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THE MONTANA MATH ENTHUSIAST

VOLUME 1, NO. 2 (October 2004)

Editorial

The Montana Math Enthusiast (TMME) has been well received by MCTM members as well as the mathematics education community. The journal website has received over 3700 hits since the release of Vol.1, no.1 in April 2004. Numerous e-mails from pre-service and practicing teachers, as well as university educators commended the first issue. MCTM hopes that the e-journal is well on its way to becoming a familiar and useful resource in the field of mathematics education. Several changes are evident in this issue such as the official ISSN number for the journal as well as the inclusion of a Book Reviews section.

The theme and spirit of this issue is quite different from the first issue. The three feature articles in this issue tackle historical and contemporary geometry content and applications. In the first article Jonathan Comes generalizes the proof for the number of possible regular polyhedra to n-dimensional regular polytopes. In the second article Grant Swicegood constructs a simple and elegant proof to the Morley Trisector Theorem using basic notions from high school geometry. The third article gives the history and mathematics of Voronoi diagrams and illustrates the wide-ranging applicability of Voronoi diagrams to solve real world partitioning problems. Michael Mumm utilizes data from the Montana Natural Resource Information Systems GIS web-service to construct a Voronoi of northwestern Montana. It is hoped that this issue will particularly appeal to high school teachers who wish to enrich the geometry curriculum with the inclusion of modules that extend the scope of the curricula. University educators and mathematicians will also enjoy the rich mathematics contained in this issue. A special thank you to the geometers who reviewed the manuscripts published in this issue.

The Book Reviews section includes two reviews. Erica Lane contributes a long thoughtful and futuristic looking review of what is now considered a mathematics education classic, namely *The Nature of Proof*. I have also included a second, shorter review of *Mathematical and Analogical Reasoning of Young Learners*, the latest addition to the Studies in Mathematical Thinking and Learning series published by LEA. We hope you enjoy this issue.

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Aims and Scope

The Montana Math Enthusiast is the e-journal of the Montana Council of Teachers of Mathematics (MCTM). This biannual e-journal provides its readers with a lively blend of mathematics content, education theory and practice. The journal primarily addresses mathematics content in addition to the role of teaching and learning at all levels. Thematic issues will focus on mathematics content and innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal also strives to introduce research based as well as historical and cross-cultural perspectives to mathematics content, its teaching and learning. Submissions, Book Reviews, and offers for reviewing manuscripts are welcome. Authors are advised to send an electronic version of the manuscript in APA style to sriramanb@mso.umt.edu

Regular Polytopes

Jonathan Comes

In the last proposition of the *Elements* Euclid proved that there are only five regular polyhedra, namely the tetrahedron, octahedron, icosahedron, cube, and dodecahedron. To show there can be no more than five he used the fact that in a polyhedra, the sum of the interior angles of the faces which meet at each vertex must be less than 360. For if these angles sum to 360 the faces would tile in two dimensions. Since the interior angles of a p-sided polygon are $180 - 360/p$, the only possible polyhedra have the property that $q(180 - 360/p) < 360$ where $q > 2$ is the number of faces which meet at each vertex. With this in mind the only possible regular convex polyhedra are given in table 1.

p	q	$q(180 - 360/p)$	name
3	3	180	tetrahedron
3	4	240	octahedron
3	5	300	icosahedron
4	3	270	cube
5	3	324	dodecahedron

Table 1. Possible regular polyhedra

In order to generalize the idea of this proof to n-dimensional polytopes we need some new terminology. We define the *n-dimensional angle* between two $(n-1)$-dimensional figures X and Y with $(n-2)$-dimensional intersection S, as the angle between the line segments x and y, where $x \in X$, $y \in Y$ and x and y are perpendicular to S with the nonempty intersection of x and y in S. For example the 3-dimensional angle (also referred to as the platonic angle) between any two intersecting faces of an icosahedron as seen in figure 1 is

$$2\arcsin\left(\frac{\sqrt{5}+1}{2\sqrt{3}}\right) \approx 138.2.$$

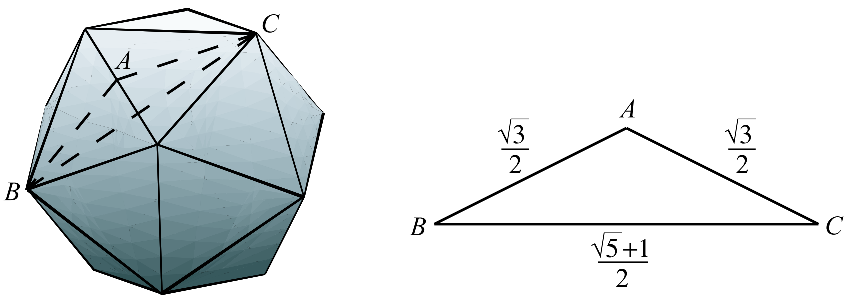

Figure 1. Properties of the icosahedron

Because of the symmetry of regular polytopes we can define the *interior n-angle* of a regular n-dimensional polytope to be the n-dimensional angle between any two intersecting $(n-1)$-dimensional figures in the polytope. Now just as before with 3-dimensional polyhedra, we know that in an n-dimensional polytope the sum of the interior $(n-1)$-angles of the $(n-1)$-dimensional polytopes meeting at an $(n-2)$-polytope must be less than 360. For if the sum of these interior $(n-1)$-angles was 360 then the $(n-1)$-dimensional polytopes would tile in $n-1$ dimensions. An example of this sort of tiling is three hexagons intersecting at one point in two dimensions. Another is four cubes intersecting at on edge as shown in figure 2. This occurs because the interior 3-angle of a cube is easily seen to be 90.

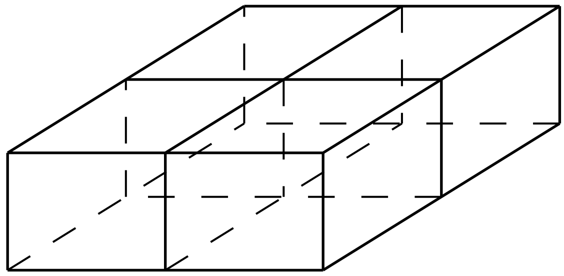

Figure 2. cubes tiling in three dimensions

Now we can find a short list of the possible four dimensional polytopes by simply finding the interior 3-angles of all the three dimensional polyhedra. We already know the interior 3-angles for the icosahedron and the cube. For the dodecahedron, using figure 3, we see the interior 3-angle is

$$2\arcsin(\frac{\sqrt{5}+1}{4\sin 72}) \approx 116.6.$$

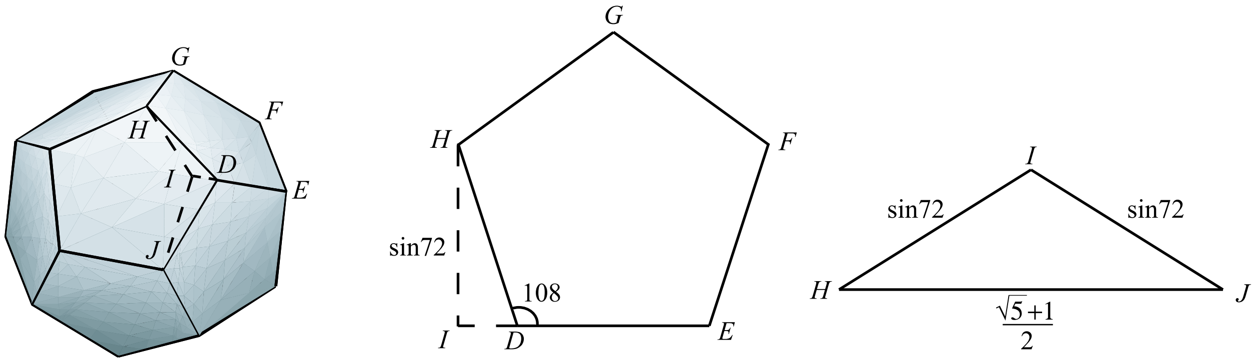

Figure 3. Properties of the dodecahedron

The interior 3-angles for the tetrahedron and octahedron will be calculated later, but it is not hard to show that they are $2\arcsin(\sqrt{3}/3) \approx 70.5$ and $2\arcsin(\sqrt{6}/3) \approx 109.5$ respectively. Instead of naming the four dimensional polytopes, we will use what is known as the "Shläfli symbol." The "Shläfli symbol" can be used to denote regular polytopes of any dimension as follows. The symbol $\{p\}$ denotes the p-sided polygon. The symbol $\{p,q\}$ denotes the polyhedron whose faces are $\{p\}$, and there are q faces meeting at each vertex. Similarly the "Shläfli symbol" $\{a_1, a_2, ..., a_{n-1}\}$ represents an n-dimensional regular polytope made up of $(n-1)$-dimensional regular polytopes $\{a_1, a_2, ..., a_{n-2}\}$ of which there are a_{n-1} meeting at each $(n-2)$-dimensional regular polytope $\{a_1, a_2, ..., a_{n-3}\}$. For example the four dimensional regular polytope known as the hypercube is made of cubes, and three cubes meet at each edge. Therefore it is represented by $\{4,3,3\}$. Now if we let $\phi_{\{p,q\}}$ denote the interior 3-angle of the polyhedron $\{p,q\}$, the only possible regular 4-dimensional polytopes have the property that $r\phi_{\{p,q\}} < 360$, where $r > 2$ is the number of polyhedra $\{p,q\}$ meeting at each edge. With this in mind table 2 lists all possible 4-dimensional regular polytopes.

p	q	r	$\phi_{\{p,q\}}$	$r\phi_{\{p,q\}}$	"Shläfli symbol"
3	3	3	70.5	211.5	$\{3,3,3\}$
3	3	4	70.5	282	$\{3,3,4\}$
3	3	5	70.5	352.5	$\{3,3,5\}$
3	4	3	109.5	328.5	$\{3,4,3\}$
4	3	3	90	270	$\{4,3,3\}$
5	5	3	116.6	349.8	$\{5,3,3\}$

Table 2. Possible 4-dimensional regular polytopes

In order to find all the possible 5-dimensional regular polytopes we must calculate the interior 4-angles of all the 4-dimensional regular polytopes. As before we let $\phi_{\{p,q,r\}}$ denote the interior 4-angle of the polytope $\{p,q,r\}$. Then it is easy to see that $\phi_{\{4,3,3\}} = 90$. Also it will be shown later that $\phi_{\{3,3,3\}} \approx 75.5$ and $\phi_{\{3,3,4\}} = 120$. To calculate $\phi_{\{5,3,3\}}$ we need to find the "angle" that two dodecahedrons meet at a pentagonal face. To do this we first let P be a pentagonal face of $\{5,3,3\}$. Also let O, N, and M be vertices of $\{5,3,3\}$ such that O is on P, N and M are not on P, but NO and MO are edges of $\{5,3,3\}$ as in figure 4a. If we let a be the perpendicular distance from M (and therefore N) to P, then

$$a = \sin(72)\sin(180 - \phi_{\{5,3\}})$$

as can be seen in figure 4b. But the distance from M to N is $(\sqrt{5}+1)/2$ since M and N are vertices of a pentagon. Therefore, as shown in figure 4c, we have

$$\phi_{\{5,3,3\}} = 2\arcsin(\frac{\sqrt{5}+1}{4a}) = 144.$$

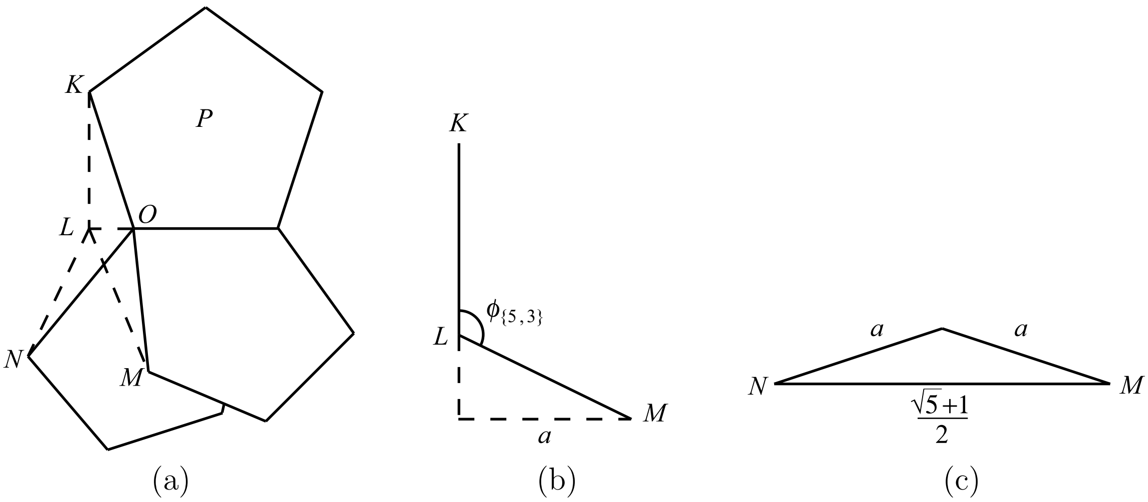

Figure 4. Properties of $\{5,3,3\}$

To calculate $\phi_{\{3,3,5\}}$ we need to find the "angle" that two tetrahedrons meet at a triangular face. Let T be a triangular face in $\{3,3,5\}$. And let R and S be vertices of $\{3,3,5\}$ which are not in T, but are connected by an edge to every vertex of T as in figure 5a. The distance between R and S is $(\sqrt{5}+1)/2$ since R and S are vertices of a pentagon, and if we let b be the perpendicular distance between R (and therefore S) and T, then

