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CATCH ME IF YOU CAN!

Steve Humble

Catch me if you can!

Steve Humble[1]
The National Centre for Excellence in the Teaching of Mathematics, UK

Learning mathematics outside the classroom is not enrichment, it is at the core of empowering an individuals understanding of the subject.

The three activities described in this article can all be used outside the classroom in a maths lesson. Teaching mathematical concepts in this way engages and reinforces learning. It puts the ideas learnt into a setting and allows time for those ideas to be developed without any of the maths hang-ups which can occur in the classroom. By taking maths beyond the classroom, we can more clearly illustrate the connections between the real world and what they are studying in school. In so doing students and teachers alike are enthused by the wealth of resources they have all around them in their own environments.

From a very young age we all play “catch me if you can!”, Tag being the most well known version, where one person chases others. When the player catches another they say “Tag, caught you, your on”. The pursuer then becomes the pursued. The 1968 classic car chase movie Bullitt, had Steve McQueen driving his Mustang GT 390 at speed through the hilly streets of San Francisco(1). This is a wonderful example of a movie car chase, but I am sure you could name others. These movies show students real life cases of pursuit. Another example is a fighter plane in battle following on a pursuit curve to shoot down a bomber aircraft. The fighter will continually point its guns and plane towards the target bomber it is trying to shoot down. As the fighter moves in, closing the gap between itself and its prey, the velocity vector will always be pointing towards the bomber.

[1] **Steve Humble** (aka Dr Maths) is a regular contributor to The Montana Mathematics Enthusiast. He works for The National Centre for Excellence in the Teaching of Mathematics in the North East of England (http://www.ncetm.org.uk). He believes that the fundamentals of mathematics can be taught via practical experiments. He is the author of the book *The Experimenter's A to Z of Mathematics,* which develops an experimenter's investigative approach to mathematical ideas. Always having had great fun playing with maths, he enjoys teaching this to others.
For more information on Dr Maths go to
http://www.ima.org.uk/Education/DrMaths/DrMaths.htm

Email: drmaths@hotmail.co.uk

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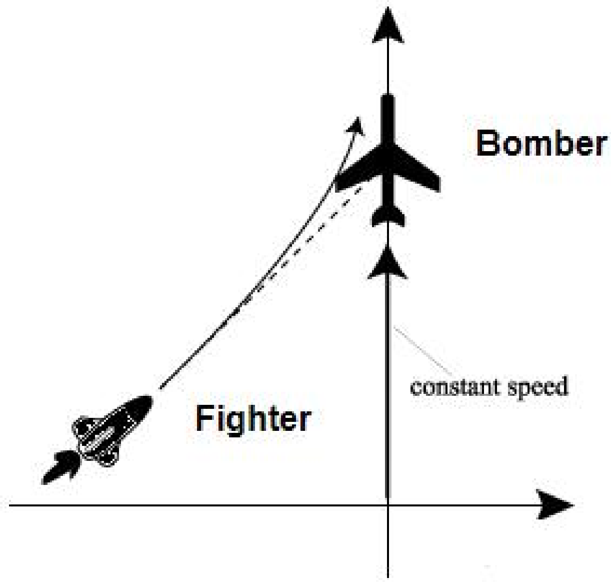
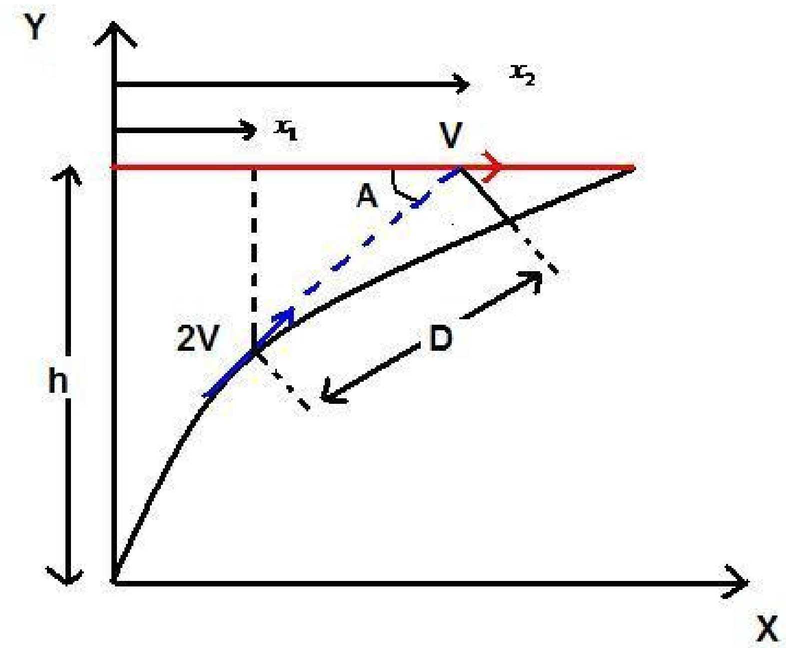


