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THE JOURNAL (WHEEL) KEEPS ON TURNING

Bharath Sriraman
The University of Montana

The title of this editorial is a spinoff on the opening lyrics of a famous Lynyrd Skynyrd song (fill in the blank: Sweet Home__________). When the song came out in the 70's the popular media misunderstood the song and took some of its lyrics to mean support for the (infamous) George Wallace's governorship of Alabama, when in fact the band sarcastically boo'ed his segregative policies that scarred the South. The song goes

In Birmingham, they love the governor ***(boo boo boo)***
Now we all did what we could do
Now Watergate does not bother me
Does your conscience bother you?

These lines bring me to the theme of this editorial, which is namely: (1) What does it take to keep the journal's wheel turning (running, moving, progressing), and (2) "Does your conscience bother you?"

Issue #3 brings volume 6 for the year 2009 of the journal to an end. It consists of 14 feature articles, one Montana feature and a book review which total ~ 250 pages and could easily constitute another double issue. Since 2008, we have increased the number of issues of the journal to 3 per year with occasional supplemental issues, but this has also increased the time and effort needed to consistently produce high quality issues that address the scope the journal purports to cover. In addition the Montana Monograph Series in Mathematics Education is also thriving. This year alone three new monographs have been produced: *Interdisciplinarity, Creativity and Learning* (Monograph 6), *Critical Issues in Mathematics Education* (Monograph 7), and *Relatively and Philosophically E^{a}rnest* (Monograph 8) which is a Festschrift to celebrate Paul Ernest's 65th Birthday this year. Several other monographs are in the works on the topics of discourse in addition to a Sourcebook on Nordic Research in Mathematics Education.

The size, breadth and depth of this issue is a good indicator that *The Montana Mathematics Enthusiast* continues to flourish thanks to support and the continual flow of manuscripts from all over the world. However, it has become increasingly difficult for us to get timely reviews on some manuscripts because the critical mass of reviewers seem to be spread thin across the numerous journals in mathematics education, and tend to be otherwise busy people. Having said that, if we as a community want to keep this journal as an outlet for diverse ideas (mathematical, educational, political, cultural), innovation, with free access, as well as consistently maintain

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quality control to ensure high standards of scholarship, then we need your time, support and conscience in the review process. As opposed to the lyric that said “Now we all did what we could do”, we all now have to do, what we can do. Readers interested in getting listed as reviewers should contact me and list areas in which they can review manuscripts. Again, we are interested in those that believe in constructive reviews and we continue to encourage researchers from under-represented regions of the world to consider the journal as an outlet for their scholarship.

In this issue, we have articles that cover a wide spectrum of mathematics and mathematics education. Two of the articles are slanted towards geometry and art, and some others build on topics covered in earlier issues such as the article by Xia and Xia that makes use of Maple to automate theorem proving in elementary geometry. As usual the articles have been written by a diverse array of authors from 9 different countries, some of whom have recently completed their doctorates, some that are on the cusp of finishing their dissertations, and others by more experienced and seasoned authors. Three of the papers happen to be from Canada- and I’m happy about our northern neighbors supporting the journal. The international reader may be unaware that Montana shares a border with British Columbia, Alberta and Sasketchwan.

Several papers in this issue relate directly to teaching and learning situations in mathematics classrooms that hopefully interest mathematics teachers that read the journal. In addition there are articles that report on research in mathematics education that cover mathematical modeling, cognition and affective issues. The Montana feature by Elijah Bodish is an expository article on the relationship between the work of the Cubists and the 4th dimension. Finally, a review of Anna Sfard’s *Thinking as Communicating* is also included in this issue.

In keeping with the theme of the journal wheel turning- 2010 promises to be another good year for the journal. Two special issues are planned, one on creativity and giftedness (vol7, no2) being guest edited by Ali Rejali (Iran) and Viktor Freiman (Canada), and another issue (vol7, no3) focused regionally on Montana and its neighbors. On a parting note, anyone interested in guest editing a special issue of the journal is encouraged to send in a proposal outlining the topic they propose to cover with a list of authors and reviewers. Again, offers to review books and commentaries on previously published papers are also welcome. Thank you for making the journal an integral part of the community. Have a great summer (or winter) depending on your hemispheric orientation.

Two Applications of Art to Geometry

Viktor Blåsjö*

Geometry and art exploit the same source of human pleasure: the exercise of our spatial intuition. It is not surprising, then, that interconnections between them abound. Applications of geometry to art, of which we shall indicate a few, go back at least to Alberti's *De Pictura* (1435). But although geometry started out, as it so often does, as a most courteous suitor in its relationship with art, it was soon to be affectionately rewarded. We shall study two of these rewards.

Geometry applied to art

Let us indicate briefly how geometry may be applied to art. A perspective painting distorts sizes and shapes. A building in the distance may be smaller than a man's head, the circular rim of a cup becomes an ellipse, etc. Lines, however, always remain lines. This simple fact is the key to drawing tiled floors (figure 1), as Alberti explained in *De Pictura*, because it guarantees that the diagonal of the first tile is also the diagonal of successive tiles. Furthermore, all lines parallel to the viewer's line of sight will meet at one point in the picture, namely the point perpendicularly in front of the viewer's eye (the so-called "centric point"). The horizon is the horizontal line through this point, because if the observer looks downwards from there, no matter how little, then the ray from his eye will hit the ground, whereas is he looks upwards it will not, so this is indeed the boundary between ground and sky (here we are assuming, of course, that the earth is flat). Thus, for example, placing the centric point close to the ground gives the viewer the impression that he is lying down. This trick is used to great effect by Mantegna in *St. James led to Execution* (figure 2). It also follows, in the words of Alberti, that the horizon is "a limit or boundary, which no quantity exceeds that is not higher than the eye of the spectator . . . This is why men depicted standing in the parallel [to the horizon] furthest away are a great deal smaller than those in the nearer ones—a phenomenon which is clearly demonstrated by nature herself, for in churches we see the heads of men walking about, moving at more or less the same height, while the feet of those further away may correspond to the knee-level of those in front." (*De Pictura*, Book I, §20, quoted from the Penguin edition, Alberti (1991, p. 58).) For more on the role of geometry in Renaissance art see, e.g., Kline (1985, ch. 10) and Ivins (1973).

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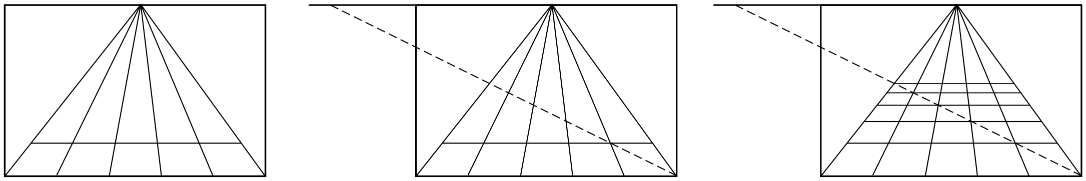


Figure 1: Drawing a tiled floor.

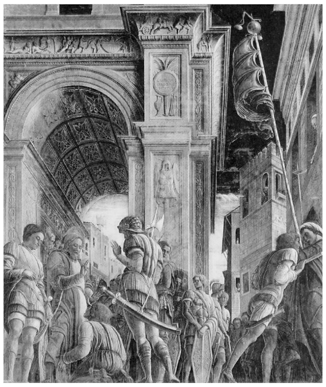


Figure 2: Mantegna's *St. James led to Execution*.

Newton's classification of cubic curves

Let us now turn to the applications of art to geometry. Our first example is Newton's classification of cubic curves. The classification of curves is the zoology of mathematics—indeed, Newton spoke of dividing curves into different "species." Art provides a picturesque criteria for whether two curves should be considered to be of the same species or not: two curves are of the same species if one is a projective view of the other, i.e., if when painting the picture of one curve you obtain the other. Newton (1695), §5, used this idea to classify cubics "by shadows," as he said, into the five equivalence classes illustrated in figure 3 (for more details see Newton (1981), vol. VII, pp. 410–433, Newton (1860), Ball (1890), Brieskorn and Knörrer (1986), Stillwell (2002)). We shall show where $y = x^3$ fits into this classification by showing that it is equivalent to $y^2 = x^3$ (the mirror image of the middle curve in figure 3). The classification of cubics is a natural setting for the use of projective ideas because cubics are the next step beyond conics, which are themselves too easy: projectively, they are all the same; any section of a double cone projected from the vertex of the cone (the eye point) onto a plane perpendicular to the axis (the canvas) comes out as a circle.

We imagine ourselves standing on top of the flat part of $y = x^3$ and painting its image on a canvas standing perpendicular to the plane of the curve (figure 5a). I say that the painting comes out looking like figure 5b. First of all, the dashed line represents the horizon. Let us focus first

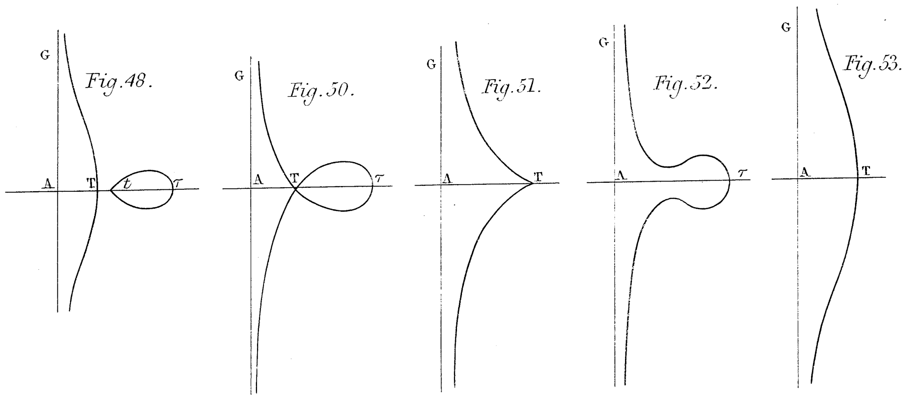
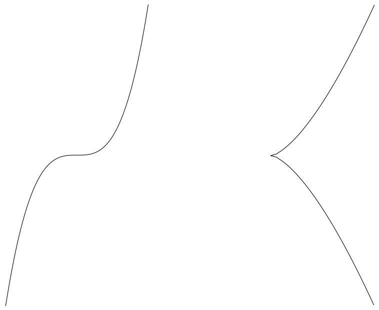


Figure 3: The five projective equivalence classes of cubic curves. (From Newton (1860).)



$y = x^3$ $y^2 = x^3$

Figure 4: Two equivalent cubic curves.

on the part of figure 5b below the horizon, which is supposed to be the image of everything in front of us. Apparently, even though the curve $y = x^3$ goes of to our right, we will see it meeting the horizon straight ahead of us. We understand why by looking at the support lines drawn in the figures. The dotted line and the brush stroke line on our right are parallel so in the picture they should meet at the horizon (like railroad tracks, if you will). Since the curve $y = x^3$ essentially stays between these two lines (almost all of it, anyway), it must stay between them in the picture as well, so it is indeed forced to meet the horizon straight ahead of us. The part above the horizon is similar, but we must allow for a mathematical eye that can see through the neck, so to speak. To draw the image of any point in front of us we connect it to our eye with a line and mark where this line intersects the canvas. To draw the image of any point behind us we use the same procedure, ignoring the fact that the canvas is no longer between the eye and the point.

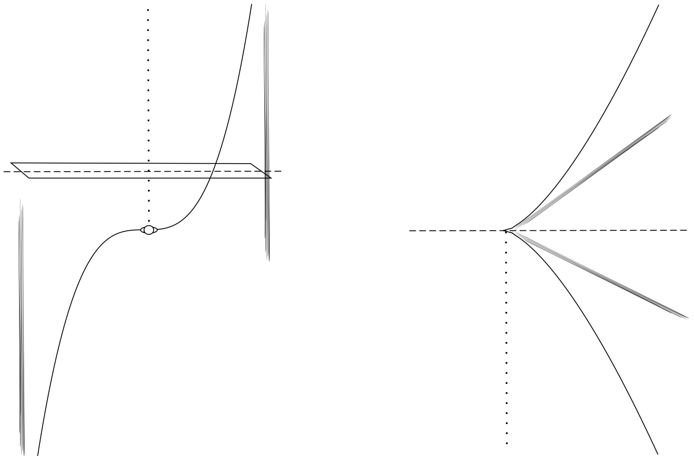

(a) (b)

Figure 5: Projective equivalence of $y = x^3$ and $y^2 = x^3$.

Desargues' theorem

We shall now see how Desargues' theorem emerges beautifully from natural ideas of perspective painting, namely the "visual ray construction" of 'sGravesande (1711) (see Andersen (2006) for a modern commentary). Desargues' theorem is one of the great results of projective geometry. Let us first look briefly at what it says and how we can think about it. The theorem says: if two triangles (ABC and $A'B'C'$) are in perspective (i.e., AA', BB', CC' all go through the same point, O) then the extensions of corresponding sides (AB and $A'B'$; BC and $B'C'$; AC and $A'C'$) meet on a line. Desargues' theorem is especially easy to think about in three dimensions, as indeed Desargues himself did (as conveyed to us by Bosse (1648); see Field and Gray (1987, chapter VIII)). Consider a triangular pyramid. Cut it with two planes to get two triangles. The three points of intersection of the extensions of corresponding sides will or course be on a line (the intersection of the two planes). By projecting the triangles onto one of the walls of the pyramid we get two plane triangles in perspective and the theorem holds for them also. So Desargues' theorem holds for any two triangles in perspective that can be obtained by projection from a triangular pyramid. We feel that any triangles in perspective can be obtained in this way so Desargues' theorem is proved. Now let us see what it has to do with art.

Visual ray construction of the image of a line. We shall draw the perspective image of a ground plane. To do this we rotate both the eye point and the ground plane into the picture plane: the ground plane is rotated down about its intersection with the picture plane (the "ground line") and the eye is rotated up about the horizon. Consider a line AB in the ground plane. The intersection of AB with the ground line is of course known. The image of AB intersects the horizon where the parallel to AB through the eye point meets the picture plane, and parallelity is clearly preserved by the turning-in process. So to construct the image of AB we turn it into the

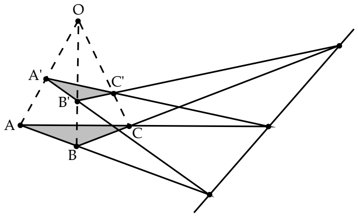


Figure 6: Desargues' theorem.

picture plane and mark its intersection with the ground line and then draw the parallel through the eye point and mark its intersection with the horizon; the image of AB is the line connecting these two points. $\square$

Collinearity property of the visual ray construction. Draw the line connecting a turned-in point A and the turned-in eye point. The image of A is on this line because if we turn things back out the eye-point–to–horizon part of the line will be parallel to the A–to–ground line part of the line, so that the image part of this line is indeed the image of the A–to–ground-line line. $\square$

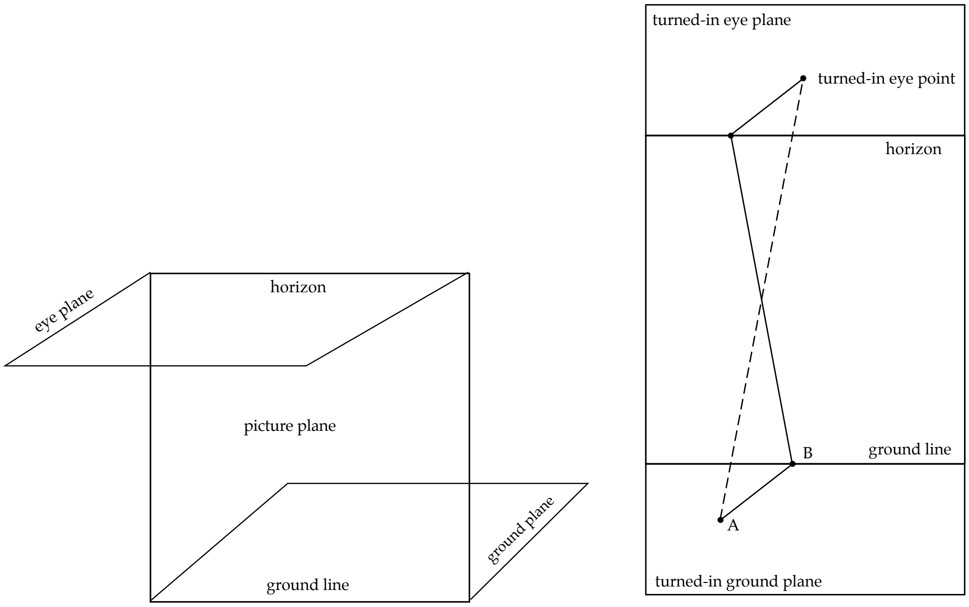

eye plane
horizon
picture plane
ground line
ground plane
turned-in eye plane
turned-in eye point
horizon
B
ground line
A
turned-in ground plane

Figure 7: The visual ray construction.

Desargues' theorem by the visual ray construction. Construct the perspective image $A'B'C'$ of a triangle ABC. By the image-of-a-line construction, intersections of extensions of corresponding sides are all on a line, namely the ground line, and by the collinearity property $A'B'C'$ and ABC are in perspective from the eye point, so we have Desargues' theorem. $\square$

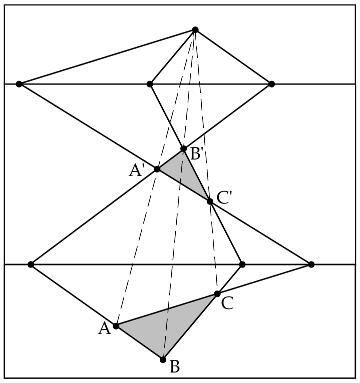


Figure 8: Desargues' theorem by the visual ray construction.

A more conventional proof of Desargues' theorem would be to use projective simplification, following Poncelet (1822, cf. §168). This proof is less directly influenced by art, but nevertheless the basic idea comes from our intuition with paintings, namely the idea of the horizon—"the line at infinity." In real life the horizon is intangible, but in a painting it is just a line like any other. And in real life parallel lines never meet, but in the painting they meet at the horizon, at a point like any other. Thus art suggests an alternative to Euclidean geometry where the line at infinity is just as real as any other line and where there is no such things as lines that never meet. Now let us use these ideas to prove Desargues' theorem. AB and $A'B'$ will meet somewhere, and BC and $B'C'$ will meet somewhere; grab the line determined by these two points and put it at the line at infinity, which is, as we said, a line like any other. This means that, in our picture (figure 9), AB will be parallel to $A'B'$, and BC will be parallel to $B'C'$. We need to show that AC and $A'C'$ meet at the same line, i.e., at the line at infinity, i.e., that AC and $A'C'$ are also parallel. Recall that AC is parallel to $A'C'$ if and only if $OA/AA' = OC/CC'$. Using this result on the two pairs of lines that we already know are parallel gives $OA/AA' = OB/BB'$ and $OB/BB' = OC/CC'$. So $OA/AA' = OC/CC'$ and thus AC and $A'C'$ are parallel.

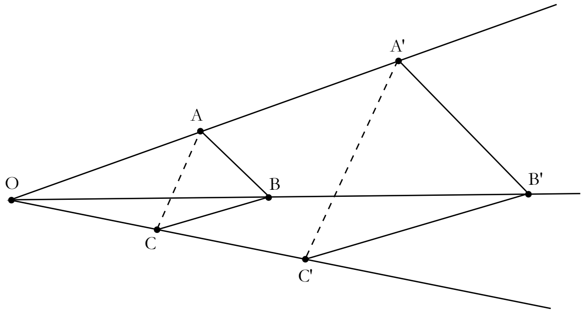


Figure 9: The simplified Desargues' configuration.

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Blåsjö

Intuitions of "infinite numbers": Infinite magnitude vs. infinite representation

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Abstract. This study examines undergraduate students' emerging conceptions of infinity as manifested in their engagement with geometric tasks. Students' attempts to reduce the level of abstraction of infinity and properties of infinite quantities are described. Their arguments revealed they perceive infinity as an ongoing process, rather than a completed one, and fail to notice conflicting ideas. In particular, confusion between the infinite magnitude of points on a line segment and the infinite representation of real numbers was observed. Furthermore, students struggled to draw a connection between real numbers and their representation on a number line.

Keywords: Infinity; Infinite numbers; Intuition; Magnitudes; Real numbers; Representations;

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> *From time immemorial, the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully* (Hilbert, 1925, p.136).

Infinity has played an important role in the historical development of mathematics and mathematical thought. From as early as 450 BC, mathematicians and philosophers have been intrigued by the ethereal dance of infinity. Over the centuries, as an understanding of infinity developed and changed, mathematics too evolved, reflecting the community's emerging understanding of a concept so heavily shrouded in mystery. With time it eventually became clear that not one, but many, concepts of infinity have a place in mathematics. This paper is concerned with two types of infinity, and the interplay between them: *potential* infinity, that which is inexhaustible, and *actual* infinity, "the infinite present at a moment in time" (Dubinsky, Weller, McDonald, & Brown, 2005, p.341).

This study is part of broader investigations regarding university students' naïve and emerging conceptions of infinity and transfinite arithmetic as they attempt to coordinate intuition and reflection with formal instruction. In what follows, students' engagement with geometric representations of infinity are described and used as a lens to their understanding of infinity and arithmetic properties of 'infinite numbers'. In particular, students' conceptions as they attended to the number of points 'missing' from the shorter of two line segments are of interest. This paper also explores what sort of connection, if any, participants made between a geometric representation of infinity and a numeric one. These can be seen as the main contributions of this study, complementing and extending prior research, which focused on learners' conceptions regarding the comparison of infinite sets.

This story of 'infinite numbers' begins with an exposition of the related literature regarding students' conceptions of infinity, as well as the theoretical perspectives that guided this study. Following that, the design of the study is described, and key findings are presented and analysed. The paper concludes with a summary of the main findings and suggestions for future avenues of investigation.

1. BACKGROUND

Students' reasoning concerning cardinal infinity has been a popular focus of current research (see among others: Dreyfus & Tsamir 2004; Tsamir, 1999, 2001; Tsamir & Dreyfus, 2002; Weller, Brown, Dubinsky, McDonald, & Stenger, 2004). The body of literature ranges from expositions of learners' intuitive understanding of infinity (e.g. Fischbein, Tirosh, & Hess, 1979) to developing pedagogical tasks that will encourage a deliberate use of formal definitions (e.g. Tsamir & Tirosh, 1999). A prominent trend has been to examine learners' conceptions through a lens of set theory – that is, students are presented with numeric sets, such as {1, 2, 3, …} and {2, 4, 6, …}, and are asked to draw cardinality comparisons. Their conceptions are then analysed based on the techniques or principles they apply to the task.

In a study conducted by Tsamir and Tirosh (1999), they noticed that visual presentations of sets had an impact on high school students' intuitive responses. For instance, one task had students compare the cardinalities of the two sets {1, 2, 3, …} and {4, 8, 12, …}. When the sets were expressed numerically, many students relied on the inclusion or 'part-whole' method for comparison and concluded that the set of natural numbers was greater than the set of multiples of four. Tsamir and Tirosh (1999) created a follow up task that presented the corresponding sets geometrically in such a way as to emphasize their one-to-one correspondence. Students were asked to consider a set of line segments with increasing lengths – i.e. {1cm, 2cm, 3cm,…} – and then to imagine constructing squares in such a way that the segments were of the same lengths as the sides of the squares. Both the set of line segments and the set of squares were depicted pictorially with the lengths and perimeters written below each segment and square, respectively. Through this analogy students could attend to the natural correspondence between a side and a perimeter of a square, and as such, they were more likely to recognise the one-to-one correspondence between the sets {1, 2, 3, …} and {4, 8, 12, …}. Tsamir and Tirosh (1999) were able to make use of the tangible nature of a geometric figure in order to emphasise correspondences between numerical sets, and also to draw students' attention to the inconsistencies of comparing infinite sets with different methods.

Inconsistencies in middle school students' intuitions about infinity were documented by Fischbein et al. (1979), who interpreted students' intuitions as they addressed issues such as the divisibility of line segments of different lengths, or the number of points on geometric figures of different dimension. The divisibility task consisted of comparing the number of times two line segments could be halved. The majority of students reasoned that although both line segments could be halved infinitely, the process would finish sooner on the shorter segment. Similarly, when comparing the set of points on a line segment with the set of points on a square, the common response alluded to infinities of different 'size'. Students appealed to 'part-whole' arguments, and reasoned that as the line segment was included as part of the square, the two sets must have different cardinalities, though both were infinite. These responses were in contrast to other observations of Fischbein et al. (1979), which suggested infinity was conceived of as a single, endless entity. Fischbein et al. concluded that the intuition of infinity is very labile and "sensitive to the conceptual and figural context of the problem" (1979, p.31).

The belief that there is only a single, endless infinite surfaced as a persuasive intuition of middle school students when they addressed set comparison tasks in a similar study by Fischbein et al. (1981). As part of the study, participants were asked to compare the cardinality of the set of natural numbers with the cardinality of the set of real numbers represented as a number line. The typical response that "there is an infinity of points on the line, and there is an infinity of natural numbers" (Fischbein et al., 1981, p.506), and so the two sets must be equinumerous is incorrect when judged by mathematical convention. Students' responses indicated that infinity was conceived of mainly as potential, that is, as an inexhaustible process. The association of infinity with inexhaustibility has also surfaced in undergraduate university students' views regarding limits in calculus (Sierpinska, 1987; Schwarzenberger & Tall, 1978; Williams, 1991). Fischbein suggested that such an association is "the essential reason for which, intuitively, there is only one kind, one level of infinity. An infinity which is equivalent with inexhaustible cannot be surpassed by a richer infinity" (2001, p.324).

2. THEORETICAL FRAMEWORK

Three inter-related frameworks are used in this study to interpret students' intuitions of infinity as well as their ideas after instruction: reducing abstraction (Hazzan, 1999), APOS: Action, Process, Object, Schema (Dubinsky & McDonald, 2001), and 'measuring infinity' (Tall, 1980).

In Hazzan's (1999) perspective, reducing the level of abstraction of a mathematical entity occurs as a learner attempts to understand unfamiliar and abstract concepts. Hazzan (1999) described several ways students make sense of new concepts by reducing levels of abstraction. For instance, Hazzan noted "students' tendency to work with canonical procedures in problem solving situations" (1999, p.80). That is, by basing arguments on familiar mathematical entities to cope with unfamiliar concepts, students lower the level of abstraction of those concepts. In the context of infinity, one such example is students' use of familiar (finite) measuring properties to interpret infinite quantities of measurable entities, such as the quantity of points on a line segment. This example of reducing the level of abstraction of infinity relates to Tall's (1980) notion of 'measuring infinity.'

Tall (1980) suggested intuitions of infinity can develop by extrapolating measuring, rather than cardinal, properties of numbers. Many of our everyday experiences with measurement and comparison associate 'longer' with 'more.' For example, a longer inseam on a pair of pants corresponds to more material. Likewise, a longer distance to travel corresponds to more steps one must walk. Tall (1980) proposed extrapolating this notion can lead to an intuition of infinities of 'different sizes.' A measuring intuition of infinity coincides with the notion that although any line segment has infinitely many points, the longer of two line segments will have a 'larger' infinite number of points. Tall (1980) called this notion 'measuring infinity' and suggested it is a reasonable and natural interpretation of infinite quantities, especially when dealing with measurable entities such as line segments. I would like to suggest that the intuition of 'measuring infinity' might develop as a consequence of learners' attempts to lower the level of abstraction of comparing the infinite cardinalities of points on line segments of different lengths.

Reducing the level of abstraction is further proposed by Hazzan (1999) to reflect a process conception of an entity. Process and object conceptions of mathematical entities are described in another of the theoretical frameworks to which I refer: that of the APOS (Action, Process, Object, Schema) theory (Dubinsky & McDonald, 2001). Dubinsky, Weller, McDonald, and Brown (2005) proposed an APOS analysis of two conceptions of infinity: actual and potential. The distinction between *potential* infinity, which can be thought of as endless, and *actual* infinity, a completed entity that encompasses what was potential, was first made by Aristotle. He, like many after him, denied the existence of actual infinity (Moore, 1995). The idea that infinitely many objects could be gathered together and thought of as a totality, was, and continues to be, very difficult. A more natural conception of infinity is that of potential, or dynamic, infinity (Fischbein, 2001). Fischbein considered dynamic infinity as “processes, which are, at every moment, finite, but continue endlessly” (2001, p.310).

Dubinsky et al. (2005) suggested that an understanding of potential infinity corresponds to a process conception in APOS terminology. That is, infinity is imagined as performing an endless action, although without having to execute each and every step. Conversely, an understanding of actual infinity develops when one is able to consider the process as a totality, i.e., when one can encapsulate it into an object. To connect this perspective to the infinite number of points on a line segment, a conception of potential infinity would correspond to, say, an action of marking or ‘creating’ points on a segment that is imagined to continue indefinitely. While actual infinity is illustrated by the idea that the infinite number of points exists as a completed entity, without needing to be marked.

Dubinsky et al. proposed encapsulation occurs once one is able to think of infinite quantities “as objects to which actions and processes (e.g., arithmetic operations, comparison of sets) could be applied” (2005, p.346). They also suggested that encapsulation of infinity entails “a radical shift in the nature of one’s conceptualisation” (2005, p.347). In terms of APOS theory, Hazzan argued that a “process conception of a mathematical concept can be interpreted as on a lower level of abstraction than its conception as an object” (1999, p.79).

APOS theory and the idea of ‘measuring infinity’ are used in my study to interpret undergraduate students’ emergent conceptions as they attempt to reduce the level of abstraction of infinity. Specifically, the questions addressed in this paper are: 1. What connections do students make between geometric and numeric representations of infinity, i.e. between points on a line and real numbers? 2. What can be learned about students' conceptions of infinity as they address properties of transfinite arithmetic?

3. SETTING AND METHODOLOGY

The participants of this study were 24 undergraduate university students in an interdisciplinary design and technology program, who had no mathematical background beyond high school. They were enrolled in the course "Foundations of Academic Numeracy", which was designed to develop quantitative and analytic reasoning. One of the objectives of the course was to provide an opportunity for students to engage in critical analysis and reflection regarding some of the fundamental ideas in mathematics. The topic of infinity was included as one of these fundamental ideas.

Data collection relied on two main sources: (i) individual written responses to "reflection activities", and (ii) follow up interviews with two of the participants. The "reflection activities" were essentially a series of written questionnaires administered over several weeks. The rationale behind these reflections was to elicit students' naïve conceptions and then to encourage them to reconsider, develop, and critique the underlying ideas through subsequent questioning. Tasks were formulated based on students' previous responses and common themes that emerged from the class. It was important, both for research and instructional purposes, that students' responses were not affected by seemingly correct solutions or the desire to appease their instructor. In order to avoid swaying students' responses, very little instruction was provided initially, and it was made clear that there was no one 'right' answer being sought. The activities reflected this in their design by, for example, recalling students' previous responses and presented them with a slight twist, so as to encourage them to challenge the issues they had unearthed. Other questions presented students with a dubious argument that claimed to be from one of their peers, in order to provoke a critique of the ideas involved. The basis for both styles of question was to avoid presenting an authoritative position. Students addressed each issue based on its appeal to their own emerging ideas.

At the end of the course, an instructional discussion on cardinality and infinite sets occurred. The discussion included comparing cardinalities of countable and uncountable infinite sets through one-to-one correspondences, or the idea of 'coupling'. Some of the specific conceptions that arose in students' reflections were also addressed. In the subsequent months, follow up interviews were conducted with two students, Lily and Jack. The interviews further explored their naïve and emerging conceptions of infinity.

The study began with two preliminary questionnaires, which included items a) and b) below. These tasks set the stage for exploring students' connection between numeric and geometric representations of infinity.

a) How many fractions can you find between the numbers $\frac{1}{19}$ and $\frac{1}{17}$? How do you know?
b) How many points are there on a line segment? How do you know?

Later questionnaires focused on the sets of points on line segments of varying lengths, and were intended to investigate ideas regarding 'infinite numbers' as well as 'infinite number properties.' Due to the contingent nature of the activities, details concerning the specific questions are developed in the following section.

The primary focus of this paper is on students' responses to two questionnaires in particular. The first (Q1, section 4.2) confronted students with an idiosyncrasy of infinite quantities and asked for an explanation. Of particular interest was the response of one participant, Lily. Her attempt to formulate an argument that was consistent with her experiences and intuitions prompted a follow up to Q1. In this follow-up (Q2, section 4.3), students were asked to respond to Lily's argument as well as to a variation of it.

4. RESULTS AND DISCUSSION

4.1 Infinite values, finite points

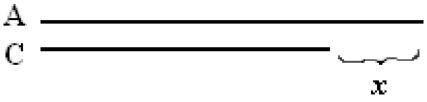
From the early stages of the study, a clear disconnect in students' conceptions of points on a real number line and numbers was observed. Typical arguments to item a), which concerned the number of fractions between $\frac{1}{19}$ and $\frac{1}{17}$, are exemplified by the following two responses:

> "Infinite. Because there are endless numbers that can be put into the numerator or the denominator and still making sure the fraction is larger than $\frac{1}{19}$ and smaller than $\frac{1}{17}$";
>
> and
>
> "You can find an infinite amount of fractions in between $\frac{1}{17}$ and $\frac{1}{19}$ because you can continue to add digits after the decimal point forever (e.g. $\frac{1}{18}$, $\frac{1.3}{18}$, $\frac{1.3625}{18}$, etc.) making the fractions a little bigger or smaller."

There are two common threads in these responses. One is the idea of potential infinity. The notions of "endless numbers" or adding "digits after the decimal point forever" imply infinity is conceived of as a process. The idea of changing the numerator or denominator corresponds to an action that is imagined to continue "forever", and is consistent with Fischbein et al.'s (1981) suggestion that infinity is intuitively thought of as inexhaustible. The second common thread in these and similar responses relates to students' conceptions of number. Both of these arguments describe processes being carried out with fractions. That is, students were attending to the rational numbers within the interval, but failed to address the irrational numbers. This might a consequence of the task itself, as the endpoints of the interval were rational numbers rather than irrational ones. However it may be more likely due to students' familiarity and comfort with rational numbers over irrational ones.

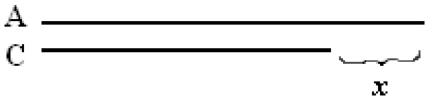
In response to item b), regarding the number of points on a line segment, the majority of participants (17 out of 24) indicated that points were either the places that a line segment begins and ends, or else they were markers that partition a line segment into equal units. These responses were surprising in light of students' responses to item a), and their ideas regarding the infinite number of 'values' on any line segment. Students' arguments supporting an infinite number of 'values' on a line segment were similar in nature to their arguments regarding item a) above. They described processes of finding "as many values as we want", however they distinguished between the finite number of points that existed on a line segment and the infinite number of points that could be "given a value" or labelled. As before, these arguments indicate a process conception of infinity. Further, the idea of 'finding values', or 'creating points' by assigning them values, may be interpreted as an attempt to reduce the level of abstraction of an infinite yet bounded quantity.

Students' distinction between point and value prompted a class discussion regarding the geometry of points and lines to establish a shared understanding (to use the term loosely) of the infinite magnitude of points (rather than 'values') on a line segment. The questionnaire following this discussion related to the number of points on line segments of different lengths, and prompted students to reflect on the number of points 'missing' from the shorter of the two segments. The following specific question was posed:

> Consider line segments A and C again. Suppose that the length of A is equal to the length of C + x, where x is some number greater than zero, as depicted below. What can you say about the number of points on the portion of A whose length is x?
>
> 
>

In order to investigate both students' rationale when comparing the number of points on line segments of different lengths, and students' intuitions regarding subtracting infinite quantities, Q_1 presented their conclusions with a slight twist.

4.2 Subtracting infinity

> **Q_1.** On a previous question, you reasoned that two line segments A and C both have infinitely many points.
>
> 
>
>
> Suppose that the length of A is equal to the length of C + x, where x is some number greater than zero. You also previously suggested that the segment with length x has infinitely many points. That is, the ∞ points on A minus the ∞ points on C leaves an ∞ number of points on the segment with length x. Put another way,
>
> $$\infty - \infty = \infty.$$
>
> Do you agree with this statement? Please explain.

Participants' responses to Q_1 revealed inconsistencies in students' conceptions, as well as a strong intuitive resistance to the idea of subtracting infinite quantities. Jack, for instance, experienced a conflict as a conception of infinity emerged that contrasted his intuition.

Previously, Jack had described infinity as a "hypothetical number" that is "the biggest number you can get", and for which "you'd have to count your whole life and you still would never get there." Intuitively, Jack seemed to conceive of infinity as an unattainable extension of 'very big'. His comment that counting your whole life "still would never get [you] there" typifies a process conception of infinity. However, this fundamental notion of infinity was challenged by the visual representation of the two line segments. In response to Q_1 Jack wrote:

> What I'm thinking is that if you got infinite points on A and if you got infinite on C, well, you're seeing that they're not equal. So how can you say that infinite points are equal? Like, visually, you're seeing that A is bigger, so therefore the infinite number has to be bigger on A than the infinite number on C. But then again, infinite is the largest you can get, so that's kind of confusing.

Jack observed that the two line segments are not equal in length, and thus concluded that the two could not have an equal amount of infinite points despite his insistence that infinity is "the largest you can get." The conflict in Jack's conceptions might be attributed to an attempt to extrapolate everyday experiences with finite measurements, where length and quantity are often directly proportional. Using familiar experiences to make sense of novel situations is considered by Hazzan (1999) as an attempt to reduce the level of abstraction of the new concept. In the case of infinity, extrapolating experiences with measurement can be deemed as a conception of 'measuring infinity'. Jack's conception of 'measuring infinity' is at odds with his intuition of a single, never-ending infinity, and his recognition of this created a cognitive conflict that he was unable to resolve.

The notion of 'measuring infinity' surfaced in several students' responses to Q_1, however most students neglected the inconsistency between it and their intuition of potential infinity. For instance, Rosemary rationalized the expression "$\infty - \infty = \infty$" by arguing that while any line segment will have infinitely many points, a longer segment would have a larger infinite number of points. She also claimed that subtracting an infinite quantity from another (albeit “larger”) infinite quantity would leave “a lot of points… extending into infinity” and “it will take forever” to count them. The inconsistency between a process conception of infinity, as exhibited by Rosemary’s description of “extending into infinity” and taking “forever”, and her measuring conception of a “larger” infinity went unnoticed.

Of the various responses to Q_1, Lily’s was unique. In her response, she disagreed with the possibility that $\infty - \infty = \infty$. She wrote:

> I disagree with this statement. For example, π is an infinite (on going) number. If we subtract $\pi - \pi$ the answer is 0, NOT ∞. But, if there is a restriction that says we can’t subtract by the same number it could still be an infinite number, but just a smaller value. For example, $\pi - 2\pi = -\pi$, is still an infinite number, only negative.

Lily appeared to conceive of infinity as potential – her use of the qualifier “on going” to describe her notion of an “infinite number” corresponds to a process conception of infinity. However, the on-going process in Lily’s conception is applied, not to the magnitude of her “infinite number”, but to its infinite decimal representation. Lily’s objection to Q_1 seems to stem from confusion between an infinite magnitude, such as the number of points on a line segment, and the infinite number of digits in the decimal representation of π. Her use of π to justify claims about infinite magnitudes is indication of a disconnect between points on a line and real numbers. Further, not only did Lily overlook the particular value of π itself, but she also failed to distinguish the differences between acting on one specific element as opposed to infinitely many. Lily reasoned that since π is an “infinite (on going) number” and $\pi - \pi = 0$, then the difference $\infty - \infty$ must also be 0. Lily’s generalization of properties of π to draw conclusions about the entire set of points can be interpreted as an attempt to reduce the level of abstraction of dealing with an infinite number of elements. The use of one number to explain properties of infinitely many coincides with Hazzan's (1999) observation that students will try to reduce the level of abstraction of a set by operating on one of its elements rather than all of them.

Another interesting aspect of Lily's response was her use of "restrictions." She proposed that the difference of two 'infinite numbers' might be another 'infinite number' if there are appropriate restrictions placed on the quantities. By restricting the 'values of infinity' she reasoned that it is possible to attain "an infinite number, it [will] just be a smaller value." Appending "restrictions" allowed Lily to conceive of 'infinite numbers' with different sizes, despite the conflict with her description of infinity as "on going". The notion of infinities with 'different values' is consistent with an intuition of measuring infinity (Tall, 1980), and serves as an example of reducing the level of abstraction. According to Hazzan, this can be seen as a case of using familiar procedures to cope with novel and abstract concepts: Lily applies the familiar procedure of subtracting real numbers to cope with the concept of subtracting transfinite ones.

4.3 'Infinite numbers'

Lily's confusion between an infinite number of elements and an infinite number of digits in one particular element emphasised the disconnect between numeric and geometric representations of infinity that appeared in the early stages of the study. The question of whether other students shared Lily's ideas regarding the magnitude of a number with infinite decimal expansion naturally arose. Thus, a follow up questionnaire (Q_2) recalled Q_1, presented Lily's argument verbatim, as well as a similar one, and asked students to elaborate on whether or not they agreed with the arguments.

Q_2. Recall [Q1 as quoted above].

Student X: [Lily's response as quoted above]

> Student Y: I disagree with this statement. You can subtract two infinite numbers and NOT end up with ∞. For example, $\frac{1}{3}$ is an infinite number, but $\frac{1}{3} - \frac{1}{3} = 0$, NOT ∞. Also, $\frac{4}{6}$ and $\frac{1}{6}$ are both infinite (on going) numbers, but if we subtract $\frac{4}{6} - \frac{1}{6} = \frac{3}{6} = ½ = 0.5$, which is not an infinite number. But sometimes it's possible to subtract two infinite numbers and get an infinite number. For example, $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$, which is infinite and smaller than $\frac{1}{3}$. So, sometimes $\infty - \infty = \infty$, but usually not.

Most participants (22 out of 24) agreed with at least one of the arguments in Q_2, which came as a surprise in light of the common description of infinity as the "largest you can get". The confusion between infinite magnitude and infinite decimal representation revealed two distinct interpretations of 'infinite numbers'. For the students who agreed with both arguments, confusion between magnitude and representation was broad: they ignored the finite magnitude of both rational and irrational numbers. For instance, Jim wrote:

> $\frac{4}{6}$ and $\frac{1}{6}$ are both infinite (on going) numbers but when subtracting them your result is $\frac{1}{2}$ which is not infinite. This proves that an infinite number subtracting by another infinite number is not always another infinite number. As a result the statement $\infty - \infty = \infty$ is not true because sometimes the result is infinite but a different value and other times the result is not infinite.

In his response, Jim readily accepted the arguments of students X and Y, neglecting the differences between a particular (finite) value and an infinite quantity. Jim used the infinity symbol to represent numbers of different magnitudes, and as such, exemplified students' notions that infinity has no 'specific value'. The dynamic nature of this conception can be interpreted as an attempt to reduce the level of abstraction of an entity that is beyond the realm of his imagination. Jim's attempt to extrapolate his experiences with finite quantities, and also to use them explicitly (though perhaps unknowingly) to justify his notions of infinity, is further indication of an attempt to reduce the level of abstraction of the expression '∞ - ∞'.

Other students held a slightly different conception of 'infinite number' – they recognized rational numbers as finite quantities and associated them with points on a number line, but did not make the same association with irrational numbers, mistaking them with infinite quantities. This interpretation was exemplified in Rosemary's response to Q_2. When addressing student X, Rosemary remarked:

> $\pi - \pi = 0$ that is correct because one is taking away the same amount of points from what they initially began with will give 0, but in the line segment question, the amount of points in x (which is ∞ amount) is much less than the amount of points in A and C. Which because of this, I agree with Student X's second statement of how there should be restrictions. In this case, points in x are less than points in A or C.

As in Q_1, Rosemary's response is consistent with the idea of 'measuring infinity', using Lily's notion of 'restrictions' to accommodate the possibility that a longer segment will have a greater number of points. Further, Rosemary identified with Lily's argument regarding $\pi - \pi$, and alludes to the possibility of a line segment having π-many points. Her remark that $\pi - \pi = 0$ is correct because "one is taking away the same amount of points from what the initially began with" illustrates participants' general confusion regarding the magnitude of irrational numbers.

Additional evidence of Rosemary's attempts to reduce the level of abstraction of subtracting transfinite numbers is seen in her response to student Y:

> Student Y states: $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$ (which is an ∞ number) but $\frac{4}{6} - \frac{1}{6} = \frac{3}{6}$ (which is only 0.5 and not an ∞ number). Well, when we represent these numbers on a number line *[drew two line segments, one from 0 to* $\frac{1}{6}$ *and one from 0 to* $\frac{1}{2}$*, and labelled the segments A*

> *and B, respectively]* then won't both line segments have ∞ points? (But of course segment B will have more than segment A)

Once again, Rosemary appealed to her intuition of 'measuring infinity' as she related student Y's numeric example to its geometric representation. In contrast to her use of π, Rosemary distinguished rational numbers from infinite quantities. Although she stated that $\frac{1}{6}$ was an "infinite number," she observed its specific value on the number line. Similarly, she remarked that though $\frac{1}{2}$ was not infinite itself (it "is only 0.5"), when represented on a number line she acknowledged there were still infinitely many points between 0 and $\frac{1}{2}$. This distinct handling of rational and irrational numbers suggests a misconception about real numbers: whereas rational numbers were associated with points, irrational numbers were not. Nevertheless, Rosemary seemed to use the words "infinite number", both to represent a number with infinitely many (nonzero) digits in a decimal representation, as well as to represent the infinite quantity of points on a line segment. It would be interesting to see if Rosemary's measuring conception would be so persuasive had she not applied the same terminology to two different notions.

4.4 After instruction: Lily and Jack

At the end of the course, the class was instructed on equivalences of infinite sets, as well as the distinction between an infinite decimal expansion and an infinite quantity. Specifics of the instruction are detailed below. In the months following the end of the course, follow up interviews were conducted with two students: Lily and Jack.

The interview with Lily took place roughly six months after instruction regarding the distinction between infinite magnitude and infinite representation, and included a discussion on the finite value of π. The interview with Lily focused on her conception of π as an 'infinite number', and since it was the number of decimal digits that gave π it's infinite quality, Lily was asked to speculate on the number of decimal digits of a rational scalar of π. She reasoned, "if we times [π] by 3 it'll just be a bigger number, with more digits." As with the line segments, Lily expressed ideas consistent with 'measuring infinity': she associated "bigger" with "more," believing that 3π would be infinite but a "bigger infinite" than π.

Lily's perception of the "infinite size" of π persisted despite instruction and also in conflict with her ideas regarding 3.14 as an approximation of π. She claimed that 3π was "3 times a number that's really big." To determine the magnitude of 3π, Lily used the familiar number 3.14, yet she was surprised to calculate that triple this number was only about 9: "let's say π is 3.14, then times 3 is going to be big. Well, not big, but (pause) well, kind of triple?" Notwithstanding Lily's attempts to reduce the level of abstraction of π by working with 3.14, it seemed difficult for her to accept π as a small number. When asked about the possibility of measuring a length of π cm, she claimed that one would need "a really big ruler" with huge spaces between each whole number to accommodate all of π's decimal digits. She argued that since π's expansion was infinite and never-ending, then any segment of length π would have to be "really long, until, if possible, there's an end to it." Lily seemed to ignore the actual magnitude of each of π's decimal digits, which, together with her process conception of a never-ending infinite, might have contributed to her notion of π as very large, despite the relatively small magnitude of 3.14.

The struggle to accommodate conflicting ideas, such as Lily faced with her conceptions of π, also surfaced in the interview with Jack. In his written responses, Jack had struggled with the conflict between his competing conceptions of potential and measuring infinity. Following instruction, Jack continued to express inconsistent notions of infinity as he attempted to reconcile his naïve understanding with a normative one. The interview with Jack, which took place two months after the end of the course, began by recalling class instruction on the correspondence between points on line segments of different lengths.

The instructional class discussion included the following well-known geometric construction of a bijection between two line segments AB and CD. The construction begins by connecting the endpoints of AB and CD with line segments that extended past the endpoints of CD to meet at a point labelled p, as depicted in Figure 1. An arbitrary point, *w*, can be labelled on AB and connected to the point *p* by a line segment. The connecting segment will intersect CD at a point *r*, as depicted in Figure 2. With this construction, it is possible to pair up each point on AB with exactly one point on CD. Conversely, a ray from *p* to any point on CD can be extended to meet a point on AB in a unique way. In this manner, every point on CD is paired with exactly one point on AB. Thus a one-to-one correspondence is constructed between the set of points on AB and the set of points on CD. Most students easily followed the construction, though there was significant resistance to the idea that the longer line segment would not have more points.

Figure 1:

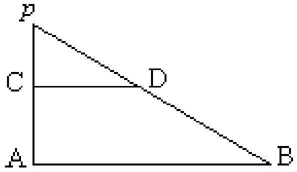
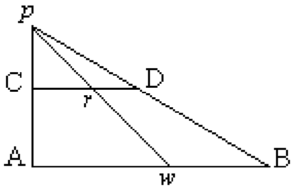
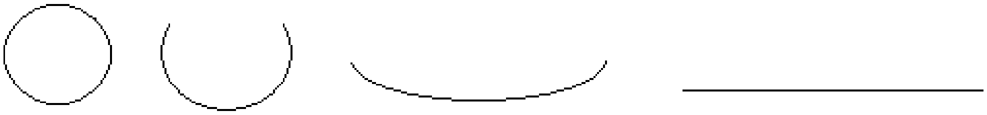


Figure 2:



Jack had no trouble recreating the above argument. However, he insisted, “that A [AB] is bigger, so therefore the infinite number has to be bigger on A [AB] than the infinite number on C [CD].” Jack’s conception of measuring infinity was very compelling, and he continued to struggle with the conflict between it and his intuition that infinity “is the largest you can get” and is “never-ending.” In an attempt to challenge his measuring intuition, Jack was asked to consider the number of points on two circles of different circumference. He claimed there were an infinite number of points because “drawing a line from the centre to the side [drew the radius of the circle], you can draw infinite of them.” Furthermore, he noted that the circles would have the same number of points because “you’re not caring about the length of the radius, which makes your circle bigger or smaller. You’re caring about the 360 degrees,” that is, the number of radii, which is the same in both circles. We then proceeded to ‘cut open’ and ‘flatten’ each circle, such as in Figure 3.

Figure 3:



Jack judged that even though the shape of the circles was now different, the number of points had not changed[2]. Jack reasoned that the two flattened circles would still have an equinumerous set of points because "you still have that imaginary [centre] point, and all the [radii] connecting to it." This construction is essentially the same as the triangle argument above: the number of rays from *p* that intersect with the longer line segment is the same as the number that intersect with the shorter line segment. The visual representation had a significant effect on Jack's perceptions. Comparing and equating the number of radii of two circles was canonical, even when they were flattened. However, Jack noted "if you go back to this [lines AB and CD], still, if you look at it this way it still doesn't make sense. The circle way kind of does. Well, not kind of, it actually does." Eventually, Jack accepted that two line segments of different lengths could have the same quantity of points, stating it was "hard to believe, but it makes sense."

5. CONCLUDING REMARKS

This paper examines undergraduate students' emerging conceptions of infinity during their efforts to coordinate intuition with conventional mathematical properties. As students grappled with properties of actual infinity, they unearthed features that were at odds with their personal experiences – participants were challenged by competing and inconsistent notions of infinity as endless or as a large number whose size was relative. In resonance with earlier work (e.g. Fischbein et al., 1979), students often remained unaware of these inconsistencies. Further, students' responses support the argument that infinity is conceived of intuitively as an inexhaustible process, rather than a completed object, in APOS terminology. However, it is notable that the conception of 'measuring infinity' which emerged in students' attempts to reduce the level of abstraction of comparing geometric infinite sets was a persuasive factor in students' reasoning, and at times overshadowed the association of infinite with endless.

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[2] Topologically, the line segment and circle do differ: an open line segment is isomorphic to $S^1 \setminus \{N\}$, for some point N. However, since the goal was to compare two circles in their 'new form' and not to compare the line segment with the circle, this fact was not addressed at that moment in the conversation.

This study sheds new light on students' emerging conceptions of infinity as manifested in their engagement with geometric tasks. Geometric representations provided a useful analogy for demonstrating qualities of transfinite arithmetic, and as such, confronted students with the property that transfinite subtraction is undefined. It has been shown that many students are tempted to treat infinity as simply a very big number (e.g. Sierpinska, 1987), however students' conceptions regarding arithmetic with transfinite numbers is lacking in mathematics education literature. This study offers a first glimpse at learners' attempts to reduce the level of abstraction of transfinite subtraction. The issue of learners' conceptions regarding transfinite arithmetic is of interest in my ongoing investigations.

Students' attempts to cope with the expression "$\infty - \infty$" revealed significant misconceptions regarding the size of real numbers. Their confusion between the infinite magnitude of points on a line segment and the infinite decimal representation of both rational and irrational numbers created an obstacle to a conventional understanding of mathematical infinity, and demonstrated a shortcoming in their understanding of number and place value. Furthermore, students' failure to identify specific numbers as points on a number line highlighted a disconnect between their conceptions regarding numeric and geometric representations of infinity. The use of finite quantities to explain phenomena of transfinite ones misguided students' intuitions and, ultimately, their understanding. Students' various attempts to reduce the level of abstraction of infinitely many points on a line segment by considering properties of a single point revealed an intuition of infinity that may be at odds with future instruction on limits and set theory.

This study opens the door for further investigation regarding some issues that may be taken for granted, such as the relationship between magnitude and representation, and the connection between points on a line and numbers. Future research will attend to the persuasive factors that can influence change in learners' emerging conceptions, as well as to the different conceptual challenges learners face when addressing properties of 'infinite numbers' and transfinite arithmetic.

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On the use of Realistic Fermi problems for introducing mathematical modelling in school

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Abstract: In this paper uses an analytical tool refereed to as the MAD (Modelling Activity Diagram) framework adapted from Schoenfeld's parsing protocol coding scheme to address the issues of how to introduce mathematical modelling to upper secondary students. The work of three groups of students engaged in solving so called realistic Fermi problems were analysed using this framework, and it was observed that the processes involved in a typical mathematical modelling cycle were richly represented in the groups' solving processes. The importance of the social interactions within the groups was noted, as well as the extensive use of extra-mathematical knowledge used by the students during the problem solving session.

Keywords: Fermi problems; Fermi estimates; Modelling cycles; Mathematical modelling; Modelling activity diagram;

1. INTRODUCTION

The study of mathematical modelling in mathematics education has been a steadily growing branch of research since at least the late 1960's (Blum, 1995). The arguments for including mathematical modelling in mathematics education have been collected under *the formative argument*; *critical competence*; *utility*; *picture of mathematics*; and *the promoting mathematics learning argument* (Blum & Niss, 1991; Niss, 1989), and mathematical modelling is nowadays explicitly included as part of the mathematics curricula in many countries all over the world.

Although mathematical modelling is getting more and more emphasised in governing curricula documents and despite extensive research efforts (e.g. ICTMA[2] publications), the adjustment and change in classroom practise on broad (national) scales are slow. Teacher-training courses, in-service courses for practicing teachers, textbooks and teaching materials, as well as ways of working with mathematical modelling in the classroom, need to be further developed and made accessible.

This paper addresses the issue of how to introduce mathematical modelling to upper secondary students in the very beginning of a modelling course or before engaging in a modelling project. In principle, such an introduction can be done using a direct or an indirect approach, where typically the direct approach is to present some sort of 'heuristics' to the students for how to model mathematically. This paper however, investigates the potential of an indirect approach, where groups of students are set to work on so called *Fermi problems*, which are open, non-

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[2] The International Community of Teachers of Modelling and Applications, http://www.ictma.net/

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standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations.

2. CONTEXT AND AIM OF THE STUDY

The study reported here is part of a bigger research project aiming to both get an overall picture of the past and present state and status of mathematical modelling in the Swedish upper secondary school, and to develop, design, implement and evaluate small teaching modules (sequences of lessons) on mathematical modelling in line with the Swedish national curriculum. In connection to the latter, the issue of how to introduce mathematical modelling in the beginning of these modules was first given priority. However, previous research and reports accounting for the designing and developing of courses and lessons on mathematical modelling (e.g. J. S. Berry, Burghes, Huntley, James, & Moscardini, 1987; Blum, Galbraith, Henn, & Niss, 2007, especially chapter 3.6) are sparse about how the actual introduction of mathematical modelling was implemented. One exception is Legé (2005), who contrasts a reductionist and a constructivist instructional approach to the introduction of mathematical modelling at high school level. Irrespective of some positive results, Legé's introductions of mathematical modelling are activities lasting over two weeks of time, which for my purpose is too long. To try to shred some light over if this could be done in a more time efficient way and to facilitate the design of the teaching modules, this pilot study was conducted.

The aim of the study reported on here was to investigate if Fermi problems could be used to introduce mathematical modelling at the Swedish upper secondary level. The question in focus can in general terms be formulated as: *What mathematical problem solving behaviour do groups of students display when engaged in solving Fermi problems?* This *preliminary* research question will be reformulated and specified in section 5 in terms of the theoretical framework that will be outlined.

The structure of the paper is as follows. First I briefly discuss perspectives on Fermi problems and mathematical modelling, before looking at previous research done in connection with this area. Then, I describe the methodology, the definition of *realistic* Fermi Problems, the concept of mathematical modelling sub-activities, and the developing of the MAD framework. The paper proceeds with accounting for the result and the analysis of the empirical study, before finally the discussion and conclusion are presented[3].

3 FERMI PROBLEMS

The term *Fermi problem* originates from the 1938 Italian Nobel Prize winner in physics Enrico Fermi (1901-1954), who was also a highly appreciated and popular teacher (Lan, 2002). He had a predilection for posing, as well as solving, problems like *How many railroad cars are there in US?* (Goldberger, 1999) or *How many piano tuners are there in the US?* (Efthimiou & Llewellyn, 2007), and by using a few reasonable assumptions and estimates, he gave astoundingly accurate and reasonable answers. Fermi was of the opinion that a good physicist as well as any thinking person could estimate any quantity quantitatively accurate just 'using one's head'; that is, just by reasoning, making some realistic and intelligent order of magnitude estimates and doing some simple calculations. Often, the questions came from everyday situations and phenomena he saw or experienced, as illustrated by Wattenberg (1988); "Upon seeing a dirty window, he [Fermi] asked us how thick can the dirt on a window pane get?" (p. 89). These types of problems are called *Fermi problems*, *back-of-envelope calculation problems* or *order of magnitude problems*. To my knowledge, Fermi himself did not define the characteristics of such problems explicitly and different authors in the literature emphasise different things.

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[3] This paper is an extend version of the paper by Berman Ärlebäck and Bergsten (in press)

3.1. Fermi Problem in physics education

Historically the ability to perform order of magnitude calculations was crucial for physicists before investing time and effort in engaging in a long and complicated calculation (Robinson, 2008). Nowadays, when today's extensive use of computers and computer packages easily and fast make these calculations for us, other arguments for including Fermi problems in physics, science and teacher education are used.

According to Chandler (1990), Fermi problems typically are intended to end up in estimates "to the nearest power of ten without using reference books or calculators" (p. 170). Carlson (1997), on the other hand, elaborates a bit more and describes the process and essence of solving Fermi problem as "the method of obtaining a quick approximation to a seemingly difficult mathematical process by using a series of 'educated guesses' and rounded calculations" (p. 308) and argues for their effective motivational potential in students. Following the same line of reasoning, Efthimiou and Llewellyn (2007, p. 254) characterize a Fermi problem as initially always seeming rather vague in its formulation, giving limited or no information on relevant facts or how to attack the problem. However, after a closer inspection and analysis, they undeniably allow an unfolding of the problem into simpler problems that eventually lead to a final answer to the original question. Another argument put forward by Efthimiou and Llewellyn (2007) is to use Fermi problems in general education and introductory science courses to foster students' critical thinking and reasoning.

