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FOSTERING CONNECTIONS BETWEEN THE VERBAL, ALGEBRAIC, AND GEOMETRIC REPRESENTATIONS OF BASIC PLANAR CURVES FOR STUDENT'S SUCCESS IN THE STUDY OF MATHEMATICS

Margo F. Kondratieva

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Margo F. Kondratieva[1] & Oana G. Radu
Memorial University of Newfoundland, Canada

Abstract: We discuss the significance of making connections between the verbal, algebraic, and geometric representations of basic mathematical objects for students' understanding of mathematical instructions. Our survey of 499 students enrolled in a pre-calculus university course reveals that such connections are not always present, even if the objects themselves are familiar to the students. We stress that the ability of making these connections needs to be specifically addressed in teaching mathematics at various levels. A proper attention to the matter contributes to the formation of students' mathematical background, which makes a difference for their success in study of calculus, in particular.

Introduction.

The words we use have different degrees of precision and clarity; they have different capacities to identify various concepts and express certain images and feelings we may experience. Consequently, some rare words may evoke fuzzy and uncertain images, if any at all. Even if a word sounds familiar it may produce nothing but a blank image in one's mind. It may also produce a poor or inadequate association featuring some restrictive interpretation or a very specific situation. The ability to retrieve a complete, adequate, and flexible image associated with a given word is essential for our communication. The development of this ability depends on the frequency of using the word in a conversation, as well as the context, personal experience and practices related to the word. In order to enhance this development it is important to reflect upon and

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[1] mkondra@math.mun.ca

adjust the image through observations of how others use the word or respond to it. The attachment of a word to an idea or object does not appear at once. There are various cognitive processes leading towards the formation of a word's meaning:

o Categorization in a very rough way.
o Recognition and assigning some meaning within a context only.
o Evocation of a related image without a context.
o Frequent use in speech or writing.
o Recollection of the word, given a definition or description of it (like in a crossword).
o Recognition of synonyms and antonyms.

Consequently, there are various levels of familiarity with a particular word:

o Never heard.
o Heard but do not know exact meaning.
o Can guess the most appropriate meaning from a variety of given descriptions.
o Can give an example or counterexample.
o Can list properties.
o Can explain the meaning with various representations and contexts.

Our everyday casual words and words used in a mathematical context do not differ much in the sense specified above: they all carry a certain meaning, which develops through their use in conversations and is accompanied by formation of certain images. A non-understanding of a sentence starts from a non-understanding or inadequate understanding of a word. However, in a mathematical conversation the situation becomes more complex due to the fact that many words have a precise formal definition, which can be expressed in mathematical symbols and formulas. The formulas may also have a geometrical or pictorial representation to accompany them and to add to the formation of a complete image. Thus, in a mathematical conversation one often needs a three-way linkage between words, formulas, and graphs. Lack or weakness of one of those associations leads to poor understanding and failure to grasp the meaning of a mathematical sentence.

In this study we worked with 499 first year university students enrolled in a precalculus course. We collected data concerning students' ability to match names, formulas, and graphs of basic planar curves, as the ability developed in high school courses. We express a concern about an unreasonable assumption, frequently occurring in teaching practices, about the presence of those three-way links in students' cognitive schemas. In order to be effective, an instruction shall not rely on the assumption about the presence of those links. Instead, it shall reinforce and strengthen the links by means of repetitive juxtaposition of the same ideas in the three different representations.

The paper is organized as follows. In section 1 we have a brief discussion about concept formation and acquisition in terms of how the concept is introduced within a field of professional knowledge and internalized by a particular learner, who is new to the field. In section 2 we describe our experimental setting, and research questions. Section 3

summarizes the expected mathematical background and abilities of the students of our sample. Section 4 contains the results of our survey. We conclude the paper with a discussion in section 5 about the importance of forming proper connections between names, images and formulas of basic plane curves, particularly for the students' future success in study at the university level.

1. Words and images in mathematics.

While it is questionable whether or not many fundamental concepts are fully expressible in words and images, an ability to do so, or at least a desire to do so with a certain precision is essential for clear communication of our understanding of them. Words and images play a dual role in this process: we use the words to define the concepts formally, but we often rely heavily on the images when it comes to internalizing the meaning.

Tall & Vinner (1981) define *concept image* as a "total cognitive structure that is associated with the concept". In their description it is important that the image must include processes and properties besides all mental pictures associated with the concept. They contrast the notion of image with *formal concept definition* as "a form of words used to specify a concept", and argue that in thinking the concept image will almost always be evoked while formal definition "will remain inactive or even forgotten".

Furthermore, it was observed that mental pictures associated with a concept contain special examples that are highly significant for the grasp of the concept. Such examples, often called prototypes, are used by the learner as "cognitive reference points". The prototypical thinking was identified in the study of natural semantic categories (Rosch & Mervis, 1975), as well as in a geometrical context (Hershkowitz & Vinner, 1983). In visual prototypical thinking "the shape of the prototype serves as a criterion for judgment" (ibid). Besides that, the thinking could be based on self-attributes of a prototype, i.e. on the features and properties this particular prototype possesses. The drawback of a prototypical judgment is that while some features of a prototype are not characteristic for the category or concept the prototype represents, they can nevertheless be considered as being essential. In this case, the student may "reject an instance as an exemplar of a concept because the instance lacks the self-attribute of the prototype" (Schwarz & Hershkowitz, 1999).

Another serious problem with prototypical thinking is that the degree of rigor is insufficient to carry on a mathematical derivation. As noted by Poincaré (1996) in his discussion on a role of the definition in mathematics, "many learners will not have understood, unless they find around them the object of such and such mathematical nature. Under each word they want to put a sensible image; the definition must call up the image, and at each stage of a demonstration they must see it transformed and evolved. On this condition only will they understand and retain what they have understood. These

often deceive themselves: they do not listen to the reasoning, they look at the figures; they imagine that they have understood when they have only seen".

A concept image appeals to a student's intuition, "but intuition cannot give us exactness, not even certainty, and this has been recognized more and more". The exactness cannot be introduced in arguments unless it is "first introduced in definitions". These observations lead us to a conclusion that formal definitions are essential for mathematical culture but often become an obstacle for mathematical teaching and learning. Initially, students need to be given an image of a concept, a prototype, an example, a framework for developing their intuition. But after this stage "they should be made to see that they do not understand what they think they understand, and brought to realize the roughness of their primitive concepts, and to be anxious themselves that it should be purified and refined" (Poincaré, 1996).

In the introduction of Polya's celebrated book *How To Solve It*, we find a similar idea: mathematics has two faces, it is presented by rigorous definitions and proofs, but it is discovered or invented by guessing and intuition. This fact is reflected in the existence of radically different approaches to its teaching and learning and in extensive discussion among educators taking opposite sides of the debate.

An analysis of the interplay between rigor and intuition brings us to the following important goal of mathematical teaching. That is, helping the learners to establish and be in control of a strong connection between the words and formulas used in mathematical reasoning, and the images produced in the learners' minds. The students ought to develop an awareness of their mental actions and the degree of adequacy of their mathematical prototypes, reinforcing their reasoning.

