctness. Similarly, participants described student presentations of solutions as *clear* or *elegant,* as opposed to *confused*, *fuzzy,* or *muddled*. Again, these characterizations of explanation seem relative to some predetermined and absolute standard. We also wish to state here what others at the institute suggested to the mathematics content specialists, namely, that they seemed to concentrate on what the students in the lab class did not know, rather than what the students did know. Interpreted another way, the participants considered mathematics from their own understanding rather than attempting to understand the subtleties of, or create a model of, each student's understanding of the problem. From a beliefs standpoint, this type of constructivist stance would indicate a viewpoint categorized by Kuhs and Ball (1986) as content-focused with an emphasis on conceptual understanding. The content specialists' responses however, with their allusions to absolutism are an indicator of content-focused with an emphasis on performance model of mathematics teaching. Further evidence of this classification is presented in the following expositions.

Each of the seven participants noted Tessa's incorrect division of the semicircle into thirds and that no one in the lab class had strongly objected to her division. Four of the participants seemed to have dropped the issue after the class discussion that, "It works for Tessa, and that's all that matters," and one of these four, Sharona, noted that Tessa's drawing was a *tool* for solving the problem, implying that a blunt tool, so to speak, was acceptable if it accomplished the job at hand. The other three participants, however, were more adamant in their objection to allowing Tessa's mistake to pass uncorrected. James, for example, questioned in his

summarizing notes for the day, "Does Tessa's misleading area drawing ever get corrected? In this course do we cultivate skepticism and constructive self-criticism? What about analysis? Or is it all just validation?" We take James' comments, along with those of other participants, as further evidence that the participants expected the use, presumably by the instructor, of some external standard of correctness and explanation.

During a focus group discussion of this episode, one of the CPTM staff suggested to James, Darryl, and John that perhaps Tessa did not imply an area model in her division of the semicircle. Rather, in her explanation Tessa stated that she needed to divide the semicircle into "three parts" and proceeded to do so, consistent with a *discrete* model of the cookie jar problem. Thought of as a discrete situation, there is no real need for the cookies, and therefore the parts of the diagram, to be of equal size. Some of the participants strongly stated that this interpretation of Tessa's solution was far beyond what she was actually thinking and that it was "not appropriate to make excuses and justifications for what Tessa did. She was a student in the class." (James, LAD discussion). We interpret James' comment that Tessa "was a student in the class" to mean that she should be judged, based on the physical and oral evidence she produced, against some preexisting standard rather than on any model of her understanding as created by an observer.

Another example of the content specialists' beliefs came about during a discussion on the role of definition. To solve the cookie jar problem, one must resolve, either explicitly or implicitly, the issue of what came to be known at the institute as *the shifting whole*. That is, the language of the problem requires that the *unit* for each fraction of cookies be redefined at successive steps in the problem. When Stan takes one third of the cookies left in the jar, and later when Nita takes one fourth of the remaining cookies, the solver must realize that as operators, these fractions do not operate on the same quantities. One might also need to consider that, with certain solution strategies such as the algebraic one shown in Figure 2, one third of the cookies remaining are equivalent to two thirds of the cookies taken, for example. In either of these situations, the unit associated with each fraction changes, or shifts.

Concern over the notion of the shifting whole initiated a discussion concerning the definition of fraction. One of the stated goals of the lab class was to explore the concept of the *unit* in understanding fractions and there seemed to be some concern among participants that this concept was not addressed explicitly, but rather left for each student to develop individually

through exploration of the cookie jar, and other problems. The data suggest that, for the institute participants as a whole, there was an issue with not providing the students in the lab class with a definition of fraction. Though this omission is a pedagogical decision, the concern over the need for a definition implies a view of mathematics as growing from given definitions and postulates rather than from experience. The participants noted this issue of the *shifting whole* and the call for a *clear* definition of fraction. We discuss this issue further in a later section.

The Cookie Jar Problem and Student Work

Participants' focus on mathematics carried over into their observations of student work. Participants noted students' fluid or hesitant use of mathematical language, correct or incorrect representations, and clarity or fuzziness of explanation. Each of the participants commented on Tessa's non-proportional division of the semicircle into thirds and five of them noted that there seemed to be little concern among the students over her mathematically incorrect representation.

In general, participants seemed surprised by students' lack of acceptance of a pictorial or geometric solution and explanation. Five of the seven mathematics content specialists noted students' focus on the presented algebraic solution to the cookie jar problem. In particular, James, John, Darryl, and Sharona each commented, in some way, that Stan's initial pictorial solution was elegant and convincing, yet, as James stated,

> I was struck by the value that they put on the mathematics. The first solution was clear, and so there was almost no need for discussion, but they seem to feel that the algebraic argument, even though they demonstrated very clearly that it was tricky and error prone [was necessary to 'prove' the solution was correct.] The thing that struck me was that they put down their initial solution and even excused it as being the kind of lame production [one might expect from] people who have their mathematical background.

One of the students actually made the comment after presenting a graphical representation of the solution that, "Of course you would need to prove it with algebra." There is no indication in the data that the participants discussed how students might have come to hold either a view of algebra as some sort of ultimate arbiter of truth or of graphical or geometric solutions as somehow inadequate. Understanding however, that one facet of a classroom teacher's pedagogical content knowledge is his understanding of what his students know and how they learn, it is important to pay attention to how the mathematical content specialists viewed the

students in the lab class. Specifically, they were surprised by how quickly the pictorial solution was dismissed by the students due to its apparent sophistication.

Discussions of classroom discourse, language and clarity, mentioned above relative to mathematics, also appeared in comments about students. Donna commented about the lab class's "impressive student interaction, especially for the first day" and Lana noted how, on the first day "important patterns [were being] established for the course". The same participants also commented in a whole group discussion about the focus of the lab class on student mathematical explanations. Donna commented on the "fuzziness" of the student explanations and what was expected in their explanations. She also took detailed notes on individual students during one of the lab classes and noted how each student explained or represented a solution. She commented in her notebook:

> I know that when I started teaching fractions some of the connections between division and other models of fractions were not obvious to me—need to admit that, so of course it's hard for the students too; it's just hard."

She seemed to understand the difficulty these students had with the mathematics content because she had her own problems with teaching it. Whether or not this was a new revelation for her as a result of this experience was not clear from the data. Two content specialists also commented on the importance of the students using correct mathematical language. Lana commented in her notebook that she was impressed by the students' mathematical language. James commented on correct language as a goal of the course, noting that getting such precision "is really, really hard and a short course like this can't achieve this."

Conclusions, Discussion, and Implications

Professional development for teacher educators beyond their initial preparation is a recent consideration in mathematics education. Those who plan and implement such professional development situations must take into account the intended consumers of their products. In this paper, we examined a particular professional development institute and identified a set of its attendees whom we designated as *mathematics content specialists*. We attempted to analyze the ways in which these participants viewed the mathematical knowledge and work of preservice mathematics teachers.

Content Specialists and Mathematical Knowledge

When the content specialists were asked to focus on the mathematical knowledge demonstrated by the students in the lesson, they focused on the correctness of the mathematics and the clarity of explanations. The content specialists were concerned that there was no explicit discussion in the class about the shifting whole in the cookie jar problem nor was a definition for fraction ever given to the students. Other participants at the institute suggested that the content specialists seemed to focus more on the mathematics that the students did not know rather than on the mathematics that they did know.

The content specialists seemed to consider the mathematics based on their own understandings instead of attempting to understand the mathematics of the students. They seemed to make dichotomous judgments based on whether a student *got it* or *didn't get it* or *understood* or *didn't understand.* More importantly, they seemed to base these judgments on some absolute standard of mathematical correctness. From one participant in particular, it seemed that as students in the class, they were meant to be judged based on this preexisting standard of mathematics that seemed to be the mathematics as understood by the content specialist.

Content Specialists and Mathematical Work

When the content specialists were asked to focus on the mathematical work of the students, their journals noted students' use of mathematical language, representations, and explanations. Each commented on Tessa's non-proportional division of the semi-circle into thirds, and five of them commented on the lack of concern among the other students about the incorrect representation. Five of the participants also commented about the elegance and convincing pictorial solution given by Stan but that the students seemed to focus on the importance of using algebra to "prove" the solution was correct even though the algebra of the problem was found to be error prone. The participants seemed to think that the students in the class did not value the pictorial representation as much as the algebraic representation because the picture did not seem to be mathematical enough.

The content specialists seemed baffled by the students' inability to accept Stan's pictorial solution as a correct mathematical approach. Instead the students made an excuse for the pictorial representation being one that someone with a lesser mathematical background would produce instead of using algebra. There was no data showing that the participants discussed or

thought about how these students would come to this conclusion; however, it seems very similar to the content specialists wanting to judge the students based on pre-existing standards. So, perhaps, the students were also aware of some pre-existing standard of which they were to be judged.

Implications for Future Professional Development of Professional Developers

With these conclusions in mind, we suggest that those who plan and implement future professional development that include mathematics content specialists as participants consider some explicit orientation and discussion of the philosophies that guided the design of the institute. We do not suggest such orientation necessarily include debate over the correctness or desirability of any particular philosophy, but merely should indicate that a particular philosophy is explicitly in play for the work of the institute and one aspect of participation is to observe and discuss where and how that philosophy operates in practice.

Second, we believe that this institute for professional development did indeed, for the participants in this study, "create a high level of cognitive dissonance to upset the balance between teachers' beliefs and practices and new information or experiences about students, the content, or learning" (Loucks-Horsley et al., 2003, p. 45). However, based on the results of this study, we suggest that planners of future professional development for teacher educators should better anticipate and take advantage of perturbations that might arise relative to the beliefs that teacher educators bring with them. Hopefully, other data from this institute will elaborate on how these and other participants viewed and interacted with the intended professional development curriculum.

Finally, we see evidence that the participants in this study were keen observers of mathematics, mathematics learning, and mathematics teaching, if from a particular viewpoint. Though there is some evidence that participants considered their own practice as they participated in the institute, what is not evident is whether or not any participant considered changes, or actually changed, either his or her teaching practices or views of student learning. This is not to conclude that participants were not reflective, but merely to note that asking participants to observe a mathematics course and to record observations in a notebook or to discuss observations with other participants may not be sufficient to open participants' own practices to the careful examination that might lead to changes in practice. We suggest that future

professional development experiences that target teacher educators should include some structure to open participants' practices to discussion and to examine those practices in a more public way. We also suggest that future professional development include some device to evaluate the effectiveness of the program, assuming that teacher educator change is one goal of professional development. This evaluation might include post-program surveys, interviews, or data from participants' own teaching, such as course syllabi, selected mathematical tasks and assessment, or classroom observations.

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Tyminski, Ledford & Hembree

Leading Learning within a PLC: Implementing new Mathematics content

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Abstract: *This paper does two things. Firstly, it examines the literature that coalesces around theoretical models of teacher professional development (PD) within a professional learning community (PLC). Secondly, these models are used to analyse support provided to two year 3 teachers, while implementing the draft Queensland mathematics syllabus. The findings from this study suggest that the development of this small PLC extended the teachers' Zone of Enactment which in turn led to teacher action and reflection. This was demonstrated by the teachers leading their own learning as well as that of their students.*

Keywords: learning communities; mathematical content; teacher professional development; Queensland; theories of teacher development

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INTRODUCTION

It is well recognised that teachers face an ongoing challenge in implementing mathematics reforms (e.g., Handal & Herrington, 2003). Given that many elementary teachers are predominantly generalist teachers with little specialist expertise in mathematics education, there is a need to support teachers to develop their mathematics teaching skills.

This paper explores the nexus between theory and practice by using the theoretical model developed by Fullan and Stegelbauer (1991) and Millett and Bibby (2004) as a way to discuss the professional development (PD) supports teachers need when they are being introduced to new mathematics content and pedagogy. To this end, this paper provides an account of a two year project in which teachers conducted a series of teaching experiments aimed at enhancing their content and pedagogical content knowledge in relation to introducing a new content area, mental computation. The following question provided the focus for this two year project:

> What supports do teachers need to enhance opportunities for a professional learning community to develop?

TEACHER CHANGE AND PROFESSIONAL DEVELOPMENT

While educational change has been of concern since the 1960s, the poor history of long term educational change has been well documented (Fullan, 2005; Miles, 2005). However, there have been lessons along the way that have influenced thinking on educational change. Fullan and Stegelbauer (1991) contributed to the discussion on three distinct phases in the change process by promoting a model of change (see Figure 1). They argued that the three phases of "initiation", "implementation" and "continuation" must also include "outcome" indicating the degree to which the school has implemented the change (p. 48). To this end the theoretical model of change they

proposed included "outcome". This position is a reflection of the understanding by the 1990s that professional learning within a community was an important component of successful educational change (Fullan, 2005; Ingvarson, Meiers, & Beavis, 2005; Smylie & Perry, 2005).

| Initiation | ↔ | Implementation | ↔ | Continuation | ↔ | Outcome |

Figure 1. A simplified overview of the change process (Fullan & Stegelbauer, 1991, p. 48)

Linking educational change and professional learning within a community, Millett and Bibby (2004) advance a model of educational change that identifies the "Zone of Enactment" (Millett & Bibby, 2004, p. 3) which extends Vygotsky's (1978) theory on the Zone of Proximal Development (ZPD). While the ZPD focused on the individual, the Zone of Enactment encompassed the professional learning community (PLC). In short, this theory sought to understand a teacher's capacity to change by examining the context and culture of the teacher's 'situation' or working environment. This situation included the professional learning community as well as external influences (see Figure 2). According to Millett and Bibby (2004), sources of support that operated within the "situation" either stimulated a teacher's "zone of enactment" leading to change and hopefully sustainable change, or inaction and ultimately failure of the intended change.
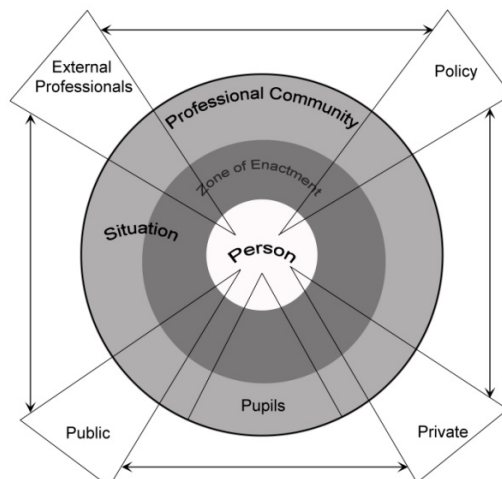
Figure 2. Theoretical model for discussing teacher change (Millet & Bibby, 2004, p. 3)

Millett, Brown, and Askew (2004) identify four conditions necessary for the realisation of Zones of Enactment: time, talk, expertise, and motivation. Firstly, they consider two aspects of *time* essential for the development of a PLC; time for teachers to engage in discussion and reflection, and time for an iterative framework of trial, reflection, discussion, modification, and retrial. Secondly, Millett et al. (2004) report that as teachers were provided opportunities to observe each other's lessons, they were encouraged to *talk* with each other about these observations as a focus for reflection and discussion. This *talk,* in turn, led to the development of *expertise* coming from within the school from teachers leading teachers. Expertise was further developed through contact with external professionals such as university researchers. Finally, *motivation* in several guises was identified as a condition for the realisation of Zones of Enactment. Some teachers were motivated by internal feelings of interest in mathematics, by a desire to improve their mathematics teaching, or from fear of mathematics teaching. Motivation might also have been external; for example, encouragement from colleagues, policy (curriculum changes), and external expert support.

This theoretical model suggests sources of support for educational change and offers a practical way to examine the teachers' Zone of Enactment within the PLC.

THE STUDY

The Context

Two Year 3 teachers, Pam and Sue, participated in a project focusing on developing young children's mental computation. In Queensland (an Australian state), the site of this study, Year 3 students are approximately 8 years of age. At the time of the study, the teachers followed the old mathematics syllabus (Department of Education, 1987), which determined that students in Year 3 should be taught specific written algorithms. In contrast, the new syllabus, which was in draft form (Queensland Studies Authority, 2004), suggested that students should develop mental computation strategies. The students in the two classes (29 students in each class) had been introduced to written algorithms, but had not been taught any mental computation strategies. One aim of the study was to develop teacher content and pedagogical content knowledge through working within a PLC as a way to enhance their agency when implementing the syllabus.

Design

The research reported here adopted a case study design (Merriam, 1998), bounded by two early years teachers from one school. The study was implemented in four phases over two years (see Figure 3). Phases 1 to 3 were repeated in each of the two years. As a way to orientate the reader each phase is described below.

Phases 1-3 repeated in second year
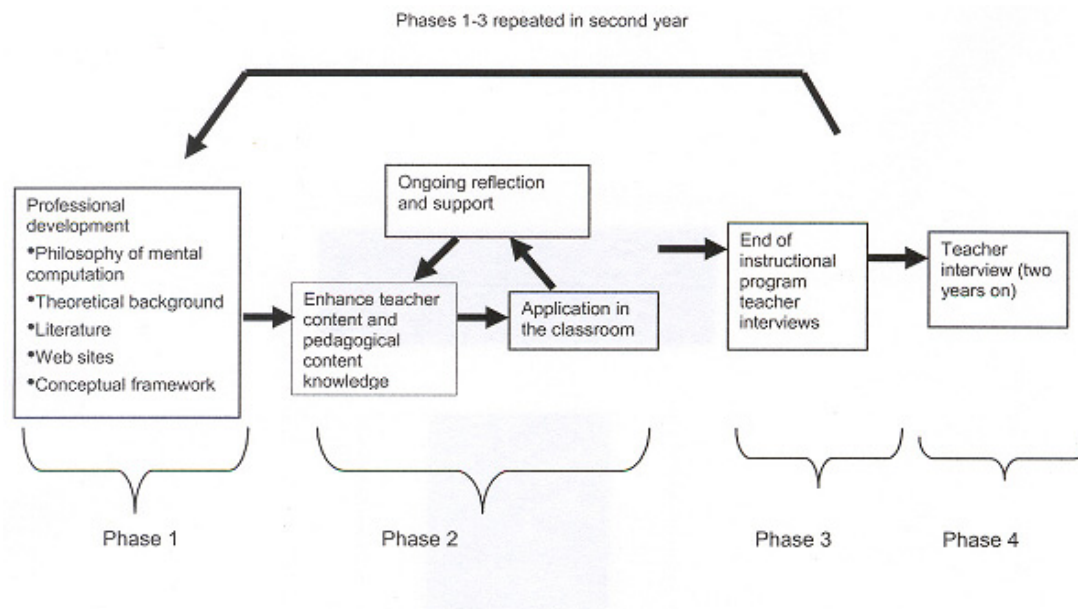
Figure 3. Phases of case study

Phase 1: Initial professional development

In the initial professional development sessions, the first author provided a conceptual

framework for mental computation (Heirdsfield, 2003a; 2003b). This framework explained the

links between mental computation and related concepts and skills. Additional material included

the project summary, relevant web sites, journal articles, the draft syllabus, explanations of

mental strategies, and suggested activities to develop mental computation strategies and

associated concepts. Pam and Sue studied the philosophy and theoretical background of mental

computation before planning the student instructional program.