Robinson (2008) puts forward a view in line with Carlson (1997), but is more specific when he writes that "[i]n order to solve a Fermi problem, one has to synthesize a physical model, examine the physical principles which are in operation, determine other constraints such as boundary conditions, decide how simple the model can be while still maintaining some realism, and only then apply some rough estimation to the problem." (p. 83). Drawing on this characterization, he argues that in the process of solving Fermi problems, the same set of skills is used which professional physicist use in their everyday work, but seldom are learned before the beginning graduate training. Thus, according to Robinson, the main argument for the use of Fermi problems in education is to introduce key skills and methodologies to students in an early stage of their schooling.

3.2. Fermi Problems in mathematics education

Turning to the field of mathematics education one can find both similarities and differences of how Fermi problems are characterized as compared to in physics education. Ross and Ross (1986) write that "[t]he essence of a Fermi problem is that a well-informed person can solve it (approximately) by a series of estimates" (p. 175), and that "[t]he distinguishing characteristic of a Fermi problem is a total reliance on information that is stored away in the head of the problem solver… Solving Fermi problems presents an artificial challenge" (p. 181). With a moderately free interpretation of the meaning of 'well-informed person' most authors agree with the first of these quotes, but concerning the latter there is diversity. Others, such as Peter-Koop (2004) and Sriraman and Lesh (2006), are of the opinion that the concept of Fermi problems is better and more useful if one allows the problems not to be purely intellectual in nature, but situated in the real world and in an everyday context.

Other characteristics ascribed to Fermi problems by some authors are their accessibility and/or self-differentiating nature, which means that the problem can be worked on and solved in different school grades as well as at different levels of complexity (Kittel & Marxer, 2005). Also, as expressed by Sowder (1992), there should not exist an exact answer: "[s]uch problems *must* be answered with an estimate, since the exact answer is not available" (p. 372, italics in original).

Some authors define the characteristics sequentially and more implicitly by describing the steps that are needed, or the understandings or insights that need to be achieved, to successfully come up with an answer. For example, Dirks and Edge (1983) list four "things typically required" when solving Fermi problems, namely "sufficient understanding of the problem to decide what data might be useful in solving it, insight to conceive of useful simplifying assumptions, an ability to estimate relevant physical quantities, and some specific scientific knowledge" (p. 602).

According to Ross and Ross (1986), the reason for teachers to use Fermi problems in teaching is twofold; first "to make an educational point: problem-solving ability is often limited *not* by incomplete information but the inability to use information that are already available" (p. 175, italics in original); and secondly, to give the students a more nuanced picture of mathematics, showing that doing mathematics is not always about getting exact answers through well-defined procedures. A more recent argument for the use of Fermi problem in mathematics education is the possibility to use them as a bridge between mathematics and other school subjects, engaging students in different interdisciplinary activities (Sriraman & Lesh, 2006).

Compared to how Fermi problems are viewed in physics education, it is notable that both disciplines to some extent describe the same procedure about how to approach and solve such problems; *making simplifying assumptions*, *estimating*, and *doing rounded calculations* are important aspects of the problem solving process. In addition, the more recent references from both fields argue for the potential inherent in the problems for foster students' critical thinking. The biggest difference between mathematics and physics education is the view of why to the use Fermi problems. In the latter, Fermi problems *in themselves* are seen to illustrate and emphasise basic and fundamental principles of physics, whereas in mathematics education, at least in the early references, they are artificial in nature, used as *tools* for teaching and learning some mathematical content. However, this view expressed in some of the references seems to be changing (e.g. Sriraman & Lesh, 2006).

3.3. Some previous research involving Fermi problems in education

Focusing on mathematics education research, Fermi problems seem to be used fairly sparse and are in principle mentioned in two different contexts at the lower educational level. First and foremost, they are mentioned in connection with (measure) estimates (sometimes called numerosity problems); and secondly, they are mentioned in connection with modelling. Recently, Fermi problems have also been suggested and used in fostering students' critical thinking (Sriraman & Lesh, 2006; Sriraman & Knott, 2009). The focus in this paper is on the use of Fermi problems in connection with modelling. However, for a review of the research on estimates see Sowder (1992), and the more recent Hogan and Brezinski (2003).

Although there exist extensive literature about mathematical problem solving, only a handful of papers explicitly use Fermi problems. Furthermore, most of the articles are theoretical in the sense that they discuss 'what one can do with Fermi problems' (see the references in the section about Fermi problems in mathematics education above). However, Peter-Koop (2003; 2004; 2009) used Fermi problems in third and fourth grade to, among other things, investigate students' problem solving strategies. She concluded that (a) Fermi problems were solved in a sensible and meaningful way by the students, (b) the students developed new mathematical knowledge, and (c) solution processes "revealed multi-cyclic modelling processes in contrast to a single modelling cycle as suggested in the literature" (2004, p. 461). The impact and use of these results on a broader scale are however unclear and remains to be researched further.

Beerli (2003) reports on Swiss teaching materials for grades 7 to 9 which emphasize Fermi problems in a thematic way throughout; and, because of the way problem solving is connected to reality, opportunities are provided to use mathematization and modelling as means to develop mathematical knowledge and skills. Beerli also suggests that Fermi problems could effectively be used in assessment (pp. 90-91).

Schoenfeld (1985b, pp. 278-281) writes about a Fermi problem called *the cell problem* used for investigating cognitive issues connected to, and the developing of, methodologies for the study of students' problem solving processes. He describes how different constellations of students try to estimate how many cells there might be in an average-sized human body, to think about criteria of what might count as a reasonable upper/lower estimate, and to decide how much confidence they have in their estimates. The result of his analysis formed part of the foundation for the research approach he used in his book *Mathematical Problem Solving* (Schoenfeld, 1985b).

4. MATHEMATICAL MODELLING

In mathematics education one can find many different approaches to and perspectives on mathematical modelling in the research literature (Blum, Galbraith, Henn et al., 2007; Haines, Galbraith, Blum, & Khan, 2007). The variety of perspectives is illustrated by Sriraman and Kaiser (2006) in their report of an analysis of the papers presented in *Working Group 13: Applications and modelling* at the CERME4 conference[4] written by European scholars. They conclude "that there does not exist a homogenous understanding of modelling and its epistemological backgrounds within the international discussion on applications and modelling" (p. 45) and argue and call for a more precise clarification of the concepts involved in the different approaches to make communication and discussions more simple and fruitful.

However, Kaiser, Blomhøj and Sriraman (2006) are optimistic about the chances for such an understanding to develop and they argue that there already in certain respects exists "a global theory for teaching and learning mathematical modelling, in the sense of a system of connected viewpoints covering all didactical levels" (p. 82), but that this "theory of teaching and learning mathematical modelling is far from being complete" (p. 82). Hence, in the last years, efforts have been made to clarify and differentiate different approaches (Kaiser & Sriraman, 2006; Kaiser, Sriraman, Blomhøj, & García, 2007) and work in this direction continues (Blomhøj, 2008). According to Kaiser and Sriraman (2006), different perspectives on, and approaches to, mathematical modelling can be classification as *realistic or applied modelling*; *contextual modelling*; *educational modelling* (with either a *didactical* or *conceptual* focus); *socio-critical modelling*; *epistemological or theoretical modelling*, or *cognitive modelling*.

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[4] The 4th conference organized by ERME, European society for Research in Mathematics Education, held in Sant Feliu de Guíxols, Spain, 17-21 February 2005.

Research on mathematical modelling in mathematics education, regardless which perspective on modelling is adapted, typically uses or develops some general description of the process of mathematical modelling (Kaiser et al., 2006). This general description is often given or summarised in a so called *modelling cycle*, which schematically and idealised illustrates how the modelling process connects the extra-mathematical world (domain) and the mathematical world (domain) (Blum, Galbraith, & Niss, 2007). Depending on the purpose and focus of the research these modelling cycles might look different and highlight different aspects of the modelling process (Haines & Crouch, in press; Jablonka, 1996).

In the Swedish context, where the present study is situated, the present mathematics curriculum and syllabus for the upper secondary level is founded in a reform from 1965. Since then, a number of reforms and revisions have been made with the affect that the emphasis on mathematical modelling in the written curricula documents governing the content in Swedish upper secondary mathematics courses has gradually been gaining more momentum (Ärlebäck, submitted). In the latest formulation from 2000, *using and working with mathematical models and modelling* is put forward as one of the four important aspects of the subject that, together with *problem solving*, *communication* and *the history of mathematical ideas*, should permeate all mathematics teaching (Skolverket, 2000). It is also stressed that "[a]n important part of solving problems is designing and using mathematical models" and that one of the goals to aim for is to "develop their [the students'] ability to design, fine-tune and use mathematical models, as well as critically assess the conditions, opportunities and limitations of different models" (Skolverket, 2000). However, a more detailed definition or description of what a mathematical model is or what it means to model mathematically, is not provided. Lingefjärd (2006) summarizes the development and situation as "it seems that the more mathematical modeling is pointed out as an important competence to obtain for each student in the Swedish school system, the vaguer the label becomes" (p. 96). The question that naturally arises is why this is the case and what can be done about it. Research in this area is lacking in Sweden, but challenges and barriers to overcome are likely to be similar to those reported by Burkhardt (2006). Some support for this assumption is found in research reports on the dominant role of the use of traditional textbooks in Swedish mathematics classrooms, greatly influencing class organisation as well as content (Skolverket, 2003).

The perspective on mathematical modelling I take in this paper is the view that mathematical modelling is a complex (iterative and/or cyclic) problem solving process, here illustrated in Figure 1. When analysing such complex problem solving process in more detail, one can do so using the notion of *competencies* (Blomhøj & Højgaard Jensen, 2007; Maaß, 2006), *modelling skills* (J. Berry, 2002), or dividing the modelling cycle into *sub-processes* or *sub-activities*. For example, Borromeo Ferri (2006) describes the modelling process in Figure 1 in terms of 6 *phases* (real situation, mental representation of the situation, real model, mathematical model, mathematical result, and real results) and *transitions* between these phases (understanding the task, simplifying/structuring the task, mathematizing, working mathematically, interpreting, and validating).

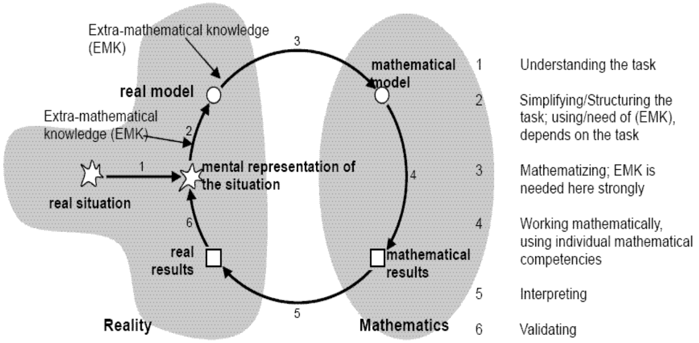
Extra-mathematical knowledge (EMK)
3
mathematical model
real model
Extra-mathematical knowledge (EMK)
2
1
mental representation of the situation
real situation
4
6
real results
mathematical results
5
Reality
Mathematics
1 Understanding the task
2 Simplifying/Structuring the task; using/need of (EMK), depends on the task
3 Mathematizing; EMK is needed here strongly
4 Working mathematically, using individual mathematical competencies
5 Interpreting
6 Validating

Figure 1: The modelling cycle by Blum and Leiβ (2007) as adapted and presented by Borromeo Ferri (2006, p. 92)

This choice of view on mathematical modelling is in line with how mathematical modelling is described in the Swedish upper secondary curriculum (see above) and other scholars have made analogous interpretations (e.g. Palm, Bergqvist, Eriksson, Hellström, & Häggström, 2004). In my interpretation of the classification of the approaches by Kaiser and Sriraman (2006) briefly mentioned above, this research is carried out in alignment with the educational perspective, where the goal to strive for is to learn students mathematical modelling. Some of the arguments of Kaiser, Blomhøj and Sriraman (2006) about different purposes for using such a view on the modelling process in research also motivate this choice of perspective; as the driving motivation is as *a didactical tool for planning* the very introduction of mathematical modelling in a modelling course or small modelling project, as *a point of reference to the particular curricular element* of modelling in governing curricula, and as the point of departure for developing *an analytical tool to retrospectively determine which part of the modelling cycle the students' have been working with.*

5. METHODOLOGY

Part of my research is done in collaboration with two teachers aiming to design a number of small *modelling teaching modules*[5] in line with the national curriculum for the Swedish upper secondary school. A natural question that arises in the early stages of the design process is how to introduce the topic of mathematical modelling in a gentle, efficient and interesting way. The paper entitled *Modeling conceptions revisited* by Sriraman and Lesh (2006), where they say in a paragraph about estimation and Fermi problems that "estimation activities can be used as a way to initiate mathematical modeling" (p. 248), caught my interest. However, not finding any reported empirical research on this specific matter, the decision was to design and conduct this small pilot study.

5.1. Developing an analytic tool

5.1.1 Schoenfeld's protocol coding scheme

For the present study I developed and used an adapted version of Schoenfeld's 'graphs of problem solving' (Schoenfeld, 1985b) to get a schematic picture of the problem solving process of students working on a Fermi problem. Originally, Schoenfeld's 'graphs of problem solving', adapted from Wood (1983, cited in Schoenfeld, 1985b), is part of "an analytic framework for the macroscopic analysis of problem-solving protocols, with emphasis on executive or control behaviour" (p. 271) in decision-making during problem-solving. His idea is to try to characterize the problem-solving process of an individual, or a group of individuals[6], by analysing transcriptions of verbal data generated during "out load problem-solving session" (p. 270). This he achieved through partitioning protocols[7], which is what Schoenfeld calls his verbal data, "into macroscopic chunks of consistent behaviour called *episodes*. An episode is a period of time during which an individual or a problem-solving group is engaged in one larger task … or a closely related body of tasks in the service of the same goal." (p. 292). Each episode was then characterized as either *reading*, *analysis*, *planning*, *implementation*, *exploration*, *verification*, or *transition*[8]. All the categories are briefly described together with so called *associated questions*, and "[a] full characterization of a protocol is obtained by parsing a protocol into episodes and providing answers to the associated questions." (p. 297).

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[5] A *modelling teaching module* is here a sequence of lessons with a planned focus on mathematical modelling.

In his original work, Schoenfeld claims that the reliability of the coding, done by three undergraduate students trained for the coding, is quite high (Schoenfeld, 1985b, p. 293), but no measure or methods of how this conclusion was drawn is presented. Scott (1994), attempting to replicate some of Schoenfeld's results, argues that "[t]here are clearly problems of interpretation in several of his [Schoenfeld's] behaviour categorisations" (p. 538), and through examples Scott illustrates that "fundamental ambiguity remains regarding the meaning of some of the parsing categories" (p. 527). However, Schoenfeld (1985b) ends the chapter on his framework for analysing the protocols in a humble way arguing that "it is best thought of as work in progress." (p. 314).

Although Scott (1994) is critical about the reliability of Schoenfeld's method, he expresses his conviction that "*concurrent verbalisation* with *no interviewer intervention* is least prone to the effects of the study environment, and to the incompleteness and inconsistency of some verbal data" (p. 537, italics in original). The debate on the suitability and usefulness of verbal data in research is reviewed and discussed in some length by Goos and Galbraith (1996). Drawing on the work by Nisbett and Wilson (1977), Ericsson and Simon (1980), Genest and Turk (1981) and Ginsburg, Kossan, Schwartz, and Swanson (1983), among others, they discuss the three different approaches of *talk/think aloud*, *concurrent probing* and *retrospective probin*g. Especially, they acknowledge differences in limitations in terms of *reactivity* (such as stress, task demands and influences from the direct environment), *incompleteness* (what is spoken out load is selective of what a subject thinks), *inconsistency* (observed behaviour do not correspond to what is verbalized), *idiosyncrasy* (the question of generalisability due to sensitivity for subjects individual differences), and *subjectivity* (the researches bias influences and interpretation).

Schoenfeld (1985a) also discusses the reliability and usefulness in mathematics education research of data in the form of "verbal reports (protocols) produced by individuals or groups" (p. 171), but at a more pragmatic level. He lists and discusses five 'variables' which might affect and reduce the limitations mentioned above when problems are solved 'in the load'. These five variables are *the number of persons being taped*, *the degree of intervention*, *the nature of instruction and intervention*, *the environment*, and *task variables* (pp. 174-176). These variables are not independent and can both reinforce or obstruct each other effects on the displayed behaviour. For example, an artificial environment (as a laboratory taped problem solving situation often is) making the students feel uneasy can be balanced with letting the students work in group constellations familiar to them. In a way, all five of Schoenfeld's variables deal with reactivity, and the 'values' of the variables will affect how comfortable the students are in a taped problem solving situation and the expectations and obligations felt by the students towards the task given them and toward the researcher. The variables controlling instruction and intervention are also connected to the limitations of incompleteness and inconsistency.

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[6] In the case when the problem-solving process of a group is under consideration, the process refers to the process of the whole group as a collective, not being constituted of each individual problem-solving process.

[7] Or rather the transcription of the verbal protocols.

[8] Some episodes were characterized as both *planning* and *implementation* (Schoenfeld, 1985b, p. 296)

The idea to build on the work of Schoenfeld (1985b) is by no means new, and other researchers have used, modified and developed his ideas. For instance, in their study of two students' collaborative problem solving activity in an applied mathematics course, Goos and Galbraith (1996) used "a selective extension of Schoenfeld's episode analysis" (p. 241) combined with another framework. Stacy and Scott (2000) follows Schoenfeld's methodology "as closely as circumstances permitted" (p.123) when studying how and to what extent students use the problem solving strategy of *trying examples* in a problem solving situation. Exploring pairs of students' problem-solving process involving functions using a graphical calculator, Brown (2003) uses a modified framework for identifying and singling out interesting "defining moments" (p. 83) on a macroscopic level, before exploring these in greater detail on a microscopic scale in the search for possible explanation of the observed behaviour. In a closer look at decision making in group solutions' involving Bayes' formula in probability also Stillman (2005) makes use Schoenfeld's protocol parsing scheme. In contrast to Scott (1994), Goos and Galbraith (1996), Stacy and Scott (2000), and Stillman (2005), who use Schoenfeld's categories to code episodes as (part of) their framework, my adaptation is more in line with the approach taken by Brown (2003). Brown modified and extended the number of categories to better suit the characteristics of the problem solving situation she studied, as I did developing the *MAD framework* used in this study (see section 5.1.3.).

5.1.2. Realistic Fermi problems

Based on the earlier overview of the meaning and use of Fermi problems, I here describe my use of the concept by giving it the following definition. What I call *Realistic Fermi problems* are characterized by:

- their *accessibility*, meaning that they can be approached by all individual students or groups of students, and solved on both different educational levels and on different levels of complexity. A realistic Fermi problem does not necessarily demand any specific pre-mathematical knowledge;
- their clear real-world connection, to be *realistic*. As a consequence a Realistic Fermi problem is more than just an intellectual exercise, and I fully agree with Sriraman and Lesh (2006) when they argue that "Fermi problems which are directly related to the daily environment are more meaningful and offer more pedagogical possibilities" (p. 248);
- the *specifying and structuring of the relevant information and relationships* needed to tackle the problem. This characteristic prescribes the problem formulation to be open, not immediately associated with a know strategy or procedure to solve the problem, and hence urging the problem solvers to invoke prior constructs, conceptions, experiences, strategies and other cognitive skills in approaching the problem;

- the absence of numerical data, that is the *need to make reasonable estimates* of relevant quantities. An implication of this characteristic is that the context of the problem must be familiar, relevant and interesting for the subject(s) working in it;
- (in connection with the last two points above) their inner momentum to *promote discussion*, that as a group activity they invite to discussion on different matters such as what is relevant for the problem and how to estimate physical entities.

Using the nomenclature of Schoenfeld (1985a), the first four of these characteristics are all task variables that taken together define a type of problem quite different from the typical problems students normally encounter in their mathematics classes. In other words, one can expect that the students, at least initially, will behave a little lost, not knowing how to proceed. To some extend the last characteristic is also a task variable with the intention to, as a group activity, counteracting the problem solving process from stalling and the group to get stuck, which relates to the variable of number of students being taped.

The characteristics of Realistic Fermi problems were used for guidance when constructing the problems used in the study. From this point on, whenever the term (Realistic) Fermi problem is used in this paper, it refers to problems with these characteristics. It can be noted that there are some similarities to the six principles used in the *Models and Modeling perspectives* (Lesh & Doerr, 2003) for designing “thought revealing activities for research, assessment, and instruction” (Lesh, Hoover, Hole, Kelly, & Post, 2000, p. 595) called *modeling-eliciting activities*. Especially *the Reality Principle* bears resemblance to the realistic character of a Realistic Fermi problem; *the Self-evaluation Principle* is similar in the sense that a Realistic Fermi problem ‘invites’ the problem solver(s) to validate assumptions, estimates and calculations performed during the solving process; an analogue to *the Simplicity Principle* is the accessibility characteristic of a Realistic Fermi problem; and, as will be clear from the formulation of the Realistic Fermi problem used, also to *the Construct Document Principle*. The six principles for constructing modeling-eliciting activities of the Models and Modeling perspectives were not taken as point for departure for the construction of the Realistic Fermi problems used in this study, since these are embedding and situated in a multi-layer research framework with a broad focus facilitating a much wider research agenda.

5.1.3 The MAD framework

Comparing the phases and transitions in the modelling process described according to Borromeo Ferri (2006), and the character of a Realistic Fermi problem as presented above, one can see that there are obvious similarities. One might therefore try to describe and analyse the process of solving Fermi problems using this framework as it is. However, since the *estimating* of different sorts of quantities when solving Fermi problems is essential, and this is usually not a typical feature of a mathematical modelling problem, it seems that this activity also needs to be incorporated into the framework to give a more nuanced picture of the problem solving process of a realistic Fermi problem.

To adopt Schoenfeld’s (1985b) categories to the present study, I started from the view of modelling presented above and included the central estimation feature of Realistic Fermi problems to identify the following six *modelling sub-activities* to be used as codes for the activities the students engage in when solving a Femi problem:

Reading: this involves the reading of the task and getting an initial understanding of the task
Making model: simplifying and structuring the task and mathematizing

Estimating: making estimates of a quantitative nature
Calculating: doing maths, for example performing calculations and rewriting equations, drawing pictures or diagrams
Validating: interpreting, verifying and validating results, calculations and the model itself
Writing: summarizing the findings and results in a report, writing up the solution

Here, the activity of *reading* is similar to Borromeo Ferri's 'understanding the task'; *making model* incorporates parts of both 'simplifying/structuring the task' and 'mathematizing'; *calculating* is the same as 'working mathematically'; and *validating* is both 'interpreting' and 'validating'. The reason for these fusions is that it is often hard to separate 'simplifying/structuring the task' from 'mathematizing' and vice versa, and that to some extent 'interpreting' and 'validating' are intertwined. The sub-activity of *estimating* is implicit in Borromeo Ferri's modelling cycle, in my understanding found both as a component of 'simplifying/structuring the task' when constructing a 'real model', and as a component of 'mathematizing' in the transition from a 'real model' towards a 'mathematical model'.

A graphical representation of the problem solving process, analogue to the graphs Schoenfeld (1985b; 1992) describes, using these categories is called a *modelling activity diagram*, and is used as an analytical tool in this study to capture the macroscopic behaviour displayed by the students engaged in the activity of solving Fermi problems[9]. Examples of how such diagrams can look like are shown in Figures 2, 3 and 4 below.

5.1.4. Research question

Using the developed vocabulary from the previous sections it is now possible to rephrase the preliminary research question presented before in more specific terms:

> *What mathematical modelling sub-activities do groups of student display when the engage in solving Realistic Fermi problems?*

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[9] I will refer to this framework as the *MAD framework*.

6. METHOD

Using the characteristics of Realistic Fermi problems, two such tasks were constructed for this pilot study. This paper reports from students' work on one of these problems (the problem the students started with), *The Empire State Building Problem*. In this problem students were asked to come up with an answer for the time it takes to go from the street level to the top observatory floor in the Empire State Building using the elevator and stairs respectively. The very formulation of the problem given to the students was the following:

> There is an information desk on the street level in the Empire State Building. The two most frequently asked questions to the staff are:
> - *How long does the tourist elevator take to the top floor observatory?*
> - *If one instead decides to walk the stairs, how long does this take?*
>
> Your task is to write a short letter answering these questions, including the assumptions on which you base your reasoning, to the staff at the information desk.

An a priori analysis was made of the two questions of the problem to identify what the students reasonably *must* estimate and model to be able to solve these problems. In addition to this scrutinizing of the problems, possible extension and more elaborated features to incorporate in the situation were also identified. As a result of this analysis, the information that students reasonably must use in the case of the Empire State Building problem are the height of the Empire State Building, the speed of the elevator and the speed when walking the stairs. As for more elaborate extensions to include in the model, we have the elevator queuing time and the capacity of the elevator. The time for getting in and out of the elevator might also be considered. In the problem on walking the stairs on the other hand one could start thinking of how to model the endurance and one's fitness.

Seven students volunteered, all from a class enrolled in a university preparatory year taking the upper secondary courses in mathematics taught by the author, and divided themselves into three groups, A, B and C. In group A these were the three male students Axel, Anders, and Axel; group B was constituted by one female, Birgitta, and one male student, Björn; and group C's members were the two male students Christer and Claes. All the names used are pseudonyms. The group constellations are by no means random; in class they normally sit together helping each other out on the problems to be covered in a specific lesson. After a short introduction, dealing with ethical issues of the study and urging the students to do their best and to think aloud, the groups were placed in different rooms equipped with videotape recorders and were set to work. The two problems were distributed one at a time, and the groups worked on each problem for as long as they wanted.

The work of all three groups on the problems captured on the videotapes was transcribed using a modified and simplified version of the TalkBank conversational analysis codes[10] as a guide for the transcription. The students' written short answers were also collected.

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[10] www.talkbank.org. In this paper '(.)' means a short pause, and '((*text*))' is a comment added by the researcher to clarify the context or meaning.

The transcriptions were coded by the categories of the six modelling sub-activities described above. The categorization was done both on the level of *utterances*, here taken to be “stretch of continuous talk by one person, regardless of length and structure" (Linell, 1998, p. 160), and on the level of *dialogues* constituted by a sequence of utterances made by the group members taking turns making utterances. The question asked to, and guiding the categorization of, each utterance and dialogue was ‘What sub-activity is the utterance/dialogue indicating that the student/group is engaged in?’. This process was repeated to refine the coding and test the reliability of the process, and the procedure was validated by looking at the video-recordings as well as the written short answers from the three groups. Examples of the coding are given in next section. The final result of this analysis was graphed in a *modelling activity diagram* for each group, showing the time spent and the moves between the different modelling sub-activities during the work with the problem. It was decided to graph the sub-activates in the modelling activity diagram using time intervals of 15 seconds to make the description as clear as possible.

7. EMPIRICAL RESULTS AND ANALYSIS

When the groups were given the Empire State Building Problem, the students’ first reactions in all three groups involved surprise and frustration of not knowing the data needed to solve the problem:

> “It’s just to estimate everything!” (Christer, group C)
>
> “It’s just to make something up!” (Axel, group A)
>
> “That’s a bad question since we don’t get to know how high it ((*the Empire State Building*)) is!” (Björn, group B)

However, after this initial shock, the groups were very active and spent approximately 30 minutes engaged in solving the two parts of the problem. The work of the groups naturally divides into three main phases. After the initial reading of the problem formulation, these are the two phases dealing with the solving of the two parts of the problem (phase one dealing with the elevator and phase two with the stairs), and a third writing phase where they compose the letter asked for in the task. All groups spent about one minute reading the problem. How the groups distributed their time on the three main phases is summarised in Table 1.

Table 1. Summary of how the groups used their time on the three main phases

Group	Time spent on first part	Time spent on second part	Time spent on writing	Total time spent on task
A	5 minutes	18 minutes	6 minutes	30 minutes
B	9 minutes	10 minutes	11 minutes	31 minutes
C	7.5 minutes	16.5 minutes	7.5 minutes	32.5 minutes

When solving the first part of the *Empire State Building Problem* regarding the use of the elevator, none of the groups elaborated their solution to incorporate the extensions of the problem suggested from the a priori analysis presented above or any other elaborations. All groups simply calculated that time equals the height divided by the average speed, a model taken for granted. However, in dealing with the second part of the problem, they developed quite different ways to approach how to model the endurance of the stair climbers, and also spent time discussing how the stairs actually where constructed (if it was a spiral staircase, if there were landings on each floor, and so on and so fourth). Group A used the same model as in the problem with the elevator (using an estimated average speed), and group C a variant of the same approach estimating the average time needed to walk the stairs of one floor and adding some additional time for resting before walking the next one. Group B developed a more advanced model where the time taken for a given floor depended on how high up in the building the floor is situated. Some of the key estimations done by the groups, and their answers to the problem, can be found in Table 2 below[11]. It could be noticed that the mathematical demands were kept at a very elementary level throughout the problem solving activity in all groups.

Table 2. Some of the groups' estimated quantities and their answers to the problems (see the Appendix)

Group	Estimated height (m)	Estimated elevator speed (m/s)	Answer, time for using the elevator	Answer, time for using the stairs
A	300	5	60 s	40 min
B	175	3,6	48,6 s	16 min 15 s
C	350	2	2 min 55 s	1 h 55 min

The *modelling activity diagram* for the solving process for group A, B and C is shown in Figure 2, 3 and 4, respectively. On the vertical axis the modelling sub-activities ***R****eading,* ***M****aking model,* ***E****stimating,* ***V****alidating,* ***C****alculating* and ***W****riting* are displayed. Time in minutes, starting from the point when the students got the problem formulation and to the time they handed in the written solution (the letter), is displayed on the horizontal axis.

7.1. Group work

7.1.1. Group A

The work of group A is driven forward by Alfred and Axel, who talk approximately twice as much as Anders[12], and it is their initiatives and ideas that form the strategies the group chooses to pursue. On the few occasions where Anders comes with a suggestion or a comment outside the black box in which Alfred and Axel are working for the moment, his ideas are given very little attention and eventually fade away. Alfred and Axel are both high achieving students in class, while Anders is just above average. The group is focused in the sense that all three students seem to be engaged in the same sub-activity most of the time. During the session a legible difference in attitude toward the problem and what the solution should look like and contain becomes apparent between Alfred and Axel. Axel takes the task to get as good estimates as possible very seriously. He is the one that initiates the group in the sub-activity of *validating* most of the times and makes it his priority to make the group's solution as realistic as possible. Alfred, on the other hand, is not interested in getting good estimates of the needed quantities, but rather more focused on the principle arguments behind how to achieve the answer. In Alfred's opinion, if the actual height of the Empire State Building is 200 meters or 300 meters is not important; what matters is how to use the estimated height to come up with an answer. This difference occasionally creates tensions in the group, but Axel is the most persistent one, and usually persuades Alfred (and Anders) to reconsider their sometimes naïve assumptions and estimates.

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[11] From http://www.esbnyc.com/ one can read that the 86th Floor Observatory is situated 320 meters above street level and that for the 102th Floor Tower the figure is 373 meters. There are in all 73 elevators in the building operating at speeds between 3 m/s and 5 m/s, and that it is possible to ride from the lobby to the 86th floor in less than a minute (20090403 one can find several video-clips on http://www.youtube.com documenting this ride). The number of stairs are 1576 and each step is approximately 19 cm in height. Normally, walking the stairs is not permitted, but there is an annual stair climbing race, "the Empire State Building Annual Run Up", and the record for running the stairs from the lobby to the 86th floor is 9 minutes and 37 seconds

[12] The number of utterances coded uttered by Anders is 113, by Axel 209, and by Alex 272.

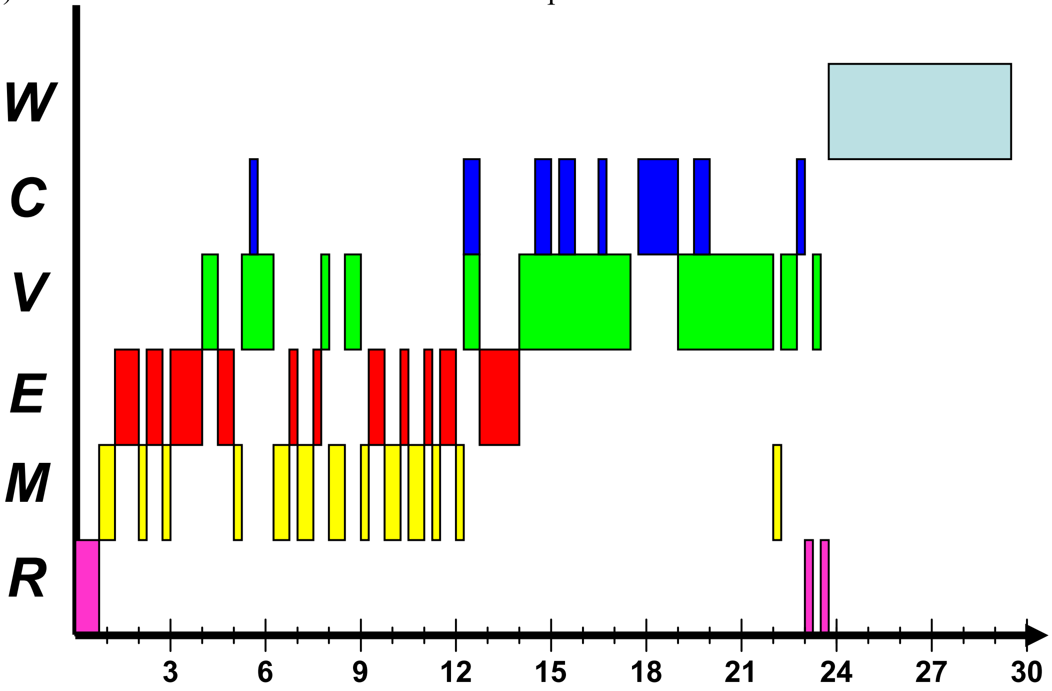

Figure 2. The modelling activity diagram for the Empire State Building problem (group A)

Figure 2 displays the modelling activity diagram of the work of group A on the Empire State Building problem. After a short initial *reading* phase, about 5 minutes are spent on the first part of the problem consisting of the dialectic interplay between *making a model* and *estimating* (with some minor elements of *validating* and *calculating*). When the group engages in solving the second part of the problem they first continue in this dialectic manner for another 8 minutes, followed by approximately 10 minutes composed of *validating* with a parallel element of *calculating*. The problem solving session ends with a 7 minutes phase of *writing*. In the writing phase all three members of the group are involved in dictating the letter with Alfred functioning as the secretary. The letter can be found in the appendix.

Alfred, Anders and Axel use the same strategy to solve both parts of the problem, namely to calculate the time using an estimated height and an average speed of the elevator. However, in the second part, they first use this model to get time 33 minutes for climbing the stairs, but adjust this calculation to 40 minutes since their model of how the stairs look includes landings on every floor.

7.1.2. Group B

Birgitta and Björn are two of the two most ambitious students in the class and they are used to be able to solve most of the textbook problems they encounter in class without too much effort. During this non-standard problem solving session they have difficulty in deciding whether or not an estimate is good enough and utterances expressing this insecureness are common (this is exemplified in the two dialogue extracts from the group found later in this paper). At one point in the beginning of their problem solving session, when I entered the room to check on the recording device, Birgitta and Björn quite obstinately wanted me to confirm whether their estimation of the height of the Empire State Building was correct, but naturally I neither confirmed nor rejected their estimate.

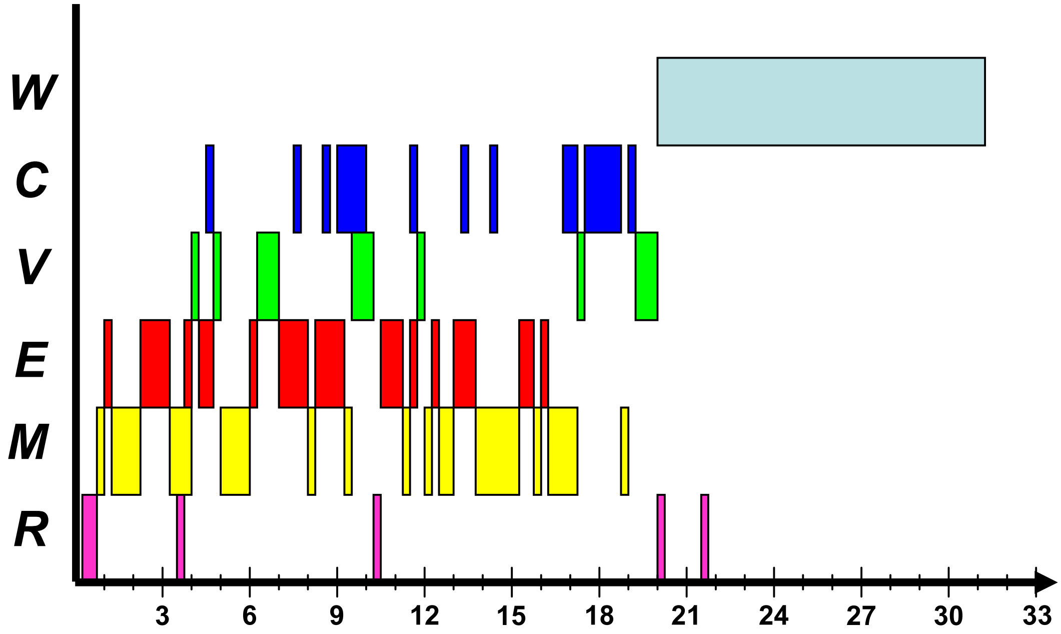

Figure 3. The modelling activity diagram for the Empire State Building problem (group B)

Group B starts with a short initial phase of *reading* (see figure 3) before Britta and Björn spend approximately 9 minutes on the first part of the problem, partly struggling to understand and make sense of what to do and how to do it. About the same amount of time, 10 minutes, is then devoted to the second part of the problem, followed by 11 minutes of *writing*.

The model activity diagram for group B looks a bit different from group A. Their problem solving process is not as concord as the process group A displayed, and Björn and Britta's modelling activity diagram appears to sprawl more than the diagram of group A. After about the first three minutes the problem solving process displayed in figure 3 jumps between *making a model*, *estimating*, *validating* and *calculating* (occasionally returning to the problem formulation for some re-*reading*) in a seemingly random but uniformed distributed manner until the phase of *writing* begins.

To answer the second part of the problem, group B develop a model of how physical tiring it is to climb the stairs in a high building. In their model they group their estimated number of floors in Empire State Building, 50, into groups of 5. They then estimate that it will take 15 seconds to walk from floor 1 to floor 5, and their model predicts that the next 5 floors will take the same amount of time that climbing the last 5 floors did plus additional as many seconds longer as the number of floors you already climbed. Hence, to climb from floor 6 to floor 10 will take 15 + 5 = 20 seconds, from floor 11 to floor 15 will take 20 + 10 = 30 seconds, and so on (see the Appendix, group B's letter for details). As a consequence of this model, the time it takes to climb the last 5 floors, from floor 46 to floor 50, is 240 seconds, meaning that climbing one floor in average takes almost a minute (58 s.). However, this is something neither noted or reflected on by Björn and Birgitta.

Of the three groups, group B is the one that spent the longest time on *writing*. One reason for this is that Birgitta starts writing the letter but has trouble in formulating their reasoning in the second part of the problem so Björn has to take over. In addition, their solution to the second problem is the most sophisticated and complicated of the groups, which contribute to the larger amount of time needed to go through and account for all the details in the writing process. Bigitta's and Björn's letter can be found in the appendix.

During the problem solving process Birgitta's beliefs about what constitutes an answer to a (school) mathematics problem surface as the excerpt below illustrates.

Birgitta: But if you think about it (.) we are supposed to give them an answer
Björn: Mm
Birgitta: But since we don't have the values we can't. We have to do like you said before ((*to give and estimate*))
Björn: Mm
Birgitta: We have to sort of (.) but how do we do that without having any values?
Björn: Mm (.) we can estimate
Birgitta: Yes but then it'll be (.) estimations
Björn: Mm
Birgitta: So that's not an answer
Björn: Mm
Birgitta: Is it (.) or?
Björn: Hm ((*indicating that perhaps it is*))

Another aspect of the same belief is the fact that they give an answer to the first part of the problem with a very high accuracy, 48.6 s, despite the rough estimates done for the calculation producing the answer.

7.1.3 Group C

Christer and Claes in group C are two lively, open, very talkative and mischievous students, which is reflected in their work on the problem. Being full of fun they now and then take more or less serious and reasonable assumption under consideration for inclusion in their solution. An example is when they work on how long it would take to climb the stairs and discuss the need for some extra time for stops and argue with the accompanying mother-in-law every now and then. Concerning their abilities in mathematics, Claes is an average student and Christer just below average. During the problem solving activity Christer shows a focus on getting an answer, whereas Claes wants to try different approaches and see if the answers these provide agree. Claes is the enthusiastic one of the two who brings many of the ideas to the table, while Christer is the one questioning their work initiating validation processes. Among the three groups, group C seems to be the one enjoying the problem solving activity the most.

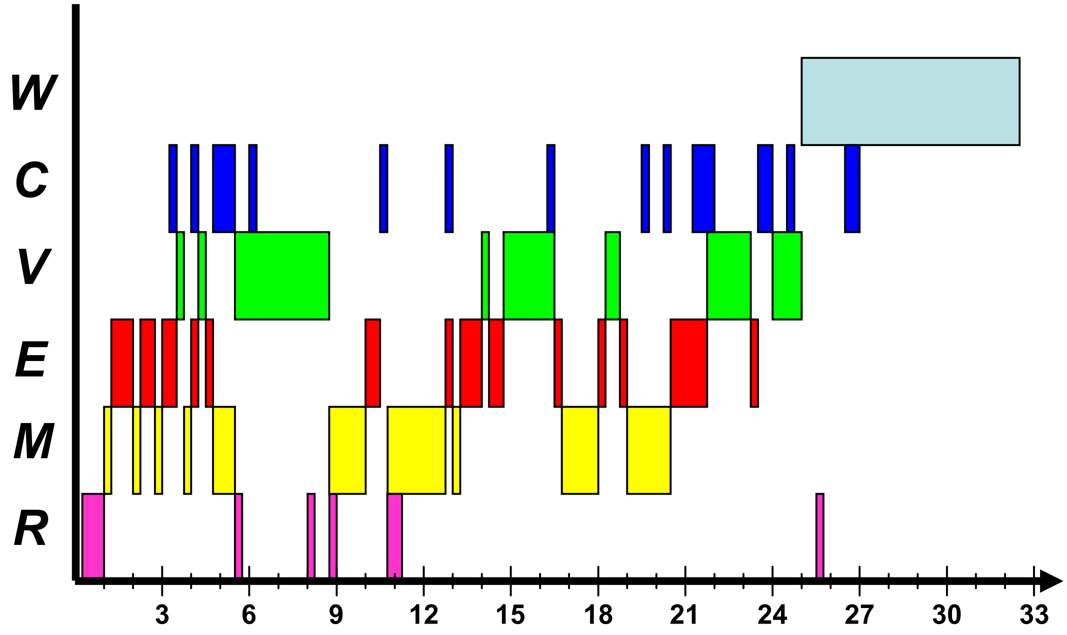

Figure 4. The modelling activity diagram for the Empire State Building problem (group C)

In Figure 4 the modelling activity diagram of group C starts with a short initial *reading* phase, followed by about 7.5 minutes spent on the first part of the problem and then approximately 16.5 minutes on the second part. The *writing* phase that ends the problem solving session is about 7.5 minutes. The modelling activity diagram of group C bears resemblance to the diagrams of both the other groups; the sub-activities Christer and Claes engage in when they struggle with the first part of the problem are similar to the one displayed by group A. However, when solving the second part of the problem, there are more similarities with the way group B engage in the sub-activities. It is Christer who under silence does all the *writing* while Claes devotes himself to drawing pictures of skyscrapers; the letter can be found in the appendix.

7.2. Examples of the coded categories

The following section will provide examples of, and comments on, how the data was coded using the categories of the MAD framework.

7.2.1. Making model

A typical segment of the group work, which was categorized as *making model* deals with negotiating and agreeing on how to structure the problem and which assumptions or idealizations to make. This is briefly illustrated in the following excerpt which takes place around 3 minutes into the discussion of group A. The group just estimated the height of the Empire State Building to be 200 meters and now turned to properties of the elevator:

Alfred: But in the case of a tourist elevator (.) shall we then assume that it

just goes from the ground to the absolute top ((*floor*))?
Anders: Yes
Axel: Without stopping
Anders: Yes

Another example of a small dialogue coded as *making model* is the following when Christer and Claes decide on with method to use to calculate an estimate for the time it takes to use the stairs. They have just spent quite some time discussing how to take into account that it is physically tiring to walk the stairs in terms of resting a given amount of time after a given interval of floors. So far, their idea has been to let the amount of time spent resting depend on how high up in the building you are, but they had not come to an agreement on the details:

Christer: Ok, shall we try to use some sort of average in this case to (.) 'case it's not interesting to calculate (.) I mean it's (.) 'case it'll (.) it'll (.) the velocity will decrease the higher up one gets
Claes: Mm, but (.) but
Christer: Let's settle for using an average all the time
Claes: ((*nodding*)) Then, then we can (.) Put it like this, we can make an estimate (.) let's say that you start to rest on every 10th floor and rest for a minute. Otherwise, if you're a fatso you'll never make it

7.2.2. Estimating

Estimation segments are often initiated with a direct question as for example "Yes, but how high is the poor building?", "How fast can an elevator run?" or "How many floors are there?". The discussions that follow such a question are all aimed to produce an estimate of some quantity, a number:

Alfred: Shall we estimate a speed by which an elevator can travel upwards? A typical elevator in a building of a regular height of say 200 meters (.) going all the way up (.) in meters per seconds
Axel: Yes, say that (.) a typical elevator might travel by the speed of 3-4, 3 meters per second
Alfred: Shall we say 5? It's a quite fast elevator
Axel: Mm, yes, let's settle for 5 ((*m/s*)).
Alfred: 5 ((*m/s*))?
Anders: Hm ((*nodding his head*))

The following excerpt comes from when Björn and Birgitta try to get an estimate of the height of the Empire State Building. They have just come to terms with the fact that they have to make some estimates to be able to solve the problem. This example also to some extent illustrates the insecureness that characterizes Birgitta and Björn's way of working:

Birgitta: Ok, sure. Let's start making up some values then
Björn: Mm
Birgitta: Mm (.) eh (.) Empire State (.) State Building (.) well, how high was the World Trade Centre for example?
Björn: How high was what?

Birgitta: World Trade Centre
Björn: What? ((*inaudible*))
Birgitta: I stink at this (.) estimating stuff
Björn: Yeah, me too ((*inaudible*))
Birgitta: ((*laughter*)) But
Björn: But (.) I don't know (.) it must (.) take 50 floors (.) it can't be more than say (.) sort of 100 meter say (.) more than 300 meter sort of (.) no (.) 150 ((*inaudible*)) (.) Say that a floor is like 3 meters
Birgitta: Yes
Björn: Or if you think about (.) they might be higher that that ((*inaudible*))

Before Björn and Birgitta use the estimated number of floors and how high each floor is to calculate the height of the building in a validation process of the suggested height of 150 meters by Björn above, the dialogue when Birgitta expresses her beliefs about what should constitute a (mathematical) answer to the problem takes place (see the section on group B above). Eventually, they agree to estimate the height of the Empire State Building to 175 meters.

7.2.3. Validating

Questions also often start up segments of *validation* ("Shall we really say 100 floors?"), as do statements of doubt ("It feels like that's too fast (.) for that building"). In these segments previous assumptions, estimates, calculations and results are looked at critically and are either made manifest or rejected in favour of better versions, as in the following example where the calculated elevator time evokes questioning of the previous estimated height of the *Empire State Building* (of 200 meters):

Axel: It feels like that's too fast (.) for that building
Anders: Hm (.) it does
Axel: I think it is (.) and now I'm stretching it (.) I think it is 300 ((*meters*)) (.) I think it is 280 meters, that I was off with a hundred ((*meters*)) before
Alfred: We can say that it ((*the Empire State Building*)) is 200 meters, that's fine
Axel: Yes ((*10 seconds of quiet mumbling, Axel is working on his calculator*)) 60 seconds, one minute to go up there. Is that reasonable?
Anders: Yes
Alfred: If we assume, how high
Axel: Yes, 'cause that's almost probable
Alfred: Yes, 300 meters (.) 300 meters
Axel: One minute then. One minute can be really long
Alfred: Yes, especially in an elevator

Although Axel's second statement expresses a re-estimation of the height of the Empire State Building, and indeed that he afterwards is doing some calculations, this excerpt is coded as the group being engaged in the activity of *validating*. This extract exemplifies the type of considerations that must be dealt with when using the codes of the MAD framework; it codes the activity on the group level and not on the individual level. Nevertheless, Axels' contribution in this excerpt (estimating) has a big influence on the groups' final result.

7.2.4. Calculating, Reading and Writing

Calculating is an activity normally preformed by one of the group members in the background of some other activity, so here the video recording is crucial for the coding. Occasionally the whole group focus on the actual calculation, but regardless of how it is obtained and by whom, the result of a calculation is important for how the solving process evolves.

The sub-activities *reading* and *writing* are rather self-explaining, but it is notable that when the group came to writing down their answers (in the form of a letter) they just reproduced and retold what they said before without any reflections or critical scrutiny.

7.3. The use of extra-mathematical knowledge

It could also be observed in the data that the students frequently used their personal extra-mathematical knowledge and experiences from outside schools in the solving process. It seems that they did this in at least three different ways: in a *creative* way to construct a model or to make an estimate, in the process of *validating* a result or an estimate, and finally in a *social* way as a narrative anecdote.

In the following excerpt Axel shares a personal experience from an amusement park in a creative way in hope to easier get an estimate of the elevator speed. However, in this specific case his reasoning makes him question the estimated height of the Empire State Building (200 m) which the group had agreed on two minutes earlier. Hence he is also using this piece of extra-mathematical knowledge to (involuntary) initiate a validating process:

Axel: Hm (.) Did anyone ride the FREE FALL[13]
Anders: ((*in unison with Alfred*)) Nope.
Axel: I was thinking that since (.) hm, it is 90 meters ((*high*)), how long does it take to get up there? (.) I think it takes 15, 20-25 seconds (.) and that's 90 meters.
Alfred: Yes.
Axel: It ((*the Empire State Building*)) must be higher than 200 meters.

As it turned out, all seven participants had a common friend living at the top apartment in a four storage apartment building. All groups used their knowledge and experiences about this apartment whereabouts as point of departure for modelling and estimating the elevator speed, the height and number of the steps of the stairs, time for walking the stairs. Other extra-mathematical knowledge invoked included experience from working as a postman, visiting a high tower in Malaysia, the whereabouts of other friends'/relatives' apartments, different elevator-experiences, climbing up in a radio tower on a Jacobs' ladder, and mounting climbing, just to mention a few.

7.4. Similarities and differences between the groups

From the constructed modelling activity diagrams (figures 2, 3 and 4) one can observe that the students engage in all of the predefined different sub-activities, that they do spend a considerable amount of time in each sub-activity, and that they go back and forth between the different types of activities numerous times. In other words, the processes involved in the mathematical modelling cycle pictured in Figure 1 are richly represented in the groups' problem solving processes.

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[13] *FREE FALL* is an attraction in the amusement park *Gröna Lund* in Stockholm, Sweden.

Looking at the problem solving process of the three groups one can observe both similarities and differences. To get an overview of the amount of time categorized as spent on the different sub-activities for each group and phase of the problem solving process Table 3 was complied. From this table it is clear the group A spent by far the least time engaged in *Making models* compared to the other two groups. One reason for this might be that group A quickly decided to base their work on the model that time equals height divided by average speed, and did not have to engage in discussing more advanced models. Group B had initial problems in coming to terms with how to attack the problem, which explains why they devote twice as much time making model during the first part of the problem than the other groups. In the second part of the problem, on the other hand, group C spend the most time *Making models* since they started to think in the same line as group B, but gave up this idea and ended up in just a slightly different model than the one adapted by group A.

Table 3. Summary of the amount of time categorized as the different sub-activities for each group and part of the problem solving process, respectively

	First part			Second part			Writing			Total		
	A	B	C	A	B	C	A	B	C	A	B	C
R	0:45	0:45	1:15	0:30	0:15	0:45	0:00	0:30	0:15	1:15	1:30	2:15
M	1:15	3:30	1:45	3:30	4:00	6:15	0:00	0:00	0:00	4:45	7:30	8:00
E	2:45	4:15	2:15	3:15	2:30	4:15	0:00	0:00	0:00	6:00	6:45	6:30
V	1:30	2:00	3:45	8:30	1:15	5:00	0:00	0:00	0:00	10:00	3:15	8:45
C	0:15	1:45	1:30	3:45	2:45	2:45	0:00	0:00	0:30	4:00	4:30	4:45
W	0:00	0:00	0:00	0:00	0:00	0:00	5:45	11:15	7:30	5:45	11:00	7:30
										31:45	34:30	37:45

On the whole the groups spent approximately the same amount of time *Estimating* and *Calculating*. However, group B is the only group who spend more time *Estimating* on the first part of the problem relative to the second part. One explanation of this behaviour might be the insecureness expressed by the group, due to not being able to relate the Empire State Problem to the types of problem they normally encounter in their mathematics classrooms. This insecureness makes the group seriously doubt their capabilities and even hard to engage in making assumptions and estimates at all. When solving the second part, on the other hand, they more or less focus on the procedure of how to come up with an answer and *Estimating* becomes secondary.

When it comes to *Validating*, group B engage by far the least in this sub-activity of the three groups. This is due to the little time group B spent validating on the second part of the problem, and the group seems to be happy when they have calculated an answer and do not express any inclination to question their numbers coming from a calculation. To a certain extent this behaviour is not surprising knowing some of the beliefs expressed by Birgitta about (school) mathematics.

In *Writing* the letter asked for in the task, the groups used between approximately six to eleven minutes; group A used the least amount of time and group B the most time. Why group B spent so much time writing in comparison to the other two groups has already been discussed (see 7.1.2.), and looking at the letter produced by group A and C (see the Appendix) one can observe that much more effort and detail is put into the letter by group C, giving a possible explanation for group C using more time than group A in writing the letter.

Finally, taking a look at how much time was spent totally on the different sub-activities, one can see that group A spends the least time, group B somewhat more time, and group C the most time. This is in line with the observed behaviours of the groups; group A works in a cohesive way where the members of the group dynamically follow each other if one starts to engage in another sub-activity. This makes group A's work very focused and thus, relative to the other groups, not so many sub-activities are being engaged upon simultaneously. The other two groups exhibit less of this behaviour. In the case of group B their issue of not having a clear way to approach the problem makes their behaviour more searching and ambivalent in trying to cope with the situation. The behaviour of group C, on the other hand, is more or less the opposite to the one found in group A, and is explained by that fact that group C is engaged in multiple sub-activities four times as much as group A.

8. SUMMARY AND DISCUSSION

The modelling activity diagram is a different way to describe students' modelling processes than has been done in many empirical studies. Borromeo Ferri (2007a; 2007b) pictured what she called "individual modelling routes" of her students by drawing arrows in the modelling cycle shown in Figure 1. In the context of picturing the processes engaged in during solving modelling tasks, the modelling activity diagram can be used to visualize a group or an individual student's modelling sub-activities in a more linear way along a timeline. Thus, it provides a simple dynamical picture of the activities involved. From the results presented above, the modelling activity diagram shows that Fermi problems might serve well as a means to introduce mathematical modelling at this school level: All modelling sub-activities are richly represented and contributed in a dialectic progression towards a solution to the task.

It is clear from the three modelling activity diagrams produced, that the view presented on modelling (see Figure 1) as a cyclic process is highly idealised, artificial and simplified. This way of conceiving mathematical modelling was useful for the developing of the MAD framework, but real authentic modelling processes are better described as haphazard jumps between different stages and activities, as is also noted by Haines and Crouch (in press).

Borromeo Ferri (2006; 2007a; 2007b) also notes that students use extra-mathematical knowledge in the modelling process when deriving the mathematical model and validating results. In the latter case, she differentiates between "intuitive" and "knowledge-based validation" (2006, p. 93) and notes that students mostly only make what she calls "inner-mathematical validation", and that validating for students means "calculating". However, the data in this study provides numerous examples where personal extra-mathematical knowledge is used by the students in the validation of both models and estimates as well as in the validation of calculations.

One of the reasons for using Realistic Fermi problems was to urge the groups into discussions about the problem setting and how to approach the problem. In my opinion this worked out nicely, but the study also suggests that to some extent such problems take the focus away from the mathematics, which I believe students experience hard to discuss. It also makes the problem available in an indirect way through the discussions about how to structure the problem and what (and how) to estimate. In this respect the realistic feature of the problem is crucial. Although the mathematics was kept at a very elementary level, one could have tried to deepen it by explicitly asking for, say, an equation relating height and time spent in the elevator or in the stairs.

Looking at the three letters produced by the groups (see the Appendix), and taking into account how much time the they spent on composing them, it is astonishing how little information they contain about the groups' activities during the 30-minutes long problem solving session. This brings to the fore the issue of how to assess modelling work and the development of modelling competencies, an active area of research where some results and methods are emerging, but which in my opinion still needs to be further researched.

In the data material one can note that the group dynamics are essential for the evolution of and activation of the different sub-activities during the problem solving process. It is the discussions and interactions in the groups, when different beliefs and opinions are confronted, that drive and shape the modelling process. Group behaviour is strongly influenced by individual preferences and group composition, making it one of the most important task variables to consider. Group B is an example of a constellation which was not optimized, whereas group C displayed a better blend of personalities, in the sense that members of group C complemented each other, bringing different attitudes and perspectives to the collaboration. The members of group B on the other hand, were in a sense too alike, with the result that the group got stuck within their expectations and way of thinking. Indeed, this social dimension of the problem solving process is something I feel the framework and methodological choices need to take into account more seriously in the future.

Schoenfeld reminds us that "any framework for gathering and analyzing verbal data will illuminate certain aspects of cognitive processes and obscure others" (1985a, p. 174). In line with Schoenfeld (1985a), I wanted to minimize my *degree of intervention* once the groups were set to work, and after giving the students the initial instructions I only briefly visited them to check on the recording devices. On some occasion I got a question from the students (see the section on group B above), but otherwise the *nature of instruction and intervention* was limited to the initial instruction and later the problem formulation, in order not to affect and interfere in the problem solving activity more than absolutely necessary. The environment in which the groups where situated was naturally superficial. Although the departmental group rooms where nicely furnished, the students had never been there before, and in addition there was a camera directed at them. They were also asked to 'think aloud' and to work on a non-standard problem. The question now is if the subjects felt so uncomfortable that it induced an atypical and pathological behaviour in the students' problem solving process? I would argue that most circumstances above had little or no effect on the students' behaviour. For one thing, they had all volunteered to participate, and more over, they all knew me very well, infusing some sort of confidence in them. In addition, working in groups of two or three students quickly seemed to make them forget the camera and the direct surroundings. However, all these *task variables* are hard to keep track of and to try to realise their consequences for the displayed behaviour.

