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SMALL CHANGE - BIG DIFFERENCE

Ilana Lavy

Atara Shriki

Small change– Big Difference

Ilana Lavy[1] & *Atara Shriki*[2]

Emek Yezreel Academic College — *Oranim Academic College of Education*

Israel — *Israel*

Abstract: Starting in a well known theorem concerning medians of triangle and using the 'What If Not?' strategy, we describe an example of an activity in which some relations among segments and areas in triangle were revealed. Some of the relations were proved by means of Affine Geometry.

Keywords: Affine geometry; Problem solving; Problem posing; Triangle theorems; Generalizations

1. Introduction

The 'What If Not (WIN) strategy (Brown and Walter, 1990) is based on the idea that modifying an attribute of a given statement could yield a new and intriguing conjecture which consequently may result in some interesting investigation. Using interactive geometrical software, let us apply the WIN strategy to the theorem: *The three medians of a triangle divide it into 6 triangles possessing the same area*. This paper presents some results obtained by

[1] ilanal@yvc.ac.il

[2] shriki@tx.technion.ac.il

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modifying the premises so that each side of the triangle is devided into *n* equal segments instead of two. More precisely, given a triangle ABC, with sides *a*, *b*, *c* each diveded into *n>2* equal segments. Each of the $\frac{1}{n}$ dividing point is connected to the opposite vertex (Fig. 1). Unlike the case of medians ($n = 2$), in this modified version, there appears to be some quadrangles as well. Is there anything particularly interesting about the new parts? – That is what we are set to examine.

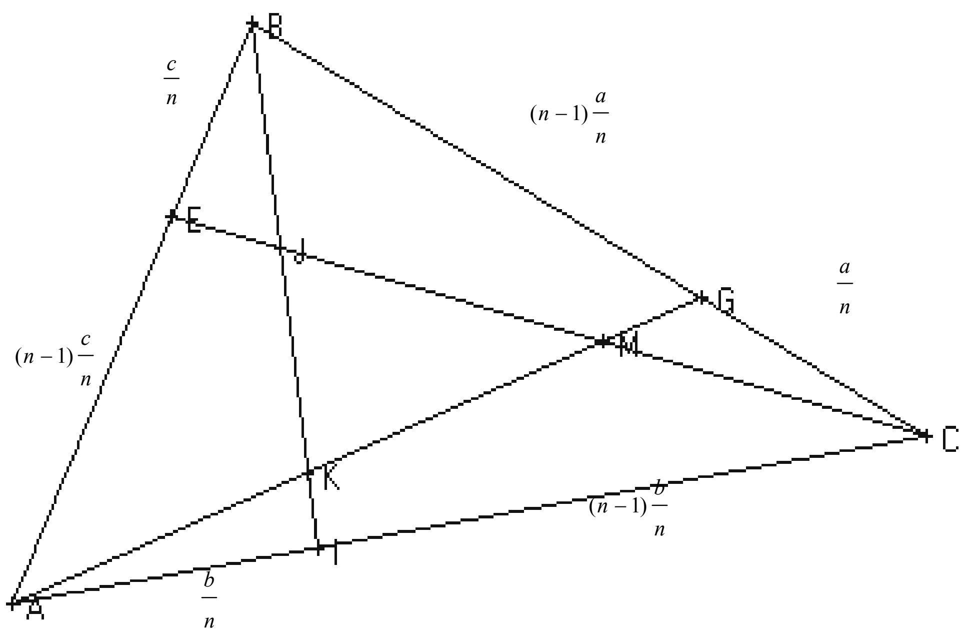

Figure 1: Schematic description of the problem

Let us first look at the particular case n=3, and then generalize it for *any* value of n.

2. The case of n = 3

Figure 2 demonstrates the case of n=3.