$$b = \frac{\sqrt{3}}{2}\sin(\phi_{\{3,3\}})$$

as can be seen in figure 5b. Therefore, as shown in figure 5c,

$$\phi_{\{3,3,5\}} = 2\arcsin(\frac{\sqrt{5}+1}{4b}) \approx 164.5.$$

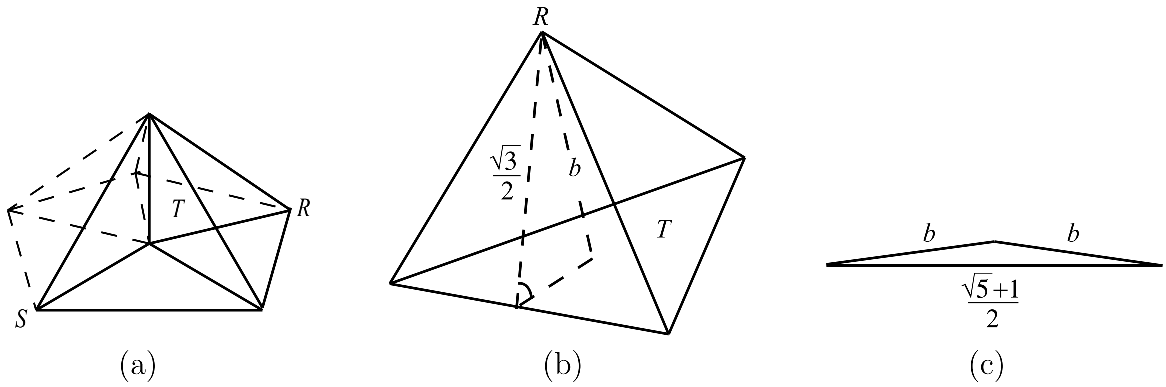

Figure 5. Properties of $\{3,3,5\}$

To calculate $\phi_{\{3,4,3\}}$ we need to find the "angle" that two octahedrons meet at a triangular face. So let U be a triangular face in $\{3,4,3\}$. And let V and W be vertices of $\{3,4,3\}$ which are not contained in U, but are connected by edges to the same two vertices of U as in figure 6a. The distance between U and V is $\sqrt{2}$ since U and V are vertices of a square, and if we let c be the perpendicular distance from V (and therefore W) to T, then

$$c = \frac{\sqrt{3}\sin(180 - \phi_{\{3,4\}})}{2}$$

as can be seen in figures 6b and 6c. Therefore, as seen in figure 6d,

$$\phi_{\{3,4,3\}} = 2\arcsin(\frac{\sqrt{2}}{2c}) = 120.$$

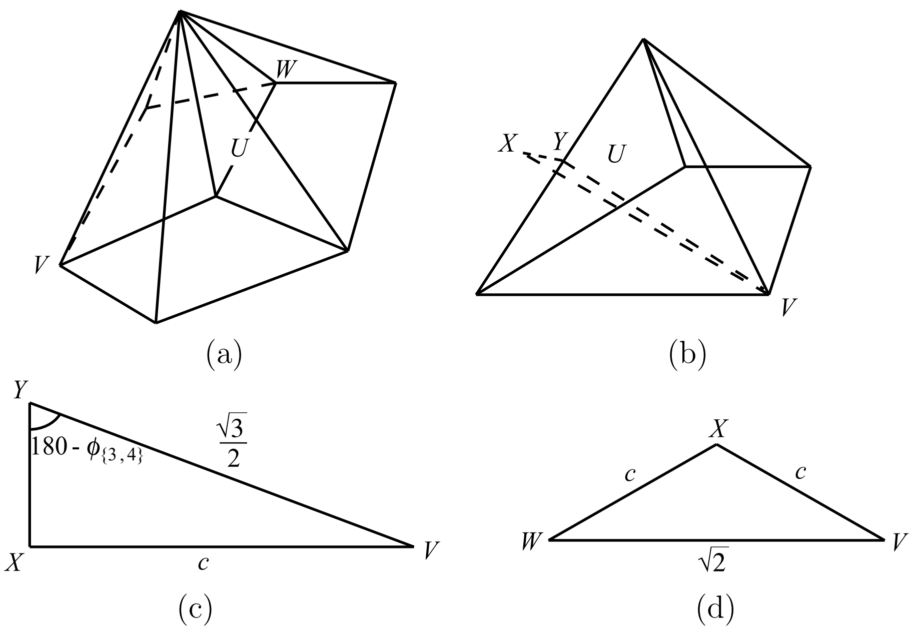

Figure 6. Properties of $\{3,4,3\}$

So now we have all the 4-angles of the regular 4-dimensional polytopes. It is interesting to notice that since $\phi_{\{3,3,4\}} = \phi_{\{3,4,3\}} = 120$, the polytopes $\{3,3,4\}$ and $\{3,4,3\}$ will tile in four dimensions. Now if we let $s > 2$ denote the number of 4-dimensional polytopes of the form $\{p,q,r\}$ meeting at each 3-dimensional polytope $\{p,q\}$, we know that the only possible 5-dimensional regular polytopes have the property that $s\phi_{\{p,q,r\}} < 360$. With this in mind table 3 lists all the possible 5-dimensional polytopes.

p	q	r	s	$\phi_{\{p,q,r\}}$	$s\phi_{\{p,q,r\}}$	"Shläfli symbol"
3	3	3	3	75.5	226.5	$\{3,3,3,3\}$
3	3	3	4	75.5	302	$\{3,3,3,4\}$
4	3	3	3	90	270	$\{4,3,3,3\}$

Table 3. Possible 5-dimensional regular polytopes

Now we will show that for all $n > 4$ there can be no more than three regular polytopes. These polytopes are of the form $\{3,3,\ldots,3\},\{3,3,\ldots,3,4\}$, and $\{4,3,3,\ldots,3\}$. The n-dimensional polytope of the form $\{4,3,3,\ldots,3\}$ is the n-dimensional cube or n-cube, and will always have interior n-angle of 90. Because of this we can never fit four n-cubes about an $(n-1)$-cube without tiling in n-dimensions. Therefore it is not possible for an $(n+1)$-dimensional polytope of the form $\{4,3,3,\ldots,3,4\}$ to exist. The n-dimensional polytope of the form $\{3,3,\ldots,3\}$ is the n-dimensional simplex or n-simplex. Let ϕ_n denote the interior n-angle of of the n-simplex. To find ϕ_n we first look at one of the properties of a simplex. Given an n-simplex we can create an $(n+1)$-simplex by placing a new vertex in our new dimension such that it is at distance one from all the vertices of our n-simplex as shown in figure 7.

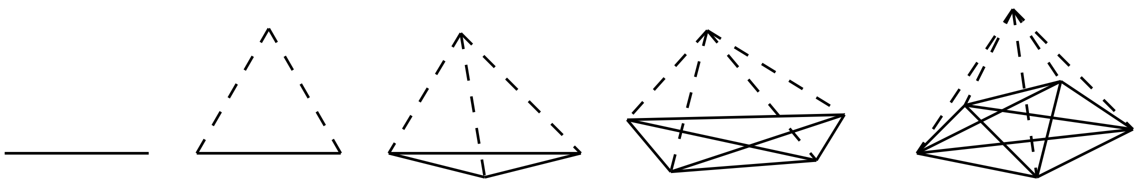

Figure 7. The n-simplex for $1 \le n \le 5$

Now if we let A and B be vertices of an n-simplex. And let S denote the $(n-2)$-simplex which is contained in the n-simplex but does not contain A or B. We know that the distance from A to B is 1. If we let t denote the perpendicular distance from A (and therefore B) to S then we know that

$$\phi_n = 2\arcsin(\frac{1}{2t}) \qquad (\star)$$

as can be seen in figure 8.

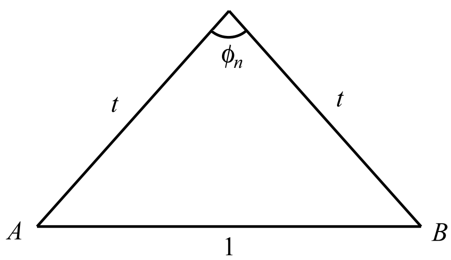

Figure 8. ϕ_n

Since A (and therefore B) is equidistant from all vertices in S, the line through A (and therefore B) perpendicular to S contains the center of S which we denote O_{n-2}. But this line contains the points which are equidistant to all vertices of S it will also contain the center of the $(n-1)$-simplex which we denote O_{n-1}. If we let C be any vertex in S, then figure 9 depicts the relationship described above.

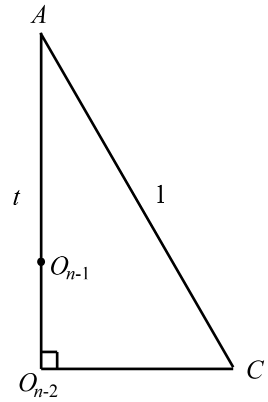

Figure 9.

Now if we let x_n denote the distance from any vertex in an n-simplex to the center of that n-simplex, and let y_n denote the perpendicular distance from the center of an n-simplex to any $(n-1)$-simplex contained in that n-simplex, then we know that $t = x_{n-1} + y_{n-1}$. Also from figure 9 we obtain figure 10 which shows the relationship between x_{n-1}, y_{n-1}, and x_{n-2}.

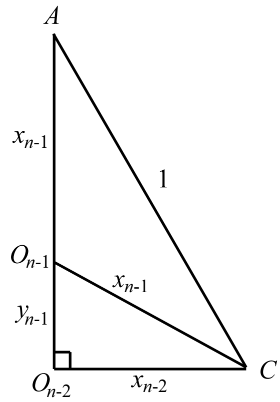

Figure 10.

From $\triangle O_{n-1}O_{n-2}C$ and $\triangle AO_{n-2}C$ we have the following equations:

$$x_{n-2}^2 + y_{n-1}^2 = x_{n-1}^2,$$

$$x_{n-2}^2 + (x_{n-1} + y_{n-1})^2 = 1.$$

Substituting the first equation into the second gives us

$$2x_{n-2}^2 + 2(x_{n-1} + y_{n-1}) = 1$$

$$\Rightarrow x_{n-1} = \frac{1}{2(x_{n-1} + y_{n-1})}.$$

Since $t = x_{n-1} + y_{n-1}$, we can now rewrite $(\star)$ as

$$\phi_n = 2 \arcsin(x_{n-1}). \qquad (1)$$

So we have reduced the problem of finding the interior n-angle of the n-simplex to finding x_{n-1}. We will find x_{n-1} recursively in the following way. First we let D be the midpoint of the edge with endpoints A and C. And let s be the distance from D to O_{n-1}. Now since $\triangle AO_{n-1}C$ is isosceles, we have the similar triangles $\triangle AO_{n-1}D$ and $\triangle ACO_{n-2}$ as can be seen in figure 11.

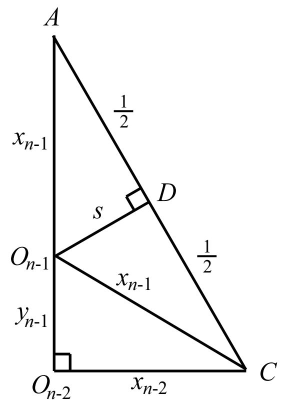

Figure 11.