Phase 2: Implementation phase

The instructional program was designed following the initial PD sessions. While the teachers

collaborated, they each implemented different programs in response to both teacher and class

differences. Their instructional programs consisted of eight weekly, one-hour lessons in each year of the two years that the project operated.

Ongoing support and reflection were integral to this implementation phase. During each lesson, the researcher took field notes on the outcomes of the lesson as a way to inform subsequent teaching episodes. During end-of-lesson meetings, feedback was provided on the content and management of the lesson as well as suggestions for further activities. The teachers reflected on student outcomes, and discussed ideas for subsequent lessons with the researcher. Discussions often continued into mid morning recess.

Phase 3: End of project reflection

At the end of each year of the study, the teachers were interviewed. They reflected on their learning, and identified the supports they believed enhanced their professional learning in this community.

Phase 4: Follow up interview two years after the completion of the project

Two years after the completion of the project one of the two teachers was interviewed to discuss the long-term effects of the project and to reflect on the supports she believed enhanced her professional learning. The second teacher had left the school.

Data Collection and Analysis

The data constituted teacher narrative interviews (Auerbach & Silverstein, 2003) conducted at the end of each year of the instructional program and two years afterwards. All interviews were transcribed and coded using a process of qualitative data analysis (Auerbach & Silverstein, 2003). Firstly, relevant text was selected. Secondly, themes were created from this text, using the "conditions" identified by Millet et al. (2004); and finally, the themes were applied to theory.

Results and Discussion

The teacher narrative interviews are examined in terms of the four conditions necessary for realisation of Zones of Enactment – time, talk, expertise, and motivation (Millett et al., 2004).

Time

This current study found that the provision of *time* was an essential source of support for professional learning. The teachers suggested that the PD offered at the commencement of each year of the project and the ongoing access to the researcher provided a very supportive structure through each of the first two phases of the study. However, allocation of time was essential to permit the teachers to engage in this PD.

> Pam: The project provided a very beneficial PD program. But we needed the teacher release time to fit it all in.

This finding has been documented regularly (Durrant & Holden, 2006; Hargreaves & Evans, 1997; Heid et al., 2006), particularly when implementing new content and pedagogy (Lamb, Cooper, & Warren, 2007) or new policy directives (Millett & Bibby, 2004).

Some aspects of time, as mentioned in Millett et al. (2004) were evident in this study – trial, reflection, discussion, and modification. However, retrial was not a factor, as Pam and Sue did not "retrial" lessons in the following year, as they indicated that all classes are different from each other, and what works in one class in one year with one teacher may not necessarily work elsewhere. In fact, Pam and Sue individually trialled new pedagogy in the following year, not as a "retrial", but rather in response to student needs.

> Pam: I love the number board. I get to do a lot with the number board, but I have just taken off with the number line this year. And I think, and I'll tell you

why I do – I bought one. It had never been done before, a lot of number line work has never been done [in this school], and this year I did something with the number line and they [the children] said to me, "but you didn't put the arrows in it. …Last year we did a lot of number line work." I said, "Right I'll do a bit more this year." I didn't have to start at scratch to teach number lines, as the children had been taught before.

Interviewer: But you did, before, didn't you.

Pam: Yes. You had to start at scratch and go through, "this is the number line, which way are you going to have to move?" And that's the result of that of um, doing that number board work I did do that to death because I loved it. …um and I was confident in the number board. The following year, I did a whole lot of activities, prelim activities, on the number board, like cutting pieces out of the number board, yes. That sort of thing – where would I go, what would come next but, give them the idea of placement on the number board. I could have done it on the number line if I was happy about it.

While the teachers were appreciative of the time allocated for PD, and some planning, they believed that they needed additional time.

Sue: We would have liked more time reading. And we needed more time for planning for sequencing [of lessons].

The time allocated to the PD was important, and the ongoing provision of time throughout the project enhanced the commitment of the teachers.

Pam: I only feel sad that we didn't do more with the mental computation as a staff but then I guess staff change a bit, staff move on and I think the sad thing is that something takes place all the time. You know like one year it's mental computation and next year – this year it's grammar. You know, we never actually see something, see it through, practise with it, like we tell the students to do. Practise it and then we think we can do it by ourselves.

Researcher: But you had two years of it so that's better than…

Pam: Yeah, better than a lot of other people… So you can say that I was a bit …spoilt

Talk

This current study also identified two aspects of *talk* that were beneficial to the teachers' professional learning: Teacher-researcher talk and teacher-teacher talk. At the end of each observed lesson, the researcher provided feedback on the conduct of the lesson. When probed about the reflective discussion at the end of each lesson, the teachers were in agreement that this period of reflection supported their ongoing development. The focus of these discussions was mostly on how to target the intent of the lesson and this usually involved encouraging the students to develop their own strategies.

Pam: And I'll tell you what else I found was the feedback we got after each teaching session.

Researcher: Did you feel you had a say in that discussion after the lesson?

Pam: Yeah, and you asked me why I did things.

The teachers also spent time reflecting together. They believed that this collaborative reflection enhanced their own understanding and helped focus their individual lesson development.

Pam: Even when [the researcher] wasn't there we would actually just sit there and say what does this actually mean? Could we go further…are we just going to add on 9 or take 9? Or are we going to go to 19 or 39? All the talking helped us to get the language of mental computation to teach it.

Contrary to the findings reported by Millett et al. (2004), Pam and Sue did not believe that viewing each other's lessons and talking about their observations would have benefited them. They stated their classes were quite different from each other, and, therefore, observations would not have contributed anything to their understanding of their own classes. Instead, collegial reflection on the intent and outcomes of their lessons was more beneficial.

When asked to reflect on the PD and how it was structured Pam commented,

Pam: It really changed my way of thinking…We worked together collaboratively. That made our lessons more successful and we were very honest with each other… All the talking helped us to get the language of mental computation to teach it…The readings and websites were good too but I tell you what was great. The conceptual framework!

Researcher: Do you feel you had ownership of the process?

Pam: Yes I did. Sometimes I wished I didn't because I mucked up a few lessons. But I guess that is what happens when you do own it … and it's new.

Expertise

This current study also identified the contribution of *expertise* to professional learning. Initially, the researcher provided teachers with background information in the form of published research papers and web sites that detailed the philosophy and theoretical background of mental

computation, mental computation strategies, and suggestions for the learning experiences aimed at promoting number sense. In addition, to this background information the researcher provided Pam and Sue with a conceptual framework to guide their practice. During the next two years of the teaching experiment, the teachers utilised the conceptual framework for mental computation (Heirdsfield, 2003a; 2003b) for planning. Pam reported that she not only viewed mathematics with the connections developed in the conceptual framework, she also viewed all learning this way. Further, she still constructed conceptual frameworks before embarking on a new topic.

Pam: The conceptual frameworks ... have become part of my planning. Your conceptual framework was used to identify where the children were at, and what each child needed. ... I now construct a conceptual framework before I teach anything new. It helps me see the links. …I also think about the links between lessons in my planning.

Pam: [The conceptual framework] changed my way of thinking. It's not so much thinking; it made me more aware of why we teach certain things. They [concepts] don't come up in isolation.

Consequently, the teachers not only developed more connected knowledge, they also developed a connectionist orientation (Askew, Brown, Rhodes, Johnson, & William, 1997), as evidenced by their developing supporting lessons for topics related to mental computation.

Pam: In other lessons, when you weren't there, we worked with the children on numeration, you know with MAB [Multibase Arithmetic Blocks – materials used to assist students develop understanding of place value]. And we worked on their number facts strategies. It all had to go together.

Thus researcher's expertise supported the teachers' content knowledge and pedagogical content knowledge. The teachers were empowered by the "materials" provided during the PD. Rhodes and Millett (2004) also reported teachers' enhanced knowledge as a result of having access to a wide range of materials. While this current study clearly identified the importance of the expertise provided by the researcher, the data also highlight the importance of the contribution of teacher expertise. In short, the researcher's expertise in relation to mental computation complemented the teachers' classroom expertise. Pam and Sue spent time together reflecting on and discussing their lessons even though they developed their lessons individually. Pam discussed her preparation of lesson plans as supporting her construction of knowledge, again being responsive to her Zone of Enactment.

> Pam: It forced me to become aware of the sequencing required to develop mental computation strategies.

Further, follow up conversations with her colleague helped Sue to refine her lesson plans. This finding is in line with Rhodes and Millett (2004) who questioned whether teachers who were not actively involved in the planning of lessons always developed a depth of understanding they would have had, had they been partners in the process. The teachers, themselves, also developed expertise, as a result of discussing and reflecting collaboratively.

> Pam: Now you take things on board that you feel confident with. And I felt confident with that so I did take that on board well.

Thus these teachers recognised both their own and each other's expertise as well as that of the researcher. While the researcher remained the "expert" in relation to mental computation, the teachers were considered "experts" in their classrooms. The researcher's expertise supported the

teachers' content knowledge and pedagogical content knowledge. The teacher-researcher relationship was deemed to be of mutual benefit.

Motivation

Finally, this current study identified the part played by *motivation* to professional learning. As indicated by Millett and Bibby's (2004) theory, the majority of the teachers were primarily stimulated to learn by external motivators such as a new policy, curriculum document or accountability requirements. Here it was claimed that without the need to meet new external requirements, most teachers would lack motivation. This certainly proved to be true for the majority of the teachers in this present study. However, the reaction of Pam and Sue to the project suggests that we should not discount internal motivation.

While external motivation in the form of curriculum change might have played a small part within this current study, internal motivation seemed to be the driving force for Pam and Sue.

> Pam: And at the time I was looking for something a little bit different in professional development too. I mean I'm only going to be teaching a couple more years, before retirement… You need to take the challenge. When the challenge goes out I'll retire.

These teachers had begun to lead their own learning. However, this did not automatically extend to the whole school community. In contrast, the other teachers in the school were not interested in pursuing PD in relation to mental computation, although mental computation was mentioned in the draft syllabus at that time.

> Researcher: During the project, was there talk throughout the school about mental computation or was no one else interested?

Pam: No, no one was interested. The syllabus was only in draft form then.

Beyond this initial internal motivation, the fact that the Pam and Sue chose to plan individually may have sustained their motivation through the study. By having to develop their own lessons, the teachers in the present study appear to have acquired a sense of ownership of the lessons supporting the work of Joyce and Showers (1995).

Conclusion

This project highlighted the importance of time, talk, expertise, and motivation, the "four key conditions necessary for the realisation of rich zones of enactment" (Millet et al., 2004, p. 250). Within this project, time and talk enabled trial, reflection, discussion, and modification (of subsequent lessons). External expertise was deemed to be essential to support teachers' learning with respect to new content knowledge and pedagogical content knowledge. It seemed that time, talk, and expertise complemented internal motivation, resulting in "deep change" (Millet et al., 2004, pp. 246-250). In the follow up interview during phase four of this study, it is worthy of note that the supports (i.e., time, talk, expertise, and motivation) offered during the conduct of the teaching experiment had not been continued. In addition, the school now has only one of these teachers to draw on as the expert. No time has been allocated for her to support her colleagues in the PLC that incorporates the whole school. However, in order for collaborative knowledge construction to extend to the full school, external motivation, prompted by policy change, is not sufficient on its own, and time becomes the critical issue. In addition, these two teachers needed two years of ongoing assistance from the external expert to provide the necessary expertise for them to develop a sense of agency and to lead their own learning and that of their students in this new content area. In contrast, opportunities for talk to assist in leading

the learning of colleagues appear to be restricted to staffroom discussions. As a way forward, it

seems that focusing on the PLC (Millett & Bibby, 2004) while at the same time recognising that

change is implemented through phases is critical for understanding educational change. For this

study, the PLC needed the ongoing supports of time, talk, expertise and motivation to carry them

through the phases of change. Of particular importance here has been the teachers' participation

in acquiring new/background knowledge so that they understood 'what' should change as well as

'why' it should change. These questions led them to vision 'how' they should go about this

change. With the support of ongoing expertise, the teachers collaboratively as well as

individually planned and then implemented a phase of action and reflection. It seems that critical

to the success of progression through the phases and them gaining in momentum has been the

provision of time, talk, expertise and motivation. This theoretical perspective combines the

features of Fullan and Stegelbauer's (1991) and Millett and Bibby's (2004) models resulting in a

new model as seen in Figure 4.
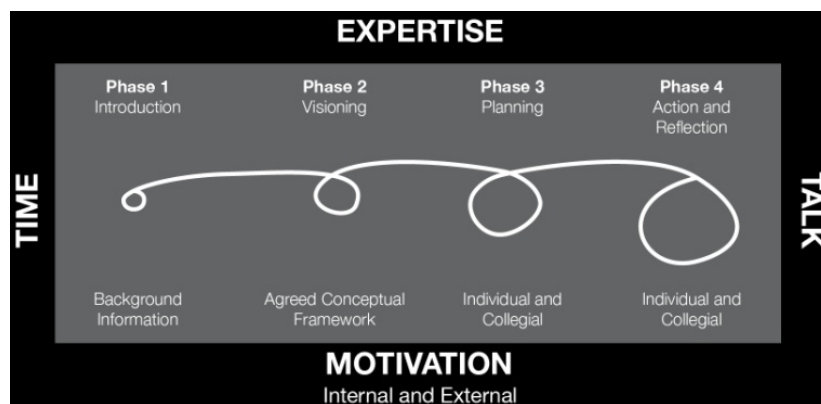
Figure 4. Model for discussing ongoing educational change within a PLC

Further research on the theoretical model developed here is necessary. It is important to work

towards an understanding of establishing a professional learning community where; time, talk,

expertise and motivation are provided or inherent. Under these circumstances careful tracking of

the teachers within the PLC through the phases of introduction, visioning, planning and action reflection would lead to a greater understanding of sustainable educational change within a PLC as it goes through phases of implementation.

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Mathematical Reasoning in Service Courses: Why Students Need Mathematical Modeling Problems

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Abstract: *In this paper we argue that conventional mathematics word problems are not aligned with the typical learning goals and expectations partner disciplines, especially business, have in requiring that their students take mathematics courses. Using the taxonomy of educational objectives presented by Anderson and Krathwohl (2001) we show how mathematical modeling problems can be used to promote the needed alignment and contrast two examples to illustrate the differences. We then demonstrate how the more conventional word problem can be rewritten as a modeling problem. Sample assessment materials and instructional activities are included to support teachers in making the transition to the use of modeling problems.*

Lynn Steen, in reference to a broad study of quantitative literacy, claims that "Most students finish their education ill prepared for the quantitative demands of informed living"

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(Steen, 2004, p. 11). If this is true of life in general, then we believe this is particularly true of student's preparation for the use of quantitative tools in the workplace, especially in the business world. Moreover, we are amongst those educators who are trying to address the concerns of employers who remark that employees who were "A" students in statistics cannot even use simple graphs to analyze day-to-day problems. The reason for this situation, we think, is not that the students are stupid or the offered courses are bad. The problem, we believe, is that the mathematics textbooks for undergraduates are not targeted toward the goal of applying quantitative methods and techniques to real world problems. Since the problem sets students work from these texts are the backbone of what students will take from the course, a course built around such a text is simply not designed to promote this kind of thinking.

Judging from our fairly extensive review of textbooks designed to support mathematics courses for business undergraduates, we feel secure in making the claim that the majority of such textbooks are designed to teach students particular mathematical techniques and procedures rather than to help students develop thinking skills necessary for analyzing the kinds of quantitative information they will encounter in their professional lives. While there are few such formal studies at the college or university level, studies of pre-college textbook problems illustrate that this imbalance is extensive and persistent (e.g., Vincent and Stacey, 2008; Kuln, 1999; Witt, 2005). The Association for the Advancement of Collegiate Schools of Business [AACSB] indicates that there exists a very different purpose in requiring these students to enroll in mathematics courses. They describe general learning goals, such as "communications abilities, problem-solving abilities, ethical reasoning skills, and language abilities" (p. 60). Throughout its accreditation standards, AACSB maintains that "accreditation does not mandate any particular set of courses, nor is a prescribed pattern or order intended." (p. 69) With regard to mathematics,

the standards make clear the expectation that students in management programs will have learning experiences in "analytic skills" (standard 15, p. 71). Moreover, the standards make reference to only a single math-specific learning outcome: " [P]rograms…will include learning experiences [such as] statistical data analysis and management science as they support decision-making processes throughout an organization." (p. 70)

In addition, a report from the Curriculum Foundations Project (Lamoureux, 2004) of the Mathematical Association of America [MAA] states that "In general, business faculty are less concerned with specific course content than with developing quantitative literacy and analytical ability in our students." The report further defines this literacy as being "comfortable with using mathematics as a tool to communicate analytical concepts" (p. 19) and clearly explains the need for such an educational objective:

> Business decisions are most commonly made under conditions of uncertainty and risk. Inferences must be drawn from data and information that is incomplete, inconclusive, and most likely imprecise. Wherever possible, math courses should attempt to illustrate this ambiguity and provide guidance in dealing with such uncertainty and variation. (Lamoureux, 2004, p. 19)

These ideas and broad goals are echoed by many of the other disciplines that participated in the Curriculum Foundations Project. Broadly phrased, these recommendations indicate that partner disciplines are interested in problem solving skills, mathematical modeling, and communication (CRAFTY, 2004, pp. 3-4). It is our argument that these objectives are not being addressed by the current slate of texts, problem sets and supporting materials. We will focus our analysis on applications to the business mathematics curriculum, as that is the area with which we are most familiar, but these ideas can, we feel, be extended to almost any of the partner disciplines.

To help support our claim, we make use of the taxonomy of educational objectives provided by Anderson and Krathwohl (2001), a revision and extension of the original Bloom's taxonomy. The taxonomy describes educational objectives, learning activities and assessment processes using a two-dimensional framework (p. 5). One dimension describes the type of knowledge being used or developed (knowledge domain) as either: factual, conceptual, procedural or metacognitive. The knowledge domain is then cross-referenced against a cognitive process domain describing what students are doing with that knowledge: remembering, understanding, applying, analyzing, evaluating, or creating. It is our experience that the textbooks, classroom activities, and assessments for mathematics courses typically offered to business students focus primarily on two categories of learning: remembering factual knowledge and applying procedural knowledge. However, the motivation described by the AACSB and the MAA for students to complete mathematics courses as a part of their business curriculum seems to fall at quite a different level of the taxonomy. While the "analytic skills" objective clearly matches up with applying procedural knowledge, the other expectations – problem solving, quantitative literacy, and communication – are quite different. When combined with the idea that such students need to understand that their information is incomplete and uncertain, we see that many more areas of the taxonomy are involved. Even by itself, problem solving in realistic settings tends to reach into the upper three levels of the cognitive process domain (analyze, evaluate and create) and touch, potentially on all knowledge domains (p. 269). Of particular importance, though, is the metacognitive domain. True problem solving requires that the problem solver engage in reflective thinking. This is typically absent in most procedural mathematical activities. The expectation that students will learn to communicate their mathematical solutions requires that students experience activities from the "understand" level of

the cognitive process domain, an area widely acknowledged as lacking in most procedure-driven approaches to teaching mathematics. These criticisms apply equally well to the idea of quantitative literacy, which, depending upon the specific definition used, could fall nearly anywhere in the taxonomy. Thus, a severe disconnect is apparent; students are expected to be able to apply their mathematical knowledge in a variety of settings as a result of taking such classes. Textbooks designed to support these classes, however, do not typically provide adequate materials to support such learning objectives. Instead, they require mostly lower-level thinking involving the use and development of particular mathematical skills.