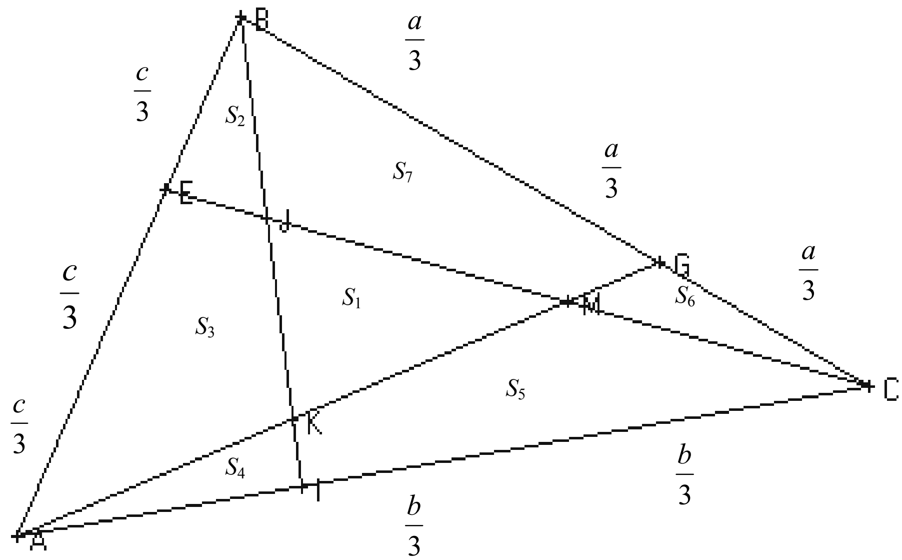

Figure 2: Schematic description of the case: n=3

$BE = ED = DA$; $AI = IH = HC$; $CG = GF = FB$.

Let S_1 = area(JMK) ; S_2 = area(BEJ) ; S_3 = area($EAKJ$) ; S_4 = area(AIK) ;

S_5 = area($KICL$) ; S_6 = area(LCG) ; S_7 = area($BJLG$)

Based on measurements taken by means of dynamic geometry software, the following conjectures as regards to areas and segments were raised:

$$KJ = JB \; ; \; LK = KA \; ; \; JL = LC \tag{1}$$

$$S_2 = S_4 = S_6 \tag{2}$$

$$S_3 = S_5 = S_7 \qquad (3)$$

$$\frac{S_1}{S_2} = 3 \qquad (4)$$

$$\frac{BK}{KI} = \frac{AL}{LG} = \frac{CJ}{JE} = 6 \qquad (5)$$

In order to prove the above conjectures, we join *KD*, *LH* and *JF* (Fig. 3) to generate triangles *BDK*, *AHL* and *CF*.

As follows we prove the claim: $EJ \parallel DK$; $IK \parallel HL$; $GL \parallel FJ$.

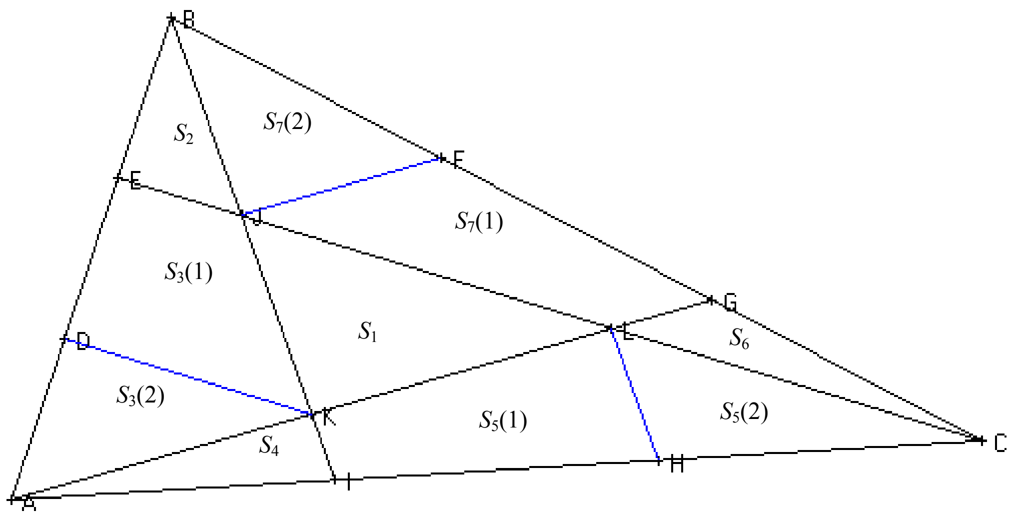

Figure 3: Constructed segments *KD*, *LH* and *JF* and related areas

Proof:

Let $S_3(1) = \text{area}(EDKJ)$; $S_3(2) = \text{area}(DAK)$; $S_5(1) = \text{area}(IHLK)$;

$S_5(2) = \text{area}(HCL)$; $S_7(1) = \text{area}(GFJL)$ and $S_7(2) = \text{area}(FBJ)$ (Fig. 3).