For the validity of the study using adult students in a special university preparatory course may be questioned. However, the mathematical pre-knowledge was the same as for adolescent students in school taking the same courses, and the increased extra-mathematical knowledge that came into play when working on the problems studied were not of a kind that could not have been experienced also by the latter group. That the participants chose the groups themselves can be considered to be an advantage for this pilot study, facilitating openness in the discussions. An alternative method of using a grounded theory approach could also have been applied to the data. However, the results of a study can only be interpreted within the research framework chosen, which in this case was linked to the mathematical modelling perspective where the pre-defined categories used make it easier to relate to and locate the results in previous research.

9. CONCLUSIONS AND FUTURE RESEARCH

Returning to the research question, one can conclude that all the modelling sub-activities proposed by the framework (*reading, making model, estimating, calculating, validating* and *writing*) are richly and dynamically represented when the students get engaged in solving Realistic Fermi problems. Thus this study shows that small group work on Realistic Fermi problems may provide a good and potentially fruitful opportunity to introduce mathematical modelling at upper secondary school level.

This research may be continued in a number of ways. First, the tool *modelling activity diagram* as an instrument of analysis has a potential to be developed further in different directions, depending on what it will be used for. One idea is to incorporate the group dynamics into the diagram by indicating in a sub-activity segment how much each group member contributes to the discussion. One could also try to modify the framework to be more general; *reading* in this study is just reading, but in a more general setting *reading* could stand for the gathering of any external information. In some situations it may be needed to try to split the sub-activity *making model* into the two sub-activities *structuring* and *mathematizing*.

Second, in a setting of teaching mathematical modelling, a joint follow-up session with the three groups in close connection to the problem solving session could serve well as a meta-cognitive activity, discussing the problem and their way of solving it, in order to highlight the processes involved in mathematical modelling. In future research the outcome of such an intervention might be fruitful to investigate.

Third, the data also pose interesting questions about how the students validate their results, models and estimates – why do they choose to do this the way they do? A (calculated) result depends on the model developed, the estimates done and the performed calculation. So, in validating a result it is desirable that all these three “influences” are looked upon critically (see Figure 5). Since all of these three types of validation are present in the data it would be interesting to investigate this phenomenon in more detail.

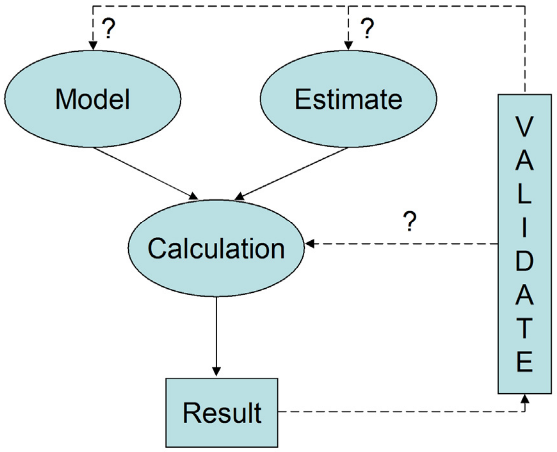
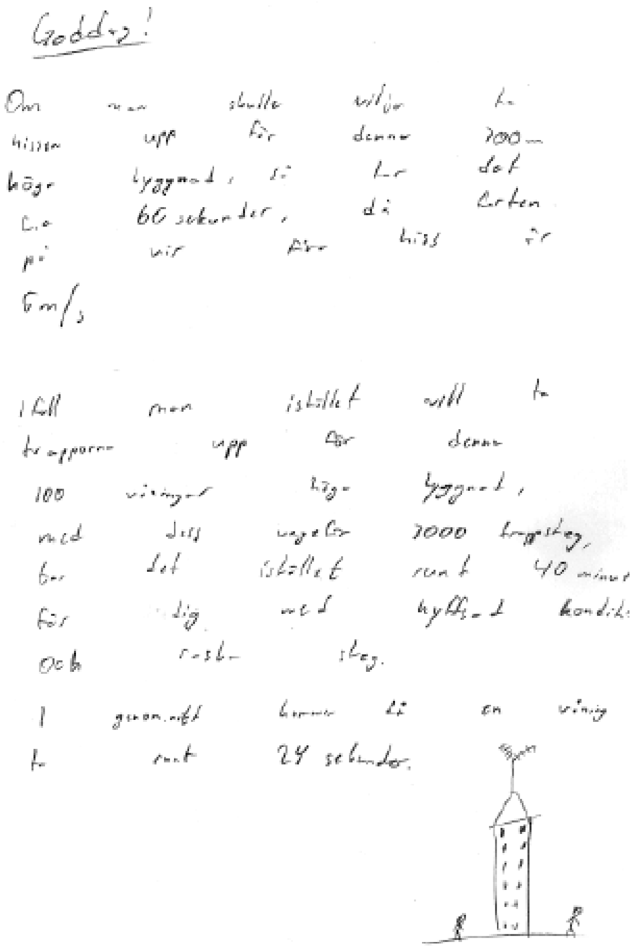


Figure 5. Possible validation routes

One of the most evident results produced by the MAD framework, illustrated in figures 2, 3, and 4 respectively, is the non-cyclic nature of the modelling process. Although the idealised view of mathematical modelling as described in terms of a modelling cycle has been much employed in mathematics education research, the discrepancy with what actually happens when students engage in modelling activities is palpable. My opinion is that this ‘inconsistency’ is something that researchers ought to take more seriously to refine current theories and methods to be able to better validate our research findings. Reference to elaborated epistemological analyses of mathematical modelling and authentic mathematical models in relation to education (e.g. Jablonka, 1996) is needed as well as discussions from the learner’s perspective. Using a competence approach (Blomhøj & Højgaard Jensen, 2007; Maaß, 2006) might provide one alternative to the commonly used modelling cycle, but surely there must be other routes as well to investigate?

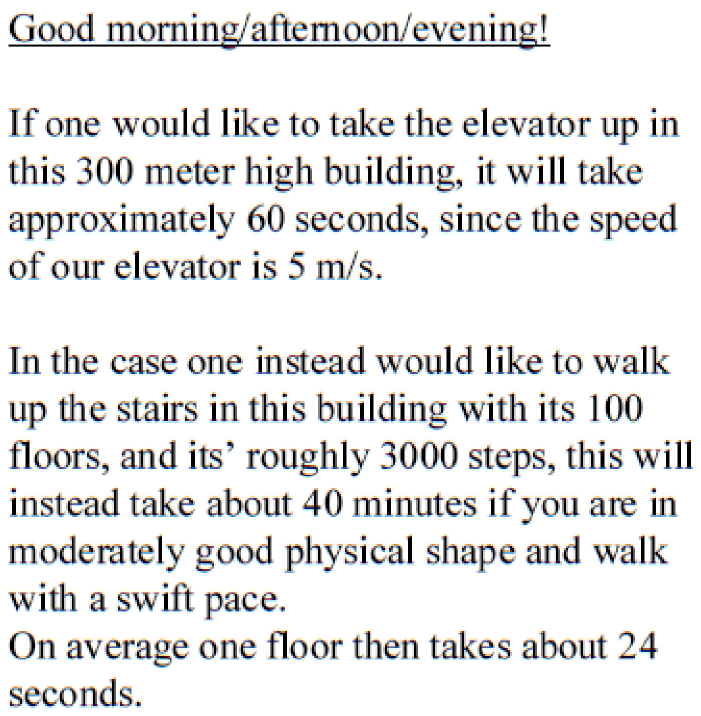
APPENDIX, Letter produced by group A



Goddag!

Om man skulle vilja ta
hissen upp för denna 300m
höga byggnad, så tar det
ca 60 sekunder, då farten
på vår [illegible] hiss är
5m/s

I fall man istället vill ta
trapporna upp för denna
100 våningar höga byggnad,
med dess ungefär 3000 trappsteg,
tar det istället runt 40 minuter
för dig med hyfsad kondition
och raska steg.
I genomsnitt kommer då en våning
ta runt 24 sekunder.

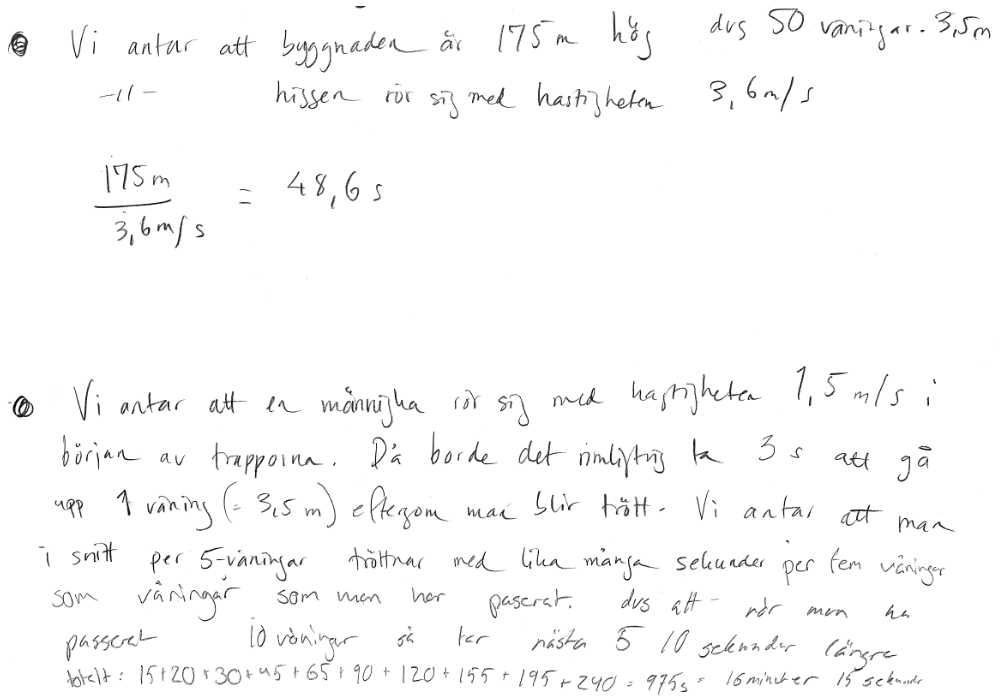


Good morning/afternoon/evening!

If one would like to take the elevator up in this 300 meter high building, it will take approximately 60 seconds, since the speed of our elevator is 5 m/s.

In the case one instead would like to walk up the stairs in this building with its 100 floors, and its' roughly 3000 steps, this will instead take about 40 minutes if you are in moderately good physical shape and walk with a swift pace.
On average one floor then takes about 24 seconds.

APPENDIX, Letter produced by group B



- Vi antar att byggnaden är 175 m hög dvs 50 våningar · 3,5 m
  -11- hissen rör sig med hastigheten 3,6 m/s

  $$\frac{175\text{m}}{3{,}6\text{m/s}} = 48{,}6\text{ s}$$

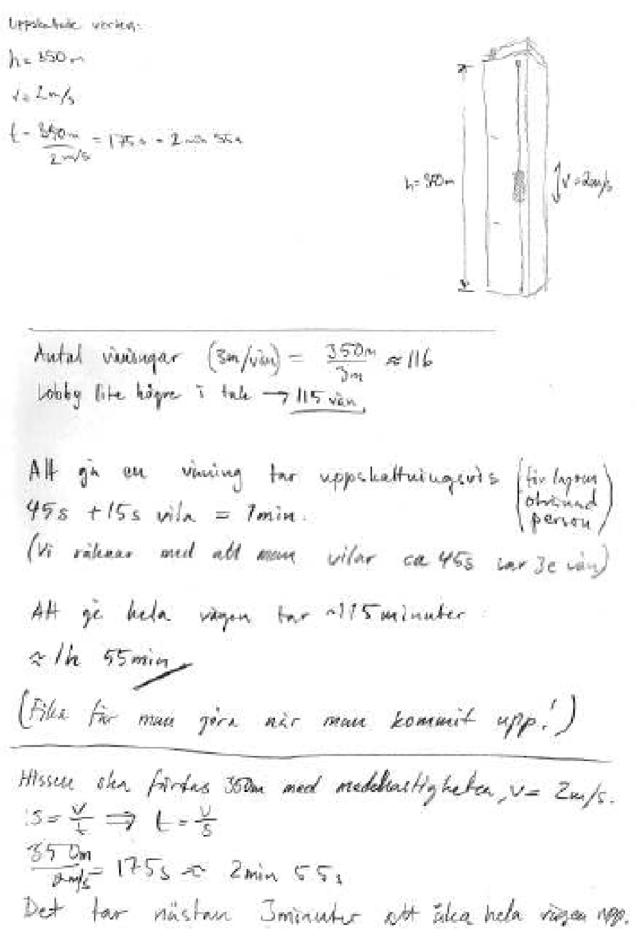
- Vi antar att en männigka rör sig med hastigheten 1,5 m/s i början av trapporna. Då borde det rimligtvis ta 3 s att gå upp 1 våning (= 3,5 m) eftersom man blir trött. Vi antar att man i snitt per 5-våningar tröttnar med lika många sekunder per fem våningar som våningar som man har passerat. dvs att när man ha passerat 10 våningar så tar nästa 5 10 sekunder längre
  totalt: 15+20+30+45+65+90+120+155+195+240 = 975s = 16 minuter 15 sekunder

- We assume that the building is 175 meters high, that is 50 floors x 3.5 m.
  We assume that the elevator travels with a speed of 3.6 m/s

  $$\frac{175\text{ m}}{3.6\text{ m/s}} = 48.6\text{ s}$$

- We assume that a person moves with the speed of 1.5 m/s in the beginning of the staircase. Then, it should reasonably take 3 s to walk one floor (= 3.5 m) as you become tired. Vi assume that you in average per every 5 floors get tired with as many seconds per every 5 floors as floors you already walked, that is when you walked 10 floors the next 5 floors will take 10 more seconds. Totally: 15+20+45+65+90+120+155+195+240=975 s = 16 minutes 15 seconds

Appendix, Letter produced by group C



Uppskattade värden:

$h = 350 m$

$v = 2 m/s$

$t = \frac{350 m}{2 m/s} = 175 s = 2 min 55 s$

Antal våningar (3m/vån) = $\frac{350m}{3m} \approx 116$

Lobby lite högre i tak → 115 vån.

Att gå en våning tar uppskattningsvis (för lagom otränad person)

45s + 15s vila = 1min.

(Vi räknar med att man vilar ca 45s var 3e vån)

Att gå hela vägen tar ~115 minuter

≈ 1h 55min

(Fika får man göra när man kommit upp!)

Hissen ska färdas 350m med medelhastigheten, v = 2m/s.

$s = \frac{v}{t} \Rightarrow t = \frac{v}{s}$

$\frac{350m}{2m/s} = 175s \approx 2min\ 55s$

Det tar nästan 3minuter att åka hela vägen upp.

Estimated values:
h = 350 m
v = 2 m/s
t = = 175 s = 2 min. 55 s.

Number of floors (3m/floor) $= \frac{350m}{3m} \approx 116$

The ceiling is a bite higher in the lobby →115 floors

To climb one floor takes approximately (if you are a moderate unfit person)
45s + 15 s rest = 1 minute
(Vi assume that one rests for roughly 45s on every third floor)

To walk the whole way takes ~115 minutes ≈1h55min.
(Have some coffee can you get when you arrive at the top!)

The elevator travels 350 m with an average speed of v=2m/s.

$$v = \frac{s}{t} \Rightarrow t = \frac{s}{v}$$

$$\frac{350m}{2m/s} \approx 175s \approx 2\,min\,55s$$

It will take almost 3 minutes to ride all the way up.

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Mathematical Beauty and its Characteristics
- A Study on the Students' Points of View

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Abstract: Based on the statement, that the experience of mathematical beauty has a positive influence on students' motivations and attitudes towards mathematics and its study, the focus of this paper is the aesthetic component of mathematics. First, the role of aesthetics for perception and education is addressed. The appreciation of the beauty of mathematics is one of the wellsprings of this subject, not only in research but also in school education. This should have implications for the teaching of mathematics. However the beauty making elements have not been very well analysed. In particular, it is not clear to what extent the criteria for aesthetics found in literature are in agreement with emotions of students. A study on this topic is presented below. It involves students out of grades 5 to 12, as well as university mathematics teacher students, and reveals similarities and differences between the views of students of different educational levels.

Keywords: Aesthetics; Affect; Attitudes; Beliefs; Emotions; Mathematical beauty;

1. THE ROLE OF AESTHETICS FOR PERCEPTION AND EDUCATION

In the literature there are many reports concerning the use of aesthetics as a guide when formulating a scientific theory, or selecting ideas for mathematical proofs (Brinkmann & Sriraman, 2009).

The first who introduced mathematical beauty as well as simplicity as criteria for a physical theory was Copernicus (Chandrasekhar 1973, p. 30). Since then, these criteria have continued to play an extremely important role in developing scientific theories (Chandrasekhar 1973, p. 30; 1979; 1987). This is especially so for truly, creative work that seems to be guided by aesthetic feeling rather than by any explicit intellectual process (Ghiselin 1952, p. 20). Dirac, for example, tells about Schrödinger and himself (Dirac 1977, p. 136):

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> It was a sort of act of faith with us that any questions which describe fundamental laws of nature must have great mathematical beauty in them. It was a very profitable religion to hold and can be considered as the basis of much of our success.

Van der Waerden (1953) reports that Poincaré and Hadamard pointed out the role of aesthetic feeling when choosing fruitful combinations in a mathematical solution process. More precisely, Poincaré asked how the unconscious should find out the right, that is fruitful, combinations among the many possible ones. He gave the answer: "by the sense of beauty, we prefer those combinations that we like" (Van der Waerden 1953, p. 129; see also Poincaré 1956, p. 2047-2048).

A similar statement is given by Hermann Weyl (Ebeling, Freund and Schweitzer 1998, p. 209):

> My work has always tried to unit the true with the beautiful and when I had to choose one or the other I usually chose the beautiful.

Thus theories, that have been described as extremely beautiful, as for example the general theory of relativity, have been compared to a work of art (Chandrasekhar 1987); Feyerabend (1984) even considers science as being a certain form of art.

Mathematics and mathematical thought are obviously directed towards beauty as one profound characteristic. Papert and Poincaré (Dreyfus and Eisenberg 1986, p. 2; Hofstadter 1979) even believe that aesthetics play the most central role in the process of mathematical thinking. The *appreciation of mathematical beauty by students* should thus be an integral component of mathematical education (Dreyfus and Eisenberg 1986). But Dreyfus and Eisenberg remarked in 1986, that developing an aesthetic appreciation for mathematics was not a major goal of school curricula (NCTM, 1980), and they suggested that "this is a tremendous mistake". However in the curricular guidelines of Northrhine-Westfalia, Germany (MSWWF 1999, p. 38), the development of students' appreciation of mathematical beauty is explicitly demanded in the context of the *fostering of long-life positive mathematical views*. The importance of this demand may be stressed by the following statement given by Davis and Hersh (Davis and Hersh 1981, p. 169):

> Blindness to the aesthetic element in mathematics is widespread and can account for a feeling that mathematics is dry as dust, as exciting as a telephone book, as remote as the laws of infangthief of fifteenth century Scotland. Contrariwise, appreciation of this element makes the subject live in a wonderful manner and burn as no other creation of the human mind seems to do.

In addition to the positive influence on students' attitudes towards mathematics, the experience of mathematical beauty would surely have as well *a positive influence on students' motivations for the study of mathematics*. Of course, this statement can only be confirmed on the basis of a classroom teaching that emphasizes students' aesthetic feelings.

2. CRITERIA OF AESTHETICS

If we want students to experience mathematical beauty, we first have to bring out the characteristics of mathematical aesthetics. What does it mean, for example, that a theorem, a proof, a problem, a solution of a problem (the process leading up to a solution, as well as the finished solution), a geometric figure, or a geometric construction is beautiful?

Although assessments about beauty are very personal, there is a far-reaching agreement among scholars as to what arguments are beautiful (Dirac 1977). Thus it makes a sense to search for factors contributing to aesthetic appeal. Before starting on this journey, Hofstadter (1979, p. 555) sounds a note of warning when suggesting, that it is impossible to define the aesthetics of a mathematical argument or structure in an inclusive or exclusive way:

> There exists no set of rules which delineates what it is that makes a peace beautiful, nor could there ever exist such a set of rules.

However we can find in the literature several indications of criteria determining the aesthetic rating.

The Pythagoreans took the view that beauty grows out of the mathematical *structure*, found in the mathematical *relationships* that bring together what are initially quite independent parts in such a way to form a unitary whole (Heisenberg 1985). Chandrasekhar (1979) names as aesthetic criteria for theories their display of "*a proper conformity of the parts to one another and to the whole*" while still showing "some *strangeness* in their proportion". Weyl (1952, p. 11) states that beauty is closely connected with *symmetry,* and Stewart (1998, p. 91) points out that *imperfect symmetry* is often even more beautiful than exact mathematical symmetry, as our mind loves surprise. Davis and Hersh (1981, p. 172) take the view that:

> A sense of strong personal aesthetic delight derives from the phenomenon that can be termed *order out of chaos*.

And they add:

> To some extent the whole object of mathematics is to create order where previously chaos seemed to reign, to extract structure and invariance from the midst of disarray and turmoil.

Whitcombe (1988) lists as aesthetic elements a number of vague concepts as: *structure, form, relations, visualisation, economy, simplicity, elegance, order*. Dreyfus and Eisenberg (1986) state, according to a study they carried out, that *simplicity, conciseness* and *clarity* of an argument are the principle factors that contribute to the aesthetic value of mathematical thought. Further relevant aspects they name are: *structure, power, cleverness* and *surprise*. Cuoco, Goldenberg and Mark (1995, p. 183) take the view that:

> The beauty of mathematics lies largely in the *interrelatedness of its ideas*. ... If students can make these connections, will they also see beauty in mathematics? We think so...

Ebeling, Freund and Schweitzer (1998, p. 230) point out, that the beautiful is as a rule connected with *complexity*; complexity is necessary, even though not sufficient, for aesthetics.

Complexity and simplicity are both named as principal factors for aesthetics: how do these notions fit together? If simplicity is named, it is mainly the simplicity of a solution of a complex problem, the simplicity of a proof to a theorem describing complex relationships, or the simplicity of representations of complex structures. It looks as if simplicity has to be combined in this way with complexity, in order to bring out aesthetic feelings (Brinkmann 2000).

The criteria for aesthetics might give us an idea of how to choose mathematical objects for presentation in classroom, if we want to bring out aesthetic feelings in the students. However, we have to consider that the criteria for aesthetics noted above, have in the main been developed by mathematicians and scientists. Sinclair (2004) suggests, according to some prevalent experience of teachers, that there are stimuli that commonly trigger also students' aesthetic response. But, it is not at all clear whether the criteria brought out above will point to worthwhile classroom activities which in turn will give rise to the looked for emotions of students.

Furthermore, the quoted criteria for aesthetics are given by qualitative characteristics[2], and hence by their nature they are fuzzy quantities. Thus aesthetic considerations will depend on individual judgements. Accordingly another point of interest will be to find out whether in mathematic classes the aesthetic sensation of students can be expected to be relatively homogenous.

3. STUDENTS' JUDGEMENTS ON MATHEMATICAL BEAUTY – A STUDY

In order to gain more insight into the aesthetic feelings of students, a study was carried out by the author in Germany.[3]

3.1 Design of the study

The participants of the study were on the one hand 168 students attending two gymnasiums. They were in grades 5 to 8 (96 students) and grades 11 and 12 (72 students). On the other hand 85 university mathematics teacher students were included in the study.

The students were asked to work on the questionnaire given in Figure 1, which had been developed by the author. (The consecutive number in the left column in Figure 1 has been added here with regard to the evaluation of the study data.)

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[2] Birkhoff (1956) made an attempt to quantifying aesthetics in a general way, but his proposal seems not to be very convincing.

[3] First results have been published in Brinkmann 2004a, 2004b and 2006.

Figure 1: Questionnaire

What is a beautiful mathematical problem?

1. Write down one ore several mathematical problems, which appeal to you, respectively write down their contents.
2. What do you think is a “beautiful” mathematical problem?
3. Which are, in your view, characteristics of a beautiful mathematical problem?

1 ☐ The problem is tricky.
2 ☐ The problem has a simple solution.
3 ☐ The problem is complicated.
4 ☐ The problem looks complicated but it has a simple solution.
5 ☐ The problem is simple.
6 ☐ The problem has an elegant solution.
7 ☐ The problem is complex.
8 ☐ The problem and its solution are easily to be understood.
9 ☐ The problem is unfamiliar for me.
10 ☐ The topic of the problem is interesting.
11 ☐ The problem has a surprising solution.
12 ☐ The solution of the problem is obvious.
13 ☐ The nature of the problem is familiar to me.
14 ☐ The nature of the problem is new to me.
15 ☐ The solution of the problem can easily be guessed.
16 ☐ The problem looks simple but it has a complicated solution.
17 ☐ There are considered regular patterns or structures.
18 ☐ The solution of the problem is complicated.
19 ☐ The problem is a puzzle.
20 ☐ The problem requires several solution steps.
21 ☐ The problem is about symmetric figures.
22 ☐ It is possible to solve the problem by logical considerations, without calculating.
23 ☐ The solution of the problem shows unexpected regularities.
24 ☐ The problem refers to realistic applications.
25 ☐ The problem has got more than one possible solution.
26 ☐ The problem respectively its solution are clearly structured.
27 ☐ The facts are presented clearly.
28 ☐ The solution of the problem is significant for further applications.
29 ☐ The problem requires a complex intellectual examination.
30 ☐ …

While the first two items of the questionnaire are of an open format, the third item uses a closed format to refer to a number of specific characteristics of mathematical problems that might be related to aesthetical feelings of students. In this item, beauty-making elements referred to in the literature are used:

- A number of statements refer to different nuances of simplicity (statements 2, 5, 8, 12, 13, 15), or complexity (statements 1, 3, 7, 9, 18, 19, 29) of a mathematical problem or its solution, or to combinations of both (statements 4, 16).
- The statements 26 and 27 refer to clarity and structure.
- The statement 6 refers to the characteristic of elegance.
- The statements 11 and 23 refer to the characteristic of surprise.
- The statements 17, 21 and 23 refer to symmetry and regularities.
- The statement 28 refers to the feature of power.

Based on experiences of the author as a teacher, as well as on advice given by colleagues, further possibly beauty making characteristics of a mathematical problem were included: the aspect of interest (statement 10), the feature of novelty / not novelty (statements 14, 13), the reference to applications (statements 24, 28), the open ended problem feature (statement 25).

Item 1 and 2 of the questionnaire provide qualitative responses that help us

- to interpret the answers given to item 3 (e. g. to become an idea of that what is denoted as a simple problem),
- to find further beauty making characteristics of problems, that are not yet considered in item 3.

In the primarily developed questionnaire, the statements 20 and 22 were not yet enclosed. They have been added later, according to first study results, as these criteria have been named by some students under item 2 (see Brinkmann 2004a). Hence, 36 of the students out of grades 5 to 8 and 72 students out of grades 11 and 12 (i.e. the participants of the first studies) had not been working on the completed questionnaire.

The students were instructed to tick as many statements in item 3 they felt were correct. Of course it might be that the statements marked by a student do not have equal weighting. However the focus of the study was to explore beauty-making elements, without emphasizing individual rankings, hence this matter was not a real issue.

Out of the students in grades 7 and 8, 36 had participated at the “Math Kangaroo” competition shortly before they completed this questionnaire, and thus could name problems out of this competition when working on task 1.

The responses given by the 11th and 12th graders were differentiated according to the students’ mathematical achievement in school lessons (belonging to the best third of the class/course or not).

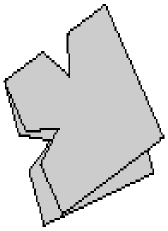
3.2 Results

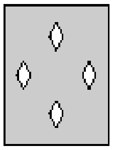
Students in grades 5 to 8:

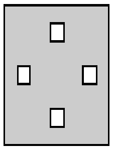
About two third of the 5th to 8th graders named *puzzles* as beautiful mathematical problems, but most of these students added that the puzzles should not be too difficult. One third of the students specified problems, that require drawing activities, as beautiful, especially geometry problems of this kind.

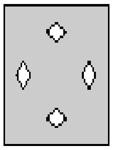
As an example, the following problem out of the Math Kangaroo competition 2003 was named by about half of the students that had participated before at this competition (see above):

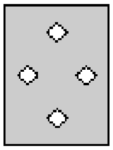
A square piece of paper is folded twice and cut in the way you can see in the picture. How will the piece of paper look after unfolding?

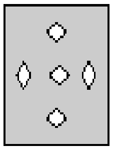


A)


B)


C)


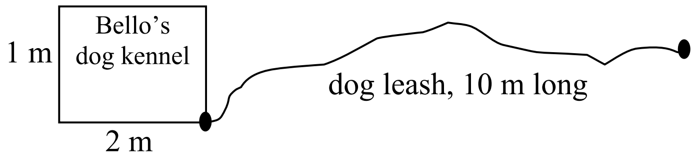
D)


E)


Further beautiful problems given by the students were mostly modelling problems, where mathematics is applied in real world contexts, e. g.:

- The problem of the garden cottage: The site plan is given on a scale of 1:500, and the allowance that the distance to the site borders has to be at least 2.5m and the distance to a street at least 5m. Find out the area where the cottage is permitted to be built.

- The dog kennel problem: Draw the area where Bello can move.



- Sparkling mineral water automat: Is it more favourable to make sparkling mineral water with a home automat, or to buy ready-made sparkling mineral water?

- The helicopter problem: There is given the map of an area where 2 helicopters have to be positioned, and also the maximal possible velocity of the helicopters. Where should the helicopters been positioned, such that every place of the area can be reached within 10 minutes?

Each of these problems had been named by some 15% to 30% of the students, which knew them.

Regarding *item 2*, the majority of the students answered that a problem and its solution must be simple to be beautiful. As examples of this point they gave among others 1x1=1, and a problem without fractions or percents. Many students made their statement of simplicity more precise by adding that a problem is *not* beautiful if they cannot solve it by themselves, or if it is so complicated that they have no idea what to do. But a simple problem is also *not* beautiful if they had to work on this problem many times, or if it is boring, and too simple. One student wrote in response to item 2 that a very difficult but solvable problem is beautiful.

In *item 3,* about two third of the students ticked *interesting topic* (statement 10), *puzzles* (statement 19), *simple solution / simple problem* (statements 2 and 5). Statement 22 (*a problem solvable by logical considerations, without calculating*), that had been added in the questionnaire after the first studies, turned out to be relevant: it has been ticked by 58% of the students that had this choice. As well the second later added statement 20 (several solution steps required) is of importance (27%). Further statements to matter are statement 4 (*a problem that looks complicated but has a simple solution*), statement 8 (*the problem and its solution are easily to be understood*) and statement 12 (*obvious solution*).

Just the reference to realistic applications plays a subordinate role, although one might expect this when regarding the examples given to item 1. Rather the interesting topic is decisive.

Table 1 shows a more detailed summary of the outcomes for item 3.

Table 1: Outcomes in item 3 of the questionnaire, grades 5 to 8[4]

Nr.	Statement	%
10	The topic of the problem is interesting.	67
2	The problem has a simple solution.	64
19	The problem is a puzzle.	61
5	The problem is simple.	60
22	It is possible to solve the problem by logical considerations, without calculating.	58
4	The problem looks complicated but it has a simple solution.	47
8	The problem and its solution are easily to be understood.	44
12	The solution of the problem is obvious.	44
11	The problem has a surprising solution.	30
1	The problem is tricky.	29
21	The problem is about symmetric figures.	28
20	The problem requires several solution steps.	27
25	The problem has got more than one possible solution.	27

The number of statements ticked by the students for item 3 differed mostly from 3 to 7. The most preferred combination was statement 2 (simple solution), statement 5 (simple problem), and statement 12 (obvious solution).

Students in grades 11 and 12:

When working on *item 1* of the questionnaire most of the students stated that they never really had come across a beautiful mathematical problem, but that it would be nice to do so. As "beautiful" examples out of the known problems, they mainly named very simple arithmetical problems (such as $1 \cdot a = a$), binomial formulas, and problems that can be solved by simple algorithms (e.g. systems of linear equations). Only in isolated cases (3 higher-achieving students) were some comparatively complex problems quoted (e.g. derivation of Pythagoras' theorem, calculation of volumes with integrals).

The answers to *task 2* were quite varied. For the *lower-achieving students* a beautiful problem is mostly a problem that can be solved easily by well-known formulas or a well-known algorithm, or a problem where one sees what is to do immediately. But for these students the solution should not be too simple because that would make it boring, and hence it should consist of several (simple) steps. As well the solution should not be too obvious; it is better to have to first think a little about the problem. Contrarily to the younger students, a connection to the real world is also important, as is the feeling that the problem could be useful for life.

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[4] There are listed only those statements ticked by at least 25% of the students. The percentages to number 20 and 22 refer only to the 60 students that had the possibility to tick the respective statements.

For the *higher-achieving students* a beautiful problem must be a problem that he or she can solve, and it must be presented in a clear way. But these students did not stress the need for the problem to be solvable by using well-known formulas and algorithms. On the contrary, the aesthetic appeal seemed to be greater if one has to think about the problem in an unconventional way, if connections within mathematics have to be seen, if more than one well-known formula has to be used, and the successful combination of these formulas has to be found out by oneself.

For *task 3* (see Table 2) there were no great differences between high-achievers and low-achievers with regards the frequency of selection of the statements. The most important characteristic of a beautiful problem for these students is an *interesting topic* (81%), followed by the *reference to realistic applications* (65%). Also important are a *simple problem solution* (60%), a *familiar problem nature* (60%, although more emphasized by low-achievers), and a *clear presentation* of the facts (58%). The characteristics of a *simple problem*, a *complicated problem with simple solution*, and a problem with a *clear structure*, were each marked by about 40% of the students (the second one more emphasized by high-achievers). Three fourth of the high-achievers ticked problems with more than one possible solution. In contrast to the younger students, only one third of the students (about half of the high-achievers) marked puzzles and tricky problems as beautiful. About half of the high-achievers ticked problems with a surprising solution, and problems that require a complex intellectual examination.

Table 2: Outcomes in item 3 of the questionnaire, grades 11 and 12[5]

Nr.	Statement	% all	% of low	% of high
10	The topic of the problem is interesting.	81	79	83
24	The problem refers to realistic applications.	65	67	63
2	The problem has a simple solution.	60	65	50
13	The nature of the problem is familiar to me.	60	71	38
27	The facts are presented clearly.	58	56	63
5	The problem is simple.	43	46	38
4	The problem looks complicated but it has a simple solution.	42	31	63
26	The problem respectively its solution are clearly structured.	39	42	33
25	The problem has got more than one possible solution.	36	17	75
1	The problem is tricky.	31	21	50
19	The problem is a puzzle.	28	19	46
28	The solution of the problem is significant for further applications.	26	27	25
11	The problem has a surprising solution.	22	10	46
29	The problem requires a complex intellectual examination.	22	8	50
15	The solution of the problem can easily be guessed.	22	23	21

The number of statements ticked by an older student for item 3 was generally greater compared to the number selected by a younger student: 22% ticked 2-5 statements, 64% ticked 6-10 statements and the remaining 14% ticked 11-14 statements. Every listed statement was ticked at least once. In spite of this, there are visible favourites named by the majority of the students.

University mathematics teacher students:

The university mathematics teacher students named under *item 1* mainly examples of

- problems that have the character of a puzzle (e. g. : "How many squares can you draw in a chessboard?"),
- problems with astonishing or unexpected solutions (e. g.: "You have a rope that is 1m longer than the equator. Imagine, you put this rope around the equator and stretch it concentrically. What distance would the rope than have from the earth surface?" Solution: $1/(2\pi)$ m, these are round 16 cm – a surprising solution, as it is contradictory to human intuition.),
- problems where mathematics is applied in real life situations, as well as using geometry, algebra or combinatorics,
- problems requiring logical thinking,

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[5] There are listed only those statements ticked by more than 20% of the students.

- problems requiring drafts or drawings.

As for *item 2*, the students characterized puzzles, tricky problems, problems that prompt thinking as beautiful. Further they mainly named as characteristics of a beautiful problem the reference to realistic applications, clearness, the possibility for creative solutions, the existence of more than one solution, complex solutions composed of several steps.

For *item 3* the mathematics teacher students indicated as most important features of a beautiful mathematical problem, similar as the elder school students, an *interesting topic,* the *reference to realistic applications* and a *clear presentation* of a problem. Further *tricky problems, puzzles* and problems with a *surprising solution* are also identified by these students as beautiful. In contrast to the younger school students, the feature of having *more than one possible solution* is relevant for these students; but simplicity is not a beauty making element, though it is the combination of a *complicated look but having a simple solution*. Similar as for scientists, *elegance* is felt as a weighty feature. More detailed information is provided by table 3.

Table 3: Outcomes in item 3 of the questionnaire, mathematics teacher students[6]

Nr.	Statement	%
24	The problem refers to realistic applications.	78
10	The topic of the problem is interesting.	76
1	The problem is tricky.	67
27	The facts are presented clearly.	58
6	The problem has an elegant solution.	54
4	The problem looks complicated but it has a simple solution.	49
25	The problem has got more than one possible solution.	41
11	The problem has a surprising solution.	32
19	The problem is a puzzle.	31
22	It is possible to solve the problem by logical considerations, without calculating.	29
20	The problem requires several solution steps.	28
14	The nature of the problem is new to me.	25
23	The solution of the problem shows unexpected regularities	21

The majority of the mathematics teacher students ticked 6-10 of the listed statements.

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[6] There are listed only those statements ticked by at least 20% of the students.

4. LIMITATIONS

It goes without saying, that the results of the study are dependent on the learning experiences of the students involved: The students' judgements can only refer on that what the students got to know. Further, the teacher's personality, especially the enthusiasm with which a teacher deals with a certain mathematical problem, might have an influence on the emotions of learners and thus also on there judgements about aesthetics.

Item 1 and 2 of the questionnaire provide qualitative statements, which may help us

- to interpret the answers given in item 3 (e.g. to get an idea of what a student means when ticking "a simple problem"),
- but also to get new ideas of what it is that makes a problem beautiful for a student.

Clearly a characteristic expressed for items 1 or 2 does not necessarily occur in the list of item 3. In this case a quantitative statement giving the weight of such a characteristic is not possible at this stage, nor was it an aim of this project. For example, one such example is the statement that a problem requires several solution steps, expressed in one of the first studies to the topic and added later to the questionnaire (see above), or the statement given by mathematics teacher students that a problem should give possibility for creative solutions. In order to explore the relevance of such an argument, further studies would have to be carried out on the pre-condition that the participating students knew already respective problems.

5. DISCUSSION

In face of existing limitations, the study allows some core statements. The aesthetic feelings of school students towards a mathematical problem seem to be strongly connected with interest, with the problem having realistic applications (especially for elder students), and also giving students *feelings of security and success*: It seems to be a necessary condition that one has got to have the feeling that you could succeed in solving a certain mathematical problem, if it is going to be perceived as beautiful. In this respect it would appear that beautiful problems have to be simple enough for the group of students under observation.

But a beautiful problem is not just a simple problem. On the contrary, it has to have a certain degree of complexity: it is more beautiful if one has to think about the problem, for example if the problem is a puzzle (especially for younger students); if the solution consists of several steps; and if one has to combine a number of formulas to get a solution. The different answers given by lower-achievers compared to those given by the higher-achievers on this point leads to the conclusion that the permissible degree of complexity for a beautiful problem depends on the mathematical ability of each individual.[7]

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[7] The feedback of three teachers, who repeated the study in their school classes/courses confirm the results reported in this paper regarding school students.

With regard to university mathematics teacher students, feelings of security and success in the solving process of a mathematical problem seem not to be a pre-condition for aesthetic feelings to arise. These students could also get pleasure from a problem which they probably could not solve by themselves. Viewing the fact, that these students will be in future mathematics teachers, an important outcome of the study are the similarities to assert when comparing the judgements of these students with those of school students expressed in item 3. For both, the notions of an interesting topic and of a reference to realistic applications, are relevant, and a great deal of every student group consider tricky problems and puzzles as beautiful. However, it is not at all clear if the idea of that what an "interesting topic" is coincides – as for this point further studies are needed.

If we want in schools to bring mathematical beauty within students' experiences, we need to use a different style of mathematical problem. We have to consider the interests of the students and choose, if possible, problems referring to realistic applications. When doing so, we have to be very aware of the abilities of our students, in order to present to them challenging problems that they can solve.

However, in my opinion it is desirable that students' aesthetic feelings are not only restricted to those problems they feel they can solve by themselves. Thus, as a further conclusion of the study, we should create phases in classrooms, which have an atmosphere that is not predominated by the demand of success. On the contrary, these phases should give the students time for leisure, time and freedom to just enjoy mathematics.

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An Application of Gröbner Bases

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Abstract. In this paper, we program a procedure using Maple's packages, with it we can realize mechanical proving of some theorems in elementary geometry.

Keywords: Gröbner base; geometry theorem; Maple.

AMS Subject Classification (2000) 68-04, 68N30

1. Introduction.

Gröbner bases have been fruitfully applied to many problems, one of them is to deal with the problem of automated geometry theorem proving (see [1], [2], [3] and [4]). Maple is a comprehensive computer system for advanced mathematics, and its Gröbner package can be used to compute a Gröbner basis of a polynomial ring. In this paper, we will discuss how to use Maple's packages to realize mechanization of theorem-proving in elementary geometry. Generally, the problem of mechanical theorem-proving can be done by the following three steps.

(1) The first step is to introduce a number system and a coordinate system such that a theorem of elementary geometry can be changed into an algebraic problem;

(2) The second step is to sort the algebraic expressions of the involved theorem's conditions, to set measures for checking if algebraic expressions of the involved theorem's result can be induced from the sorted algebraic expressions;

(3) The last step is to compile a program according to the above measures, and carry out it on a computer.

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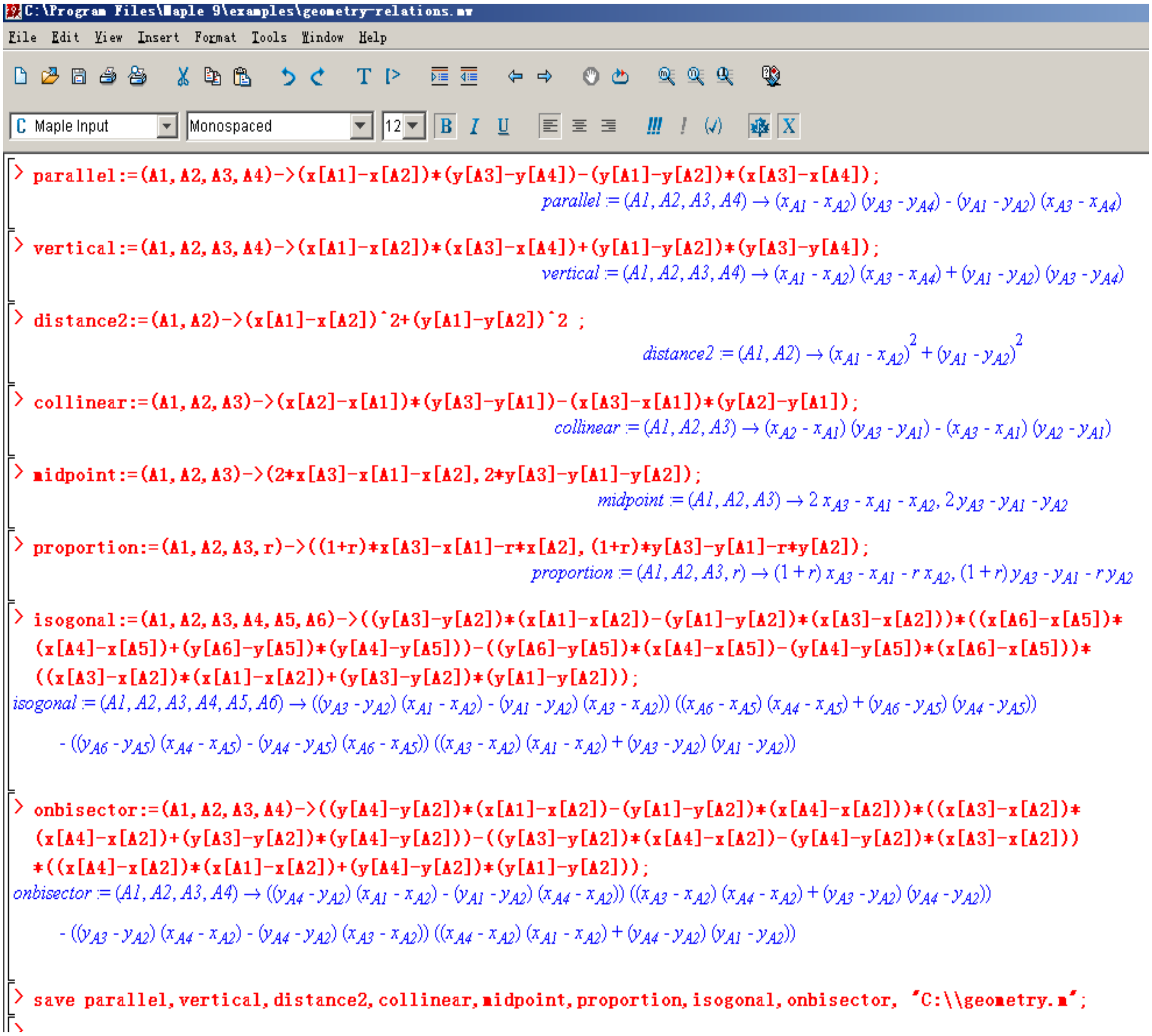
2. Some Algebraic Expressions of Common Geometric Relations.

The method of Gröbner bases depends on the construction of a particular type of polynomials to represent given geometric relations. To illustrate the translation of geometric statements into a suitable system of polynomials, we consider a simple example: the line A_1A_2 is parallel to the line A_3A_4. Let the coordinates of points A_j be (x_{Aj},y_{Aj}), $j=1,2,3,4$, in a coordinate system. In an analytic setting two lines are parallel if and only if they have the same slope. We can translate $A_1A_2 // A_3A_4$ into an equation by relating their slopes,

$$\cdot \quad \frac{y_{A_1}-y_{A_2}}{x_{A_1}-x_{A_2}}=\frac{y_{A_3}-y_{A_4}}{x_{A_3}-x_{A_4}}.$$

In the form of a polynomial equation this condition is $(x_{A_1}-x_{A_2})(y_{A_3}-y_{A_4})-(y_{A_1}-y_{A_2})(x_{A_3}-x_{A_4})=0$.

Although it may seem as if each polynomial function needs to be set equal to zero, this is not required for the Gröbner basis method. Hence in order to generalize these Maple functions for use in the method of Gröbner basis each Maple function returns only a polynomial in x_i and y_i. In a Maple command window, we input common geometric relations, and save the functions as a Maple internal file for conveniently using them later.



```
> parallel:=(A1,A2,A3,A4)->(x[A1]-x[A2])*(y[A3]-y[A4])-(y[A1]-y[A2])*(x[A3]-x[A4]);
```

$$parallel := (A1, A2, A3, A4) \rightarrow (x_{A1} - x_{A2})(y_{A3} - y_{A4}) - (y_{A1} - y_{A2})(x_{A3} - x_{A4})$$

```
> vertical:=(A1,A2,A3,A4)->(x[A1]-x[A2])*(x[A3]-x[A4])+(y[A1]-y[A2])*(y[A3]-y[A4]);
```

$$vertical := (A1, A2, A3, A4) \rightarrow (x_{A1} - x_{A2})(x_{A3} - x_{A4}) + (y_{A1} - y_{A2})(y_{A3} - y_{A4})$$

```
> distance2:=(A1,A2)->(x[A1]-x[A2])^2+(y[A1]-y[A2])^2 ;
```

$$distance2 := (A1, A2) \rightarrow (x_{A1} - x_{A2})^2 + (y_{A1} - y_{A2})^2$$

```
> collinear:=(A1,A2,A3)->(x[A2]-x[A1])*(y[A3]-y[A1])-(x[A3]-x[A1])*(y[A2]-y[A1]);
```

$$collinear := (A1, A2, A3) \rightarrow (x_{A2} - x_{A1})(y_{A3} - y_{A1}) - (x_{A3} - x_{A1})(y_{A2} - y_{A1})$$

```
> midpoint:=(A1,A2,A3)->(2*x[A3]-x[A1]-x[A2],2*y[A3]-y[A1]-y[A2]);
```

$$midpoint := (A1, A2, A3) \rightarrow 2\,x_{A3} - x_{A1} - x_{A2},\ 2\,y_{A3} - y_{A1} - y_{A2}$$

```
> proportion:=(A1,A2,A3,r)->((1+r)*x[A3]-x[A1]-r*x[A2],(1+r)*y[A3]-y[A1]-r*y[A2]);
```

$$proportion := (A1, A2, A3, r) \rightarrow (1+r)\,x_{A3} - x_{A1} - r\,x_{A2},\ (1+r)\,y_{A3} - y_{A1} - r\,y_{A2}$$

```
> isogonal:=(A1,A2,A3,A4,A5,A6)->((y[A3]-y[A2])*(x[A1]-x[A2])-(y[A1]-y[A2])*(x[A3]-x[A2]))*((x[A6]-x[A5])*
  (x[A4]-x[A5])+(y[A6]-y[A5])*(y[A4]-y[A5]))-((y[A6]-y[A5])*(x[A4]-x[A5])-(y[A4]-y[A5])*(x[A6]-x[A5]))*
  ((x[A3]-x[A2])*(x[A1]-x[A2])+(y[A3]-y[A2])*(y[A1]-y[A2]));
```

$$isogonal := (A1, A2, A3, A4, A5, A6) \rightarrow ((y_{A3} - y_{A2})(x_{A1} - x_{A2}) - (y_{A1} - y_{A2})(x_{A3} - x_{A2}))((x_{A6} - x_{A5})(x_{A4} - x_{A5}) + (y_{A6} - y_{A5})(y_{A4} - y_{A5}))$$
$$- ((y_{A6} - y_{A5})(x_{A4} - x_{A5}) - (y_{A4} - y_{A5})(x_{A6} - x_{A5}))((x_{A3} - x_{A2})(x_{A1} - x_{A2}) + (y_{A3} - y_{A2})(y_{A1} - y_{A2}))$$

```
> onbisector:=(A1,A2,A3,A4)->((y[A4]-y[A2])*(x[A1]-x[A2])-(y[A1]-y[A2])*(x[A4]-x[A2]))*((x[A3]-x[A2])*
  (x[A4]-x[A2])+(y[A3]-y[A2])*(y[A4]-y[A2]))-((y[A3]-y[A2])*(x[A4]-x[A2])-(y[A4]-y[A2])*(x[A3]-x[A2]))
  *((x[A4]-x[A2])*(x[A1]-x[A2])+(y[A4]-y[A2])*(y[A1]-y[A2]));
```

$$onbisector := (A1, A2, A3, A4) \rightarrow ((y_{A4} - y_{A2})(x_{A1} - x_{A2}) - (y_{A1} - y_{A2})(x_{A4} - x_{A2}))((x_{A3} - x_{A2})(x_{A4} - x_{A2}) + (y_{A3} - y_{A2})(y_{A4} - y_{A2}))$$
$$- ((y_{A3} - y_{A2})(x_{A4} - x_{A2}) - (y_{A4} - y_{A2})(x_{A3} - x_{A2}))((x_{A4} - x_{A2})(x_{A1} - x_{A2}) + (y_{A4} - y_{A2})(y_{A1} - y_{A2}))$$

```
> save parallel,vertical,distance2,collinear,midpoint,proportion,isogonal,onbisector, `C:\\geometry.m`;
>
```

3. Definitions and a Basic Principle.

The method in this paper is based on the theory of Gröbner bases, so we introduce some concepts and a theorem about Gröbner bases.

Definition 1. Let I be a nonzero ideal of a ring A, $G=\{g_1, \ldots , g_s\}$ be a nonzero finite set of polynomials. The G is called a Gröbner basis of the ideal I, if and only if for each polynomial f in I, there exists j, $1\le j\le s$, such that $lp(g_j)|lp(f)$. Where $lp(f)$ denotes the leader product of power of f.

Definition 2. For polynomials f, g, h in a ring A, $g\neq 0$, f is called one-step reduce to h by module g, denoted by $f \xrightarrow{g} h$, if and only if $lp(g)$ is a factor of nonzero monomial expression X of f, and , $h=f-\frac{X}{lt(g)}g$, where $lt(g)$ denotes the leader of g.

Definition 3. Let f, $f_1, \ldots, f_s$, and $f_j\neq 0$ $(1\le j\le s)$, set $F=\{f_1, \ldots, f_s\}$, f is called reduce to h about module F, denoted by $f \xrightarrow{F}_{+} h$, if and only if $f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \to \ \ldots \xrightarrow{f_{i_t}} h_t = h$, where $f_{ij}\in F$, $h_{ij}\in A$ $(j=1,\ldots,t)$.

Theorem A. (Buchberger's Theorem [5]) Let I be a nonzero ideal of a ring $A=[x_1, \ldots, x_n]$, $G=\{f_1, \ldots, f_t\}\subseteq I\backslash\{0\}$, then the followings are equivalent:

(a) G is a Gröbner basis of I;

(b) $f\in I$ if and only if $f \xrightarrow{G}_{+} 0$.

4. The Main Procedure.

We have seen that we can translate conditions and the conclusion of a geometric theorem into polynomials: $f_1,\ldots,f_m$ (the hypotheses) and g (the conclusion) in the ring $K[x_1,\ldots, x_j\,; y_1,\ldots, y_j\,]$. In what sense then does our conclusion g, follow from the hypotheses $f_1,\ldots,f_m$? An algebraic formulation of the problem is as follows:

$$\forall(x_1,...,x_j,y_1,...y_j),\ (f_1=0\wedge...\wedge f_m=0)\Rightarrow g=0.$$

Let I be the ideal generated by the set $\{f_1,\ldots,f_m\}$ in $K[x_1,\ldots, x_j\,; y_1,\ldots, y_j\,]$, then the conclusion g follows from the hypotheses $f_1,\ldots,f_m$ means that $g\in I$.

Using the command **gbasis** of Maple, we can calculate a Gröbner basis of the ideal I, and a Gröbner basis of an ideal has the property that every polynomial in the ideal reduces to 0 with respect to the basis. Hence, to determine if the conclusion g is in the ideal I, we need only to use the command **normalf** of Maple to calculate the remainder of the conclusion polynomial g after division by a Gröbner basis of the ideal I.

By above algorithmic principle, we compile a procedure **provegeo** in Maple language, with it we can realize mechanization of some geometry theorems proving. The source codes of the procedure 'provegeo' are shown in Maple sheets of following examples. To use the procedure conveniently, we may save it as a Maple package.

5. Examples

Example 1. The diagonals of a rhombus are mutual perpendicular**.**

Establishing a coordinate system as the figure 1, then we have the following conditions:

1. *AD*//*BC*,

2. *AD*=*AB*,

3. The parallelogram *ABCD* is non-degenerate.

Our conclusion is $AC \perp BD$. The mechanical proof is as following Maple sheet:

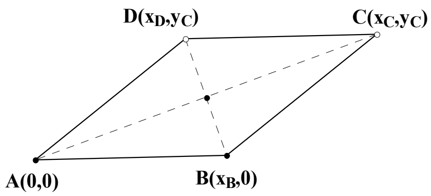
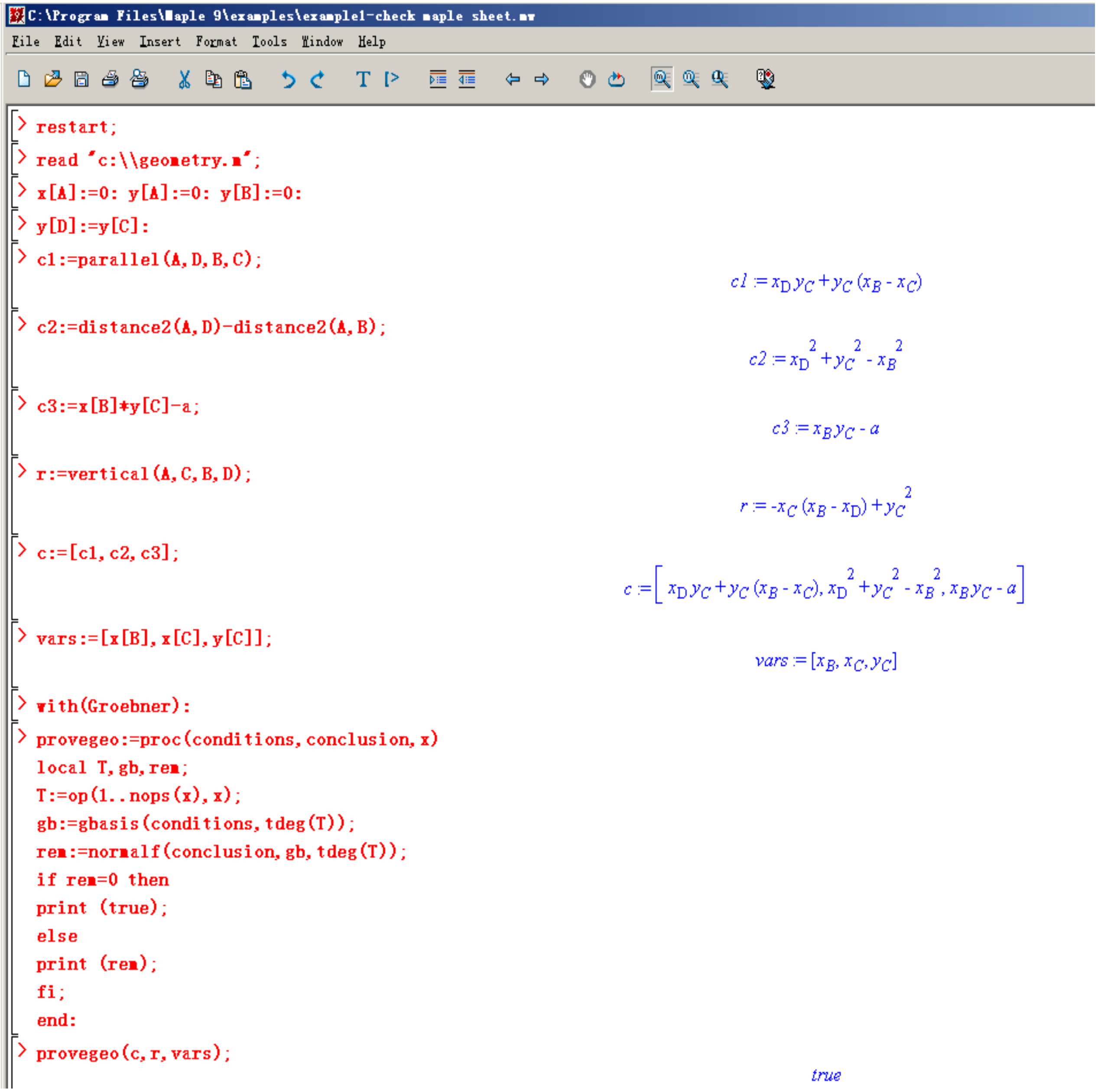

D(x_D,y_C)
C(x_C,y_C)
A(0,0)
B(x_B,0)

Figure 1


C:\Program Files\Maple 9\examples\example1-check maple sheet.mw
File Edit View Insert Format Tools Window Help

```
> restart;
> read `c:\\geometry.m`;
> x[A]:=0: y[A]:=0: y[B]:=0:
> y[D]:=y[C]:
> c1:=parallel(A,D,B,C);
```

$$c1 := x_D y_C + y_C (x_B - x_C)$$

```
> c2:=distance2(A,D)-distance2(A,B);
```

$$c2 := x_D^2 + y_C^2 - x_B^2$$

```
> c3:=x[B]*y[C]-a;
```

$$c3 := x_B y_C - a$$

```
> r:=vertical(A,C,B,D);
```

$$r := -x_C (x_B - x_D) + y_C^2$$

```
> c:=[c1,c2,c3];
```

$$c := \left[x_D y_C + y_C (x_B - x_C), x_D^2 + y_C^2 - x_B^2, x_B y_C - a \right]$$

```
> vars:=[x[B],x[C],y[C]];
```

$$vars := [x_B, x_C, y_C]$$

```
> with(Groebner):
> provegeo:=proc(conditions,conclusion,x)
  local T,gb,rem;
  T:=op(1..nops(x),x);
  gb:=gbasis(conditions,tdeg(T));
  rem:=normalf(conclusion,gb,tdeg(T));
  if rem=0 then
  print (true);
  else
  print (rem);
  fi;
  end:
> provegeo(c,r,vars);
```

true

Remark: In the above proof, the non-generate condition x_By_C-a (i.e. $x_By_C \neq 0$) of the parallelogram *ABCD* should be looked as a given condition, otherwise the proposition could not be checked correctly. The variable x_D is not an independent variable, it is as a parameter. If x_D is listed in 'vars', the proposition is also checked to be true.