We propose that this misalignment can be best addressed through a curriculum centered on mathematical modeling problems, rather than a curriculum focused on mathematical procedures. When solving mathematical modeling problems, the modeler begins with a real world situation. The modeler then constructs a model world by making assumptions. This involves the tasks of abstraction, simplification, and quantification of real-world phenomena and events. The modeler analyzes the problem in the model world using mathematical tools and techniques, and then transitions back to the real world by making meaning through interpretation, evaluation and communication. Often, this process is repeated through many cycles, with each cycle informing the next. In a classroom context, using the real world as a source of problems presents a variety of serious challenges. For this reason, we, in accord with Lamoureux (2004, p. 22,) advocate using *realistic* rather than *real world* contexts, problems and data. A realistic context is one that is ill-defined, requires the use and interpretation of information in a variety of forms, both quantitative and qualitative, and a need for communicating the results to an authentic, appropriate audience. The data provided to students for such realistic problems is typically invented, rather than genuine, but should have a great deal

of variability and noise built into it. This allows the creator of the data to include the features needed for teaching specific concepts while still providing some of the ambiguity of real data. As such, these realistic problems stand as educational proxies for what students will encounter on the job in real world contexts.

At our college, we have spent quite a few years developing a new course in modeling to help business students develop deeper mathematical understanding through applying this understanding to the business world (Green & Emerson, 2008a). We have three primary objectives in this course:

1. That students develop facility with certain modeling and technological techniques that are useful in dealing with real world data,
2. That students learn to analyze and interpret the results of those techniques in realistic contexts, and
3. That students learn to communicate their findings in a realistic context.

In order to align our teaching and our assessment tools with these objectives, then, we must have students explore problems that are typically ill-defined and are contextually rich. Such problems are distinct from the kinds of "word problems" present in most mathematics texts, which tend to present closed, template problems that are only superficially contextual. By way of illustration, we will contrast two problems and their solutions. Both problems are relatively elementary in their mathematical demands; indeed, they were selected for this apparent simplicity. One problem is drawn from a typical textbook designed for an undergraduate mathematics course for business students. The other is an example of a mathematical modeling problem drawn from our course. After comparing the two problems, we will discuss the features of the modeling problem that make it more appropriate for meeting the goals of courses like ours, and then illustrate how the more standard problem could be adapted to this modeling framework. We conclude with practical recommendations for teaching and assessing in a mathematical modeling framework.

These practical matters are important to consider, since the modeling framework often opens up multiple solution paths using many different approaches. Ensuring student learning with respect to one's goals then requires slightly more consideration than the assessment of student learning in a mathematical procedures course.

I. A TALE OF TWO PROBLEMS

The problem shown in figure 1 is an example of a mathematical modeling problem that is closely aligned with the recommendations of AACSB (2006) and the MAA (Lamoureux, 2004). Notice that it is stated as a memo from a boss at a fictional consulting firm. This was done for a variety of reasons. For one, we found that we could not get the depth of reasoning we desired from our students using the standard textbook method of presenting problems. Good mathematical modeling requires that students explain their thinking and make explicit their assumptions and inferences. In order to do this, we had to form a realistic rhetorical context that provided an appropriate audience for communication. In this sense, we agree with the definition of a "mathematical problem" as presented in Falsetti and Rodriguez (2005) which focuses not only on the format of the problem and its difficulty, but also on the relationship between the statement of the problem and the problem solver. Roughly speaking, they claim that without a motivation to solve the problem, there is no problem to solve. Our memos present a realistic context for solving the problem. This involves the extrinsic motivation of trying to keep the boss happy as well as a more intrinsic motivation derived from the problem's relation to the business world. This format also allows us to present a real world problem while still providing some initial filtering and scaffolding without setting up the model world for them. In contrast, the authors of most mathematics texts for business students begin in the model world and neglect or omit the transition from the real world to the model world needed to solve the problem, thus

depriving students of the experience of having to make such a transition. Without experiences of this kind, there is no reason to expect students to be able to transfer their knowledge from one context to another.

Memo

To: Job Placement Staff

From: Project Management Director

Date: January 14, 2010

Re: Placement of Managerial Clients

Since our company does management consulting, we have two middle-management clients who have come to us looking for new positions. Each of the clients is aware of and is qualified to work for each of the four large companies in the local region. I need you to analyze the four companies in the attached data file and make a recommendation to each client as to which company each would be better suited to. The data file contains a list of the management salaries at each of the four companies. There are about the same number of managers in each company with roughly the same ratios of middle- to upper managers in each.

Each of our clients has just moved out of the lower 25% management salary rank in his or her previous position. They are, however, quite different. Manager A is a confident go-getter who enjoys leaving the competition behind. Manager B, on the other hand, prefers to run with the pack. He wants to do well, of course, but stability and security are important.

To get started, you might consider generating comprehensive summary statistics and side-by-side box plots for these four companies. Based on what you learn from this information make a recommendation of a company for each client. Be sure to provide as much evidence as possible.

Attachment: Excel data file "C04 Companies.xls"

Figure 1. An example of a mathematical modeling problem for business students.

We devised the conventional mathematical problem shown in figure 2 based upon presentations of similar material in a variety of standard textbooks designed to support mathematics courses for undergraduate business students. This problem stands in stark contrast to the modeling problem (figure 1). We see that very little evidence is given in order to solve the modeling problem. There is mention of personality types and files listing salaries (the data file provided contains salaries for 100 employees at each of the four companies), but it is not clear

how this is sufficient to answer the question. The only clue given as to how to go about answering this question is provided in a suggestion about using boxplots; however, there is nothing to indicate how to interpret these so as to arrive at a final decision. The only aid students have for making such interpretations are some thumbnail psychological descriptors that must be translated into mathematical constructs. Furthermore, student assumptions about the time horizon play a critical role in balancing the different factors and arriving at a final decision. The memo clearly states that we are looking for an evidence-based decision, but the particular decision, the rationale for it, and the organization of the results are open-ended. We can also see that there is no generic procedure that will lead to a conclusion; if different descriptions of each manager were given or if the salary data were different, students would follow different solution paths. Once students have developed a solution to the memo, they could be encouraged to think more deeply about the problem and consider other factors or interpretations of the psychological factors. This could also offer students a chance to correct any mis-readings of the graphs or statistics, which in turn could influence their interpretations of the problem. Finally, the memo states that each of the managers is aware of these four companies, so that students will have to support their choice of one company over the others by making well-justified decisions, using mathematical tools to support their judgments.

In contrast, the conventional problem (figure 2) provides a small sample of data, and except for the necessary formulas, everything needed to solve the problem is provided in the problem statement. These formulas would be found in the chapter in which this problem appears, although the problem statement clearly identifies which techniques to use and how to decide what the results of the calculations mean. During the solution of the conventional problem, students do not need to make any assumptions. Furthermore, unlike the memo problem, there is

one set of mathematical procedures applicable to the problem and one way to interpret their results. In the solution, instructors would look only for the correct use of these two mathematical procedures. These procedures are also generic, in the sense that they could be used on any problem like this; although the results would change if the data were different, the process would be identical. Asking students to revise such work would result in correcting a calculation, but would not necessarily lead to any deeper understanding of the problem or the context. Finally, we see that the conventional problem expects nothing more than a set of calculations; the answers to each part of the problem are numbers, with one identified as being larger than the other. At no point are students required to re-contextualize these answers and make a judgment about the original question concerning the choice of an investment.

Investment Analysis. Paul Hunt is considering two business ventures. The anticipated returns (in thousands of dollars) of each venture are described by the following probability distributions:

Venture A		Venture B	
Earnings	Probability	Earnings	Probability
-20	0.3	-15	0.2
40	0.4	30	0.5
50	0.3	40	0.3

a. Compute the mean and variance for each venture.
b. Which investment would provide Paul with the higher expected return (the greater mean)?
c. In which investment would the element of risk be less (that is, which probability distribution has the smaller variance)?

Figure 2. An example of a conventional mathematical problem for business students.

II. SOLUTION AND ANALYSIS OF THE MODELING PROBLEM

The memo clearly suggests that students look at the salary structures of the companies by way of side-by-side boxplots. The results of generating these boxplots from the salary data are given below (see Figure 3).
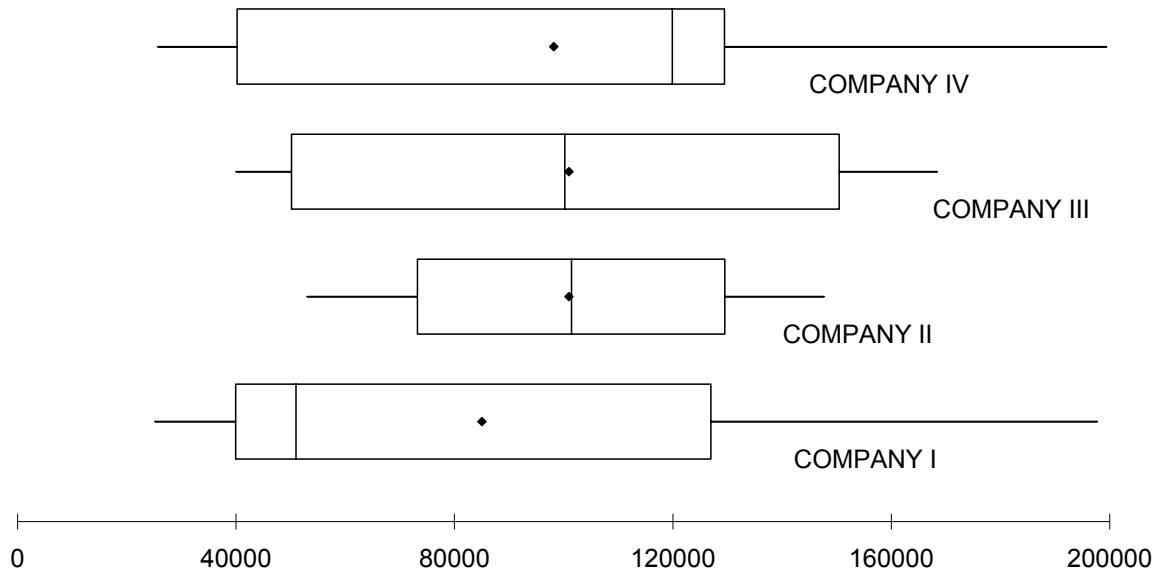
Figure 3. Salary structure of the four companies in the modeling problem.

Now that we have these graphs, what can we do? Obviously, we need to separate the problem into a solution for each managerial candidate. In looking at the two managers, we see that we cannot jump to a final solution for each candidate without deeper analysis; we need to consider what each candidate likes about each company. For this, we will need to look at all the features of the boxplots, and then, finally, assemble some coherent picture of what each candidate likes and dislikes about each company in order to make a final judgment regarding the placement of each manager. We will start by analyzing candidate A. For this candidate, notice that the graphs indicate that each company has something to offer her. For this reason, we seek to find reasons to eliminate companies from the mix, rather than produce supportive arguments.

We assume that a "confident go-getter" will like the highest maximum salary possible. This leaves manager A with either company I or IV. Because manager A "likes to leave the competition behind" we assume that she would prefer to be in a company where the salaries are spread out at her rank; thus, we look for a company with a wide range of salaries between the

first quartile and the median, so that A can stand out from the other managers. We also make the assumption that one's position within the salary structure, not one's absolute salary, is directly related to one's power in the company. Thus, although the salaries are quite different for a person in the middle of the second quarters at companies I and IV, a manager at this salary position would have the same responsibility at each company. A will reach $120,000 at the end of the 2nd quarter in IV, whereas she will reach less than $65,000 at the end of the 2nd quarter in I. We submit that A would prefer to make small increments on high salaries in the 3rd quarter at IV than large increments on much smaller salaries in the 3rd quarter of I. By the time A reaches the third quartile in each company, she will be making about $130,000 at either one, but she will have earned far more in IV during the 2nd and 3rd quarters combined than in I. Thus, we eliminate company I from consideration.

Although company II has the highest entry salary for our manager (it has the highest first quartile) and its third quartile is higher than at companies I and IV, the fact that the maximum salary at company II is about $55,000 below the maximum salaries at I or IV would appear to be a strong negative for such a go-getter, and its short interquartile range does not provide much scope for her competitiveness. Taken together, we are left with either company III or company IV as recommendations for manager A. On the other hand, if we were to make a different assumption about manager A's preferences (e.g., the short-term goals are more important than the long-term goals) then company II might become a more viable solution.

Company III has a higher starting salary than company IV, which is in its favor. The spread of the second quarter also allows manager A an opportunity to distinguish herself. That III has a longer interquartile range (in fact, the longest of any of the companies) in conjunction with the fact that it is situated higher on the salary scale than is IV's (both Q1 and Q3 of III are

respectively higher than Q1 and Q3 of IV) might lead one to conclude that A would earn more money over her entire career at III, even though III's maximum is about $35,000 less than IV's. However, it is not clear that such is the case. If we look at A's position at the half-way point in the 2nd quarter of each of III and IV, we would see that A would actually make more money in company IV in this quarter, assuming that she moves at approximately the same rate throughout a quarter. When we regard A's position at the half-way point in the 3rd quarter in each of III and IV, we see that although A would make more money in III it is not clear that the amount she would gain in this quarter would compensate for what she lost in the 2nd quarter. We judge, therefore, that in the light of this stand off, that the greater maximum in IV (about $35,000) has sufficient weight for us to recommend IV over III for A.

Analyzing manager B involves slightly more interpretation from the real world to the model world. Manager B's two preferences, running with the pack and stability, have direct implications for the boxplot characteristics he would prefer. We argue that running with the pack implies that B would prefer short quarters over longer ones and stability implies that B would prefer a company whose salary boxplot has the most symmetric quarters throughout.

A short quarter means that salary differences between the managers within this quarter are relatively smaller than in longer quarters; that is, the salaries tend to cluster or pack together much closer, which is what we mean by running with the pack if you are a manager in this quarter. In contrast, the salaries in longer quarters are more disparate so that there is no pack to run with. In a boxplot whose quarters are about the same length, we can assume that the salaries tend to be distributed equally in each quarter with about the same increments. The path for advancement, then, progresses steadily and predictably throughout the company, which would signal stability for our manager B. That is, there are no sudden jumps or huge differences in

salaries to upset B's boat, as would be the case if there were a mix of short quarters and significantly larger ones as are present at companies I and IV..

Company II is the clear cut choice for B. It has relatively short, symmetric quarters, as well as the highest entry salary by far (B wants to do well). Furthermore, II's relatively high third quartile (same as companies I and IV) would suggest that II would be very competitive for accumulating money throughout the 2nd and 3rd quarters combined.

We see then that the modeling problem has many different aspects which fit into different areas of the taxonomy. Obviously, students must correctly construct the boxplots in order to begin the problem-solving process. This appears in the taxonomy under "Apply Procedural Knowledge." The problem also requires factual knowledge, in that the students must be able to read the features of the boxplots, which falls under "Remember Factual Knowledge." Students must also make comparisons throughout their analysis; this fits under "Understand Factual Knowledge". But students must also differentiate between features of the plots that are important and those that are not; this higher level skill falls under "Analyze Factual Knowledge." At an even higher level, students must make a judgment regarding the placement of each manager. Judgments fall under "Evaluate" in the taxonomy; these specific judgments relate to balancing out different features at each company based on the profile of each manager, so students must "Evaluate Conceptual Knowledge." Students are also required to hypothesize about how the personality features play into the decision-making process; this appears in the taxonomy under "Create Conceptual Knowledge." Finally, the students must explain their thinking process and clearly describe the assumptions, comparisons, and judgments being considered. This clearly falls under the metacognitive knowledge domain: "Understand Metacognitive Knowledge."

Thus, in the modeling problem, students are dealing with almost all levels of the cognitive domain and every aspect of the knowledge domain.

III. SOLUTION TO CONVENTIONAL PROBLEM

In solving the conventional problem, we are first instructed to compute the mean and variance for each venture. To compute the mean of the investments, we simply multiply each expected return by its associated probability, then add these products. For venture A, we find the mean to be 0.3(-20) + 0.4(40) + 0.3(50) = -6 + 16 + 15 = 25. For venture B, we find the mean to be 0.2(-15) + 0.5(30) + 0.3(40) = -3 + 15 + 12 = 24. Computing the variance requires multiplying the squared deviation of each investment outcome from the mean by its associated probability and then summing these products. For venture A, we get a variance of 0.3(-20 – 25)^2 + 0.4(40-25)^2 + 0.3(50-25)^2 = 885. For venture B, we get a variance of 0.2(-15 – 24)^2 + 0.5(30-24)^2 + 0.3(40-24)^2 = 339. We have now completed part (a) of the problem.

Part (b) of the conventional problem asks which venture has the higher expected return. Since the expected return is estimated by the mean, we see that venture A has the higher mean and thus, the higher expected return (25 is greater than 24). The venture with the lower risk (part (c)) is that which has the lower variance, since it has the lesser spread from the mean, indicating less chance of returns that are extremely low. Thus, we see that venture B has the lower risk, and we have completed the solution of the stated problem using simple calculations and comparisons.

Part of the simplicity of this problem derives from its form. In the framework presented by Falsetti and Rodriguez (2005), this problem is not a true problem. Rather, it has already been presented as an "associated mathematical problem" that is phrased verbally. This removes the need for making assumptions and generating hypotheses. In effect, this problem requires only that students implement particular mathematical procedures. The problem even provides the

translation of these procedural results back into the real world with its parenthetical cues regarding the higher mean and smaller variance.

The conventional problem stands in stark contrast to the modeling problem with regard to the taxonomy. The conventional problem clearly requires the application of procedural knowledge in the use of the formulas for the mean and the variance of the investments. The problem also asks for a simple comparison of the values, an instance of understanding factual knowledge. But this is all the problem requires. However, the modeling problem shows the depth of thinking possible with simple mathematical tools, in direct contrast to a procedural problem like the conventional one presented here. Thus, simple problems do not have to be confined to lower-level thinking or to single domains of exploration. It is these instances of higher order thinking using quantitative information that the MAA (Lamoureux, 2004) and the AACSB (2006) seem to be interested in developing in business students completing mathematics courses.

IV. TRANSFORMING A CONVENTIONAL PROBLEM

Clearly, the conventional problem begins with a realistic motivation – deciding between two different business ventures. But it never requires students to make a judgment supported by their computations. And, quite clearly, the content of the two problems is different. This choice was intentional, since it gives us an opportunity to compare the two approaches by transforming the conventional problem into a modeling problem. Fortunately, this problem can be made more realistic without much difficulty.