We employ Affine Geometry to prove this claim:

Let *AC* be on the *x*-axis, and *AB* on the *y*-axis, while the unit scale on the *x*-axis is the length of *AC* and the unit scale on the *y*-axis is *AB*. Hence, the coordinates of the vertices are:

$A(0,0)$; $B(0,1)$; $C(1,0)$; $E(0,\frac{2}{3})$; $I(\frac{1}{3},0)$; $G(\frac{2}{3},\frac{1}{3})$; $F(\frac{1}{3},\frac{2}{3})$

For the equation of line *CE* we get:

$$\frac{x-x_E}{x_C-x_E}=\frac{y-y_E}{y_C-y_E}\Rightarrow\frac{x-0}{1-0}=\frac{y-\frac{2}{3}}{0-\frac{2}{3}}\Rightarrow y_{CE}=\frac{2}{3}-\frac{2}{3}x$$

And for line *BI*:

$$\frac{x-x_B}{x_I-x_B}=\frac{y-y_B}{y_I-y_B}\Rightarrow\frac{x-0}{\frac{1}{3}}=\frac{y-1}{0-1}\Rightarrow y_{BI}=1-3x$$

Thus the coordinates of J ($BI\cap CE$) are:

$$\left.\begin{array}{l} y=\frac{2}{3}-\frac{2}{3}x \\ y=1-3x \end{array}\right\}\Rightarrow\frac{1}{3}=\frac{7}{3}x\Rightarrow x=\frac{1}{7}\Rightarrow y=\frac{4}{7}\Rightarrow J(\frac{1}{7},\frac{4}{7})$$

Vectors $\vec{JF}$ and $\vec{AG}$ are:

$$\vec{JF}=(x_F-x_J,y_F-y_J)=(\frac{1}{3}-\frac{1}{7},\frac{2}{3}-\frac{4}{7})=(\frac{4}{21},\frac{2}{21})\ ;$$
$$\vec{AG}=(x_G-x_A,y_G-y_A)=(\frac{2}{3}-0,\frac{1}{3}-0)=(\frac{2}{3},\frac{1}{3})$$

Therefore: $\vec{JF}=\frac{2}{7}\cdot\vec{AG}$, and hence vectors $\vec{JF}$ and $\vec{AG}$ are parallel.

By symmetry considerations: $IK\parallel HL$ and $EJ\parallel DK$. This proves

(1) *KJ* = *JB* ; *LK* = *KA* ; *JL* = *LC*.

Notice that parallelism is not affected by affine transformations.

Referring to the notations in Fig. 3 we shall now prove that:

$$\frac{S_3(1)}{S_2}=3 \qquad (6)$$

$$\frac{S_3(2)}{S_2} = 2 \tag{7}$$

$$S_3(1) = S_5(1) = S_7(1) \tag{8}$$

$$S_3(2) = S_5(2) = S_7(2) \tag{9}$$

Since $EJ \| DK$ and $IK \| HL$ and $GL \| FJ$ we get:

$\Delta BEJ \approx \Delta BDK$; $\Delta AIK \approx \Delta AHL$; $\Delta CGL \approx \Delta CFJ$. The similarity ratio is 2.

Consequently, $S_3 = 3 \cdot S_2$ and similarly: $S_5(1) = 3 \cdot S_4$; $S_7(1) = 3 \cdot S_6$. As a result, (6) is proved,

In addition, since area(BDK) = 2·area (DKA) we get:

$S_2 + S_3(1) = 2 \cdot S_3(2) \Rightarrow S_2 + 3 \cdot S_2 = 2S_3(2) \Rightarrow S_3(2) = 2 \cdot S_2$.

Similarly $S_5(2) = 2 \cdot S_4$ and $S_7(2) = 2 \cdot S_6$.

Thus (7) is proved.

The above relations are summarized in Fig. 4.