2. The sample, the procedure and the research questions.

2.1 The sample.

The sample consisted of 499 students enrolled in a precalculus undergraduate course at a large Atlantic Canadian University. This course is offered by the Department of Mathematics and Statistics for those students who, according to their Mathematical Placement Test scores, need to improve their mathematical skills in order to study calculus and other courses offered by the department. These students have previously studied mathematical concepts tested in our questionnaire in senior high school. The questionnaire was administered before these concepts were reviewed and used in the precalculus course.

According to the provincial curriculum, the most advanced mathematical course, which is not required for graduation but is desirable for students planning mathematics related university study, is Mathematics 3207. Students in the advanced stream normally graduate from high school with Mathematics 3205, and students in academic stream – with Mathematics 3204. The same core curriculum and textbook is used for both

Mathematics 3204, and Mathematics 3205, but the latter course covers the material in more depth.

Upon labelling what was the highest-level senior high school mathematics course and year of graduation, the students were divided into representative categories. There were 73 students graduated with Mathematics 3207, 52 students with Mathematics 3205, 222 students with Mathematics 3204, and a mixed sample of 152 students who did not specify the highest-level mathematics course taken.

2.2 The survey and the procedure.

The questionnaire shown in Appendix A was administered in English to the subjects of the sample. The students were not asked to provide their names, but they were asked to state the highest mathematics course taken in high school and the year of completion.

There was no review or any special activity aimed at refreshing students' memory about the subject of the survey. The students did not know prior to the survey what types of questions are going to be asked and were not specifically prepared for them. The students were asked to perform to the best of their ability, but they were not motivated by any reward for showing good results. We speculate that many of them were working at the level of knowledge recall and did not try to analyze in any way the information given. In this sense, the results of the survey reflect the true state of the concepts' knowledge as they were formed and retained by the students.

The questionnaire was administered for 25 minutes, during regular class time. The first question was designed to reveal the students' concept images. Within the first question, six words were provided: line, circle, semicircle, ellipse, parabola, and hyperbola. The students were asked to draw what first comes to their mind upon reading the given words. The Cartesian coordinate axes with no division scale were given. The second question asked the students to state how many functions can be drawn through three given points. The Cartesian coordinate system was provided and did not contain a division scale. The three points were positioned in the first quadrant. This question is the same as in a study of Schwarz & Hershkowitz (1999). We do not provide results for this question, as its purpose was to act as a separator between the first and the third question. The third and last question was designed to test the students' understanding of correspondence between algebraic and graphical transformations. Within the third question, the provided graphs incorporated scaled axes. The students were asked to match the formulas and names with the provided images. The questionnaire specifically addressed the fact that there might be several correct formulas for one graph, e.g. $x = -2$ and $x + 2 = 0$; $xy = 1$ and $y = \dfrac{1}{x}$; $y = |x|$ and $y = \sqrt{x^2}$ for line, hyperbola, and absolute value, respectively. Students could have matched one of two or both formulas for the same graph.

2.3 The research questions.

Based on the results obtained from the survey, we aim to address the following questions:

1. What are the students' prototypes associated with the words: line, circle, semicircle, parabola, hyperbola, and ellipse?

2. What is the most frequently encountered example in each case?

3. How well are the students able to recognize and name the graphs of the curves listed in question one?

4. How well are the students able to match the graphs of the curves with the associated algebraic equations, and to recognize the corresponding algebraic and geometric transformations, such as shifts and stretching, applied to the standard form of a curve?

3. Mathematical context to be tested in the survey.

3.1 General principles and approaches introduced in high school.

The objects we work with have a strong visual aspect: they all are plane curves, which can be defined as a locus of points in the plane with certain geometric properties. While the curves can be introduced through those characteristic properties, or otherwise as conic sections, they are also graphs of algebraic equations in the Cartesian coordinate plane. According to the high school curriculum, the students we surveyed were supposed to be familiar with only the latter aspect of the curves. Needless to say, this reduces the richness of the concepts along with the broadness of possible applications, but we leave this matter for another discussion.

The important fact that should be known to students is that behind each of the tested mathematical object such as line, circle, parabola, etc., there is a whole family of curves. Usually one can talk about the principal member of the family equipped with a number of parameters. Varying the parameters, one can describe all other members of the family, including some degenerate or untypical cases, and even bifurcations of the family. This fact can be viewed as an application of a more general principle: starting from an arbitrary curve one can transform it by stretching and shifting to another curve of the same algebraic kind. Alternatively one talks about rescaling and shifting the system of coordinates while leaving the curve unchanged. In any case the core of the general principle is the correspondence between the algebraic and geometric transformations: the horizontal/vertical translations of the curve produce the shift of the arguments in the algebraic equation of the curve $F(x, y) = 0 \rightarrow F(x - a, y - b) = 0$, while the horizontal/vertical stretching of the curve corresponds to the rescaling $F(x, y) = 0 \rightarrow F(ax, by) = 0$. Note that both operations are linear with respect to the arguments x and y.

Our first question aims to find out whether or not the name of a curve evokes any graphical images in the minds of the students. Considering that there is an infinite number of possible responses, we are also interested whether some of them are more popular than others, and how broad or narrow is the set of all produced examples in the case of each curve.

On a separate page we tested the students' ability to name an algebraic curve given in the Cartesian plane and to choose an appropriate formula from a pool of algebraic equations. Besides knowing the prototypical shape of curves, another principle appears to be very helpful for matching a Cartesian graph with a formula, i.e. the curve consists of those and only those points whose coordinates satisfy the algebraic equation of the curve. Consequently, it helps to look at some special points, such as the *x*- and *y*-intercepts and the origin, as well as to investigate the boundaries of a curve, and to identify special features of the domain and range.

Thus, besides the basic knowledge and comprehension, this task requires analysis and synthesis to some degree. The latter comes into play particularly when a students is asked to recognize an algebraic formula for a non-traditional (for senior high school) but intuitively familiar curve, such as a semicircle. Acquiring the skills of analysis and synthesis is possible if "elements are not presented as meaningless statements to be learned at the level of Knowledge, but where emphasis is on "why" of each point" (Whilhoyte, 1965 as cited in Furst, 1981). "Thus, the student may not know what a principle means until understanding occurs at least at the next level (Comprehension). But even under knowledge of specific there is necessarily embedded a variety of intellectual abilities and skills" (Pring 1971, & Sockett, 1971 as cited in Furst, 1981).

For the purpose of illustration we present few examples of reasoning useful for matching the equation $y = \sqrt{1-x^2}$ with corresponding graph (see Appendix).

Method 1. Analyzing domain and range of function $y = \sqrt{1-x^2}$ students notice that $y \geq 0$ and $y \leq 1$ and that $x^2 \leq 1$. Thus, the entire curve is constrained by the rectangle $-1 \leq x \leq 1$, $0 \leq y \leq 1$. This makes the choice of graph obvious.

Method 2. If the students start from the graph, they notice that the following integer points $(1,0), (-1,0)$, and $(0,1)$ belong to the graph. Thus, they can choose $x = 1$ and $y = 0$ and substitute these values into the provided equations, until one gives an identity. If more than one graph are selected this way, then other integer points will help to single out the answer.