To begin, the problem can be made to involve more than just computational procedures by requiring students to make and justify a decision: "Paul Hunt is considering two business ventures. Which should he choose?" This immediately opens up a variety of solutions, since students have to balance choosing between the venture with the higher expected return and the

venture with the lower risk. In justifying their decision, however, students will need to make and articulate assumptions about Paul Hunt's investment goals. The instructor is then evaluating not just the computations the student completes, but the argument the student makes and the support provided in that argument. By eliminating the explicit references to means and variance, it then becomes the student's task to determine which computational procedures will help compare the two investments. Students who only compute the expected returns could then be encouraged to think about risk upon revision, and students choosing other methods altogether (graphical methods, say) could be pointed toward the expected value and risk. Students using both tools in the computations, but failing to consider their implications could then be asked to revise their thinking. In all cases, the revision can both correct computational errors and direct students to deepen their thinking.

Some mathematics professors might object to our transformed problem for at least two reasons. First, mathematics students should not have to know what constitutes good business risks (that's the job of the business department, isn't it?). Second, students are not directed toward a unique answer; the problem is ambiguous since we are looking for more than one thing, so how do students know where to begin? For these reasons, these professors may say that we have removed the problem from the domain of mathematics. Our response to the first objection is simply that this course is designed to help a certain group of students, namely business students, learn how mathematics is related to their intended discipline. In response to the second objection, we argue that students should have the opportunity to learn how and when to make assumptions in order to be able to make reasonable decisions under conditions of uncertainty. This is the critical ability they need in order to transfer what they have learned in this particular context to other problems. Rarely will they encounter a problem in the real world with an

obvious statement about computing means and variances for a set of numbers, but they will often be required to make decisions among various options, each of which has its own benefits. In completing a problem such as the modified one above, students are practicing both their computational skills and their application of those skills. It is this type of knowledge that is most transferable, regardless of whether the student is majoring in mathematics business.

The above problems illustrate, in part, the contrast between mathematical and mathematical modeling approaches to teaching undergraduate business students. We would like to use these examples as a springboard to compare, in more general terms, the two approaches. It seems to us that there are at least nine aspects that differentiate them. These can be divided into two broad areas: those that refer to the statement of the problem itself and those that relate to the solution of the problem. Relating to the statement of the problem, we can explore the *nature of the evidence available*, the *connections to mathematical procedures*, the *types of assumptions needed*, and the *complexity* of the problem. When examining the solution of the problem, we look at its *uniqueness*, how it is to be *assessed*, its *robustness*, the *transferability of the techniques used*, and its *amenability to revision*. We summarize our comparison of the two problems using these nine dimensions in figure 4.

Aspect	Math modeling approach for preparing business students	Mathematical approach for preparing business students
Nature of evidence needed to solve the problem	Insufficient, possibly contradictory evidence provided	All "evidence" given in either the problem statement, that section of the book, previous math background, or teacher's examples
Connections to mathematical procedures	Unclear, not proscribed	Almost directly given
Types of assumptions made	Required to bridge the real world and the model world, can't make progress or a final decision without them	Only allowed to assert mathematical postulate or theorems (e.g. a quantity is either bigger than 1, equal to 1, or less than 1)
Complexity	Comes from the interaction between the mathematical world, the context/real world and the need to communicate the information	Comes from layering mathematical techniques on top of other mathematical techniques
Uniqueness	More than one possible solution path, each may have a separate justifiable final decision	One solution possible
Solution features being assessed	Does the solution make sense? Are the steps justified and supported?	Were the mathematical techniques performed correctly?
Robustness of strategy for solving the problem	Problem is sensitive to changes in the data or context, often resulting in a completely new problem strategy	Problem is insensitive due to its templated nature
Transferability	High for the general method of solving a problem, but not necessarily for specific solution techniques	Low, both on method and specified techniques because students didn't have to make the connections
Revision	Allows for a deeper understanding of the context, deeper reflection, and learning to occur	Amounts to "correcting mistakes"

Figure 4. Comparison of mathematical and math modeling approaches.

V. HOW DO WE ASSESS LEARNING IN MATHEMATICAL MODELING?

Assessing such modeling assignments has required us to change the focus of our grading practices. In part, this was to give students credit for the kind of work they were doing, rather than only penalize them for not doing the work in a particular way. In part it was to provide additional feedback to encourage students to rethink the assignments and submit revised versions of their work. After several attempts to use more conventional methods for grading, we developed our own system. It is flexible and easily adapted to other assignments or courses. Most importantly, it directly links assignment feedback and grading to course objectives. The

system, referred to as COGS (Categorical Objective Grading System), is explained in detail in Green and Emerson (2007).

Broadly, though, we would like to see a student's memo contain an introduction to the problem, an analysis of the situation that weaves judgment and reasoning together with mathematical statements and data analysis, and a concluding statement that reads something like "Assuming X and based on Y1, it seems that manager A would fit well at company Z; however, if Y2 were weighed more heavily, A might have a better fit at company IV." In other words, the student should qualify each piece of the claim. Some people might say this is not the way a mathematics problem should conclude. We argue, however, that this is the kind of quantitative decision making, along with explicit discussions of assumptions and alternatives, that people are willing to pay good money for in our uncertain world. Further, this is exactly the sort of logic and thinking that is expected of mathematicians and statisticians.

The difference between a really good and an adequate response to the modeling problem is that a merely adequate response assumes that the reader makes the same leaps of logic and assumptions as the writer, while a good response spells out, explicitly, the assumptions being made and how they help. For example, what does "stability" mean in this problem? Many students can interpret it to mean that a boxplot that is narrow and has equal-sized small quarter lengths is stable, but cannot or do not explain why without further prompting.

VI. HOW DO YOU TEACH STUDENTS TO DO THIS?

We hope to have shown in part that fairly high level mathematical modeling does not necessarily involve complex mathematics. In fact, we teach the mathematical procedures as needed, rather than teaching a fixed sequence of topics. Teaching for mathematical modeling then is not solely about having students reproduce complicated mathematical techniques, which

is the goal of many mathematics courses, even those designed explicitly to connect mathematics to a particular application area, such as business.

The focus in teaching mathematical modeling must shift then from ingraining mathematical techniques in students to teaching students how to negotiate the transitions between the real world and the model world. This includes helping them learn to identify useful data to collect and proper tools for analyzing and interpreting that data. In the conventional problem, the teacher's role is to teach students the analysis useful in the model world, and the transition from the real to the model world has taken place behind the scenes. In the modeling approach, the teacher's role is to challenge students to make the transitions between the two worlds. This allows for revision of thinking and deeper learning, which results in a greater opportunity for transfer of knowledge. Indeed, it is these negotiations that Dias (2006) most descries the absence of, calling for more emphasis on the "validation phase" of the modeling process. She describes the distinct lack of this aspect of modeling in the work of a student who was motivated to find a solution to a real business problem. The student brought much prior knowledge and provided many connections to the real world initially, but failed to use this effectively once she had mathematized the problem and gotten an answer. It seems, then, that it is not very natural for students to worry about the validation phase or the cyclic nature of the process. And while there are many possible explanations for this – including the obvious critique that most of the time, it has not been needed in their mathematical courses, so they have no practice with it, nor do they see a need for it – what remains is a valid need for helping students advance their thinking, focusing on these transitions. The format of the modeling problems we have presented provides one way for instructors to bring this component of modeling into their teaching and assessing more explicitly.

Another component useful for shifting the focus is found in technology. By letting technology take much of the burden of the mathematical procedures, students can focus their attention on the other aspects of modeling. This does not mean that you avoid teaching mathematical manipulations and calculations; it means teaching students to think effectively through the technological tools available. We provide in-class activities to help students learn to use spreadsheet technology as a tool through which to think. We also teach how the technology fulfills its supporting role in carrying out the mathematical procedures. This, in turn, helps them to understand the mathematical procedures themselves, but from a different perspective than is traditional. In addition, we spend class time in small group discussions on the differences between a perceived and actual problem, on the possible causes of a problem, on how data might help us to understand the problem and its causes, on asking questions to more deeply explore the problem situation, and on the data that might be collected and how it could be organized to promote certain types of analysis. This is the first step in modeling – making the transition from the real world problem situation to a model world. Explicit discussion and assessment of these aspects of modeling are required if students are to value them, so we have memo assignments that specifically target these components of the process.

The memo problem format also helps students learn how to close the loop and attempt a genuine validation phase for their modeling also. Because each memo assignment requires that the response be readable by "the boss" students must include some sort of executive summary of their work. And because our evaluation of this work is formative, we provide feedback and encourage revision of the work. The feedback itself is often sufficient to point out gaps between the solution and the explanation or between the problem and its mathematization, but class discussions also follow each memo assignment. With our process of memos, feedback and

revision, a student with a solution like that in Dias (2006) would receive targeted feedback and questions encouraging deeper thought and re-thinking of the problem and the proposed solution so that the final version of the solution contained a deeper level of analysis.

Teachers should expect that students will find this difficult. They are on unfamiliar ground. A central example we use to help students relates to age discrimination at a company. Students are presented with a memo from the CEO of the company that he has been hearing complaints that the company is unfriendly to older workers. As a result, it is believed that older workers are leaving the company in large numbers. Students then receive three responses from different managers describing how they would solve this problem. The responses are filled with unstated assumptions and hidden beliefs about the situation. Students work in small groups during class to analyze the responses and identify these assumptions and beliefs. Eventually, they discover that each of the managers is making a huge assumption: that a problem actually exists at the company. Students next work together to identify ways to collect data to determine whether there is a problem at the company and if so, the cause of the problem and steps to remediate the situation.

This process deals with a critical concern of many instructors: teaching for transferability. We believe that the process of having students interrogate the problem situation to identify assumptions, possible causes, and useful data is what students can learn to transfer to other contexts. A fundamental component of this is that students must learn to make and refine their assumptions about the problem. This is a matter of teaching priority. Conventional mathematical problems focus primarily on model analysis with a minor emphasis placed on interpretation of the results. While differing from problem to problem, mathematical modeling problems taken as a whole place roughly equal emphasis on all three aspects of realistic problem solving:

constructing the model world, analyzing the model world with mathematical tools, and interpreting the results of the analysis. Notice that when the emphasis is placed on the techniques and procedures of analyzing the model, as is the situation with conventional problems, students have little intrinsic motivation for doing the procedures correctly. However, when the interpretation and analysis of the problem, the really crucial aspects of problem solving, are dependent on the procedures, students are more invested in correctly selecting and implementing procedures appropriate to the problem and the data at hand.

An example that further illustrates the nature of all three aspects of mathematical modeling relates to one of the later memos from our course where students are asked to investigate a company in order to determine whether there is evidence of gender discrimination. If there is, they are required to determine the extent of the problem. Students need to decide what data to collect and compare on each employee, how to code this data for analysis, and what kinds of variables (categorical, numerical, and/or interaction) to include in their models. Once they have collected and organized their data, they build and analyze models (typically multiple regression models) to determine which of their variables are significant, as well as determining the accuracy of the model in comparison to the predicted gender differences. After homework and classroom discussions about the process of collecting and organizing the data, we present the students with a common database of employee salaries to use in their analysis and reporting. Students can then construct their models to predict the salary of an employee, and implement their process of reducing the model by eliminating variables that are not significant. This refinement involves several iterations. Once a final model is constructed, students interpret the model and provide a judgment regarding the strength of the gender discrimination case against the company, based on their analysis. The data we provide, in particular, is very ambiguous

about the case, allowing students the freedom to emphasize different aspects of their models and the data in order to support their claims.

We simulate this cyclic process with a similar problem on the final exam for the course. Students are provided with the context of the problem and the variables. They then receive output for three different multiple regression models using different combinations of variables and are asked to first explain how each model was produced and why, and then to select one model as the most useful or accurate with a detailed explanation. The last part of the problem requires them to use their selected "best model" to determine the likely outcome of the gender discrimination suit. By providing all the procedural components, students can focus entirely on the interpretation and validation aspects of modeling.

VII. CONCLUSION

Teaching can, ultimately, be broken into three components. First, one sets instructional goals for learning. Then, one develops instructional activities to promote this learning. Finally, one must assess what the students have learned. Problems occur when these three components are out of alignment, as is the case in many mathematics courses for business students which use problems like the conventional problem discussed above. We have seen how the taxonomy introduced by Anderson and Krathwohl (2001) helps to place our instructional activities in order to provide the needed alignment. The general principle that we have presented here is that mathematical modeling problems promote a stronger alignment between the goals of a course compatible with MAA (Lamoureux, 2004) and AACSB (2006) expectations and the assessment of student learning in such courses.

However, creating better prepared students requires more than just changing the types of problems to which they are exposed. Instructional design must also change to align with the goals and assessment tools. This involves, at the least, the following:

1. Incorporating technology to support the implementation of mathematical procedures,
2. Having students articulate their assumptions and judgments through writing,
3. Providing feedback so that students may re-think the problem and revise their work, and
4. Providing classroom activities that support the learning and mimic the authentic problems being used.

One tool that we have found invaluable in implementing this change is the introduction of a memo as a way of providing a rhetorical context (Green & Emerson, 2008b). This gives students permission to make assumptions. In order to develop quantitatively literate students, the professed reason for putting business students through a mathematical experience, educators must give students this permission. In this way students practice and prepare for the real world, making judgments and selecting paths through a problem, using quantitative information to support their analysis. They are forced to make a choice among various possibilities and to justify their reasons for doing so.

It is easy to see why mathematical modeling problems have not been more commonly used. Most teaching involves making some distinction between the subject matter being taught and everything that is outside of this. Teachers compartmentalize and focus in order to attempt to control for different background knowledge outside of the subject area. Mathematics teachers are no more guilty of this than any other teacher, but it is necessary to show students how to (and convince them that they can) connect the compartmentalized pieces of knowledge to the real world. Despite this necessity, in business mathematics we tend to see mostly mathematical, not mathematical modeling, problems used. Ultimately, the alignment of our teaching practices and our assessment tools with our curricular goals requires that we make such a change.

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Randomness: Developing an understanding of mathematical order

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Is randomness merely the human inability to recognise a pattern that may in fact exist?

The three activities described show how order can be found in seemly random activities. The author has found that by using these activities on randomness, his students have developed a greater understanding of mathematical pattern and sequence. The teaching mathematical concepts in this way, engages and reinforces learning. It puts the ideas learnt into a setting and allows time for those ideas to be developed without any of the maths hang-ups which can sometimes occur in the classroom. By taking the maths beyond the classroom, we can more clearly illustrate the connections between the real world and what they are studying in school. In so doing students and teachers alike are enthused by the wealth of resources they have all around them in their own environments.

Understanding if events are random or have some underlying structure is a fascinating area of mathematics, filled with great discoveries. To understand whether the present spread of swine flu throughout the world has some structure, or is just random pockets of disease, will save lives. If there is a pattern, then finding this could enable countries to stop its spread. In the 1920s mathematicians Kermack and McKendrick (1) pioneered work into understanding if a set of results was randomly generated or had some underlying pattern. One of the first uses of these techniques was to predict the spread of disease.
As humans we find it very hard to deal with randomness. Psychologists call it Confirmation of Bias – with a new idea we attempt to prove it correct not wrong.
For example given the data 2,4,6 and asked to guess the rule, most people would say the numbers go up in 2`s in this pattern and so the next would be 8 and 10. Yet equally well it could be that the pattern is increasing and so the next numbers are 7 and 12.
The Philosopher Francis Bacon said "the human understanding, once it has adopted an opinion, collects any instances that confirm it, and though the contrary instances may be more numerous, either does not notice them or else rejects them, in order that this opinion will remain unshaken" (2)

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The concept of randomness is merely an attempt to characterise and distinguish types of sequences which confuse most people. It seems almost irrelevant to think about how it has been generated: flip of a coin, Geiger counter or practical joker. What matters is the effect on those who see it.

Which is more random, a string of heads or tails, or alternating heads and tails?

A game that was created by Walter Penney in 1969 (3), and is based on observing the occurrence of groups of heads and tails when repeatedly throwing a coin.

Let your opponent (1st player) select any sequence of three coins and then, referring to the table below, you choose the relevant 2nd players choice next to it according to the chart. You then record a sequence of coin throws looking for one of your three coin sequences in the long chain of throws, such as HTHTHHHHTHHHTTTTHTHH. The winner is the person whose pattern appears first.

1st player's choice	2nd player's choice	Odds in favour of 2nd player
HHH	THH	7 to 1
HHT	THH	3 to 1
HTH	HHT	2 to 1
HTT	HHT	2 to 1
THH	TTH	2 to 1
THT	TTH	2 to 1
TTH	HTT	3 to 1
TTT	HTT	7 to 1

At first glance you would think that the game is completely fair and not biased in any way, but in fact whatever sequence is selected by your opponent, you can always select a sequence which is more likely to appear first.

The maths of this order from apparent randomness can be seen by looking at the following three cases:

- If your opponent chooses HHH, you then choose THH (as in the table). The one time in eight that the first three tosses of the coin is HHH, your opponent wins straight away. Yet in all other cases, if HHH is not in the first three tosses of the coin, then THH will occur first.
- If your opponent chooses HHT, you then choose THH. The chance that HHT occurs first is conditional on either getting HHT or HHHT or HHHHT etc

$$P(\text{ HHT before THH }) = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = \frac{1/8}{\left(1 - 1/2\right)} = \frac{1}{4}$$

Hence $P(\text{THH before HHT}) = 1 - \frac{1}{4} = \frac{3}{4}$

- If your opponent chooses HTH, you then choose HHT
  let x=P(HHT comes before HTH).
  Ignore any leading Ts, and if we look to see what happens after the first H. Half the time the next throw is H, and then HHT is more likely to occur before HTH.
  Half the time the next throw is T, but if this is followed by another T, we are back to the beginning, hence you can write $x = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot x$

  This gives, $\frac{3}{4}x = \frac{1}{2}$, so $x = \frac{2}{3}$

The reasoning behind the other cases follows in a similar way, by putting the first last. You can see from the table that your choice as the second player has a greater chance of appearing before your opponents in each case. This is why on average the second player should win over a group of say ten games.

As well as looking at the theory, students should be encouraged to play the game. This practical aspect of mathematical development is often overlooked in education and often leads to a richer understanding of the subject. (4)

A variation on Penney's Game is **The Humble Randomness Game** and uses a pack of ordinary playing cards. The game follows the same format using Red and Black cards, instead of Heads and Tails. Yet due to the finite number of cards in a pack you can show that the second players chance of winning is greatly increased.

When we know that the event is random how can we deal with choices?

The Game of Googol was invented by John Fox in 1958 (5). This game is played by asking someone to take as many slips of paper as he pleases, and on each slip write a different positive number. The number can be as small or large as they please, hence the name googol.

The slips are then turned face down and shuffled. You then, one at a time, turn the slips face up. The aim is to stop turning when you come to the number that you guess

to be the largest of the series. At no point can you go back and pick a previously turned slip. If at the end you have turn over all the slips, then you must select the final slip.