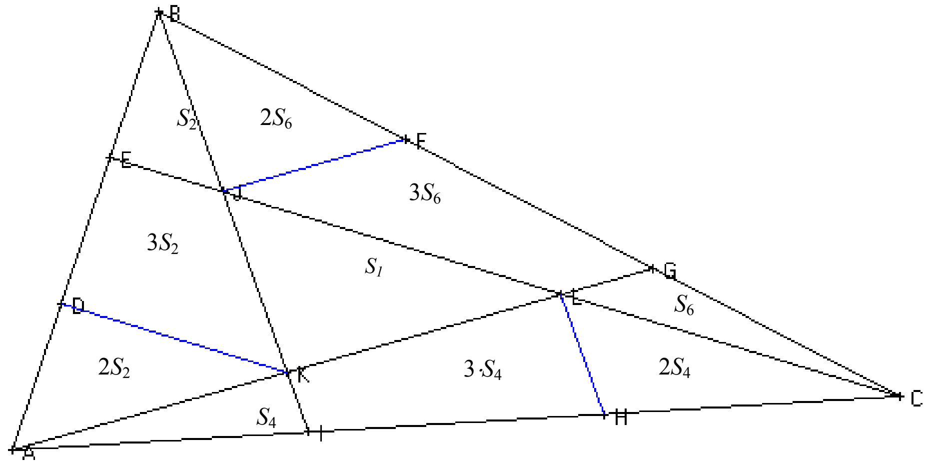

Figure 4: Relations among areas

We shall now prove that: (2) $S_2 = S_4 = S_6$

Since $\text{area}(BAI) = \text{area}(ACG) = \text{area}(CBE) = \frac{1}{3}\text{area}(ABC)$ it follows that:

$6 \cdot S_2 + S_4 = 6 \cdot S_4 + S_6 = 6 \cdot S_6 + S_2 \Rightarrow 5 \cdot S_4 = 6 \cdot S_2 - S_6$; $6 \cdot S_4 = 5 \cdot S_6 + S_2$

Thus: $S_4 = 6 \cdot S_6 - 5 \cdot S_2$. Therefore:

$6 \cdot S_4 + S_6 = 6 \cdot (6 \cdot S_6 - 5 \cdot S_2) + S_6 = 6 \cdot S_6 + S_2 \Rightarrow$

$36 \cdot S_6 - 30 \cdot S_2 + S_6 = 6 \cdot S_6 + S_2$; $31 \cdot S_6 = 31 \cdot S_2 \Rightarrow S_2 = S_6$.

Now: $S_4 = 6 \cdot S_6 - 5 \cdot S_2 = 6 \cdot S_6 - 5 \cdot S_6 = S_6$.

Hence (2) $S_2 = S_4 = S_6$ is proved.

Following the above we obtain: $S_3(1) = S_5(1) = S_7(1)$, $S_3(2) = S_5(2) = S_7(2)$, which imply that we also proved (3) $S_3 = S_5 = S_7$,

We shall now show that: (4) $\frac{S_1}{S_2} = 3$.

Proof:

$\Delta ADK \approx \Delta AEL \Rightarrow 4 \cdot \text{area}(ADK) = \text{area}(AEL)$, hence if $\text{area}(ADK) = 2 \cdot S_2$ than $\text{area}(AEL) = 8 \cdot S_2$. Thus: $\text{area}(AEL) = 2 \cdot S_2 + 3 \cdot S_2 + S_1 = 8 \cdot S_2 \Rightarrow S_1 = 3 \cdot S_2$, and (4) is proved.

The relations obtained are summarized in Fig. 5.

It is now left to prove (5) $\frac{BK}{KI}=\frac{AL}{LG}=\frac{CJ}{JE}=6$. Clearly:

$$\frac{\text{area}(BKA)}{\text{area}(AKI)}=\frac{\frac{BK\cdot h}{2}}{\frac{KI\cdot h}{2}}=\frac{6\cdot S_2}{S_2}=6\Rightarrow\frac{BK}{KI}=6.$$

And $\frac{AL}{LG}=\frac{CJ}{JE}=6$, stems from symmetry considerations.

Thus we complete the proof for all the connections that were discovered for the case of n = 3.