Method 3. Students square both sides of the equation $y = \sqrt{1-x^2}$ and obtain the familiar equation of the unit circle. Then, observing that $y \geq 0$, they choose the graph of the upper semicircle.

3.2 Particular notions introduced in high school.

This section gives a brief overview of when and how the curves of our interest are introduced in the textbooks currently used in the province. In this respect, we refer to *Mathematical Modeling, Book 1* (Barry, Small, Avard-Spinney, & Wheadon, 2000) used for study Mathematics 1204, which is a level-one senior high school course, normally taken by students in grade 10, and *Mathematical Modeling, Book 3* (Barry, Besteck-Hope, Hope, Pilmer, Small, Avard-Spinney, & Wheadon, 2002), used for both Mathematics 3204 and Mathematics 3205, which are graduation level courses. For the most advanced mathematical course Mathematics 3207, *Mathematical Modeling, Book 4* (Barry, Besteck-Shaw, Brown, & Avard-Spinney, 2002) is used.

1. The line

The line is formally introduced in Mathematics 1204 in the slope y- intercept form $y = mx + b$, where m represents the slope and b is the y-intercept. In Book 1, the concept of line is mainly used in applications of linear behaviours, e.g. economy-cost issues.

2. The circle

The name and the shape of the circle are introduced as early as elementary school. However, neither the equation nor the coordinate axes are present until Mathematics 3204/3205. In Book 3, the circle is defined as "the set of points in a plane that are at the same distance (radius) from a fixed point called the centre" (Barry et al., 2002). A unit circle is introduced via equation $x^2 + y^2 = 1$ as a circle with radius 1 and centered at the origin. Any circle is viewed as an image of the unit circle under one of the following mapping rules $(x, y) \rightarrow (rx, ry)$ and $(x, y) \rightarrow (x + h, y + k)$, or their combination. As a result, the general equation of a circle in standard form is $(x - h)^2 + (y - k)^2 = r^2$. It can be rewritten in the transformational form as $\left[\dfrac{x - h}{r}\right]^2 + \left[\dfrac{y - k}{r}\right]^2 = 1$.

3. The absolute value function

In the high school course Mathematics 1204, the notion of absolute value $|x|$ is introduced as the distance between a number x and the origin. The algebraic description of this function is $y = |x|$. In Book 1, students are encouraged to "construct a table of values for this function using x-values between -4 and 4" (Barry et al., 2000), to graph the function and to describe the shape of the function in their own words. A variety of examples are listed and the theoretical results of their investigations are summarized succinctly as vertical and horizontal translations. For example, in Book 1 "the graph of $y - q = |x|$ is the image graph of $y = |x|$ after a vertical translation of q units; and the graph of $y = |x - p|$ is the image graph of $y = |x|$ after a horizontal translation of p units" (Barry et al., 2000). Reviewing the absolute value in Book 4, a more elaborate image is

presented, i.e. "the graph is composed of two segments, each described by a different linear equation" (Barry et al., 2002). The notion of a piecewise-linear function and algebraic formula, $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ are introduced. Therefore, a complete connection between name, algebraic definition and graphical image is established in Mathematics 3207.

4. The parabola

In Book 1, the investigation technique is used in introducing the concept of a graph of a quadratic function. Students are being asked to "construct a table of values" for $y = x^2$ "using x-values between -4 and 4", and then they are asked to graph the function (Barry et al., 2000). Oftentimes, the emphasis is placed on the study and recognition of elementary functions, e.g. "if you can recognize the graphs of the basic functions like $f(x) = x$ or $f(x) = x^2$, you can often use these basic shapes to sketch the graphs of more complex functions" (Barry et al., 2000). With reference to the material studied before, the term parabola is introduced in Book 3, as "the graph of any quadratic function" (Barry et al., 2002). Details pertaining to the vertex, axis of symmetry and the transformational form are discussed. The transformational form of a quadratic function is expressed as $a(y - k) = (x - h)^2$, where parameters a, k, h are real numbers and $a \neq 0$. The transformational form is used as early as Mathematics 1204, together with the standard form $y = a(x - h)^2 + k$, where $a \neq 0$. In both Book 1 and Book 3, the first example introduced is $y = x^2$ and is often used for further comparison with transformed shapes.

5. The ellipse

We notice that the shape of the ellipse appears as early as Mathematics 1204 (Barry et al., 2000), but no proper identification is attached to the shape. During Mathematics 3204/3205, the name oval is used for the first time in conjunction with the shape of an ellipse (Barry et al., 2002). Further along Book 3, the ellipse is explored as being a stretching transformation of the unit circle with possible translation. The transformational form of the equation of the ellipse is given as $\left[\dfrac{x - h}{a}\right]^2 + \left[\dfrac{y - k}{b}\right]^2 = 1$.

6. The hyperbola

As early as Mathematics 1204, students have the opportunity to see hyperbolas, although the actual name of the curve is not revealed in Book 1. The shape of a hyperbola occasionally appears, e.g. in the "equipping your function toolkit" section (Barry et al., 2000). In Mathematics 3207, the simple rational functions are formally introduced. The

first example of such function appears in Book 4 and has the form $f(x) = \dfrac{c}{x}$ (Barry et al., 2002). In the same book, the hyperbola is defined as follows. "These functions (i.e. $y = \dfrac{c}{x}$) are examples of rational functions and their graphs can form a conic section called a *hyperbola*" (Barry et al., 2002). The notions of horizontal and vertical asymptotes are also introduced and discussed at this level.

4. Results.

4.1 Evoking images.

The first question of our survey stated: "Draw what comes to mind when you read the following words: line, circle, semicircle, ellipse, parabola, and hyperbola". The data collected address our first two research questions: what are the students' prototypes and what is their frequency? The results obtained for the first question are presented in the following charts.
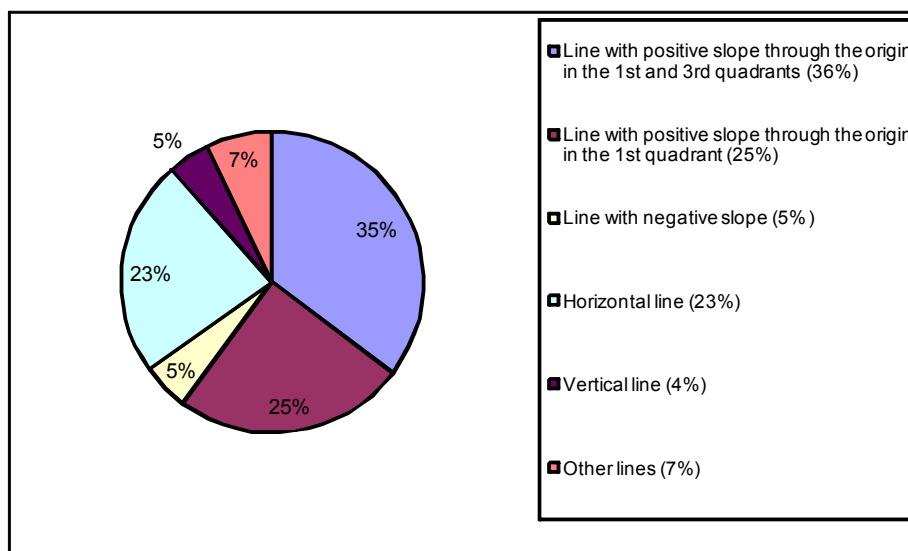
Figure 1. The variety and frequency of images of *line* evoked by the entire sample of the precalculus students.