These similar triangle give us $s = x_{n-1}x_{n-2}$. So $\triangle ADO_{n-1}$ gives us the following relation between x_{n-1} and x_{n-2}:

$$x_{n-1}^2 = (x_{n-1}x_{n-2})^2 + \frac{1}{4}. \qquad (\star\star)$$

To find a recursive formula for x_{n-1} in terms of x_{n-2} we look at $\triangle AO_{n-2}C$ to see that the angle at vertex A is $\arcsin(x_{n-2})$. And $\triangle ADO_{n-1}$ gives us

$$\cos(\arcsin(x_{n-2})) = \frac{1}{2x_{n-1}} \Rightarrow$$

$$x_{n-1} = \frac{1}{2\cos(\arcsin(x_{n-2}))}. \qquad (2)$$

From $(\star\star)$ we can find the limit of x_n as n approaches infinity as follows.

$$x^2 = x^4 + \frac{1}{4} \Rightarrow x = \frac{\sqrt{2}}{2} \Rightarrow \lim_{n\to\infty} x_n = \frac{\sqrt{2}}{2}.$$

And since we know $\phi_n = 2\arcsin(x_{n-1})$ we have

$$\lim_{n\to\infty} \phi_n = 90.$$

But x_n (and therefore ϕ_n) is a strictly increasing sequence, so $\phi_n < 90$ for every n. Thus we can always fit three or four n-simplexes about an $(n-2)$-simplex without tiling in n dimensions. Therefore it is still possible for an $(n+1)$-simplex and an $(n+1)$-dimensional polytope of the form $\{3, 3, \ldots, 3, 4\}$ to exist. But since ϕ_n is increasing, it is not possible for an n-dimensional polytope of the form $\{3, 3, \ldots, 3, 5\}$ to exist when $n > 4$. Also equations (1) and (2) give us a way to compute the interior n-angle for the n-simplex. Because the one dimensional simplex is a line segment, we know $x_1 = 1/2$. From this we can recursively find ϕ_n for any n. Table 4 lists the interior n-angle of the n-simplex for $2 \leq n \leq 8$.

n	x_{n-1}	ϕ_n	"Shläfli symbol"
2	$1/2$	60	$\{3\}$
3	$\sqrt{3}/3$	70.5	$\{3,3\}$
4	$\sqrt{6}/4$	75.5	$\{3,3,3\}$
5	$\sqrt{10}/5$	78.5	$\{3,3,3,3\}$
6	$\sqrt{15}/6$	80.4	$\{3,3,3,3,3\}$
7	$\sqrt{21}/7$	81.8	$\{3,3,3,3,3,3\}$
8	$\sqrt{7}/4$	82.8	$\{3,3,3,3,3,3,3\}$

Table 4. interior n-angles for the n-simplex

If we let ϕ'_n denote the interior n-angle for the n-dimensional polytope $\{3, 3, \ldots, 3, 4\}$, then one can similarly show

$$\phi'_n = 2\arcsin(\sqrt{2}x_{n-1}).$$

Using this equation along with (2) we can find ϕ'_n for all n. Table 5 lists ϕ'_n for $2 \leq n \leq 5$.

n	x_{n-1}	ϕ'_n	"Shläfli symbol"
2	$1/2$	90	$\{4\}$
3	$\sqrt{3}/3$	109.5	$\{3, 4\}$
4	$\sqrt{6}/4$	120	$\{3, 3, 4\}$
5	$\sqrt{10}/5$	126.9	$\{3, 3, 3, 4\}$

Table 5. interior n-angles for the n-dimensional polytope $\{3, 3, \ldots, 3, 4\}$

Since ϕ'_n is an increasing sequence, $\phi'_n \geq 120$ for all $n > 3$. So the n-dimensional polytopes of the form $\{3, 3, \ldots, 3, 4, 3\}$ can not exist for $n > 4$. Thus there can be no more than three regular n-dimensional polytopes for $n > 4$.

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The Morley Trisector Theorem

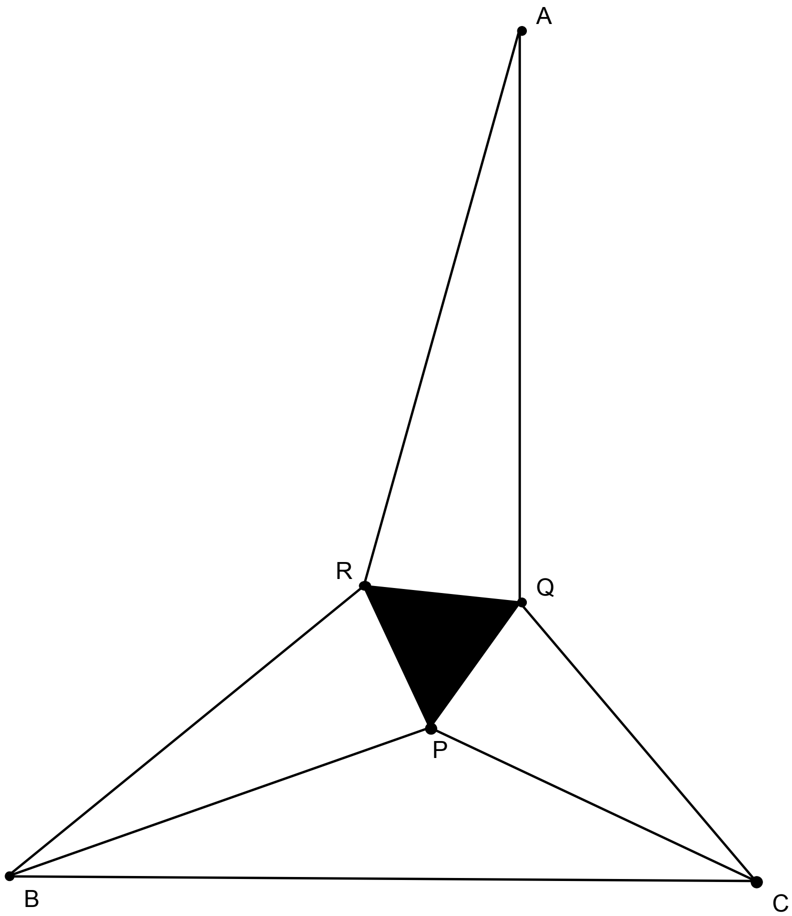
Grant Swicegood

This paper deals with an unannounced theorem by Frank Morley that he originally published amid a collection of other, more general, theorems. Having intrigued mathematicians for the past century, it is now simply referred to as Morley's trisector theorem:

> *The three intersections of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle.*

An example construction of this theorem can easily be made using Geometer's Sketchpad® (see Construction #1), but the purpose of this paper is to construct a formal proof of this theorem for any triangle. The importance of this problem is perhaps not so much in its applications to other areas of geometry, but in the ingenuity behind it. For thousands of years people had worked with compasses and straight edges in attempts to trisect the angle, but for some reason no one had ever noticed the equilateral triangle that forms when all of the trisectors of a triangle are constructed. It is also a great exercise in complex applications of simple geometry and truly reveals the power of just a few basic theorems. While some methods of proving this problem involve trigonometric techniques, the proof presented here is based solely on rudimentary high school geometry.

This problem first appeared in Morley's (1900) paper entitled "On the Metric Geometry of the Plane *n*-line" in the first issue of the *American Mathematical Society Translations* (Baker, 1978, p.737). However, Morley included it as a very specialized case of one of his more general theorems, so for quite a few years it was rather overlooked by much of the mathematical community. Morley's youngest son, Frank V. Morley believed his father's unwillingness to declare his result was due to a certain amount of uncertainty on Morley's part as

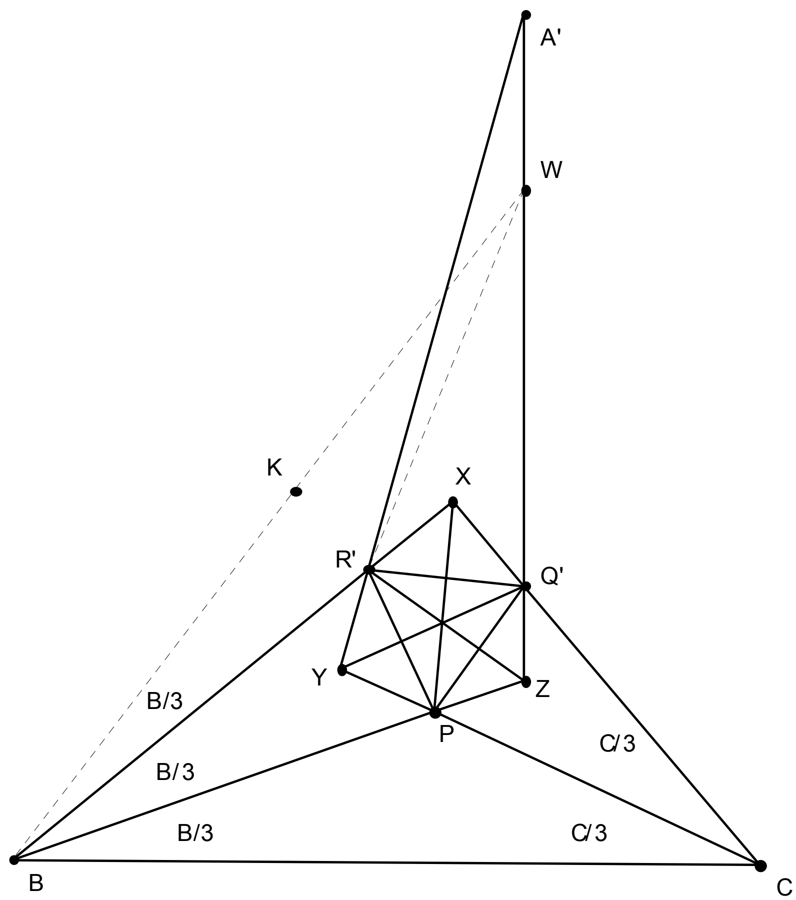


Construction #1 of Morley's Trisector Theorem

to his original discovery of the theorem. (740). Morley had trouble believing that such an eye-catching result could have been overlooked for the past 2,000 years and didn't want to be overly confident about his ingenuity until he was sure no one else had already observed it. For this reason, he used only "quiet, semi-private mentions" of the theorem to colleagues in the U.S.A, Britain, Europe, and the Far East in an effort to see if it had already been noticed by someone else before him (Baker, 1978, p.741). After no promising leads, he was eventually forced to put his name to the trisector theorem as its discoverer.

Despite his reluctance in advertising his "jewel," Morley did tell a few friends, "such as Richmond at Cambridge and Whittaker at Edinburgh," about his result "and by 1904 it had become public" (Baker, 1978, p.738). For the next decade the problem spread throughout Europe and intrigued mathematicians everywhere, but it was not until 1914 that a formal proof was offered up to the world (Kay, 2001, p.8). This first proof by Taylor and Marr would be followed by other more elegant proofs by Satyanarayana and Naraniengar, and by 1920 the problem had created enough interest to be set in St. John's group of Entrance Scholarships (738). To this day the problem is a challenging exercise for any mathematician, since there has been no overly simple proof of its claim. As with any good problem, a fair amount of creativity and work is needed to come up with the answer.

This proof of Morley's theorem is based on the figure in Construction #2, borrowed from David C. Kay (2001, p.223). It is based on ΔABC, which we assume is an existent triangle with points X', Q', and R' constructed as shown, where $m\angle XPQ' = m\angle XPR' = 30$ and $\Delta PQ'R'$ is equilateral. Note that P is the in-center of ΔBCX and $\overrightarrow{XP}$ bisects $\angle BXC$, and let Y and X be the intersections of the altitudes of $\Delta PQ'R'$ from Q' and R' with rays $\overrightarrow{CP}$ and $\overrightarrow{BP}$. From the angle measures to be obtained later in this proof, it can easily be seen that $m\angle YR'Q' + m\angle ZQ'R' > 180$, so rays $\overrightarrow{YR'}$ and $\overrightarrow{ZQ'}$ meet at some point A'.



Construction #2

First, observe that since $\overline{R'Z}$ is the perpendicular bisector of $\overline{PQ'}$, then $m\angle PQ'Z = m\angle Q'PZ = 180 - m\angle Q'PB = 180 - (m\angle XPB + 30)$, since $m\angle Q'PX$ = 30. Now, look at ΔXPB. Since $\overrightarrow{XP}$ bisects $\angle BXC$, then

$$m\angle XPB = 180 - B/3 - \frac{1}{2}\left(180 - \frac{2}{3}B - \frac{2}{3}C\right)$$
$$m\angle XPB = 180 - B/3 - 90 + B/3 + C/3$$
$$m\angle XPB = 90 + C/3$$

So going back, $m\angle PQ'Z = 180 - (m\angle XPB + 30)$, now implies that

$$m\angle PQ'Z = 180 - (m\angle XPB + 30)$$
$$m\angle PQ'Z = 180 - (90 + C/3 + 30)$$
$$m\angle PQ'Z = 60 - C/3$$

and similarly, it can easily be seen that $m\angle PR'Y = 60 - B/3$.

We are now ready to establish the measure of angle A'. Using the concept of supplementary angles and the fact that $m\angle PQ'R'$ = 60, we can see that $m\angle A'Q'R' = 180 - (60 + m\angle PQ'Z)$, and using what we have just found,

$$m\angle A'Q'R' = 180 - (60 + m\angle PQ'Z)$$
$$m\angle A'Q'R' = 180 - 60 - 60 + C/3 \quad .$$
$$m\angle A'Q'R' = 60 + C/3$$

Similarly, we find that

$$m\angle A'R'Q' = 180 - (60 + m\angle PR'Y)$$
$$m\angle A'R'Q' = 180 - 60 - 60 + B/3$$
$$m\angle A'R'Q' = 60 + B/3$$

and using the sum of the interior angles of triangle $\Delta A'Q'R'$, we see

$$m\angle A' = 180 - m\angle PQ'Z - m\angle PR'Y$$
$$m\angle A' = 180 - (60 + C/3) - (60 + B/3) \quad .$$
$$m\angle A' = 60 - C/3 - B/3$$

Now from our assumption that ΔABC exists we know that

$$m\angle A = 180 - m\angle B - m\angle C$$
$$\frac{m\angle A}{3} = \frac{180}{3} - \frac{m\angle B}{3} - \frac{m\angle C}{3},$$

and we write $A/3 = 60 - B/3 - C/3$, which should look familiar. Thus $m\angle A' = A/3$.