Example 2 (Apollonius' theorem). Given a ΔABC, if *D* is any point on *BC* such that it divides *BC* in the ratio *n*:*m* (*mBD*= *nDC*), then $mAB^2 + nAC^2 = mBD^2 + nDC^2 + (m+n)AD^2$. When *m*=*n*(=1), that is, *AD* is the median falling on *BC*, the theorem reduces to $AB^2 + AC^2 = BD^2 + DC^2 + 2AD^2$ $= \frac{1}{2}BC^2 + 2AD^2$.

Establishing a coordinate system as the figure 2, then we have the following conditions:

1. *D* divides *BC* in the ratio *n* : *m*,
2. Δ*ABC* is non-generate.

Our conclusion is $mAB^2 + nAC^2 = mBD^2 + nDC^2 + (m+n)AD^2$.

The mechanical proof is as following Maple sheet:

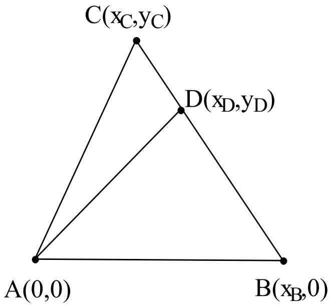
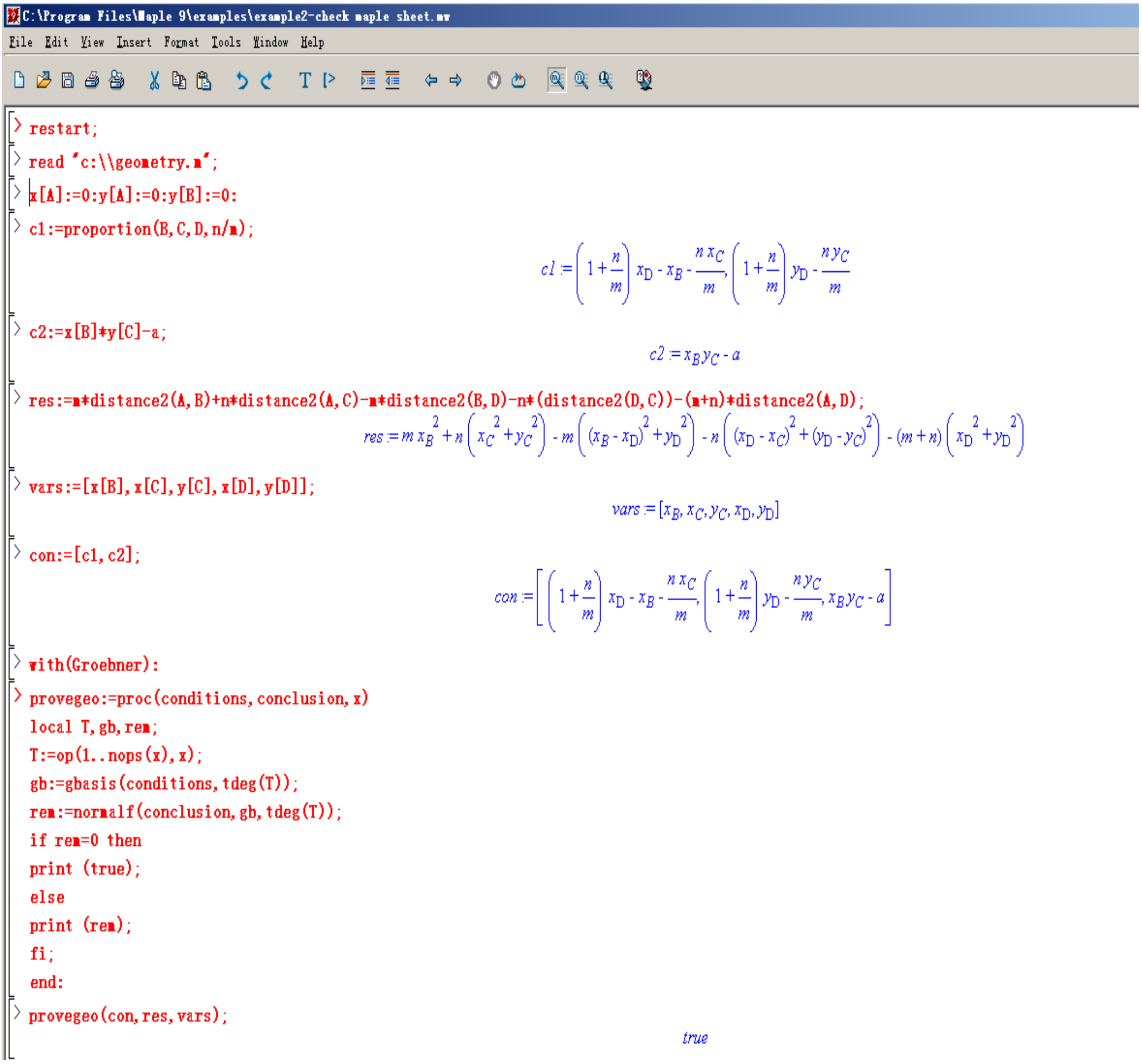


Figure 2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File Edit View Insert Format Tools Window Help

```
> restart;
> read 'c:\\geometry.m';
> x[A]:=0:y[A]:=0:y[B]:=0:
> c1:=proportion(B,C,D,n/m);
```

$$c1 := \left(1+\frac{n}{m}\right) x_D - x_B - \frac{n\,x_C}{m},\ \left(1+\frac{n}{m}\right) y_D - \frac{n\,y_C}{m}$$

```
> c2:=x[B]*y[C]-a;
```

$$c2 := x_B\,y_C - a$$

```
> res:=m*distance2(A,B)+n*distance2(A,C)-m*distance2(B,D)-n*(distance2(D,C))-(m+n)*distance2(A,D);
```

$$res := m\,x_B^2 + n\left(x_C^2 + y_C^2\right) - m\left((x_B - x_D)^2 + y_D^2\right) - n\left((x_D - x_C)^2 + (y_D - y_C)^2\right) - (m+n)\left(x_D^2 + y_D^2\right)$$

```
> vars:=[x[B],x[C],y[C],x[D],y[D]];
```

$$vars := [x_B, x_C, y_C, x_D, y_D]$$

```
> con:=[c1,c2];
```

$$con := \left[\left(1+\frac{n}{m}\right) x_D - x_B - \frac{n\,x_C}{m},\ \left(1+\frac{n}{m}\right) y_D - \frac{n\,y_C}{m},\ x_B\,y_C - a\right]$$

```
> with(Groebner):
> provegeo:=proc(conditions,conclusion,x)
  local T,gb,rem;
  T:=op(1..nops(x),x);
  gb:=gbasis(conditions,tdeg(T));
  rem:=normalf(conclusion,gb,tdeg(T));
  if rem=0 then
  print (true);
  else
  print (rem);
  fi;
  end:
> provegeo(con,res,vars);
```

$$true$$

Example 3. Supposed that *CD* bisects ∠*ACB*, and *AE*//*CD*, then ΔACE is an isosceles triangle.

Establishing a coordinate system as the figure 3, then we have the following conditions:

1. *CD* bisects ∠*ACB*,
2. *A*, *B*, *D* are collinear,
3. *AE*//*CD*,
4. ΔABC is non-generate.

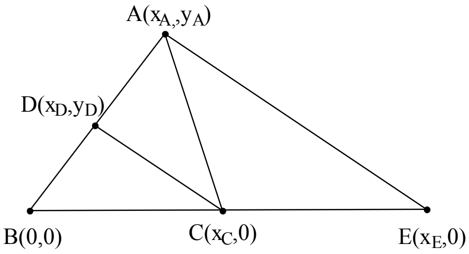
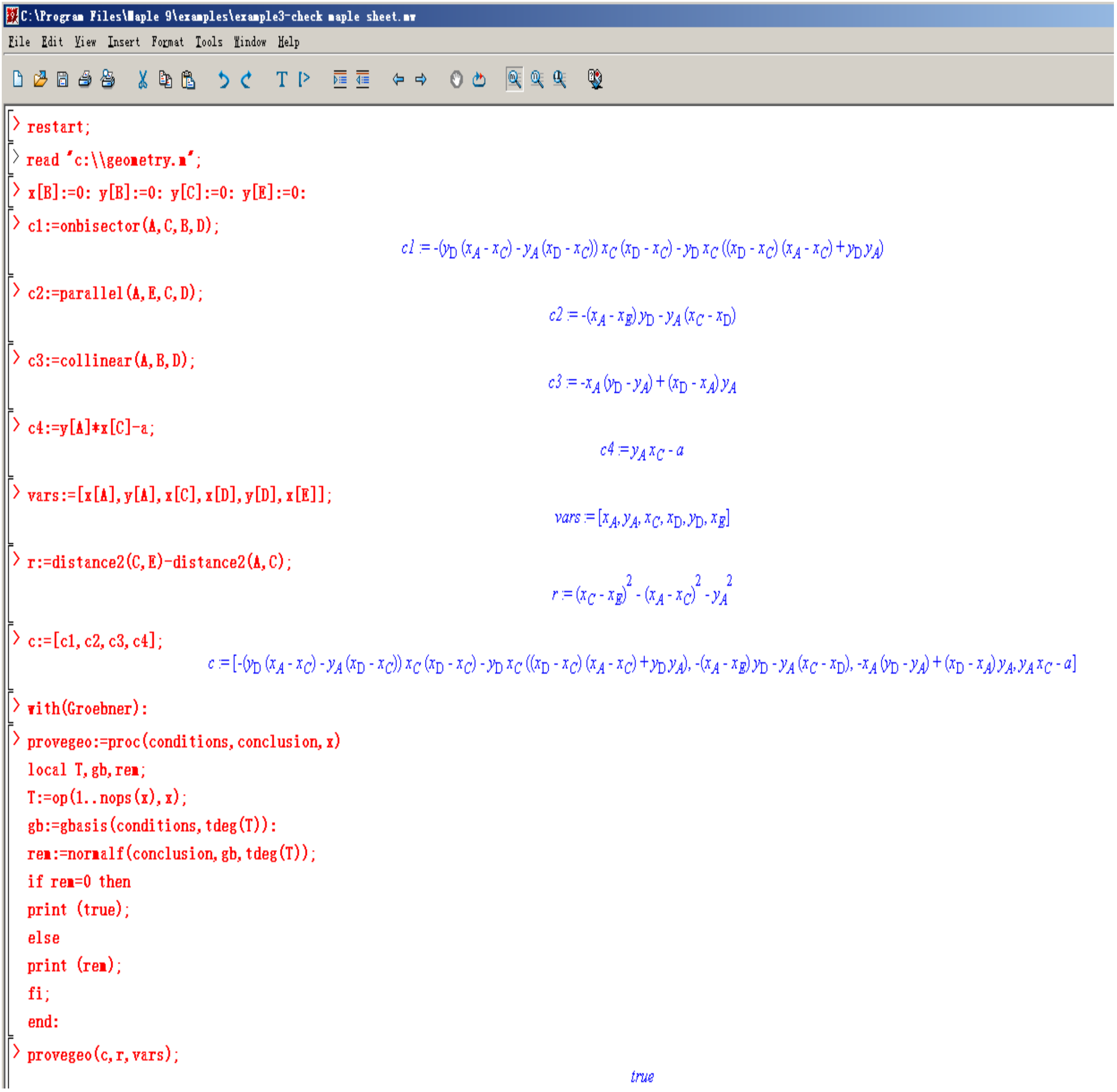

A(xA,,yA)
D(xD,yD)
B(0,0)
C(xC,0)
E(xE,0)

Figure 3

Our conclusion is $AC = CE$.   The mechanical proof is as following Maple sheet:


C:\Program Files\Maple 9\examples\example3-check maple sheet.mw
File Edit View Insert Format Tools Window Help

```
> restart;
> read 'c:\\geometry.m';
> x[B]:=0: y[B]:=0: y[C]:=0: y[E]:=0:
> c1:=onbisector(A,C,B,D);
```

$$c1 := -(y_D (x_A - x_C) - y_A (x_D - x_C)) x_C (x_D - x_C) - y_D x_C ((x_D - x_C) (x_A - x_C) + y_D y_A)$$

```
> c2:=parallel(A,E,C,D);
```

$$c2 := -(x_A - x_E) y_D - y_A (x_C - x_D)$$

```
> c3:=collinear(A,B,D);
```

$$c3 := -x_A (y_D - y_A) + (x_D - x_A) y_A$$

```
> c4:=y[A]*x[C]-a;
```

$$c4 := y_A x_C - a$$

```
> vars:=[x[A],y[A],x[C],x[D],y[D],x[E]];
```

$$vars := [x_A, y_A, x_C, x_D, y_D, x_E]$$

```
> r:=distance2(C,E)-distance2(A,C);
```

$$r := (x_C - x_E)^2 - (x_A - x_C)^2 - y_A^2$$

```
> c:=[c1,c2,c3,c4];
```

$$c := [-(y_D (x_A - x_C) - y_A (x_D - x_C)) x_C (x_D - x_C) - y_D x_C ((x_D - x_C) (x_A - x_C) + y_D y_A), -(x_A - x_E) y_D - y_A (x_C - x_D), -x_A (y_D - y_A) + (x_D - x_A) y_A, y_A x_C - a]$$

```
> with(Groebner):
> provegeo:=proc(conditions,conclusion,x)
  local T,gb,rem;
  T:=op(1..nops(x),x);
  gb:=gbasis(conditions,tdeg(T)):
  rem:=normalf(conclusion,gb,tdeg(T));
  if rem=0 then
  print (true);
  else
  print (rem);
  fi;
  end:
> provegeo(c,r,vars);
```

$$true$$

Example 4 (Simson line). Given any ΔABC and a point P in the plane of the triangle, if perpendiculars from P on to the sides AB, AC, BC, meet those sides at U, V, W respectively, then U, V, W are collinear if and only if P lies on the circumcircle of ΔABC.

Establishing a coordinate system as the figure 4, for the sufficiency, we have the following conditions:

1. P, B, C lie on the circumcircle of ΔABC,
2. $PU\perp AB$, $PV\perp AC$, $PW\perp BC$,
3. V lies on the line AC, W lies on the line BC,
4. ΔABC is non-generate.

Our conclusion is that U, V, W are collinear. The mechanical proof is as following Maple sheet:

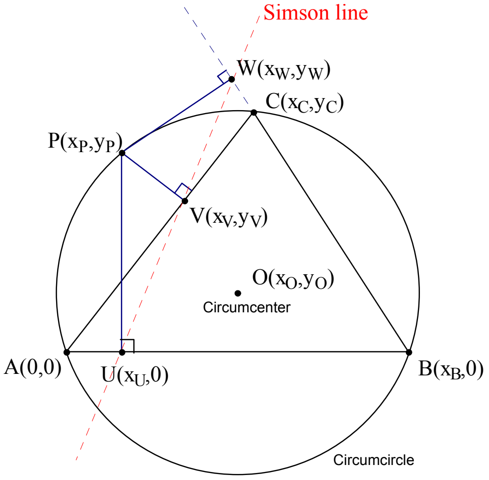
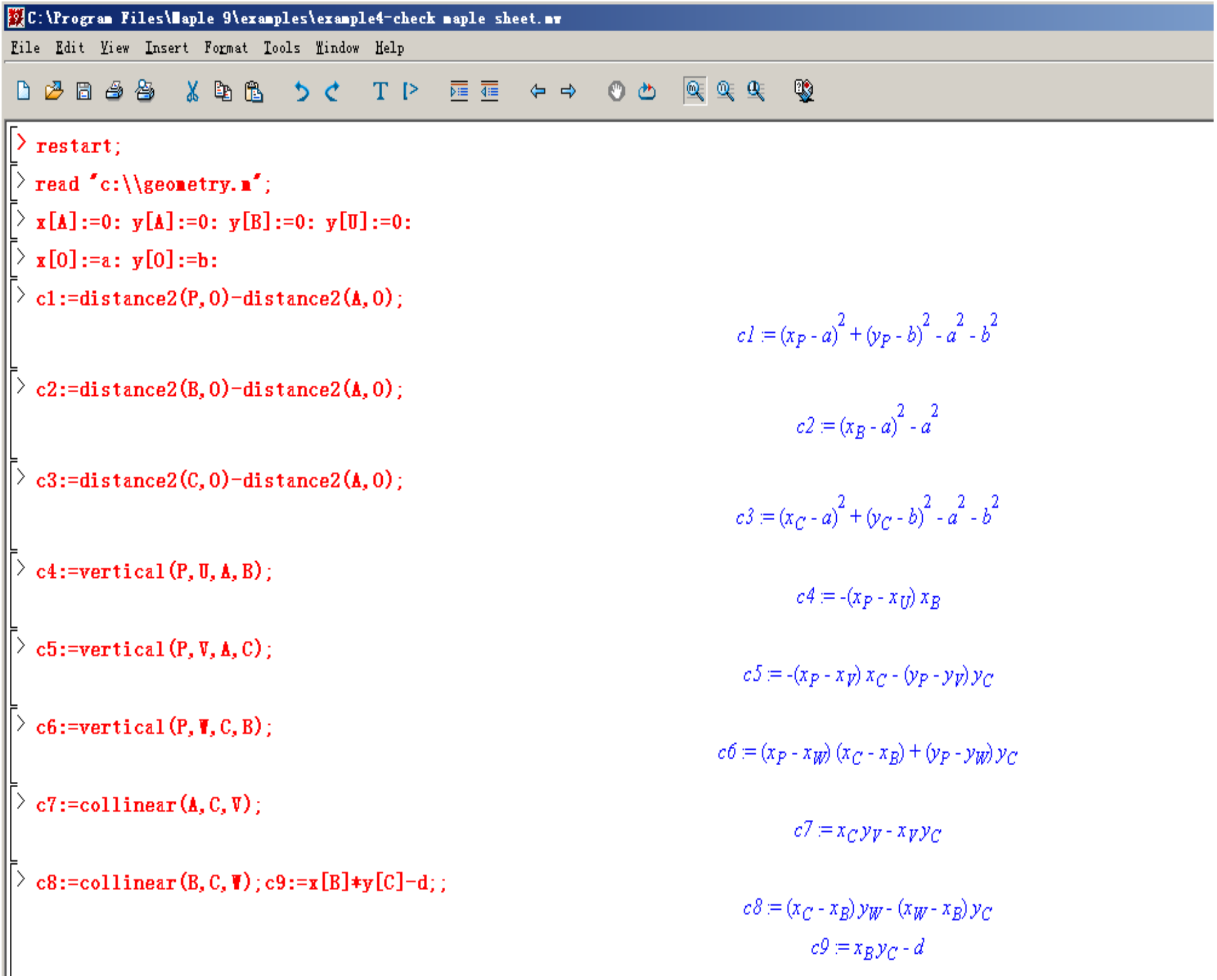

Simson line
W(xW,yW)
C(xC,yC)
P(xP,yP)
V(xV,yV)
O(xO,yO)
Circumcenter
A(0,0)
U(xU,0)
B(xB,0)
Circumcircle

Figure 4


C:\Program Files\Maple 9\examples\example4-check maple sheet.mw
File Edit View Insert Format Tools Window Help

```
> restart;
> read `c:\\geometry.m`;
> x[A]:=0: y[A]:=0: y[B]:=0: y[U]:=0:
> x[O]:=a: y[O]:=b:
> c1:=distance2(P,O)-distance2(A,O);
```

$$c1 := (x_P - a)^2 + (y_P - b)^2 - a^2 - b^2$$

```
> c2:=distance2(B,O)-distance2(A,O);
```

$$c2 := (x_B - a)^2 - a^2$$

```
> c3:=distance2(C,O)-distance2(A,O);
```

$$c3 := (x_C - a)^2 + (y_C - b)^2 - a^2 - b^2$$

```
> c4:=vertical(P,U,A,B);
```

$$c4 := -(x_P - x_U)\, x_B$$

```
> c5:=vertical(P,V,A,C);
```

$$c5 := -(x_P - x_V)\, x_C - (y_P - y_V)\, y_C$$

```
> c6:=vertical(P,W,C,B);
```

$$c6 := (x_P - x_W)\,(x_C - x_B) + (y_P - y_W)\, y_C$$

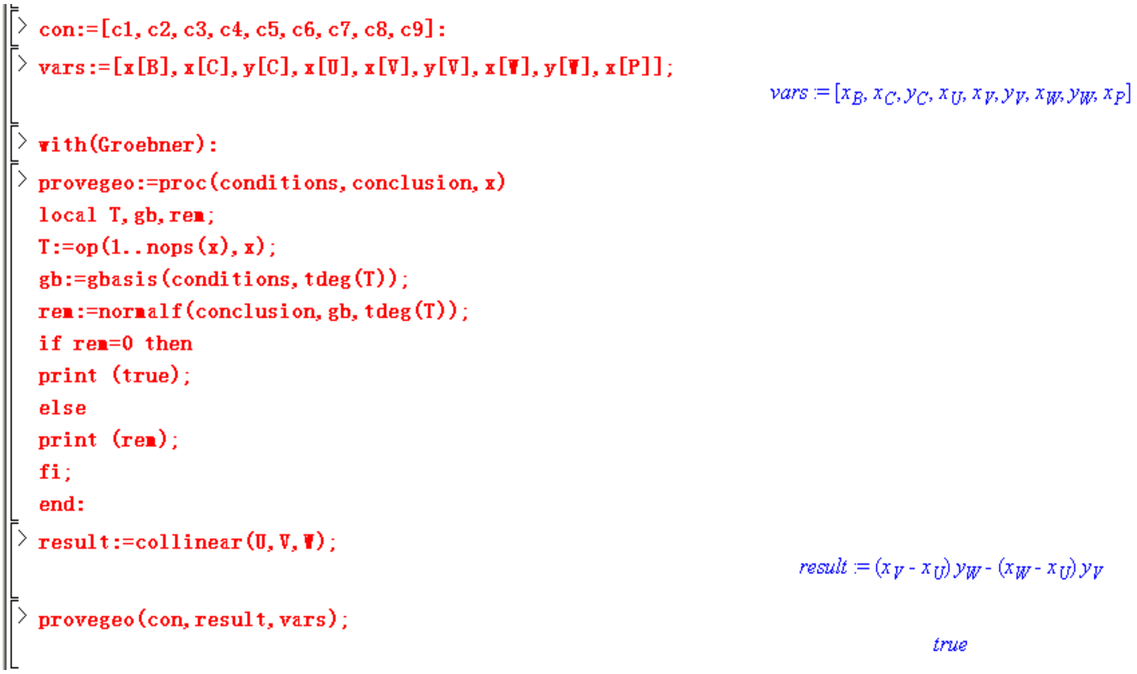
```
> c7:=collinear(A,C,V);
```

$$c7 := x_C\, y_V - x_V\, y_C$$

```
> c8:=collinear(B,C,W);c9:=x[B]*y[C]-d;;
```

$$c8 := (x_C - x_B)\, y_W - (x_W - x_B)\, y_C$$

$$c9 := x_B\, y_C - d$$



```
> con:=[c1,c2,c3,c4,c5,c6,c7,c8,c9]:
> vars:=[x[B],x[C],y[C],x[U],x[V],y[V],x[W],y[W],x[P]];
```

$$vars := [x_B, x_C, y_C, x_U, x_V, y_V, x_W, y_W, x_P]$$

```
> with(Groebner):
> provegeo:=proc(conditions,conclusion,x)
  local T,gb,rem;
  T:=op(1..nops(x),x);
  gb:=gbasis(conditions,tdeg(T));
  rem:=normalf(conclusion,gb,tdeg(T));
  if rem=0 then
  print (true);
  else
  print (rem);
  fi;
  end:
> result:=collinear(U,V,W);
```

$$result := (x_V - x_U)\, y_W - (x_W - x_U)\, y_V$$

```
> provegeo(con,result,vars);
```

$$true$$

In fact, since P lies on the circumcircle of ΔABC, y_P is not an independent variable, y_P is not listed in 'vars', in this case, the computation time is about 27 seconds. If y_P is listed in 'vars', the proposition is correctly checked, but the computation time is about 103 seconds.

For the necessity, one condition is that U, V, W are collinear, and the conclusion is that P lies on the circumcircle of ΔABC. Exchanging $c1$ for *result* in the proof of the sufficiency, we can similarly check that the necessity is true.

Example 5 (Euler line).

In any triangle ΔABC, the orthocenter H, the centroid G and the circumcenter O are collinear, and $GH=2OG$. The line passing by these points is known as the Euler line of ΔABC.

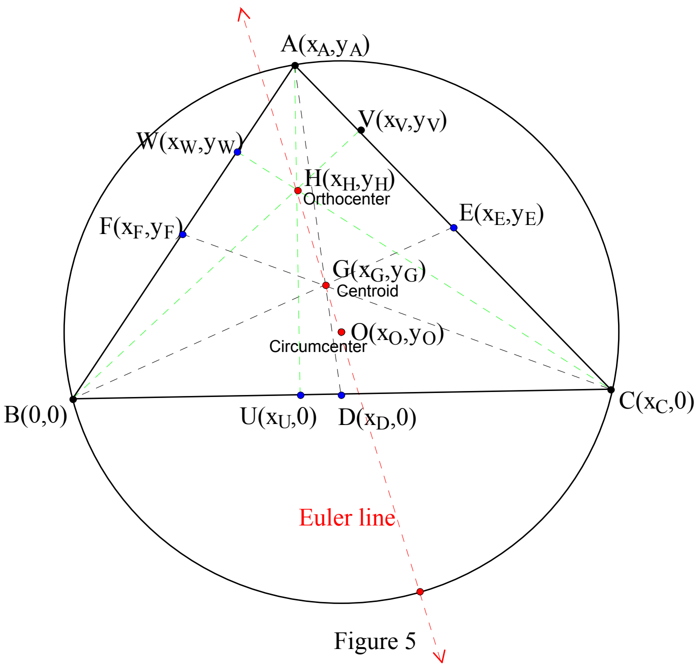
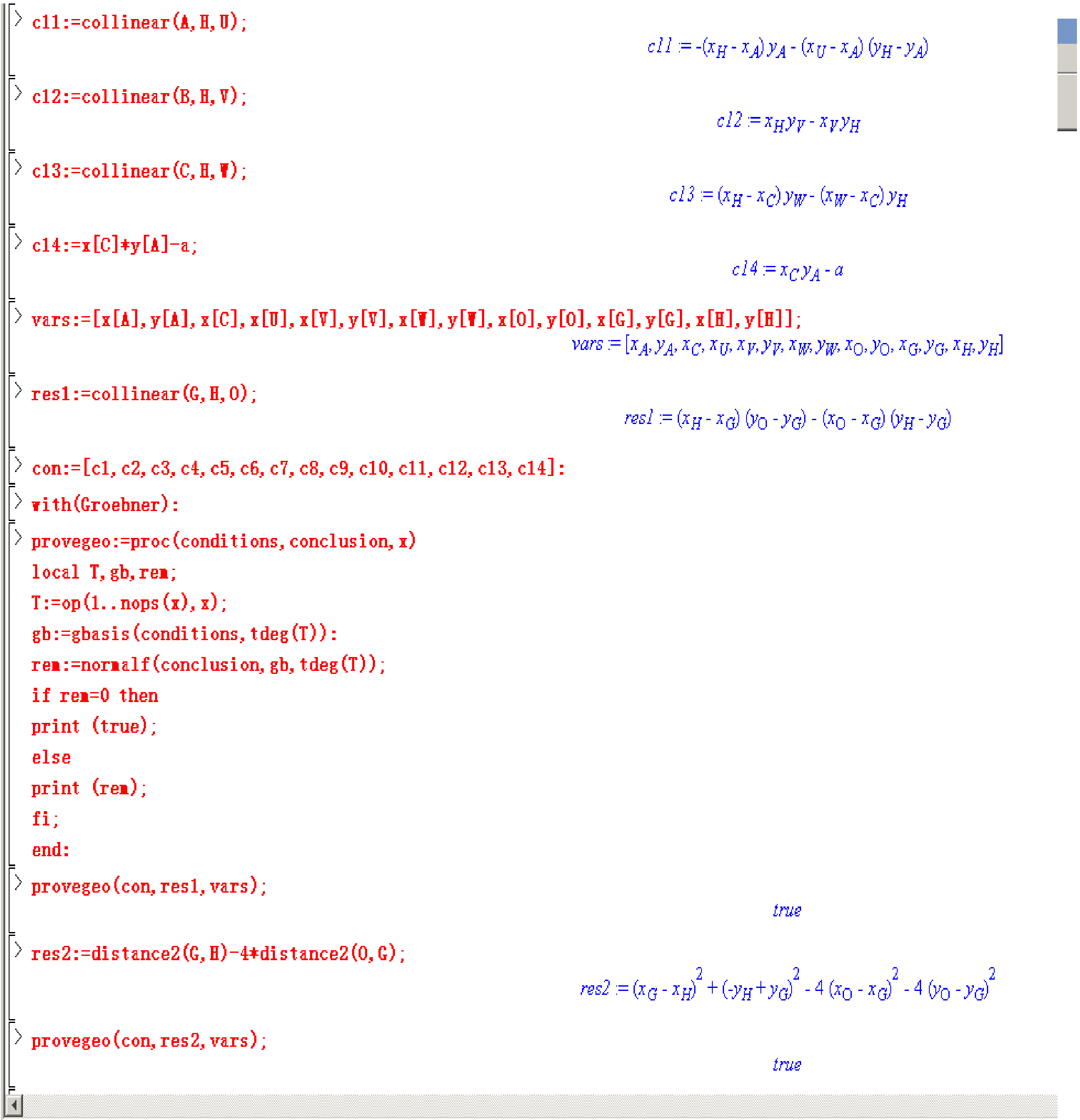

A(x_A,y_A)
V(x_V,y_V)
W(x_W,y_W)
H(x_H,y_H)
Orthocenter
E(x_E,y_E)
F(x_F,y_F)
G(x_G,y_G)
Centroid
O(x_O,y_O)
Circumcenter
B(0,0)
C(x_C,0)
U(x_U,0)
D(x_D,0)
Euler line

Figure 5

Establishing a coordinate system as the figure 5, we have the following conditions:

1. O is the circumcenter of ΔABC,
2. G is the centroid of ΔABC,
3. H is the orthocenter of ΔABC,
4. ΔABC is non-generate.

Our conclusion is that O, *G*, *H* are collinear and *GH* = 2*OG*. The mechanical proof is as following Maple sheet:



```
> c11:=collinear(A,H,U);
```

$$c11 := -(x_H - x_A)\, y_A - (x_U - x_A)\,(y_H - y_A)$$

```
> c12:=collinear(B,H,V);
```

$$c12 := x_H y_V - x_V y_H$$

```
> c13:=collinear(C,H,W);
```

$$c13 := (x_H - x_C)\, y_W - (x_W - x_C)\, y_H$$

```
> c14:=x[C]*y[A]-a;
```

$$c14 := x_C y_A - a$$

```
> vars:=[x[A],y[A],x[C],x[U],x[V],y[V],x[W],y[W],x[O],y[O],x[G],y[G],x[H],y[H]];
```

$$vars := [x_A, y_A, x_C, x_U, x_V, y_V, x_W, y_W, x_O, y_O, x_G, y_G, x_H, y_H]$$

```
> res1:=collinear(G,H,O);
```

$$res1 := (x_H - x_G)\,(y_O - y_G) - (x_O - x_G)\,(y_H - y_G)$$

```
> con:=[c1,c2,c3,c4,c5,c6,c7,c8,c9,c10,c11,c12,c13,c14]:
> with(Groebner):
> provegeo:=proc(conditions,conclusion,x)
  local T,gb,rem;
  T:=op(1..nops(x),x);
  gb:=gbasis(conditions,tdeg(T)):
  rem:=normalf(conclusion,gb,tdeg(T));
  if rem=0 then
  print (true);
  else
  print (rem);
  fi;
  end:
> provegeo(con,res1,vars);
```

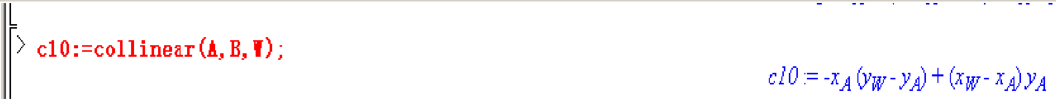
$$true$$

```
> res2:=distance2(G,H)-4*distance2(O,G);
```

$$res2 := (x_G - x_H)^2 + (-y_H + y_G)^2 - 4\,(x_O - x_G)^2 - 4\,(y_O - y_G)^2$$

```
> provegeo(con,res2,vars);
```

$$true$$



```
> c10:=collinear(A,B,W);
```

$$c10 := -x_A\,(y_W - y_A) + (x_W - x_A)\, y_A$$

The above five examples have been checked correctly on a microcomputer, in a similar way, other geometric propositions may be proved by the Maple internal file ‘geometry.m’ and the procedure ‘provegeo’. An appropriate coordinate system should be chosen to reduce variables as possible, and only those independent variables are listed in 'vars', the less variables in 'vars', the less time in the computation.

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Small change– Big Difference

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Abstract: Starting in a well known theorem concerning medians of triangle and using the 'What If Not?' strategy, we describe an example of an activity in which some relations among segments and areas in triangle were revealed. Some of the relations were proved by means of Affine Geometry.

Keywords: Affine geometry; Problem solving; Problem posing; Triangle theorems; Generalizations

1. Introduction

The 'What If Not (WIN) strategy (Brown and Walter, 1990) is based on the idea that modifying an attribute of a given statement could yield a new and intriguing conjecture which consequently may result in some interesting investigation. Using interactive geometrical software, let us apply the WIN strategy to the theorem: *The three medians of a triangle divide it into 6 triangles possessing the same area*. This paper presents some results obtained by

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modifying the premises so that each side of the triangle is devided into *n* equal segments instead of two. More precisely, given a triangle ABC, with sides *a*, *b*, *c* each diveded into *n>2* equal segments. Each of the $\frac{1}{n}$ dividing point is connected to the opposite vertex (Fig. 1). Unlike the case of medians (*n* = 2), in this modified version, there appears to be some quadrangles as well. Is there anything particularly interesting about the new parts? – That is what we are set to examine.

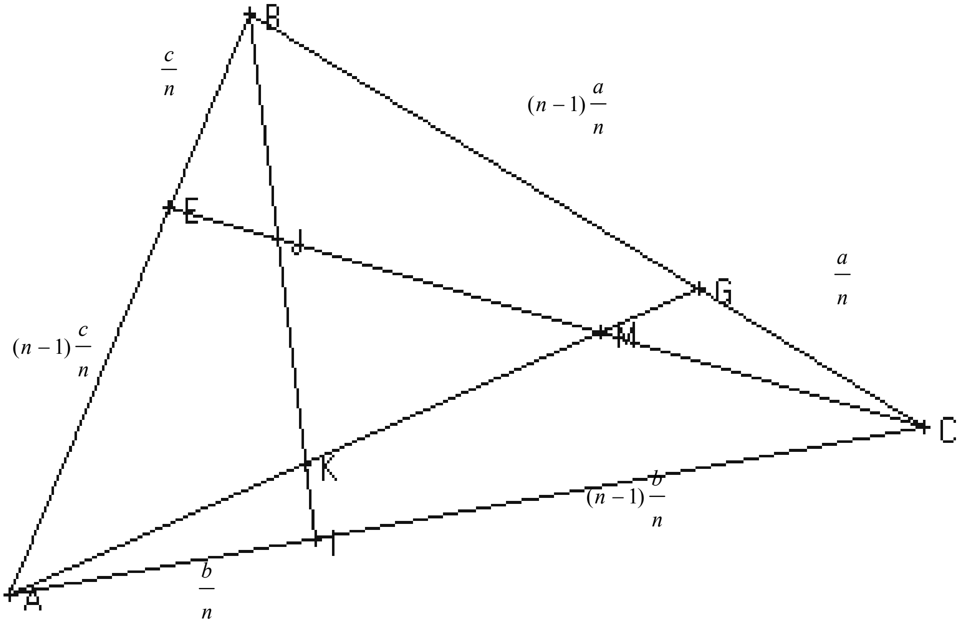
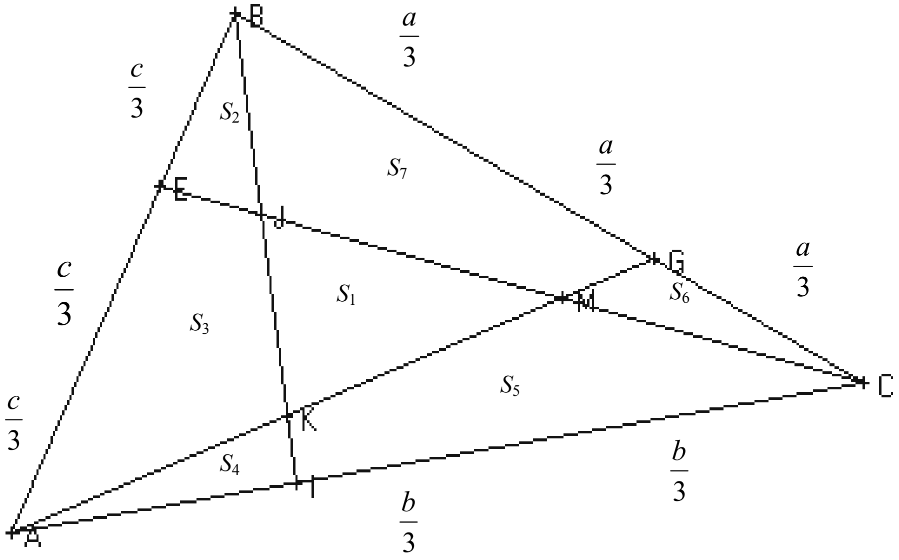

B
$\frac{c}{n}$
$(n-1)\frac{a}{n}$
E
J
G
$\frac{a}{n}$
$(n-1)\frac{c}{n}$
M
C
K
$(n-1)\frac{b}{n}$
I
A
$\frac{b}{n}$

Figure 1: Schematic description of the problem

Let us first look at the particular case n=3, and then generalize it for *any* value of n.

2. The case of n = 3

Figure 2 demonstrates the case of n=3.



B
E
J
G
M
C
K
I
A
$\frac{a}{3}$
$\frac{a}{3}$
$\frac{a}{3}$
$\frac{c}{3}$
$\frac{c}{3}$
$\frac{c}{3}$
$\frac{b}{3}$
$\frac{b}{3}$
S_1
S_2
S_3
S_4
S_5
S_6
S_7

Figure 2: Schematic description of the case: n=3

$BE = ED = DA$; $AI = IH = HC$; $CG = GF = FB$.

Let S_1 = area(JMK) ; S_2 = area(BEJ) ; S_3 = area($EAKJ$) ; S_4 = area(AIK) ;

S_5 = area($KICL$) ; S_6 = area(LCG) ; S_7 = area($BJLG$)

Based on measurements taken by means of dynamic geometry software, the following conjectures as regards to areas and segments were raised:

$$KJ = JB \,;\, LK = KA \,;\, JL = LC \qquad (1)$$

$$S_2 = S_4 = S_6 \qquad (2)$$

$$S_3 = S_5 = S_7 \tag{3}$$

$$\frac{S_1}{S_2} = 3 \tag{4}$$

$$\frac{BK}{KI} = \frac{AL}{LG} = \frac{CJ}{JE} = 6 \tag{5}$$

In order to prove the above conjectures, we join *KD*, *LH* and *JF* (Fig. 3) to generate triangles *BDK*, *AHL* and *CF*.

As follows we prove the claim: $EJ \parallel DK$; $IK \parallel HL$; $GL \parallel FJ$.

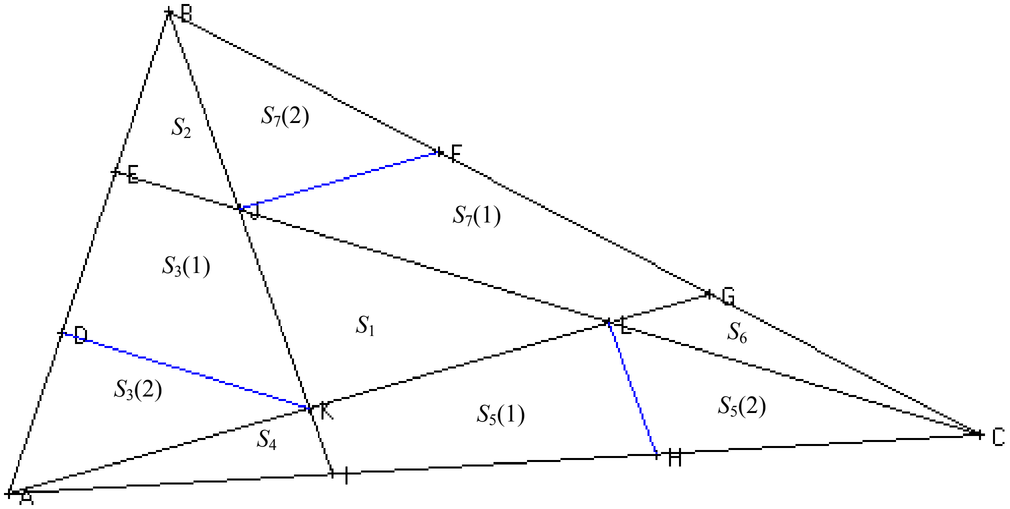


Figure 3: Constructed segments *KD*, *LH* and *JF* and related areas

Proof:

Let $S_3(1) = \text{area}(EDKJ)$; $S_3(2) = \text{area}(DAK)$; $S_5(1) = \text{area}(IHLK)$;

$S_5(2) = \text{area}(HCL)$; $S_7(1) = \text{area}(GFJL)$ and $S_7(2) = \text{area}(FBJ)$ (Fig. 3).

We employ Affine Geometry to prove this claim:

Let *AC* be on the *x*-axis, and *AB* on the *y*-axis, while the unit scale on the *x*-axis is the length of *AC* and the unit scale on the *y*-axis is *AB*. Hence, the coordinates of the vertices are:

$A(0,0)$; $B(0,1)$; $C(1,0)$; $E(0,\frac{2}{3})$; $I(\frac{1}{3},0)$; $G(\frac{2}{3},\frac{1}{3})$; $F(\frac{1}{3},\frac{2}{3})$

For the equation of line *CE* we get:

$$\frac{x-x_E}{x_C-x_E}=\frac{y-y_E}{y_C-y_E}\Rightarrow\frac{x-0}{1-0}=\frac{y-\frac{2}{3}}{0-\frac{2}{3}}\Rightarrow y_{CE}=\frac{2}{3}-\frac{2}{3}x$$

And for line *BI*:

$$\frac{x-x_B}{x_I-x_B}=\frac{y-y_B}{y_I-y_B}\Rightarrow\frac{x-0}{\frac{1}{3}}=\frac{y-1}{0-1}\Rightarrow y_{BI}=1-3x$$

Thus the coordinates of J $(BI\cap CE)$ are:

$$\left.\begin{array}{l} y=\frac{2}{3}-\frac{2}{3}x \\ y=1-3x \end{array}\right\}\Rightarrow\frac{1}{3}=\frac{7}{3}x\Rightarrow x=\frac{1}{7}\Rightarrow y=\frac{4}{7}\Rightarrow J(\frac{1}{7},\frac{4}{7})$$

Vectors $\vec{JF}$ and $\vec{AG}$ are:

$$\vec{JF}=(x_F-x_J,y_F-y_J)=(\frac{1}{3}-\frac{1}{7},\frac{2}{3}-\frac{4}{7})=(\frac{4}{21},\frac{2}{21})\ ;$$
$$\vec{AG}=(x_G-x_A,y_G-y_A)=(\frac{2}{3}-0,\frac{1}{3}-0)=(\frac{2}{3},\frac{1}{3})$$

Therefore: $\vec{JF}=\frac{2}{7}\cdot\vec{AG}$, and hence vectors $\vec{JF}$ and $\vec{AG}$ are parallel.

By symmetry considerations: $IK\|HL$ and $EJ\|DK$. This proves

(1) *KJ* = *JB* ; *LK* = *KA* ; *JL* = *LC*.

Notice that parallelism is not affected by affine transformations.

Referring to the notations in Fig. 3 we shall now prove that:

$$\frac{S_3(1)}{S_2}=3 \qquad (6)$$

$$\frac{S_3(2)}{S_2} = 2 \tag{7}$$

$$S_3(1) = S_5(1) = S_7(1) \tag{8}$$

$$S_3(2) = S_5(2) = S_7(2) \tag{9}$$

Since $EJ \| DK$ and $IK \| HL$ and $GL \| FJ$ we get:

$\Delta BEJ \approx \Delta BDK$; $\Delta AIK \approx \Delta AHL$; $\Delta CGL \approx \Delta CFJ$. The similarity ratio is 2.

Consequently, $S_3 = 3 \cdot S_2$ and similarly: $S_5(1) = 3 \cdot S_4$; $S_7(1) = 3 \cdot S_6$. As a result, (6) is proved,

In addition, since area(BDK) = 2·area (DKA) we get:

$S_2 + S_3(1) = 2 \cdot S_3(2) \Rightarrow S_2 + 3 \cdot S_2 = 2S_3(2) \Rightarrow S_3(2) = 2 \cdot S_2$.

Similarly $S_5(2) = 2 \cdot S_4$ and $S_7(2) = 2 \cdot S_6$.

Thus (7) is proved.

The above relations are summarized in Fig. 4.

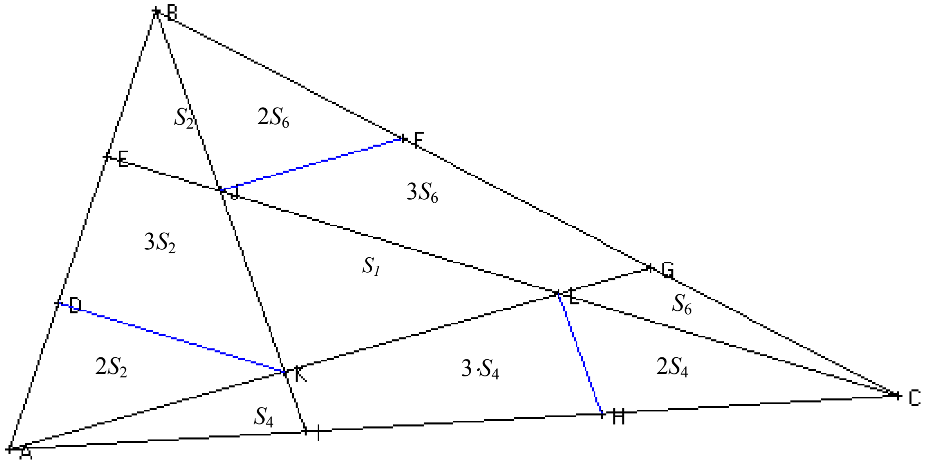


Figure 4: Relations among areas

We shall now prove that: (2) $S_2 = S_4 = S_6$

Since $\text{area}(BAI) = \text{area}(ACG) = \text{area}(CBE) = \frac{1}{3}\text{area}(ABC)$ it follows that:

$6 \cdot S_2 + S_4 = 6 \cdot S_4 + S_6 = 6 \cdot S_6 + S_2 \Rightarrow \ 5 \cdot S_4 = 6 \cdot S_2 - S_6 \ ; \ 6 \cdot S_4 = 5 \cdot S_6 + S_2$

Thus: $S_4 = 6 \cdot S_6 - 5 \cdot S_2$. Therefore:

$6 \cdot S_4 + S_6 = \ 6 \cdot (6 \cdot S_6 - 5 \cdot S_2) + S_6 = 6 \cdot S_6 + S_2 \Rightarrow$

$36 \cdot S_6 - 30 \cdot S_2 + S_6 = 6 \cdot S_6 + S_2; \quad 31 \cdot S_6 = 31 \cdot S_2 \Rightarrow S_2 = S_6.$

Now: $S_4 = 6 \cdot S_6 - 5 \cdot S_2 = 6 \cdot S_6 - 5 \cdot S_6 = S_6$.

Hence (2) $S_2 = S_4 = S_6$ is proved.

Following the above we obtain: $S_3(1) = S_5(1) = S_7(1)$, $S_3(2) = S_5(2) = S_7(2)$, which imply that we also proved (3) $S_3 = S_5 = S_7$,

We shall now show that: (4) $\frac{S_1}{S_2} = 3$.

Proof:

$\Delta ADK \approx \Delta AEL \Rightarrow 4 \cdot \text{area}(ADK) = \text{area}(AEL)$, hence if $\text{area}(ADK) = 2 \cdot S_2$ than $\text{area}(AEL) = 8 \cdot S_2$. Thus: $\text{area}(AEL) = 2 \cdot S_2 + 3 \cdot S_2 + S_1 = 8 \cdot S_2 \ \Rightarrow S_1 = 3 \cdot S_2$, and (4) is proved.

The relations obtained are summarized in Fig. 5.

It is now left to prove (5) $\frac{BK}{KI} = \frac{AL}{LG} = \frac{CJ}{JE} = 6$. Clearly:

$$\frac{\text{area}(BKA)}{\text{area}(AKI)} = \frac{\frac{BK \cdot h}{2}}{\frac{KI \cdot h}{2}} = \frac{6 \cdot S_2}{S_2} = 6 \Rightarrow \frac{BK}{KI} = 6 .$$

And $\frac{AL}{LG} = \frac{CJ}{JE} = 6$, stems from symmetry considerations.

Thus we complete the proof for all the connections that were discovered for the case of $n = 3$.

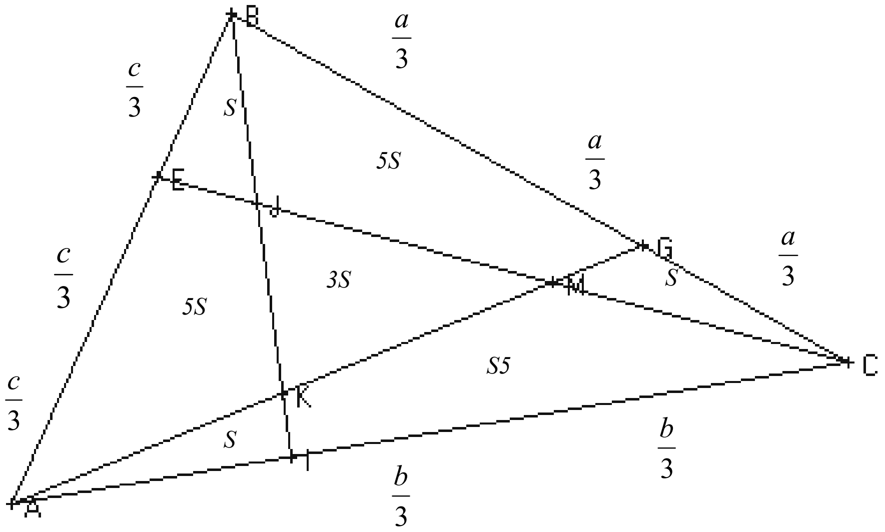

B
$\frac{a}{3}$
$\frac{c}{3}$
S
5S
$\frac{a}{3}$
E
J
G
$\frac{a}{3}$
$\frac{c}{3}$
3S
M
S
5S
C
S5
$\frac{c}{3}$
K
S
I
$\frac{b}{3}$
A
$\frac{b}{3}$

Figure 5: relations among areas of the case: n=3

3. The general case

We shall now examine the general case, in which $n = k$.

For the general case we will show that the following patterns hold:

$$\frac{JK}{BJ} = k - 2 \tag{10}$$

$$S_2 = S_4 = S_6 \tag{11}$$

$$S_3 = S_5 = S_7 \tag{12}$$

$$\frac{S_1}{S_2} = k(k\text{-}2)^2 \tag{13}$$

$$\frac{BK}{KI} = k(k-1) \tag{14}$$

The terminology refers to Figure 6 and Figure 3.

In order to prove (10)-(14) we join *KD*, *LH* and *JF* to generate triangles *BDK*, *AHL* and *CF*.

E is the $\frac{1}{k}$ – point of *BD*, *I* is the $\frac{1}{k}$ – point of *AH* and *G* is the $\frac{1}{k}$ – point of *CF* (Fig. 6).

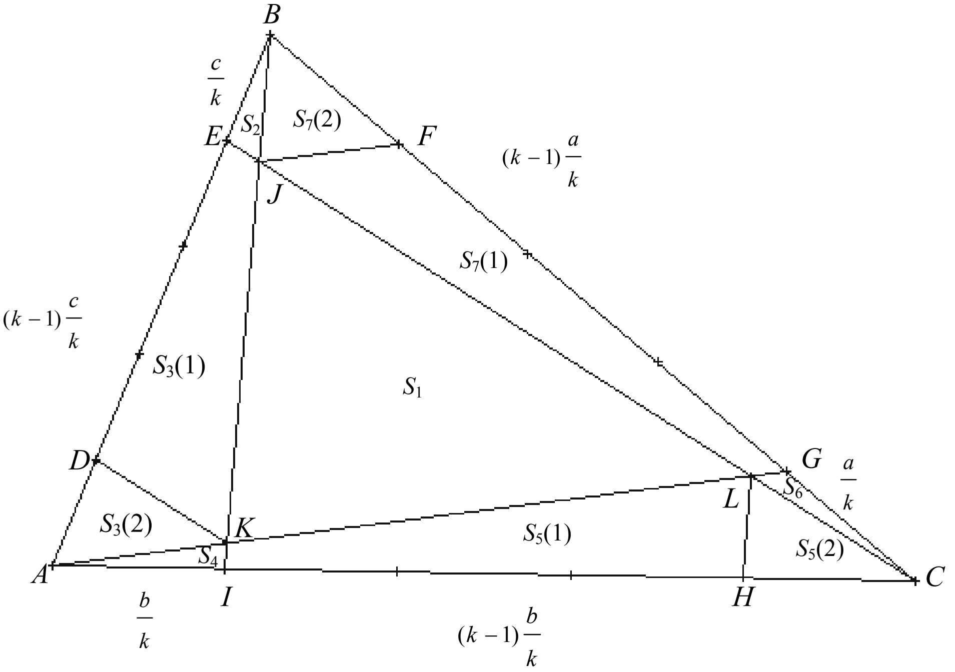

B
$\frac{c}{k}$
E
S_2
$S_7(2)$
F
$(k-1)\frac{a}{k}$
J
$S_7(1)$
$(k-1)\frac{c}{k}$
$S_3(1)$
S_1
D
G
$\frac{a}{k}$
L
S_6
$S_3(2)$
K
$S_5(1)$
$S_5(2)$
A
S_4
C
$\frac{b}{k}$
I
H
$(k-1)\frac{b}{k}$

Figure 6: Schematic description of the case of *n*=*k*

We first prove that: $EJ \| DK$; $IK \| HL$; $GL \| FJ$.

Proof:

We employ Affine Geometry to prove this claim:

Let AC be on the x-axis, and AB on the y-axis. The unit scale on the x-axis is the length of AC and the unit scale on the y-axis is AB. Consequently, the coordinates of the vertices are:

$A(0,0)$; $B(0,1)$; $C(1,0)$; $E(0,\frac{k-1}{k})$; $I(\frac{1}{k},0)$; $G(\frac{k-1}{k},\frac{1}{k})$; $F(\frac{1}{k},\frac{k-1}{k})$

For the equation of CE we get:

$$\frac{x-x_E}{x_C-x_E}=\frac{y-y_E}{y_C-y_E}\Rightarrow\frac{x-0}{1-0}=\frac{y-\frac{k-1}{k}}{0-\frac{k-1}{k}}\Rightarrow y_{CE}=\frac{k-1}{k}-\frac{k-1}{k}x$$

And for BI:

$$\frac{x-x_B}{x_I-x_B}=\frac{y-y_B}{y_I-y_B}\Rightarrow\frac{x-0}{\frac{1}{k}}=\frac{y-1}{0-1}\Rightarrow y_{BI}=1-kx$$

Therefore, for the coordinates of J ($BI\cap CE$) we get:

$$\left.\begin{array}{l} y=\frac{k-1}{k}-\frac{k-1}{k}x \\ y=1-kx \end{array}\right\}\Rightarrow x=\frac{1}{k^2-k+1}\Rightarrow y=\frac{(k-1)^2}{k^2-k+1}\Rightarrow J(\frac{1}{k^2-k+1},\frac{(k-1)^2}{k^2-k+1})$$

Vectors $\overrightarrow{JF}$ and $\overrightarrow{AG}$ are:

$$\overrightarrow{JF}=(x_F-x_J,y_F-y_J)=(\frac{1}{k}-\frac{1}{k^2-k+1},\frac{k-1}{k}-\frac{(k-1)^2}{k^2-k+1})=(\frac{(k-1)^2}{k(k^2-k+1)},\frac{k-1}{k(k^2-k+1)})\ ;$$
$$\overrightarrow{AG}=(x_G-x_A,y_G-y_A)=(\frac{k-1}{k}-0,\frac{1}{k}-0)=(\frac{k-1}{k},\frac{1}{k})$$

Thus we get: $\overrightarrow{JF}=\frac{k-1}{k^2-k+1}\cdot\overrightarrow{AG}$, and hence vectors $\overrightarrow{JF}$ and $\overrightarrow{AG}$ are parallel.

By symmetry considerations $IK\|HL$ and $DK\|EJ$. Thus we have proved that:

(10) $JK=(k-2)\cdot BJ$. Similarly $KL=(k-2)\cdot AK$; $LJ=(k-2)\cdot CL$.

We will now prove (11) $S_2=S_4=S_6$.

From the parallelism it follows that $\Delta BEJ \approx \Delta BDK; \Delta AIK \approx \Delta AHL; \Delta CGL \approx \Delta CFJ$ with a similarity ratio $\frac{1}{(k-1)}$

In addition, since area (BAI) = area (ACG) = area (CBE) $= \frac{1}{k}$ area(ABC)

we get: $\frac{S_3(1)+S_2}{S_2} = \frac{S_5(1)+S_4}{S_4} = \frac{S_7(1)+S_6}{S_6} = (k-1)^2 \Rightarrow S_3(1) = (k^2-2k)\cdot S_2$.

Similarly, $S_5(1) = (k^2-2k)\cdot S_4;\ S_7(1) = (k^2-2k)\cdot S_6$

Furthermore, $(k-1)\cdot S_3(2) = S_2 + S_3(1) \Rightarrow S_3(2) = (k-1)\cdot S_2$.

