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Abstract:
In this paper we propose some reflections on Wittgenstein's ideas about grammar and rules; then we shall consider some consequences of these for the foundations of set theory and, in particular, for the introduction of major concepts of set theory in education. For instance, a community of practice can decide to follow a particular rule that forbids the derivation of arbitrary sentences from a contradiction: since, according to Radford's perspective, knowledge is the result of thinking, and thinking is a cognitive social praxis, the mentioned choice can be considered as a form of real and effective knowledge.

Keywords: Foundations of mathematics; thinking; Set theory; Wittgenstein

1. Introduction

In this paper we shall propose some reflections on the main ideas of Ludwig Wittgenstein (1889-1951) about grammar and rules; then we shall consider some consequences for the foundations of set theory, with regard to mathematics education too.

Wittgenstein's fundamental reflections on the "grammar" and on the meaning of "following a rule" will allow us to approach the focus of this paper. Of course we shall not try to expound "what Wittgenstein really meant", but rather we shall try to see some implications of Wittgenstein's views, in particular for mathematical and educational practice. In order to do this, we are going to refer both to some Wittgensteinian ideas and to some well known interpretations.

First of all, it is necessary to highlight the important sense of the term "grammar": according to Wittgenstein, the grammar is a particular philosophical discipline by which it is possible to describe the use of words in a language (Wittgenstein, 1969, § 23). The importance of the grammar is crucial in Wittgenstein with regard to reflections on mathematics, too: as a matter of fact, a theorem, like every other analytical (true) statement, expresses a grammatical rule (Gargani, 1993, p. 99).

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Let us now consider a rule (in particular, a grammatical rule, with reference to "grammar" in the aforementioned sense) and let us analyse the practice that we identify as "obeying a rule". Wittgenstein notices: «And hence 'obeying a rule' is a practice. And to *think* one is obeying a rule is not to obey a rule. Hence it is not possible to obey a rule 'privately': otherwise thinking one is obeying a rule would be the same thing as obeying it» (Wittgenstein, 1953, § 202). These words, frequently quoted, are very important: as a matter of fact, the distinction between thinking one is obeying a rule and obeying it is the difference between the behaviour that a person recognises to be in accordance, by definition, with the considered rule, and the corresponding grammatical decisions of the community (Frascolla, 2000, pp. 135-136).

It follows, once again, that the collective aspect clearly assumes a primary importance (Wittgenstein, 1953, § 206), although different interpretations attribute different roles to this aspect: for instance, according to C. McGinn, in order to follow a rule a public reference is not strictly necessary; the main point is the presence of support of behaviour that can be observed from outside (Messeri, 2000, pp. 184-185; McGinn, 1984, pp. 43-45).

According to S. Kripke (1982), Wittgenstein's position must be interpreted from a sceptical viewpoint: there is no such fact as "obeying a rule" and Wittgenstein's sceptical solution can be summarized in the (common) possibility of ascribing to someone the behaviour indicated by the expression "obeying a rule" (Messeri, 2000, p. 174). Of course, once again, a public reference is unavoidable (Kripke, 1982, pp. 27-49). According to C. Wright (1980), Wittgenstein suggests that the use of a concept cannot be completely fixed by one's past experience; hence who learns a language does not try to acquire an objective system of applications defined in the teacher's mind; better, the learner looks for the teacher's approval (as stated in: Messeri, 2000, p. 176; this points us towards an interesting educational reflection on the didactical contract).

An important element to be highlighted is that in order to allow comprehension of the expressions employed in a language, it is necessary to describe "obeying a rule" as a fact that can be acknowledged from outside, so a community is the only background that can give sense to the habit according to which we consider and treat individual answers as correct or wrong (Messeri, 2000, pp. 176-177). In addition, a very important position in both Kripke and Wright is worth mentioning: both of them place great emphasis on Wittgenstein's idea, defined as *community view*, according to which the sense of the discussion about "obeying a rule" is to be framed within a collective practice (Messeri, 2000, p. 177).

2. Frege's system and Russell's paradox

The original "ingenuous set theory" is based upon the so-called Comprehension Principle of Gottlob Frege (1848-1925), according to which, if we consider any property $P(x)$, we can define the set: $\{x \mid P(x)\}$ (and it is unique, according to another Fregean principle, the Extensionality Principle).

The subject leads us to remember a very important historical reference: on 16 June 1902, in a celebrated letter, Bertrand Russell (1872-1970) communicated to Frege that he found a contradiction in the theoretical system proposed by the great German logician. The Comprehension Principle, as noted, states the possibility of assigning a set by a characteristic property: given the property $P(x)$ there is a set I such that $x \in I$ if and only if $P(x)$. Nevertheless, if we consider the property $P(x)$ as $x \notin x$ and define the set $H = \{x \mid x \notin x\}$ (and this is allowed by the aforementioned Principle), then we are dealing with the famous Russell's paradox: from H∈H it follows H∉H and from H∉H it follows H∈H.

From the logical viewpoint, as it is well known, the presence of a contradiction in a theory causes a lot of problems. *Ex falso quodlibet*, so from a false proposition (absurd) proposition, e.g. "H∈H ∧ H∉H", we can derive any proposition Q:

- H∈H ∧ H∉H, hence H∈H;
- H∈H ∧ H∉H, hence H∉H;
- from H∉H it follows H∉H ∨ Q;
- H∉H ∨ Q, and being H∈H, it follows Q.