With respect to drawing lines, 61% of the students draw lines with positive slope; while only 5% draw lines with negative slope. We infer that the apparent prototype is the line with positive slope, passing thought the origin. The lines with positive slopes drawn followed the pattern of $y = x$, or small variations of it, e.g. $y = cx$ and $c > 0$. We believe that the observed apparent prototype is influenced by both the frequency of similar

examples and the nature of the very first example students encountered while the concept was introduced.
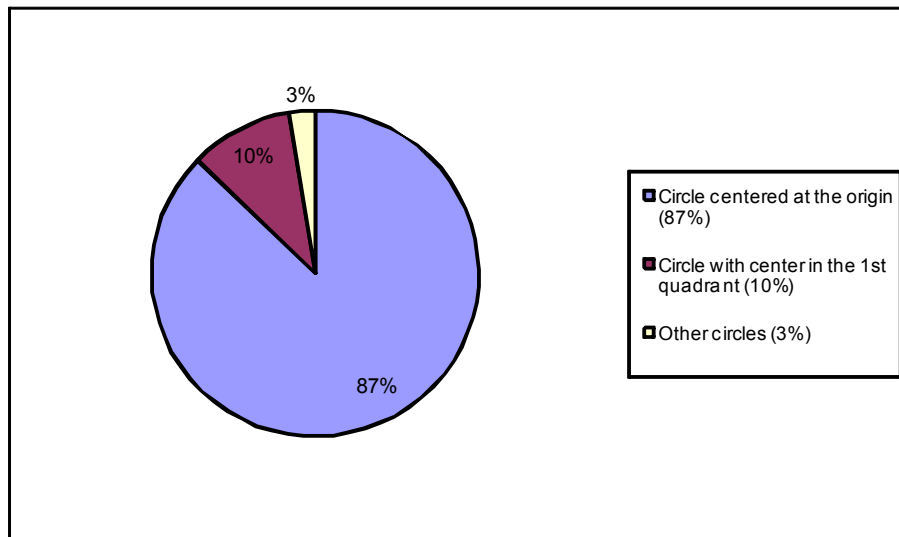

Figure 2. The variety and frequency of images of *circle* evoked by the entire sample of the precalculus students.

With respect to drawing the circle, 87% of the entire sample did draw a circle centered at the origin. We infer that the obvious prototype is the circle centered at origin.
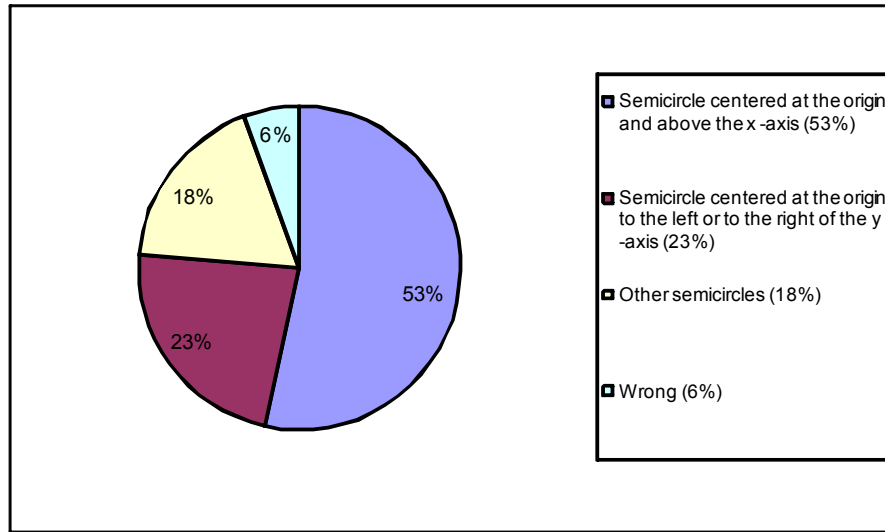
Figure 3. The variety and frequency of images of *semicircle* evoked by the entire sample of the precalculus students.

In terms of the semicircle concept, 76% of the entire sample decided to split in half the prototype circle either above the x-axis or to the left or right of the y-axis. Therefore, we infer that the semicircle prototype is directly connected to and derived from the circle prototype. Only 18% of the entire sample decided to draw other types of semicircles.
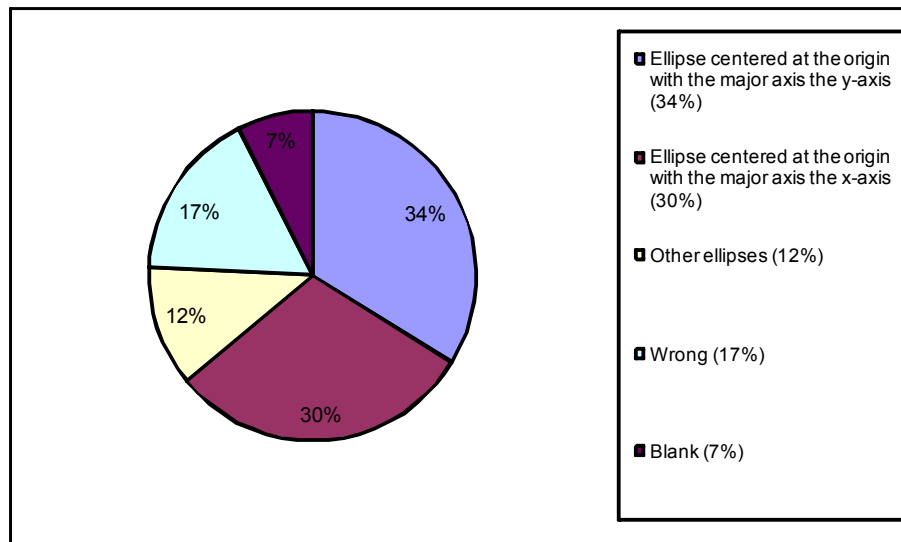

Figure 4. The variety and frequency of images of *ellipse* evoked by the entire sample of the precalculus students.