Now let us consider $m\angle A'ZB$. Since $\overline{R'Z}$ is the perpendicular bisector of $\overline{PQ'}$, then

$$m\angle A'ZB = 2 \cdot m\angle R'ZQ'$$
$$m\angle A'ZB = 2 \cdot (90 - mPQ'Z) \quad .$$
$$m\angle A'ZB = 2 \cdot (90 - (60 - C/3)) = 60 + \frac{2C}{3}$$

Similarly,

$$m\angle A'YC = 2 \cdot m\angle R'YQ'$$
$$m\angle A'YC = 2 \cdot \left(90 - m\angle PR'Y\right)$$
$$m\angle A'YC = 2 \cdot \left(90 - \left(60 - B/3\right)\right) = 60 + \frac{2B}{3}$$

Now construct ray $\overrightarrow{BK}$ such that $\overrightarrow{BC} - \overrightarrow{BX} - \overrightarrow{BK}$ and $m\angle KBC = B$. Consider that $m\angle A'ZB = 60 + \frac{2C}{3}$ and $m\angle KBZ = \frac{2B}{3}$, so $m\angle A'ZB + m\angle KBZ = 60 + \frac{2C}{3} + \frac{2B}{3}$. So now there are two cases: either $\overrightarrow{BK}$ and $\overrightarrow{ZA'}$ are parallel or they intersect at some point. However, if $\overrightarrow{BK}$ and $\overrightarrow{ZA'}$ are parallel, then, $m\angle A'ZB + m\angle KBZ = 60 + \frac{2C}{3} + \frac{2B}{3} = 180$, so

$$60 + \frac{2C}{3} + \frac{2B}{3} = 180$$
$$C + B = 180$$

which gives a contradiction since we assume that ΔABC exists. Therefore $\overrightarrow{BK}$ must intersect $\overrightarrow{ZA'}$ at some point which we will label W.

Now consider the line $\overleftrightarrow{RZ}$ were we to extend the segment indefinitely. We can easily see that $\angle WBR'$ and $\angle R'BZ$ slice off the same sections of $\overleftrightarrow{RZ}$ as $\angle BWR'$ and $\angle R'WZ$, respectively. This implies that the ratio of $\frac{\angle WBR'}{\angle R'BZ}$ must be the same as the ratio of $\frac{\angle BWR'}{\angle R'WZ}$. However, we know that $\angle WBR' = \angle R'BZ = B/3$. Thus, $\angle BWR'$ and $\angle R'WZ$ must be equal as well. Since we know that their sum is equal to $2(A/3)$, then we can easily see that $\angle BWR' = \angle R'WZ = A/3$. This implies that $m\angle R'WZ' = m\angle R'A'Q' = A/3$ and therefore $W=A'$. Given this we must conclude that $m\angle A'BC = B$ as well.

Similarly, we can find $m\angle A'CB = C$, which when combined with $m\angle A'BC = B$ implies that $m\angle BA'C = A$. Thus $A' = A$, $Q' = Q$, $R' = R$, and therefore $\Delta PQR \equiv \Delta PQ'R'$, which by assumption is equilateral. So we see that for any triangle ΔABC we can use the angle trisectors to construct an equilateral triangle ΔPQR, and that Morley's Trisector Theorem is indeed true.

In conclusion, we have shown the validity of Morley's Theorem and that its proof need not involve any mathematical knowledge past that one might find in a high school geometry class. It is really just a matter of complexity and being able to look at the problem in new and creative ways. It would be interesting to look at the more general theorems under which Morley published this trisector theorem and to see what other generalizations of it he worked out. Another possible extension would be to see if an analogous construction might exist for other polygons with greater numbers of sides.

If nothing else has been learned from paper, we can at least taste the excitement and amazement Morley experienced whenever he first found this result. It seems preposterous that mathematicians could have missed such a simple and straightforward construction for so many years, but it is really quite reasonable when we consider the restraints Euclid imposed on the world of geometry. By not being able to construct angle trisectors with a compass and straightedge, people never really considered working with them too awfully much. Perhaps the genius in Morley's talent was not for the mathematics itself, but more in being able to see the world in new and different ways.

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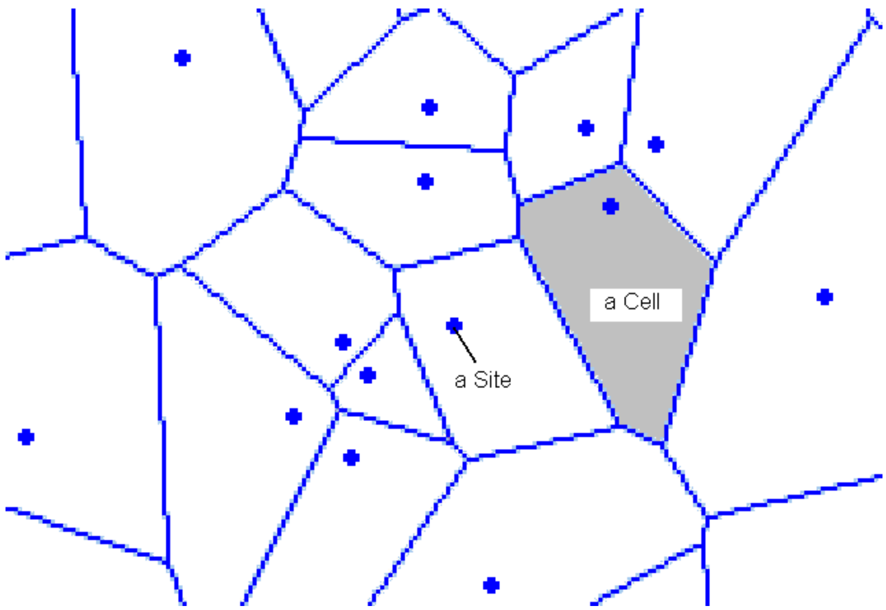
Voronoi Diagrams

Michael Mumm

Statement of the Problem:

Suppose we have a finite number of distinct points in the plane. We refer to these points as *sites*. We wish to partition the plane into disjoint regions called *cells*, each of which contains exactly one site, so that all other points within a cell are closer to that cell's site than to any other site.

An example of a Voronoi diagram:



Stated more formally, suppose P: = { $p_1, p_2, \ldots, p_n$ } is a set of distinct points (sites) in the plane. We subdivide the plane into n cells so that each cell contains exactly one site. An arbitrary point (x,y) is in a cell corresponding to a site p_i with coordinates (x_{pi} , y_{pi}) if and only if $\sqrt{(x - x_{pi})^2 + (y - y_{pi})^2} < \sqrt{(x - x_{pj})^2 + (y - y_{pj})^2}$ for all p_j with $j \neq i$, $1 \leq j, i \leq n$. That is, the Euclidean distance from (x,y) to any other site is greater than the distance from (x,y) to p_i.

It turns out that the boundaries of the cells defined in this way will be composed of straight lines and segments forming convex polygons and will be defined by the perpendicular bisectors of segments joining each pair of sites. This method of partitioning a plane is called a *Voronoi diagram*.

Although this paper deals chiefly with two dimensional diagrams and the Euclidean distance metric, it should be noted that the concept of Voronoi diagrams can be generalized to n dimensions and to an arbitrarily defined distance metric. In addition, general geometric primitives such as line segments or curves may be used as sites instead of ordinary one-dimensional points. In the course of our discussion we will describe techniques of constructing

Voronoi diagrams, provide some historical background on the subject, and discuss the multitude of applications that utilize Voronoi diagrams.

Rationale:

Voronoi diagrams have countless applications in nearly all of the major sciences. Whenever one has a discrete set of data distributed in such a way that the concept of 'distance' has some meaning, a Voronoi diagram may be useful. With a Voronoi diagram as a reference, it is unnecessary to calculate the distance to each site in order to determine which site is closest to a particular point. The site corresponding to the cell that contains the point will always be closest. In applications which have a fixed set of sites, a Voronoi diagram need only be constructed once and then all subsequent distance calculations become unnecessary. Even if more sites are eventually added to a system, the basic structure of the Voronoi diagram remains intact. It is relatively easy to modify an existing diagram to accommodate new sites without reconstructing the entire system. For large scientific projects which use computers, this reduction in basic operations may result in a dramatic increase of algorithmic efficiency.

Voronoi diagrams may be extremely useful in the business world as well. A typical example is a pizza delivery franchise which has a network of restaurants servicing a large city. When an order comes into the central office, the operator, or computer, can use a Voronoi diagram to determine which restaurant will be able to deliver the pizza quickest relative to the location of the caller. It's easy to imagine similar applications arising in a large mail-order company with many distribution warehouses like amazon.com, or in the postal service itself.

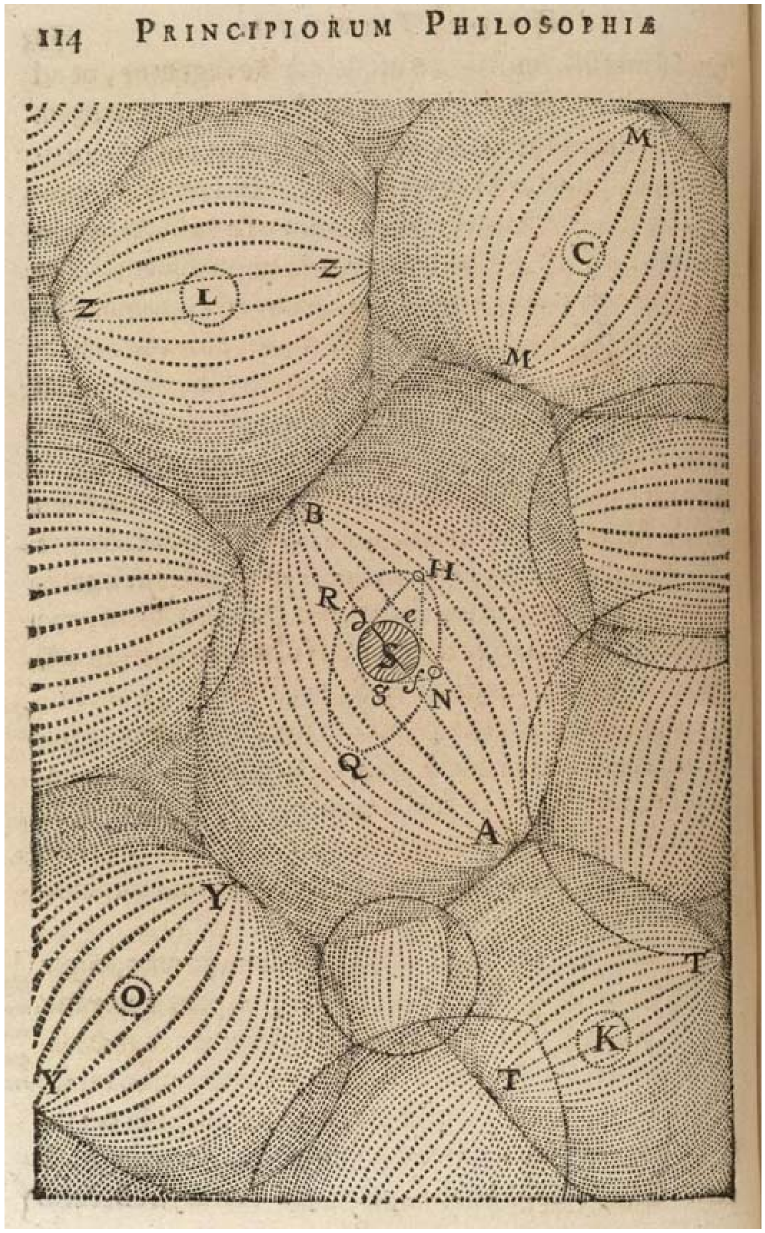
We continue the discussion of applications throughout the paper.

History/Background:

History:

Voronoi diagrams have a long history, dating back as early as the 17^{th} century. Work by Descartes on a partitioning of the universe into 'vortices' is one of the first known references to the subject. Even though Descartes does not explicitly define his vortices in the same way as Voronoi cells, his work is conceptually very similar [3].