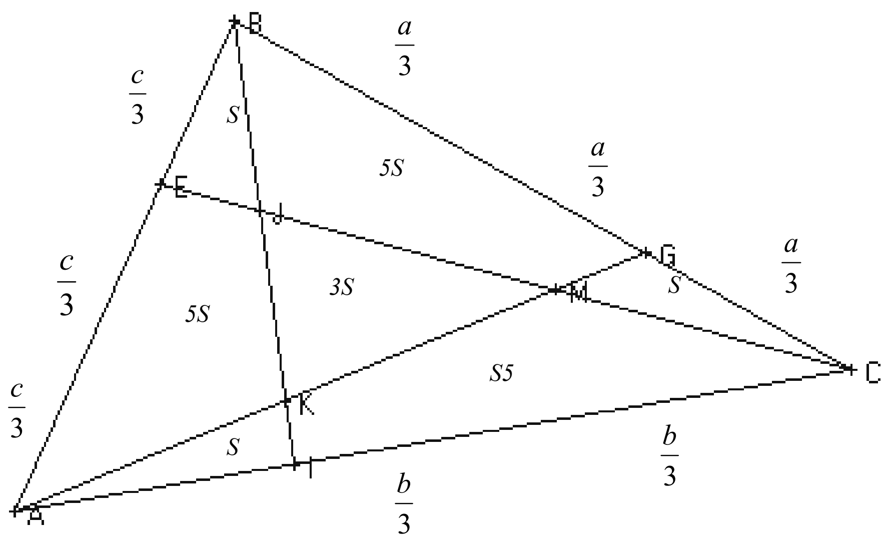

Figure 5: relations among areas of the case: n=3

3. The general case

We shall now examine the general case, in which $n = k$.

For the general case we will show that the following patterns hold:

$$\frac{JK}{BJ}=k-2 \qquad (10)$$

$$S_2 = S_4 = S_6 \tag{11}$$

$$S_3 = S_5 = S_7 \tag{12}$$

$$\frac{S_1}{S_2} = k(k\text{-}2)^2 \tag{13}$$

$$\frac{BK}{KI} = k(k-1) \tag{14}$$

The terminology refers to Figure 6 and Figure 3.

In order to prove (10)-(14) we join *KD*, *LH* and *JF* to generate triangles *BDK*, *AHL* and *CF*.

E is the $\frac{1}{k}$ – point of *BD*, *I* is the $\frac{1}{k}$ – point of *AH* and *G* is the $\frac{1}{k}$ – point of *CF* (Fig. 6).

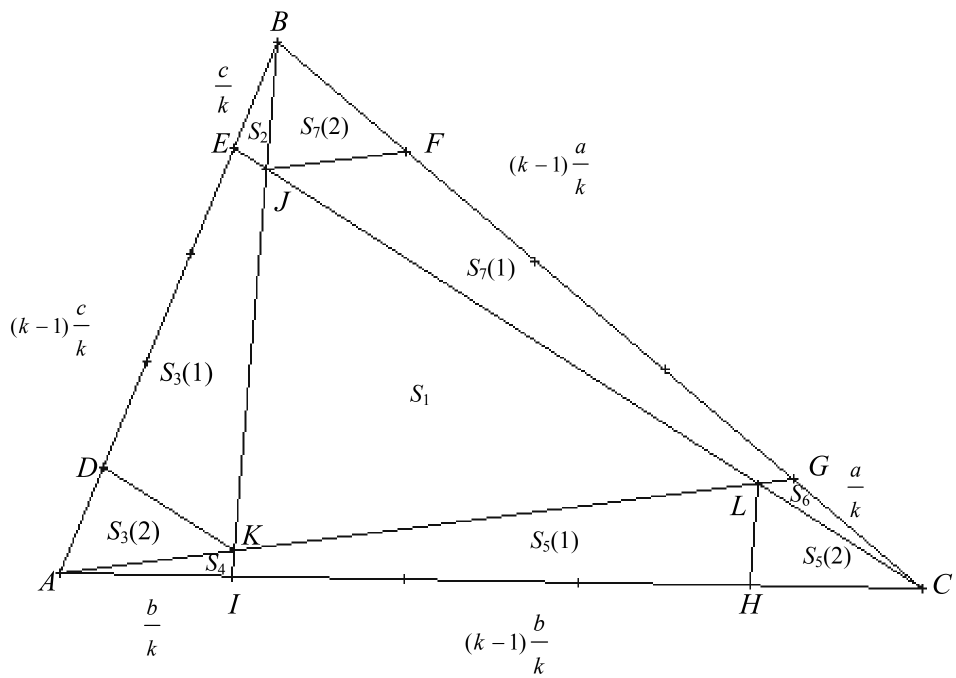

Figure 6: Schematic description of the case of *n*=*k*

We first prove that: $EJ \| DK$; $IK \| HL$; $GL \| FJ$.