Since area(BKD) = (k-1)· area(DKL) then $\dfrac{\frac{BD\cdot h}{2}}{\frac{AD\cdot h}{2}} = k-1$. ($k \geq 2$)

As a result: $(k-1)\cdot S_5(2) = S_4 + S_5(1) \Rightarrow S_5(2) = (k-1)\cdot S_4$ and

$(k-1)\cdot S_7(2) = S_6 + S_7(1) \Rightarrow S_7(2) = (k-1)\cdot S_6$

From the above relations we get: $(k^2-k)\,S_2 + S_4 = (k^2-k)\cdot S_4 + S_6 = (k^2-k)\cdot S_6 + S_2$.

Thus: $S_4 = \frac{(k^2-k)\cdot S_2 - S_6}{k^2-k-1} \Rightarrow \frac{(k^2-k)\{k^2-k)\cdot S_2 - S_6\}}{k^2-k-1} + S_6 = (k^2-k)\cdot S_6 + S_2$

$(k^4-2k^3+k+1)\cdot S_2 = (k^4-2k^3+k+1)\cdot S_6 \Rightarrow S_2 = S_6$

By symmetry considerations $S_2 = S_4$. Hence (11) $S_2 = S_4 = S_6$.

Therefore (12) $S_3 = S_5 = S_7$ is also proved.

Consequently $S_3(1) = S_5(1) = S_3(1);\ S_3(2) = S_5(2) = S_3(2)$

We shall now prove that: (13) $S_1 = k(k-2)^2\cdot S_2$

Proof:

$$\frac{area\,(AEL)}{area\,(ADK)} = (k-1)^2 \Rightarrow \frac{S_3(1)+S_3(2)+S_1}{S_3(2)} = \frac{(k-1)\cdot S_2 + (k^2-2k)\cdot S_2 + S_1}{(k-1)\cdot S_2}$$

$$S_1 = (k-1)^3 \cdot S_2 - (k-1) \cdot S_2 - (k^2 - 2k) \cdot S_2 \Rightarrow S_1 = (k^3 - 4k^2 + 4k) \cdot S_2 = k(k-2)^2 \cdot S_2$$

Finally we prove that: (14) $\frac{BK}{KI} = \frac{AL}{LG} = \frac{CJ}{JE} = k(k-1)$.

Proof:

We will use the connection:

$\text{Area}\,(ABK) = S_2 + S_3(1) + S_3(2) = S_2 + (k^2 - 2k) \cdot S_2 + (k-1) \cdot S_2 = (k^2 - k) \cdot S_2$

area (AKI) = S_2

Hence: $\frac{area(ABK)}{area(AKI)} = \frac{\frac{BK \cdot h}{2}}{\frac{KI \cdot h}{2}} = \frac{(k^2 - k) \cdot S_2}{S_2} = k(k-1) \Rightarrow \frac{BK}{KI} = k(k-1)$.

From symmetry considerations we get: $\frac{AL}{LG} = \frac{CJ}{JE} = k(k-1)$.

The findings can be summarized as follows:

The three *k*-ians of a triangle divide the it into seven sections. The relations between the measures of the areas are described in Figure 7.

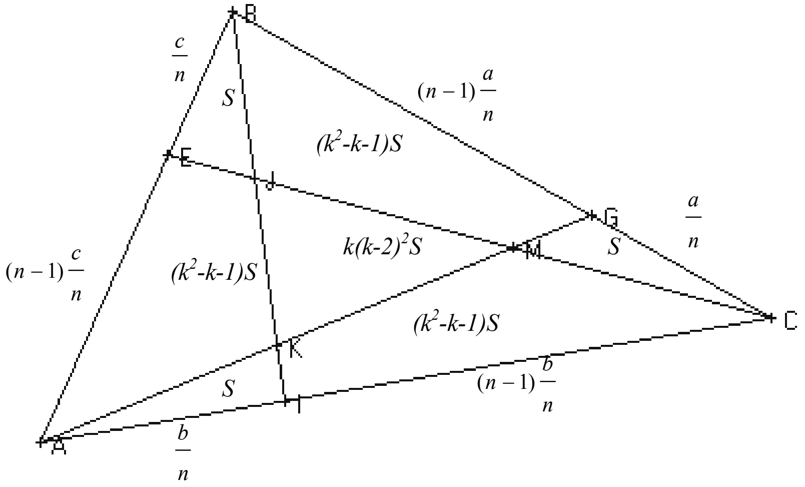

B
$\frac{c}{n}$
S
$(n-1)\frac{a}{n}$
$(k^2\text{-}k\text{-}1)S$
E
J
G
$\frac{a}{n}$
$(n-1)\frac{c}{n}$
$k(k\text{-}2)^2S$
M
S
$(k^2\text{-}k\text{-}1)S$
C
$(k^2\text{-}k\text{-}1)S$
K
S
$(n-1)\frac{b}{n}$
I
A
$\frac{b}{n}$

Figure 7: relations among areas of the case n=k

5. Theorem concerning *k*-ians of triangle

Employing the WIN strategy once again, each side of the triangle can be divided into any number, p, q and r, of equal segments. Vertex A is connected to the $\frac{1}{p}$-point, vertex B is connected to the $\frac{1}{q}$-point, and vertex C is connected to the $\frac{1}{p}$-point, as shown in Fig. 8.

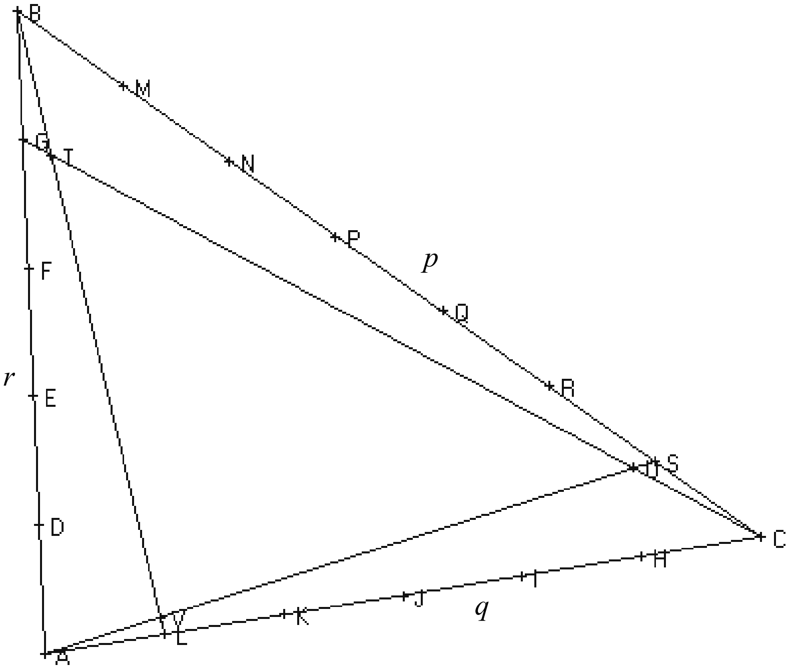


Figure 8: schematic description of the *k*-ians

In this case we get: $\frac{BY}{YL}=(p-1)\cdot q$; $\frac{AU}{US}=(r-1)\cdot p$; $\frac{CT}{TG}=(q-1)\cdot r$.

Proof:

Employ again Affine Geometry to prove this claim:

Let AC be on the x-axis, and AB on the y-axis. The unit scale on the x-axis is the length of AC and the unit sale on the y-axis is AB. Consequently, the coordinates are:

$A(0,0)$; $L(\frac{1}{q},0)$; $B(0,1)$; $C(1,0)$

The vectors: $\vec{AC}=\{1,0\}$, $\vec{CB}=\{-1,1\}$; $\vec{CS}=\{-\frac{1}{p},\frac{1}{p}\}$; $\vec{AS}=\vec{AC}+\vec{CS}=\{1-\frac{1}{p},\frac{1}{p}\}$

The equations of lines BL and AS: $BL: y-1=-qx$; $AS: y=\frac{x}{p-1}$.

The coordinates of Y: $Y=BL\cap AS=(\frac{p-1}{1+q(p-1)},\frac{1}{1+q(p-1)})$

The vectors $\vec{B}Y$ and $\vec{Y}L$ are:

$$\vec{BY}=\{\frac{p-1}{1+q(p-1)},\frac{-q(p-1)}{1+q(p-1)}\}=q(p-1)\cdot\{\frac{1}{q(1+q(p-1))},\frac{-1}{1+q(p-1)}\}$$

$$\vec{YL}=\{\frac{1}{q}-\frac{k-1}{1+q(p-1)},\frac{-1}{1+q(p-1)}\}=\{\frac{1}{q(1+q(p-1))},\frac{-1}{1+q(p-1)}\}$$

The last two results imply that $\vec{BY}=q(p-1)\cdot\vec{YL}$. We leave to the reader to verify that $AU=(r-1)p\cdot\vec{US}$; $CT=(q-1)r\cdot\vec{TG}$.

In Addition, we urge the reader to look for relations among areas that are formed as a consequence of the new division.

Implication for class activities

In this paper we describe a process which can be implemented on various well known mathematical theorems. Utilizing the WIN strategy, which is a useful tool which can easily be applied, combined with the working in an interactive computerized environment, enables the formulation of various inquiry activities such as the example given in this paper. Such an activity could be given as a long term project for developing inquiry skills and mathematical knowledge.

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Mathematical curiosities about division of integers

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As mathematics educators, our focus of attention is mainly placed on the learning and teaching of mathematics. But, as we study phenomena of mathematical learning and teaching, we often come across intriguing *mathematical* phenomena that capture our interest. We find ourselves often bouncing mathematical ideas back and forth, not just looking for (new/better) ways of teaching or presenting a mathematical concept, but also of uncovering and discovering potential understandings of the concept. These mathematical issues we encounter represent for us a significant aspect of our work, and are also very stimulating. One of these issues arose for us as we were tackling issues of division of numbers and of conventions relating to the remainder; issues that are, mathematically speaking, as we hope to communicate, very interesting and thought provoking. Thus, we explore four different avenues/curiosities about division, where operations with positive *and* negative numbers are considered, as well as the meaning one can draw out of these operations.

Curiosity 1: Division, integers and conventions

Let's take a very simple division, like $18 \div 4$. One answer to this operation is "4 remainder 2." That said, what about $3r6$, $2r10$, $5r^{-}2$? The usual answer when dividing numbers requires one to ask how many times does 4 go into 18, and then describe what is leftover as the remainder after having taken out all the 4's you can from 18. Thus, in this case, $18 \div 4 = 4r2$. However, one could argue that all four answers given above are equivalent and make sense mathematically. Indeed, they are all mathematically correct and represent an understanding that division represents a partitioning of a number (the dividend) into equally sized parts (the divisor), where in some answers the dividend has not been fully partitioned. For $3r6$, the number 18 has had three groups of size four taken out with six parts remaining, which can be represented by $18 = 3 \cdot 4 + 6$. This is correct, but the idea of taking out as many 4's as possible is not yet complete.

With this in mind, as Brown (1981) explains, the division algorithm respects some conventions, given by its definition, since the remainder (r) is defined as, and needs to be, between zero and the divisor (i.e., $0 \leq \textit{remainder} <$ divisor). Thus, 18 ÷ 4 gives $4r2$ and not $3r6$, even though both are conceptually acceptable. With infinitely many possibilities in any division

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problem, and in order to use the procedure appropriately, some conventions have to be respected. The issue of convention and definition play a significant role in the answer. Therefore, the last three answers ($3r6$, $2r10$, $5r\,^{-}2$) would be ruled out because of the mathematical convention, just as $16 \div 4 = 3r4$ would be ruled out even if it is conceptually adequate.[3]

With this rule or convention established for the remainder, what would now happen if negative numbers are used? If we attempt to calculate $^{-}18 \div 4$ by using the above convention, we obtain $^{-}18 \div 4 = {}^{-}5r2$ (and not $^{-}18 \div 4 = {}^{-}4r\,^{-}2$). This seems counterintuitive in comparison to our calculations for the previous example 18 ÷ 4. As we could not go "over" or beyond 18 when calculating $18 \div 4$ (e.g., with $18 \div 4 = 5r\,^{-}2$), in the case of $^{-}18 \div 4 = {}^{-}5r2$ we do. Again the mathematical convention guides the way in which division and its algorithm are to be used. In that sense, one needs to know the adequate mathematical convention in order to obtain a mathematically acceptable answer, even if alternatives are conceptually meaningful.

But, again, what would happen for $18 \div {}^{-}4$? If we attempt to follow the convention, we need to have the remainder lying between 0 and the divisor. Hence $0 \leq remainder < {}^{-}4$, which is mathematically impossible. So, analyzing two potential answers, we obtain $18 \div {}^{-}4 = {}^{-}4r2$ or $18 \div {}^{-}4 = {}^{-}5r\,^{-}2$. In both answers the remainders 2 or $^{-}2$ are bigger than $^{-}4$. The only answer that could satisfy the requirement that the remainder be smaller than $^{-}4$ would be $18 \div {}^{-}4 = {}^{-}6r\,^{-}6$, an answer that clearly goes "over" 18 and that appears conceptually acceptable, but would still be inadequate because $^{-}6$ is smaller than 0, the lower bound for the remainder.

Through browsing and searching different definitions for the remainder, one way that we have found to step away from this inconsistency for various cases of signed numbers is to redefine the remainder in terms of the divisor's absolute value: $0 \leq remainder < |\text{divisor}|$.[4] In this case, $18 \div {}^{-}4 = {}^{-}4r2$ where the remainder is both bigger than 0 and smaller than $|{}^{-}4|$. But this step, as often happens in mathematics (Hersh, 1987; Lakatos, 1976), requires reworking the definition; in this case for what a reminder is. Notice also in this case that the answer does not require us to go "over" 18 as was done for $^{-}18 \div 4$. That said, what about $^{-}18 \div {}^{-}4$? With the new definition of the remainder, we obtain $^{-}18 \div {}^{-}4 = 5r2$, which goes "over" $^{-}18$. We therefore obtain two cases where the product of the quotient and divisor go "over" or beyond the dividend, and two cases where the product stays "under" or below the dividend.

On an interesting note, one could argue that each time we claimed to go "over" $^{-}18$ in the divisions, we in fact obtained a number that was "under" $^{-}18$ (by attaining a smaller number than it). For example, $^{-}18 \div 4 = {}^{-}5r2$ resulted in $^{-}20 + 2$, and $^{-}18 \div {}^{-}4 = 5r2$ resulted also in $^{-}20 + 2$. We explore this issue in the next sections, as we attempt to understand what these computations mean conceptually and how we can contextualize them.

This sort of interplay of convention and concepts is often hidden within the procedures we use, or even is taken for granted as part of the conceptual understanding of it. In this case, we are able to see the mathematical richness in digging deeper to understand the role that the conventions and algorithms are playing in the answers we obtain, both in regard to the concept itself and the conventional way of reporting it.[5] This raises, we think, the interest in pulling these mathematical notions apart and exploring them in depth, and represents, as Brown (*ibid.*) suggests, "one way in which we can relate elementary and advanced knowledge of the discipline so that both perspectives are enriched rather than destroyed by the linkages" (p. 13).

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[3] Brown (1981) explores the meaning of these various possibilities after one child, Sharon, offered him a similar answer to a division procedure.

[4] Another option could have been to remove the lower bound, as Brown (1981) does. But, in this case with integers, we wanted to explore the décision of taking the divisor's absolute value as a lower bound.

Curiosity 2: Conceptualizations of division

Several metaphors or conceptualizations exist for dividing numbers. Here we work with three of these ideas in the context of the examples presented above. In particular, we look at division as a measurement concept and as a partitioning concept. Then, as division is the inverse of multiplication, we explore the connections between multiplication as repeated addition and division as repeated subtraction and how these conceptualizations can become quite difficult to make sense of when working with integers.

One of the first conceptualizations of division is that of a measurement problem, where for the problem 18 ÷ 4 we can think of asking 'how many groups of size four can be found in 18 things?' Here we are trying to find the *number of groups* when we already know the size of each group. In the second conceptualization, partitioning, we can think of 18 ÷ 4 as asking 'if four people were to share 18 things equally, how many things would each person get?' Here we are trying to find the *size of each group* when we know how many groups we have. The results will be the same for each conceptualization, but represent something different depending on how one approaches the problem (see, e.g., Hart, 1981; Simon, 1993). However, both of these conceptualizations can be difficult to make sense of as they sometimes break down when working with integers while at the same time continuing to adhere to the convention for the remainder.

Case 1: 18 ÷ 4: In this case, where both the dividend and the divisor are positive, both conceptualizations are simple to apply. For the measurement concept, as was mentioned, we can ask 'if we have 18 things and 4 things are given out at once, how many people/groups will be given four things?' Here the answer $4r2$ tells us that four people will receive four things and we will have two things left over. For the partitioning concept, we ask 'if we have 18 things to be given equally to four people, how many things will each person have?' The answer $4r2$ denotes that each person will be given four things and we will have two things left over. In both of these conceptualizations, going "over" the quantity of 18, as was done previously to satisfy conventions, does not make sense because we cannot give out more things than we have.

Case 2: $^{-}18 \div 4$: In this case, the conceptualizations of division become problematic. In particular, from the measurement perspective, we ask the question 'if we have $^{-}18$ things and four things are given out at once, how many people/groups will be given four things?' It is difficult to imagine having $^{-}18$ things. And, more importantly, when we complete the problem with the result $^{-}18 \div 4 = {}^{-}5r2$, the $^{-}5$ represents the number of groups that have each been given four things. But how can we have $^{-}5$ groups? This is hard to imagine, and it has haunted mathematicians for years in the historical developments of negative numbers!

Under the partitioning concept, we can make a bit more sense of this problem. From this approach, we ask 'if we have $^{-}18$ things to be given equally to four people, how many things will each person have?' We can alter this question slightly to be a financial question, as we often do to ease one's understanding. We ask 'if four people owe $18 and they are to split the debt evenly, how much money will they each owe?' This makes sense on a conceptual level, where debt is represented by a negative value. However, when we look at the answer ¯5*r*2, again ¯5 could be said to be problematic since the debt is overpaid, though it helps to make sense division-wise, and it acts as a viable option to cover the entire expense! That said, for the next two cases, things become even trickier.

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[5] Something reminiscent of Davis (1973) and Brown (1981) exploration of non-standard ways of children for subtracting and dividing numbers, as well as Kieren's (1999, 2004) "missing fraction mysteries" task where children had to find fractions between ¼ and ¾ and began writing fractions like 5.9/6, 5.99/6, 5.999/6, etc.

Case 3: 18 ÷ ¯4: What are we to do in this case? Under the idea of division as measurement, the question becomes 'how many groups of size ¯4 can we take out of 18 things' and within the idea of division as a partitioning, we ask 'how will negative four people share 18 things?' Taking negative things out of positive ones does not make sense, nor does the idea that we have negative people. In the case of a positive dividend and negative divisor, the conceptualizations we are working with here do not help to make sense of the calculations and are unreasonable.

Case 4: ¯18 ÷ ¯4: The measurement conceptualization is interesting with the example ¯18 ÷ ¯4. The question we ask here is 'how many groups of size ¯4 can be found in ¯18 things?' Here we find that ¯18 can be divided up into groups of ¯4. When we do this, we see that we have four groups of ¯4. If we stop there, however, we have ¯2 remaining. And, as we mentioned previously, while this is mathematically correct, it is not appropriate in regard to the convention for the remainder, giving 4*r*¯2. So, we have to take out another group of ¯4, so this leaves us with five groups of ¯4 and a surplus remainder of 2 (i.e., 5*r*2). Unfortunately here, however, going "over" makes the conceptualization a bit hazy, whereas 4*r*¯2 makes more sense.

The partitioning concept in this case is a bit more difficult to see. We ask 'if we have ¯18 things to be given to negative four people, how many will each person receive?' Again, the idea of having negative people or a negative entity that is supposed to receive something is hard to imagine. The partitioning concept appears limited in helping to make sense of this case.

Turning our attention to the connection between multiplication and division, we obtain additional, yet different ways of making sense of division. Multiplication is often presented as repeated addition. If we have 3 · 4, we can write this as 4 + 4 + 4. Since multiplication and division are closely connected as inverse operations of each other, if multiplication is repeated addition then division can be seen as repeated subtraction. This way of seeing division, fruitful in cases when the dividend and the divisor are of same sign, becomes quite complicated in the other cases presented above.

For example, 18 ÷ 4 can be solved in the following way 18 – 4 = 14; 14 – 4 = 10; 10 – 4 = 6; 6 – 4 = 2 at which we stop because we can not take out another 4 (because it would lead to ¯2). Our result is then 4*r*2. A similar thing happens when we have ¯18 ÷ ¯4. Here we have ¯18 – (¯4) = ¯14; ¯14 – (¯4) = ¯10; ¯10 – (¯4) = ¯6; ¯6 – (¯4) = ¯2; and finally ¯2 – (¯4) = +2 (if we accept going "over" ¯18).

However, when the dividend and divisor are of opposite signs, this conceptualization of division becomes problematic. Let's look at these two possibilities. Under the idea of repeated subtraction ¯18 ÷ 4 becomes ¯18 – 4 = ¯22; ¯22 – 4 = ¯26; ¯26 – 4 = ¯30; and so on. The result becomes more negative and we can 'pull out' infinitely many 4's from the number ¯18 without ever closing in on an answer. 18 ÷ ¯4 appears similarly troublesome. 18 – (¯4) = 22; 22 – (¯4) = 26; 26 – (¯4) = 30; and so on. In this case, our answer becomes more positive as we 'take out' ¯4's from 18 and again do not succeed on closing in an answer. Additionally, aside from the computations here, this appears difficult to conceptualize as how is one to remove negative quantities from a positive quantity, and vice-versa?

Thus, what the three different conceptualizations offer us is the fact that each of them can help to some extent in making more sense of the operations of division of integers, but they are also limited. There does not seem to exist a definitive conceptualization working for all of these examples as all have their limits and need to be reflected upon – and we believe it is in the thinking through that they become mathematically interesting and significant. That said, other aspects to look into concern the play with numbers, independently of any context or constraints. We look at this through the use of simple calculators in the next section.

Curiosity 3: Division and calculators

As we saw above, a purely conceptual approach made some aspects of division difficult to make sense of for different cases of integers. This raises the issue of exploring the numbers themselves and ideas of dividing integers and the remainder with a calculator. Again, in this calculator-context, we look at some of the previously explored outcomes and how these sometimes connect and sometimes don't connect to results obtained through using a calculator. For example, in the case of $18 \div 4$ the calculator produces the result 4.5, which makes sense for the remainder. One can look at it in the following way: $18 \div 4 = 4.5 = 4 + 0.5 = 4 + \frac{2}{4}$, where the 2 of $\frac{2}{4}$ was our remainder from the previous exploration of this case. Thus, the decimal number result coincides with the remainder result.

In the case of ${}^{-}18 \div 4$, the calculator produces the result ${}^{-}4.5$. Compare this to our previous result that followed the convention for the remainder where we had ${}^{-}18 \div 4 = {}^{-}5r2$. Clearly the algorithm for division in the calculator is not following the convention for the remainder as the decimal portion of the result represents ${}^{-}0.5$, which is ${}^{-}\frac{2}{4}$, where ${}^{-}2$ is the remainder. Thus, a question arises 'are these the same result numerically?' Our previous answer of ${}^{-}5r2$ can be re-written as ${}^{-}5 + \frac{2}{4} = {}^{-}5 + 0.5 = {}^{-}4.5$. So, these different approaches yield the same answer numerically, yet they go about finding and representing the solution differently.

Similar to the example above, for $18 \div {}^{-}4$ a calculator gives ${}^{-}4.5$ as a result. Applying the convention for the remainder to this problem yields ${}^{-}4r2$. This is quite interesting in that while the calculator's results are identical for both $18 \div {}^{-}4$ and ${}^{-}18 \div 4$, the algorithm for division does not give the same result for them. Again, looking at the result ${}^{-}4r2$ one can wonder if the result is the same as ${}^{-}4.5$. Note here that ${}^{-}4r2$ can be written as ${}^{-}4 + \frac{2}{-4}$, where 2 is the remainder and ${}^{-}4$ is the divisor. This then becomes ${}^{-}4 + ({}^{-}0.5) = {}^{-}4.5$, which is indeed the same as our calculator's calculation.

For the final case of ${}^{-}18 \div {}^{-}4$, the calculator offers 4.5 as a result. Our previous work with this problem and the remainder gave us the result $5r2$. These results don't appear to be the same. But if we look at $5r2$, this can be written as $5 + \frac{2}{-4} = 5 + ({}^{-}0.5) = 4.5$, leading to the same numerical value in the end, but coming from different answers. These issues for calculators and of considering numbers only for themselves have in fact led us to consider issues about the long division algorithm, how it can function in these cases and how one can make sense of it in relation with integers. We explore this as our next and last curiosity.

Curiosity 4: Long division algorithm

This final curiosity about the long division algorithm ties back to the conventions about the remainder, and has an obvious connection to the above explorations on calculators. The long division algorithm is peculiar in the sense that there are not necessarily conventions attached to it, but rather there are specific steps that one needs to follow to obtain the answer. For example, with two positive numbers the steps to solve 18 ÷ 4 with long division looks like the following. Step 1 (**Figure 1a**): How many times does 4 go into 18? 4 times. Then, we multiply 4 by 4 and obtain 16. 18 minus 16 gives 2. There are two options here for Step 2: one is to obtain the answer in terms of remainder, which gives 4*r*2, where the remainder 2 leads to $\frac{2}{4}$ or ½ (**Figure 1b**). The second choice is to opt for a decimal representation, leading one to place a decimal point after the 4 and add a zero after the two (**Figure 1c**).

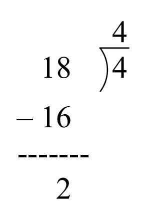

Figure 1a

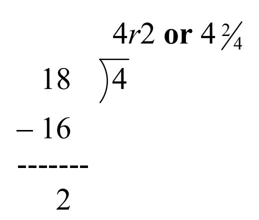

Figure 1b

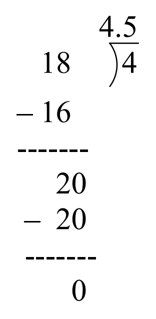

Figure 1c

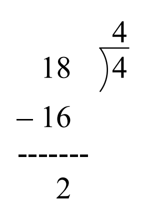
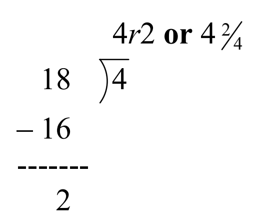
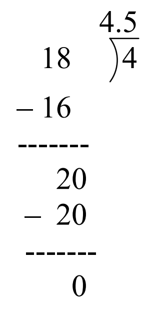
Figure 1a

Figure 1b

Figure 1c

Then, the question is ‘how many times does 4 go into 20?’ 5 times. 5 multiplied by 4 equals 20, 20 minus 20 gives 0. We could spend time explaining the ins and outs of this procedure, but because we want to underline other dimensions for the division, we assume the reader is aware of these rationales and the reasons why it functions. Obviously, the aspects we want to emphasize concern the play with integers and how it causes us to take a step back and question the steps we are taking and the coherence of these steps. We approach this in a similar fashion as with the conventions in the first section.

Looking at $^{-}18 \div 4$ and using the same steps as above, we obtain the following. Step 1: How many times does 4 go into $^{-}18$? Right here, at Step 1, we have also two options. One option is to follow the same reasoning as for the convention and opt for $^{-}5$ and the other is to opt for $^{-}4$. Let’s have a look at the former (**Figure 2a**). $^{-}5$ times 4 gives $^{-}20$. $^{-}18$ minus $^{-}20$ gives $^{+}2$. At this stage, again, there are two options: stopping with the remainder or continuing on with decimals. If we stop with the remainder (**Figure 2b**), it gives $^{-}5r2$ where the remainder 2 leads to $^{2}\!/_{4}$. But, then the question becomes ‘is it $^{+}\,^{2}\!/_{4}$ or $^{-}\,^{2}\!/_{4}$?’ Taking $^{+}\,^{2}\!/_{4}$ seems counter-intuitive, as the quotient and its value created with the remainder would not be of the same sign ($^{-}5$ and $^{+}\,^{2}\!/_{4}$). However, taking $^{-}\,^{2}\!/_{4}$ would mean $^{-}(5\,^{2}\!/_{4})$ and this is clearly wrong. The same thing happens if we opt for decimals (**Figure 2c**), as it gives $^{-}5$ and “.5”. Is the “.5” positive or negative? In other words, is it “$^{-}5 + {}^{+}.5$” giving $^{-}4.5$ or is it “$^{-}5 + {}^{-}.5$” giving $^{-}5.5$? The former, $^{-}4.5$, is definitely the answer, which means that the various quotient values need to be computed (added) in order to find the final answer. Therefore, the decimal point “$^{+}.5$” does not belong to the quotient $^{-}5$, but stands on its own and has its own sign (in this case, positive). This obviously does not happen when it is only positive numbers, as all quotients have the same sign – it also illustrates the mathematical richness underlying these operations as we address later. Therefore, the answers to $^{-}18 \div 4$ are “$^{-}5 + {}^{+}.5$” or “$^{-}5 + {}^{+}\,^{2}\!/_{4}$”, leading to $^{-}4.5$ or $^{-}(4\,^{2}\!/_{4})$.

¯5 ¯18)4 – ¯20 ------- 2	¯5*r*2= ¯5 ¯2⁄4 **or** ¯5 +2⁄4 ¯18)4 **??** – ¯20 ------- 2	¯5.5 **or** “¯5 + .5” ¯18)4 **??** – ¯20 ------- 20
Figure 2a	**Figure 2b**	**Figure 2c**

This leads us to the second route in solving this problem, which is taking ¯4 for the quotient (**Figure 3a**). ¯4 times 4 gives ¯16. ¯18 minus ¯16 gives ¯2. Again there are two options at this point: stopping with the remainder or continuing with decimals. The remainder option gives ¯4*r*¯2 which means ¯4 and ¯2⁄4 (**Figure 3b)**. Here, because there is a sign attached to it, we know directly that both parts of the quotient obtained are the same sign and can be added together, giving ¯(4 2⁄4), the same answer we had above. For the decimals (**Figure 3c**), the question becomes ‘how many times does 4 go into ¯20?’ giving ¯5 as an answer. Here again, both parts of the quotient obtained are of the same sign, making it easy to see how they add and leading to the answer “¯4 + ¯.5” or ¯4.5. However, it appears quite unfamiliar to see a sign before the tenths place after the decimal point. Also, some could raise the issue, with reason, that we have not respected the procedure, since ¯16 is bigger than ¯18 and therefore we would have taken too many 4’s from ¯18; the impact of which is that we obtain ¯2 as the result of ¯18 – ¯16, something that should not happen as one is not supposed to obtain a negative number at this stage since it indicates to the solver that the number taken is too big. This is a very interesting argument because it requires that one rethink what it means “to take all there is to be taken from the dividend.” In this case, again, what appears important is the understanding and the mathematical rationale one develops, and not the steps one follows.

$\overline{^{-}4}$	$\overline{^{-}4r\,^{-}2\ ;\ ^{-}4\ ^{-}\frac{2}{4}\ ;\ ^{-}(4\frac{2}{4})}$	$\overline{^{-}4.5}$
$^{-}18\ )4$	$^{-}18\ )4$	$^{-}18\ )4$
$-\ ^{-}16$	$-\ ^{-}16$	$-\ ^{-}16$
-------	-------	-------
$^{-}2$	$^{-}2$	$^{-}20$
Figure 3a	**Figure 3b**	**Figure 3c**

An issue arises again if we opt for a division that takes one additional step, for example in the case of $^{-}19 \div 4$. We won't go into all the details but we look into the subtleties that could happen if one decides to go "over" $^{-}19$. Thus, as in **Figure 4a**, our answer to the question 'how many times does 4 go into $^{-}19$?' is $^{-}4$ times, which leads to $^{-}19$ minus $^{-}16$ giving $^{-}3$. Here, one can take directly the remainder and obtain $^{-}4r\,^{-}3$ and then $^{-}(4\frac{3}{4})$, albeit of course the convention for the remainder is not respected. A curious aspect, however, resides in the decimal answers (**Figure 4b to 4e**). Here, after having positioned the decimal point and added the 0 to $^{-}3$ (giving $^{-}30$), one still has to consider two decisions: continuing to go "over" the number and then choosing $^{-}8$ to give $^{-}32$, or staying "under" and going with $^{-}7$ to give $^{-}28$. Of course, one could continue with steps similar to those previously taken with the quotient of $^{-}4$, that is, to not go "over." But, as we have seen, what appears most important is the meaning one gives to each step rather than the taking of these steps. In the case of going "over" (**Figure 4b**), we obtain $^{-}4$ and $^{-}.8$, and with $^{+}2$ as a resultant of the operation. In the other case (**Figure 4c**), we obtain $^{-}4$ and $^{-}.7$, with $^{-}2$ as the resultant. The next step is interesting but tricky, since in the case of **Figure 4b** the question is 'how many times does 4 go into 20?', and in the **Figure 4c** the question is 'how many times does 4 goes into $^{-}20$?' Thus, in the latter case, as is reported in **Figure 4e**, we obtain $^{-}.05$ as an answer, leading to $^{-}4$ with $^{-}.7$ and $^{-}.05$ giving $^{-}4.75$ as the result for the division; all values obtained to form the final quotient being of the same sign. But, in the former case, as is reported in **Figure 4d**, we obtain $^{+}.05$ as the second decimal answer. This leads to the following sequence to obtain the resulting answer to the division: $^{-}4 + {}^{-}.8 + {}^{+}.05 = {}^{-}4.8 + {}^{+}.05 = {}^{-}4.75$. Both cases offer the same resulting value, albeit in different formats, but also require a different way of processing them as it could be easy to end up with $^{-}4.85$ for the answer in the case of **Figure 4d**. These represent insightful subtleties inherent to these operations that require one to pay important attention to the meaning of each step and calculation.

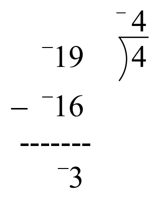

Figure 4a

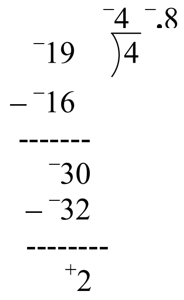

Figure 4b

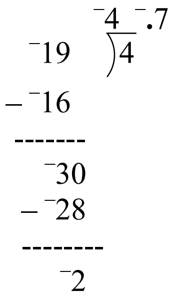

Figure 4c

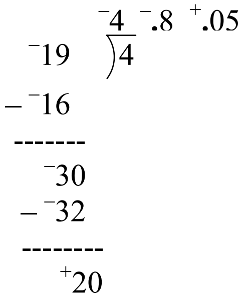

Figure 4d

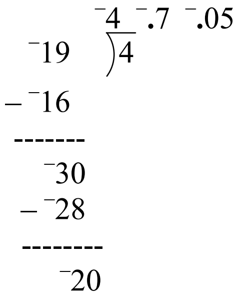

Figure 4e

What also appears fascinating and that emerges from issues of long division, as well as the play with calculators in the previous section, is the fact that the remainder is not considered alone in the production of an additional quotient, but gets assigned a "negative sign" when combined with the divisor. This leads to the realization that the sign of the numerical value produced by the combination of the remainder and the divisor needs to be reflected upon and is often taken for granted as giving a positive result. In these cases, as we have seen, the remainder is always connected to a divisor and the value of that additional part of the quotient takes a sign in relation to both. This is reminiscent of work done on comparison of fractions where a fraction can only be compared and understood in regard to its referent. Hart's (1981) study is famous for having asked students a question of the type: *If Mary spends ½ of her amount and Johnny spends the ¼ of it, who spent the most?* (see p. 72), leading students to consider that ½ and ¼ are in relation to something (½ of a small amount can be smaller than the ¼ of a large amount). Thus, as well, in the case of the remainder and divisor, the value produced that completes the division quotient is always in relation not only to the divisor and the remainder but also to the sign of both of these.

At this point, we have looked at two possibilities: $18 \div 4$ and $^{-}18 \div 4$. What happens with $18 \div {}^{-}4$ and $^{-}18 \div {}^{-}4$? Similar issues appear to pop-up as the play with the remainder requires that the solver pay attention to the signs attached to them, as well as consciously making the decisions to opt for going "over" or staying "under" for the first quotient when beginning the division. As a way of pushing your thoughts and developing your own ways of making sense of these, we do not explore these options here and leave them for you to try them out with both $18 \div {}^{-}4$ and ${}^{-}18 \div {}^{-}4$. As Descartes was famous for doing and announcing in his writing, we leave you the joy of working through these illuminating ideas on your own…

Concluding remarks

This paper raises an intriguing phenomenon that is present within other mathematical topics – that depending on the aspect we pay attention to (convention, conceptualizations, calculator, long division), the orientations taken sometimes make sense and sometimes do not. A fascinating aspect here is that, for the case of dividing integers, there does not appear to be a pattern present in the difficulties: each orientation helps to make sense of different type of division or hinders it (e.g., the conceptualization of measurement helped to make sense of ${}^{-}18 \div {}^{-}4$ but partitioning did not, whereas it was the opposite for ${}^{-}18 \div 4$; other simplification and difficulties emerged for long-division or conventions). What this means is (1) each operation can be clarified by some orientations but blurred by others. It does not appear that one sort of division was easier to make sense of through all the means and conceptualizations explored (except, of course, cases of positive divided by positive). And, (2) it illustrates all the attention one needs to pay to, and the mathematical richness one can draw from, these operations and ways of approaching them. These mathematical explorations of division with integers cannot be taken care of in a machine-like manner without deep mathematical thinking; they require important mathematical investments in the ideas by the solver. These are, therefore, rich mathematical contexts and situations to probe into.

All this makes us rethink issues of understanding of division, as often one will offer bigger and bigger numbers to verify one's understanding, assuming that if a person is able to operate on big numbers, then that person surely understands or even has demonstrated understanding of the concept at hand. We have offered here a different view in our explorations: that of staying with small numbers if one wishes to, but of digging into the concept itself through analysing its functioning and the meaning of the answers one obtains with integers.

These issues raise for us the significance of working on the exploration of mathematical concepts as a genuine activity of mathematics educators. Albeit this is not research *per se* in its traditional sense, yet these explorations have something to offer to our understanding of the very concepts that we work on with students in classrooms. We see it important to delve deeply into mathematical concepts and ideas, to understand the concepts, to make sense of what is happening, to gain a stronger footing in our own understanding of seemingly simple ideas. These sorts of mathematical developments of school mathematics appear here as initiatives driven to enhance our understandings of mathematics, a clear intention of all work being done in mathematics education.

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Helping Teachers Un-structure: A Promising Approach

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Abstract: The amount of overt structure in the presentation of a task affects students' engagement, creativity, and willingness to tolerate frustration. In a professional development project, with algebra teachers from nine American schools, we tried to help teachers make judicious decisions in their use of structure by having them facilitate low-structure tasks, remove structure from overly structured tasks, and observe "at-risk" students engaged in learning through low-structure tasks. Project schools that worked on structuring generally improved their algebra passing rates, both overall and for African-American students.

Keywords. Professional development, task structure, underrepresented minority students, US teachers, algebra, teacher change

Generally, people become teachers because they want to help others. They enjoy their work when through their effort and ingenuity, students actually learn and can do things they could not do before. Their instinct is to try to make things easier for their students; however, this desire to help can have the unintended effect of creating boring classrooms full of disengaged students. One of the major themes of our teacher development work has been to find effective ways to help teachers to structure their classroom tasks just enough so students are able use their creativity and inventiveness to reason their way to solutions.

Three of us were the co-directors of a National Science Foundation Math Science Partnership project, REvitalizing ALgebra, (REAL), which aimed to improve the performance of secondary students in elementary algebra and college students in remedial elementary algebra (Hsu *et al*, 2007a and 2007b). Our fourth author was the outside evaluator of the REAL program. Our specific goal was to improve the performance of students from underrepresented populations. We worked intensely with some lead teachers from six high school and two middle school math departments. There were two groups of about nine teachers who came together in successive years. Each group met for three hours a week during their first academic year and daily for three weeks the following summer. During their second academic year they met daily with other department members at their schools in an extra preparation period paid for by the NSF grant. There were about nine graduate students and nine undergraduate mathematics majors who also met with each group of teachers during their first year, but we will focus on the pre-college teachers.

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Six months before we began work with secondary teachers, we spent time in classrooms at their schools as "flies on the wall," putting ourselves in the shoes of the students. During those initial classroom visits, most teachers were asking questions that required short computations and little or no reasoning. On further inspection the problems, even those originally designed to be open to multiple solution methods, had been augmented with "scaffolding" which reduced the tasks to a series of small steps that required little thinking. For example, directions were added to tell students to make a chart and look for a pattern. Sometimes the directions even gave the column headings for the chart. In other cases explorations were limited. For instance, in a problem that originally asked students to come up with many kinds of function output patterns, the directions gave only linear patterns to "discover" and then broke down the process of finding the linear patterns into a to a step-by-step algorithm.

Based on our observations and on the work of researchers on engagement and success (National Research Council and the Institute of Medicine, 2004), we knew we needed to help teachers to get their students more engaged in learning and doing mathematics. We concluded that one way to get more students to succeed in algebra was to convince the teachers to structure their assignments differently, and generally to use less structure in the tasks they assigned both in class and for homework. Advantages of less structure can include:

1. more student creativity, flexibility, active problem solving, and the sense that mathematical struggle is an essential part of math and not something shameful;
2. more and higher quality mathematical discourse in student groups;
3. more student exploration, and assumption of responsibility for learning;
4. student belief that math is more than a small number of computations to be done quickly and a large number of problems whose solution methods must be memorized; and
5. more student engagement and interest in the mathematics!

On the other hand, we realized that by restricting the choices and creativity of students and directing their thinking to a prescribed solution method, teachers often felt:

6. more certainty about the mathematics being used and less anxiety about the complexity of managing different groups working in different ways;
7. more control over any resulting whole class interaction and greater ease of grading student work;
8. more certain of student confidence as they succeed at tasks a teacher thinks they can accomplish;
9. more control of the class, as students feel certain of how to start and which "direction" their thinking should take.

Our concern was that most of the lower-level algebra classrooms, where students from underrepresented populations had been tracked and where lessons were highly structured, lacked all of the benefits of (1)-(5) above and showed none of the positive aspects of (6)-(9) anticipated by teachers. We saw students who were unsure of themselves, unable to begin to work, and unable to take any risks. Their goal was to get the right answer, and they were unwilling or afraid to try something and learn from the consequences. Students hid their work from each other and gave up quickly when they didn't find answers right away. Teachers would explain how to do many of the problems, classes were boring, and students often exhibited their lack of engagement through disruptive behavior.

We were concerned that the heavy structure used in their mathematical tasks did not reflect a careful balance of advantages and disadvantages, but instead stemmed from:

a. teachers' fear of or aversion to letting their students struggle with a problem;
b. an unexamined belief that their students were not capable of succeeding at less structured tasks;
c. lack of awareness of the resulting gains and losses from structure choices;
d. teachers' own lack of experience with good, creative problem solving in a less structured task.

Based on our observations, we decided to make "structure" an important theme. In most meetings we discussed readings and movies that addressed (a) and (b), and to a lesser extent (c). In addition to reading about the advantages of reducing the amount of structure in problems teachers worked on several assignments dealing with questions of how much structure. They worked on low-structure mathematics problems themselves and then team-taught those problems to the other participants. We gave them overly structured problems to redesign using less structure, and they discussed a provocative movie of a lesson study where teachers improved a task by redesigning it with less structure for the students. In addition, some observed their peers who were using less structured tasks in their classes.

Un-Structuring Tasks, First Try

Experiencing low-structure tasks. Participants worked on low-structure math problems in groups every week. We used problems from the Interactive Mathematics Program, from other sources, and of our own invention. The key features of the problems were that they required some inventive thinking and exploration and rewarded multiple approaches. We acted as group work facilitators, challenging groups to justify their work and to explain their work to each other, and asking key questions when groups were stuck.

Teaching low-structure tasks. Once each semester, we divided into groups and gave each group a low-structure math problem that they would "teach" to subgroups of their classmates. They would, of course, first have to work on the tasks themselves. After the problem was sufficiently explored, one of us would facilitate their planning of what outcomes to aim for, how to guide the exploration, and what to anticipate. Then they would "teach" their problem to subgroups of 6 to 12 classmates. Finally, we helped them to reflect on their teaching experience.

These tasks were not solely about un-structuring, but we definitely wanted people to notice and enjoy the benefits of mathematical exploration in safe and encouraging environments. After a month of getting to know each other, teachers were enjoying doing math together. However, two issues worried us. First, many participants failed to grasp important aspects of facilitating the solution of unstructured problems. When some teachers taught their lessons to their peers, they added a lot of scaffolding to the problem, even though they themselves had enjoyed a less structured version. Other teachers erred in the opposite direction. They had not noticed the teaching moves we had been making as facilitators, but interpreted "un-structuring" as doing nothing. They had missed the fact that we had carefully considered in advance probable student reactions and had appropriate questions in our pockets, we were monitoring issues of status differences and work imbalance, and we were looking for excellent ideas and different strategies that groups might share with others. Second, we were concerned that the teachers who had gained some new awareness were not adjusting their classroom practice. This lack of progress could be seen in classroom observations and in the exceedingly structured lesson plans teachers submitted when we worked on planning. Several people commented that low-structure activities were fine for well disciplined groups (like ours), but were not possible in their classrooms.

Figure 1

Consider the following problem:

The Statue of Liberty in New York City has a nose that is 4 feet 6 inches long. What is the approximate length of one of her arms?

1. Solve the problem. (Think about your own nose and arms.)
2. Pick two other body parts and find the approximate length that these parts should be on the Statue of Liberty.
3. Examine what you did with the three examples from Questions 1 and 2. How was your work the same in the three cases? How did it change from case to case?

A Fairly Open Version

We want to solve the following problem:

The Statue of Liberty in New York City has a nose that is 4 feet 6 inches long. What is the approximate length of one of her arms?

1. How long is the Statue's nose in inches?
2. Estimate how long your nose is, in inches.
3. What is the ratio of the length of the Statue's nose to your nose?
4. Estimate how long your arm is.
5. Multiply the answers from (4) and (3). What is the relationship between this number and the length of the Statue of Liberty's arm?
6. Write down a brief explanation of why you gave the answer you did in (5).
7. Pick two other body parts on the Statue of Liberty, and using the strategy from (1) through (5), figure out their lengths.
8. Explain this strategy for figuring out lengths on the Statue of Liberty. Make sure a classmate can understand it.

A Closely Structured Version

Un-Structuring Tasks, Follow-up

Explicitly removing structure from a task. We decided we were being too subtle and that we should directly call attention to the issue of structure. Midway through the second semester, we gave an assignment, the Statue of Liberty's Nose, (See Figure 1) with two versions of two different activities, one version was fairly open and one closely structured. We then asked for a list of the pros and cons of the more structured approach. As a homework assignment, they were given another structured activity and were asked to rewrite it so it would to be more open. In the following class, we discussed the pros and cons of each approach and how to prepare questions to use with the less structured activity to get the benefits that the more structured activities promised. Our goal was to drive home the idea that the support that students might need could come from sources other than breaking the problem down into little, tiny directed steps. Methods for facilitating effective small group mathematics discussions were discussed and sometimes illustrated in video clips, and teachers worked on what we called "pocket" questions for specific problems. These are questions teachers have ready to ask, to challenge a complacent group, restart a frustrated group, or to give a gentle hint to a group. Pocket questions are often the questions that would be written out in an overly-structured task; we argued that it is better to structure as the need arises.

In addition we asked participants to teach a specific problem in their own classes. ("What can you say about the repeating decimal expansion of fractions?") We gave them the problem to work on themselves, and then they worked in groups on plans for teaching the problem in their own classes.

Responding to provocative videos. We showed several videos. One was a movie, "Brown Eyes" about a Korean boy on his first day at a new elementary school. Both teachers and students made many assumptions because he did not speak. It was unclear whether he was just shy or did not know English. Before and after school, the film showed him at home where he was a resourceful and responsible problem solver. The discussion following the showing of this film was emotionally charged as participants examined their own assumptions about the problem solving and reasoning abilities of their students.

We also arranged a viewing of a video of a group of elementary teachers working on lesson study (Lewis, 2005). The task is for students to analyze a changing geometric pattern of triangles. For the initial lesson, students are told to fill out a pre-made table with data in a specific order and then to identify the pattern. When it is taught, the students fill in the table without thinking much about it, and the teachers notice that the over-structuring sabotages the students' understanding. They revise the lesson in a perfect example of "un-structuring" by asking individuals to collect different data and to have them synthesize it as a group. The students who participate in the new lesson display deeper engagement and more creativity along with better reasoning in their explanations.

Peer observation-live unstructured tasks. Project teachers still needed to see that it was possible to teach classes of "at-risk" students using problems with less structure. In each of the two cohorts there were a few teachers, who were already moving in the direction of using less structured problems, and we arranged for some teachers to be able to visit them.

Results

Immediate results were mixed, but over time the teachers continued to change their practices. Through interviews with teachers, through reading their written reflections and listening during their planning sessions with their home departments, we observed an increased appreciation of the amount of structure in a mathematical task as a choice that can have important consequences such as those described in (1) - (9). Their discussions indicated that they realized there were careful choices and teaching moves required when leading low-structure activities, though they were not always sure what they were. Deconstructing the roles of students and teachers in low-structure classrooms to understand how their roles differ from a traditional classroom was not something most participants did on their own. Participants began to focus on questioning as a key to facilitating low-structure tasks. It takes time, practice, and thoughtful reflection to become a good questioner who can create a safe learning environment where students can risk showing what they know and explaining their reasoning. For many teachers becoming a good questioner seems to be a fairly advanced developmental stage of teaching. In one high school, where teachers had already embraced many of the principles advocated by REAL, it took two years for teachers to fully realize the importance of the questions they ask when facilitating groups.

Many of the teachers seemed to believe that using low-structure activities was an ideal to aspire to, and they realized that they must teach their students how to work together, take risks, make mistakes, look for multiple strategies, and explain their thinking, but they still needed to learn how to make these things happen.

We asked teachers to respond to questionnaires before and after the program began. Unfortunately, before it began we did not realize how crucial restructuring would be in improving teacher practice. So, we did not ask whether they considered that a factor when planning lessons. However, in the post questionnaire, we can compare two groups of respondents: a "fully REAL" group of teachers that were at the school during all of the REAL program and “newer” teachers who came to the school sometime after REAL began. When asked to what extent “Unstructuring lessons so students can use their own strategies for solving problems” was a key consideration in their planning math lessons, the "fully REAL" group had 65% rank it 4 or 5 (the highest two scores) out of 5. The "newer" group had only 43% do so. We also asked teachers to check three top considerations out of a list of ten, and 23% of the "fully REAL" group picked unstructuring in their top three considerations, while only 4% of the newer group did.

The REAL Project has collected data on academic achievement by algebra students in the partnership schools. Our original project involved two years work with two cohorts. For five sites, we continued working actively with their teachers and funding teacher projects. For three other schools we gave no continuing support. Two departments were not interested in continuing, and when the district moved the third school to a new site, all of the REAL leadership left, leaving only first year teachers at the new location.

In general, the "continuing work" schools gave above-average ratings for unstructuring in terms of importance in their planning. They also showed gains in algebra performance by African-American students and overall. This is in contrast to the "no continuing work" schools which showed no change in student performance and rated unstructuring as 3.3 and 3.4 (below the school average of 3.6) in terms of importance in their planning. On the one hand, it is satisfying to feel that our work can lead to improved results in schools. On the other hand, one of our hopes was to create lasting change in all our departments.

The table shows data for the "continuing work" schools in more detail.

Table 1

REAL support after second year	Algebra Failure 2003 or 2004	Algebra Failure 2007	Importance of Structure
High School (1)	Total: 35%	Total: 25%	4.1
	AfrAm: 40%	AfrAm: 22.3%	
High School (2)	Total: 56.5%	Total: 45.0%	3.2
	Afr.Am: 64%	Afr.Am: 49%	
High School (3)	Total: 39.3%	Total: 43.3%	3.6
	Afr.Am: 62.2%	Afr.Am: 48.5%	
	2003 Percent of all 8th Graders Passing Algebra	2007 Percent of all 8th Graders Passing Algebra	
Middle School (1)	20.2%	26.5%	4.3
Middle School (2)	11.2%	27.2%	4.7

The data from High School 1 shows a truly impressive drop in algebra failure rates both collectively (25% down from 35% at project start), and disaggregated by ethnicity. Especially notable was the failure rate of African-Americans who are the second largest ethnic subgroup, and about a quarter of the student body (22.3% down from 40% at project start). Teachers at that school rated the post-survey question about unstructuring an average score of 4.1 for importance on the 5 point scale, where the average for all schools was 3.6. Note that this school was unusual for several reasons. First, the school had started changing their practices and improving their success with algebra classes before they joined the REAL program. Second, we funded it directly for two years of co-teaching following the first year work with teacher leaders, instead of one year of teacher meetings as the other schools had.

In High School 2, we met our targets for reducing absenteeism in every ethnic group, and we met our passing rate targets for every ethnic group (except Asian students). The absentee rates are down remarkably from the project start (down to 4.0 yearly absences/student from 12.3). Most encouragingly, the failure rate of African-American students has shown a big drop from 64% to 49% since the start of the program. This rate is still unacceptably high, but we are encouraged that significant change has occurred in the right direction. The overall failure rate dropped from 56.5% to 45.0%. That school did not rate unstructuring as a key concern in planning lessons.

Their average was 3.2, which is below average, but note that most of their teachers are using a reform curriculum whose activities did not need unstructuring.

In High School 3, the African-American failure rate has gone from 62.2% in the normal first year math course to 48.5%, which is very good progress. However, notice the curious overall drop in the passing rate. Statistically, this is due to the failure rate for Latino students increasing from 30.6% to 45.6%. This presents a very mixed picture that we cannot explain but that could be related to changes in the Latino student population. This school rated unstructuring an average of 3.6 in importance.

The challenge in middle school is to give more students a chance to take algebra, while maintaining healthy passing rates. For this reason, our main benchmark for Middle Schools 1 and 2 is the number of students passing algebra as a percentage of the total student population (not just a percentage of the number of students taking algebra). At Middle School 1 the passing rate of students in their algebra classes has met our target and improved to 26.5% of all eighth grade students taking and passing algebra from an initial rate of 20.2%. of all eighth graders. At Middle School 2 the rate went from 11.2% to 27.2%. The two middle schools rated unstructuring as 4.3 and 4.7 on the average.

Conclusion

On the whole, there was much more discussion of reducing structure than action in the classroom. Some lead teachers (teachers who directly participated in the REAL professional development class) did loosen up their activities with some excellent results. One lead teacher mentioned that she now assigns the book's enrichment problems, whereas in the past she skipped over them. Other lead teachers began trying low-structure problems as supplemental activities or as introductions to new topics. But many teachers reported that low-structure tasks were beyond their reach as teachers because of (1) their own limitations (fear of losing control, feeling mathematically unprepared to handle spontaneous questions, fear of a lack of skill in bringing tasks to resolution), (2) the limitations of their students, and (3) the limitations of their schedules by the demands of state standards and testing programs.

We do take some consolation in the broad acceptance among our lead teachers that low-structure, exploratory tasks are a positive idea. Of the psychological obstacles (a) - (d) listed above, we think we successfully addressed all but (b) the belief that their minority and remedial students were not capable of succeeding at less structured tasks. A proclaimed change of heart has yet to be matched by a change in teaching practice; however, with further support the latter change may occur. When teachers described their work in the second year, they would usually contrast their work against an ideal of lowering the amount of structure and express some guilt (with reasons) for falling short.

Participants in REAL repeatedly said that their honors track or Advanced Placement students could work in groups on unstructured problems quite successfully. However, they believed the students in their lower track algebra courses, who had previously struggled with math, could not. These students, teachers said, couldn't work together collaboratively and resisted thinking out loud and explaining their strategies to each other. By high school, students have learned that right answers rather than well reasoned solution processes lead to success in school. When asked about their reactions to less structured activities students tended to react negatively, complaining that their teachers weren't explaining enough.

In one sense, the behavior of the REAL teachers paralleled the behavior of their students. The teachers in REAL, like their students, resisted when pushed to think deeply and share with their peers their thinking about what math was important to teach and why, and how they might best teach it. Some teachers came to REAL thinking that the project directors had already worked out those answers. Similarly, students generally don't question the norm that giving the right answer is what it takes to be successful in school. In fact, what REAL advocated for both teachers and students was the value of the process of thinking and struggling with questions. We could be more explicit in our future work about getting teachers to reflect on this parallel experience in their own learning as a preliminary step to shaping the learning experiences they create for their students.

We did influence the classroom practice of most of our lead teachers. We also had significant effects on the work and culture of many departments, especially in how they spend their time together. In particular, departments indicated increased communication about math course content, discussing common instructional strategies, and reflecting on lessons together, and attributed REAL with these increases. The project seemed to have given many partners the courage to undertake changes they had quietly hoped for in the past such as revision of curriculum, regular peer observations and collaboration, and even elimination of some tracking.

On the other hand, we had to learn patience. First, we had hoped for more movement and growth during the first school year. This proved unrealistic, and in retrospect it seems difficult to ask teachers to change their practice dramatically in the middle of a school year. Indeed it was difficult for teachers to change their intellectual perspectives on their teaching during the year. The fall and spring semesters were marked by slow, cautious change and an opening of minds, followed by great leaps of attitude and ambition during the subsequent summer session, followed by slower but more visible change during the following school year. There is no easy way around this dynamic, which was clear in both groups and across all schools.

Second, we needed patience in the second year waiting for our lead teachers' visible changes of heart, mind, and talk to result in changes in their classrooms. A few teachers were not influenced by the program, but for the majority of participants changes of attitude were visible in their work, in their rhetoric, in their discussions with their peers, and in their private interviews with the outside evaluator. Indeed, most teachers did try different things in the classroom, and as discussed above, some were profoundly influenced. But many of them would try to teach differently in one class, then return to their comfort zone for a few classes, and then try something different again (often when one of us came to visit). This is probably a very natural way for change in teacher practice to occur when the teacher gets to choose the pace. And in fact, we intentionally set up the program's structure and incentives to allow teachers to change at their own pace. Nonetheless, it taxed our patience to see the difference in the verbalized hopes and intents and the classroom reality.

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Who Can Solve 2x=1? – An Analysis of Cognitive Load Related to Learning Linear Equation Solving

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Abstract: Using $2x=1$ as an example, we discuss the cognitive load related to learning linear equation solving. In the framework of the Cognitive Load Theory we consider especially the intrinsic cognitive load needed in arithmetical, geometrical and real analytical approach to linear equation solving. This will be done e.g. from the point of view of the conceptual and procedural knowledge of mathematics and the APOS Theory. Basing on our observations, in the end of the paper we design a setting for teaching linear equation solving.

Keywords: conceptual knowledge; cognitive load theory; linear equations; procedural knowledge

1. Introduction

Cognitive Load Theory, as Sweller (1988) defined it, proposes that optimum learning assumes conditions that are aligned with human cognitive architecture. While this architecture is not yet known precisely, there already exists consensus among cognition researchers that learning happens the easier the less short time working memory – the part of our mind that provides our consciousness and enables us to think, to solve problems, and to be creative etc. – is encumbered. The term cognitive load refers to the total amount of mental activity by which the working memory is oppressed at an instance in time. The most important factor that contributes to cognitive load is the number of knowledge elements that must be employed simultaneously. Basing on Miller (1956), Sweller suggests that most human beings can hardly deal with more than seven (plus minus two) elements in tandem. An immediate consequence of Cognitive Load Theory is that when we design instructional material or our action in mathematics class, we should try to minimize the working memory load by paying extra attention to choosing problem solving methods, how we represent background information, how we put forward exercises and so on.