With regards to the ellipse, there is no clear winner in terms of the prototype used; since 34% draw an ellipse stretched along the y-axis, while 30% draw an ellipse stretched along the x-axis.
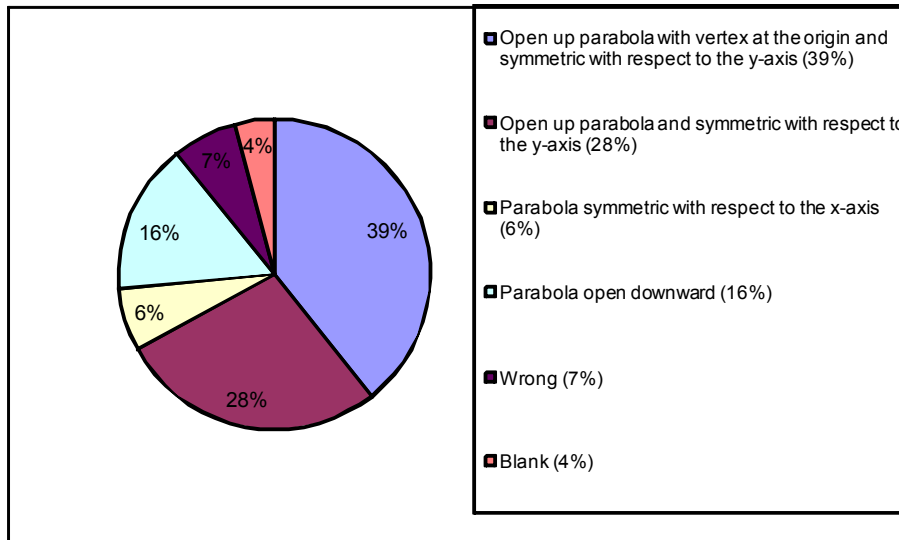

Figure 5. The variety and frequency of images of *parabola* evoked by the entire sample of the precalculus students.

With respect to drawing parabolas, 67% of the entire sample's preference was related to drawing an open upward parabola, while 16% of the students draw open downward parabolas. We infer that the evident prototype is an open upward parabola, passing thought the origin. The drawn open upward parabolas followed the pattern of $y = ax^2$, $a > 0$; while the open downward parabolas followed the pattern of $y = ax^2$, $a < 0$. In other words both types of parabolas had vertex at the origin. We believe that the observed prototype coincides with the first example of the graph of a quadratic parabola presented in Mathematics 1204.
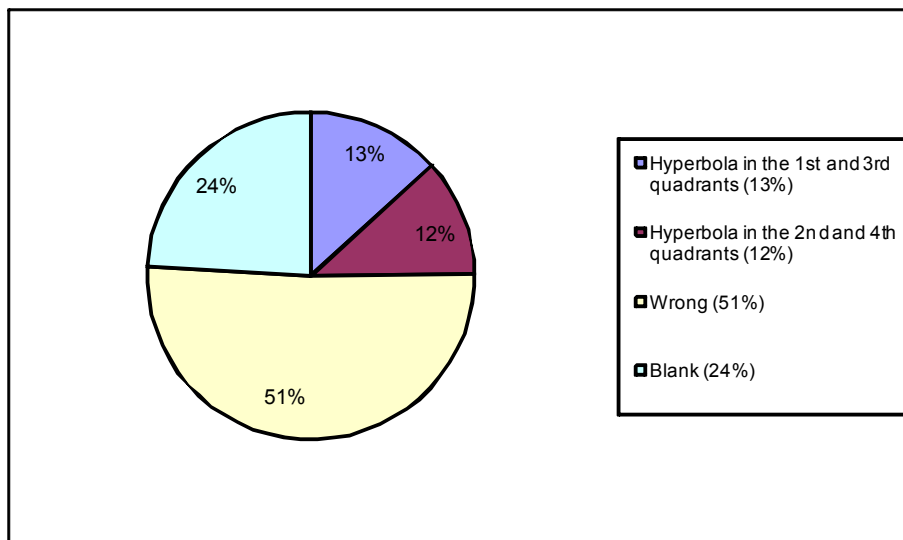
Figure 6. The variety and frequency of images of *hyperbola* evoked by the entire sample of the precalculus students.

The diagram on Figure 6 clearly reflects the absence of the hyperbola from the high school curriculum. As we pointed out earlier, only students completed Mathematics 3207 receive proper knowledge in relation to this curve. Such students constitute about 15% of our sample, so the fact that 25% of the sample nevertheless is familiar with the curve, is an evidence of random occurrence of this object in earlier mathematical courses.

It is noticeable that the majority of the graphs produced by the students are either centered around the origin (circle, semicircle, ellipse and hyperbola) or pass through the origin (line and parabola). It is hard to say whether this is an evidence of the rigidity of students' prototypes having an irrelevant feature such as reference to the origin of the Cartesian plane. Probably, this is just a natural result of frequent exposition of the students to the origin-centered graphs, so that images having this attribute are indeed "what comes to mind first" but this does not exclude the familiarly of the students with other less typical examples. Having said that, we still see a potential danger of the frequent use of the origin centered examples, as this may cause the formation of a distorted view and restricted prototypes, and is particularly undesirable for students planning to study future mathematical courses that require more flexibility and adaptability of the images. The students' ability to juggle with the visual and graphical aspects of basic curves will be essential in grasping more elaborate mathematical objects.

But what makes understanding of the curves flexible? The whole idea that a parabola remains a parabola even if it is translated and rotated in a plane is not difficult. Far less obvious is the connection of a curve transformation with corresponding algebraic manipulations, and we claim that this very connection is often not well established as it will follow from the results of the second page of our questionnaire.

4.2 Matching graphs and formulas with names.

In this section we report the results obtained from the responses occurred on the second page of our questionnaire where students were provided with twelve graphs and were asked to match them with equations and names from a given list (see Appendix). The following table contains information on each curve for entire sample as well as for each category of students who identified their highest mathematical course as Mathematics 3207, Mathematics 3205, or Mathematics 3204. In the last column, for a purpose of comparison, we also give data collected for a group of randomly selected students who had completed six or more undergraduate mathematical courses including calculus stream at least two years prior to the survey date. We call them the senior math group. This group of 27 students also was not specifically prepared or informed about the types of questions prior to the survey, so their performance is, in the same way as with the precalculus students, a true measurement of the students current state of knowledge.

	Entire Sample		3207 Sample		3205 Sample		3204 Sample		Mixed Sample		Senior Sample	
Number of Students	499		73		52		222		152		27	
	Name	Formula	Name	Formula	Name	Formula	Name	Formula	Name	Formula	Name	Formula
Vertical line	80%	27%	86%	44%	88%	27%	77%	17%	77%	32%	100%	85%
Horizontal line	86%	46%	94%	66%	84%	42%	88%	40%	80%	45%	100%	92%
Line with positive slope	82%	17%	94%	33%	81%	17%	83%	10%	75%	20%	100%	96%
Line with negative slope	84%	7%	94%	12%	86%	13%	85%	3%	77%	7%	100%	89%
Parabola opened upward	80%	27%	86%	44%	88%	27%	77%	17%	77%	32%	100%	85%
Parabola opened downward	66%	17%	76%	22%	77%	15%	69%	12%	54%	20%	96%	74%
Hyperbola Quadrants 1 & 3	42%	6%	50%	7%	48%	8%	45%	4%	31%	9%	93%	71%
Hyperbola Quadrants 2 & 4	66%	17%	76%	22%	77%	15%	69%	12%	54%	20%	96%	74%
Circle	85%	31%	92%	37%	88%	50%	88%	26%	78%	29%	100%	93%
Semicircle	72%	2%	81%	5%	82%	2%	74%	2%	61%	2%	81%	67%
Ellipse	61%	24%	70%	31%	77%	38%	65%	21%	47%	21%	96%	89%
Absolute Value	63%	51%	78%	65%	69%	57%	64%	47%	52%	47%	89%	89%

Some of the observations from the table are:

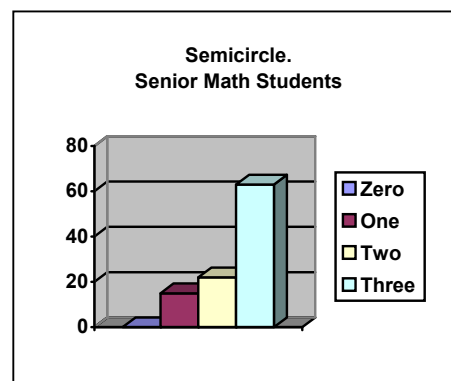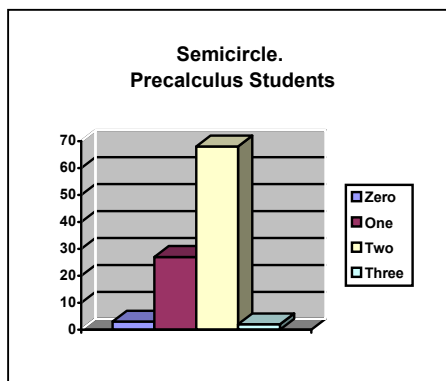- o Students recognize the names of curves much better than their formulas.

o   Students more correctly recognize formulas for lines with positive slope than for lines with negative slope, and parabolas opened upward than parabolas opened downward.

o   There is a noticeable increase in percentage 10-17-33% for students enrolled in Mathematics 3204-3205-3207 in recognizing the formulas for the lines with positive slope. However, we would expect a much better match for a line with equation $y = x$. The best match was done for the formula corresponding to a horizontal line, i.e. 40-42-66% for Mathematics 3204-3205-3207.

o   The order of preference in recognizing the line formulas is: horizontal, vertical, line with positive slope and line with negative slope.

o   Some matching assignments were less straightforward than others because they require a few algebraic manipulations in order to be compared to standard forms. Consequently, the performance in such cases was less successful. Particularly, recognition of the line with negative slope and the semicircle presented difficulty for many students.

o   The parabola with positive leading coefficient is a preferred example over the parabola with negative leading coefficient for both formulas and names. This is in accordance with the way parabola was introduced in high school. We conclude that the prototype is the parabola with positive leading coefficient.

o   Although hyperbola does not belong to the Mathematics 3204 or Mathematics 3205 curriculum, we found out that a significant percentage of students 45%, respectively 48% know the name of the hyperbola in quadrants 1 and 3, and that 69%, respectively 77% know the name of the hyperbola in quadrants 2 and 4.

o   Mastering the formula for ellipse shows less successful performance than mastering the formula for the circle.

o   The absolute value function proved to have relatively good results in terms of terminology, matching formula and graphs.
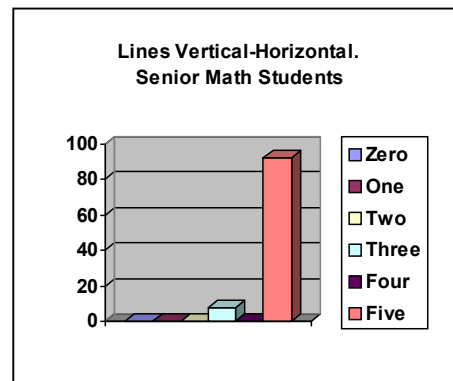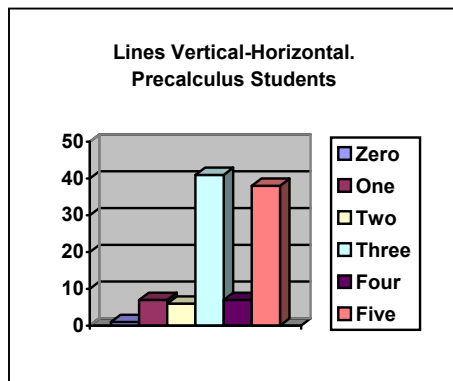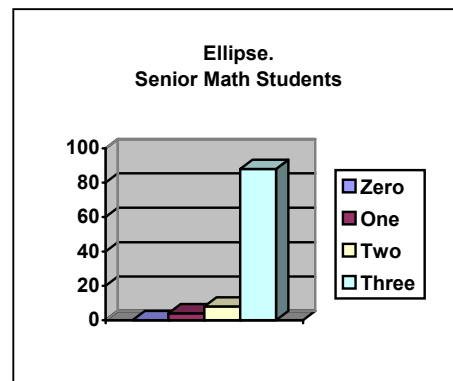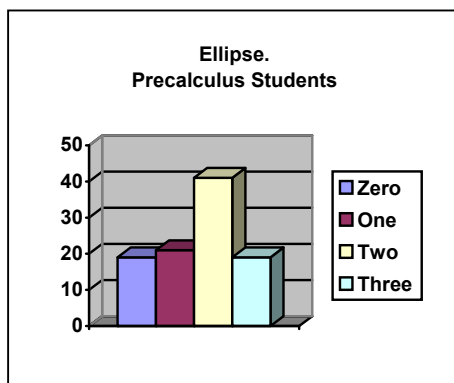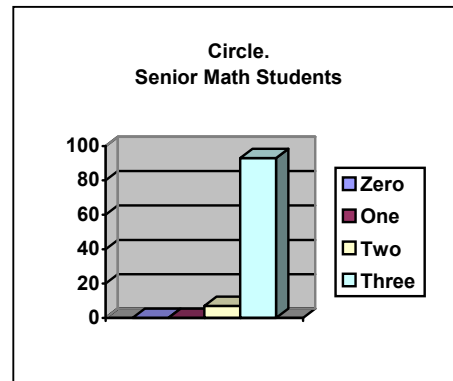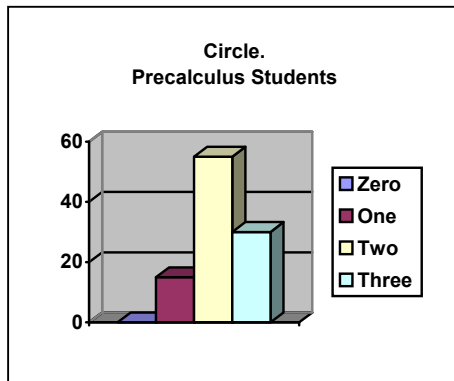
In order to characterize the level of students' knowledge about each particular mathematical object we use a graphical bar-diagram representation of the results collected. For this purpose we used the following marking schema: if both equation and name were written correctly under a graph on page 2, the student was given 2 points; if only name or only formula were identified correctly, the student was given 1 point; zero points were given for either incorrect or no answer; an additional point was given for a correct image of the same object drawn on the first page. This way for each of circle, ellipse and hyperbola a student could collect at most three points (two on the second page and one on the first), and for parabola – at most five points (four on the second page and one on the first). We separated lines in two subcategories: vertical or horizontal, and lines
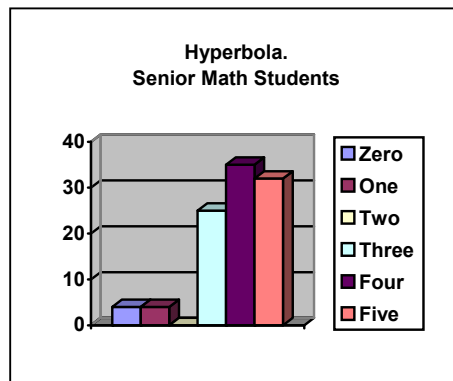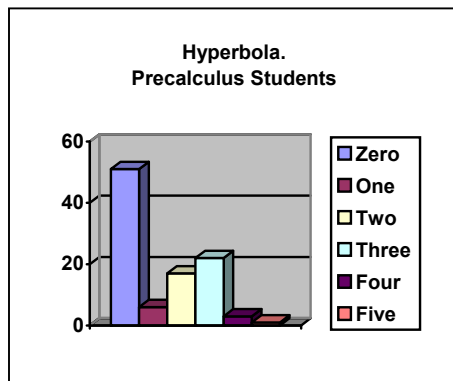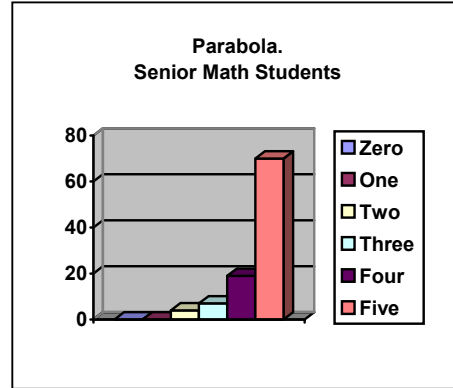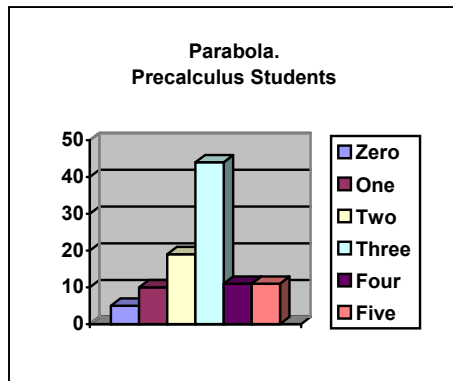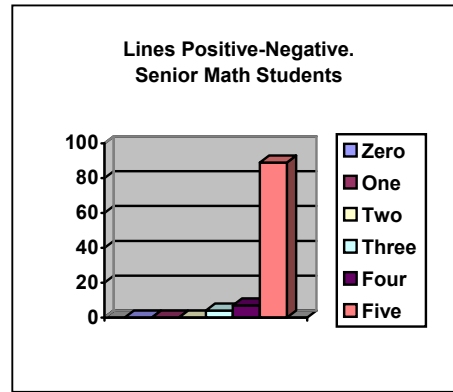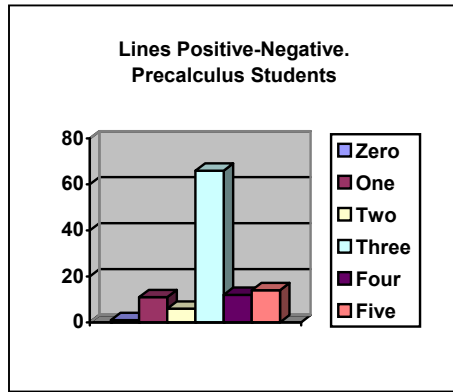
with positive or negative slope. In this way at most five points were collected for each subcategory of lines (four on the second page and one point on the first page; any image of line drawn on the first page contributed one point into each subcategory of lines).