Proof:

We employ Affine Geometry to prove this claim:

Let AC be on the x-axis, and AB on the y-axis. The unit scale on the x-axis is the length of AC and the unit scale on the y-axis is AB. Consequently, the coordinates of the vertices are:

$$A(0,0)\ ;\ B(0,1)\ ;\ C(1,0)\ ;\ E(0,\frac{k-1}{k})\ ;\ I(\frac{1}{k},0)\ \ ;\ \ G(\frac{k-1}{k},\frac{1}{k})\ ;\ F(\frac{1}{k},\frac{k-1}{k})$$

For the equation of CE we get:

$$\frac{x-x_E}{x_C-x_E}=\frac{y-y_E}{y_C-y_E}\Rightarrow\frac{x-0}{1-0}=\frac{y-\frac{k-1}{k}}{0-\frac{k-1}{k}}\Rightarrow y_{CE}=\frac{k-1}{k}-\frac{k-1}{k}x$$

And for BI:

$$\frac{x-x_B}{x_I-x_B}=\frac{y-y_B}{y_I-y_B}\Rightarrow\frac{x-0}{\frac{1}{k}}=\frac{y-1}{0-1}\Rightarrow y_{BI}=1-kx$$

Therefore, for the coordinates of J ($BI\cap CE$) we get:

$$\left.\begin{array}{l} y=\frac{k-1}{k}-\frac{k-1}{k}x \\ y=1-kx \end{array}\right\}\Rightarrow x=\frac{1}{k^2-k+1}\Rightarrow y=\frac{(k-1)^2}{k^2-k+1}\Rightarrow J(\frac{1}{k^2-k+1},\frac{(k-1)^2}{k^2-k+1})$$

Vectors $\vec{JF}$ and $\vec{AG}$ are:

$$\vec{JF}=(x_F-x_J,y_F-y_J)=(\frac{1}{k}-\frac{1}{k^2-k+1},\frac{k-1}{k}-\frac{(k-1)^2}{k^2-k+1})=(\frac{(k-1)^2}{k(k^2-k+1)},\frac{k-1}{k(k^2-k+1)})\ ;$$

$$\vec{AG}=(x_G-x_A,y_G-y_A)=(\frac{k-1}{k}-0,\frac{1}{k}-0)=(\frac{k-1}{k},\frac{1}{k})$$

Thus we get: $\vec{JF}=\frac{k-1}{k^2-k+1}\cdot\vec{AG}$, and hence vectors $\vec{JF}$ and $\vec{AG}$ are parallel.

By symmetry considerations $IK\|HL$ and $DK\|EJ$. Thus we have proved that:

(10) $JK=(k-2)\cdot BJ$. Similarly $KL=(k-2)\cdot AK$; $LJ=(k-2)\cdot CL$.

We will now prove (11) $S_2=S_4=S_6$.

From the parallelism it follows that $\Delta BEJ \approx \Delta BDK; \Delta AIK \approx \Delta AHL; \Delta CGL \approx \Delta CFJ$ with a similarity ratio $\frac{1}{(k-1)}$

In addition, since area (BAI) = area (ACG) = area (CBE) = $\frac{1}{k}$ area(ABC)

we get: $\frac{S_3(1)+S_2}{S_2} = \frac{S_5(1)+S_4}{S_4} = \frac{S_7(1)+S_6}{S_6} = (k-1)^2 \Rightarrow S_3(1) = (k^2-2k)\cdot S_2$.

Similarly, $S_5(1) = (k^2-2k)\cdot S_4$; $S_7(1) = (k^2-2k)\cdot S_6$

Furthermore, $(k-1)\cdot S_3(2) = S_2 + S_3(1) \Rightarrow S_3(2) = (k-1)\cdot S_2$.

Since area(BKD) = $(k$-$1)\cdot$ area(DKL) then $\frac{\frac{BD\cdot h}{2}}{\frac{AD\cdot h}{2}} = k-1$. $(k \geq 2)$

As a result: $(k-1)\cdot S_5(2) = S_4 + S_5(1) \Rightarrow S_5(2) = (k-1)\cdot S_4$ and

$(k-1)\cdot S_7(2) = S_6 + S_7(1) \Rightarrow S_7(2) = (k-1)\cdot S_6$

From the above relations we get: $(k^2-k)\, S_2 + S_4 = (k^2-k)\cdot S_4 + S_6 = (k^2-k)\cdot S_6 + S_2$.