This paper has got two purposes. We shall first study the cognitive related to a few approaches to solving linear equations. More precisely, we aim to clarify what kind of intrinsic cognitive load a learner encounters in arithmetical, geometrical and real analytic approaches to linear equations. This will be done e.g. by analyzing what conceptual and procedural knowledge (Hiebert & Lefevre, 1986; Haapasalo & Kadijevich, 2000; Star 2005) is required in these approaches. Further, we also refer to

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the APOS Theory (Asiala et al, 1997) when we consider the complexity of the learning processes related to these approaches.

The term intrinsic cognitive load refers to the load that is due to the content to be learned. The intrinsic cognitive load cannot be modified by instructional design but, of course, it must be acknowledged, for instance, in order to be able to customize the total cognitive load when designing teaching and instructional material etc. However, we shall also discuss the extraneous cognitive load, which is due to, for example, teacher's activity in the class. This will be done in the last section. For the more detailed description of the intrinsic and extranous cognitive load, we refer to Sweller (1988).

Another purpose of this paper is to give some aid in designing teaching linear equations. Modern technology makes possible to use illustrative methods also in teaching of arithmetic and algebra. Therefore geometrical aspect plays nowadays more essential role than in the past also on those fields of school mathematics where its potential has traditionally been seen very limited. Hence it is important to clarify, whether geometrical approach lightens – and if yes, then how – the cognitive load related to learning linear equation solving.

For the sake of perfection, we shall also shortly discuss the amount of the cognitive load that is related to mathematically complete understanding of linear equations of one real variable. We shall see, among other things, that solving $2x = 1$ in ordered field with the least-upper-bound property requires much more than one might think at first glance. Of course, this real analytical approach cannot be taken into school as such but, in the last section, we shall ponder the pros and cons of all three approaches and then relying on our observations we shall design a more optimal approch for teaching linear equations both at school and in mathematics teacher training.

Naturally, linear equations have already appeared in several mathematics educational research. The most of these however seem to concentrate not on the challenge itself that lies in learning to solve linear equations but, if anything, on measuring the development of learners' arithmetical skills, or on the question how pupils learn to solve real life problems using linear equations, or they are some how related to the comprehension of the concept of equation, function etc. Nevertheless, some papers consider linear equations also from the perpective of cognitive sciences. For example, MacGregor and Stacey (1993) studied cognitive models underlying students' formulation of linear equations. Qin et al (2004) and Anderson (2005) focus merely on neuroscientific issues but are based on the data of a 6-day experiment in which children learned to solve linear equations and perfect their skills. Having browsed the ISI Web of Knowledge and Google Scholar, it seems that the present paper provides a new perspective on teaching linear equations.

One easily thinks that, for example, $2x = 1$ is so simple equation that finding its solution hardly encumbers our cognition. On second thought, this is not the whole truth. There are several contexts in which this equation bears remarkably different content, e.g. mathematical models of rational numbers and real numbers differ from each others fundamentally, and on some more complicated occasions even the perception of the meaning of the symbols "2" and "1" may be an untrivial task. Indeed, the expression $ax = b,\ a \neq 0,$ is reasonble in some contexts even thought symbols a, b and x were not numbers, vectors or any other numeric variables. Nevertheless, we shall confine ourselves to dealing only with rational or real numbers.

Using $2x = 1$ as an example we now study linear equation solving and what kind of cognitive load is related to solving this equation with profound understanding in arithmetical, geometrical and real analytical (i.e., in the contexts of the ordered field that has the least-upper-bound property) approaches.

What constitutes a single knowledge element or cognitive load unit? It depends on both the learner's familiarity and expertise on the subject to be studied and the content itself. According to the APOS Theory, an expert can handle several concepts, procedures etc. as a single schema whereas a novice may already be confused about the details related to a single concept. Therefore we consider only the relative intrinsic cognitive load of different approaches and do not give any quantitative measure of the load for each approach. That would require a large empirical data because the cognition research has already revealed that human brains can digest illustrated data easier than data given in form of lists, tables etc. In a theoretical paper like this one, it is not possible to realize a reliable quantitative comparision of the total cognitive load that an individual learner actually experiences in geometrical and other approaches and thus we only can reveal and discuss the mathematical details that constitute the intrinsic cognitive load.

2. Arithmetical approach

Lithner (2003) has noticed that even at university students most often base their reasoning and problem solving strategies on the identification of similarities. Since linear equations are easily identifiable, it is also very probable that most mathematics teachers in their teaching – and along them their pupils, too – strongly aim at constructing *one* general algorithm for linear equation solving. Such an arithmetical algorithm apparently presumes that $ax = b,\, a \neq 0,$ is solved by applying the equivalence

$$ax = b \Leftrightarrow x = \frac{b}{a}.$$

Applying this division-based rule is eventually a routine procedure and, therefore, the intrinsic cognitive load required to produce a correct solution for $2x = 1$ and other such linear equations may seem to be quite limited. However, from the point of view of conceptual knowledge, linear equations are not only related to division but also to multiplication and rational numbers. It is well-known that these concepts are not at all trivial for most pupils at school. Hence it is not so surprising to notice that, e.g., only 45 percent of the eight-graders who took part in TIMMS 2003 gained full credits in "If $4(x+5) = 80$, then $x =$" (Gonzales et al, 2004).

Moreover, many pupils, and even some university students, find it difficult to perceive that the division is actually carried out by the multiplication by the inverse of a:

$$ax = b \Leftrightarrow a^{-1}ax = a^{-1}b \Leftrightarrow 1x = a^{-1}b \Leftrightarrow x = \frac{1}{a}\cdot b = \frac{b}{a}. \qquad (1)$$

This is quite natural, since the chain of equivalences in (1) consists of several arithmetical operations and equalities and there are at least two ways to denote the inverse. Hence the number of knowledge elements that all must be pieced together in order to fully understand the operational equivalence between division and multiplication by an inverse is significant.

Looking at the conceptual knowledge related to solving both $2x = 1$ and $ax = b$ deeper, a natural question arises: Do we have to understand what rational numbers really are in order to be able to comprehend the division-based solving procedure of linear equations or is it vice versa: we learn the concept of rational number through solving linear equations? According to Haapasalo & Kadijevich (2000) both orders appear. The comprehension of the concepts of division and rational numbers cannot thus be separated from the deeper appreciation of the solving algorithm of linear equations.

To be exact, solving $2x = 1$ using division-based algorithm does not necessarily require complete understanding of rational numbers and their arithmetics because in this case the division needs be applied only on integers. Since calculating the ratio of two non-integer rationals is eventually multiplication of a rational number by an inverse of a rational number, i.e.,

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \left(\frac{r}{s}\right)^{-1} = \frac{p}{q} \cdot \frac{1}{\left(\frac{r}{s}\right)} = \frac{p}{q} \cdot \frac{s}{r},$$

the cognitive load related to conceptual understanding of the division-based solving algorithm is in the case of $2x = 1$ considerably lower than in the general case. More precisely, in this case, a learner can produce the correct answer $x = \frac{1}{2}$ with reasonable conceptual understanding if he or she does not know the arithmetics of non-integer rationals but only perceives that $\frac{1}{2}$, and more generally any rational number, is a ratio of two integers.

Of course, it is possible to solve $2x = 1$ also without using division but by simply observing that $2 \cdot \frac{1}{2} = 1$ or $\frac{1}{2} + \frac{1}{2} = 1$. These approaches are clearly less burdening in the sense of intrinsic conceptual cognitive load than the division-based one above but, on the other hand, they rely on intuitive knowing or guessing the correct answer and then representing the left-hand side of the original equation as a suitable product or sum and thus are not as general as the one based on division.

All in all, there are several acceptable arithmetical methods that may provide the correct solution for $2x = 1$ and similar linear equations. What can we say about the eventual cognitive load related to this approach?

According to the APOS Theory, on the higher level a learner is, the more and more versatilely he or she exploits automated and routine procedures. For a learner at the level of Scheme (S) or Object (O), the division-based algorithm may constitute only a one single knowledge element and for a learner on the level of Action (A) or Process

(P) already the number of details in (1) may exceed the capacity of his or her perceptive skills (for the definition of the APOS levels, see Asiala et al, 1997). And as the TIMMS 2003 results show, learners with same educational background can be in very different stage of their learning process. This complicates even further giving any quantitative estimate of the cognitive load.

On the other hand, it is very plausible that linear equations are in most cases introduced at school in such a way which we classify belonging to the arithmetical approach in this paper. Therefore we think that, instead of giving any numeric estimate of the cognitive load, it is more reasonable to compare the load of the other two approaches to the one of the arithmetical approach and then design, if possible, an optimal approach piggybacking onto pros of each three approaches.

We conclude this section by observing that all procedures considered above share at least one fundamental problem: they do not explicitly say why there are no other solutions but $x = \frac{1}{2}$ for $2x = 1$.

3. Geometrical approach

Presumably only few mathematics teachers have applied, at least until the existence of modern computers and mathematical softwares, illustrations as a principal tool for finding the solution for linear equations but maybe a little more often they have used images for convincing their pupils of the fact that there are no other solutions. On the other hand, the more central role computing machinery takes in mathematics education, the more central geometrical approach also in solving equations may become.

Before discussing the details, it is worth to consider shortly what solving equations in geometrical context really means. In geometry we first and foremost deal with geometrical objects. Straight lines, curves etc. are geometrical objects; equations, expressions etc. are primarily not. Lines and curves intersect, coincide and so on; equations and expressions have roots, factorize and so on. In other words, we ask different questions about geometrical objects and non-geometrical objects. Analytical geometry is the field of mathematics that relates these different kind of worlds to each others and hence it is possible to solve arithmetical problems also geometrically. For example, in the xy-plane solving $2x = 1$ is reasonable and it means finding the x-parameters of the intersection points of the curves $y = 2x$ and $y = 1$. In Euclidean or other non-analytic geometry, we could speak only of the intersection points of curves without any chance to join this action to arithmetical concepts.

Mathematically most natural and the only reasonable setting to study the solution of $2x = 1$ in an illustrative way thus is the xy-coordinate plane. By presenting the both sides of the equation as straight lines and then studying the set of points where these lines intersect we find the complete solution of the equation. The illustration is given in Figure 1.

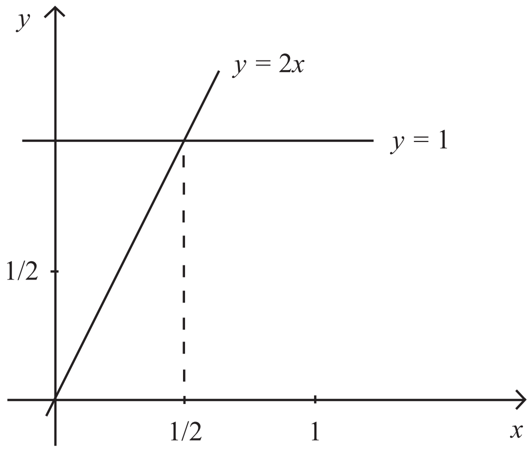


Figure 1. The illustration of $2x = 1$.

In order to be able to solve $2x = 1$ completely in this setting, a learner should at the minimum know that two non-parallel straight lines always intersect exactly at one point. Again, at first glance, this may seem to be a piece of cake but does a learner really know that? Is it only an intuitive conclusion justified by a prompted observation from elementary Euclidean geometry or can it been explained in any other way than by solving linear equations? Being punctilious, it seems that prerequisities to use this approach are more challenging than the problem itself to be solved or we must fool ourselves and accept at least one of the fundamental and non-trivial features of the machinery for granted. After all, mathematical reasoning should be beyond everyday facts!

Let us now consider in more detail the load on working memory needed in understanding the relationship between the illustration in Figure 1 and the solution of $2x = 1$. First a learner must transform a single equation $2x = 1$ into a pair of equations

$$\begin{cases} y = 1, \\ y = 2x, \end{cases}$$

then construct the graphs of these equations, find the intersection point of the lines in the plane, identify the value of the parameter x of this point, and then finally go these steps backwards in order to be able to interpret this value as the only solution of the original equation. The number of operations and processes to be controlled simultaneously in the working memomy seems to exceed the magical seven easily if a learner has not yet gained, with respect to the APOS Theory, O- or S-level capacity in using the coordinate system.

It is worth observing that a learner must go through all of the above steps also in that case if computers are applied. The most remarkable difference is that computers can provide ready-made operations for some of the subroutines, e.g. for finding the x-parameter of the intersection point. In other words, computers can only lighten the arithmetical load but not provide an escape from understanding the relationship of the original problem and the illustration which constitutes the core of the cognitive load of the whole manoeuvre.

It seems that also the necessary conceptual knowledge in this manoeuvre readily exceeds the knowledge needed in the arithmetical approach; for example, can we assume the facility to create or read graphs of straight lines in the coordinate system without good knowledge of arithmetics of (at least) rational numbers? Already finding the correct slope requires good understanding of proportions.

On the other hand, human brains can receive and manipulate data better in an illustrated than in a pure arithmetical form. Most propably, human brains can group larger data as a single schema or an information element for working memory if the data is given figuratively. Hence, let us look at Figure 1 once again. If it were, say, a Java applet based dynamic figure such that using it a learner easily perceived how the straight lines and the expressions $2x$ and 1 are related to each others, and the figure automatically produced the cutted line and the value for the x-coordinate of the intersection point, this setting could provide all tools for controlling the geometrical solving of $2x = 1$ as a single schema. From this point of view, at least procedurally the geometrical approach is not more burdening than the arithmetical approach.

Using the similar thinking as above, one may conclude that illustrations *always* makes mathematics easier. Counterexamples do however exist, as the following one related to elementary algebra verifies.

Even at college and university one can meet every now and then student who claim that $\frac{x}{2}+\frac{x}{3}=\frac{2x}{5}$. Having asked other students how this student could be corrected, a common answer has been that teacher should equip the example with an image like the one in Figure 2.

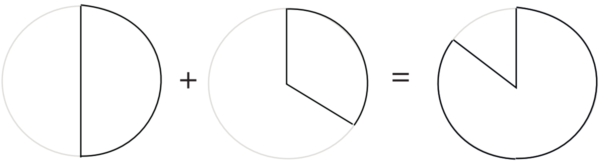


Figure 2. The illustration of $\frac{x}{2}+\frac{x}{3}=\frac{5x}{6}$.

Now, what is the point in this image? A half and a third of a disk is not equal to $\frac{2}{5}$ of the disk but $\frac{5}{6}$ of the disk. But do we really think that understanding *this* is problematic to our student? Obviously not but more propably he or she does not sense any meaning for $\frac{x}{2}$ and $\frac{x}{3}$ and hence cannot apply proper arithmetical rules for them. For the same reason the cognitive load that student must take over in order to be able to understand the correspondence between the image and polynomial expressions is greater compared to the aid that the image can provide.

All in all, the cognitive load related to linear equation solving in the geometrical approach depends remarkably both on the learner's capacity to use the coordinate system and also on the computing tools that are available. For a learner at A- or P-level in using the coordinate representations this approach is propably more burdening than the arithmetical approach and for an advanced user, the load is quite the same as in the arithmetical case.

Nevertheless, geometrical approach provides a somewhat sufficient explanation for the uniqueness of the solution, at least in the context of school mathematics; the complete explanation would take good knowledge of the algebraic structure called group, which already belongs to university mathematics and to the real analytical approach in this paper.

4. Real analytical approach

Now we study $2x = 1$ in the context of real numbers as they are ultimately defined in real analysis, i.e., in the context of an ordered field that has the least-upper-bound property (e.g. Rudin, 1976, 8). To solve $2x = 1$ means then finding the sequence of necessary axioms to establish the chain of equivalences (or implications) between the equation $2x = 1$ and the solution. This is quite typical conventional problem in academic mathematics; it is to be solved using so-called means-ends analysis (see e.g. Larkin et al., 1980) whose principal idea is reducing differences between the current problem state and the goal state. Although this strategy is forceful in obtaining answers, unfortunately, it unavoidably induces high levels of cognitive load. This is because the strategy requires attention to be directed simultaneously to the current state, the goal state, differences between them, procedures to reduce those differences and any possible subgoals that may lead to solution. (Sweller, 1988).

As a matter of fact, $2x = 1$ must be read so that it is the abbreviation for $x \oplus x = 1$ since it is not stated in the axioms that the natural numbers belonged to such an algebraic structure we are dealing with. This implies, for example, that we can solve the original equation by multiplying the equation by the inverse of 2 only if we are able to show that the natural number 2 belongs to the algebraic structure. Following this method – and there hardly are any other available – we soon run into a surprising challenge: there exist examples of fields, e.g. $\{0, 1\}$ equipped with the usual (mod 2) –arithmetic, where $x \oplus x = 1$ does not have any solution! Hence we deduce that in the field of real numbers, in addition to the axioms related to addition and multiplication, we need at least the axioms of order – in other words, the properties of inequalities! – in order to be able to solve this seemingly simple equation $2x = 1$. The same holds again for the general case, too. Ultimately, as anyone familiar with axioms of real numbers can withness, it takes several hours of lectures to provide all necessary details and hence in most mathematics teacher training programs students never see them.

As one could assume beforehand, in this setting both the conceptual and procedural knowledge required are of much greater dimension than in arithmetical or geometrical approaches. But this is the only approach that provides mathematically complete answer to $2x = 1$. It is also self-evident that one cannot use this approach at school. A classical dilemma follows: the more advanced mathematical education we give to mathematics teacher students the less they benefit from it in the pedagogical sense.

5. A cognitive load generated effected approach to teaching linear equation solving

As a summary of the previous sections, we can say that the cognitive load related to learning linear equation solving is quite the same in the arithmetical and geometrical approaches and remarkably heavier in the real analytical approach. Taking into account also the discussion in the beginning of Section 3, one is easily led to think that the most suitable educational arrangement is such that pupils are first put to solve linear equations in the arithmetical context and then they proceed to studying the graphs of linear functions in analytical geometry, and then finally, those few who wish to be real mathematicians, study axioms of real numbers at university.

On the other hand, the arithmetical approach has least tools for motivation of the uniqueness of the solution and the geometrical approach provides at least a plausible solution to that. Moreover, the analysis in the previous sections merely deals with the intrinsic cognitive load and the total cognitive load that a learner experiences is remarkably affected also by the extraneous cognitive load, which is due to e.g. how the instructional materials is used to present information in actual teaching. Clever instructional solutions may smooth the peaks of the intrinsic load in minimizing the total load.

So, could we enhance learning linear equation solving by modifying the traditional practice? Especially, if we evaluate the capacity to study problems in whole higher than the capacity to produce single solutions quickly, the uniqueness of the solution of linear equation should be emphasized right from the beginning. Representing this point of view, we now present the keynotes of an approach to teaching linear equations in which we try to apply as many cognitive load generated effects, i.e., instructional techniques that have been developed in Cognitive Load Theory to facilitate learning, as possible. In Table 1 the most typical effects are listed and compared to standard practice by Cooper (1998). The term ‘goal free effect’ refers to generating goal free problems which is just the opposite to generating problems that require the means-ends analysis. This effect should automatically induce forwards working solution paths and thus impose low leves of cognitive load (Cooper, 1998 and the references therein). **See Table 1 in Appendix**

In our view, an ideal setting for learning general linear equation (i.e. $ax+b=cx+d$) solving is a dynamic two-part figure which combines the arithmetical and geometrical approaches so that

1. In the arithmetic window, as the equation to be solved have been entered, the left-hand side of the equation of is displayed, say, in blue color and the right-hand side in red color. The original equation and the current equivalent equation on which a learner performs arithmetical operations are both shown;

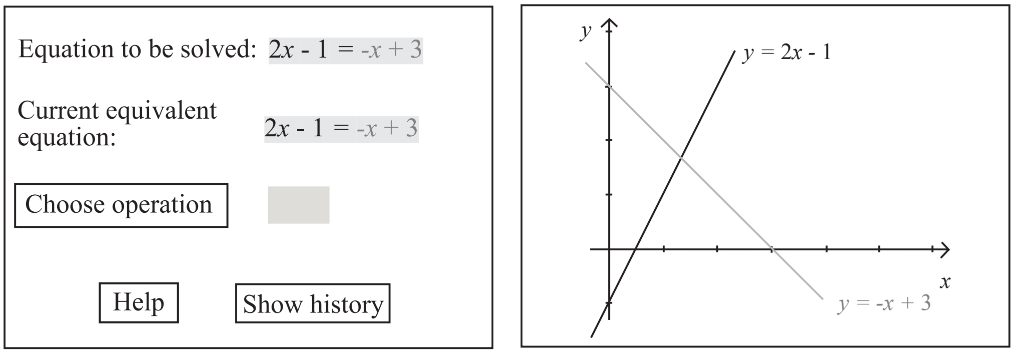
2. The figure automatically generates in the graphics window (the xy-coordinate plane) the graphs of $y=ax+b$ and $y=cx+d$ with the corresponding colors displaying also the equations of these straight lines;

3. In the arithmetics window, a learner can choose and perform any arithmetical operation, e.g. “Divide by 5”, and the figure performs the corresponding operation for both straight lines and their formulas in the graphics window;

4. A learner is auditorily guided to manipulate (using arithmetical operations) the original equation first into form $ex = f$ and then finally to divide this by e so that it becomes $x = \frac{f}{e}$;

5. Especially in the last stages of the process, a learner is encouraged to pay attention to the positions of the straight lines and notice that one of the lines is horizontal and the other one goes through the origin;

6. When the solution is found, i.e., when the current equivalent equation takes the form $x = x_0$, the figure automatically generates an extra vertical line through $x = x_0$, the line through the intersection point of the blue and red lines, marking the solution. The figure also displays this value numerically.



Equation to be solved: $2x - 1 = -x + 3$

Current equivalent equation: $2x - 1 = -x + 3$

Figure 3. An exemplar view of a dynamic figure for learning linear equation solving.

Clearly, this setting exploits the split attention and the modality effetcs. Also the redundancy effect is made good use of although two equation are shown at every turn. If the original equation is not shown, a learner may have a greater cognitive load in remembering the original task and in checking whether he or she got the right answer. And while a single arithmetical operation performed by a learner induces several changes in both arithmetic and graphics windows, it is necessary to display all these expressions in order to indicate the correspondence between the aritmetical and geometrical viewpoints. It is a little more difficult to say whether the straight lines corresponding to the original equation should be displayd throughout. On the other hand, it was logical and informative, on the other hand, it may be redundant. A possible solution is that these lines are displayed shadowedly in background after the first non-trivial arithmetical operation is performed or a learned is encouraged to use Show history –function so that an extra attention is paid to the position of the straight lines.

How well the goal free effect and the worked example effect are made use of depends merely on the expertise of instructor. A pro of this setting is that all arithmetical operations induce a single simple geometrical action. In other words, geometrically multiplication and division are not more complicated processes than addition or subtraction. Therefore it possible to head to solving general linear equations right after having worked a few examples of type $x+a=b$ and $ax=b$. This fact, in our view, is perhaps the most significant advantage of this setting compared with the traditional practices in which several weeks may be spent on solving only $x+a=b$ and $ax=b$. Anyway, this kind of dynamics should easily allow using goal free problems and studying versatile worked examples also collaboratively.

A few critical questions may also arise: for example, should we allow a learner also to move straight lines in the graphics window and let the figure automatically perform the corresponding arithmetical operations in the arithmetics window? Or should the figure somehow underline the coordinate values of the intersection point of $y=ax+b$ and $y=cx+d$ from the beginning? The answer to the first question is: No. Although freedom to move these lines may help a learner to understand the correspondence between the sides of the original equation and the lines, it easily leads to misconceptions and diversion, e.g. if a learner translates the lines in the graphics window so that the intesection point of lines remains fixed, a learner may think that he or she is still solving an equation equivalent to the original one. The latter question may also be answered negative while it is not so obvious. Namely, in this process the x-coordinate of the intersection point remains, of course, fixed. Hence, there is no urgent educational need to emphasize this value until the geometric solution is in its most visible form especially if this multiplied the cognitive load in perceiving the actions in the graphics window. On the other hand, seeing the coordinates of the intersection point at every turn would be of some relevance. Thus the best solution might be such that a user could choose whether the coordinates are displayed or hidden.

The setting described above also facilitates so-called trialogical approach to learning which is related to innovative knowledge communites and especially to the knowledge-creation metaphor of learning. The term “trialogical” refers to the fact that in this approach the emphasis is not only on individuals or on community but also on the way people collaboratively develop mediating artifacts. (Paavola & Hakkarainen, 2005).

More precisely, if the dynamic figure is equipped with saving function, a learner can always trace back with his or her instructor or other learners the steps that he or she has performed. Moreover, since there are only a limited number of possible operations that lead to correct solution, it is possible to program the figure to interactively help a learner to perform necessary steps correctly. It is important to notice that although learners may adopt using the means-ends analysis in linear equation solving, it is also possible to program the help function of the figure so that the goal free effect and thus more communicative learning is applied.

Finally, are there any elements in the real analytic approach that could be utilized in this approach, too? Perhaps, there is. First, the help function can be programmed so that in the arithmetics window it actively motivates a learner to pay attention to that subtraction and division are, respectively, addition of opposite number and multiplication by inverse. Moreover, if the figure allows a learner to enter also combinations of linear expressions to both sides of the equation to be solved, also the need to operate properly with the distributive laws can be discussed within the figure's interactive interface. Second, the need to solve also the existence of the solution can be discussed easily in this framework if the figure is also programmed to generate equations to be solved in varied domains.

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Appendix

Table 1. Cognitive load generated effects (Cooper,1998)

Standard Practice	Cognitive load generated effect
1. Use conventional problems which specify the goal so that students "know what they have to find"	The goal free effect Use goal free problems
2. Students need to solve many problems to learn because "practice makes perfect"	The worked example effect Students learn by studying worked examples. Problem solving is used to test if learning has bee effective
3. Instructional materials which require both textual and graphical sources of instruction should be presented in a "neat and tidy" fashion where the text and graphics are located separately	The split attention effect Instructional materials which require both textual and graphical sources of instruction should integrate the text into the graphic in such a way that the relationships between textual components and graphical components are clearly indicated
4. The same information should be presented in several different ways at the same time	The redundancy effect Simultaneous presentations of similar (redundant) content must be avoided
5. Similar to-be-learned information should be presented using an identical media format to ensure consistency in the instructional presentation	The modality effect Mix media, so that some to-be-learned information is presented visually, while the remainder is presented auditorily

If mathematics is a language, how do you swear in it?

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Swears are words that are considered rude or offensive. Like most other words, they are arbitrary symbols that index meaning: there is nothing inherently wrong with the letters that spell a swear word, but strung together they conjure strong meaning. This reminds us that language has power. This is true in mathematics classrooms too, where language practices structure the way participants understand mathematics and where teachers and students can use language powerfully to shape their own mathematical experience and the experiences of others.

When people swear they are either ignoring cultural norms or tromping on them for some kind of effect. In any language and culture there are ways of speaking and acting that are considered unacceptable. Though there is a need for classroom norms, there are some good reasons for encouraging alternatives to normal behavior and communication. In this sense, I want my mathematics students to swear regularly, creatively and with gusto. To illustrate, I give four responses to the question: If mathematics is a language, how do you swear in it?

Response #1: To swear is to say something non-permissible.

I've asked the question about swearing in various discussions amongst mathematics teachers. The first time I did this, we thought together about what swearing is and agreed that it is the expression of the forbidden or taboo. With this in mind, someone wrote $\sqrt{-1}$ on the

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whiteboard and giggled with delight. Another teacher reveled in the sinful pleasure of scrawling $^{n}/_{0}$ as if it were graffiti. These were mathematical swears.

Though I can recall myself as a teacher repeating “we can’t have a negative radicand” and “we can’t have a zero denominator”, considering the possibility of such things helps me understand real numbers and expressions. For example, when the radicand in the quadratic formula is negative ($b^2 - 4ac < 0$), I know the quadratic has no roots. And when graphing rational expressions, I even sketch in the non-permissible values to help me sketch the actual curve. Considering the forbidden has even more value than this.

Though it is usually forbidden to have a negative radicand or a zero denominator, significant mathematics has emerged when mathematicians have *challenged* the forbidden. Imaginary numbers opened up significant real-world applications, and calculus rests on imagining denominators that approach zero. This history ought to remind us to listen to students who say things that we think are wrong, and to listen to students who say things in *ways* we think are wrong (which relates to response #3 in this article). We can ask them to explain their reasoning or to explain why they are representing ideas in a unique way.

Knowing what mathematical expressions are not permitted helps us understand the ones that are permitted. Furthermore, pursuing the non-permissible opens up new realities.

Response #2: Wait a minute. Let’s look at our assumptions. Is mathematics really a language?

Good mathematics remains cognizant of the assumptions behind any generalization or exploration. Thus, in this exploration of mathematical swearing, it is worth questioning how mathematics is a language, if it is at all.

It is often said that mathematics is the universal language. For example, Keith Devlin has written a wonderful book called “The Language of Mathematics”, which is a history of mathematics that draws attention to the prevalence of pattern in the natural world. The book is not about language in the sense that it is about words and the way people use them. Its connection to language is more implicit. Humans across cultures can understand each other’s mathematics because we share common experiences of patterns in the world and of trying to make sense of these patterns. We can understand each other. Understanding is an aspect of language. There are other ways in which mathematics can be taken as a language, and there can be value in treating it as a language, as demonstrated by Usiskin (1996).

However, it would not be so easy to find a linguist who calls mathematics a language. Linguists use the expression ‘mathematics register’ (e.g. Halliday, 1978) to describe the peculiarities of a mainstream language used in a mathematical context. David Pimm (1987) writes extensively about aspects of this register. It is still English, but a special kind of English. For example, a ‘radical expression’ in mathematics (e.g. “$3\sqrt{2}+\sqrt{5}$”) is different from a ‘radical expression’ over coffee (e.g. “To achieve security, we have to make ourselves vulnerable.”) because they appear in different contexts, different registers.

Multiple meanings for the same word in different contexts are not uncommon. Another example significantly related to this article is the word ‘discourse’, which has emerged as a buzzword in mathematics teaching circles since reforms led by the National Council of Teachers of Mathematics in North America. The word is often used as a synonym for ‘talking’ (the practice of language in any situation) and also to describe the structure and history of mathematics classroom communication (the discipline of mathematics in general), which, of course, has a powerful influence on the practice of language in the classroom. Both meanings have validity, so it is up to the people in a conversation to find out what their conversation partners are thinking about when they use the word 'discourse'. It is the same for the word 'language'.

Who has the right to say mathematics is a language, or mathematics is not a language? Language belongs to all the people who use it. Dictionaries *describe* meanings typically associated with words more than they *prescribe* meaning. By contrast, students in school often learn definitions and prescribed meanings – especially in mathematics classes. This is significantly different from the way children learn language for fluency.

When we are doing our own mathematics – noticing patterns, describing our observations, making and justifying conjectures – language is alive and we use it creatively. When we make real contributions to a conversation it is often a struggle to represent our ideas and to find words and diagrams that will work for our audience. For example, I have shown some excerpts from students' mathematical explorations in Wagner (2003). The students who worked on the given task developed some new expressions to refer to their new ideas and in the article I adopted some of these forms, calling squares '5-squares' and '45-squares' (expressions that have no conventional meaning). When we are doing our own mathematics we try various words to shape meaning. By contrast, mathematical exercises – doing someone else's mathematics repeatedly – are an exercise in conformity and rigidity.

One role of a mathematics teacher is to engage students in solving real problems that require mathematical ingenuity, which also requires ingenuity in communication because students have to communicate ideas that are new to them. Once the students have had a chance to explore mathematically, the teacher has another role – to draw their attention to each other's mathematics. When students compare their mathematical ideas to those of their peers and to historical or conventional mathematical practices, there is a need to standardize word-choice so people can understand each other's ideas. In this sense, the mathematics register is a significant language phenomenon worth attending to. However, there is also value in deviating from it with awareness. Teachers who resist the strong tradition of pre-reform mathematics teaching are swearing, in a way, by deviating from tradition.

Response #3: Swear words remind us of the relationship between language and action.

There are connections between inappropriate words (swearing) and inappropriate actions. For example, it is inappropriate to use swear words publicly to refer to our bodies' private parts, but it is even less appropriate to show these private parts in public. It is taboo.

This connection between action and words exists for appropriate as well as inappropriate action. Yackel and Cobb (1996) describe the routines of mathematics class communication as 'sociomathematical norms.' These norms significantly influence students' understanding of what mathematics is. Because teachers use language and gesture to guide the development of these norms, this language practice relates to conceptions of what mathematics is and does. Thus, I suggest that there is value in drawing students' attention to the way words are used in mathematics class, to help them understand the nature of their mathematical action. This goes beyond the common and necessary practice of helping students mimic the conventions of the mathematics register. Students can be encouraged to investigate some of the peculiarities of the register, and to find a range of ways to participate in this register.

For example, we might note that our mathematics textbook does not use the personal pronouns 'I' and 'we' and then ask students whether (or when) they should use these pronouns in mathematics class. When I asked this question of a class I was co-teaching for a research project, most of the students said personal pronouns were not appropriate because mathematics is supposed to be independent of personal particularities, yet these same students continued to use personal pronouns when they were constructing their new mathematical ideas. A student who said, “Personally, I think you shouldn’t use ‘I’, ‘you’, or ‘we’ or ‘me’ or whatever” also said later “I’m always thinking in the ‘I’ form when I’m doing my math. I don’t know why. It’s just, I’ve always thought that way. Because I’m always doing something.” (The research that this is part of is elaborated in Wagner, 2007.) The tension between students’ personal agency in mathematical action and their sense of how mathematics ought to appear is central to what mathematics is.

Mathematical writing tends to obscure the decisions of the people doing the mathematics. Students are accustomed to word problems like this: “The given equation represents the height of a football in relation to time…”. The reality that equations come from people acting in particular contexts is glossed over by the structure of the sentence. Where did the equation come from? The perennial student question, “Why are we doing this?”, may seem like a swear itself as it seems to challenge the authority of classroom practice. However, it is the most important question students can ask because even their so-called applications of mathematics typically suggest that equations exist without human involvement.

Though I find it somewhat disturbing when mathematicians and others ignore human particularities, it is important to recognize that this loss is central to the nature of mathematics. Generalization and abstraction are features of mathematical thinking, and they have their place in thoughtful human problem solving. There is value in asking what is always true regardless of context. There is also value in prompting mathematics students to realize how mathematics obscures context and to discuss the appropriateness of this obfuscation. Mathematics students should make unique contributions (using the word ‘I’) *and* find ways of generalizing (losing the word ‘I’).

This connection between agency-masking language form and mathematics’ characteristic generalization, is merely one example of the way language and action are connected. Whenever we read research on discourse in mathematics classrooms we can consider the connections between mathematics and the aspects of discourse described in the research. As with the example given here, talking about these connections with students can help them understand both the nature of mathematics and the peculiarities of the mathematics register. A good way of starting such a conversation is to notice the times when students break the normal discourse rules – the times that they ‘swear’. We can take their mathematical swears as an opportunity to discuss different possible ways of structuring mathematical conversations.

Response #4: I’m not sure how to swear mathematically, but I know when I swear in mathematics class!

The connection between human intention and mathematics reminds me of one student’s work on the above-described investigation that had ‘5-squares’ and ’45-squares’. For the research, there was a tape recorder at each group’s table. Ryan’s group was working on the task, which is described in the same article (Wagner, 2003). Ryan had made a conjecture and was testing it with various cases. Listening later, I heard his quiet work punctuated with muffled grunts of affirmation for each example that verified his conjecture, until he exclaimed a loud and clear expletive, uttered when he proved his conjecture false.

Linguistic analysis of swearing practices shows how it marks a sense of attachment (Wajnryb, 2005). Ryan swore because he cared. He cared about his mathematics. He cared about his conjecture and wanted to know whether it was generalizable. His feverish work and his frustrated expletive made this clear. I want my students to have this kind of attachment to the tasks I give them, even if it gets them swearing in frustration or wonder (though I'd rather have them express their frustration and wonder in other ways). The root of their frustration is also behind their sense of satisfaction when they develop their own ways of understanding. As is often the case with refuted conjectures, finding a counterexample helped Ryan refine the conjecture into one he could justify.

To help my students develop a sense of attachment to their mathematics, I need to give them mathematical investigations that present them with real problems. They may swear in frustration but they will also find satisfaction and pleasure.

Reflection

Swearing is about bucking the norm. The history of mathematics is rich with examples of the value of people doing things that others say should not be done. Thus there is a tension facing mathematics teachers who want both a disciplined class and one that explores new ideas.

Though my own experiences as a mathematics student were strictly discipline-oriented, I try to provide for my students a different kind of discourse – a classroom that encourages creativity. I want my students to swear mathematically for at least four reasons. 1) Understanding the non-permissible helps us understand normal practice and to open up new forms of practice. 2) Creative expression casts them as participants in the long and diverse history of mathematical understanding, which is sometimes called the universal language of mathematics. 3) Attention to the relationship between language and action can help students understand both. 4) The student who swears cares: the student who chooses a unique path is showing engagement in the discipline.

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From Trapezoids to the Fundamental Theorem of Calculus

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Abstract: The philosophy of Mathematics Education undergoes changes from the school to college level and students generally have a tough time coping with the transition. It is our endeavor to impress the importance of introducing college level topics at an early stage, so that students are not lost in the transition. Keeping this in mind, we suggest an early exposure to an important topic from Calculus; approximating the area of a planar region. Traditionally this topic is introduced using Riemann Sums but in this paper we try to follow a student's natural inclination in approximating areas and explain how this approach can be adopted at the middle school or high school level. It is our belief that using suitable technology like TI- 83/84 or Maple, this approach can be adapted to various other college level topics providing the student with a sound footing to cope with college level mathematics.

Keywords: Calculus; Collegiate math teaching; Fundamental theorem of Calculus; Riemann sums; Trapezoids; Teaching with technology

Introduction via rectangles

Every introductory calculus textbook, at some point or another, investigates the problem of finding the area of a region lying between two vertical lines, a curve, and the x-axis. An examination of

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several popular textbooks ([6];[7];[4]) follow similar approaches when introducing students to the solution of this problem.

- Remind students that for rectangles area = length × width
- Draw a picture showing an approximation using left-handed rectangles
- Draw a picture showing an approximation using right-handed rectangles
- Suggest that more rectangles result in a better approximation
- Introduce Riemann Sums
- Introduce limits and define the definite integral
- Demonstrate the need for the Trapezoidal Rule and Numerical Integration

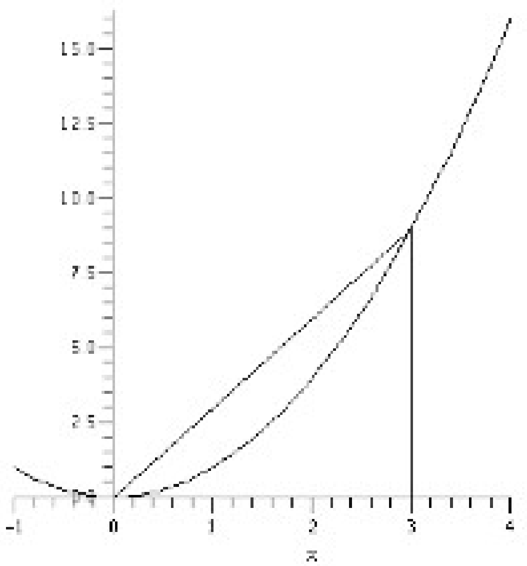
This approach, with minor additions, subtractions, and other alterations serves as the standard approach to this topic. This approach, based on a student's ability to recognize the utility of rectangles in the approximation process, works superbly. However, it has been the authors' experience that not all students are inclined to use rectangles to arrive at their first approximation of the area. This paper will examine what can happen when students are allowed a different starting point. During this proposed mathematical journey students will discover a slope-area connection while discovering the need for mathematically rich concepts such as mathematical induction.

Introduction via trapezoids

Consider the following problem asked to a class of students studying calculus for the first time: Approximate the area between the x-axis, the curve $y = x^2$ and the lines $x = 0$ to $x = 3$. It has been the authors' experience that many student's first impulse is to use a triangle (see figure 1) to find the required approximation. This is not surprising since the hypotenuse of the triangle closely approximates the curve and students have known and used the formula for the area of a triangle is

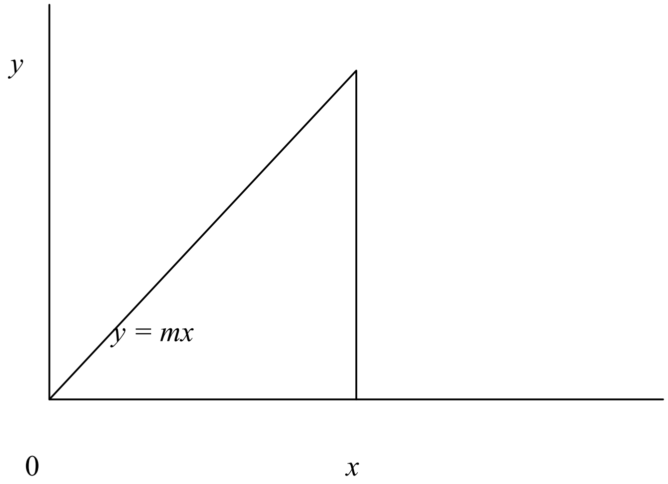
A=*1/2bh,* where *b* is the length of the base of the triangle and *h* is the height of the triangle, for many years. Students rarely approximate this area with the rectangle of base 3 and height 9, after all it is obvious that the area of the rectangle is much larger that the requested area. This paper will investigate what can happen if students are allowed to explore their initial impulse rather than be immediately redirected to rectangles. Before students proceed they must agree that a fixed triangle can be moved through two-dimensional space without changing its area. After this agreement is reached students can be presented with the following question:

Figure 1
y=x^2



Find the area of the triangle below:

Example 1



Given this graph students will quickly state that $Area = \frac{1}{2}(x)(y)$ square units.

The authors suggest students be guided through the following derivation leading to an alternative form of this area formula. The first step in this alternative vision is to ask students to observe that the equation $y = mx$, defines the hypotenuse of the triangle in this position on the coordinate plane. Knowing this students see that these triangles have base = x and height = mx. Thus the area of such a triangle is $1/2\ (x)(mx) = 1/2mx^2$. This "new" formula suggests, that the area the area of a triangle can be thought of as dependent on the slope of the hypotenuse and the length of the base. Thus a link between slope and area has been established. At this point the line can be moved and the boundaries changed and the following graph (Figure 2) and question can be presented:

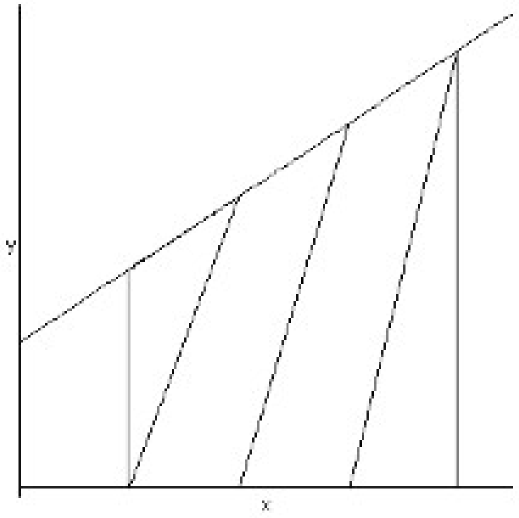


Figure 2
y= m*x + kT

Find the area of the shaded region bounded by the x-axis, the line $y = mx + k_T$ and the vertical lines $x = x_0$ and $x = x_1$.

Students will recognize that the shape created in figure 2 is a trapezoid. Some students will remember the formula for the area of a trapezoid to be $A = \frac{h(b_1 + b_2)}{2}$, where b_1 and b_2 are the lengths of the two parallel sides of the trapezoid and h is the height (distance separating the parallel sides) of the figure. Thus in the figure presented $b_1 = mx_1 + k_T$, $b_2 = mx_0 + k_T$. and $h = x_1 - x_0$. Thus :

$$\begin{aligned}
Area &= \frac{(x_1 - x_0)[(mx_1 + k_T) + (mx_0 + k_T)]}{2} \\
&= \frac{x_1(mx_1 + k_T) + x_1(mx_0 + k_T) - x_0(mx_1 + k_T) - x_0(mx_0 + k_T)}{2} \\
&= \frac{mx_1^2 + x_1k_T + mx_0x_1 + x_1k_T - mx_0x_1 - x_0k_T - mx_0^2 - x_0k_T}{2} \\
&= \frac{mx_1^2 + 2x_1k_T - mx_0^2 - 2x_0k_T}{2} \\
&= \frac{m(x_1^2 - x_0^2) + 2k_T(x_1 - x_0)}{2} \\
&= \frac{m}{2}\left(x_1^2 - x_0^2\right) + k_T\left(x_1 - x_0\right) \qquad (1)
\end{aligned}$$

Again students see a "new" formula in which the area of the figure is calculated using the slope of the line which forms one of its boundaries and the values of *x* that dictate its height and position on the x-axis.

Using Technology and Algebra

The following program can be used on the TI 83/84 to calculate areas using equation (1).

```
Input "slope",M
Input "Y Intercept",K
Input "Lower Limit",L
Input "Upper Limit",U
(M/2)*(U^2-L^2)+K*(U-L)→A
Disp A
```

The following program can be used on Maple 11 to calculate area using equation (1).

```
M: =; U: =; L: =; K: = ;    (Input values of M, U, L, K)
Area: = (M/2)*(U^2-L^2)+K*(U-L);
```

```
Area;
Evalf(%);
```

The area can be computed using the traditional trapezoidal rule as follows.

```
with (student);
M: =; U: =; L: =; K: =;     (Input values of M, U, L, K)
trapezoid (Mx+k, X=U..L, 1);
evalf(%);
```

What of students that do not remember the formula for the area of a trapezoid, are they left in the dark unable to create an argument that can lead to the insight above? No, they can approach this problem by dividing the figure into two more recognizable shapes, a triangle and a rectangle, calculating their areas and adding the results. Their work may look like what is found below.

$$\begin{aligned}
Area &= (mx_0 + k_T)(x_1 - x_0) + \tfrac{1}{2}[(mx_1 + k_T) - (mx_0 + k_T)](x_1 - x_0) \\
&= (x_1 - x_0)\left[(mx_0 + k_T) + \left(\frac{mx_1 + k_T}{2}\right) - \left(\frac{mx_0 + k_T}{2}\right)\right] \\
&= (x_1 - x_0)\left[\frac{mx_0 + k_T}{2} + \frac{mx_1 + k_T}{2}\right] \\
&= \frac{x_1 - x_0}{2}[mx_0 + mx_1 + 2k_T] \\
&= \frac{1}{2}[mx_1x_0 + mx_1^2 + 2k_Tx_1 - mx_0^2 - mx_0x_1 - 2k_Tx_0] \\
&= \frac{1}{2}[mx_1^2 - mx_0^2 + 2k_Tx_1 - 2k_Tx_0] \\
&= \frac{m}{2}[x_1^2 - x_0^2] + k_T[x_1 - x_0] \qquad (1)
\end{aligned}$$

This is the same result that was found when the direct rule for the area of a trapezoid was used.

It is the authors' belief that such an exposition serves three purposes:

First, students clearly see that in mathematics there can be different ways to view the same concept.

It illustrates the general slope-area connection central to the calculus.

This introduction and some new notation:

Area under $y = mx + k$ between x_0 and x_1 is equal to

$$\frac{m}{2}[x_1^2 - x_0^2] + k[x_1 - x_0] = \frac{m}{2}\left[x^2\right]_{x_0}^{x_1} + k\left[x\right]_{x_0}^{x_1}$$

exposes students to an example of the Fundamental Theorem of Calculus which they will encounter in the near future.

Since not all graphed function result in lines, it is not inappropriate at this time to suggest students explore shapes with curved tops and learn the lessons they have to teach. A natural starting point for this journey is our starting point $y = x^2$ with a vertical shift $y = x^2 + k_p$. Since students probably have not learned a formula for the area under such a curve the first step in the discovery process is to find an approximation of the answer sought. Examination of the graph found below (figure 3) suggests that a trapezoidal approximation (using a single trapezoid) is a better first approximation than a rectangular approximation (using a single rectangle).

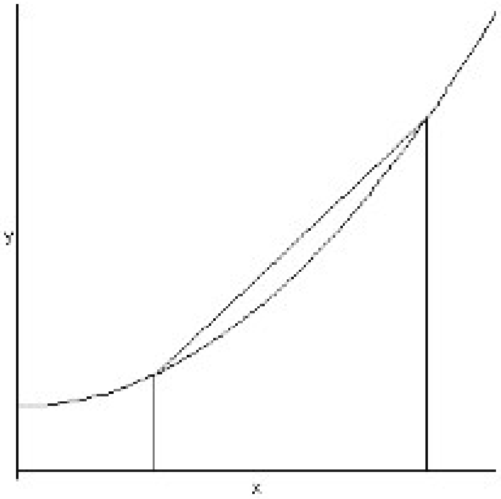


Figure 3
$y = x^2 + k$

Thus, after the equation of the line containing the points $\left(x_0, x_0^2 + k_p\right)$ and $\left(x_1, x_1^2 + k_P\right)$is found formula (1), derived above, can be used to find a *"good"* approximation of the desired area under the parabola. The slope, *m*, of the line is:

$$m = \frac{(x_1^2 + k_P) - (x_o^2 + k_P)}{x_1 - x_0}$$
$$= \frac{x_1^2 - x_0^2}{x_1 - x_0}$$
$$= x_1 + x_0.$$

The y-intercept, k_T, of the line is found as follows:

$$x_0^2 + k_P = (x_1 + x_0)(x_0) + k_T$$
$$x_0^2 + k_P = x_1 x_0 + x_0^2 + k_T$$
$$k_P - x_1 x_0 = k_T$$

Thus, the equation of the non-perpendicular line connecting the two parallel sides of the trapezoid is $y = (x_1 + x_0)x + (k_P - x_1 x_0)$ and the approximate area of this figure is:

$$\frac{x_1 + x_0}{2}\left[x_1^2 - x_0^2\right] + \left[(k_P - x_1 x_0)(x_1 - x_0)\right]. \quad (2)$$

From this we see that:

$$Area \approx \frac{x_1 + x_0}{2}\left[x_1^2 - x_0^2\right] + \left[(k_P - x_1 x_0)(x_1 - x_0)\right]$$

$$(x_1 - x_0)\left[\left(\frac{(x_1 + x_0)^2}{2}\right) + (k_P - x_1 x_0)\right]$$

$$\left(\frac{x_1 - x_0}{2}\right)\left[\left((x_1 + x_0)^2\right) + 2(k_P - x_1 x_0)\right]$$

$$\left(\frac{x_1 - x_0}{2}\right)\left[x_1^2 + 2x_1 x_0 + x_0^2 + 2k_P - 2x_1 x_0\right]$$

$$\left(\frac{x_1 - x_0}{2}\right)\left[x_1^2 + x_0^2 + 2k_P\right]$$

$$\left(\frac{x_1 - x_0}{2}\right)\left[(x_1^2 + k_p) + (x_0^2 + k_P)\right]$$

$$\left(\frac{x_1 - x_0}{2}\right)(y_1 + y_0)$$

which is the traditional trapezoidal rule for one subdivision. One can now observe that

$$\left(\frac{x_1 - x_0}{2}\right)(y_1 + y_0) = \left(\frac{h(b_1 + b_0)}{2}\right).$$

which demonstrates that the trapezoidal rule for one subdivision is simply a functional form of the elementary school formula for the area of a trapezoid. Once again emphasizing to students that the same mathematical concept can be viewed in various ways.

Those wishing to program equation (2) into the TI 83/84 may use the following:

```
Input “Constant”,K
Input “Lower Limit”,L
Input “Upper Limit”,U
((U+L)/2)*(U^2-L^2)*((K-U*L)*(U_L))→A
Display A
```

Also a student interested in programming in Maple 11 may use the following:

```
T := ((U + L)/2)*(U^2 − L^2)) + ((K − U * L)*(U − L));
K :=;U :=;L :=;
evalf(T);
```

The area may be also computed by using the traditional trapezoidal rule by using following code:

```
with(student);
K :=;U :=;L :=;
trapezoid(x^2 + K, x = L..U, 1);
evalf(%);
```

It quickly becomes obvious to students that breaking the region up into two trapezoids, as seen in figure 4 below, results in an even better approximation of the area desired.

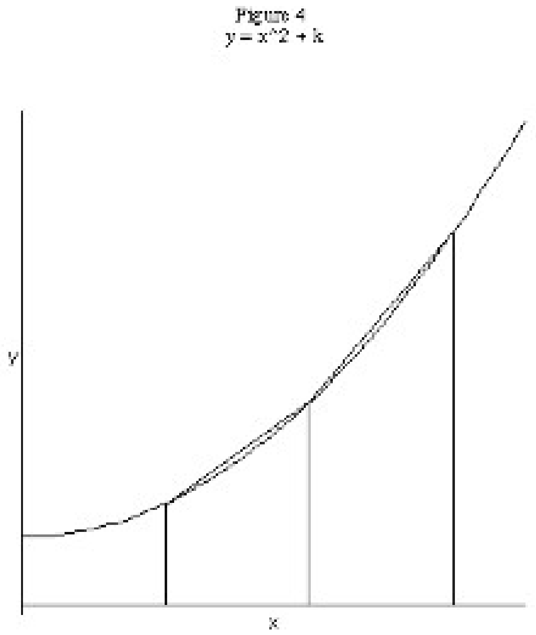


Figure 4
y = x^2 + k

$$Area \approx \left\{ \frac{x_1 + x_0}{2} \left[x_1^2 - x_0^2 \right] + \left[(k_p - x_1 x_0)(x_1 - x_0) \right] \right\} + \left\{ \frac{x_2 + x_1}{2} \left[x_2^2 - x_1^2 \right] + \left[(k_p - x_2 x_1)(x_2 - x_1) \right] \right\}$$

Continuing this line of reasoning they see that three is better than two and one hundred would be better that three etc. A TI 83/84 program for this alternative formula for this approximation using n subintervals is found below.

```
Input”Constant”, K
Input “Number of Subintervals”, N
Input “Lower Limit”, L
Input “Upper Limit”, U
(U-L)/N→W
0→S
For (I,1,N,1)
```

```
L+W→U
((U+L)/2)*(U^2-L^2) + ((K-U*L)*(U-L))→T
S+T→S
U→L
END
Disp S
```

Again a Maple program for the above is found below:

```
N: =; U: =; L(0):=; W:=$\frac{U-L(0)}{N}$;R=0;
For j from 0 by 1 to n-1 do
U(j)=L(j)+W;
$A(j) := R + \left( \left( \frac{U(j)+L(j)}{2} \right) * \left( U(j)^2 - L(j)^2 \right) + \left( K - U(j) * L(j) \right) * (U(j) - L(j)) \right)$;
$R = A(j)$;
od;
```

(Readers wishing to use MS Excel to execute the trapezoidal rule are referred to [3].)

Mathematical Induction

Let us look at the number of subdivisions used to approximate the area under a curve. For n=1, we have the area of a trapezoid. For n=2, with subdivisions, x_0, x_1, x_2 we have the area approximated by the sum of two trapezoids i.e

$$Area \approx \frac{x_2 - x_0}{2 \times 2}(y_0 + y_1) + \frac{x_2 - x_0}{2 \times 2}(y_1 + y_2) = \frac{x_2 - x_0}{2 \times 2}(y_0 + 2y_1 + y_2).$$

The authors suggest that the students be asked to verify that for *n*=3, you get $Area \approx \frac{x_3 - x_0}{2 \times 3}(y_0 + 2y_1 + 2y_2 + y_3)$. Now, for the problem with 100 sub-divisions the student has most likely made an educated guess and written down the formula $Area \approx \frac{x_{100} - x_0}{2 \times 100}(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{99} + y_{100})$. At this point one can conjecture that the general formula based on the number of subdivisions, say *n*, it will be $Area \approx \frac{x_n - x_0}{2 \times n}(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{n-1} + y_n)$.

Although it seems to be true from the pattern, it is just a guess and should not be considered a rule at this stage. In order to elevate this conjecture to the status of a rule we have to provide a convincing argument. The convincing mathematical argument is called a proof. There should be no loopholes in the proof. There are different methods of proof. One of them is called the method of induction. The logic behind this method is follows.

Suppose: P(n) represents some sort of argument involving natural numbers. Example: P(n) can represent the statement: The area under a curve y=f(x) using n subdivisions is approximately $\frac{x_n - x_0}{2 \times n}(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{n-1} + y_n)$. We can use an inductive argument as follows: P(1) is verified to be true: i.e. the result is true for n=1. In our situation P(1) would represent the statement, Area ≈ $\left(\frac{x_1 - x_0}{2}\right)(y_1 + y_0)$ which is true. Now, suppose the result is true for some natural number *k* i.e. P(*k*) is true. This statement is known as the induction hypothesis. The induction hypothesis in this problem is the assumption that the area under a curve y = f(x) with k subdivisions on the interval [a, b] is approximately $\frac{b-a}{2 \times k}(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{k-1} + y_k)$. We will use this to prove that the result is true for next integer n = k + 1 or P(k + 1). At that point we would have clearly shown that if the result is true for any given integer, then it is also true for the next integer.

1. The result is clearly true for n=1 since it is the well-known formula for the area of a trapezoid.
2. Assume the induction hypothesis: with *k* subdivisions, each of length $l = \frac{b-a}{k}$, the area is approximately $\frac{l}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{k-1} + y_k)$.
3. Let us work out the last step: Show that P(k+1) is true. We need to somehow use the induction hypothesis to prove P(k + 1). This can be done easily. For k + 1 subdivisions, each of length $l = \frac{b-a}{k+1}$, we can sum up the area of the trapezoids corresponding to the first k subdivisions and then to this add the area corresponding to the subdivision $[x_k, x_{k+1}]$. By the induction hypothesis, the Area corresponding to the first k subdivisions is approximated by $\frac{l}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{k-1} + y_k)$. The area corresponding to the subdivision $[x_k, x_{k+1}]$ is $\frac{x_{k+1} - x_k}{2}\left(y_k + y_{k+1}\right)$. But $x_{k+1} - x_k = l$. So the area with *n* =*k* + 1 subdivisions is approximately

$$\frac{l}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{k-1} + y_k) + \frac{l}{2}(y_k + y_{k+1})$$

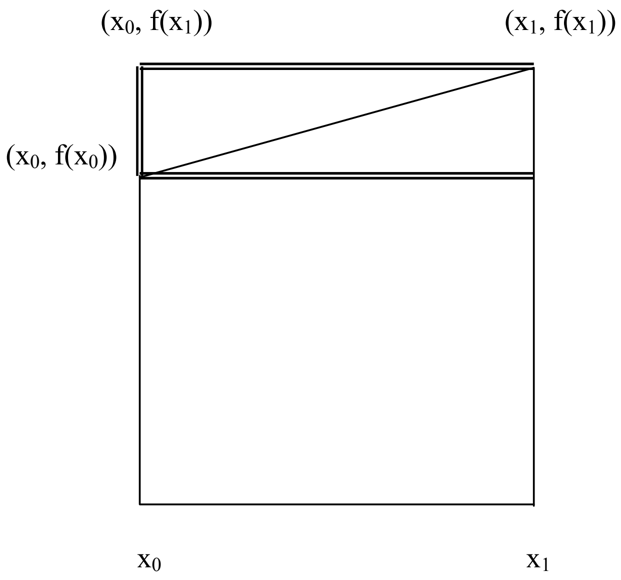
$$= \frac{l}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{k-1} + 2y_k + y_{k+1})$$

It should be clear to students that the formula is independent of the equation of the curve and so that the above discussion will work for any continuous curve.