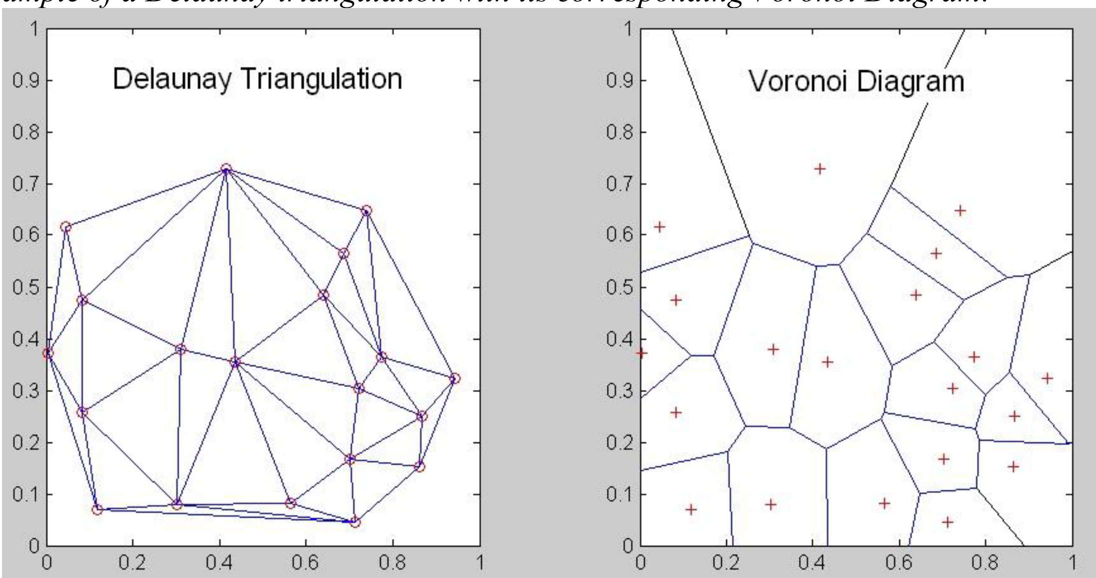
A drawing from Descartes which describes the partitioning of the universe into vortices – Notice that the vortices closely resemble Voronoi cells[11]:



Two German mathematicians, Lejeune Dirichlet and M. G. Voronoi, were credited with formalizing the modern concept of the Voronoi diagram [5]. Dirichlet was born in 1805 and in his work on quadratic forms he made some of the first significant contributions to the field of Voronoi diagrams. Indeed, it is because of him that Voronoi diagrams are also well known as Dirichlet tessellations. Before his death in 1859, Dirichlet had formalized the concept of the Voronoi diagram in the two and three dimensional cases [5]. Work by M. G. Voronoi in 1908 formalized the n dimensional case and gave Voronoi diagrams the name we commonly use today.

The two dimensional dual of the Voronoi diagram in a graph theoretical sense is the Delaunay triangulation [2]. Work on Delaunay triangulations (or, alternately, Delaunay tessellations) was done by French mathematician Charles Delaunay before 1872. In a Delaunay triangulation, any two sites are connected if they share a Voronoi diagram cell boundary, as shown below. An alternate definition, more in accordance with Delaunay's original work, is that two sites are connected if and only if they lie on a circle whose interior contains no other sites [3].

An example of a Delaunay triangulation with its corresponding Voronoi Diagram:



Even before Voronoi diagrams were formalized mathematically, they were developed independently in other sciences. In 1909, BT Boldyrev a Russian scientist, used "area of influence polygons" in his work in Geology [10]. Voronoi diagrams were used in Meteorology by Thiessen in 1911 to help model average rainfall [5]. Influential work in crystallography was done utilizing Voronoi diagrams by a German named Paul Niggli in 1927. In 1933, physicists EP Wigner and F. Seitz did important research using Voronoi diagrams in physics. Voronoi diagrams continued to play a key role in research done in Physics, Ecology, Anatomy, and Astronomy throughout the 1900's.

Applications and algorithms:

As mentioned earlier, applications of Voronoi diagrams are by no means confined strictly to mathematics. They go by many names as they relate to various fields of science. We provide a small glossary below [2,3,16].

Field of science:	**Term used:**
Mathematics:	Voronoi diagram, Dirichlet tessellation
Biology and Physiology:	Plant polygons, Capillary domains, Medial axis transform
Chemistry and Physics:	Wigner-Seitz zones
Crystallography	Domains of action, Wirkungsbereich
Meteorology and Geography:	Thiessen polygons

In the Work/Solutions section of this paper, we describe a way of constructing a Voronoi diagram using a geometric algorithm. It turns out that although this kind of algorithm is very easy to intuitively understand, it is not as computationally efficient as other known techniques. In [3], Aurenhammer and Klein classify some of these algorithms. The geometric technique we describe is an example of an *incremental construction* algorithm and it has a relatively poor algorithmic efficiency of $O(n^2)$. Using this notation, we assume the reader has a basic understanding of algorithmic efficiency classification. In these examples, n is the number of sites in the system.

Another category of algorithm used for Voronoi diagrams is *divide and conquer*. This technique works by recursively dividing the set of sites in order to decrease the problem size. Eventually the subsets of sites are small enough that diagrams are easily constructible. These sub-diagrams then must be merged back together, up the recursive tree, into the complete diagram for the system. Although this merging process is complicated, and care needs to be given as to how the set of sites is split each time, the result is a total algorithmic efficiency of O(n*log(n)). This is a significant improvement over the incremental construction algorithm.

Another algorithm of O (n*log(n)) efficiency is the *sweep* algorithm. This algorithm works by sweeping a vertical line across the plane horizontally. As this line passes through various Voronoi cells, their boundaries are constructed. Although the efficiency of this algorithm is the same as the *divide and conquer*, it can have some additional desirable characteristics if the sites are distributed in a certain way. There is no universally superior algorithm; ultimately one must choose an algorithm based on the particulars of the data and of the application.

Work/Solutions:

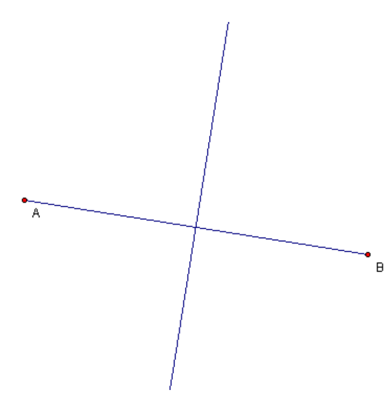
Our first task is to describe a simple geometric algorithm for constructing a Voronoi diagram. We start with a system containing only two sites and build to the four site case. From here it is easy to generalize to any system with a finite number of sites.

A simple geometric construction algorithm:

As mentioned in the first section, perpendicular bisectors have a fundamental role in the construction of a Voronoi diagram. When our system contains only two sites it is a very easy matter to construct a Voronoi diagram; the perpendicular bisector of the segment joining the two sites is all that is necessary (Figure 2).

Fig 1:

Fig 2:



To build a Voronoi diagram for three sites (points A, B, D below) we first construct a triangle with vertices at the three sites. From here we must rely on the fact that the perpendicular bisectors of the three sides of a triangle meet at a single point. This is illustrated in Figure 3 below. To complete the diagram we simply remove the superfluous rays, line segments, and points which were used in construction (Figure 4).

Fig 3:

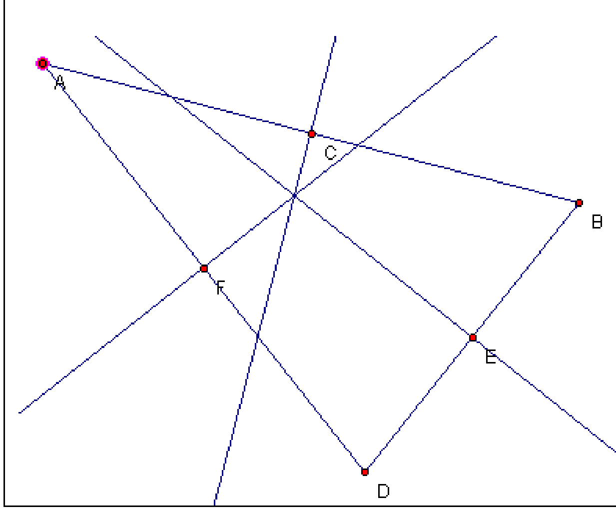
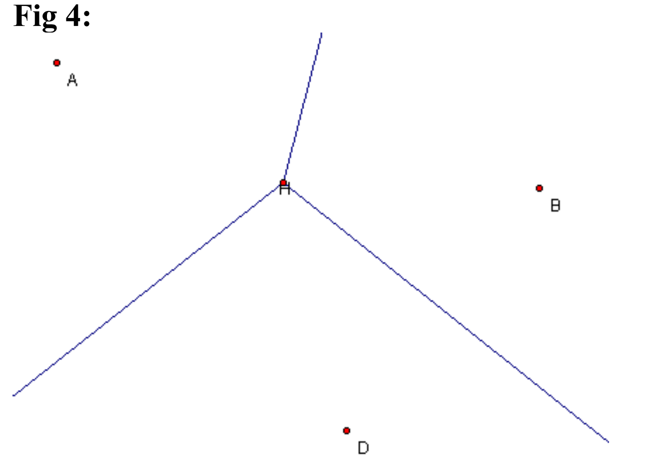


Fig 4:



Things become a bit more complicated when we have four sites. We begin with the existing diagram for three sites, then add another site (point I below) and adjust the diagram accordingly. To do this we must construct perpendicular bisectors between I and each previously existing site. These bisectors allow us to determine which parts of the diagram need to be adjusted.

We begin by constructing the perpendicular bisector between I and D (Figure 5). From this, we see that everything to the left of the line KJ belongs to I's cell. We adjust the diagram accordingly and remove the superfluous parts (Figure 6).

Fig 5:

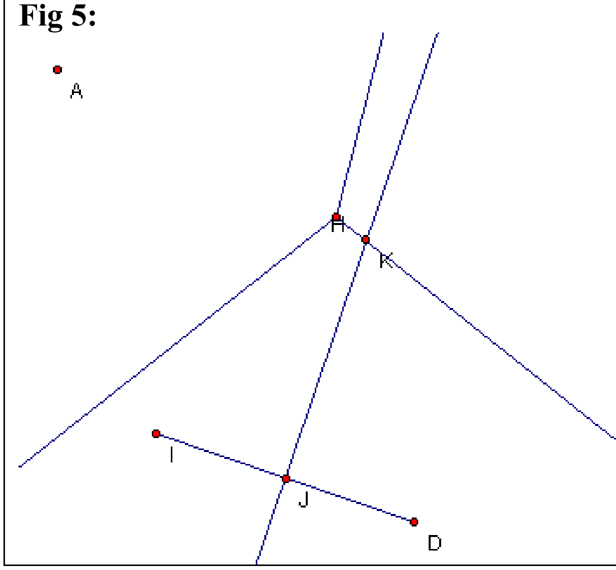
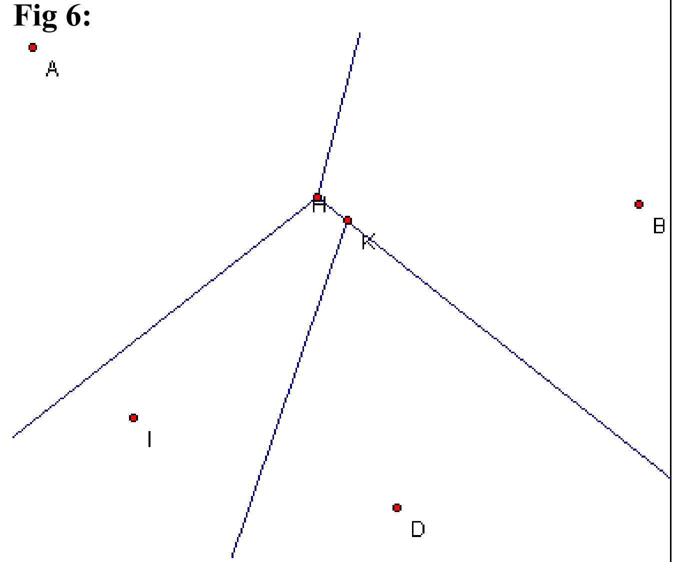


Fig 6:



Now we construct the bisector between I and A (Figure 7). From this we see that everything below the line LN belongs to I's cell. Now we remove the superfluous parts (Figure 8).

Fig 7:

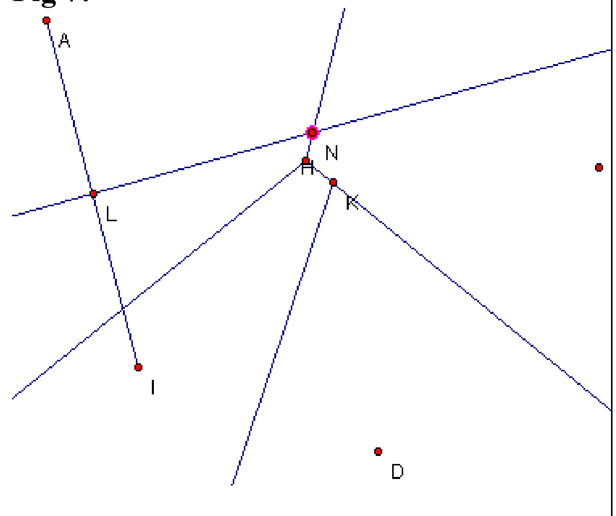
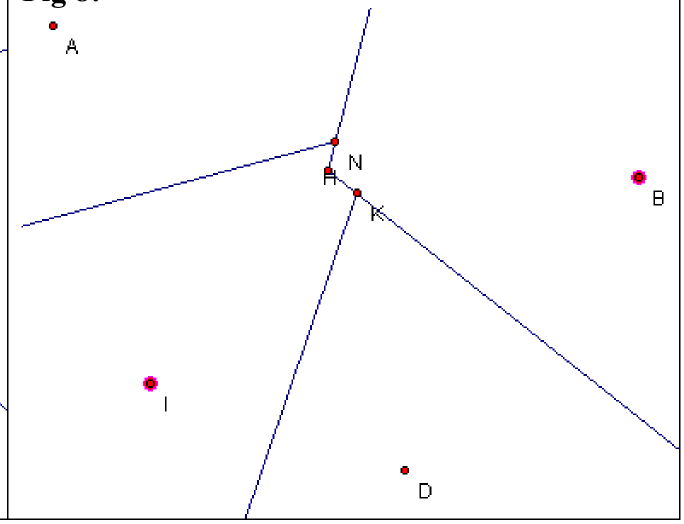


Fig 8:



Finally, we construct the bisector between I and B, our last pair of sites (Figure 9). From this we see that everything to the left of segment NK belong to I's cell. After removing the remaining superfluous parts, we have our completed Voronoi diagram for four sites (Figure 10).