For each object we created a bar-diagram which shows the percentage of the total number of students who collected zero points, one point, two points, or three points (extended to four points and five points in case of parabola and the two subcategories of lines). Obviously, there are two extreme profiles with 100% of a sample at zero points, and 100% of a sample at the maximum possible points, which correspond to complete non-familiarity and perfect performance, respectively. In reality, the profile of the bar-diagram is somewhat in-between the extreme shapes, but closeness to one extreme or another characterizes the degree of success in performance with respect to a particular object (curve). The profile also shows the degree of homogeneity of a particular group of students in terms of their familiarity with a particular object of study. For example, it turned out that the sample in our study was more homogeneous in performance with circle, and lines with positive or negative slope, compared to their performance with the ellipse, the horizontal or the vertical lines.

For a comparison purpose, we give bar-diagrams created for the senior math group described above. We observe that, while for this latter group of students with stronger mathematical background the bar-diagrams are closer to the perfect shape, the profiles for different notions (curves) still show a difference. They signal a possibility of improvement in performance with the same notions (curves) that present a challenge for the group of freshmen. Thus, despite the performance of students, taking calculus courses, improves the statistical difference between the levels of knowledge in each category remains.

**Circle.
Precalculus Students**

**Circle.
Senior Math Students**

**Ellipse.
Precalculus Students**

**Ellipse.
Senior Math Students**

**Lines Vertical-Horizontal.
Precalculus Students**

**Lines Vertical-Horizontal.
Senior Math Students**

**Lines Positive-Negative.
Precalculus Students**

**Lines Positive-Negative.
Senior Math Students**

**Parabola.
Precalculus Students**

**Parabola.
Senior Math Students**

**Hyperbola.
Precalculus Students**

**Hyperbola.
Senior Math Students**

The goal of our discussion was not to provoke a search for a reason or examine how good or poor the freshmen's performance is, but rather to attract the instructors' attention to the following observation. If, during a lecture for this group of students an equation $x + y = 2$ was given as a simple example, then 93% of the audience would not evoke an image of line with negative slope, although at least 84% of the group know what the line with negative slope is! Even if the line is drawn on the board, many mathematically inexperienced students will not make a connection between the equation and the graph unless it is explicitly explained. The explanation may only take minutes, but could make a big difference in the clarity of the example. Systematicity in such explanations leads to students' development of the ability of making necessary connections themselves.

5. Demands of the undergraduate mathematics curriculum: calculus.

Calculus is a major and important component of the introductory undergraduate university level mathematics. More senior courses such as real, complex and functional analysis, differential geometry, integral and partial differential equations, and many applications in physics, biology, economics and business build up their content on the solid ground of differential, integral and vector calculus. In the calculus sequence, the courses focus on general notions such as limit, as well as on the differentiation and integration techniques for finding such quantities as rates of change, areas, volumes etc. Students often find themselves being able to follow the explanations of general ideas but experience difficulties when the ideas are applied to concrete examples. This is indeed a paradoxical situation: the examples which ought to be illustrative are instead confusing. One of the major reasons is a non-flexibility of students' knowledge concerning some basic mathematical examples, e.g. fundamental curves such as parabola, ellipse and hyperbola, but often times even lines and circles, and their algebraic equations.

Criticizing Bloom's taxonomy of educational objectives, where Knowledge and Comprehension are regarded as two distinct levels, Pring (1971) remarks that "it does not make sense to talk about knowledge of terms or symbols in isolation from the working knowledge of this terms and symbols, that is, from the comprehension of them and thus the ability to apply them". The familiarity with terminology, without working knowledge and comprehension, is certainly not the final pedagogical objective. But in the reality of the learning processes this is a clearly observable stage of cognitive development, when some images start to be attached to the terms (words), but they are so fragile and rough, they are so "not a precise idea such as reasoning can take hold of" (Poincare, 1996).