The Trapezoidal Rule

It can now be pointed out that the area of a trapezoid is simply the average of the areas of two rectangles as demonstrated on figure 5 below.

Figure 5



$$\text{Area of Trapezoid} = \frac{(x_1 - x_0)(f(x_1) + f(x_0))}{2}$$

$$= \frac{(x_1 - x_0)f(x_1) + (x_1 - x_0)f(x_0)}{2}$$

$$= \frac{\text{Area of rectangle with side } (x_1 - x_0) \text{ and side f}(x_1) + \text{Area of rectangle with side } (x_1 - x_0) \text{ and side f}(x_0)}{2}$$

At this point students have been led to the idea of left and right, or upper and lower, rectangular sums which then leads to the concept of the Riemann Sum and finally the definition of the definite integral. At this time teachers can introduce any student interested in the historical development of mathematics to the ancient Greek method of area by exhaustion used by, among others, Archimedes (interested readers are referred to [6];[1];[2];[5]).

Conclusion

The journey suggested above can be summarized as follows:

- Allow students to use a triangle to approximate a desired area
- Allow students to discover the connection among triangles, slope, and area
- Have students use a trapezoid to approximate a desired area
- Allow students to discover the connection among trapezoids, slope, and area
- Suggest that more trapezoids result in a better approximation
- Deduce the trapezoidal rule and validate using mathematical induction
- Show that the area of a trapezoid is the mean of the areas of two rectangles
- Draw a picture showing an approximation using left-handed rectangles
- Draw a picture showing an approximation using right-handed rectangles
- Suggest that more rectangles result in a better approximation
- Introduce Riemann Sums
- Introduce limits and define the definite integral.

This discovery path is interesting in several respects. It begins with a problem involving the use of low degree polynomials functions, things students are familiar with and have mastery over. It then allows them the freedom to follow a path of discovery suggested by their intuition. This path leads them to some interesting discoveries: there is a connection between area and slope, for special cases the "slope formula" can take different and interesting forms, there is a trapezoid rectangle connection, examination of simple problems can lead to generalizations from which the need for mathematical induction arises naturally. Simply stated this demonstration follows a natural progression that prepares students for both the Fundamental Theorem of calculus and future work in numerical analysis while introducing them to the need for proof.

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Graph isomorphisms and matrix similarity: Switching between representations

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Abstract: A proof whether two graphs (possibly oriented graphs or multigraphs, etc.) are isomorphic or not can be derived by various methods. Some of them are reasonable for small numbers of vertices and/or edges, but not for larger numbers. Switching from iconic representation to a matrix representation transforms the problem of Graph Theory into a problem in Linear Algebra. The support provided by a Computer Algebra System is analyzed, in particular with regard to the building of new mathematical knowledge through a transition from graphical to algebraic representation. Moreover two important issues are discussed: a. the need for more than one representation; b. the direction of the switch between representations, which is non standard, from graphical to algebraic.

Keywords: Computer Algebra systems (CAS); Collegiate mathematics; Graph theory; Linear Algebra; Matrices; representations; isomorphisms;

I. Introduction.

Undergraduate mathematics is often taught as a collection of stand-alone courses, and students are not always aware of the bridges that exist between different areas of mathematics. Geometry and Linear Algebra are taught in separate courses (a nice exception is Dieudonné's book, 1969). Sometimes, Linear Algebra and Ordinary Differential Equations are taught together in one course, but usually not. Moreover, numerous topics relevant to applications of Analysis to Geometry disappeared from syllabi a long time ago. Thom (1962) expresses strongly his opposition to this trend.

In the present paper, we show and explore a bridge between two other mathematical fields, Graph Theory and Linear Algebra. Graph Theory is part of Discrete Mathematics, a branch of Mathematics which deals with objects that can be described by either finite or countable sets. In regular courses, Linear Algebra is presented over the real and the complex fields, in which cases it is understood as belonging to the continuous part of Mathematics, not to the discrete part. Linear Algebra over finite fields is taught in advanced courses, not aimed to every student. The discrete point of view provides numerous methods for proving theorems, different from the methods used in a continuous setting (see Grenier 2008). Switching from Graph Theory to Linear Algebra gives an opportunity to use other methods than the typical methods of Discrete Mathematics, i.e. exhaustion of cases (enumeration), induction, and so on. The technology is not responsible for the discovery of the bridge, but it helps to explore it, and then helps to study cases which would be unilluminating with only hand made computations.

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Moreover, we will show that the activities presented here lead to develop new mathematical knowledge simultaneously in two domains. The situation at the course starting point is as follows:

1. Matrix similarity is a standard topic in any course in Linear Algebra. But, as this topic appears at the end of the course, applications to other fields are rarely shown. This was the case for the students whose work is presented in this paper.
2. Algebraic graph theory is absent from numerous textbooks in Discrete Mathematics and from the syllabus of courses.

In one class, the teacher decided to have his students learn at least a few topics of algebraic graph theory, outsourcing to a Computer Algebra System (CAS) part of the operative knowledge. Later, the author had a discussion with a colleague teaching a parallel course. This colleague valued the introduction of these activities which enhance an important mathematical knowledge, but he said that doing the same thing with his own class was impossible, in particular because of the lack of CAS literacy of his students.

"Technology can be used to compute, ..., to reinforce, clarify, anticipate, or get acquainted with ideas, and to discover and investigate phenomena" (Selden, 2005). As showed by Dana-Picard (2005), the exploration of a cognitive neighborhood[2] for a given mathematical topic is mainly concerned by the last two components, discovery and investigation. How investigation can be fostered by switching between registers of representations has been studied by Duval (1999), Arcavi (2003), Presmeg (2006), Dana-Picard and Kidron (2008), etc... We elaborate on this issue in the last section. Not only the mathematical fields are different, but also the ways to use a CAS are different.

Various kinds of technological tools have been introduced into the mathematics classroom and into the researcher's lab, ranging from a graphical hand-held device to an interactive (non user-programmable) website and to a CAS. In particular, their graphical features are emphasized in order to provide visualizations, either fixed or animated, but all the other features, algebraic, numerical, etc., are important and they are used in classroom. In some CAS, algorithms specific to Graph Theory have been implemented, which enable the drawing of a picture of the graph from the abstract definition of the vertices and the edges. For the classroom activities the Derive software has been used. It has no implementation of specific features for Graph Theory and only the Linear Algebra algorithms were used. Note that the Linear Algebra packages of other CAS can assist this activity, sometimes with specific outputs. We address this issue in Section V.

II. The mathematical situation.

An important problem in computational complexity theory is determining whether, given two graphs G_1 and G_2, it is possible to re-label the vertices of one graph so that it is identical to the other, or not. This re-labeling is called *a graph isomorphism* and we denote $G_1 \cong G_2$ .In simple words, two graphs are isomorphic if they can be represented with identical drawings. For example, see Figure 1: the permutation of vertices $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$ preserves the existence (resp. the non-existence) of an edge between vertices, whence ensures the fact that the two given iconic representations correspond to isomorphic graphs. A formal definition of a graph isomorphism can be found in Rosen's book (1999, p. 460).

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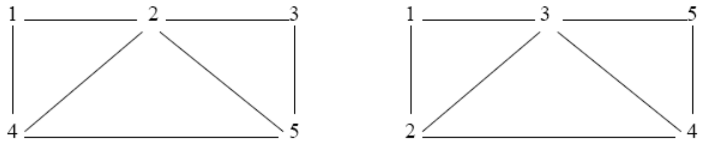
[2] Recall that a mathematical domain A is said to belong to a cognitive neighborhood of another mathematical domain B if theorems and/or methods from B can be applied to solve problems in A.

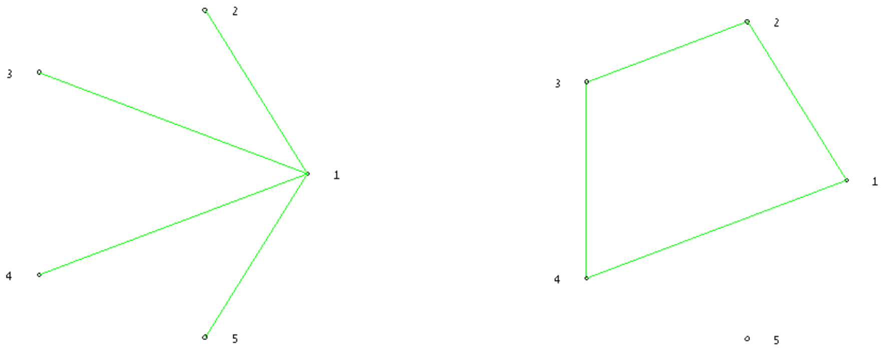


(a) First labeling **(b) Second labeling**

Figure 1: One picture, two sets of labels.

Adjacency matrices are used to describe graphs in a computational way. For a given graph, label the rows and the columns of a square matrix $A = (a_{ij})$ by the vertices of the graph. For a non-oriented graph, a_{ij} is the number of edges between vertices *i* and *j*. For an oriented graph a_{ij} is the number of arrows from vertex *i* to vertex *j*. Thus, the adjacency matrix of a non-oriented graph is symmetric and for an oriented graph the adjacency matrix can be either symmetric or non-symmetric. Relabeling the vertices of the graph changes the adjacency matrix in the same way reordering the vectors of a basis of a *n*-dimensional vector space changes the matrix of a linear operator: the original matrix *A* and the new one *B* are similar, i.e. there exists an invertible square matrix *P* of order *n* such that $B=P^{-1}AP$.

Using adjacency matrices, we translate a problem in Graph Theory into a problem in Linear Algebra. The second one is not easier than the first one. To determine whether two given square matrices of the same order are similar is easy when both are diagonalizable. If they have the same eigenvalues, with the same respective multiplicities, then they have the same diagonalization, up to a re-ordering of the chosen eigenvectors. The set of eigenvalues (each one is written a number of times equal to its multiplicity; for example we write {1,1,2} if 1 is a double eigenvalue and 2 a simple eigenvalue). Suppose that diagonalizations of the matrices A_1 and A_2 exist and are given by $D = P_1^{-1} A_1 P_1$ and $D = P_2^{-1} A_2 P_2$, for appropriate invertible matrices P_1 and P_2 , then $A_2 = (P_1 P_2)^{-1} A_1 (P_1 P_2)$ i.e. A_1 and A_2 are similar. If the matrices are not diagonalizable, similarity is harder to check. Of course, if one matrix is diagonalizable and the other is not, they are non similar. Note that the theorem sustaining the classroom activities is a "if ... then ..." theorem, not a "if and only if" theorem. If the graphs G_1 and G_2 are isomorphic, then their adjacency matrices have the same eigenvalues, but the converse is not true (see Cvetkovič et al. 1995, pages 61 sq.). The smallest known pair of non isomorphic graphs with the same spectrum is given by Skiena (1990, page 85); see Figure 2. Both graphs have two simple eigenvalues 0 and -2, and a triple eigenvalue equal 0, and their adjacency matrices are similar. The non existence of an isomorphism can be found at first glance: the graph in (a) is connected and the graph (b) is not.



(a) **(b)**

Figure 2: Non isomorphic graphs with the same spectrum.

The volume of the computations increases very fast with the number of vertices of the graphs. Here a Computer Algebra System reveals useful for technical assistance on computing. But not only for this assistance. Outsourcing of the computations to the CAS and careful observation of the output may yield a better understanding of the mathematical situation and enhance understanding of older knowledge. "Technology can be used to compute, to reinforce, clarify, anticipate, or get acquainted with ideas, and to discover and investigate phenomena" (Selden, 2005).

III. The study frame.

The Jerusalem College of Technology (JCT) is an Engineering School for High-Tech and Orot College is a Teacher Training College. In both institutions students learn a one-year course in Linear Algebra and an introduction to Graph Theory is given as part of a subsequent course in Discrete Mathematics. Matrix similarity belongs to the Linear Algebra syllabus. For various reasons, this topic has been taught at the very end of the course and quite no application to other fields of mathematics has been shown, beyond the fact that a basis change transforms the matrix of a linear transformation into a similar matrix.

Isomorphisms of graphs are an important topic in the syllabus. Conversations with colleagues teaching parallel courses revealed that students learn generally existence theorems related to degrees of vertices. Several textbooks do not mention more than this and the exercises are based either on the definition only or on such theorems about degrees of vertices (or in-degree and out-degree for directed graphs). Students are often reluctant to use adjacency matrices beyond writing the adjacency matrix of a given graph, or conversely drawing a picture of a graph whose adjacency matrix is given. "The computations are heavy", they say (for example, recall that paths of a given length n are counted using the nth power of the adjacency matrix). Therefore a Computer Algebra System (namely Derive) has been used in the classroom activities, in particular the Linear Algebra algorithms. Note that the Linear Algebra packages of other CAS can assist this activity, sometimes with specific outputs. More than two thirds of the students in the class had a good CAS literacy, as a result of two previous courses strongly based on CAS use. Other students had an opportunity to improve their knowledge and their know-how with regards to software.

The central topic of the activities is new for the students. It has to be introduced and developed explicitly by the teacher, using *strategic scaffolding*, one of the scaffolding categories detailed by Hobsbaum et al (1996); Anghileri (2006, p. 36) elaborates on this issue. The main characteristics are:

- A measured amount of teacher support;
- A careful selection of the tasks and of their difficulty level;
- Students' ability to build a mathematical meaning from the given tasks;
- Explicit strategies.

The extended literature about scaffolding emphasizes the fact that scaffolding is relevant for one student-one teacher situations. Here the teacher had to provide such a scaffolding separately to every student, but also a form of "global" scaffolding to the class as a whole.

Within the global frame of the group, each student can have his/her own learning process. Therefore consolidation of knowledge (Dreyfus and Tsamir, 2004) has to be observed on an individual basis. As the work described in this paper is based on classroom activities, i.e. not in an individual frame, we will elaborate only briefly on the consolidation issue, in the last section.

For the CAS assisted part of the work, we refer to Fischer's (1991) didactical principle of *outsourcing operative knowledge and operative skills*. Peschek and Schneider (2001) regard operative knowledge as a means to generate new mathematical knowledge (see also Peschek 2005). In fact they distinguish three *fields of competence*: basic knowledge, operative knowledge and skills, and reflection. In the following activities, the needed basic knowledge is matrix similarity, acquired at the end of the Linear Algebra course. Because of a lack of time in this course, the topic has been shown but not applied in concrete situations. Students have now an opportunity to manipulate this knowledge in an applied situation. The operative skills are outsourced to the CAS. Most students (but not all of them) had already good operative skills for matrix computations using the CAS, including computation of eigenvalues and eigenvectors. During the sessions, they could improve these skills and discover new commands of the CAS. Moreover, new mathematical knowledge has been constructed, new CAS literacy being part of it.

IV. Classroom activities with CAS.

We present here classroom activities which took place with a group of 15 students (about 20 years old). Their course in Graph Theory comes one semester after the course in Linear Algebra. This enables them to use eigenvalues and diagonalization of matrices in a situation very different to what has been met either in Linear Algebra or in other courses with some geometric flavor. The students were already used to switch from iconic representation to algebraic representation and from algebraic representation to iconic representation.

1. First activity.

Consider the graph with two different vertex labeling given in Figure 1. The respective adjacency matrices are

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

`Operative knowledge:` using Derive's command **eigenvalues**, the students found that both matrices have the same five distinct real eigenvalues. Thus, the matrices A_1 and A_2 are diagonalizable, and for suitable eigenvector orderings, both matrices have the same diagonalization. It follows that the matrices are similar, whence $G_1 \cong G_2$.

`Reflective thinking:`

Mina: This is not new; we knew already that the graphs are isomorphic!
Vered: So what?
Mina: Why did we do all this work?
Vered: We are now convinced that our way of working is right. Not?

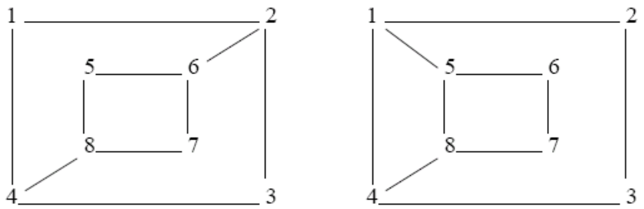
Silence for a while. The second student sees that something still "disturbs" the first student. So she adds:

Vered: We see always a trivial example when learning something new. So we are really sure that the theorem is right. I will do the same thing when I"ll teach.

This remark was important for the teacher. It shows that Vered is aware not only of the new mathematical knowledge she is currently leaning, but also of the structure of the educative sequence.

2. Second activity.

We consider the two graphs shown in Rosen's book (1999, p. 461, example 10); see Figure 3. The vertices of the graph G_1 will be denoted by u_k and the vertices of G_2 by v_k, $k = \mathbf{1,...,8}$.



G_1 G_2

Figure 3: Two non isomorphic graphs

`Reflective thinking:`

Teacher: Let us check whether these graphs are isomorphic or not.
Vered: Easy! We check the degrees of the vertices.

Short silence, everybody computes.

Vered: These are the same degrees.
Teacher: So, what is your conclusion?

Leah: The graphs are isomorphic.
Short silence.
Hadas: Maybe not.
Vered: Why not?
Hadas: The degrees are not at the same place in the two graphs.
A couple of students, together: She is right!

The teacher asks for a clearer explanation of what happens. One student explains that the degree 3 vertices compose a connected subgraph in G_1, but not in G_2. This convinces the class that the two graphs are not isomorphic, but more than 10 students demand what they call "a stronger algebraic proof".

Operative knowledge:
Vered: Let's use matrices as we did before!
Teacher: Good idea, do it. Please write down the adjacency matrices.

The adjacency matrices of the two given graphs are

$$A_1=\begin{pmatrix}0&1&0&1&0&0&0&0\\1&0&1&0&0&1&0&0\\0&1&0&1&0&0&0&0\\1&0&1&0&0&0&0&1\\0&0&0&0&0&1&0&1\\0&1&0&0&1&0&1&0\\0&0&0&0&0&1&0&1\\0&0&0&1&1&0&1&0\end{pmatrix}\quad\text{and}\quad A_2=\begin{pmatrix}0&1&0&0&1&0&0&0\\1&0&1&0&0&0&0&0\\0&1&0&1&0&0&0&0\\0&0&1&0&0&0&0&1\\1&0&0&0&0&1&0&1\\0&0&0&0&1&0&1&0\\0&0&0&0&0&1&0&1\\0&0&0&1&1&0&1&0\end{pmatrix}$$

With Derive's command **eigenvalues**, the students determine the eigenvalues of A_1.

The output is: 0, -1, 1, $\frac{1}{2}+\frac{\sqrt{17}}{2}$, $\frac{1}{2}-\frac{\sqrt{17}}{2}$, $-\frac{1}{2}+\frac{\sqrt{17}}{2}$ and $-\frac{1}{2}-\frac{\sqrt{17}}{2}$.

Reflection:
Teacher: Any comments?
Short silence.
Shira: There are not enough.
Teacher: Not enough what?
Shira: Not enough eigenvalues. There are only 7.
Teacher: What did you expect?
Myriam: Eight.
Teacher: So, what happened?
Short silence.
Vered: There must be one double.
Teacher: Why?
Vered: The matrix is symmetric, it must have a diagonalization.
Teacher: Very nice. How can we know who is the double eigenvalue?
Short silence.
Yael: (with a short hesitation) how can we compute the characteristic polynomial?
Myriam: It's a determinant, there must be a command.
Teacher: Right. Who knows?

Operative knowledge:

A couple of students answer that the command is **charpoly.** The teacher recalls the syntax. With this command, the result is

$$P(\lambda) = \lambda^{8} - 10\lambda^{6} + 25\lambda^{4} - 16\lambda^{2} = \lambda^{2}\left(\lambda^{6} - 10\lambda^{4} + 25\lambda^{2} - 16\right)$$

Vered: Here it is; it's 0.
Teacher: Vered, what is 0?
Vered: The double eigenvalue.
Myriam: How nice!!
Tehila: OK, but what do we do now?
Vered: The same thing with the other matrix.

The computation for A_2 is performed the same way, using Derive. The eigenvalues are

$$\frac{1}{4}\left(3+\sqrt{5}+\sqrt{22-2\sqrt{5}}\right), \frac{1}{4}\left(3+\sqrt{5}-\sqrt{22-2\sqrt{5}}\right), \frac{1}{4}\left(3-\sqrt{5}+\sqrt{22-2\sqrt{5}}\right),$$
$$\frac{1}{4}\left(3-\sqrt{5}-\sqrt{22-2\sqrt{5}}\right), -\frac{1}{4}\left(3+\sqrt{5}+\sqrt{22-2\sqrt{5}}\right), -\frac{1}{4}\left(3+\sqrt{5}-\sqrt{22-2\sqrt{5}}\right),$$
$$-\frac{1}{4}\left(3-\sqrt{5}+\sqrt{22-2\sqrt{5}}\right), -\frac{1}{4}\left(3-\sqrt{5}-\sqrt{22-2\sqrt{5}}\right).$$

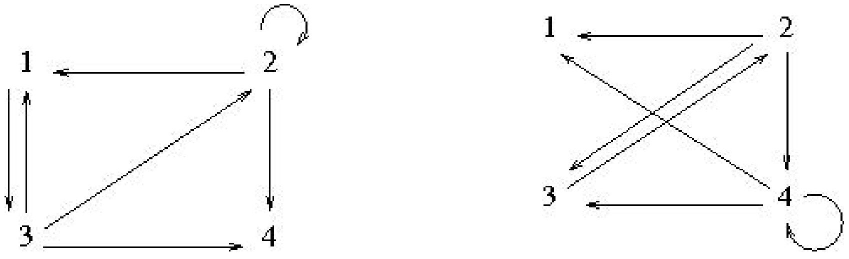
Reflection:
Teacher: What do you see?
Shira and Vered: (at the same time) they are different.
Teacher: Different from what?
Shira: From A_1 .
Vered: We did it! The matrices are not similar.

3. Third activity.

After the second activity, students received homework assignments (check whether couple of pairs of graphs are isomorphic or not). The next meeting took place one week later, with a third classroom activity. The task was to show that two given graphs with the same number of vertices and the same number of edges are non isomorphic. The teacher could let the students work on their own, and almost no intervention was necessary.

The next step in the same meeting consisted in turning students' attention towards similar situations, either with oriented graphs or with multigraphs. Precise definitions are given by Rosen (1999). One example is shown by Figure 4. The class dealt with the new situation using the same algebraic and technological tools, but in a different algebraic situation. The same CAS commands were used as in the previous sessions.

First example. The graphs in Figure 4 are given and the students are asked to check whether they are isomorphic or not.



The graph G_1. **The graph G_2.**

Figure 4 : Two oriented graphs - first example .

The graphs are oriented graphs, and their respective adjacency matrices are not symmetric. We have:

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

A few students note that the theorem on the diagonalizability of symmetric matrices cannot apply, and do not know how to proceed. A couple of students propose immediately to use the CAS. They determine the eigenvalues of A_1 and the eigenvalues of A_2: for matrices, the eigenvalues are $0, \frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

Myriam: There are only three.
Shira: Yes, one is double.
Myriam: (Asks the teacher) We look for eigenvectors?
Vered: Yes, with the computer.

Most of the students determine the eigenvectors with the CAS and conclude that both matrices are diagonalizable, with the same diagonalization, whence the graphs are isomorphic.

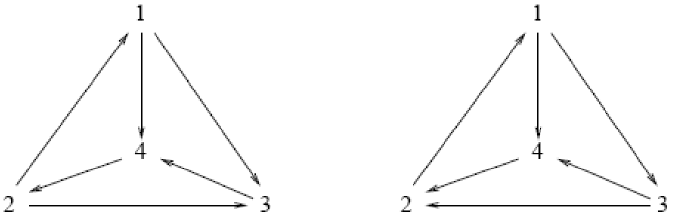
At this point, something interesting happens.

Yael: I computed the characteristic polynomial of the matrices. It is the same. So they are surely similar.

At the same time, one student says ``yes!!", and another one says ``No! you don't know!". A discussion follows, recalling that having the same characteristic polynomial is a necessary condition for matrices to be similar, not a sufficient condition. The student who said ``no", named Rachel, explains that they must look for eigenvectors.

Rachel: There exists a basis of eigenvectors for each matrix, therefore they are similar (she means ``the graphs are similar").

Second example. Now the students are given the graphs displayed in Figure 5.



The graph G_1. **The graph G_2.**

Figure 5: Two oriented graphs - second example .

Their respective adjacency matrices are

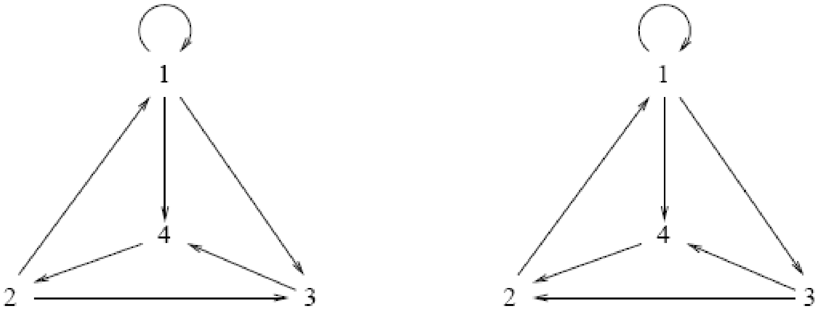
$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Here all the students follow Yael's way and compute the characteristic polynomials. In both cases they obtain $P(\lambda) = \lambda^4 - 2\lambda - 1$. Using once again the software, they determine the eigenvalues. Most of them appear with very complicated expressions (complex numbers whose real part and imaginary part are given by non rational expressions).

Vered: It's ugly!
Teacher: Why?
Vered: Impossible to understand.
Teacher: Why?
Rachel: Complex numbers.
Teacher: Is this a problem, from an algebraic point of view?
Vered: But they are four.
Teacher: So, what is your conclusion?
Vered: We did not learn matrices with complexes, but ... (She waits a few seconds) this means that the matrices are diagonalizable?

Finally, the teacher has to explain that here the algebraic properties (for determinants, characteristic polynomial of a matrix, etc.) are the same over the reals and over the complex numbers. The class concludes that the two matrices are similar, whence the two graphs are isomorphic.

Third example. Finally, the teacher modifies slightly the graphs and gives to the students the graphs displayed in Figure 6



The graph G_1. **The graph G_2.**

Figure 6: Two oriented graphs - third example .

The respective adjacency matrices are

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

All the students but two compute immediately the characteristic polynomials. They are respectively $P_1(\lambda) = \lambda^4 - \lambda^3 - 2\lambda$ and $P_2(\lambda) = \lambda^4 - \lambda^3 - 2\lambda - 1$. The conclusion is shouted by three students at the same time: ``they are not similar!"

Teacher: Who are not similar?
Yael: The matrices.
Teacher: Why?
Naomi: (speaking for the first time) The polynomials are different, so the eigenvalues are different.
Teacher: All of them?
Naomi: No. At least one. And here we have 0 for A_1 and not for A_2.
Teacher: Remind us what the question was?
Yael: If the graphs are isomorphic.
Vered: OK, the graphs are not isomorphic.

3. Brief description of further activities.

Further work and activities have been done with the same class. After the second activity, students received homework assignments. The next meeting took place one week later. A central task was to show that two given graphs with the same number of vertices and the same number of edges are non isomorphic. For this, almost no teacher intervention was necessary. Another task was devoted to understanding the non-reversibility of the theorem described at the end of the first section (see Cvetkovič et al. 1995, pages 61 sq.); its description does not fit in this paper.

Part of the third meeting was devoted first to oriented graphs because they may provide non-symmetric adjacency matrices, i.e. matrices which can be non diagonalizable. Then the students had to study a pair of non isomorphic graphs having the same set of eigenvalues. The goal of this last example was to convince the students that the whole study relied on a one-way theorem: if two matrices are similar, then they have the same set of eigenvalues, but the converse is not true.

After the last meeting, the teacher had an informal discussion with the students. He asked for remarks about the CAS assisted work. Here are a few excerpts from the discussion between students (the first one was not previously quoted):
Student A: I'm sure that I would not have worked out all these examples by hand.
Student B: And so? You would not have learnt this?
Student A: No, I would have waited to see the answer from somebody else.
Student B: And so you would not have learnt the topic!
Student A: (hesitating) Maybe you are right, …, not so well.

V. Discussion.

1. The classroom activities.

In the first activity, the teacher chose an example where the isomorphism between the graphs is trivial. The graphical display itself proposes an invertible mapping between the sets of vertices. This enabled the students to discover how to work, in a situation where they have control on the results. Vered expressed this clearly. During this first activity, the students gained conviction that the working pattern is suitable. Therefore they were more independent from the teacher during the second activity. He helped somehow with passing from one step of reflective thinking to the next one, or with providing some new operative knowledge, such as an appropriate command of the CAS. The teacher's support was gradually faded; it was limited to questions. Reflection and interpretation were made by the students.

Teacher's support has been gradually removed during the third activity. At the end all the students but two were totally independent of teacher assistance. This has been checked with an assignment which included the study of one pair of non oriented multigraphs and of one pair of oriented multigraphs. Finally, the educative segment has been spread over a little more than two weeks, and gradually developed, meeting Anghileri's request (2006). The strategy has been made clear already from start:

- A progressive choice of examples: non oriented graphs, in order to have benefit of the theorem on the diagonalizability of symmetric matrices, then non oriented multigraphs and oriented graphs for which the theorem does not apply.
- Translation into notions from Linear Algebra and use of a CAS.

2. Switching between representations.

The original definition of a graph as a pair of sets (V, E), where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices and $E = \{e_1, e_2, \ldots, e_p\}$ is the set of edges, contains in itself a first kind of representation. Let us call this an *enumerative representation*. For small n and small p, it is possible to prove that two given graphs are isomorphic by *construction* of a specific isomorphism. Such a proof by construction becomes quickly unilluminating when the number of vertices and/or the number of edges increases. In a situation where the graphs are not isomorphic, besides the "boring" aspect of enumeration, there is a need to prove that all the cases have been considered (proof by exhaustion of cases). This is a formal proof, using combinatorial formulas, i.e. the point of view has been partly switched towards Combinatorics. Of course switching from the *enumerative representation* (V, E) to an *iconic representation* helps, but increasing *n* and/or *p* has a similar effect in the new setting as in the old one.

Considering only *iconic representations* of graphs does not yield enough insight into the concept of an isomorphism of graphs beyond simple examples, as suggested by Leah's reaction, and more by Student A in the last discussion. The matrix representation and its companion algebraic tools provide a possibility to have a more profound insight. "In some cases the current representations may prove an obstacle to the full development of a concept" (Ferrari 2003). Out of the record, students claimed that iconic representation is more readable for them, but others said that they felt more comfortable with matrix representation, as "they can do computations". A foreign colleague of the author said (free translation): "I observe everyday researchers in Discrete Mathematics and in its Teaching. I see that, most of the time, they work with iconic representations, and not with matrices. They find numerous theorems. And also a lot of consistent situations for the students to work (colors, Euler paths, etc.). But it's different from what you do, and one completes the other".

Despite the fact that matrix representation is more abstract than the graphical one, it opened the way to new mathematical knowledge, through manipulation both of old knowledge coming from another field and of the usage of a CAS. A great diversity of situations could not have been presented using graphical representation only (see Lesser and Tchoshanov 2005). Actually we may view the working sequence as a two-step activity:

a. Switching from the iconic representation to the matrix representation, according to Peschek (2005), as "one abstracts relationships from the (reference) context and presents them with symbols, thus outsourcing the problem in the formal-operative system of mathematics".
b. Outsourcing (part of the) operative knowledge to the computer.

Graphs, multigraphs (whether oriented or not) are defined as abstract objects, namely a pair of sets with a suitable property linking them (see Rosen 1999). We have here two presentations for a graph:

- The graphical presentation is visual/iconic (Lesser and Tchoshanov 2005) and acts as "stimuli on the senses" (Janvier et al. 1993).
- The other representation is algebraic. It is a symbolic representation enabling manipulations.

As noted by Lesser and Tchoshanov (2005), a single type of representation does not insure student learning and performance. In many occurrences, a graphical representation is used to "encode" more abstract properties. That is the case with the study of a function: the first steps, namely finding the domain and the possible symmetries, computing limits, derivatives, checking domains where the function is monotonous, where there are (eventually) extremal points and/or points of inflection, are then encoded into a graphical representation. It happens that this representation has to be fractioned into pieces, because an impossibility to represent all the special features in one graph (see Dana-Picard 2005). Here the *symbolic-algebraic representation* is a useful tool for the student to *understand the graphical situation* and to gain a more profound insight.

3. The switching direction.

We wish to emphasize an interesting aspect of the work. As mentioned in first section, similarity of matrices is a topic which had been taught in a previous course, but a lack of time enabled the teacher to give only a small number of examples. Students' personal work suffered also of this lack of time for practice. A similar situation occurs generally for the study of graph isomorphisms, but for a slightly different reason. The amount of necessary computations increases very fast with the number of vertices in the graph. So the teacher may decide either to limit himself/herself to examples of graphs with only a few vertices, or to present larger graphs but showing only the results. In both cases, students do not acquire practical skills; they have no real opportunity to improve their operative knowledge. Activities built on the switching between representations, iconic and algebraic, supported by CAS, enabled to really build new mathematical knowledge in both domains, Graph Theory and Linear Algebra, simultaneously. This enhances the fact that each topic can be viewed as belonging to a cognitive neighborhood of the other. Generally bridges are built in one direction, from topic A to topic B, but here the bridge between the two topics is traveled in both directions when switching from iconic representation to algebraic representation and conversely.

The CAS provided the help ``for reasoning by fostering the development of ... experimental reasoning style'' (Sinclair et al. 2006). This appears through the intertwining of reflective thinking and application of operative knowledge during the sessions. A difference appears with the human support: not only the CAS assistance does not fade with time, but the new computing skills become an integral part of the new mathematical knowledge.

In the second activity, different CAS may give different outputs when displaying the eigenvalues. For the given square matrices of order 8, Derive gives seven different eigenvalues, inviting the student to understand that one of them must be a double eigenvalue. Note that other packages may give a more detailed output, including the multiplicities of the eigenvalues. We have here an example of the *double reference* evoked by Artigue (1997, page 152): on the one hand, the computer ``understands'' the input in a way which can be different from the students' intention, on the other hand the mathematical meaning of the output can be different of what the student expects when he/she writes the same thing. See also (Lagrange, 2000).

Students working with a CAS become progressively acquainted with swapping between various representations: algebraic, numerical and graphical. For a given object, different representations can be provided by the CAS itself. Functions of one real variable are a well documented example, with numerical representation (a table of values), graphical representation and generally algebraic representation (a "closed form" such as $f(x) = \ldots$). The main problem is developing students' ability to link representations; see Pierce (2001). Prior to the activities described in this paper, the students had to solve a couple of exercises in reversed directions: a) write the adjacency matrix of a graph (resp. directed graph, given in iconic form, b) draw a picture of a graph whose adjacency matrix is given.

The activities performed here by the students have a different aspect. The main originality lies in the fact that the link is not oriented from an algebraic representation towards a graphical one as in most problems on one-real-variable functions, but in the other direction. Graphs are given by graphical representations and the representation used for checking the existence of an isomorphism between two graphs is purely algebraic. This may be technically trivial, but from a conceptual point of view, it is not trivial for the students: the work requires reversing the direction of the switch between representations. The students' hesitations reveal their level of ability to deal with a matrix representation instead of a graphical one. Previous working sessions revealed the difficulty for students to link matrices to graphs and graphs to matrices (including oriented graphs, i.e. links towards non-symmetric matrices) but helped with removing the obstacles. The CAS provided assistance, and students showed increasing operative knowledge.

4. Consolidation and routinization of previous knowledge.

In the same fashion we had to be careful when speaking about scaffolding, we must be careful if we wish to deal with *consolidation*. Both are very personal and apply to individuals, one student at a time. Here our study relies on the dynamic of a group of students. Each student has his/her own pace of acquisition of new mathematical knowledge, and consolidation should be checked with each student separately. The above classroom activities do not provide enough individual data.

Nevertheless, the observation of the group reveals various components of consolidation among those enumerated by Dreyfus and Tsamir (2004): immediacy, self-evidence, confidence, flexibility and awareness. For example, along the different activities, there were more and more immediate reactions to questions, either immediate answers (revealing also self-evidence) or immediate and correct outsourcing of the work to the computer. This last point is part of the ability to switch between different representations of the graphs (flexibility). From the beginning, Vered showed enough self-confidence to answer and ask, but for others like Naomi, the first intervention appeared during the third activity.

The activities revealed also the following fact: at the beginning, the students did not achieve for symmetric matrices and their diagonalization the routinization mentioned (and requested) by Artigue (1997). In Section II, we mentioned the lack of time at the end of the Linear Algebra course, which provoked a shortage in solved examples. Even for low dimensions, a lot of computations are needed, looking for eigenvalues and eigenvectors, inverting matrices, and so on. Hand computations are very unilluminating and both educators and students are reluctant to do them. The students had here an opportunity to make full computations of eigenvalues and eigenvectors, and sometimes of the diagonalization of a matrix. An important progress towards the requested routinization has been made as a byproduct of the activities. Moreover they had an opportunity to deal with a concrete problem involving these tools. The CAS was a facilitator, making examples of higher dimension possible to treat, thus enabling students to acquire an extended operative knowledge, and at the same time more mathematical insight.. The CAS has not been used as a black box, but rather as an assistant in a process of *reasoned instrumentation*. We meet Elbaz-Vincent's requirements (2005) about ``the necessity of developing specific classroom activities and specific exercise sheets, ..., showing clearly the value of the CAS either as a platform for experimentation or as an assistant ..."

CAS-assisted work had another side effect. For non isomorphic graphs, the following cases can appear:

a. The adjacency matrices have different characteristic polynomials.

b. The adjacency matrices have the same characteristic polynomial, whence the same eigenvalues with the same multiplicities, but one of the matrices is diagonalizable and the other one is not.

At the beginning, the command **eigenvalues** has been used without reference to the characteristic polynomial. The necessity to obtain more information, and to know how to interpret the output, has revealed the necessity of another command. During the activities, a black box has been opened and examined.

5. The role of CAS: further characteristics.

The assistance provided by the CAS is useful only if the students are able "to plan correct operations and to interpret results intelligently" (Fey 1990, quoted by Pierce 2001). Two remarks made by students emphasize this issue:

(i) In the second activity, Shira's remark on the number of eigenvalues is important. It has been provoked by Derive's output, where the eigenvalues are given, without mention of their respective multiplicities.

(ii) The meaning of Vered's claim "we did it" is non trivial. She noted that, despite the regular usage of a CAS to provide explicit numerical results, this time the actual eigenvalues of the matrices were quite irrelevant. The important issue was the comparison between the two sets of eigenvalues. Vered has understood that the fact that the eigenvalues are not the same is the important issue.

There are not so many opportunities to convince students that either the precise or approximate values of results are not the only interesting output. In this study, we found a couple of occurrences where the precise values of the matrix eigenvalues were not interesting. The point was in the comparison between the sets of eigenvalues. The outsourcing of the computations has an effect beyond the computations themselves. The CAS assisted activities described in section III are an example of the claim by Cuoco and Goldenberg (1996): "...we are talking about using technology in support of the hard thinking, not for performing the low-level details". More than acting as a calculator, the CAS worked here as an *assistant to reflection*.

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Sum of n Consecutive Numbers

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Theorem
For all n, it is always possible to find at least one sum of n consecutive numbers with an equivalent sum of $n-1$ consecutive numbers**?**

Until recently I did not realise that this wonderful pattern existed.

1+2=3
4+5+6=7+8
9+10+11+12=13+14+15
16+17+18+19+20=21+22+23+24
etc

My first thoughts on reading this connection in O'Shea's[1] book about number curiosities, was why had I not read about it before? It is simply beautiful. O'Shea writes but a few lines on it, and then moves on to his next strange fact. This made me wonder if the pattern would always be true, and here is my proof that it is.

Note that the LHS always starts with a square number. This will always be true, as square numbers occur in the natural number system as follows; 1, 4, 9, 16, 25, 36… with a common difference of 3, 5, 7, 9…..which you can see fits the pattern above. Therefore we can say that each line will always start with a square number. The first few lines in the pattern can be shown to be true, hence it can be proven that the pattern is true for all natural numbers, by considering that the next k th line in the pattern is true.

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$$k^2 + (k^2 + 1) + (k^2 + 2) + ... + (k^2 + k) = (k^2 + k + 1) + (k^2 + k + 2) + ... + (k^2 + k + k)$$

To show that LHS equal RHS, collect the k, k^2 terms on both sides

$$k^3 + k^2 + (1) + (2) + ... + (k) = k^3 + (k + 1) + (k + 2) + ... + (k + k)$$

Then collect $(1 + 2 + 3 + ... + k) = \frac{1}{2}k(k + 1)$ on both sides, giving

$$k^3 + k^2 + \tfrac{1}{2}k(k + 1) = k^3 + \tfrac{1}{2}k(k + 1) + (k) + (k) + ... + (k)$$

$$k^3 + k^2 + \tfrac{1}{2}k(k + 1) = k^3 + \tfrac{1}{2}k(k + 1) + k(k)$$

Therefore true for k, hence true for all and proof of the above theorem.

Corollary

In each line in this natural number pattern, we find a triangular, square and cube number sequence.

Angel Proof – "as if at a glance"

Between each pair of square numbers there are $2n$ numbers, n on the LHS and n on the RHS of the above pattern. By adding n to each of the n numbers on the LHS we obtain the RHS.

Reference

[1]The magic numbers of the professor by Owen O'Shea & Underwood Dudley [Page 135]
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The Contributions of the Comprehension Tests to the Cognitive and Affective Development of Prospective Teachers: A Case Study

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Abstract: The aim of this study was to investigate the role of comprehension tests while teaching algebra and its effects on students' success. This study was carried out with 108 third year undergraduate students enrolled in math education in faculty of education. Several data collection instruments were used for gathering data from the participants such as; comprehension test, written documents, semi-structured interviews schedule, and participant and nonparticipant observations sheets. Collected data were subjected to content analysis and triangulation among the data was ensured. Results indicated that three different major categories emerged from the content analysis of the data: (1) measure of comprehension test(s), (2) positive and (3) negative impacts of the test on learning (the development of cognitive and affective skills). Further recommendations and implications about the use of comprehension tests is given at the end of the study based on the findings.

Key words**:** Algebra teaching, comprehension tests, undergraduate mathematics students.

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Introduction

Assessment is a complicated process which may influence the beliefs on the nature and knowledge of mathematics, and educational process, on instruction and the connection among individual, school and society. This viewpoint regarding assessment is crucially important for reform attempts to be realized in educational process (Ridgway&Passey, 1993). Today, the tests are usually used for assessing outcomes and give more emphasis to products and rote learning rather than students' progress. They have only interested in the products of the learning. These tests do not provide adequate evidences for teacher to design their one lesson plans based on the needs.They seem to be limited while determining students' thinking process, strategies and learning potentials (Ginsburg, Jacobs&Lopez, 1993). For this reason, these tests need to be supported with several other assessment methods to assess students as a whole and needs to be re-designed according to new educational approaches and development in the world (Romberg, 1993). Essentially, NCTM (1989) see that standardized achievement tests do not reflect and measure students' general success, readiness or measure any program you want to measure. In traditional way of assessment in maths, quotations of "… explain the theory and prove it" or "… prove the theory" include these kinds of questions and help try to understand and to assess the students' knowledge about the theories. However, there are some deficiencies of these types of assessment and they are as follows; (Conradie&Frith, 2000):

a) Whichever the students have ability, students should need to memorize some facts in order to respond the types of questions given above.

b) The students can only concentrate on the concepts stressed in the lessons as a result of studying and memorizing the results and theorems taught by the instructor/teacher.

c) The answers of the questions mentioned above cannot be easily assessed.

d) The feedback to be gathered from the students with low success will be quite limited when they are testes in this way and it would be hard to diagnose their deficiencies of understanding or learning.

The insufficient results of the assessments activities performed by use of the traditional methods direct the educators to seek for alternatives. In this sense, the comprehension tests are proposed as alternative tests to the traditional ones. These tests were first used at the beginning of 1990s. It has been believed with the use of these tests that the traditional methods does not seem to adequately measure the students who have low math success, but have potential to learn maths (Frith, Frith & Conradie, 2006). The comprehension tests are dynamic tests that provide chances the instructors/teacher to add several things to the questions in order to help students find the correct answers. The test also includes instruction regarding the concepts to which the students are not familiar. The concept of dynamic test assesses the learning potential of the learners and concentrates how the students learn rather than what they learn (Feuerstein, 1979; cited in Frith, Frith&Conradie, 2006). For example, in these tests, students' ability is directed from teaching to learning; students generalize the results, make interpretations and practice the definitions. In the construction of the comprehension tests, the basic points to be paid more attention are that the test should include the basic maths concepts and the language used in the test should be simple and clear (Frith, Frith&Conradie, 2006).

Purpose of the Study

It is known that the traditional teaching methods don't affect the students' maths learning, not develop their attitudes towards maths and also not improve the students' academic success (Alanis, 2004). Moreover, traditional measurements and assessment instruments do not show the students' knowledge level and also how they apply their knowledge (Hiebert&Carpenter, 1992). Due to the limitations of traditional assessment methods, researchers have been tried to find out the alternative methods for assessing students' performance. Furthermore, the researchers who are dealing with maths teaching are trying to design alternative methods to the traditional teaching methods. The comprehension tests are observed to be one of these methods. However, the learning and teaching environment should be re-designed in order to implement the comprehension test effectively.

The purpose of the present action research study was to investigate the 3 year undergraduate math students' perceptions of the usage of comprehension tests and their effectiveness in math teaching. Following research question guided the overall study.

What are the perceptions of undergraduate math education students' regarding comprehension tests and their effectiveness?

2. METHOD

2.1. Research Design

This action research was carried out by making use of both qualitative and quantitative research techniques. The action research is a type of investigation of reel world functions and their effects, and is used such areas as for up-dating instructional methods and evaluation procedures and for increasing instructional effectiveness of teachers (Cohen, Manion & Morrison, 2000). According to McKernan (2000) the action research is a systematic self-reflective scientific investigation used by either researcher or teacher for developing personal understanding and application regarding an emerged problem. This action research was realized in the fall semester of 2007-2008 academic year in the class of "Introduction to Algebra".

2.2. Participants

This sample of the study comprised 3rd year undergraduate students who took the course titled as Introduction to Algebra from Department of Math Education in Faculty of Education. The participants will become a math teacher one year later and will be appointed to second level of primary education. Until this semester, the participants have taken the following math related-classes; Abstract Mathematics, Calculus I, Calculus II, Geometry, Calculus III, Linear Algebra I, Calculus IV, Linear Algebra II. This semester, they took Analytic Geometry in addition to Introduction to Algebra.

It is believed that since a comprehension test mainly includes basic mathematical knowledge due to its nature (Frith, Frith & Conradie, 2006), the classes that the participants took up to know are prerequisite for using comprehensive tests. In the research, Introduction to Algebra was preferred because this lesson includes basic mathematics knowledge and also is sufficient for the comprehension tests because the course is based on the theory. The course of Introduction to Algebra is offered in one semester (14 week) in 3 hours. The participants of the study were selected among the students taking this class. 108 volunteered to participate in the study.

2.3. Data Collection Instruments

Several data collection instrument were used for gathering data from the participants. These instruments are as follows; (1) the comprehension test, (2) written documents, (3) semi-structured interviews, (4) participant and non-participant observation.

2.3.1. Development of the Comprehension Test and its Administration

The maths comprehension tests aim at assessing the university students' math knowledge as well as revealing undergraduate students' potential of success (Frith, Frith & Conradie, 2006). In other words, they entail process rather than product. Introduction to Algebra lesson was considered in this manner and re-designed for effective administration of these tests. Instead of asking "… prove the theory?" or "show…?" which are traditional teaching methods questions, the questions and theorems were given to the students with solutions and proofs and sometimes with their solutions and proofs together but without theory and problem explanations. In this way, the students' ideas about the theory and problems were aimed to be investigated. The theory proofs and the solutions of the problems and the critical points, and transition points were numbered as 1, 2,…etc, the students were asked to consider these numbers while answering the questions. During the implementation of the course, a discussion among the students was created with the questions of "where does it come?", "why do we write it?" "if it was like that, what would happen?". Students' were randomly selected for the asked questions. This way enables the students to be awake all the time during the instruction and raise their attention to the discussion. Through the instruction, informal feedbacks were regularly gathered from the students. Also, the students who don't answer the question correctly were protected by the researcher, (the researcher and the students both in the lessons and without the lesson activities, they know each other) and the researcher create a safe and convincing environment to have a confidence. After a 2,5 months teaching and learning period, a first comprehension test with four questions were given to the students and they were asked to answer the questions. These four questions were all regarded as group theory and proofs. Two of these questions' proof were already shown in the instruction, the rest were given as an assignment. However, some parts of the proved theories were not stated in the comprehension test. Moreover, students were not informed about the test day, they only knew that test will be administered at any time within three weeks. This situation enabled the students to get high motivation toward the class..

Comprehension Test Examples

Two of the questions of comprehension test are given below. Whereas 3rd theorem was not proved in the class and given to the students as an assignment, 4th theorem proved in the class. Simple and clear language was used while writing test items. Below example was designed to reveal the strengths and weaknesses of the students while proving the theorem. Conradie and Frith's (2000) study was taken as a base, 3(a), 3(d), 3(e), 4(d), 4(e), and 4(h) were concepts used for the proof, 3(b), 3(c), 4 (b), and 4(c) are important steps in proof, 3(f), 3(h), 3(i), 4(f), and 4(g) are regarded as the structure of proof, 3(g) and 4(e) are regarded as sensitive points, and 4(i) is for investigating students' knowledge of proof methods.

Item 3.Read the theorem below and prove it.

Every group is isomorphic to a permutation group (Cayley's Theorem).

Proof: Let g be a fixed element of G and consider the mapping λ_g: G →G defined by $\lambda_g(x)=gx$ for all x in G **(1)**. $\lambda_g(x)=\lambda_g(y)\Rightarrow gx=gy\Rightarrow g^{-1}(gx)=g^{-1}(gy)\Rightarrow x=y$ for all x and y in G **(2)**.For $z\in G$, $\lambda_g(g^{-1}z)=g(g^{-1}z)=z$ **(3)**. Let S_G be the set consisting of the mappings λ_g. Now consider the mapping $\lambda:G\rightarrow S_G$ defined by $g\rightarrow\lambda_g$ **(4).** $\lambda_{(g_1g_2)}(x)=(g_1g_2)x=g_1(g_2x)=\lambda_{g_1}(\lambda_{g_2}(x))=(\lambda_{g_1}\lambda_{g_2})(x)$ for all x in G, then which implies that $\lambda(g_1g_2)=\lambda_{g_1g_2}=\lambda_{g_1}\lambda_{g_2}=\lambda(g_1)\lambda(g_2)$ **(5)**. In addition,

$$g_0\in Ker\lambda\Leftrightarrow\lambda(g_0)=\lambda_{g_0}(x)=x=g_0x \qquad \textbf{(6)}$$

$$\Leftrightarrow g_0=e.$$

Answer the questions below.

a) What is the aim of the function defined in (1)?

b) What is shown in (2) ?

c) What is shown in (3) ?

d) Is S_G a group? Why?

e) Why is it needed to define of λ in (4)

f) What is it shown in (5)? Why is it needed?

g) What is the identity element of S_G and how is it used in (6)?

h) Is λ surjective? Why?

i) How can the proof be ended with the start of (6)?

(4) Read the proof below and answer the following questions.

Write $<a>=\{...,a^{-2},a^{-1},e,a,a^2,...\}$. Denote this set by H. Then, for $\forall i\in Z$, $a^i,a^{-i}\in<a>$ **(1)**. However, $H\neq\phi$ **(2)**. Let $x,y\in H$ then there exist $k,l\in Z$ such that

$x = a^k, y = a^l$ **(3)**. We get $xy^{-1} = a^k (a^l)^{-1} = a^k a^{-l} = a^{k-l} \in H$ **(4)**. Hence, we get the result we look for **(5)**.

a) What is the method of the proof here?

b) How was it ($\forall i \in Z$ için $a^i, a^{-i} \in < a >$) found in (1)?

c) What was show in (1)?

d) Why is $H \neq \phi$ in (2)?

e) How were they ($x = a^k, y = a^l$) equated? Is $k, l \in R$ taken? Why?

f) What was showed in (4)?

g) It is given $a^{k-l} \in H$ in (4) What is the reason of this?

(h) How is ($(a^l)^{-1} = a^{-l}$) in (4) written?

(i) What was found in (5)?

2.3.2. Written Documents

After the comprehension test including above items was completed, the students were asked to write a composition including their opinions regarding the test and the instruction. Following questions were asked to students for guidance purpose; “What do you remember when you hear comprehension test?” and “What are your opinions about the instruction based on the comprehension test?” at the very beginning, the students were informed that their composition will be used for formative purpose and will shape the instruction. And further their writings will be evaluated and the results will be shared with them. The students wrote their compositions in 20-25 minutes.

2.3.3. Semi-Structured Interviews

The students interviewed were selected with the purposeful sampling method. Further, their responses in the comprehension test also play crucial role. Selection was done according to their answers in the test. They were grouped into three parts with regard to success in the test as lower, medium and higher. This helped observe the effects of comprehension test on the students with different achievement level and get more sound results. For ethical concern, the students were left free to attend the study. At the very beginning, 22 students who were suitable for the study were selected, but 5 of them indicated that they didn't want to participate in the interview. Thus, the interview was realized with a total number of 17 students; 9 females and 8 males. 5 of the participants were from high level, 4 of them from medium level and 8 of them from lower level. Individual interview was conducted with four of them, and the rest was grouped into three and groups interview was performed with these groups separately. It took nearly 60 minutes. Students were asked not to give their name during the interview because of confidentiality reason. At the beginning of the interviews, the purposes of the interview were explained and the researcher used the questions of "why?", "explain", "how?" to get in-depth perspectives of participants. The clinical interview method was used (Gingsburg, 1981) and tried to obtain details of the students ideas regarding interview questions. All these interviews were carried out in a room of the researcher and the place was observed to be quite safe and comfortable. In the group interviews, the researcher used the tape-recorder and later transcribed verbatim and further all participants approved their own transcripts. On the other hand, the researcher only took notes while doing an individual interview since no permission was obtained from the participants.

2.3.4. Participant Observation

Since the researcher was the administrator of the instrument at the same time, he closely observed the students during the instructions and test administration. The researcher obtained students opinions through his informal discussions with the students and recorded these opinions into his note-book. He used them for up-dating the instructions.

2.3.5. Non-participant Observation

Total three lessons, each of which was randomly selected from different group, were video-typed. Total record time was about 50 minutes for each video-record. These records were analyzed by three experts: the researcher himself and two other educational specialist experts. Also, these specialists came to one of the classes to examine the one hour lesson. One of the specialists (educational sciences expert) focused more on the relationship between students and researcher (instructor), the attitudes of the students to the instruction, the teaching-learning environment. Based on what the specialist indicated, the instructor started to observe some of the students more closely and detect their learning difficulties. The other specialist (science education expert) focused more on the transitions between concepts and operations.

2.4. Data Analysis

The analyses of the data collected were continued until they reached saturation. In that way, the data was defined, explained and classified. The constant comparative method was used while analyzing the written documents, the interview transcripts and observations. The constant comparative method consists of open, axiel and selective coding steps (Glaser & Strauss, 1967; Strauss & Corbin, 1998). In the open coding step, firstly, the participants written answers and the responses in the interviews were read more then one without considering any theory so as to understand the data logic, and then they were coded. 229 open coding were observed at the end of the coding process. Some of the open coding examples are, “to be directed to search” (open coding: 10), “I didn’t say that the teacher asked a hard question, it was hard to say that it was teacher’s fault and I thought I memorize the many knowledges.” (open coding: 16), “When I solved the test, I understood some of what I didn’t understand before.” (open coding: 173). In axial coding step, after the researcher examined the details of the opening codes, three categories and corresponding sub-categories were emerged. These categories were (1) What does the comprehension test measure?, (2) The positive effects of the comprehension test?, and (3) the negative effects of the comprehension test? In the selective coding step, the relationship between the sub-categories and the major categories and other data were investigated and a central (core) category, which cover all major categories and would explain the phenomenon, were tried to be revealed.

2.5. Trustworthiness of the Study

A triangulation was ensured among the data collected through the comprehension test, written documents, semi-structured interviews, participant and non-participant observations. No changes (wording, sentences…etc) were done over the interview transcripts and later the participants were asked to confirm what he wrote in the written documents and what the talked during the interview. Also, students' ideas and their perspectives were given in the text without any change. The analysis of the data were done during a process and until it reached the saturation, and it was observed that different participants reported similar results. These all showed that the study can be replicable. Moreover, the class activities and video records were analysed by the specialists and their ideas were considered. The content of the comprehension test was examined by an expert (math education expert) on algebra. The categories, their sub-categories and their appropriateness were given two educational specialists who know the qualitative research and coding procedures very well. Test items and codes emerged were revised by considering the feedback taken from experts. One of the taken feedbacks was given below.

Example (feedback): In category 2, whereas "the communication" the sub-category was explained by the researcher as "to realize the deficiency of mathematical communication", the maths education specialist indicated that this communication referred to the differentiation among the table, graphics, verbal statement and symbolic representation, and it should be rewritten and find out the students ideas, or needed to be determined what they referred to communication. After that, this category was re-defined as “explaining the problems in verbal format” and “using the symbolic language of maths”.

At the end of the examination of all categories, the concordance correlation coefficient between the researcher and the maths education specialist was calculated.86 between the researcher and the science education specialist was calculated .88.

3. FINDINGS

After data analyses,, three categories were emerged. These categories were explained below:

Category 1. What does the comprehension test measure?

This category consisted of three sub-categories such as, “knowledge”, “individual differences” and “teaching-learning process”. Furthermore, the sub-category of knowledge was broken down into knowledge level, use of knowledge and self-assessment; the sub-category of individual differences included the capacity of understanding, learning differences, readiness and concentration; and the sub-category of the teaching-learning included learning in the process and the continuing the lesson. Some of the quotations about the descriptions of students’ written answers are as follows.

> “In our exam system, learning was until the exam. But, in comprehension test, even in the test, I observed that I educated myself in the way of learning and thinking in the process” (open coding: 188; sub-category: teaching-learning process, description: learning in process).
>
> “It helped not to forget the previous subjects. It helped both teachers and students look ahead, and helped the students construct their own learning foundation” (open coding: 165; sub-category: individual differences, description: readiness).
>
> “Actually, I realized that I have not learned, but I assumed that I have learned. Although I have studied math, I realized that I am insufficient in the theory part of math for three years, and at the same time I understood that I am little clumsy in the this subject” (open coding: 159; sub-category: knowledge, description: knowledge level).
>
> “It requires using all of your math knowledge that you earned during your life” (open coding: 78, sub-category: knowledge, description: use of knowledge).

When the quotations above were examined, the comprehension tests provides the students with the opportunities to determine their own knowledge level, to what extend they use their knowledge and their readiness level. Two of the quotations showing that the comprehension test measure the knowledge level of students are given below.

…

R: ...For example, there was a broken off in last two lessons (he mentioned that he didn't come to the lesson. I was lost since a new lesson was based on the previous one When I go to home, I do the test again.

M: What is the relation of this with the comprehension test?