You may think that the chance of finding the correct slip is $\dfrac{1}{N}$, with N being the number of candidate slips. Yet this is far from true if you use a clever strategy.

Regardless of the number of slips in the Game of Googol, the probability of picking the largest number, using a good strategy, is around 37% or $\dfrac{1}{e}$

The game has generated many interesting applications, such as how to optimise the selection of your BEST partner, how to select the BEST job applicant and which is the BEST motel to stay at. (6)

A new application suggested by the author is **How to Optimise Your BEST Buy in the Sales**. The work of Psychologists and experimental economists has shown that people tend to stop searching too soon. Given you have 20 shops to visit how do you know when to make the purchase?

With the Game of Googol strategy you would first visit 7 shops, making a note of the BEST bargain up to this point. Then use this "BEST bargain so far" as a reference for future shopping. Once you find a better BEST bargain than the one you found in the first 7 shops, you buy it.

Here is how the theory works. Given N shops, select the BEST bargain in each shop you visit. Reject an initial number of r shops, and then choose the first BEST bargain which is better than all of the ones so far.

BEST bargain is in one of N shops each with a chance $\dfrac{1}{N}$

If BEST bargain is in the first r th shops, it is rejected, but if it is in the r+1th shop it is certain to be selected.

If in the r+2 th shop, you cannot be sure if it is selected or not. It will only be chosen BEST, if best so far is in the initial r shops.

Chance of this is $\dfrac{r}{r+1}$

If in the r+3 th shop, it will only be selected if best so far is in initial r, the chance of this is $\dfrac{r}{r+2}$ and so on.

$$P(r) = \sum_{k=r}^{N} \text{P(K th bargain is the BEST).P(K th bargain is selected/it is the BEST)}$$

$$= \frac{1}{N}\left(1 + \frac{r}{r+1} + \frac{r}{r+2} + \frac{r}{r+3} + ... + \frac{r}{N-1}\right)$$

$$= \frac{r}{N}\sum_{k=r}^{N-1}\frac{1}{k}$$

Since $\displaystyle\sum_{k=r}^{N-1}\frac{1}{k} \approx \int_{r}^{N-1}\frac{1}{k}dk = \ln\left(\frac{N-1}{r}\right)$ due to Euler-Maclaurin with approximate error of

$\dfrac{1}{2}\big(f(N-1) + f(r)\big)$

Hence $P(r) \approx \frac{r}{N} \ln\left(\frac{N-1}{r}\right)$

To find the maximum, differentiate and set equal to zero

$\frac{d}{dr} P(r) = \frac{1}{N} \ln\left(\frac{N-1}{r}\right) - \frac{r}{N}\left(\frac{1}{r}\right) = 0$

$\Rightarrow e = \frac{N-1}{r}$ or $\frac{r}{N-1} = e$

In the short run, chance may seem to be volatile and unfair. Considering the misconceptions, inconsistencies, paradoxes and counter intuitive aspects of probability, it is not a surprise that as a civilization it has taken us a long time to develop some methods to deal with this. In antiquity, chance mechanisms, such as coins, dice and cards were used for decision making and there was a strong belief in the fact that God or Gods controlled the outcome. Even today, some people see chance outcomes as fate or destiny – "that which was meant to be"

It is in this world that the magician lives and it is these beliefs that he uses to help to create illusions. One such magic trick is to claim to know the position of all the cards in a random pack. A famous version was created by Si Stebbins in 1898. Stebbins was an American vaudeville performer who developed a system which requires you to arrange the cards and suits in a sequence. Each subsequent card in the sequence has a value three more than the previous one and the suits rotate in Clubs, Hearts, Spades and Diamonds order (known as "CHaSeD" order). This arrangement allows the magician to know the identity of a chosen card by glimpsing the next card, or determining the exact position of any card in the pack by a mathematical calculation, although many other properties of the system are known and have been applied to different card tricks.

3c 6h 9s Qd 2c 5h 8s Jd Ac 4h 7s 10d Kc
3h 6s 9d Qc 2h 5s 8d Jc Ah 4s 7d 10c Kh
3s 6d 9c Qh 2s 5d 8c Jh As 4d 7c 10h Ks
3d 6c 9h Qs 2d 5c 8h Js Ad 4c 7h 10s Kd

Given that
1=Ace
11=Jack
12=Queen
13=King

Notice that these four groups of thirteen cards have a number of patterns. When you put these groups together as a pack you can then cut the deck as many times as you wish as the cyclic order is still persevered.
Stebbins admitted, in one of his books, entitled Stebbins' Legacy to Magicians (1935) to having developed the system from the Spanish magician Salem Cid. Yet if we look back in history there have been examples of similar tricks to this being used. Variations in which the cards progress by five, four or three have been seen as far

back as early 17th century in books by the Portuguese writer Gaspar Cardoso de Sequeira and the Spanish writer Minguet. (7)

There are many developments you can make to this trick such as:

- When a spectator chooses a card from the deck the magician can easily find their card by looking for a break in the pattern
- Asking a spectator to cut the pack at any point and then by glancing at the bottom card you can know their chosen card.

The problem with the Stebbins system from a mathematical point of view is that it is very predictable and does not look like a random collection of cards.

1s, 4d, 3s,10c, 7c, 11s, 8s, 12c, 13d, 4c, 2d, 10h, 6s, 6d,
9d, 5c, 5h, 4s, 13h, 2c, 9c, 4h, 1c, 6h, 7h, 10d, 8d, 2s, 7s,
9s, 2h, 8h, 13c, 3d, 13s, 1h, 5s, 3h, 11d, 11h, 9h, 3c, 12s,
11c, 10s, 5d, 6c, 8c, 1d, 7d, 12h, 12d

s=spade (1)
h=heart   (2)
c=club   (3)
d=diamond (4)
1=Ace
11=Jack
12=Queen
13=King

This collection of cards is also in a sequence and by knowing the previous card you can determine the next just as with Stebbins arrangement. This sequence is better if you want to show the spectator the cards before you start the trick as it looks random. Good questions to ask students are:

Can you see the pattern?
How could you find a pattern, in such a seemly random collection of cards?

This is an interesting question from the point of view of trying to discover hidden secrets in our world. When a mathematician first sets out to try and discover how something works they may start from just this point with a collection of data which they believe holds a pattern and yet looks completely random.

To start to solve this problem a good thing to do is to look at small numbers in the pattern first.

1s gives 4d
2s gives 7s
3s gives 10c

Three times the card value plus one gives the next card value. Then can you find similar patterns with other suits?

Once you have found the connections between the valves you will then need to discover what the pattern is between the suits.

1s gives 4d Spade gives Diamond
1h gives 5s Heart gives Spade

1c gives 6h Club gives Heart

Students enjoy discovering hidden patterns and this work develops naturally into why we need to find order in a seemly random world.

Maurice Kendal points out that, man is in his childhood and is still afraid of the dark. Few prospects are darker than the future subject to blind chance! (8)

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Humble

The Constructs of PhD Students about Infinity: An Application of Repertory Grids[i] [ii]

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Abstract: *Infinity has been one of the more difficult concepts for humanity to grasp. A major component of the research on mathematics education related to infinity has been the study of student's conceptions and reasoning about calculus subjects, particularly limits and series. Some related studies are about Cantor's ordinal and cardinal infinity. However since most students at the high school and college level are unfamiliar with symbolic representations and terminology, such as a set theoretic approach, a context (generally geometric) is used for investigating notions of infinity indirectly. In this paper we report on a study on the constructs of PhD Students about the notion of infinity. The aim of our study was to gain insight into the constructs about infinity held by PhD students and to investigate the effects of a graduate level set theory course on their informal models. We also propose repertory grid methodology as a way of capturing the constructs of students and argue that this methodology can help us to learn further details about the understanding of infinity.*

Keywords: Infinity; Potential infinity; Actual Infinity; Set Theory; Repertory Grid techniques; Teaching and learning set theory; Ordinality; Cardinality.

"*It seems that the various strategies that were used in the learning unit "Finite and Infinite Sets" did indeed enable the students to progress towards acquiring intuitions which are consistent with the theory they learned. However, we lack the means to evaluate these effects systematically. In order to proceed in devising instructional strategies that take into account the intuitive background of the learners we need to develop means to measure "degrees of intuitiveness*" Dina Tirosh (1991)

DIFFERENT CONCEPTS OF INFINITY

Today there are different concepts of infinity which are generally accepted by the mathematical community in spite of the rejections of many mathematicians since Aristotle.

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From the early history of mathematics two basic concepts of infinity have been actual and potential infinity. Fischbein (2001) identified actual infinity as what our intelligence finds difficult, even impossible to grasp: the infinity of the world, the infinity of the number of points in a segment, the infinity of real numbers as existing, as given etc. According to him when we mention the concept of potential infinity, we deal with a dynamic form of infinity. We consider processes, which are, at every moment, finite, but continue endlessly. For example we cannot conceive the entire set of natural numbers, but we can conceive the idea that after every natural number, there is another natural number.

Dubinsky et al. (2005) proposed an APOS analysis of conceptions of infinity. They suggested that interiorizing infinity to a process corresponds to an understanding of potential infinity, while encapsulating an object corresponds to actual infinity. For instance, potential infinity could be described by the process of creating as many points as desired on a line segment to account for their infinite number, whereas actual infinity would describe the infinite number of points on a line segment as a complete entity. Tall (2001) categorized the concepts of infinity as natural and formal infinities. He wrote: "Concepts of infinity usually arise by reflecting on finite experiences and imagining them extended to the infinite."

Tall (2001) referred to such personal conceptions as natural infinities. Natural concepts of infinity are developed from experiences in the finite world. He suggested that in the twentieth century to rationalize inconsistencies different formal conceptions of infinity are built by formal deduction. Contrary to the Cantorian counting paradigm that leads to Cardinal number theory, Tall (1980) suggested an alternative framework for interpreting intuitions of infinity by extrapolating the measuring properties of number. When the cardinal infinity asserts that any two line segments have the same number of points, regardless of length, a measuring intuition of infinity asserts that the longer of two line segments will have a "larger" infinite number of points. For example, if the line segment CD is twice as long as the line segment AB then CD has twice as many points as AB. Tall (1980) called this notion "measuring infinity" and suggested that it is a reasonable and natural interpretation of infinite quantities for students. He stated that "Infinite measuring numbers are part of a coherent number system extending the real numbers, including both infinitely large and infinitely small quantities. So it is consistent with non-standard analysis." Tall argued that experiences of infinity that children encounter are more related to the notion of infinite measuring number and are closer to the modern theory of non-standard analysis than to cardinal number theory. However Monaghan (2001) suggested that this can only be described as a possible trend in older children's thought.

Since measuring infinity can be thought in the non-standard view, a well-organized view to different conceptions of infinity comes from Tirosh. In her study about teaching Cantorian theory, Tirosh (1991) categorized the term infinity in different contexts as potential infinity (representing a process that is finite and yet could go on for as long as is desired), actual infinity (in the sense of the cardinal infinity of Cantor or ordinal infinity, also in the sense of Cantor, but this time representing correspondences between ordered sets) and non-standard infinity (which arises in the study of non-standard analysis, and unlike the others, admits all the operations of arithmetic, including division to give infinitesimals). These themes about the conceptions of infinity which were observed before guided this study and repertory grid methodology was applied in the light of these identified conceptions.

RESEARCH ON UNDERSTANDING INFINITY

In most of the research about infinity, students have been given some tasks generally related with calculus and geometry subjects (limit, series, straight line, point etc.) and researchers have looked for students' difficulties in understanding different mathematical

concepts as infinitesimal calculus, series, limits etc. Particularly the focus is on the difficulties in understanding actual infinity (Borasi, 1985; Falk, 1986; Fischbein, Tirosh & Hess 1979; Jirotkova & Littler, 2003; Jirotkova & Littler, 2004; Monaghan, 2001; Petty, 1996; Sierpinska, 1987; Taback, 1975; Tall, 1980)

Petty (1996) attempted to explore the role that reflective abstraction plays in the individual's construction of knowledge about infinity and infinite processes. He gave four undergraduate elementary education students some tasks involving iterative processes generally from calculus and geometry. Comparing 0.999… and 1, folding a rectangular sheet of paper continuously and drawing polygons with an increasing number of sides inside a circle are some examples of the tasks. Students are interviewed as they attempted to resolve problematic situations involving infinity and infinite processes. Based upon four detailed student case studies, he found that as students begin to relate their solution activity to previous tasks, higher levels of reflective activity could be observed.

In Piagetian-constructivist parlances some models have been developed to determine the understanding levels of limit and infinity. In order to clarify the role of reflective abstraction in his study Petty (1996) used one of these models developed by Robert (1982) and Sierpinska (1987). This model involves three stages:

Stage 1. Static Concept of Limit
   a. The individual's perception of a limit is in finite terms.
   b. For the individual, infinity does not exist; everything is finite and definite.
   c. If infinity does exist, all that is bounded must be finite and definite.

Stage 2. Dynamic Concept of Limit
   a. The individual has a perception of limit as a continuous unending process; the limit is a value which is approached yet never attained.
   b. For this individual, infinity exists and involves recognition of potential infinity as opposed to actualized infinity. It has a contextual aspect and may involve a transitional phase.

Stage 3. Actualized Infinity
   a. The individual has conceptualized infinity as a mathematical object.
   b. Infinity is treated as a whole and definite object.
   c. The individual believes that it is possible to predict the outcome of an infinite process.

Based on the study, Petty altered the three-stage model to a four-stage model by separating third stage. He wrote:

"If a student acknowledges that the infinite repetition of 0,999… is identical to 1, this individual is accepting that an infinite process has a predictable outcome. However, this does not infer that this same individual has conceptualized infinity as a mathematical object. Whereas, this individual will perhaps have attained the Actualized Infinity level, it would not be sufficient evidence to suggest that this person is able to perform operations on or with their conception of infinity."

The four stages of this model are: the Static Level, the Dynamic Level, the Actualized Infinity Level, and Infinity as a Mathematical Object Level. He found that none of the students interviewed for this study had attained either the third or fourth level of conceptual development. Again in the Piagetian view Dubinsky et al. (2005) applied APOS theory to suggest a new explanation of how people might think about the concept of infinity. They proposed cognitive explanations, and in some cases resolutions, of various dichotomies, paradoxes, and mathematical problems involving the concept of infinity. These explanations are expressed in terms of the mental mechanisms of interiorization and encapsulation. As we mentioned before, they characterized the process and object conceptions of infinity by APOS

theory that correspond to an understanding of potential and actual infinity. In his article about Cantors' cardinal and ordinal infinities, Jahnke (2001) emphasized that Cantor was the first to use the concept of pair wise correspondence to distinguish meaningfully and systematically between the sizes of infinite sets. Dependent on this usage, many studies involve Cantors' cardinal and ordinal infinity concern the comparison of infinite quantities (Fischbein, Tirosh & Hess 1979; Duval, 1983; Tirosh, 1985; Martin & Wheeler, 1987; Sierpinska, 1987; Tirosh and Tsamir, 1996; Tsamir, 2001).

One such study is Tirosh's (1985) dissertation in which she gave 1381 students (in the age range 11-17 years) 32 mathematical problems related with a comparison of infinite quantities. In each of these problems two infinite sets, with which the students were relatively familiar, were given. The students were asked to determine whether the two sets were equivalent and to justify their answers. She found that students' responses to the problems were relatively stable across the age group.

The main argument used by the students to justify their claim that two sets have the same number of elements was "All infinite sets have the same number of elements". For example 80 % of the students claimed that there is an infinite number of natural numbers and also an infinite number of points in a line. The claim that two infinite sets were not equivalent was justified by one of the following three arguments:

1. "A proper subset of a given set contains fewer elements than the set itself." For example, 51% of the students used this argument claimed that there are fewer positive even numbers than natural numbers.

2. "A bounded set contains fewer elements than an unbounded set." For example, 12 % of the students used this argument claimed that the number of the points in a square is greater than that in a line segment.

3. "A linear set contains more elements than a two dimensional set." For example, 38 % of the students used this argument claimed that there are more points in a square than in a line segment.

She found that a small percentage of the students (less than 1 %) intuitively employed the notion of 1-1 correspondence and the intuitive criteria that the students used to compare infinite quantities and theorems of set theory were inconsistent with each other.

In order to examine the impact of the "Finite and infinite sets" learning unit on high-school students' understanding of actual infinity, Tirosh and her colleagues taught this unit to students aged 15-16 in four tenth-grade classes. Questionnaires employed after the lessons showed that 86 % of the students acquired the basic concepts in the learning unit. Based on the study, she claimed that learning unit on infinite sets may be introduced without particular difficulty starting from the tenth grade.

Duvals' (1983), Tirosh and Tsamirs' (1996) observations were related to representations of the same comparison-of-infinite-sets task. They studied students by giving different representations concerning the equivalency of two infinite sets and evaluated their responses. They found that students decisions as to whether two given infinite sets have the same number of elements largely depended on the representation of the infinite sets in the task.

Tirosh (1991) also summarized the results of some late psycho-didactical studies about Cantorian set theory. According to her research, she found that:

1. There are profound contradictions between the concept of actual infinity and our intellectual schemes, which are naturally adapted to finite objects and finite events. Consequently, some of the properties of cardinal infinity, such as the fact that
$\aleph_0 + 1 = \aleph_0$ and $2 \aleph_0 = \aleph_0$ are very difficult for many of us to swallow.

2. Intuitions of actual infinity are very resistant to the effects of age and of school-based instruction.

3. Intuitions of actual infinity are very sensitive to the conceptual and figural context of the problem posed.
4. Students possess different ideas of infinity which largely influence their ability to cope with problems that deal with actual infinity. These ideas are usually based on the notion of potential infinity.
5. The experiences that children encounter with actual infinity rarely relate to the notion of transfinite cardinal numbers. But they do have increasing experiences in school of quantities which grow large or small.

In her study, Tsamir (2001) described a research-based activity which encourages students to reflect on their thinking about infinite quantities and to avoid contradictions by using only one criterion, one-to-one correspondence, for comparing infinite quantities. According to several studies that involve tasks concerning the comparison of infinite sets, Tsamir (2001) says that when students are presented with tasks involving comparison of infinite sets, they often use four criteria to determine whether a given pair of infinite sets are equivalent: the part-whole criterion (e.g. "A proper subset of a given set contains fewer elements than the set itself"), the single infinity criterion (e.g. "All infinite sets have the same number of elements, since there is only one infinity"), the 'infinite quantities-are-incomparable` criterion and the one-to-one correspondence criterion (as in Fischbein, Tirosh & Hess 1979; Duval, 1983; Borasi, 1985; Martin & Wheeler, 1987; Moreno & Waldegg, 1991; Sierpinska, 1987; Tirosh & Tsamir, 1996)

Even if Cantorian set theory is the most commonly set theory of infinity, it has a limited place in math curricula. Topics in Set theory require the full knowledge of the concept of (actual) infinity and they have been thought to be difficult for students. Thus pedagogical studies in this area have been generally with upper secondary and university students and even in this setting it has been hard for researchers to investigate students' understanding about all concepts of infinity, potential, actual, non-standard etc. In the remainder of this paper, we report on our study with mathematicians who are PhD students, in which we look for their constructs according to all known concepts of infinity as reported in extant studies. We claim that we can get more clear results about understandings of infinity by studying with mathematicians who can live in the actual world of infinity.