[Fighter and bomber]

The first mathematician to work on the idea of pursuit analysis was the French mathematician Pierre Bouguer in 1732. One way in which his analysis may be illustrated is using the analogy of a cat and mouse chase, with the mouse moving towards its hole in the wall in a straight line at constant speed. The velocity vector of the cat always moves directly towards the mouse. There are two possible areas of mathematics to investigate: how long does it take to catch the mouse? and what is the curve of pursuit the cat follows? Many books (2) tend to look at the latter question which younger students often find difficult. Alternatively by looking to find the point at which the cat and mouse meet, makes the problem more accessible to a wider range of age and ability.

Here is a method to find the rendezvous point, when the cat moves twice as fast as the mouse.



[Cat and mouse]

h is the initial distance between the cat and the mouse.
D is the distance between the cat and the mouse.
x_1 is the horizontal distance moved by the cat.
x_2 is the horizontal distance moved by the mouse.
v is the speed of the mouse, at some general point.
A is the angle between the cat and the mouse, at some general point.

The following three equations define a general point on the pursuit of the mouse.

$$\frac{dx_1}{dt} = 2v\cos A$$
$$\frac{dx_2}{dt} = v$$
$$\frac{dD}{dt} = v\cos A - 2v$$

Note that these show that the distance D is reducing.

Using $\cos A = \frac{1}{2v}\frac{dx_1}{dt}$ gives $\frac{dD}{dt} = \frac{v}{2v}\frac{dx_1}{dt} - 2v$
and $\frac{dD}{dt} = \frac{1}{2}\frac{dx_1}{dt} - 2\frac{dx_2}{dt}$

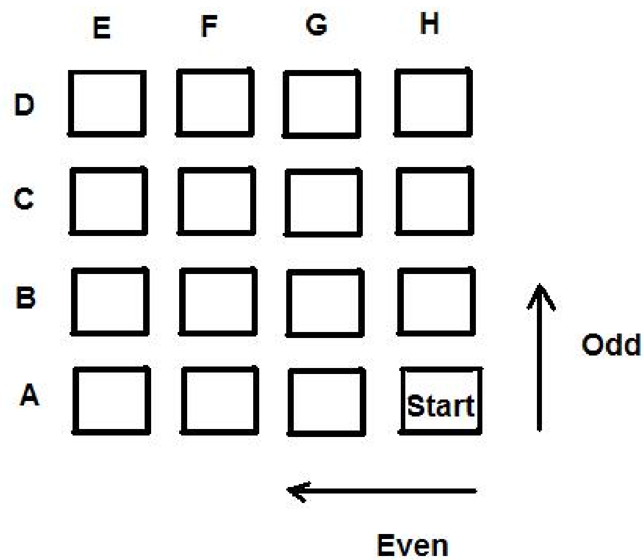
Integrating this equation gives $D = \frac{x_1}{2} - 2x_2 + c$
Using the initial conditions when $t = 0$, $D = h$, $x_1 = 0$, $x_2 = 0$ gives $c = h$

Hence $\qquad D = \frac{x_1}{2} - 2x_2 + h$

When $t = T$, $D = 0$, $x_1 = x_2 = x \Rightarrow 0 = \frac{x}{2} - 2x + h$

Solving this equation gives the point at which the mouse is caught as $x = \frac{2}{3}h$.

Students who have not yet learnt Calculus could still tackle this problem and solve the algebra for various cat and mouse speeds. For example, if the cat moves k times as fast as the mouse then the equation describing the point at which the mouse is caught can be written as
$0 = \frac{x}{k} - kx + h$ and solved for various values of k and h.