R: Sometimes, it is just written on a note-book. I am writing without knowing. Because of this test, I close up the writings and I can be able to think "That is like that, or like this?", "Does it happen like that? After that, I write. But, when I do not understand a note, I directly passed. Also, I could not write a reason in the comprehension test. However, in regular exam, I directly write from my memory because its reason is not asked.

The other individual interview quotation is as below:

İ: ...Comprehension test measure the students' knowledge cumulation more effectively. If you directly ask the proof, it can be half; some knowledge of the students inside can be measured. It can be fairer exam.

These two interviews showed that students have done rote learning and memorization. It is observed that comprehension test can be assumed to provide an opportunity for the students to be far from the rote learning. Further, students' partial knowledge can also be measured with these tests.

Category 2.The Positive Effects of the Comprehension Test

This category includes two sub-categories such as the effects on development of cognitive and affective skills. The details about these sub-categories are as below.

Sub-category 2.1.Cognitive skills to be developed with the comprehension test

This sub-category further includes four themes which are problem solving, reasoning, communication and connections. The codes of inquiring the knowledge, step by step solution logic, paying attention to the smallest details, the importance of definition and theories, intuitive thinking, determining the critical points, using the previous knowledge are the descriptions of the problem solving sub-category. The codes of being away from the memorization, the power of interpretation, doing all steps, and mathematical thinking are the descriptions of reasoning sub-category. The codes of expressing the presentation of a problem in a verbal form and using the mathematical symbolic language are in the communication sub-category. Some of the quotations of these themes are given below

"I learn why what I had done did. At least, I have started to pay attention." (open coding: 111; sub-category: problem solving, description: inquiring the knowledge).

"I have learned that it is quite hard to put forward an idea on what I do not know. You can't know that I used my imagination not to submit to give empty paper. Also I noticed that I have a huge imagination power." (open coding: 123; sub-category: problem solving, description: intuitive thinking).

"I got confused when (you) asked "according to what did we write this?" I do not know according to what I wrote this, but I had a headache because of thinking of how I should express this" (open coding: 134; sub-category: communication, description: expressing the presentation of a problem in a verbal form).

"… we need to use more scientific, more algebraic expressions. We understand how the transitions occur, but we had a difficulty while expressing." (open coding: 93; sub-category: communication, description: use the mathematical symbolic language).

"I never think before where an object comes from with the relation of the last step." (open coding: 96; sub-category: connections, description: relations between the mathematical concepts).

"The comprehension test is similiar to the text given and the questions asked regarding this text in the Turkish lesson. Some of the questions are based on the text while some others are based on your interpretation." (open coding: 196; sub-category: reasoning, description: the power of interpretation).

The examples above showed that students believed that their skills on problem solving, communication, reasoning and connections have not been adequately developed. However, they reported that the comprehension test seemed to be quite useful to develop their four basic skills. A quotation of an individual interview showing the development of these skills is given below.

M: How did you feel when you are studying for the test and teaching the lesson according to the test?

M1: With a student psychology, students don't want to be randomly selected (He refers to this that students are randomly selected for the asked questions). *But he thinks at that moment. We need to make a research and connection. We suggest to make connections with the past.*

…

M: What did you fell while answering the comprehension test?

M1: I feel that I don't have adequate knowledge on the subject. I understand that I don't answer the questions by considering my whole knowledge. I understand that it is important to know the concepts with comprehension test.

M: Don't you understand it before?

M1:No.

M. Why?

M1: We do not use them that much. Application was more important in other classes. We never looked at definitions and even theorems.

This interview showed that a student had difficulty with making connections among the concepts and had no adequate knowledge for subject. Also, the more important one can be that the interview revealed that the student seemed to have no idea about the effects of definitions and theorems on the mathematical problem solving and thinking. A quotation from

Group interview -2 about the effects of comprehension test on development of mathematical communication skills is given below.

> ...
>
> *H: I believe that I can be able to show how to express mathematical terms in verbal form with a comprehension test. As I said before 2x2 is equal to 4, but why?*
>
> *M: Is it mathematical speaking?*
>
> *H: Yes, abstract thinking will be more concrete with written explanations*

A quotation showing students' lack of mathematical communication skills from an individual interview is as below.

> *M: What does the comprehension test make you feel? What kinds of things come to your mind?*
>
> *P: ... I think these tests will be more successful when they can be started at the primary schools and the students will be grown up with these tests. The students who talk about mathematics and make interpretation on the math will be trained.*
>
> *M: What are the contributions to mathematical talking?*
>
> *P: Firstly, a student needs to understand the question in the test and make a comment. This situation will motivate the student among his/her friends. Maths is not a subject talked among the students it is frightened. Even though I'm in the maths department, I still don't speak about it.*
>
> *M: Why?*
>
> *P: Especially, the classes are based on the theory. Until now, we have memorized all the theorems before exam, but the comprehension test is different.*
>
> *M: Do you think that it is good or bad?*
>
> *P: When I see the comprehension test, I understand that although I have been studying maths, I could not be able to make comments. I thought that I don't give right of my department.*

The open coding 134 given above, when the interview quotations and the observations during the lessons are considered, it is understood that students had difficulty with and are insufficient in the mathematical communication skills such as transferring their thinking to writing, interpreting the expressions, and symbolizing the verbal expressions. As reported by the students, the comprehension tests are seen as an alternative to solve the mentioned problems.

Category 2.2.Affective skills to be developed by the comprehension tests

This category is further divided into six themes such as self-confidence, interest, anxiety, belief, motivation, and value. The self-confidence theme mainly concentrates on developing self-confidence. The interest theme concentrates on showing interest and making the students become curious. Decreasing exam anxiety was emerged under the anxiety theme. Believing is under the theme of belief. Enjoying of what has been done and motivating to study are the main codes of the motivation. Ensuring the equity/justice and showing respect to the teacher are the main codes of the value. Some of the quotations taken form students' writings are given below.

"I didn't understand what it was and why it was like that. Now, I really feel that I have learned. This gives me an enormous confidence." (open coding: 126; sub-category: self-confidence, description: developing a self-confidence).

"I left blank when the proof of the theorem was asked as an memorization. But, in this application (test), I am thinking of some parts even I do not know the theorems…In this way, my interest stays alive." (open coding: 153; sub-category: interest, description: becoming curious).

" … I took the exam without having any fear. I don't have any anxiety about when I forget the proofs of the theorems…" (open coding: 229; sub-category: anxiety, description: decreasing the exam anxiety).

" … I really enjoyed the lesson, because I tried to make interpretation." (open coding: 215; sub-category: motivation, description: enjoying).

"While answering the comprehension test, I believe that I can be able to solve the problems even if I did not see the theory before. I really believe myself." (open coding: 194; sub-category: belief, description: believing to learn).

"The differences between the students can easily be observed; the ones who don't follow the lesson and find the notes of the lesson and the ones who take his/her own notes. This is a big problem to be overcome for ensuring and bringing the justice." (open coding: 169; sub-category: value, description: ensuring the justice).

The examples stated above indicated that the comprehension test help the students increase the affective skills such as interest, self-confidence, motivation toward the algebra in particular and learning in general, and also decrease the exam anxiety. A quotation from group interview-1 pertaining to the themes of the self-confidence and motivation:

...

M: Will you use the comprehension test when you become a teacher?

H: Yes. Students' self-confidence increases. When you ask a question with a proof, he/she becomes depressed when he/she doesn't solve. But, let's say s/he solves three by saying a, b, and c. S/he will think her/his motivation will increase by saying that I know something.

B: Especially for the questions we don't know. Well, I don't know, everything comes on your eyes in a moment. By saying" s/he may do like this form here, it can be possibly done like that", everything comes true ... S/he says that I really learn".

M: What increase?

F: Confidence

M: Okay, what happen when it increases?

F: The person's performance increases. If it increases, a person automatically studies lesson, listen the lesson.

An individual interview showing that the comprehension tests reduce the exam anxiety is given below:

...

M: Do you have feedbacks from the other exams?

G: We partially have, but mostly for appraising the learning. In here, comprehension test is so different. The students are calmer and, more relax and because of it, there is no exam stress. They can take the more real feedback.

These examples and two quotations from the interviews above are so important since they show the positive effects of the comprehension tests on development of the students' affective skills. This situation was also supported with the classroom observations. It is observed that the majority of the students' interests and curiosity were increased toward the class and they started to frequently ask "why is it like that?", "how does it happen?" and also started to discuss with their peers about the connections between mathematical concepts and operations. This situation makes the teacher-student and student-student interaction possible and active.

Category 3.The Negative Effects of the Comprehension Tests

This category further consists of two main themes; "negative effects on individuals" and "the negative effects as a result of the test construct". The first theme is regarded as (1) directing the students to study on some other parts and (2) students' describing of the hardness of the exam. The second theme is more regarded as the following codes; (1) the practical level of the test is so low, (2) it includes the basic knowledge, (3) preparation of the test is quite hard and (4) it is hard to use the test with the students in the second level of primary school (e.g. 6th, 7th, and 8th grades).Some of the quotations related to these themes and descriptions are given below.

"It doesn't give a change to make operation. The level of operation is very limited." (open coding: 154; sub-category: the negative effects as a result the test itself, description: the practical level of the test is quite low).

"Even though I don't know the topic, I could write something by using my only basic mathematical knowledge." (open coding: 170; sub-category: the negative effects as a result of the test itself, description: it includes the basic knowledge).

"You need to think more and complicated. Due to this, it becomes tiring." (open coding: 207; sub-category: negative effects on individuals, description: the hardness of the exam).

"The structure of the questions should be well prepared. The questions the teacher asks should be same with what the students understand. If a student understand in a different way, her/his answer can be wrong." (open coding: 188; sub-category: the negative effects as a result of the test itself, description: preparation is hard).

"…successive questions make a students tired I think I answered 30 questions in the exam." (open coding: 160; sub-category: negative effects on individuals, description: the hardness of the exam).

"Since (the teacher) did not require us to prove a theorem entirely, it helps us to understand the important parts; it protects us to waste our time for the unnecessary parts. We need to search more than one source for understanding the important parts." (open coding: 161; sub-category: negative effects on individuals, description: directing the students to work some parts).

When looking at the quotation above, students reported that practice level of the comprehension test is so low, preparation of the test is hard the answering part is hard and the test directs them to some basic parts. Another quotation from a group interview-3 reveals the hardness of the preparation part.

...

G: The application needs courage. That is, even if the students complain about the memorization, you can possibly here some voices about the opposite ideas. But this point of view should not be change.

N: It is hard for the teacher. It requires to work more for making a transitions, going back and also preparing the questions.

The quotation above shows us that it is hard to prepare a comprehension test, but still it should be used. Most of the students believed that when they become a teacher, they would use the comprehension test in many places (at the end of the subject, as a preparation for the exams, or as an exam; open coding: 8, 217). In addition, eight of the students claimed that the comprehension test would bring a workload when they become a teacher and for that reason they do not tend to use it (open coding: 17). An individual interview excerpt that reflects the effects of the comprehension test on the working style is as below:

…

F: In here, I did not look at what we proved or what we did? I was even unaware of what the theorem was. I just focused on the transitions. Why did we pass from here to there?

M: But, there are in questions?

F: I was not aware of the questions.

M: For example in the, option f of the question 2 it was asked that "what was proved?". What did you do, then?

F: I didn't do it.

M: Did you stop there?

F: Yes

M: Doesn't it reveal something about the students for the evaluator?

F: Yes, but if there was not that option, I would never understand.

The interview excerpt above shows the hardness of using the comprehension test. For this reason, the questions should be carefully designed and the transition points should be very well identified.

DISCUSSION

The main goal of all curricula in general and of maths curriculum in particular is to realize meaningful learning for all levels of the students. The professional literature indicates that traditional teaching methods do not increase students' interests toward maths (Peng, 2002), meet their needs (Saye, 1997) and improve their academics achievement (Dede, 2003). The first aim in traditional teaching is to develop students' operational skills. In the standard test, to get a good grade, it is enough to choose the correct answer. The answer key or the teacher is the means which control the students' answers. Speed in the test is more important than thinking of finding the correct answer(s). Several behavior (e.g. comprehending the logic and the process of the test, explaining the process and criticizing) that contribute to development of students' mathematical thinking should not be ignored (Burns, 1985, as cited in Montgomery, 1987).

In this action research, the impact of use of comprehension test in Algebra class in mathematic teaching department on the 3rd year prospective teachers' opinions was investigated. The data was collected from the participants though the use of the comprehension test, written documents, individual and group interviews and classroom observations. All data was subjected to the content analyzed and they were grouped into three categories.(1) what does the comprehension tests measure?, (2) the positive effects of comprehension tests (thinking that they develop the cognitive and affective skills) and (3) negative effects of the comprehension tests.

At the end, it was found as a results of students' reports that comprehension tests measure knowledge, individual differences and also some components regarding teaching-learning process. To be able to use the knowledge and to make a self-assessment are more observable than the others. Also, in Turkey, the application of knowledge has been emphasized in new primary and high school maths curriculum. Moreover this point has taken place under the general goals of maths education as understanding the mathematical concepts, systems, and establishing a the relationship between concepts and systems, and integrating them into the daily life and other learning environments (The Ministry of National Education

(MoNE), 2005a, 2005b). Similarly, self-assessment method has taken place in new maths curriculum under the title of self-management proficiency “as questioning his or her works related with maths.” (p. 13). Self-assessment is a process including an individual aims, performance, and self -judgement, and also self-reactions to these judgement which are unacceptable and significant (Schunk, 1995).One of the examples showing this situation is as follow, *“I understand that I memorize the subjects. I understand that I could not pass the lesson with memorization.”* (open coding: 90). Positive self-assessment which helps the student to see their own capacities will motive them and feel themselves capable (Schunk, 1995).

Another result drawn from the research is that the comprehension tests have a positive effect on development of students’ cognitive skills. Under the category of cognitive skills, problem solving, making connections, communication and reasoning skills were observed and it was appearent that the test had an impact of development of these skills. Also National Council of Teachers of Mathematics (NCTM) (1989) always stresses the importance of improving these four skills. In Turkey, the new primary and high school maths curricula developed by considering the principles of NCTM put serious emphases on improving these four skills (MoNE, 2005a, 2005b). In that way, it has been aimed to develop and/or improve these skills of students. To this point, this action research points that the comprehension test is an effective mean.

Another important result of this research is regarded as the positive effects of the comprehension test on developments of affective skills of students. One of the important aims of the math education is to keep the affective skills of the students at the high level. Affective skills and features includes several components such as motivation, interest, attitude, self-efficacy, anxiety, belief and value. In the literature several research studies pointed out that affective factors have positive impact on learning and academic success of students (Bloom, 1979, Cates&Rhymer, 2003; Clute 1984; Dede, 2006, Hackett &Betz, 1989; Jingsong, 2003; Lumsden, 1999). In this regard this study is quite significant because it shows the valuable contributions of the comprehension test to development of affective skills.

Another result of the research showed the negative effects of the comprehension tests. They are related to individual and due to the test construct. Negative impact of the test on the individual is more regarded as directing the students to the special points (especially to transitions). A good example of this situation was reported in open coding: 161.This result and the hardness of the preparing the test was also observed in the study of Conradie and Frith's (2000). Another factor is related to the impractical level of the test. However, this can be easily overcome. Since the comprehension tests are dynamic in nature, the students can be required to indicate the next step while proving and/or solving any question and to complete the blanks. Further, in the test, the teacher can even ask students to produce new ideas about the new conditions. With this way, students' habits of working only some parts can be dealt with. Although there are practical items in the test, the students didn't pay adequate attention to these questions. For example, the questions given in the option *d* of the 3rd question, "Is S_G a group? Why?" and in option *h*, "Is λ surjective? Why?" are the practical questions, because these weren't given in the text and wanted students to find by their own. With the question in the option *e* of the question 4, "How do they ($x = a^k, y = a^l$) take? Are $k,l \in R$ taken? Why? ", a condition of the theorem was changed and the students were asked to take students' ideas about the new situation.

When the categories, the sub-categories, and themes emerged in the research are assessed, descriptions regarding memorization and reasoning skill can be found under each category. For this reason, the central phenomenon (or core category) of the research was named as "development of mathematical reasoning skill"

CONCLUSIONS and IMPLICATIONS

The most significant point of the results is that the students reported their memorization of the mathematical proofs and concepts. At the same time, the students reported that their mathematical reasoning and interpretation skills were not developed. This situation is, of course, so bad, but the good point is that they would like to deal with this deficiency and try to find alternatived for developing their own skills. In the research, the class observations, written documents of the students and also the interviews with the students showed that even though the comprehension test has some negative points, it can be proposed an alternative to overcome these problems. The most important advantage of the comprehension test is that this test helps reveals which point is very well understood by a student and which point isn't. For example, at a result of this research, students understood the injective function (option b of the question 3, the question was correctly answered by 90 % of the student), but did not understand surjective function (option c of the question 3; the question was correctly answered by only 10% of the students, and there was no correct answer for the option h of the question 3). Another advantage of the test is that it helps teachers obtain feedback from the students with low success (look at the interview coded with I. This student's mathematics achievement was so low). It was observed by the researcher and the experts (observers) that majority of the students participated in all sessions of the classes since the instructions were designed in line with the items of the comprehension test and they were active during the lessons. Further, it was observed that students' interest to the class become higher day by day. As a result of this, the students started to solve the questions regarding as the theorems which were proved in the lessons (question 2, 4). On the other hand, similar success was not observed for the questions regarding the theorems which were not proved in the lessons. Of course, it is hard to change the habits. In the light o f the feedbacks gathered from the students and the experiences of the researcher during the research, despite the hardness of the preparation and assessment of the comprehension test, it is believed that this test should patiently be used and it is suggested to the math instructors and teachers use their own instruction and assessment process. In addition, since the comprehension tests include basic mathematical concepts, these tests should be started to be used at the first class of the university in order to use them more effectively and efficiently.

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Cubism and the Fourth Dimension

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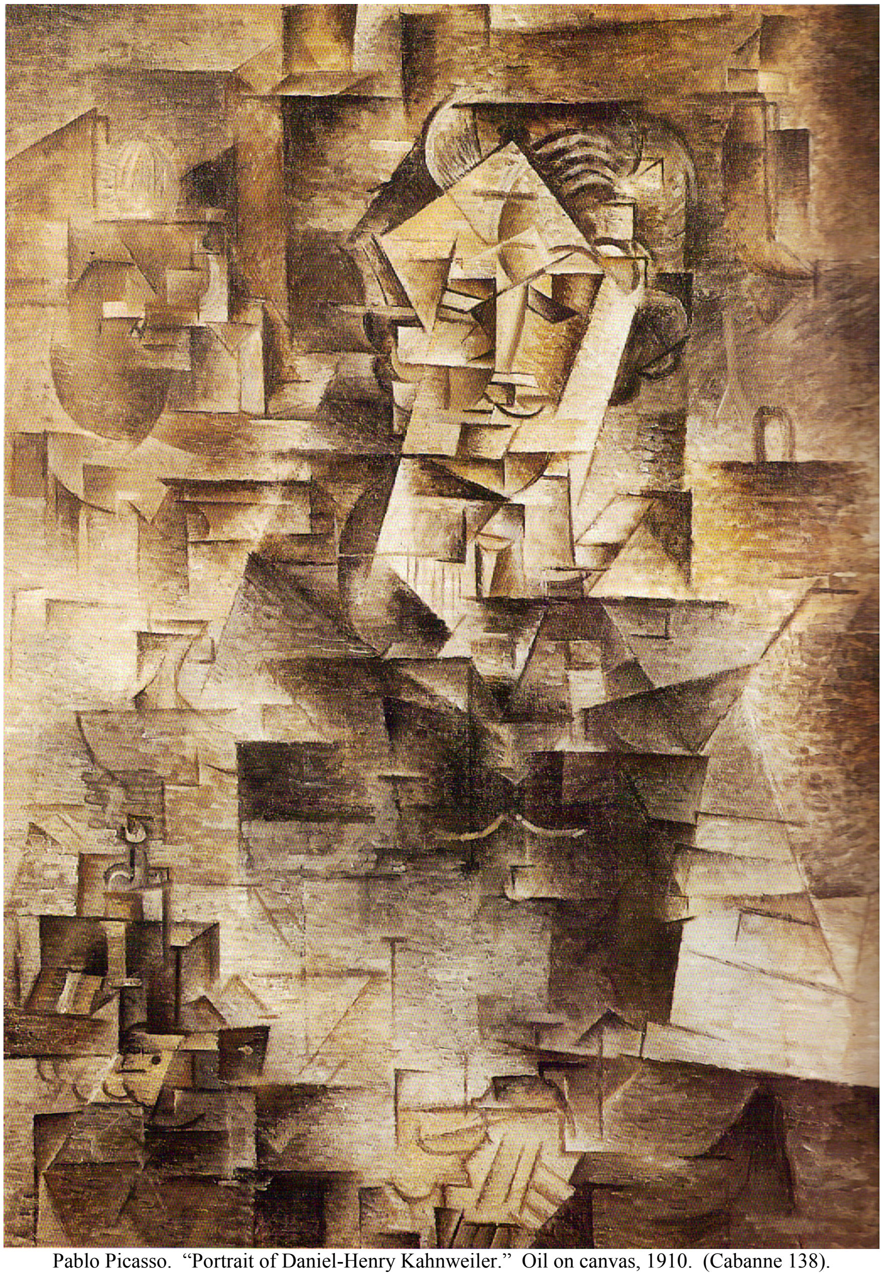

Pablo Picasso. “Portrait of Daniel-Henry Kahnweiler.” Oil on canvas, 1910. (Cabanne 138).

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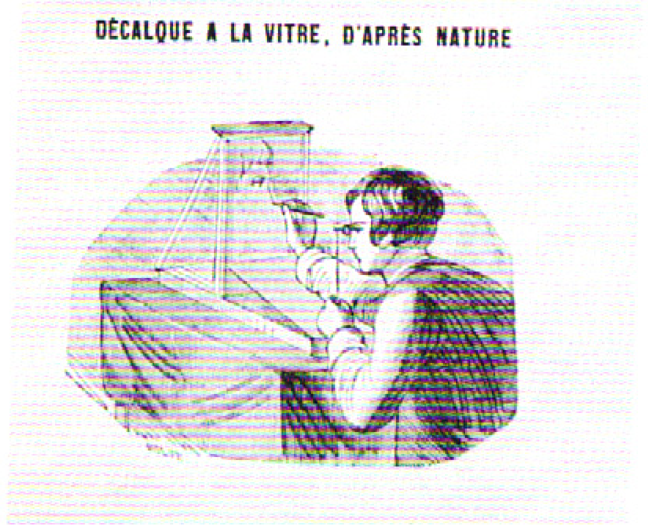
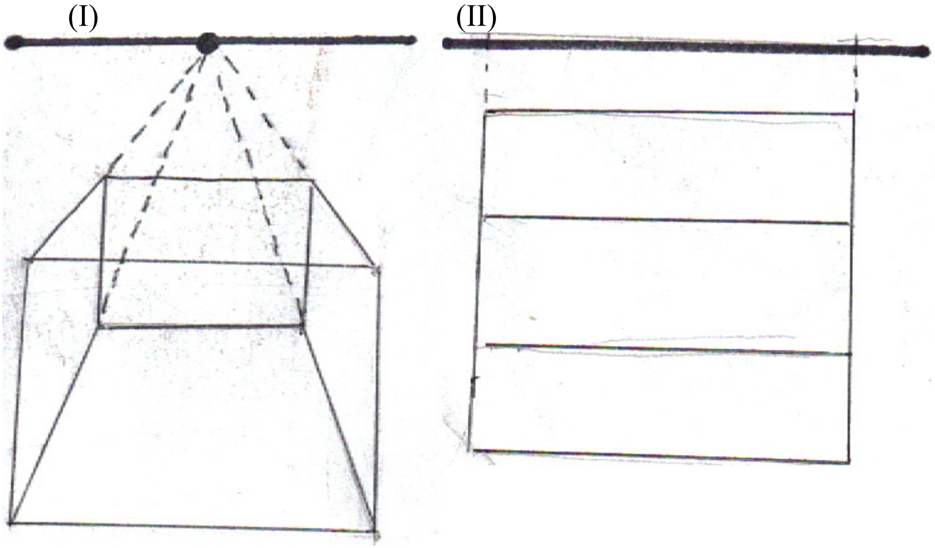
When one looks into the subject of geometries that attempt to explain fourth-dimensional space, it is inevitable that one encounters references to Cubism. The purpose of this paper is to find what the similarities between this mathematical concept and cubism are. There are many historical arguments as to how the cubists encountered literature about the fourth-dimension, and whether they were exposed to it at all, which I will for the most part omit and instead let the art speak for itself. It is important to see how two fields are interrelated in order to gain a better understanding of both fields, in this case art and geometry. In addition, visualizing things that the human eye cannot immediately perceive, that must be left up to the mind is important to people who want to gain a better understanding of their reality.

Leone Batista Alberti, in 1435, wrote the first book that discussed central projection and section, the process in which an artist would transfer an object onto a canvas by imagining that the image is traced onto a window, parallel to the artist's eye, which is looking out onto the subject.


DÉCALQUE A LA VITRE, D'APRÈS NATURE

"Tracing on Glass, After Nature", from Frederic Goupil, *La perspective experimentale, artistique, methodique et attruyante ou l'orthographie des formes* (1860) (Henderson, plate 41).

When one considers the space around oneself, as only perceived visually, all lines appear to converge away from the observer. This notion is adapted into the technique of perspective drawing, an attempt to render an image that visually makes sense on a canvas but spatially is inaccurate in its representation. In a one-point perspective drawing, there is a horizontal horizon line, which lies at infinity. On the horizon line there is a vanishing point in which all lines parallel to the z-axis intersect.



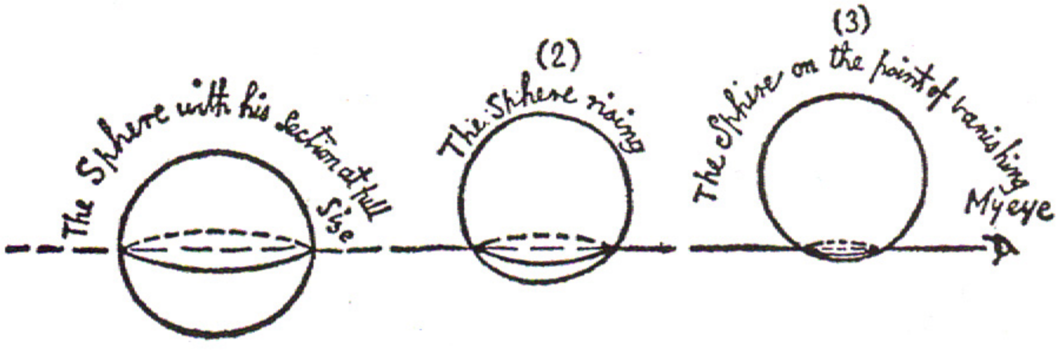
When a cube is projected onto a two-dimensional surface using a perspective technique (I) there is much less confusion as what the image is representing in space. However, if the cube is drawn on a two-dimensional surface and is not distorted in any way (II), all vertices are of equal length and no lines intersect at the horizon, in this case it is much more difficult to determine what the image represents in space.

In Plato's "Allegory of the Cave", he discusses with Socrates a hypothetical world where people are born chained in a cave where they would only see the shadows of reality. Then at a certain time, they would be unchained and upon leaving the dark cave and approaching the light, the former prisoners would initially be blind to reality. Now imagine that humans have been similarly "chained" in the fourth-dimension so that they can only see the shadows cast into a third dimension and are blind to the fourth-dimension. This hypothetical idea is part of what created theories and geometries concerning the fourth-dimension, and is part of what made it popular since a better understanding of extra dimensions would bring a more enlightened understanding of reality.

The fourth-dimension is built from the similarities found in the geometry we are accustomed to visualizing. Beginning with a zero-dimensional point, and then by moving that point in any direction for any length creates a line (and an x-axis). Moving the line perpendicular to the x-axis creates a plane (and an x and y-axis). Then moving the plane perpendicular to both the x and y-axis creates a space (and an x, y, and z-axis). This is the space we are accustomed to with a left-right, forward back, and up-down, it is easy to grasp what images of objects, in three-dimensional space, represent, regardless of how distorted they are because it is intuitive to us. By analogy of the previous transformations, moving a space perpendicular to the x, y, and z-axis creates a fourth dimension. However, due to our being stuck in three-dimensional space, we cannot visualize a fourth-dimensional coordinate system, or what an object in the fourth-dimension would look like. Two main methods of representing four dimensional objects, the slicing method and the projection method, have developed in an attempt to make the unseen seen.

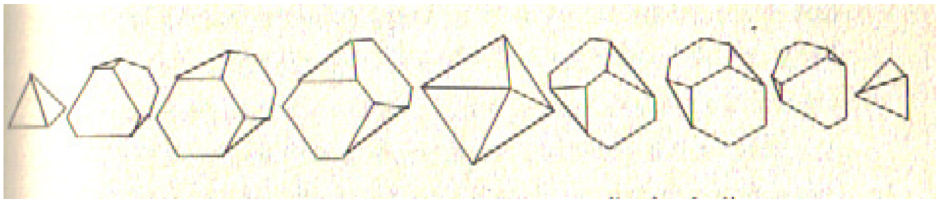
The slicing technique may go as far back as 1846, when Gustav Theodor Fechner, in his book *Vier Paradoxa*, "may have published the first discussion of two-dimensional beings being unaware of the third dimension that surrounds them" (Henderson 18). The technique was popularized mainly by Edwin Abbots book Flatland, which "E Jouffret discusses…in his 1903 *Traite elementaire de Geometry a quatre dimensions*, a book known to Duchamp and certain to his cubist friends" (Henderson 25). In Flatland, a two-dimensional being known as A. Square is visited by a sphere from three-dimensional spaceland. A. Square then proposes to the sphere that maybe spacelanders could be unaware of a surrounding fourth dimension. The sphere is infuriated by the idea of higher dimensions, but Abbot gets across the message that the ideas he proposes are not impossible to grasp.

When A. Square first encounters the sphere, from A. Square's perspective it is a series of circles, starting with a point, increasing in size, then reducing in size back to a point, and finally disappearing, in the same fashion it appeared.

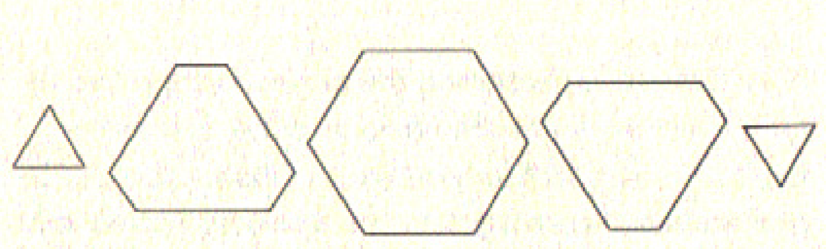

The Sphere with his section at full size
(2)
The Sphere rising
(3)
The Sphere on the point of vanishing
My eye

A. Square is observing the sphere passing through flatland. (Abbott 143).

The slicing model of visualizing the fourth dimension stems from these notions in that a fourth dimensional being passing through spaceland would appear to be a three-dimensional object gradually increasing and then decreasing in size. If a four-dimensional sphere were to enter spaceland, then it would look like a regular sphere of three-dimensions that appears from seemingly nowhere, increases and decreases in volume, and then disappears. Due to the regularity of a sphere, this is not a very compelling example, since it doesn't provide much information as to how the sphere would look in the fourth dimension. A more interesting example is the hypercube (a fourth-dimensional cube) passing through the third dimension at right angles to the main diagonal of the hypercube. According to Ian Stewart this is Charles Hinton's, a late nineteenth century British physicist and mathematician, favored method of viewing a hypercube (Abbot 175).



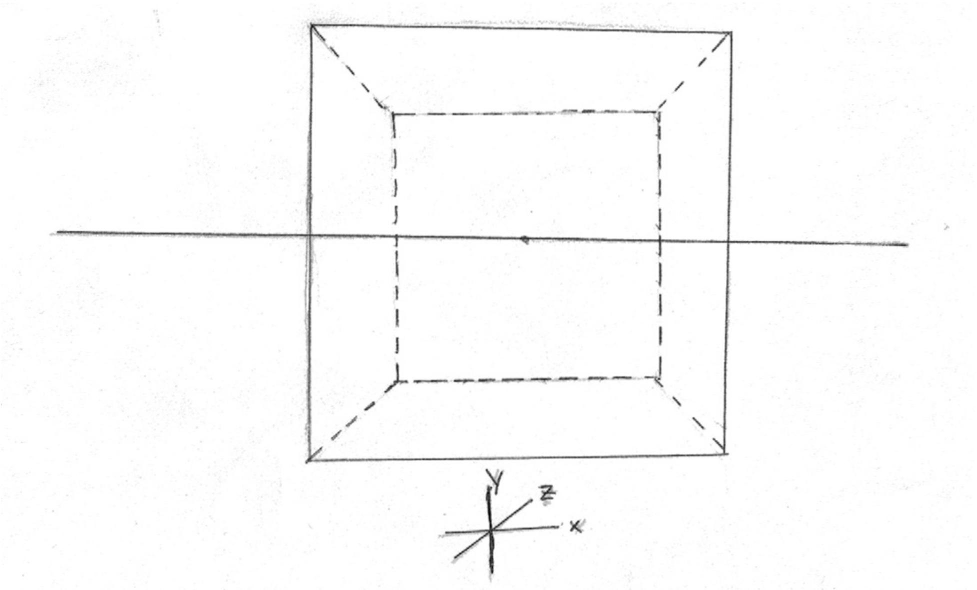
Slices of a hypercube perpendicular to its main diagonal (Abbot 175)



Slices of a cube perpendicular to its main diagonal (Abbot 166)

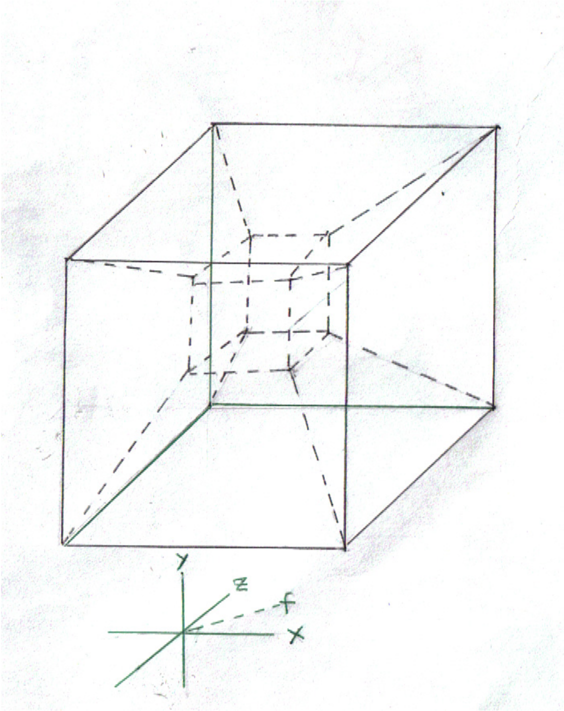
In order to get a better understanding of the fourth dimension, these slices of our perception of the object must be viewed separately but considered as a whole.

The projection method of visualizing the fourth-dimension utilizes projective geometry, a product of perspective drawing. A cube can be drawn on a two-dimensional surface using projection techniques to appear as a square within a square, in which the cubes vertices are distorted in their actual length and the location of the smaller square on the z-axis is not readily discernible unless it is known that the object is a cube.



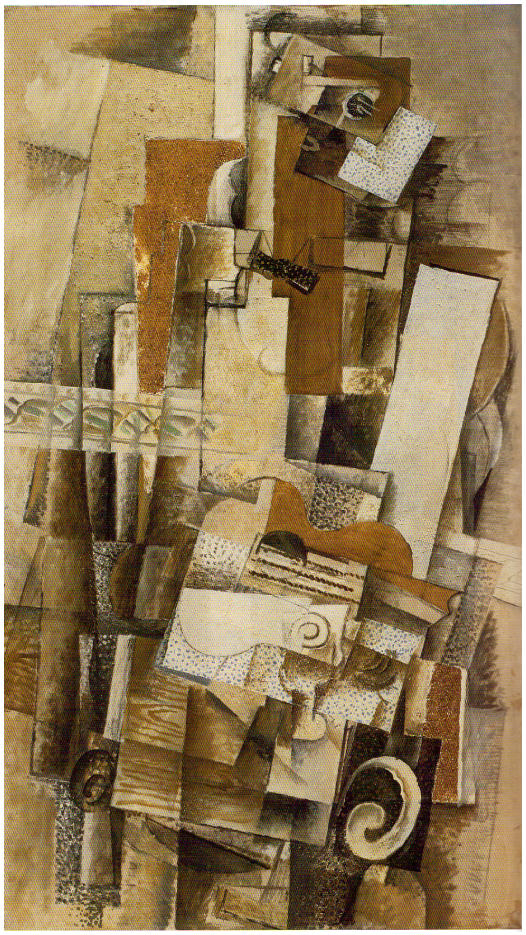
A projection of a cube onto a plane

Analogously, a hypercube can be projected into three-dimensional space as an object containing eight cubes, including a surrounding cube. However, similar to the projected image of the cube the projection of the hypercube distorts the lengths of the vertices and the location of the eight cubes, in the fourth dimension, in respect to each other.



A drawing of a projection of a hypercube into space

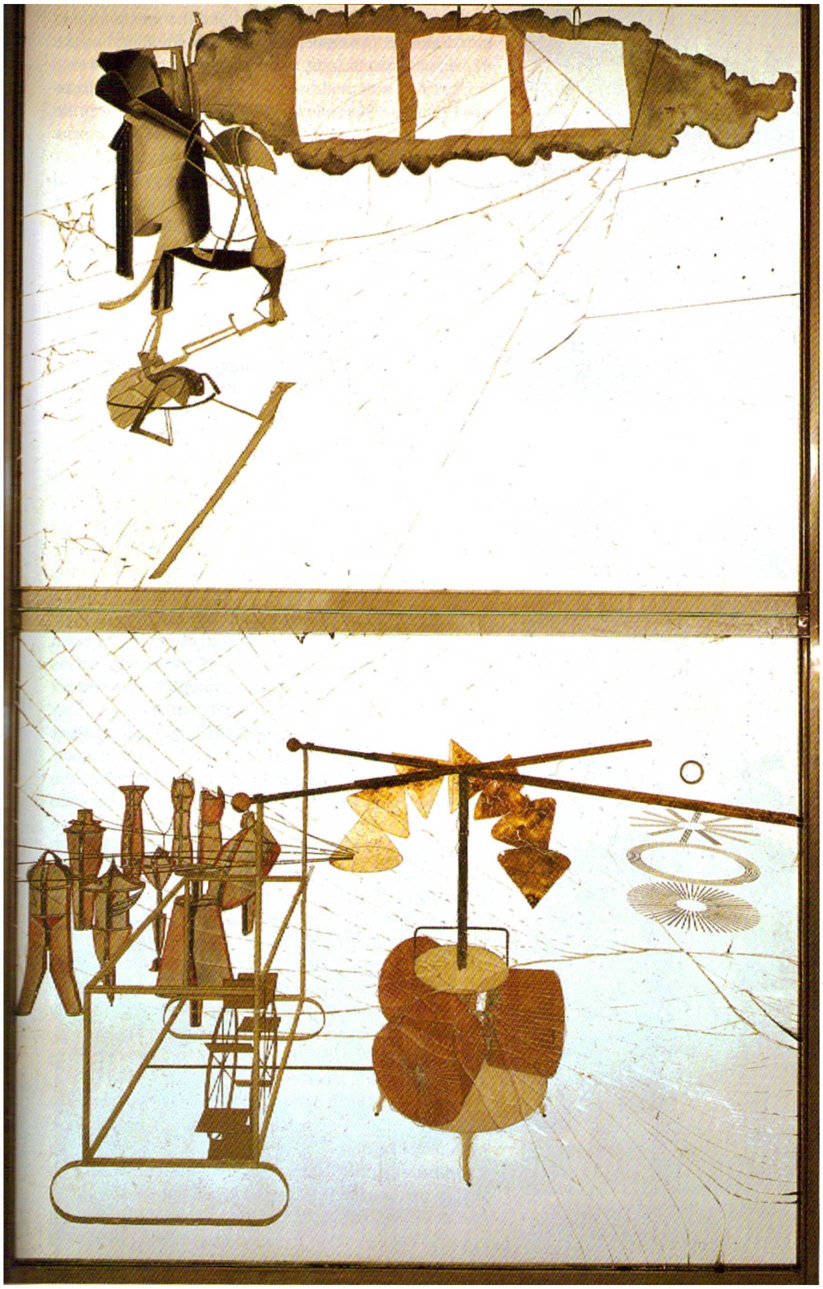
Cubism was born out of the paintings made by two friends, Georges Braque and Pablo Picasso, in France during the early twentieth century. They were both attempting to move in a direction that opposed traditional perspective drawings of the world around them. Guillaume Apollinaire, an art critic and poet, wrote that Braque and Picasso were "moving toward an entirely new art which will stand, with respect to painting as envisaged theretofore as music stands to literature. It will be pure painting as music is pure literature" (Stokstad 1077). Picasso suggested, "the viewer should approach the painting the way one would a musical composition…by analyzing it but not asking what it represents" (Stokstad 1077). "Pure Painting" may be a representation of the fourth dimension, a more complete way of looking at space, but also a way of seeing it that cannot be understood completely until each part of it is analyzed one by one. This method of viewing cubist paintings is similar to the method employed when one considers the slicing model of the fourth dimension.



Georges Braque. "Man with a Guitar." Oil and sawdust, 1914. (Cabanne 50).

Despite the obvious similarity between Picasso and Appolinaire's statements about cubism's relations to music and the subject matter of Braque's painting being a guitar, this piece also provides evidence for the cubist's influence by the ideas of fourth dimensional geometry. The painting is made up of various slices of space reduced to a two-dimensional image, and then represented together to imply their relation to one another. The head at the top of the painting is shaped like a cube and floats separately from the rest of the figure, adding another element to the fragmentation of the painting. Braque's painting could be seen as a time lapse of a higher dimensional figure passing through a lower dimension.

Marcel Duchamp, an important member of two early nineteenth century art movements Dada and Cubism, seems to have the most technical knowledge of fourth dimensional geometry compared to the rest of the artists involved in the cubist movement. Using his painting from 1912, "The Bride," as a staring point Duchamp set out to create a pictorial representation of the fourth dimension that surpassed all his earlier attempts. From 1915 to 1923, Duchamp worked on this piece, which ended up being titled "The Bride Stripped Bare by Her Bachelors, Even (The Large Glass)."



Marcel Duchamp. “The Bride Stripped Bare by Her Bachelors, Even (The Large Glass).” Oil, varnish, lead foil, lead wire, and dust on glass panels encased in glass, 1915-1923. (Stokstad 1103).

In his notes for the painting, Duchamp stressed the, “distinction between the ‘fluidity’ of the bride’s domain (top panel), and the strictly measured three-dimensional perspective of the bachelor apparatus (bottom panel)” (Henderson 134). Part of Duchamp's understanding of the fourth dimension is related to the idea that fourth-dimensional objects, due to their fluidity, are immeasurable, which is illustrated by the cloudy abstraction that makes up part of the bride. Duchamp seemed to relate to the notion of fourth-dimensional objects being the more complete image of the object, and that the part of the object we are accustomed to seeing is only a small piece in a larger body. Similar to the projection technique of viewing fourth-dimensional objects, Duchamp wrote in his notes on the painting that in order to grasp the fourth dimension one must "construct all the three-dimensional states of the four –dimensional figure the same way one determines all the planes or sides of a three dimensional figure" (Henderson 140). "A Large Glass" was painted on a glass pane because Duchamp believed that in order to "permit an imaginative reconstruction of the numerous four-dimensional bodies" (Henderson 141), the viewer must be able to see the images from multiple perspectives. By painting on glass, it allows the viewer to wander around the piece and get a better understanding of a higher dimensional object. Similar to how a sphere appears as a circle from only one perspective and it is not apparent that the object is a sphere until it is observed as a circle from all perspectives by wandering around it. Another interesting reason why Duchamp may have used glass as a canvas is its similarity to the sphere looking into flatland. The observer sees on the bottom panel a three-dimensional image made into a two-dimensional slice of the observer's space.

Fourth-dimensions of space are often misinterpreted as time being the fourth dimension. Although both theories are represented in cubism, they are distinctly separate. Time has been described as a fourth dimension since Joseph Lagrange's *Theories des function analytiques*, from 1797. However, Charles Hinton "repeatedly turned to the notion that time could be defined as a fourth spatial dimension of geometry, not simply another number necessary to describe a place at a certain time" (Robbin 25). Hinton often used the example of a spiral being pulled through a plane, which from the point of view of a Flatlander appears to be a point moving around in a circle. "According to Hinton the spiral is the complete static model of the events…it has a greater philosophical reality than the moving point, and thus it should be the object of our consideration" (Robbin 25). The event is as dependant upon space as it is upon time, and could therefore be said to be an invariant model of reality. This notion of time as an extra dimension came before the current status-quo understanding of space-time dictated by Einstein's theory of special relativity and Minkowski's geometry that goes along with it.

Special Relativity is derived from two principles. Both are experimental facts boldly assumed to hold universally. The first says that physical laws are the same for all observers. The second states that it is a law that light travels at 300,000 km/sec [c] (Kennedy 17).

From these assumed principles, one can infer that the time and space of an object vary depending upon the velocity of that object. This occurs in Einstein's mind because if a ship is traveling at 200,000 km/sec, 100,000km/sec slower than (c) and another ship is traveling in the same direction at 100,000 km/sec, 200,000 km/sec slower than (c), and light as observed from both ships travels at a constant speed of (c) then both ships must be measuring the speed of light with different perceptions of time and space (in this case each ship would have a different set interval for what a second and a kilometer looked like).

It would be less surprising to the astronauts in both ships if ship (I) measured (c) to be 100,000 km/sec and ship (II) measured (c) to be 200,000 km/sec since the velocity of an object moving alongside another object would logically be observed to change, depending upon the velocity of the measuring object. However, since both ships actually measure (c) to be the same, then it can be assumed that on ship (I) time passes slower and distances are smaller than ship (II) which has longer time intervals and shorter special measurements than a ship at rest. For clarification, the shortening in space of lengths "contracts only in the direction of travel, and its diameter remains the same" (Kennedy 19).

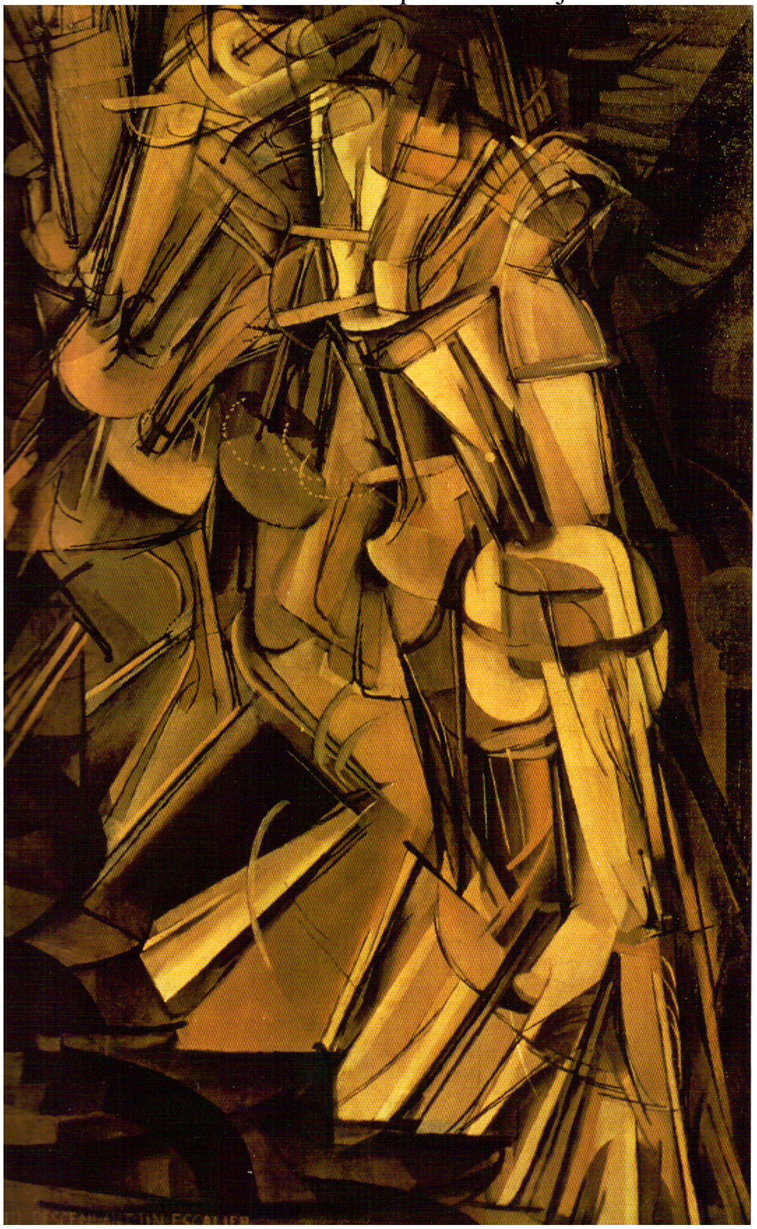
Minkowski referred to his four-dimensional space-time formulation of the special theory of relativity as entailing that "space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of unity of the two will preserve an independent reality" (Joseph 426).

Since three-dimensional geometry only deals with space, it is not able to explain the implications of special relativity. In order to graph four dimensions, which is what Minkowski and Einstein believed our universe was, on a Cartesian plane the z and y-axis must be removed, it is much easier to look at the two variables that are subject to change, x and t. Since time is always flowing through our universe, and length contractions only affect the x-axis, the z and y-axis can be taken out of immediate consideration. Using Minkowski's geometry a graph of something at rest would look like a vertical line, since time continues to pass and the objects length is constant at a single speed. The faster something moves through space the steeper the slope of the graph becomes. In Minkowski's mind, since time and space are both subject to change in the form of either dilations or contractions respectively, the only way to understand the true nature of an event is to combine the two variable things into an invariable space-time interval. The space-time interval is constant because according to the Lorentz transformation length and time are inversely related. The smaller the length of something is the larger the time is. As a time interval increases the length decreases, and as a length increases the time interval decreases, but the space-time interval stays the same, since the ratio between the two is an inverse relation.

Although it is likely that the ambiguity of both topics of the fourth dimension caused them to be interrelated and subsequently used synonymously with each other by many people, including some Cubist painters. The idea of a more complete understanding of the surrounding world still is prevalent in both space-time geometry, and the geometry of four spatial dimensions. Many cubist paintings contain more resemblances to the slice and projection models of the fourth-dimension. However, one major piece, Marcel Duchamp's "Nude

Descending the Staircase," displays his views of the importance of time in its relation to space.

Partially inspired by early photographs of objects in motion, Duchamp, instead of producing a rendering of a static image captures, the motion of the object, and paints an event instead. Regardless of the distortions of the subject, since the painting takes the fourth-dimension of time into account it is in a way a more true to life rendering of reality than a static image, which is known to be distorted on a canvas as well even if the artist tries to paint the subject as it is seen.



Marcel Duchamp. "Nude Descending a Staircase #2." Oil on canvas, 1912. (Cabanne 85).

The search for what is really real and how it can be represented honestly by an artist seems to be a major driving force for the cubist's interest in the fourth-dimension. Whether it is the invariance of space-time or the complete representation of an object in a fourth spatial dimension as opposed to the three-dimensional slice of reality we see from day to day. Many people tend to interpret cubism as the artists fragmented interpretation of the world, a representation of the fragmentation of Europe after WWI. However, it may actually be a much more optimistic art movement than people realize, while it is a rejection of the past, it also may show hope for humans' ability to find the true nature of the surrounding world. In 1908 Henry Poincare wrote, in his book *Science et Methode*, "One who devoted his life to it could perhaps eventually be able to picture the fourth dimension" (Krauss 85). Similar to the optimism of Einstein and other scientists during the early nineteenth century, cubism also put a great amount of faith into there being an order to the universe that humans can understand. It is also important that people don't view mathematics as a dull unchanging subject that is only of help to engineers and scientists. When someone brings up art and mathematics in the same sentence usually it leads to a detracting statement towards one of the topics, or M.C. Escher enters the conversation. Cubism cannot be understood completely unless it is looked at through an artist's, mathematician's, and historian's perspective. Combining these different slices of the whole interpretation is the best way to look at not just the nature of space but also the nature of people and the surrounding world.

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What's all the commotion over Commognition?

A review of Anna Sfard's (2008) ***Thinking as Communicating***, Cambridge University Press, ISBN 978-0-521-86737-5,. (xxiii + 324 pp), $99.

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If straight edge and compass constructions are the so-called "atoms" of Euclidean geometry, if sequences are the "atoms" of Analysis, then what are the "atoms" (if any) of mathematics education? Arguably mathematics education is a much wider field than Euclidean Geometry or Elementary Analysis, however there are several fundamental things that the field purports to study, chief among which is mathematical thinking or more generally "thinking". The book under review, though it appears in a Cambridge University Press series entitled *Learning in Doing: Social, Cognitive, and Computational Perspectives*, is in my view situated at the intersection of Consciousness Studies, Linguistics, Philosophy and Mathematics Education. One does not come across books within the mathematics education genre that take on the tasks of operationalizing thinking and defining consciousness. This review began a year ago when an excerpt from the book was included in vol5, nos2&3 [July 2008] of the journal. My personal interest in the contents of the book lay in the promise that the book would tackle existing dichotomies in the current discourses on thinking with the aim of showing they are resolvable or even transcend-able?

To do so, the author Anna Sfard coins the concept of commognition- a dissected juxtaposition of cognition and communication in order to remove the duality between thinking and communicating, and to resolve the four quandaries that have plagued existent discourses on thinking, namely-the quandary of number, the quandary of abstraction (and transfer), the quandary of misconceptions, the quandary of learning disability. Each quandary is illustrated and explained to the naïve reader in the form of discourse transcripts in chapter 1. The transcripts are presented as episodes from a larger data set. Chapter 1 sets the tone for the rest of the book. Even though there are many new terms that constitute the concept of "commognition", these terms are explained in the glossary towards the end of the book.

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Chapter 2 entitled Objectification problematizes the ineffectiveness of (existing) research which does not recognize that Research (capital R) ultimately is a form of communication defined by cogent narratives, with different disciplines according different rules of endorsement and engagement. Sfard warns of the dangers of unifying labels used in dominant research discourses that stand for many different phenomena and thus impede any form of clear communication to occur as well as impede the formulation of common definitions necessary to operationalize mathematical thinking without creating irresolvable dichotomies. Dichotomies invariably arise when attempting to objectify human activities (involving thinking and learning) and when attempting to communicate it. Chapter 3, Commognition: Thinking as Communication begins with the famous words of Richard Rorty "The world does not speak, we do" and goes on to give a short history of Disobjectification of Discourses on Thinking [pp. 68- 76]. The crux of this chapter is to reveal to the reader linguistic traps inherent in the way language in structured, especially when we accept that language is culturally oriented and dependent.

The most compelling chapter of the book in my opinion is chapter 4: Thinking in Language, in which an interesting definition of "consciousness" is found. Sfard explains the dilemmas arising when we try to separate thinking from speaking, awareness from consciousness, and often invokes Vygotsky and Wittgenstein to drive home the point that paradoxes are bound to occur in any attempt to carve thinking into micro-components. As a reader one actually finds oneself within the stream of thought that Sfard carefully wades into, to arrive at her eureka(!) discovery of recursivity (of reflexivity at ever deepening levels) to be the elementary particle of commognition. At least to me, this was a new presentation of something well known within the canon of consciousness studies that occurs at the intersection of theology, science, psychology and linguistics. For instance in an article I wrote together with the philosopher Walter Benesch on the topic of consciousness and science (see Sriraman & Benesch, 2005), we analyzed non-dual traditions, particularly the Advaita tradition of Shankara from the 9th century (AD) in India. In this paper we defined human consciousness as the possibility of attending/intending, and described specific experiences and their interpretations as possibilities for consciousness as attentions and intentions. Experiencing is a synthesis of *of* and *for*. Alternatively, from the position of Shankara and Advaita-Vedanta: the possibility of superimposing and the possibilities for superimposition. We gave an example of this synthesis by trying to explain and/or define 'self' or 'world'.

Any explanation, interpretation, definition, etc. is an attending/intending flow with at least five aspects.

1. The 'observer, interpreter, explainer';
2. The 'interpreted, observed, explained' or experienced object which is the context to which the interpreter refers;
3. The process of 'interpreting, observing, explaining';
4. The 'interpretation, observation, explanation' that emerges from 1 – 3; and
5. The 'awareness' of and ability to distinguish the preceding four aspects of this continuum and to focus upon them individually and collectively, assigning each significance and value.

It is within this fifth aspect that perspectives occur on the other four and upon number five itself. Every aspect of this continuum provides a vast number of possibilities for consciousness, while consciousness as the possibility of the totality is not reducible to any particular aspect, and is the source most clearly reflected in the fifth aspect. This five-aspect continuum seems to us implicit in all subject-object-process language- understanding relationships. The challenge is to preserve the totality of ''consciousness as possibility'' while utilizing and/or emphasizing particular aspects within it as possibilities for consciousness. Otherwise, we confuse the aspect with the whole.It is the processing of ''consciousness as possibility'' that is the source of exploring, explaining, defining—the possibility for theorizing, theologizing, biologizing, cosmologizing, psychologizing. It is the processing of ''consciousness as possibility'' that discusses the ''possibilities for consciousness'' in the contexts of the sciences, arts, and humanities (Sriraman & Benesch, 2005).

At the end of chapter 4, and the culmination of part I of the book, Sfard takes an evolutionary view of the human linguistic communication and claims that it is characterized by "unbounded recursivity", a claim that I agree with. In her words: "Our unbounded ability to communicate about communication was also said to play a crucial role in the phenomenon of consciousness" (p. 124).

Part II of the book consists of 5 chapters (chps 5-9) which focus specifically on mathematics as discourse. Sfard puts forth her thesis that mathematics is a form of communication and presents copious examples from the historical development of mathematical objects to substantiate the argument that discursive objects are a natural outcome of mathematical communication (viewed from a lengthy time span). These chapters cohesively use commognitive grammar (pun intended) to put forth the claim that mathematics is an autopoietic system. Episodes continually interspersed in the second part of the book lend credence to the claims. Ultimately the book clearly identifies mechanisms that underlie the historical development of the subject and how commognition becomes central to how thinking and learning progress within shared communities of learning. It would be particularly interesting for the radical constructivist camp within mathematics education to read this book and analyze whether their position can be subsumed as an extreme case within the commognitive framework- after all we do talk to ourselves! This could well be the goal of a graduate course.

The reader is bound to ask whether the four quandaries are resolved in the book? My slant on this, one way or another would take away the intellectual tension that arises when reading this book. So I urge the interested reader to answer this for themselves by reading the book. Given the generality and universality of the part I of the book, Sfard carefully annotates the book with footnotes that explain her rationale, motivation and warrants for statements made, in addition to listing instances/disclaimers in which certain claims are not applicable. This is very masterfully done and allows one to enter her stream of "commognition".

Caveat emptor: The book is not an easy read by any means, but well worth one's time and efforts if one is active as a researcher in mathematics education, and constantly stumped by the inability to clearly communicate about the same research problems, or the same research concepts, or the same "things" that are being operationalized differently. Thinking as Communicating provides the grammar by which communication can be better fostered between researchers analyzing the same discursive "mathematical" objects in teaching and learning situations. I highly recommend the book.

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