REPERTORY GRIDS

Repertory Grid Techniques is a methodology originally developed by Kelly as a research tool to explore people's personalities in terms of his Personal Construct Psychology (Kelly, 1955). It can be thought as a type of structured interview technique that guides individual's ability to compare elements to elicit constructs (attitudes, category making, assessing criteria and probably some personal tacit knowledge). This technique attempts to elicit constructs which the subject uses to give meaning to his or her world, and to have the subject rate items (elements) of their experience in terms of these elicited constructs (Williams, 2001). Although its original use was to investigate constructs about people, recent applications have included events, situations and abstract ideas. Pope and Keen (1981) point out that the technique has evolved into "methodology involving high flexible techniques and variable applications" (p. 36) Many researchers and writers of handbooks have explained these techniques and applications (Cohen et al, 1994; Fransella & Bannister, 1977).

Williams (2001) used repertory grid methodology together with a predicational view of human thinking to describe the informal models of the limit concept held by two college

calculus students. In his thesis, he explained the three aspects that answer the question "Why Repertory Grid Methodology is appropriate for his study?"

1. The underlying theory, namely personal construct psychology, is essentially Kantian in general orientation and dependence on a dialectical viewpoint and hence in harmony with logical learning theory.
2. The method attempts to have subjects themselves express the constructs they use, rather than having an observer interpret from their protocols the thought processes they employ.
3. Repertory Grid Techniques retain the flavor of qualitative methodology while at the same time providing the opportunity for quantitative analysis. Techniques for such analysis are widely used and easily available.

This methodology is particularly based upon the notion of construct. Methodologically, Kelly (1955) described a construct as "a way in which two elements are similar and contrast with the third" (p.61). To elicit a construct we need at least two elements (or statements). Elements can be people at students' environment or events, things that are related to student. Relatives of student, mathematical meanings are some examples. Elements are chosen to represent the area in which construing is to be investigated. If it is interpersonal relationships, the elements may well be people (Fransella & Bannistar, 1977). A common method of eliciting a construct is to present subjects with three items chosen from a list of interests and ask them to designate how two of the items are similar, and therefore different, from the third. In other cases, subjects may be asked to simply compare and contrast two items as in this study. In either case the result is a construct with two poles, each one represented by one of the contrasting items (Williams, 2001). To fill the rating form of the repertory grid, after the elicited constructs are written on the grid, the subject rates the elements. The Grid is a matrix of elements (items) by constructs with ratings of the elements in terms of how well the constructs apply to each other.

OVERVIEW OF THE STUDY

In this study 4 mathematicians with experience in their understanding of the notion of infinity from Gazi University, Turkey were chosen as the subjects. These mathematicians were PhD students enrolled in set theory course in the first semester of the 2006/2007academic year. All students except one of them graduated from math departments of education faculties in Ankara. One of them is graduated from the math department of science and art faculty again in Ankara. Students had no prior experience in Cantorian set theory but they had many experiences concerning infinite sets and infinity. Although we tried to report the common constructs between all of the students, this report focuses on 2 of the 4 students from the original study whom we will call the first and the second student. We thought that they are good examples of the study.

The set theory course consisted of 30 lessons spread over 15 weeks and was subdivided into sections finite and infinite sets, countable and uncountable sets, Zorns' lemma, cardinality, ordinality etc. A special attempt was made, through out the lessons, to interact with the students' intuitive background in regard to infinity and to change their primary intuitive reactions. During each lesson, students' definitions of infinity were explored and discussed, so that their personal constructs of infinity modified and refined over 15 weeks. During the lessons, students were asked to respond to a series of tasks aimed at moving their informal concepts of potential infinity toward an actual and cantorian infinity. The tasks students worked on included quizzes that are composed of set theory infinity problems in a typical textbook (Schaums & Halms) that assess students' technical competence; discussing contrasting opinions about infinity as voiced by the students and the instructor; working

problems which are also asked to picture different concepts on a paper, for example picturing a 1-1 correspondence or cardinality on a paper. The focus was on meeting students with different concepts of infinity. Beside the interviews, all of the lessons were recorded by the researcher and the questioning and interviewing that surrounded these tasks formed another corpus of data against the repertory grids from which we elicited construct relationships.

In order to investigate the changes of informal models initial and final repertory grids were elicited before and after the course. The data reported here is primarily from these repertory grids with clarifying evidence coming from transcripts of the lessons.

Eliciting elements (statements): A series of 12 written statements about infinity were used in the elicitation of a repertory grid. They were based on themes obtained from students' statements and past studies about infinity. We emphasized four themes in this study and prepared the statements (elements) according to these themes. These themes are; potential infinity (**P**), actual infinity (**A**) which can be Cantors' cardinal (**Card. A)** or ordinal **(Ord. A)** infinity, non-standard infinity (**N**) and measuring infinity (**M**).The aim was to provide students with statements in which each of different concepts of infinity were present. They were made as broad as possible to encourage the elicitation of constructs. The statements were also designed to differ in their degree of showing the concepts of infinity; some more closely approximated the potential infinity and others are more actual … They were subjected to pilot testing with other mathematicians, master degree students and were substantially rewritten. They were also reviewed by two university lecturers (Prof. Dr. and Assoc. Prof.) who give set theory course to ensure that all different concepts of infinity occurred in the statements and no important themes were missed. The statements are listed below.

==========================**ELEMENTS**==============================
Statement 1 (Themes -A, P):
Since the number of elements in any set is bigger than the number of elements in its proper subset, the number of elements in Natural Numbers Set☐ is bigger than the number of elements in Even Numbers Set **E**.
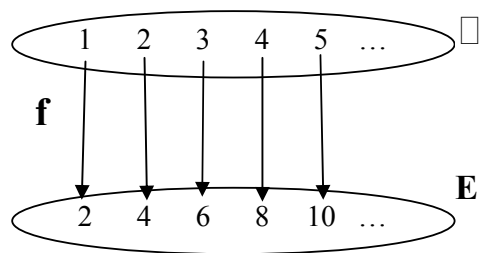
However, because of there exists an injection $f:N \rightarrow E$, $f(x) = 2x$ between N and E, it can be thought that the number of elements in N is equal to the number of elements in E which is a proper subset of N. From this point of view $S(E) = S(☐)$ and both sets have infinite number of elements.

Understanding infinity as a number causes this contradiction here. Whereas infinity is not a number, it is a concept that shows the inexhaustibility of elements in the set. The symbol ∞ is used in subjects like limit instead of numerical quantities that we can not express, in fact there is no number which is called as infinite.

Statement 2 (Themes P, M):

$\lim\limits_{n\to\infty}\left(\dfrac{n^2}{n^2+1}\right)=1$. Because as n tends to infinity $\left(\dfrac{n^2}{n^2+1}\right)\to\dfrac{\infty}{\infty}=1$. In the same way, it can be found a similar result for $\lim\limits_{n\to\infty}\left(\dfrac{n^5}{(1,1)^n}\right)$. The only difference here is that the denominator is growing faster and goes to infinity more quickly. '$n\to\infty$' expression states that n becomes bigger continuously, however '$n\to-\infty$' expression states that n becomes smaller continuously. Infinity is used to express the greatness of an inexhaustible quantity, but it can not be mentioned about any infinite number or different infinities.

Statement 3 (Theme A):
If the element m which is not a natural number is added to Natural Numbers Set, there will be a 1-1 mapping g between Natural Numbers Set and the new constructed set **N′** = {m,1,2,3,…} that is defined at the following. So the number of elements in N′ is equal to the number of elements in □ .

$$g(x)=\begin{cases} m\ ,\ \text{if } x=1 \\ x\text{-}1\ ,\ \text{if } x>1 \end{cases}$$
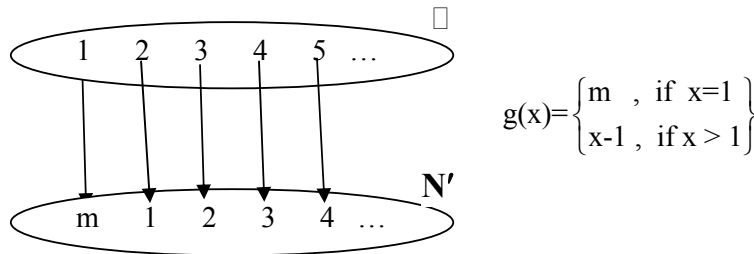
If number of elements in these sets are added we come to a conclusion S(□) = S(**N′**) = S(□) + 1 =1+S(□). If the number of elements in the infinite set N is shown by **a**, the statement will be **a** = **a**+1 = 1+**a**. We must not think infinite numbers like finite ones. Maybe this statement seems to us strange because of our prejudices belong to finite numbers.

Statement 4 (Theme P):
There are an infinite number of points on a line. Also Natural Numbers Set has an infinite number of elements. Since it can not be mentioned about different infinite numbers, the number of points on a line is equal to the number of elements in Natural Numbers Set. We can show this by a 1-1 mapping between natural numbers and all of points on a line

Statement 5 (Theme A):
In the open interval (0, 1) there is infinite number of real numbers and infinite number of rational numbers. Since A={x│x ∈(0, 1) and x∈□ } and B = (0, 1), A ⊂ B. So the infinite number that shows the number of real numbers in the interval (0, 1) is bigger than the infinite number that shows the number of rational numbers. Even if there are different finite numbers, also there are different infinite numbers. But infinite numbers must not be thought as finite

numbers. The arithmetic operations valid for finite numbers are not valid for infinite numbers in the same way.

Statement 6 (Themes N, M):
Since [0, 1] ⊂ [0, 2] for A=[0, 2] and B=[0, 1], S(A) > S(B). Same thing is valid for the numbers of points that are exist on different line segments which have different lengths. The number of points on a line segment of length 4 cm is bigger than the number of points on a line segment of length 3 cm.

Statement 7 (Themes N, M):
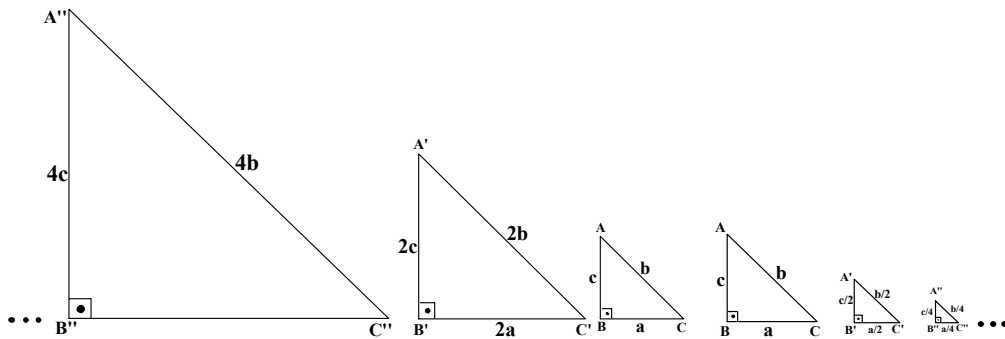
   As above, suppose that new right triangles are continuously formed by doubling or getting half of the lengths of the sides of $\overset{\Delta}{ABC}$ right triangle. If this process lasts forever in both ways, we obtain two triangles which have areas infinitely small and infinitely large at the end. Areas of these triangles would be never zero or very big number. These kind of quantities are expressed by infinite and infinitely small numbers that are obtained by the extension of real numbers.

Statement 8 (Theme Ord. A):
Suppose that the ordered set **N′** = {1, 2, 3, … ; m} was formed by adding a non natural number m to the Natural Numbers Set and in **N′** m is in order after all natural numbers. At the following **I: N → N′**, **I (x) = x** identity function is the insertion mapping of **N** into **N′** that preserves order.
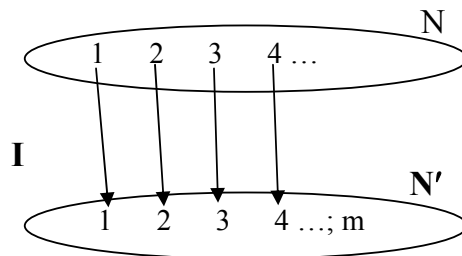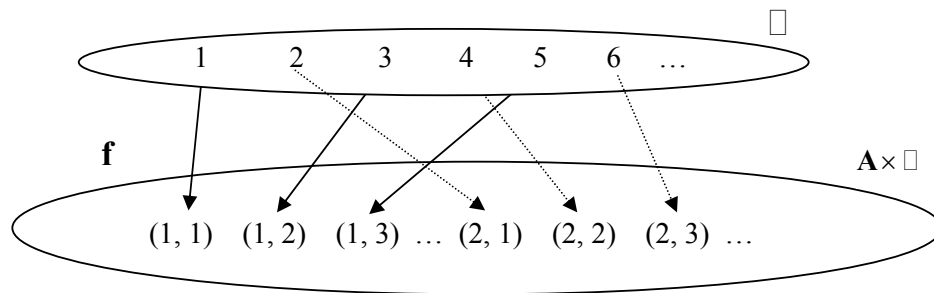

   Although these sets are equivalent infinite sets, they are different. Any mapping between these sets that preserves order as x < y → f(x) < f(y) can not be both 1-1 and surjective. When an element is added to an infinite set, although the number of elements (cardinality) does not

change, functions that preserves order give different meaning to these sets. Thus it can be said that different infinite sets that have different orders and different numbers represent these sets but they have the same number of elements (cardinality).

Statement 9 (Theme Card. A):

For A={1, 2} and B={a, b, c}, A×B={(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)} and S (A×B) = S(A).S(B). In the same way for A={1, 2} and □ ={1, 2, 3, 4,…}, A× □ ={(1, 1), (1, 2), (1, 3), …; (2, 1), (2, 2), (2, 3), …}. According to the mapping f defined at the following, it can be seen that the number of elements in (A× □) is equal to the number of elements in □ .

$$f(x)=\begin{cases} (1, x) & , \text{eğer } x=2k-1 \text{ ve } k \in \Box^{+} \text{ ise} \\ (2, x) & , \text{eğer } x=2k \text{ ve } k \in \Box^{+} \text{ ise} \end{cases}$$

Since S(Ax □) = S(A).S(□), **a = 2.a**

Statement 10 (Themes -N, P):

A number α which is smaller than all positive real numbers is called infinitely small. But no number α on the real line could be 'arbitrarily small'. If $r > \alpha > 0$ for all positive real numbers r, then $(\alpha/2) > 0$ is positive and even smaller than α and we can not call α an infinitesimal. As a result, infinitesimal and infinite concepts which are used to express only very small and very big numbers. We can not mention about these kind of numbers.

Statement 11 (Theme A):
For circumferences K= {(x, y) │x² + y² = 1} and L= {(x, y) │x² + y² =4}, even if their radiuses' lengths are different, the numbers of points that construct these circumferences are equal to each other. At the following, for every point on the small circumference you can find a point on the big circumference to map. In polar coordinates as you see, you can map every point (cosα, sinα) on K to the point (2cosα, 2sinα) on L.

f: K → L,
f(x, y)={(2cosα, 2sinα)│x=cosα, y=sinα, 0<α<2π}
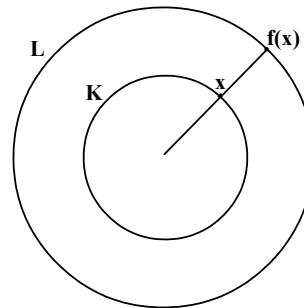
  By a 1-1 mapping f, you can see that numbers of the points on two circumferences are equal to each other. That is to say the number of elements in K is equal to the number of elements in L. They have the same cardinality.

Statement 12 (Themes Card. A, Ord. A):
At the following, **I(x) = x** identity function is a 1-1 mapping between the infinite sets ☐ = {1, 2, 3, 4, 5, 6, … }and ☐ =N′ = {1, 3, 5,…; 2, 4, 6, …} which has a different order.
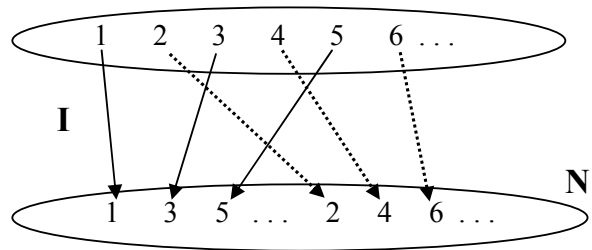
    Thus, both of sets have infinite elements and numbers of elements in these sets are equal. But, there can be also an injective mapping between these sets that preserves order which is defined as g:☐ → N′, g(x) = 2x -1.
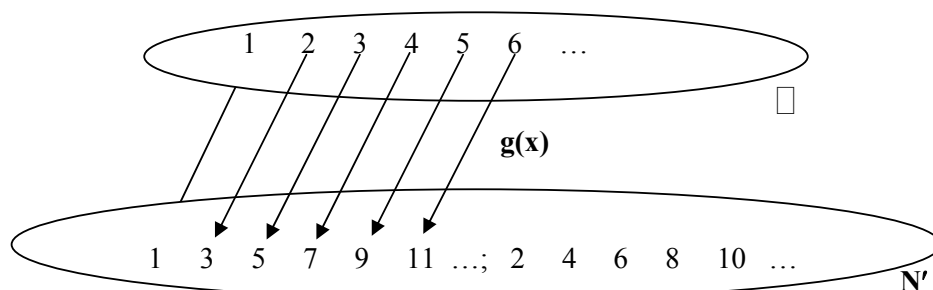
In this condition, although both of them are infinite sets, **S(N′) > S(□)** because there are enough elements until 2 in **N′** for 1-1 mapping with N. In these examples, f does not preserve order but g preserves order as $x < y \Leftrightarrow g(x) < g(y)$. As a result, according to conservation of order, you can meet with different results that are related to infinite sets.

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Eliciting Constructs: At the beginning of the course and at the end of the course, constructs were elicited by structured interviews using these 12 infinity statements. A subject was given two of the statements and asked to describe how the two statements were alike or different. Since the statements (elements) are very long, dyad method was chosen for this study. The subject responded with verbal descriptions of what was alike or different, which the interviewer recorded. Clarification was asked for as needed such as when the verbal descriptions of two constructs seemed to be the same. The interviewer chose a few of the subject's words to use as label for the emergent construct and, when offered by the subject, a label to stand for the opposite construct. Also subjects were allowed to omit some constructs which are meaningless for them. Every attempt was made to ensure that the construct elicited were not suggested to the student. When all similarities and differences between one pair of statements were written as constructs, all 12 statements were rated by the subject on a 5-point scale. One pole is arbitrarily assigned a rating of 1, it's opposite 5. This procedure was then repeated for a second pair of statements, and the process continued until the interviewer and the subject both agreed that they were unlikely to elicit new constructs.