Fig 9:

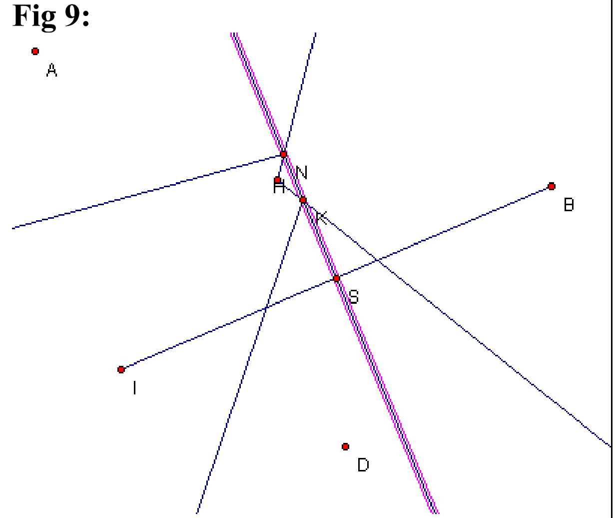
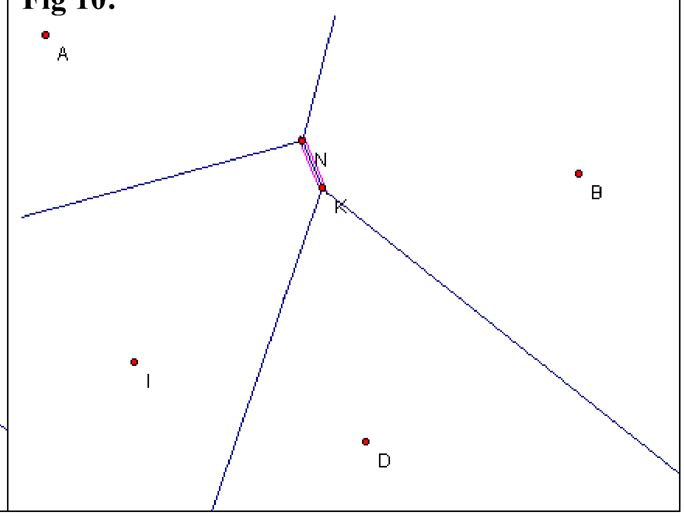


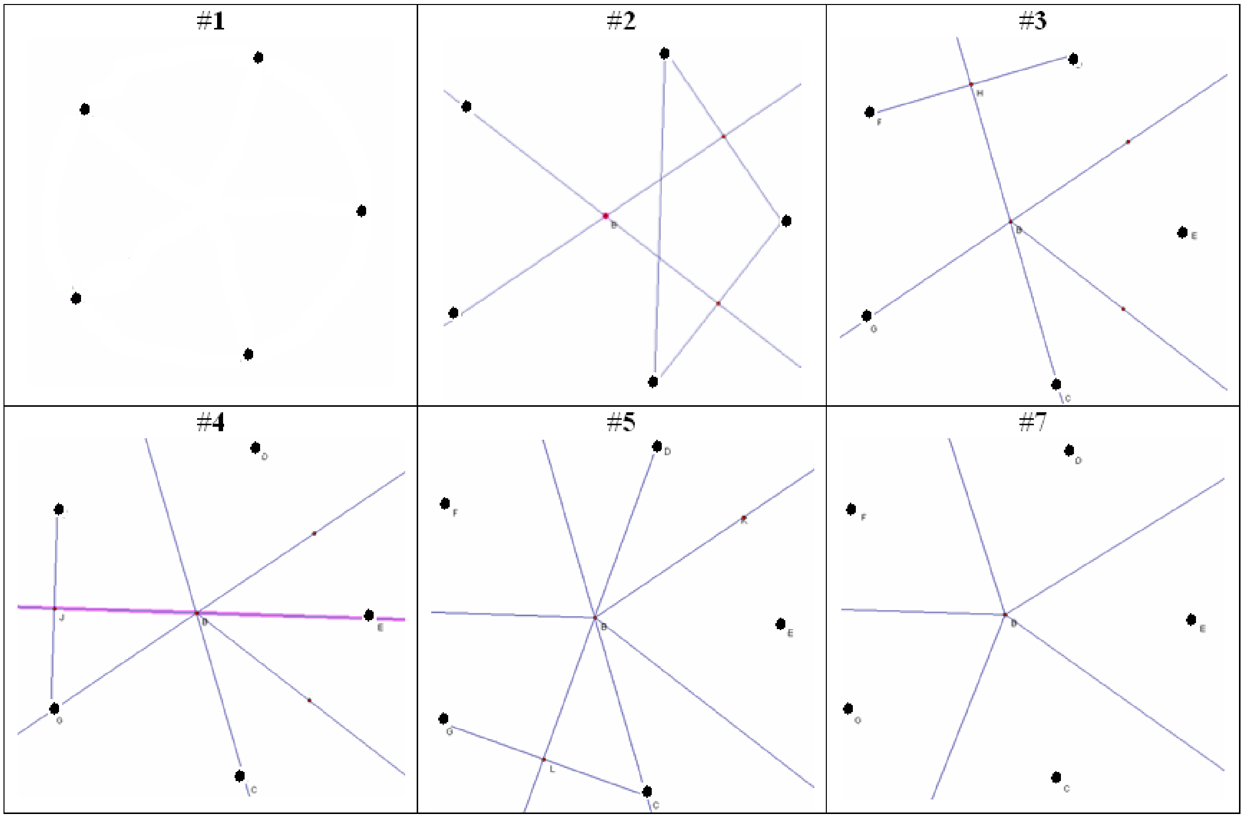
Fig 10:



This kind of approach is sufficient for any number of sites. From the four site system above we can construct a five site system by adding a site and considering which boundaries need to be adjusted for each new pair of sites. Continuing in this way we can build our system as large as we wish.

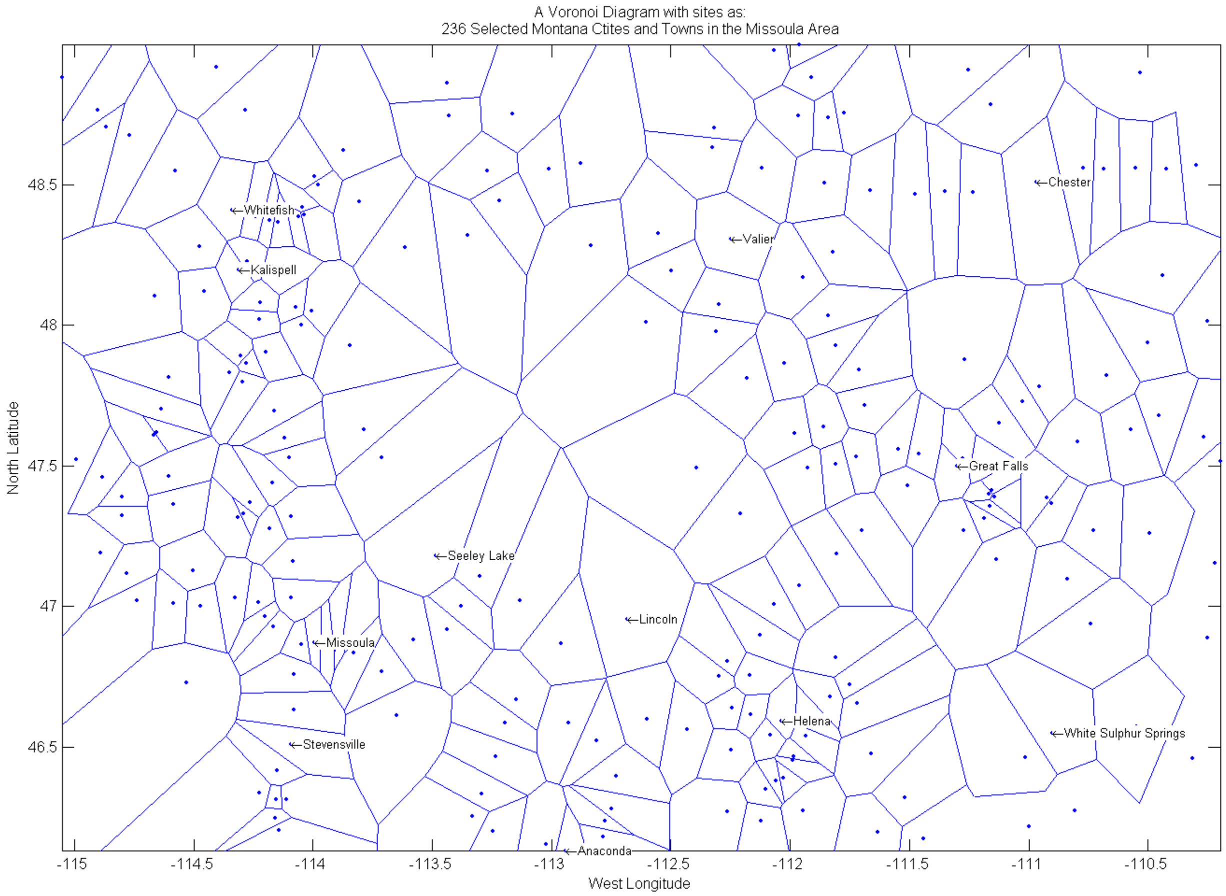
An example problem:

Construct a Voronoi diagram using the vertices of a regular pentagon as sites.



Voronoi of the Missoula area:

I decided it would be interesting to construct a Voronoi diagram of sites with local geographical relevance. After a good deal of searching, I was able to track down some data through the Montana Natural Resource Information System's GIS web service [7]. Using their service I was able to produce a list of the longitude and latitude of some 236 cities and towns in the Missoula, Montana area. This data then needed to be formatted so that it could be plotted in a Euclidean space. After that, MATLAB's Voronoi package was used to produce the actual diagram. Finally, I labeled a few key communities based on size of population and range of disbursement, so that one could get a good geographic impression of how the data was distributed. The result would have been too cluttered if I had labeled every site.



One thing the diagram provides is a sense of how remote certain sites are. If a city or town has few neighboring communities, its Voronoi cell is generally large. For example, Seeley Lake has a large Voronoi region corresponding to its relative remoteness. Even though the diagram gives us no true indication of population, we can make a rough extrapolation based on the fact that, in general, urban areas of high population densities have numerous nearby neighbors. Thus, we would expect larger cities to occur in areas of the diagram with smaller cells. This is more or less consistent with what the diagram indicates.

A more practical application of a diagram like this might be related to school districting. If each site on our diagram corresponded to a community with a school, the boundaries of the cells would be effective districting lines. If I child lived in a particular cell, they would go to school in the community that corresponded to that cell because it would be the shortest distance away.

Conclusions and Implications:

As we have seen, Voronoi diagrams have been around a long time and have undergone a good deal of study. In fact, Aurenhammer and Klein claim that about one out of 16 papers in computational geometry have been on research concerning Voronoi diagrams! More than 600 papers on the subject are listed in [8]. Ever since Descartes partitioned the universe into vortices in 1644, research in Voronoi diagrams has been done by some of the brightest minds in science and mathematics.

Because of the importance of Voronoi diagrams, the efficiency of computer algorithms used in diagram construction is equally important. There has been a good deal of progress made in algorithm design already and popular modern algorithms are extremely efficient. Even so, there is room for improvement by adapting algorithms to take advantage of known characteristics (about distribution etc.) in a set of data or a type of application.

We have hinted at a number of general applications of the Voronoi technique, but it should be made clear that there are vastly more that have gone without mention. Because of their generality, Voronoi diagrams are useful in an extremely diverse array of situations, in many different aspects of science and of everyday life. Here are a just few other applications, listed in [9]:

- Voronoi diagrams may be used in computer GUI applications that need to determine which link is closest to the mouse cursor when a user clicks on the screen.
- Voronoi diagrams have been used to correspond distribution of Bark Beetle attacks on trees to the beetle's known territorial behavior. The beetles tend to feed in isolated Voronoi regions, in a way that discourages intra-species competition for the same resources [12].
- Voronoi diagrams are used in some graphics software to create "Non-Photorealistic" images. The software distorts the original image around Voronoi cells of selected points.
- Voronoi diagrams have been investigated as a means of grouping words and multi-part symbols in documents [13].
- Voronoi diagrams have been used to increase the accuracy in the "river-mile" positioning system often employed along rivers and in reservoirs. Given an arbitrary position on the water, Thiessen polygons are used to determine which known river mile position is closest [14].
- Voronoi diagrams of a collection of randomly distributed points are being used to model some kinds of steel[15].