Thus: $S_4 = \frac{(k^2-k)\cdot S_2 - S_6}{k^2-k-1} \Rightarrow \frac{(k^2-k)\{k^2-k)\cdot S_2 - S_6\}}{k^2-k-1} + S_6 = (k^2-k)\cdot S_6 + S_2$

$(k^4-2k^3+k+1)\cdot S_2 = (k^4-2k^3+k+1)\cdot S_6 \Rightarrow S_2 = S_6$

By symmetry considerations $S_2 = S_4$. Hence (11) $S_2 = S_4 = S_6$.

Therefore (12) $S_3 = S_5 = S_7$ is also proved.

Consequently $S_3(1) = S_5(1) = S_3(1)$; $S_3(2) = S_5(2) = S_3(2)$

We shall now prove that: (13) $S_1 = k(k-2)^2 \cdot S_2$

Proof:

$$\frac{area\,(AEL)}{area\,(ADK)} = (k-1)^2 \Rightarrow \frac{S_3(1)+S_3(2)+S_1}{S_3(2)} = \frac{(k-1)\cdot S_2 + (k^2-2k)\cdot S_2 + S_1}{(k-1)\cdot S_2}$$

$$S_1 = (k-1)^3 \cdot S_2 - (k-1) \cdot S_2 - (k^2 - 2k) \cdot S_2 \Rightarrow S_1 = (k^3 - 4k^2 + 4k) \cdot S_2 = k(k-2)^2 \cdot S_2$$

Finally we prove that: (14) $\frac{BK}{KI} = \frac{AL}{LG} = \frac{CJ}{JE} = k(k-1)$.

Proof:

We will use the connection:

Area $(ABK) = S_2 + S_3(1) + S_3(2) = S_2 + (k^2 - 2k) \cdot S_2 + (k-1) \cdot S_2 = (k^2 - k) \cdot S_2$

area $(AKI) = S_2$

Hence: $\frac{area(ABK)}{area(AKI)} = \frac{\frac{BK \cdot h}{2}}{\frac{KI \cdot h}{2}} = \frac{(k^2 - k) \cdot S_2}{S_2} = k(k-1) \Rightarrow \frac{BK}{KI} = k(k-1)$.

From symmetry considerations we get: $\frac{AL}{LG} = \frac{CJ}{JE} = k(k-1)$.

The findings can be summarized as follows:

The three *k*-ians of a triangle divide the it into seven sections. The relations between the measures of the areas are described in Figure 7.

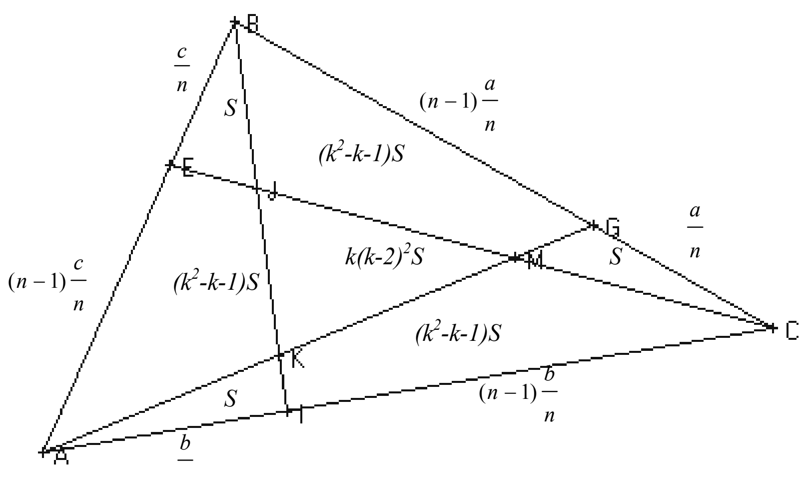

Figure 7: relations among areas of the case n=k

5. Theorem concerning *k*-ians of triangle

Employing the WIN strategy once again, each side of the triangle can be divided into any number, p, q and r, of equal segments. Vertex A is connected to the $\frac{1}{p}$-point, vertex B is connected to the $\frac{1}{q}$-point, and vertex C is connected to the $\frac{1}{p}$-point, as shown in Fig. 8.