Ironically, many students taking calculus courses have this precise kind of knowledge of the basic algebraic curves. This is a deceiving situation for students themselves as well as for their instructors relying on students' ability to comprehend while they often have just an illusion of knowing.

For instance, when it comes to visualizing 3D surfaces, such as an elliptic or hyperbolic paraboloid given by an algebraic formula, the students know that the task can

be approached by the slicing method, i.e. by identifying the curves occurring as the vertical and horizontal slices of the surface and then mental gluing the curves together. Note that the first task is to recognize the curves algebraically and then imagine their graphs, including the shifting and stretching aspects. If the students are not flexible in doing this part, the rest of the exercise is meaningless for them regardless of how extensive was the explanation. This is where the notion of the family of parabolas, ellipses or hyperbolas becomes essential, and the whole idea of correspondence between the algebraic and geometric transformations. Specifically, let the students analyze the equation $z = a(x - b)^2 + c(y - d)^2$, where a, b, c, d are the parameters of the surface in the (x, y, z)-coordinate space. Students are instructed to fix the value of $y = s$ in order to get a vertical slice of the surface in a plane parallel to the (x, z) coordinate plane. While keeping in mind that for different values of s there will be a different curve, they ought to see algebraically that the curve is always a parabola $z = a(x - b)^2$ shifted at a different height $c(s - d)^2$.

Similarly, the students shall identify the other family of vertical slides, $x = t$, as being a family of parabolas $z = c(y - d)^2$ shifted vertically by $a(t - b)^2$. The horizontal slides of the surface appear to be either a family of ellipses (case $ac > 0$) or a family of hyperbolas (case $ac < 0$), which gives either an elliptic or a hyperbolic paraboloid.

A special remark concerns two different forms of equation of a hyperbola. For example, a hyperbola in the form $u^2 - v^2 = k$ (where $k \neq 0$) never appears in the senior high school books. Therefore, a special effort is required to make a connection with the standard form $y = \dfrac{1}{x}$, using a $45°$ rotation of the coordinate system (u, v) such that $x = \dfrac{u + v}{\sqrt{2}}$ and $y = \dfrac{u - v}{\sqrt{2}}$. Then we have $1 = xy = \dfrac{(u + v)(u - v)}{2} = \dfrac{u^2 - v^2}{2}$.

The task of visualization in 3D space is by itself a difficult one, especially if the solid has a composite description that is typically bounded by several standard surfaces of the second order: cone, sphere, paraboloid etc. When students are instructed how to find a volume of a solid by evaluating a multiple integral, the most difficult part for them is to set up the limits of integration based on the algebraic description of the surface. Oftentimes, the problem is that they cannot visualize the boundaries of the solid and translate this image into the proper algebraic inequalities. Once again, the root of such difficulty lies in non-flexibility of their knowledge of elementary curves and surfaces.

An instructor who systematically fosters and reinforces the connection between algebraic and geometric manipulations, using elementary but fundamental mathematical examples, will see a remarkable difference in the students' performance at all complexity levels encountered in calculus problems.

Appendix. The questionnaire.

1. Draw what comes to mind first when you read the following words.

(a)   *line*   (b)   *circle*
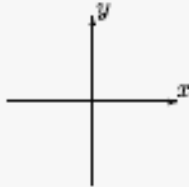
(c)   *semicircle*   (d)   *ellipse*
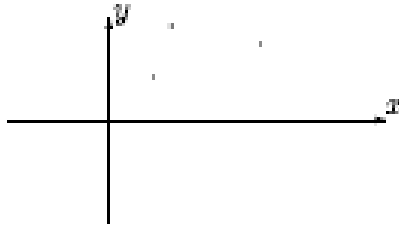
(e)   *parabola*   (f)   *hyperbola*

2. Draw an arbitrary graph of a function which passes through three points. How many different graphs can be drawn?

3. Pick the name which you think corresponds to each graph from the following list. Pick a formula which you think corresponds to each graph (there might be more than one correct answer).
Please write the name and the corresponding formula below each image.

Names: horizontal line;   line with positive slope;   parabola;   ellipse;   line with negative slope;   circle;   vertical line;   absolute value;   hyperbola; semicircle.

Formulae: $(x-2)^2 + (y-2)^2 = 1$; $y = -2$; $y = |x|$; $y = x$; $x = -2$; $y = \sqrt{x^2}$; $x + y = 2$; $xy = 1$; $y = x^2 - 4$; $y = \sqrt{1 - x^2}$; $y = \dfrac{1}{x}$; $2 + x = 0$; $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$; $y = 1 - x^2$; $xy = -1$.
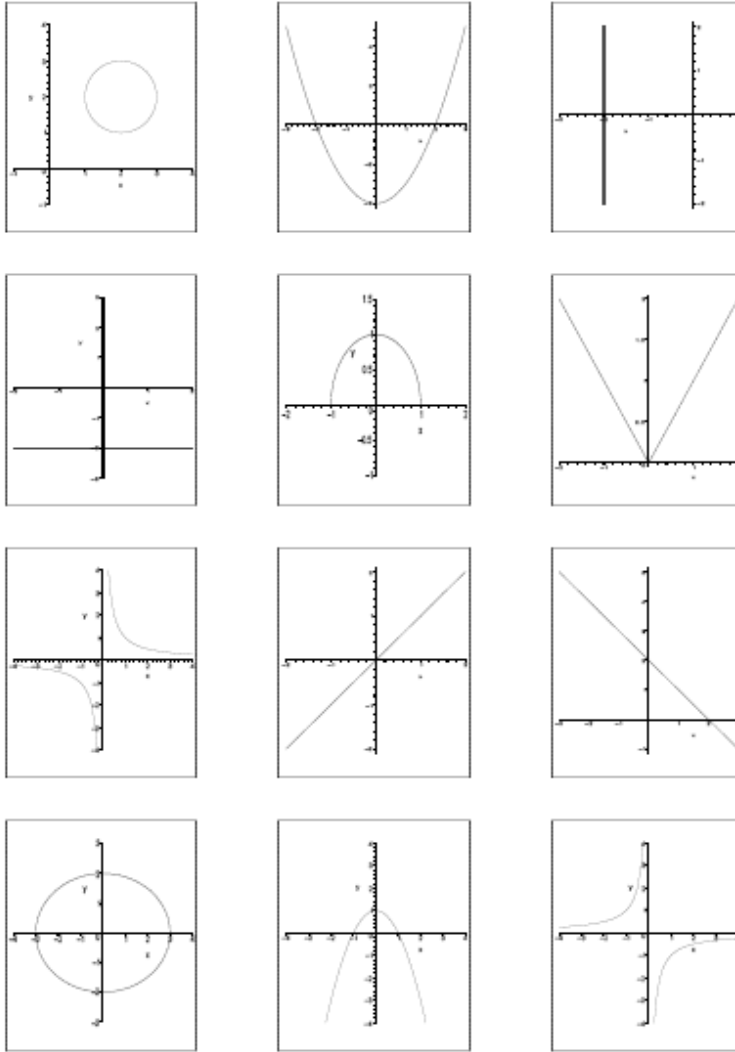
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