It is possible to generalize Voronoi diagrams to three or more dimensions and to define our distance metric however we wish. For instance, we could use a taxicab distance metric to produce a more useful diagram of sites in a city. Since it is seldom possible to travel directly to a destination in a city, a Voronoi diagram using the taxicab metric would more accurately reflect the amount of time it takes to travel along city streets to a particular site. In addition, we could weigh the 'distance' of certain busy streets more than streets in other less trafficked areas so that our diagram would even more accurately reflect travel time. The exceptional adaptability of the Voronoi diagram concept is what makes it so versatile and widely used throughout the sciences.

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Analogies and Mathematics: What is the connection?
Bharath Sriraman

Lyn English (Ed). ***Mathematical and Analogical Reasoning of Young Learners***. New Jersey: Lawrence Erlbaum & Associates, 2004. ISBN 0-8058-4945-9.

In the last decade and a half mathematics education literature has shown a rapid increase in books and articles that focus on the social and cultural issues related to mathematics learning and teaching. Although the social and cultural dimensions are important and relevant, the cognitive dimension of mathematical learning is equally important and received less attention. Mathematical and Analogical Reasoning of Young Learners takes us back to the very roots of learning and investigates foundational questions on the nature of and the evolution of reasoning in young children. The book also partially addresses cross-cultural themes in that it reports the results of a 3-year longitudinal study whose participants were children in Australia and the United States. This naturally leads to the question of the variance or the invariance of the findings across these two cultures, and an explanation for the nature of the similarity or dissimilarity in reasoning patterns.

Reasoning by analogy is a fundamental human trait. One encounters excellent examples of this propensity to "analogize" in ancient Greek philosophy. If an ancient Greek philosopher were asked: why do we create analogies? The answer would simply be to create a framework by which we could better understand the dimensions of human experience. By analogy (pun-intended) the book seeks to understand the relationship between mathematical reasoning and children's natural tendency to create analogies. For instance, attribute blocks have been used for along time to help children distinguish shapes spatially. If a child is able to distinguish between and triangle and circle, mathematics educators need to further ask what was the reasoning process that enabled the child to make the distinction. Again by analogy this is a fundamental question, which permeates research on the cognitive dimensions of mathematical learning.

The research reported in the book includes an exposition and analysis of the measures used to study the aforementioned questions. The researchers found that the development of mathematical and analogical reasoning in young children followed a similar path. Although there were individual differences, the cross-cultural differences were minimal. This is attributable to the similar socio-economic demographics in the Australian and U.S schools. Classroom instruction naturally played an important role in how the measures changed over the period of three years and children in the U.S fared slightly better than their Australian counterparts. More importantly it was heartening to note that the research indicated improvement over time in both countries.

Lest the reader of this review assumes that the book simply contains the reports of an empirical study, several chapters are devoted to classroom discourse and case studies, which qualitatively analyze the role of discourse in the development of mathematical and analogical reasoning. The vignettes provide a rich glimpse at the remarkable abilities of children to reason in natural and non-contrived contexts. Several chapters are also devoted to investigating the "analogous" research questions of the study with the teachers of the young participants. This in my opinion balances the book and presents a very different perspective. That is, it presents the beliefs of the teachers about the development of mathematical and analogical reasoning of their students. The concluding chapters of the book contain commentaries that analyze and critique the studies reported in the previous chapters of the book in addition to the possibilities for future research.

This book is a very useful resource not only for mathematics education researchers, cognitive psychologists and teacher educators but would also benefit classroom teachers of young children. Understanding the development of children's reasoning processes would help teachers tailor instruction that facilitates and nurtures the natural "mathematical" tendencies of young children. I recommend it very highly.

The Nature of Proof in Today's Classroom
Erica Lane

Harold Fawcett. ***The Nature of Proof***. New York City: Bureau of Publications, Columbia University, 1938. Re-printed by the National Council of Teachers of Mathematics in 1995.

Throughout the world, children are taught new ideas and concepts in a variety of ways, and the advocates of the different approaches claim that their method is the most effective and profitable for the students. In addition to the different teaching styles, each individual learns the material in a distinct manner. Teachers, therefore, must be aware of the diverse techniques of both teaching and learning in an effort to optimize the learning experience for the greatest number of students. This endeavor is even more difficult than it sounds. Imagine a classroom with twenty-five students. Some students learn better in a lecture setting while others learn best through visuals. Still others learn by working in pairs or groups, and some students need hands-on lessons to encourage the thought process. Numerous other learning styles exist, including the child who just does not want to learn. Everyday, teachers across the globe are faced with these situations, and researchers have attempted to find a teaching style and classroom atmosphere that is conducive to all students' needs and abilities.

An idealist would ask: Is there one approach to teaching that will cover all of the learning styles and engage the most (if not all of the) students? If there is, does this style work for all subject areas? Unfortunately, there is not one set method in which to teach all subjects nor is there one best method of teaching a single subject. If one existed, all teachers would be using it and there would be no need for further research. So, researchers today continue to conduct experiments that explore different teaching techniques. However we have a classic example of an effective teaching technique in mathematics education literature, which is both progressive and innovative by modern standards, namely Fawcett's (1938) two-year study regarding the teaching of geometry. In this book review, I will investigate Fawcett's research, citing specific examples and methods of teaching and offer new ways to apply his theories in today's classroom. In doing so, I hope that teachers will re-examine this mathematics education classic and consider its applicability to today’s classroom.

In high school geometry classes in the U.S, students are presented geometry in generally one of two styles. Traditionally, geometry has been a class separate from algebra and offered in between and Algebra I and Algebra II courses. Some schools still present mathematics in this sequence, but many schools have turned to “integrated” mathematics in which geometry, algebra, and trigonometry are combined into a series of four books. Recent examples of integrated curricula are the NSF-funded SIMMS, Core-Plus etc. Students typically take Integrated I – IV and possibly Calculus. In these classes, students are encouraged to see mathematics as a whole and use algebra and geometry to solve problems. Students take these classes in sequence, unless they need a slower curriculum or and accelerated program, at which time other texts are available.

Although these are the methods employed in the United States, both have apparent shortcomings that are causing the students to lose information that could be otherwise gained. For example, in the separated classes, students have little opportunity to connect geometry to algebra and algebra to geometry. To students, these areas are separate entities, and bridging the gap between the two subjects seems unrealistic. Even though these are specialized topics in mathematics, the two are connected, and students should explore more links between the two subjects. Recently, mathematics teachers and researchers felt the same way, so the implementation of the integrated books relieved the unconnected-ness of algebra and geometry, supposedly. The original NSF funded, research based and standards aligned integrated curricula fulfilled the promise of an authentic integration of the various strands of mathematics. However the "mainstream" K-12 textbook industry was quick to jump on the new mantra of integrated mathematics and to produce curricular materials, which tries to appease the traditionalists as well as the reformists. This hodge-podge "integrated" curriculum is very different from the NSF-funded curricular material (such as SIMMS and Core-Plus) and is now found in numerous school districts across the nation, especially those in which the administrators strive for innovation but face resistance from parents and "didactically" traditional teachers. The unfortunate consequence of this attempt to appease everybody is units and sections in the integrated books choppy and disconnected, far-removed from the vision of creating this new curriculum. The books do contain algebra, geometry and some trigonometry, but there is little relationship between the different sections, which generates confusion for the students. Without connecting one idea to the next, mathematics can be a very difficult field to learn and master. In general, these are the different ways that schools in the United States teach high school students geometry. The question then is do other techniques exist that may prove to be more beneficial in the teaching of geometry and proofs to students?

In 1932, Harold P. Fawcett, designed an experiment to test the usefulness of geometric and proof practices and how extensive work in geometry can lead to proficient and successful "transfer" of ideas and thought processes as they relate to the world outside of the classroom. The word "transfer" here is used very differently from the behaviorist conception of this construct. During Fawcett's study, only one method of teaching geometry existed in the United States, for the integrated boom had not yet hit schools. For his experiment, Fawcett established two groups, the experimental group and the traditional group. Placement of students did not depend on mathematical ability, standardized test scores, age, or gender. Students from grades nine through eleven were randomly placed in one of the classes and then asked their opinions regarding mathematics and the class that they were about to take. Each class met four times a week for forty minutes each day.

In the traditional classroom, the teacher taught the students demonstrative geometry using the standard method of the 1930s. The students learned from a geometry textbook that gave them definitions, axioms, and Euclid's theorems (in today's terms), and then they proved other theorems using the givens. Step by step, students would prove new theorems, and then use these to prove future theorems. Strictly guided by the text, students had little freedom to explore geometry on their own, and the curriculum had a rigorous structure, unlike the experimental group. This geometry was presented as a separate subject from algebra (which was the trend for decades after the study, as well), just as in the experimental group.

In the experimental group, students had the vast world of demonstrative geometry to explore at their own pace and with their own ideas. Conventional textbooks, much to the surprise of the students, became obsolete, and the students used their abilities and experiences over the two-year period to create their own textbooks. With nothing more than a little direction from the teacher, students worked individually, in pairs, and as a class to define terms, create assumptions, and prove theorems. The class became <u>almost like a democracy</u> (pun-intended for today's times!). All, or the majority of students, had to agree with the newly defined terms (or terms deemed "undefined") and proven theorems before they could be added to the textbooks of each student. Fawcett (1938) described the "principles and methods" that the teacher of the experimental group followed as:

1. No formal text is used. Each pupil writes his own text as the work develops and is able to express his own individuality in organization, in arrangement, in clarity or presentation and in the kind and number of implications established.
2. The statement of what is to be proved is not given the pupil. Certain properties of a figure are assumed and the pupil is given an opportunity to discover the implications of these assumed properties.
3. No generalized statement is made before the pupil has had an opportunity to think about the particular properties assumed. This generalization is made by the pupil after he (sic) has discovered it.
4. Through the assumptions made the attention of all pupils is directed toward the discovery of a few theorems, which seem important to the teacher.
5. Assumptions leading to theorems that are relatively unimportant are suggested in mimeographed material, which is available to all pupils but not required of any.
6. The major emphasis is not on the statement proved, but rather on the method of proof.
7. The extent to which pupils profit from the guidance of the teacher varies with the pupil and the supervised study periods are particularly helpful in making it possible to care for these variations. In addition individual conferences are planned when advisable. (p.62)

Using these teaching techniques, the teacher (who happened to be Fawcett, himself) acted more as a moderator than an instructor and allowed the students to teach themselves and each other throughout the two years.

After two years, both classes took tests on the fundamentals of geometry and their ability to analyze non-mathematical. The students were evaluated on six criteria: "record of scores made by pupils on the Ohio Every Pupil Test in plane geometry; results of paper-and-pencil tests on the nature of proof applied to non-mathematics situations; contributions of students illustrating situations to which habits of thought developed in their study of the nature of proof had transferred; parents' observations concerning improvement in the critical thinking of their child; record of six observations made by college seniors; and students' observations concerning improvement in their ability to think critically" (p.102).

On the *Ohio Every Pupil Test* as well as the nature of proof test, the experimental group scored well above the average of all students who took the tests and improved greatly from the first time they took the nature of proof test. The experimental group showed a significant increase in their ability to “transfer” critical thinking skills to other environments, and the majority of parents agreed that the logical and critical thinking skills of their children had been enhanced through this class. Finally, the reports from the college seniors as well as the pupils themselves represent a positive, intellectually stimulating experience from the two years in the experimental, demonstrative geometry course. These students were placed in a situation that varied from the usual classroom setting and rose to the occasion by expanding their abilities and skills and applying these to their lives after the class ended. The students in this class, like the mathematician Euclid did two thousand years ago, increased their own knowledge while opening doors for their peers at the same time. Their individual textbooks may never be published, but their experience has been regarded as a most beneficial way for students learn demonstrative geometry.

Since Fawcett's approach to teaching students geometry and methods of proof demonstrated a great method via which students became involved with the material, it is important to explore specific aspects of his techniques and attempt to find ways in which this can be used in today's classrooms. It seems that if this method is as effective as Fawcett claims it is, teachers should try to incorporate it into their classrooms. However, since Fawcett published the results of his study 1938, little change is evident in the teaching of geometry until the integrated units in the 1990s. One may wonder why his methods did not catch on in the United States and if it is even possible to teach with such ambiguity in the structure of geometry curriculum.

To begin his class, Fawcett and his students discussed definitions and the need for defining terms so that each student, as well as the reader, has a firm basis and understanding of the material in the new textbooks. Before jumping into words such as point, line, circle, etc., students were asked to define "school," "outstanding achievement," "foul ball," and "tardy." Although these are words that individuals use everyday without specifically defining, the students had a difficult time agreeing on these and other definitions (p.31-34). It is important to note that the class spent four weeks establishing the fact the need for precise definitions on which all students agree.