Pairs of statements were presented to all subjects in the same order. The pairs **((1-7), (2-5), (3-8), (6-11), (7-10), (4-11), (9-12), (4-5))** were chosen because they seemed to represent different concept of infinity in twos. The number of constructs elicited varied across subjects, and not all the statement pairs elicited new constructs. Subjects were also asked to rate all 12 statements on two constructs supplied by the researcher: whether the statement was true and whether the statement is good according to subject. The interviews typically lasted about an hour, with the subjects examining five to eight pairs of statements. At the end of the procedure a matrix of comparisons was produced, usually with scored ratings for each element in terms of the elicited constructs. It is a matrix in which each row represented one construct and consisted of a series of 12 rankings (one of each statement) on a 5-point scale. A rank of "5" for a statement indicated that the emergent (left-hand) pole of the construct applied strongly to that statement, whereas a rank of "1" implied that the opposite pole of the construct applied strongly to the statement (Williams, 2001).

Analysis Of Grids: There are myriad methods of analyzing the grids by hand or by computer. We analyzed the grids by a simple hand method, comparing rating patterns in successive rows by calculating the difference scores. The ratings for each element were subtracted and the differences for the row were added to provide a difference score in which low numbers indicate close relationships between constructs and high scores indicate negative relationships. The actual range of possible difference scores would depend on the number of elements and the size of the rating scales used (Bannister and Mair, 1968). In this instance the

possible range of difference scores is from 0 to 48. The construct pairs which have the difference scores 0 and 48 are at given below.

Emergent Construct	ELEMENETS (STATEMENTS)												Opposite Construct
	1	2	3	4	5	6	7	8	9	10	11	12	
True	1	1	4	3	4	3	3	3	5	2	3	5	False
Infinity is a concept	1	1	4	3	4	3	3	3	5	2	3	5	Infinity is not a concept
Differences	0	0	0	0	0	0	0	0	0	0	0	0	*Difference Score:0*

Emergent Construct	ELEMENETS (STATEMENTS)												Opposite Construct
	1	2	3	4	5	6	7	8	9	10	11	12	
True	1	1	1	1	1	1	1	1	1	1	1	1	False
Infinity is a number	5	5	5	5	5	5	5	5	5	5	5	5	Infinity is not a number
Differences	4	4	4	4	4	4	4	4	4	4	4	4	*Difference Score:48*

After we had obtained a construct relationship score for each construct pair, we used elementary linkage analysis to identify the conceptual organization. It is a cluster analysis technique that is one method of grouping or clustering together correlation coefficients which show similarities among a set of variables (Cohen et al., 2000). At the end, we prepared a construct relationship figure for each student, involving clusters and difference scores between construct pairs which indicate strengths of the relations. These figures gave us information about the understanding features of different concepts of infinity in an organized way.

RESULTS AND DISCUSSION

In this section, we evaluate the grids of two students as an example and discuss in detail the understandings of infinity held by the students. We also discuss how their understandings evolved over the set theory course. With all of the students, a clear image for infinity emerged during the interviews.

THE FIRST STUDENT

First Student's Initial View Of Infinity: The first Student had graduated from a Faculty of Education in Turkey. He was a successful student like others. His last course, related with our research subject, was the construction of real number system which had been given one semester before. He participated in the full activity. Some related comments are provided to

give the reader a sense of how the constructs were elicited which were eventually used to rank all 12 items.

Infinity is a number- Infinity is a concept (S7 - S10) "In fact I think that there is something as an infinity concept. But Should I think it as a number? ... According to me infinitesimal numbers exist. If I can not call something infinity then also I can not call something infinitesimal."

Densities of infinite sets are different (S1-S7) "Their densities are different. In the same way there are two infinities. $\infty / \infty = 1$... (S3-S8) (Are there different infinities?) Eee .Their densities can be different. To be countable or not to be. B is denser. Infinity is again the same infinity. Statement 8 is different."

1-1 corespondence can be made between infinite sets (S3-S8) "I thought Why can not I map them, I believe Statement 3. Same thing must be done in Statement 8. Here although their numbers of elements are different ... (Do natural numbers finish?) No...I don not know that 1-1 correspondence can be made between them."

Infinite sets can be classified as countable and uncountable (S2-S5) "Their element numbers are infinite. Infinity means that I can count it in a way. I can count rational numbers, but I can not count real numbers. I separate them in this way"

Infinity can be ruled (S11) "... It likes infinity that I map with R but I can not hold it, it slides."

The element numbers of infinite sets are same : If I subtract infinity from infinity the result will be again infinity, infinity does not end. In fact that I think being element number infinite and being cardinal set infinite are the same thing. I think that their number of elements are equal.

In the statement 3, we asked the students what can be written instead of "**a**". When he was asked about "**a**" that shows the number of natural numbers, he said "∞".Maybe the limit

$\lim\limits_{n\to\infty}\left(\dfrac{n^2}{n^2+1}\right)=1$ effected him that he found the result of $\lim\limits_{n\to\infty}\left(\dfrac{n^5}{(1,1)^n}\right)$ as 1. He showed his

doubts sometimes. For example, at the beginning he had said true for S6. After he had compared it with S11, he said, "it is not true". Since that was the first time that he met with cardinal and ordinal infinity, the interviewer described the meaning of ";" that is used to describe ordinality of elements and the condition of m (limit ordinal) which has no predecessor. At the end of the grid elicitation session, the following grid was formed by rating constructs.

First Student's Initial Repertory Grid

Construct Emergent Pole "5"	1	2	3	4	5	6	7	8	9	10	11	12	Construct Opposite Pole "1"
Infinity is a number	1	5	1	3	4	5	2	5	1	1	2	1	Infinity is not a number
Densities of infinite sets are different	1	3	1	5	2	1	3	2	1	4	1	1	Densities of infinite sets are not different
1-1 correspondence can be made between infinite sets	5	3	5	1	2	4	3	1	5	2	5	2	1-1 correspondence can not be made between infinite sets
Infinite sets can be classified as countable and uncountable	3	3	3	5	3	3	3	3	3	3	3	3	Infinite sets can not be classified as countable and uncountable
Infinity can be ruled	1	1	1	1	3	3	1	1	1	1	3	1	Infinity can not be ruled
The element numbers of infinite sets are same	5	3	5	5	5	3	2	3	3	5	5	2	The element numbers of infinite sets are not same.
Infinity is a concept	5	2	5	1	4	1	5	1	2	3	5	4	Infinity is not a concept
Logical	5	4	5	1	1	1	5	2	3	1	5	2	Illogical
True	5	3	5	1	1	1	5	2	2	2	5	3	False
Good	5	4	5	3	2	1	5	3	3	2	5	4	Bad

Although the primary focus of this report is on the relationship figure, we can note a few obvious relationships. He ranked the statements 1, 3, 7, 11 as "true, logical, good and representing that infinity is a concept". In addition these statements have generally high rankings on the construct "1-1 correspondence can be made between infinite sets". Whereas the elicited constructs in a repertory grid give an idea of the major meanings used to understand a subject, the relationship figure of the constructs shows how those meanings are related. First student's initial construct relationship figure is given next, the numbers in the figure show the difference scores.

First Student's Initial construct relationship figure
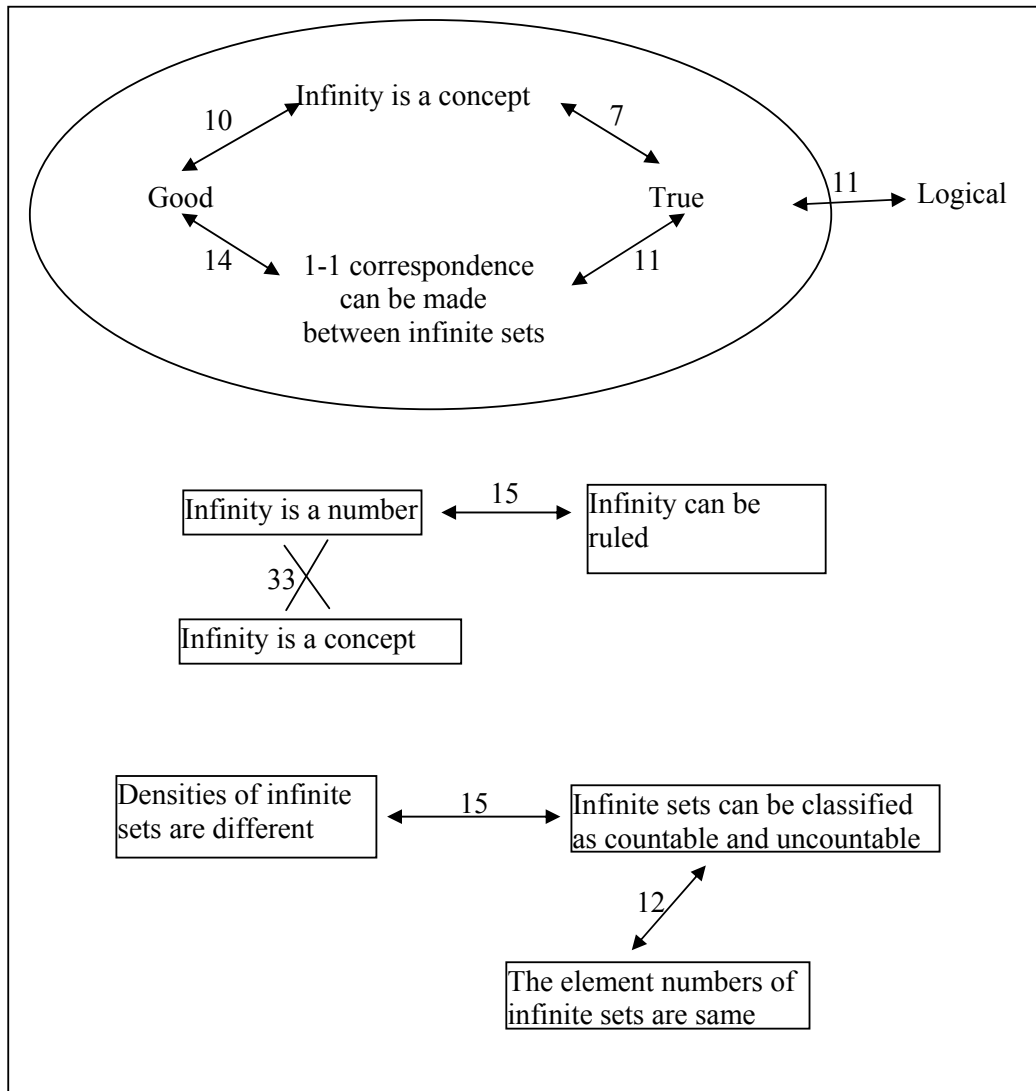
As anyone can understand from the first cluster at the top, his constructs "Infinity is a concept" and "1-1 correspondence can be made between infinite sets" are near the "good" and "true" constructs. He finds them logical.

In comparison to other students who participated in the study, "Infinity is a number" and "Infinity can be ruled" constructs are very near. According to students' constructs, "ruling infinity" and "being infinity a number" are two core meanings which are related to each other. In this study generally we see this kind of cluster in all of the figures. This is an important intuition of human being. According to the researchers, it shows the thinking of controlling infinity like using numbers to control and to understand other things. And according to him, being infinity a number and being infinity a concept are opposite constructs.

We think that the last relation between the constructs "Densities of infinite sets are different" and "Infinite sets can be classified as countable and uncountable" depends on his mathematical background. According to our observations in the lessons, students try to deal with the contradictory nature of infinite sets by using the "countable" and "uncountable" statements. For example, first student tried to explain different infinities by classifying infinite sets as "countable" and "uncountable" sets. Jahnke (2001), in his article about Cantor's cardinal and ordinal infinities, pointed out that "The discovery of countable and non-countable sets was a motive for Cantor to define the concept of cardinal number." In conjunction with his view, our findings also pointed the importance of countability and uncountability to swallow the idea of different infinities.

His construct relationship figure represents the view of a mathematician who did not meet with the set theory and different concepts of infinity. We see the effects of the real number theory and signs of the potential infinity in it.

First Student's Emerging View Of Infinity: First student attended all of the lessons and participated in the discussions. His answers to some questions during the lessons are in the following.

> Question: How many cardinals are there?
> Answer: **C**
> Question: What about cardinality of natural numbers and cardinality of real numbers ?
> Answer: …It is logical to think that they are equal.
> Question: What about the equality $\aleph_0 \cdot \aleph_0 = \aleph_0$?
> Answer: There is multiplication operation in the set of natural numbers. It is closed under multiplication. I find the equality $\aleph_0 \cdot \aleph_0 = \aleph_0$ logical, because it does not jump to **C**.
> (Thinking $\aleph_0$ as a natural number. It is hard to consider natural numbers totally)

The lessons offered the first student to refine his thinking on different concepts of infinity and give him the opportunity to meet with many examples of cantor's cardinal and ordinal infinity. It is observed that he had some information about axiom of choice and the definition of infinite sets from the previous courses. But at the beginning he was very mindful because of the statements like "$\aleph_0 + 1 = \aleph_0$" or "$2.\aleph_0 = \aleph_0$". As he encountered different conflicting conditions, he gained the opportunity to examine and to refresh his knowledge about infinity.

We assessed the effects of the learning unit on students' formal and intuitive understandings of infinity. After the lessons finished, the researcher interviewed with him again and a second repertory gird was elicited. During the final interview the following constructs were elicited.

There are different infinities - Infinity is a magnitude: (S4 – S5) "I see from this statement that there are different infinite numbers….(S6 – S11) I say that it is bigger. But it seems to me that [0, 1] and [0, 2] same. As a concept 3 cm and 4 cm states different infinities? … (S2-S5) According to me infinity and different infinities exist. I consider them like a magnitude not like a number.

Infinity is a concept - Infinity is a number: (S1) Yes, there is no a thing named infinity. I understood infinity as a representative, as a statement or a concept that represents something…. It seems to me like a concept… (S3 – S8) Infinite numbers should not be thought as finite numbers. They should be thought as a different concept.

Infinity can be ruled: (S1 - S7) There is something that I can not rule. It has no end, but, I …

There is conservation of order: (S9 – S12) In our minds, It begins from the smallest and goes towards the bigger one. In our minds there is natural ordering… Here, since we do not care conservation of order there is cardinality. Because of this, a = 2.a

1-1 correspondence can be made between infinite sets: S4 is not true, 1-1 correspondence can not be made …

Cardinality exists: (S1 – S7) According to their number of elements there are more natural numbers than even numbers. We skip 3, 5, 7, … but they have the same cardinality.

Ordinality exists: (S3 – S8) Sets have the same ordinality and cardinality …

Countable infinity exists: (S2 – S5) I can count rational numbers but I can not count real numbers. In one of them there is uncountable infinity. In fact same operations are not valid but same operations are valid for infinite numbers partially…

In the statement 3, again we asked the students what can be written instead of "**a**". When he was asked about "a" that shows the number of natural numbers, he said that "In the first interview maybe I used to write "∞" but now I prefer "ℵₒ"." His final repertory grid and construct relationship figure are on the following.

First Student's Final Repertory Grid

Construct Emergent Pole "5"	1	2	3	4	5	6	7	8	9	10	11	12	Construct Opposite Pole "1"
There are different infinities	1	1	5	1	5	3	4	5	2	1	2	5	There are not different infinities
Infinity is a concept	5	5	1	1	1	3	3	2	1	5	3	2	Infinity is not a concept
Infinity can be ruled	2	3	4	4	5	3	3	3	3	2	3	4	Infinity can not be ruled
Infinity is a number	1	1	5	5	5	3	5	5	5	1	3	5	Infinity is not a number
Infinity is a magnitude	5	4	2	3	2	3	4	2	2	4	2	3	Infinity is not a magnitude
There is conservation of order	1	1	1	1	1	1	3	5	1	1	1	5	There is not conservation of order
1-1 correspondence can be made between infinite sets	5	3	5	4	3	3	3	5	5	3	5	5	1-1 correspondence can not be made between infinite sets
Cardinality exists	5	3	5	5	4	5	3	5	5	3	5	5	Cardinality does not exist
Ordinality exists	1	1	1	1	1	1	3	5	1	1	1	5	Ordinality does not exist
Countable infinity exists	4	3	5	4	3	3	3	3	4	3	1	5	Countable infinity does not exist
True	1	2	5	1	5	1	3	5	5	3	5	5	False
Good	2	2	5	2	5	1	3	5	5	3	5	5	Bad

As it can be seen from the rankings in the grid, polarization between the constructs increased. The numbers used to rank are generally 1, 3 and 5.

First Students' Final construct relationship figure
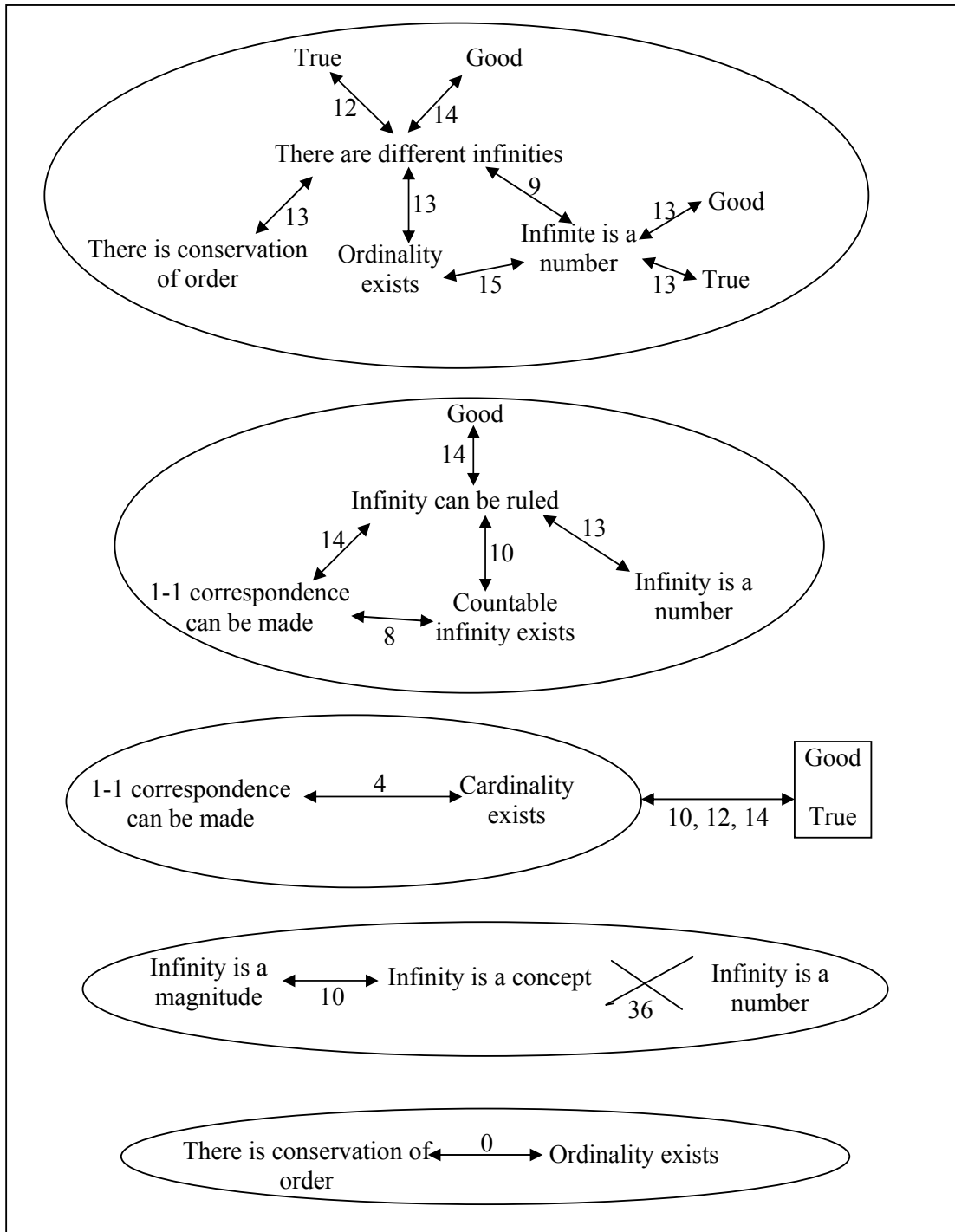
One of the major changes in his construct relationship figure is the presence of a new cluster (There are different infinities, Infinity is a number, There is conservation of order, Ordinality exists, Good, True) emerged which attaches the constructs "true" and "good" with the constructs "There are different infinities" and "Infinite is a number". And as a natural consequence of the set theory lessons the constructs "1-1 correspondence can be made" and "Cardinality exists" were linked with each other which are very near to "good" and "true" constructs. As we saw in other students' construct relationships, it was seen through his constructs that, "There is conservation of order" and "ordinality exists" have the same meaning for him. Another observation about ordinality is that there is no relationship directly between the constructs related with ordinality and the constructs "true" and "good". In the lessons we observed that understanding ordinality was more difficult than understanding cardinality for students. Anyway the cardinal numbers are generally before the ordinal numbers in set theory books as Cantor's published papers. Again the relation can be seen between the constructs "Infinity can be ruled" and "Infinity is a number". We think that intuitionally, he tries to rule infinity by undertaking it as a number. Maybe this intuition was also motive for Cantor. At the end of the set theory lessons this cluster was developed by the constructs "1-1 correspondence can be made" and "Countable infinity exists". Again we observe the opposition between the constructs "Infinity is a concept" and "Infinity is a number". Generally for all students being infinity a concept and being infinity a number were different things.