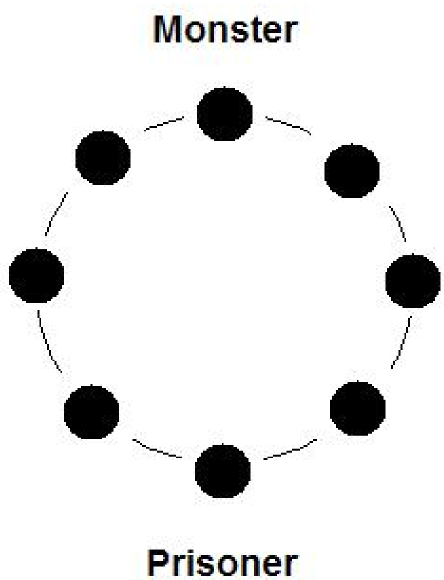


[Grid catchers]

In the next game the pursued has to traverse a 4 by 4 grid to escape the grid catchers. This game is played with a four sided dice or you can use a normal dice, throwing it again if you get five or six on the first throw. Alternatively use a spinner numbered one to four or throw two dice numbered {0,0,0,2,2,2} and {1,1,1,2,2,2}.
Standing in the bottom right hand corner, throw a dice and if you get an odd number move straight up the column the number of squares indicated. An even number indicates a move to the left. If you are still on the grid after the first move throw again, and repeat until you escape the grid.
Before the game starts everyone else has to make a prediction about which point you will exit the grid. They are called the catchers and have to pick A to H marked on the diagram above and stand by this point to catch you as you exit the grid.

Activities to try:

a) Most likely exit point.
b) At which points will you never come off the grid?
c) Least likely exit points (other than those found in (b))
d) If you had to pick 3 places to stand to catch, which would you pick?
e) Calculate the probabilities of where you will come off the grid
f) What happens if you use a dice numbered 1 to 6?



[Monster and Prisoner]

In 1965 Rufus Isaacs(3) created a pursuit-and-evasion game which he called The Princess and Monster Problem. The chase takes place in a pitch black circular tunnel with neither pursuer nor the evader being able to see each other. They both move at the same speeds on a stepping stone type grid around a circular path. In 1972 D Wilson (4) solved the problem mathematically to find the most useful game strategies when played on a discrete interval.

A variation is to play this game with 8 discrete points marked evenly around the circle with the 2 players starting an even number of points apart. I call it Monster and Prisoner. With the Monster and Prisoner game you throw a coin to decide which way you move. In one "move", each player moves one step left (Heads) or right (Tails), each with the probability of a halve . As they always start an even number of steps apart, throughout the game they will always be an even number of steps apart.

Let $E(2)$ and $E(4)$ be the mean number of moves starting at 2 or 4 apart respectively, until they meet on the same stepping stone. These are the only possibilities on an 8-position circle. You can consider one move as the following equations

$$E(4) = 1 + \frac{1}{2}E(4) + \frac{1}{2}E(2) \;\Rightarrow\; E(4) = 2 + E(2) \text{ and}$$
$$E(2) = 1 + \frac{1}{4}E(4) + \frac{1}{2}E(2) \;\Rightarrow\; 2E(2) = 4 + E(4)$$

Solving these simultaneous equations gives $E(4) = 8$ and $E(2) = 6$.

Therefore on average the game last 6 or 8 moves depending on your starting position. Students can play this version of the game and make predictions about how long it will take to get caught. Possible extensions ideas are to play the game with more or less stepping stones and find the mean number of moves until they are caught.

The following BASIC code allows you to simulate the Monster and Prisoner game for an 8 point stepping stone circle

```
10 T = 1
20 S = 0
30 M = 0: E = 4: I = 0
50 I = I + 1
55 REM ***Monster movement***
60 X=RND
70 IF X > .5 THEN M = M + 1: IF M = 8 THEN M = 0
80 IF X < .5 THEN M = M - 1: IF M = -1 THEN M = 7
90 Y=RND
95 REM ***Prisoner movement***
100 IF Y > .5 THEN E = E + 1: IF E = 8 THEN E = 0
110 IF Y < .5 THEN E = E - 1: IF E = -1 THEN E = 7
115 REM ***Check to see if caught***
120 IF M = E THEN PRINT M, E, I ELSE GOTO 50
130 S = I + S
140 PRINT "Average"; S / T ; "after "; T ; " turns"
150 T = T + 1
160 IF T < 10000 THEN GOTO 30
```

References

(1) Bullitt, had Steve McQueen http://www.youtube.com/watch?v=GMc2RdFuOxI
(2) Paul J. Nahin, Chases and Escapes. [Princeton University Press]
(3) R. Isaacs, Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization, John Wiley & Sons, New York (1965).
(4) D. Wilson, Isaacs' Princess and Monster Game on a Circle by (Journal of Optimization Theory and Applications, 9 1972 (no 4). pp.265-288]

Acknowledgements

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