After struggling with defining these terms, Fawcett allowed his students to brainstorm and determine how they wanted to approach defining geometric terms and the study of geometry. The class continued in this way for weeks as the students created lists of undefined terms and defined terms. These ideas followed this structured path (p.42):

1. Undefined Terms
    a. "…selected and accepted by the pupils as clear and unambiguous."
    b. "No attempt was made to reduce the number of undefined terms to a minimum."
2. Definitions
    a. "The need for each definition was recognized by the pupils through discussion…"
    b. "Definitions were made by the pupils."
3. Assumptions
    a. "Propositions which seemed obvious…were accepted as assumptions…"
    b. "These assumptions were made explicit by the pupils…"
    c. "No attempt was made to reduce the number of assumptions to a minimum."
    d. "The detection of implicit or tacit assumptions was encouraged…"
    e. "The pupils recognized that…the formal list of assumptions is incomplete."

Many would agree that this is a great way to get students involved in their own education, an idea that was missing in classrooms in the 1930s and is still missing in many classrooms today. However, as stated above, the students took four weeks just to establish the relevance of definitions in the world around them as well as in the world of mathematics. With the new stipulations on student learning and success, this seems like an unrealistic amount of time to apply to ideas that are non-mathematical. Nevertheless, I believe that teachers could use such an approach in teaching geometry today.

In a graduate level course in which I recently enrolled, ten students worked to define the first terms in Book I of Euclid's *Elements*. This endeavor proved to be a challenge for my class, and we labeled many terms such as point, line, and ray undefined, just as the students in Fawcett's study. The task was time consuming, but we decided that much of our struggle came from the fact that we had all been introduced to the geometric terms, assumptions, and theorems in previous classes influencing the creation of our geometry. I would argue that students in a beginning geometry class in high school will not have the same bias, and therefore will be able to reach conclusions regarding some terms and definitions. The lists of definitions and assumptions to memorize and the perfectly proved, textbook theorems have not yet spoiled their minds. These students have the ability to start from the ground and create a solid structure of geometry or other mathematical emphases with guidance from a teacher who has a firm grasp on the subject.

It is not reasonable to expect a teacher to cover an entire geometry curriculum in this way while conforming to the recently passed *No Child Left Behind Act*, but I encourage a trial period in which students in geometry (or algebra, trigonometry, calculus or even in other disciplines) are allowed to explore the specifics of the mathematics with little direction from the teacher and more influence from their peers and their previous experiences. Instead of spending a month defining terms to use for the entire two-year curriculum, I have created a lesson plan that incorporates just slight direction from the teacher and depends highly on the participation of students and their interactions with one another. This plan, or other related plans, are recommendations and may not be productive in every class.

The teacher should act as a moderator. The students should be in control of the lesson. Make sure books are closed; this is entirely from their previous knowledge.

1. *Introducing Angles*
    a. *Ask students how they define an angle. Do all the students agree? Put all of the suggestions on the board. Now, ask the students what words on the board should be defined. For example, with the definition of an angle, words like "point," "ray," and "distance" probably fall into the definition.*
    b. *Break students into groups of three or four and have them create the "ideal" definition of an angle while defining all of the ambiguous terms in their definition. Moderate the time based on students' involvement in the assignment.*
    c. *Each group will present their definitions to the class, and the other students will have a chance to agree or disagree with the group's rationale.*
    d. *Put the words acute, obtuse, straight, vertical, complementary, and supplementary on the board. As a class, have the students brainstorm the definitions to these different angles. Find common ground among the students to establish one list of definitions (and pictures) in which most (all) students agree.*
    e. *The most important part of this lesson (I think), is asking the students why this is important and relevant and how they think it is going to be used in mathematics. When and where are angles used in mathematics? Outside of mathematics? What professions use angles in their everyday routine? How have angles been used in historic mathematics? Have students write a paragraph or two using the defined terms and applying angles to the world.*

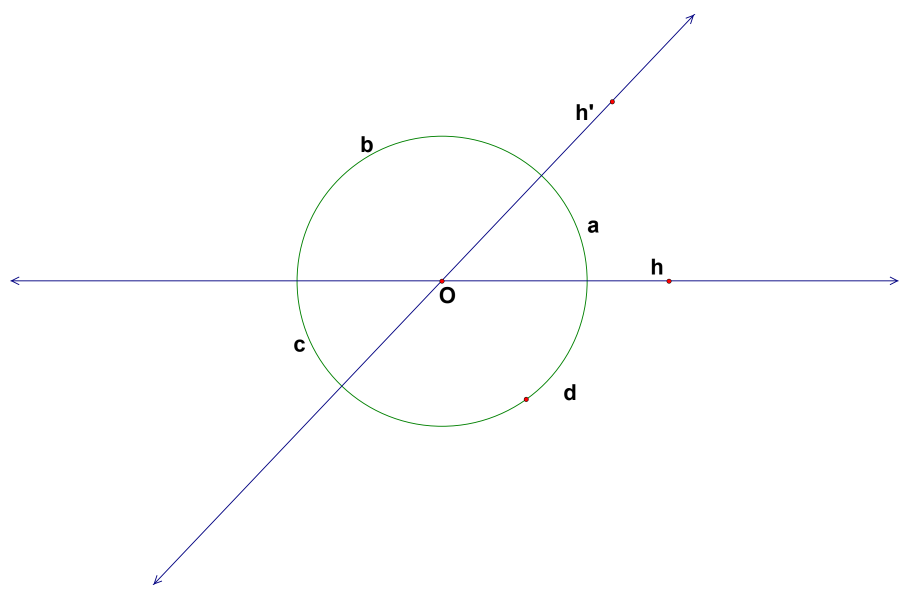
In Fawcett's study, the students used their experience of defining terms and applied their new knowledge to situations that were non-mathematical. How, for example, can the word restaurant be defined? It was important for the state of Ohio, at one point, to use the word restaurant in a law, but what is a restaurant? The state said that a restaurant is "a place of business where 50 per cent or more of the gross sales accrue from the sale of food-stuffs consumed on the premises" (p.46). Students were then asked to consider five questions, two of which are presented below (p.47):

1. The gross sales of one of the White Castles in Columbus is approximately $12,000 a year, all of this coming from the sale of food. Some of this food is eaten where purchased while the remainder is eaten elsewhere. If the amount eaten elsewhere is $6158.40, is the White Castle rightly called a restaurant?
2. How would you decide whether or not a combined ice cream parlor and soda fountain which also serves light lunches is a restaurant?

The reader should bear in mind that Fawcett's study took place over a two-year period, so lengthy discussions such as the previous example were possible. Rarely in mathematics class is there time for discussing the definition of a restaurant. However, defining terms in laws may prove to be beneficial to students in history, political science, or English classes. It is important to note that although this study was done with a geometry class, Fawcett's techniques can be modified to fit a variety of disciplines. Fawcett used these examples to emphasize the importance and defining terms so that all readers agree with the definitions. His students established an appreciation for definitions and assumptions, which enhanced their participation in the geometry portion of the class.

Finally, Fawcett's students jumped into the realm of geometry and were presented with scenarios often in the form of sketches in which they had to draw conclusions and create a list of assumptions to add to their texts. The book has a wealth of examples, I will present only one. At this point in the curriculum, teacher ability to lead the students without making conclusions for them becomes crucial. Fawcett had to introduce topics to his class in a fashion that encouraged a thoughtful and profitable discussion and led the students to discover something new.

For example, students were asked to state the properties of this figure when "h is in a fixed position and h' revolves about O in a counterclockwise direction." (p.52)



Students proposed the following properties (and others):
h and h' can be extended indefinitely.
Vertical angles are always equal.
There is a time when the four angles formed are equal.
Angle *a* is never smaller than 0°.

Some students agreed with these statements and others, and some students would not support these claims without further investigation and possibly proof. Fawcett continued in this manner for weeks until students had made significant additions to the assumptions portion of their texts. Then, students were introduced to the method of induction, which promotes forward thinking in mathematics and all disciplines. Students used induction to reason geometric proofs and to argue about non-mathematical statements including the Declaration of Independence and the assumptions used in this work. Working through these non-mathematical situations as well as geometric scenarios, students completed their textbooks in the two-year class and were tested, as described previously in the paper, against the traditional class.

In the end, students in the experimental group labeled twenty-three terms undefined, defined ninety-six terms, made fifty-five assumptions, and proved twenty-one theorems and an additional thirty-three, although not all students agreed with these others. Individually, students were rewarded by having a constructed a geometry text that was their own creation! After 182 hours (since little if no outside work was necessary), the students had achieved phenomenal goals and scored much higher on the tests than the group which was taught in the traditional manner.

Fawcett provided his students with an amazing technique to learn geometry and method of proof, but can his structure (which was funded and not under the scrutiny of *No Child Left Behind*) be used in today's classroom settings? Can teachers allow students to create their own mathematics without the aid of textbook and still meet the standards set forth by the school districts, state government, and federal government? Although recreating this setting is not impossible, it would be difficult. Seventy years ago the standards for mathematics differed from those of today, and despite the fact that students would receive an outstanding education from a teacher who could present the information in this fashion, the current standards would be more difficult to meet. It is noble of any teacher to have the desire and drive to attempt such an endeavor, with the completeness of Fawcett's study.

I do think that teachers can pull important techniques and activities from the study in an effort to put both teaching and learning into the hands of the students. The most beneficial part of Fawcett's class was that the students depended on resources that they usually use sparingly: namely <u>their own experiences and knowledge and the ideas, thoughts, and conclusions of their peers</u>. So often students sit and listen to a teacher lecture day in and day out and work most of their problems out of a previously written textbook. If students feel that they have ownership in their learning, many of them will rise to the occasion without knowing that they are becoming directly involved their own education. In traditional classrooms, how often do students get to lead a discussion or disagree with an assumption or proof on the board? According to this book, they need more of this time in order to feel passionate about the discipline.

The ability to recall geometry in the experimental group, although different from the traditional class, was enhanced through the creation of their own geometry. Even though these students did not memorize axioms or proofs, they are more likely to remember the methods that they used to determine certain aspects of the geometry and can apply these techniques to future mathematics classes. If a problem is presented that the traditional class only memorized, this will probably not remain in their long-term memory. The experimental class, on the other hand, could take the methods they used to prove such a problem and work through it and various other problems. Since they were using these methods in a variety of ways throughout life, these techniques were less likely to be lost compared to the memorized proof.

One fundamental aspect of mathematics is communication. The students in the experimental group had to establish top communication skills in order to explain their ideas, rationale, and proofs to the other students. They were forced (without knowing) to learn the vocabulary of a mathematician and use it everyday. Without the proper communication skills, other students had the opportunity to disagree with their peers. In high school, teenagers find it essential to be accepted by their peers, and this encouraged being prepared for classroom discussions. Outside of the mathematics portion of the class, students also worked with governmental, school, and various other topics in which communication through reading and explanation was necessary. In the end, students had created their own textbooks. What a great way to communicate with other mathematicians!

Harold Fawcett created an "ideal" classroom in which students had the freedom to explore geometry and methods of proof at their own rate, collaborate information in groups and as a class, and learn through teaching, not lecture. Seventy-two years later, teachers of geometry in the public high schools have not attempted this method or even taken his main ideas in an effort to give more meaning to a child's education. Recreating the entire study is an unrealistic goal, but his approaches to teaching in general can be carried to all disciplines. After reading *The Nature of Proof*, I encourage both mathematicians and mathematics educators to take the pieces of Fawcett's study that will benefit each individual classroom to open the minds of the students. If teachers give students reasons for learning and an ownership in their education, the students will retain mathematical processes and become more engaged and interested in the study of mathematics. This may come across as an idealistic vision but as John Lennon once said: "You may say I'm a dreamer, but I'm not the only one…"

For more information regarding:

- No Child Left Behind visit http://www.ed.gov/nclb/landing.jhtml
- Montana Standardized Tests 2002-2003 visit http://www.opi.state.mt.us/PDF/Superintendent/HowToInterpretScores.pdf
- The Nation's Report Card visit http://nces.ed.gov/nationsreportcard/sitemap.asp

***TMME, Volume 1, No. 2:* AUTHOR INFORMATION**

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Michael Mumm was born in Alaska and grew up in the small coastal community of Homer. Michael's early interest in mathematics was instilled in great part by his father. Upon graduation from High School he went north to the University of Alaska Fairbanks where he spent a total of two years. During and after his time in Fairbanks, he traveled extensively to Central and South America as well as to New Zealand, Australia, South-East Asia, India, Nepal, and Europe. Eventually Michael returned to school, transferring to the University of Montana, and finished his degree in Math. His undergraduate emphasis has been in Algebra. Michael is currently on a job in Antarctica, and expects eventually to return to graduate school.

Erica Lane grew up in a family of teachers in the small town of Sycamore, Illinois. After spending time in a summer camp in Montana during high school, she decided to pursue her university education "out west". Erica will graduate with the bachelors' degree in mathematics education from the University of Montana in December 2004. During her undergraduate experience here, she has tried to spark a greater understanding and a joy of mathematics in young minds by teaching at GEMS (Girls Enjoying Math in the Summer) Camp, tutoring high school and university students and student teaching. She is currently completing her student teaching experience in New Zealand. Erica plans on teaching at the high school level and eventually pursuing advanced degrees in mathematics education.*"I have always enjoyed my time in a classroom (more as a teacher than a student, of course!). Mathematics has always been my passion, and I cannot imagine a better way to apply this passion than to teach others."*