In summary, in his final construct relationship figure we observed the effects of the set theory lessons to his mental model. We also saw that some of the relations between his constructs did not change but they were developed with new constructs. We see the evidence of actual infinity in his figure. But as it can be seen from the distances between the constructs, even after a fifteen-week learning unit we can not say that his new figure is totally appropriate to actual or any concept of infinity. In the following quote from his interview he refers to an obstacle about the ordinality of infinite sets.

"It is hard to think ordered infinite sets. What is in our mind is that, it begins from the small, goes towards the big. There is the natural ordering in human's mind."

THE SECOND STUDENT

Second Student's Initial View Of Infinity: In order to illustrate the relationships between constructs about infinity more, we will look briefly at a second student. He had graduated from another faculty of education in Turkey. He had a less knowledge about infinite sets and it was difficult for him to explain the statements. The following constructs were elicited from the second student during the initial interview:

The elements of infinite sets can be compared as a big or small: (S1 – S7) In the first the infinite sets are compared. We are trying to compare the according to their numbers. Then it is said that since 1-1 correspondence can be made, the number of elements is infinite but infinite is not a number we must not mix them. Ooo but we say infinite at the end…

Infinity is a number: (S-7) In the 7th statement it is said that it can never zero or a very big number… It is mentioned about very big and very small numbers…

Infinity is a concept: (S7 – S10) We see here infinite and infinitesimal statements as a concept…

Infinity is a magnitude that never ends: (S2 – S5) There is a contradiction here… In the second infinite is described as a magnitude that never ends. Then in the other it is said that there are not any different infinite numbers or infinities.

1-1 correspondence can be made between infinite sets and natural numbers - Infinite sets have the same number of elements: (S6 – S11) " So both of these sets have the same number of elements. We see this by 1-1 correspondence… They have the same number of elements. There is no contradiction… Maybe we can make a 1-1 correspondence between line segments which have different lengths…

There are different infinite numbers: (S2 – S5) "Here It is said that as if there are different finite numbers also there are different infinite numbers… Both of the sets are infinite, but there are also irrational numbers in R…"

In the statement 3, we asked the students what can be written instead of "**a"**. When he was asked about "a" that shows the number of natural numbers, he said "∞" like the first student.

He had no clear idea about the result of $\lim\limits_{n\to\infty}\left(\dfrac{n^5}{(1,1)^n}\right)$. During the first interview his comments

were generally repeating of the statements and gave a little information about his understanding of this contradictory subject. But his elicited initial construct relationship figure was more confirmative. It is on the following.

Second Student's Initial construct relationship figure
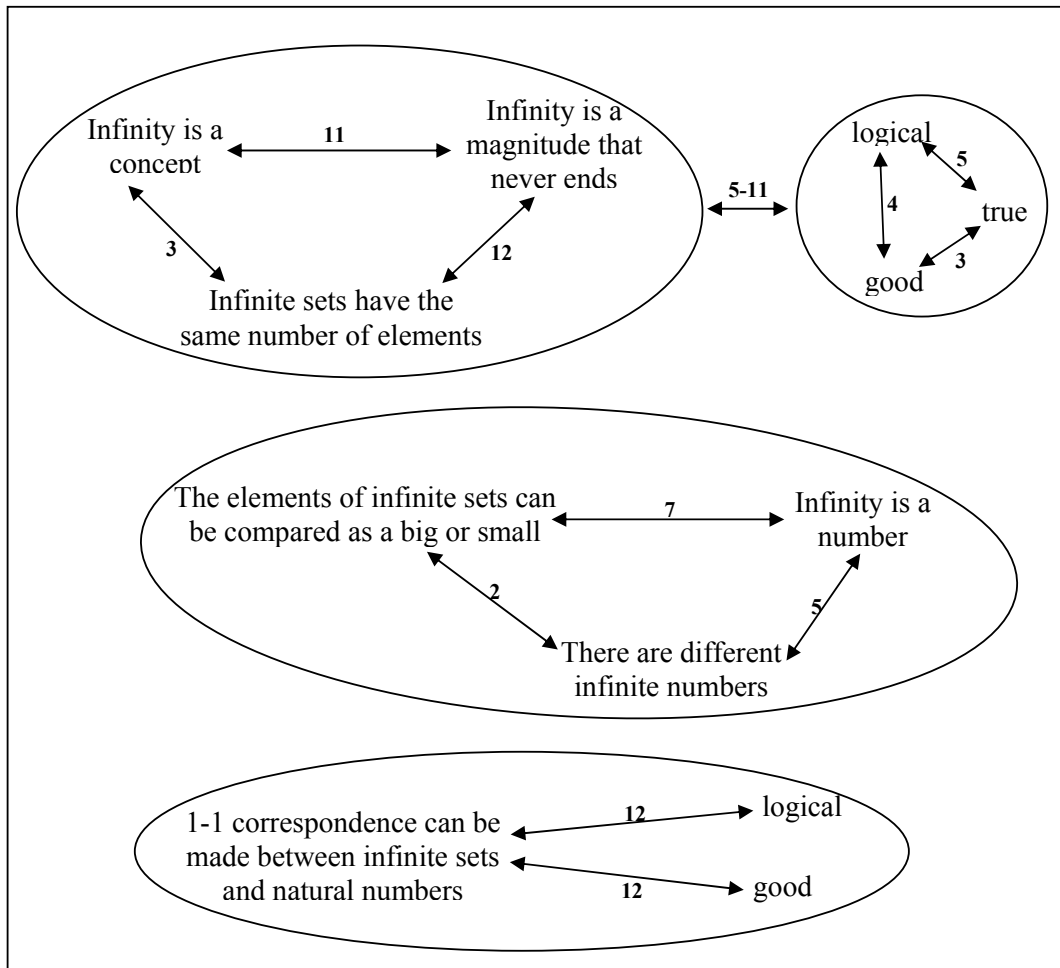
First cluster on the top ("Infinity is a concept"- "Infinity is a magnitude that never ends"- "Infinite sets have the same number of elements") is a clear sign of potential infinity which are also linked with the constructs "logical", "true" and "good". This cluster also indicates the "inexhaustible capacity of infinity" very well that Fischbein (2001) discussed.

"A second aspect, related to the intuitive interpretation of infinity refers to what one may call the "inexhaustible capacity of infinity" Infinity appears intuitively as being equivalent with inexhaustible, that is, if one continues the process of division indefinitely, all the points can be reached. In our opinion, this interpretation of infinity is the essential reason for which, intuitively, there is only one kind, one level of infinity."

According to the figure, it is also an important detail that the constructs "logical", "true" and "good" do not have the same meaning for him. For him, a thing that is true can not be fully logical. The constructs "infinity is a concept" and "Infinite sets have the same number of elements" are closer to one another than other constructs. That kind of relation is also observed in other student's figures. Thinking infinity as a concept also implies the one infinity which indicates the number of elements in all infinite sets. In the second cluster the constructs "The elements of infinite sets can be compared as a big or small", "Infinity is a number" and "There are different infinite numbers" are linked. As it can be understood from the lessons and other student's construct relationships the researcher thinks that discussing infinity as a total quantity or a number is directly related with accepting different infinities. It can be seen that the construct "1-1 correspondence can be made between infinite sets and natural numbers" is the indication of countability that are close to the constructs "logical" and "good". In summary the second student differs from the first in having a clear image of potential infinity.

Second Student's Emerging View Of Infinity: First student attended all of the lessons like the first student. He had not spoken much in the lessons but the dialog on the following can give us a reason about how did he participate in the discussions.

Question: How many countable infinite set are there?
 Student: Infinite.
Question: How many countable infinite cardinality are there?
 Student: One
Question: How many uncountable infinite cardinality are there?
 Student: Infinite, aah I do not know.
Question: What about the cardinals $\aleph_0 < c < $ Card P(R) < Card P(P(R)) <… What can
        be the union of these cardinalities? (P(R) indicates the power set of real
        numbers) Is this the biggest one?
 Student: I can take the power set of union…
Question: Then, how many cardinality are there?
 Student: Infinitely many, uncountable and infinitely many.

The following constructs were elicited from the second student during the last interview:

Infinity is a concept - Infinity is a number: (S1-S7) "...when we think it as a number or a concept we can think different sets…" (S7 - S10) It is mentioned about infinitesimal and infinite big. These are also concepts like anything we found. Yes, I can not say it is only this

… In the statement 7 it means that it can not be zero or very big number… (S4-S10) It is said that It can not be mentioned about different infinite numbers …

There are different infinities: (S1-S7) I agree with statement 7, there is not only one infinity, there can be bigger infinities. When we mention about infinity, we should not think the same thing … (S2) In the statement 2 it is said that we can not mention about different infinities, I do not agree with it. (S4 – S5) I do not agree with the statement " It can not be mentioned about different infinities". Statement 4 is wrong."

There is conservation of order: (S9-S12) "If I take into account order, this is like an excess…" (S3 – S8) "In the statement 8 I add this to the end. I see here ordinality. Since I add it to the end, since I made it w+1, even if I never see m, I can make 1-1 correspondence, I see the importance of conservation of order…"

*Ordinality exists-Cardinality exists:* (S3 - S8) "…since its' cardinality is infinite , sorry we do not say infinite anymore, since its' cardinality is $\aleph_0$, nothing change even if I add one element. Of course ordinality is also different…In the statement 8, I add this to the end. I see here ordinality…

Infinitesimal or infinite big exists: (S7-S10) "Extension of real numbers can be obtained by infinitesimal and infinite big. Thus…"

There is 1-1 correspondence: (S1-S7) "Card (R) = Card (C). Because we can make 1-1 mapping between them. The paragraph at the below is not but the part that point the equality of natural numbers and even numbers is okey. Because we can make 1-1 correspondence between them…" (S6-S11) "I can set up 1-1 correspondence between them…"

He answered the question about "a" that shows the number of natural numbers by "$\aleph_0$" like the first student. He had still no idea about the result of $\lim_{n\to\infty}\left(\dfrac{n^5}{(1,1)^n}\right)$. Even if the constructs are not very close to each other, His construct relationship figure on the following indicates the movement to the non-standard and actual infinity.

Second Student's Final construct relationship figure
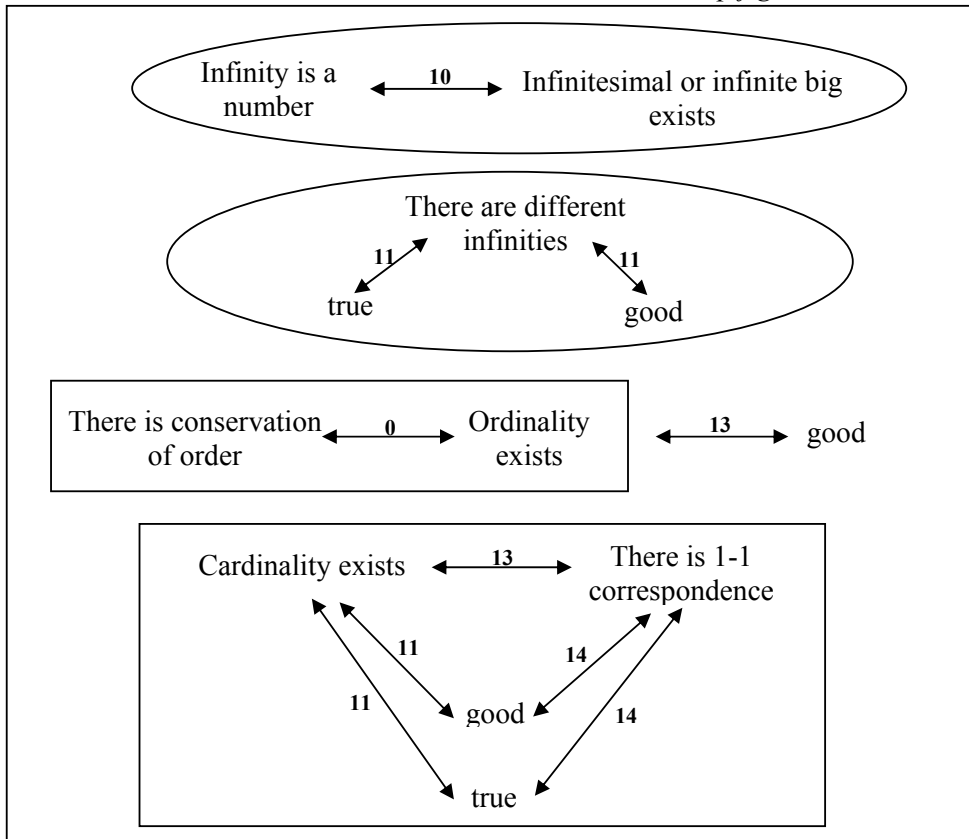
First cluster indicates the relationship between the constructs "Infinity is a number" and "Infinitesimal or infinite big exists" that refers the statements about non-standard infinity. We can see the result of meeting him different concepts of infinity during the fifteen week period by looking the second cluster ("There are different infinities"- "true" – "good"). The relationships between these constructs are not very strong but we think that the idea of different infinities occurred in his mental model about infinity. Like the first student's constructs after the lessons, the construct "Ordinality exists" is linked with the construct "There is conservation of order". They have nearly the same meaning for him. The cluster at the bottom indicates the main requirement of cardinality very well, even if the relationships between the constructs are not very strong. Since the cardinality of a set is defined by its equivalence class which is built with 1-1 correspondence between equivalent sets, it is a natural cluster that shares the conceptual space of a mathematician.

CONCLUSION

This paper presents the elicited constructs of PhD students concerning infinity and the effects of the set theory lessons to their constructs and concept images. It also offers insights into individual thinking of young mathematicians about infinity.

For the Cantor, the transfinite ordinals and the actual infinite are creations just as legitimate as the complex numbers had been ages ago (Jahnke, 2001). Although Cantor tried to legitimize his findings by arguing the modern theory of complex functions and others, his theory is not very easy to swallow intuitively even for mathematicians. Understandings that came from their past and environment are continuing to be cognitive obstacles for students. As Tirosh says "…our primary intuitions are not adapted to the notion of cardinal infinity. Thus, it would seem to require a considerable effort to develop appropriate "secondary intuitions" (i.e., intuitions which are acquired through educational intervention) of the notion of cardinal infinity."

Students generally have an understanding of potential infinity and after the lessons we saw emerging evidence of actual infinity. According to their model, Sierpinska (1987) and Petty (1996) had found that undergraduate students generally are at static and dynamic levels. We suggest that this is not also different for graduate students. Considerable effort is required to reach the higher levels of the model. One important result is the tendency of interpreting different infinities which indicate element numbers of different infinite sets by countability and uncountability of sets. We suggest that students intuitively think this when they think of the total numbers of real and natural numbers. Of course it is also directly related with their mathematical background as a mathematician. This finding is considerably parallel with the suggestion of Jahnke (2001) that is "The discovery of countable and non-countable sets was a motive for Cantor to define the concept of cardinal number."

Another important result is about teaching of set theory subjects. We observed that students found ordinality harder than cardinality as studies at the past (Tirosh, 1991). We suggest that when students meet with Cantorian theory, even if they are mathematicians, meeting them with cardinality will be more suitable. Since there is not a single concept of infinity, our goal was not to impose the Cantorian infinity as a single one. But we were expecting them to

distinguish the nature of thinking intuitively and mathematically. After the study we observed that they were wary of responding intuitively to our questions.

In his paper on "Predications of the Limit Concept: An Application of Repertory Grids", Williams (2001) argues that "Certainly no one method can capture the richness and variety that characterizes human thinking. Any method will necessarily obscure some aspects as it highlights others". However we found that as in the William's study, the repertory grid methodology was useful to capture students' understandings about the related subject. Looking for construct relationships by analyzing repertory grids provided us fundamental models and metaphors about infinity which gives further details. We think that it is a very suitable method, particularly in this subject to pass students' cognitive defenses. For example, even if the second student was not an active participant, we could learn something about his understanding by eliciting his constructs. Instead of giving some questions as a task or forcing them to discuss about any subject, evaluating and ranking different statements which everybody can do was more applicable. The technique also allowed us to capture the growth after the lessons. Of course a change had to be expected after a fifteen-week period, but observing the changes on the constructs about infinity systematically was the outcome of the methodology. We should also note the verbal data obtained by the analysis of interviews was utilized. We think that the combination of infinity concept and repertory grid technique was an original and useful in relation to existing studies on infinity. It can help us to determine the place of different concepts of infinity in our math curricula and looking at different age groups can provide us a model for the understanding of infinity concept.

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[i] This article was independently reviewed by two reviewers external and not affiliated with the editors or the editorial board of the journal.

[ii] The syntax of the English in this paper varies occasionally in its usage of tense. The reader should bear in mind that the study was conducted in Turkish and the data was then translated and reported in English.