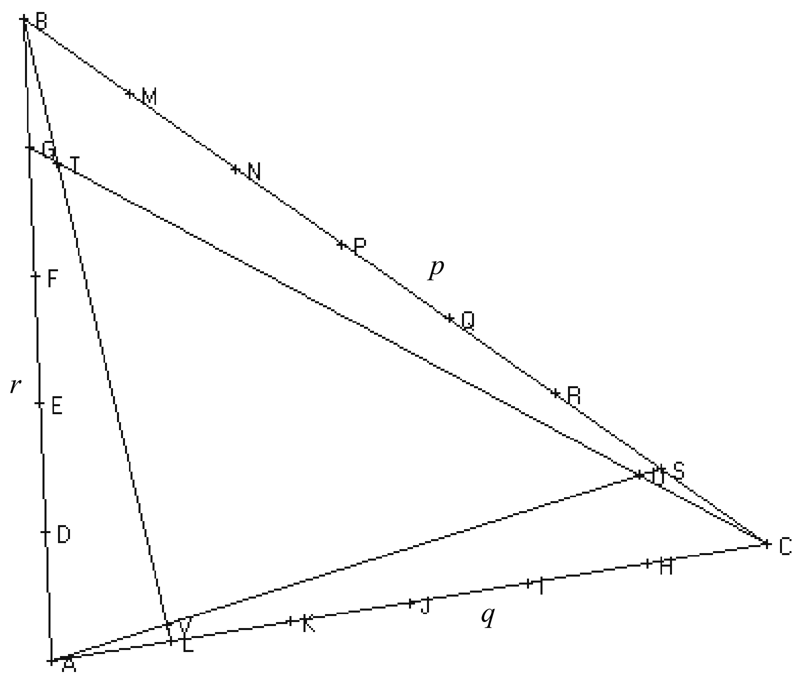

Figure 8: schematic description of the *k*-ians

In this case we get: $\frac{BY}{YL}=(p-1)\cdot q$; $\frac{AU}{US}=(r-1)\cdot p$; $\frac{CT}{TG}=(q-1)\cdot r$.

Proof:

Employ again Affine Geometry to prove this claim:

Let AC be on the x-axis, and AB on the y-axis. The unit scale on the x-axis is the length of AC and the unit sale on the y-axis is AB. Consequently, the coordinates are:

$A(0,0)$; $L(\frac{1}{q},0)$; $B(0,1)$; $C(1,0)$

The vectors: $\vec{AC}=\{1,0\}$, $\vec{CB}=\{-1,1\}$; $\vec{CS}=\{-\frac{1}{p},\frac{1}{p}\}$; $\vec{AS}=\vec{AC}+\vec{CS}=\{1-\frac{1}{p},\frac{1}{p}\}$

The equations of lines BL and AS: $BL: y-1=-qx$; $AS: y=\frac{x}{p-1}$.

The coordinates of Y: $Y=BL\cap AS=(\frac{p-1}{1+q(p-1)},\frac{1}{1+q(p-1)})$

The vectors $\vec{B}Y$ and $\vec{Y}L$ are:

$$\vec{BY}=\{\frac{p-1}{1+q(p-1)},\frac{-q(p-1)}{1+q(p-1)}\}=q(p-1)\cdot\{\frac{1}{q(1+q(p-1))},\frac{-1}{1+q(p-1)}\}$$

$$\vec{YL}=\{\frac{1}{q}-\frac{k-1}{1+q(p-1)},\frac{-1}{1+q(p-1)}\}=\{\frac{1}{q(1+q(p-1))},\frac{-1}{1+q(p-1)}\}$$

The last two results imply that $\vec{BY}=q(p-1)\cdot\vec{YL}$. We leave to the reader to verify that $AU=(r-1)p\cdot\vec{US}$; $CT=(q-1)r\cdot\vec{TG}$.

In Addition, we urge the reader to look for relations among areas that are formed as a consequence of the new division.

Implication for class activities

In this paper we describe a process which can be implemented on various well known mathematical theorems. Utilizing the WIN strategy, which is a useful tool which can easily be applied, combined with the working in an interactive computerized environment, enables the formulation of various inquiry activities such as the example given in this paper. Such an activity could be given as a long term project for developing inquiry skills and mathematical knowledge.

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