Many 20th-century logicians worked to obtain a new theoretical framework in order to avoid contradictions: Frege's Comprehension Principle has been weakened by Ernst Zermelo (1871-1953), who replaced it with a new axiom allowing the introduction of sets by elements such that they are characterised *both* by a given property P(*x*) *and* by belonging to another (existing) set. As regards Russell's paradox, given a set A we can consider the set of the sets I belonging to A and not belonging to themselves, but it is impossible to consider the set of "all" the sets I not belonging to themselves (Zermelo, 1908). Some years later, Fraenkel, Skolem and Von Neumann revised Zermelo's system and proposed a new version, the Zermelo-Fraenkel theory, ZF (Fraenkel, 1922).

It can then be seen that according to Wittgenstein an is different from a natural law (also) because of the different role of the experience («Does experience tell us that a straight line is possible between any two points?»: Wittgenstein, 1956, III, § 4); he notices that «by accepting a proposition as a matter of course, we also release it from all responsibility in face of experience» (Wittgenstein, 1956, III, § 30). He underlines moreover that «it is quite indifferent why it is evident. It is enough that we accept it. All that is important is how we use it» (Wittgenstein, 1956, III, § 2). Hence the crucial point we must take into account when we consider two different theoretical approaches (e.g. the approaches based upon Frege's Principle or upon Zermelo's) is "how we use" them: and the considered «proposition is not a mathematical axiom if we do not employ it precisely *for that purpose*» (Wittgenstein, 1956, III, § 3).

But what do we mean by that? What do we mean when we say that a proposition is employed "precisely" for the purpose of being a mathematical axiom? Let us quote Wittgenstein once again: «It is not our finding the proposition self-evidently true, but our making the self-evidence count, that makes it into a mathematical proposition» (Wittgenstein, 1956, III, § 3).

So when we accept an axiom we really make "the self-evidence count" and recognise implicitly the character of the considered proposition: in that moment «we have already chosen a definite kind of employment for the proposition» (Wittgenstein, 1956, III, § 5). As we shall see, the main problem is to decide if a particular use is included in this "definite kind of employment".

Let us now turn back to our old "ingenuous set theory" in order to highlight how mathematics, for instance from the educational viewpoint, takes into account the famous letter by Russell. Theoretically speaking, ZF axioms can be effectively considered as the basis of the introduction of the concept of set (see for instance: Drake, 1974); but from the educational point of view, generally the concept of set is not introduced by an axiomatic presentation: it may seem that the concepts of set and belonging are easy concepts to learn. Although it is true that they bear an intuitive meaning, their corresponding mathematical meanings entail a precise conceptualization (Bagni, 2006-a).

These concepts attained a mathematical formulation only in the 19th century (Kline, 1972, p. 995), when Georg Cantor (1845-1918) introduced the concept of set through some synonyms (in *Über unendliche lineare Punktmannigfaltigkeiten*, 1879-1884); Frege also made reference to the concept of set through verbal descriptions (van Heijenort, 1967, pp. 126-128). As a consequence, we can say that introductions of the concept of set in education are still rather close to the aforementioned "ingenuous theory".

Now one can propose a problematic issue (Bagni, 2006-b): *what* are we going to teach when we introduce major concepts of set theory according to an ingenuous perspective? If we introduce sets through general descriptions that cannot be considered "rigorous" (e.g. based upon Frege's Comprehension Principle), we provide our pupils with a potentially dangerous mathematical tool: for instance, a pupil could decide to derive the Russell's contradiction and, from that (the ancient *ex falso quodlibet* should once again be remembered) he or she could state an arbitrary sequence or proposition trivially "true".

Nevertheless, another quotation of Wittgenstein is very relevant to our problem: «'Contradiction destroys the Calculus' – what gives it this special position? With a little imagination, I believe, it can certainly be demolished. (…) Let us suppose that the Russellian contradiction had never been found. Now – is it quite clear that in that case we should had possessed a false calculus? For are theren't various possibilities here? And suppose the contradiction had been discovered but we were not excited about it, and had settled e.g. that no conclusions were to be drawn from it. (As no one does draw conclusions from the 'Liar'.) Would this have been an obvious mistake?» (Wittgenstein, 1956, V, § 12).

Wittgenstein's reference to the Liar is interesting: although using the verb "to lie" can lead us to (unavoidable) contradictions, our language *works* (Wittgenstein, 1956, III, § 3). No one "draw conclusions from the 'Liar'", i.e. no one uses this celebrated contradiction to produce arbitrary results (*ex falso quodlibet*). Wittgenstein recognises that «if it is consistently applied, i.e. applied to produce arbitrary results», a contradiction «makes the application of mathematics into a farce, or some kind of superfluous ceremony» (Wittgenstein, 1956, V, § 12). But it is important to underline this issue: a contradiction would be "consistently applied" (consistently, we mean, with respect to logical calculus) when it is applied "to produce arbitrary results"; this is just *one* possible choice: perhaps it is a "consistent" choice, but it is not the one and only.

Of course it is possible to object that the rigor of our logical calculus (or, better: the rigor we traditionally ascribe to our logical calculus) is very different from the features of our everyday language: «'But in that case it isn't a proper calculus! It loses all *strictness*!' Well, not *all*. And it is only lacking in full strictness, if one has a particular idea of strictness, wants a particular style in mathematics» (Wittgenstein, 1956, V, § 12).

This is the point: does an "ingenuous set theory" lose "all strictness"? As a matter of fact the presence of a contradiction does not block the *construction* of a grammar completely (let us remember once again that according to Wittgenstein every Platonic approach is excluded). Clearly Fregean logical calculus has been put into a very critical position by Russell's paradox: but this fact must be considered with reference to a "particular idea of strictness", and hence to a "particular style in mathematics" (related to a "proper calculus"): «'But didn't the contradiction make Frege's logic useless for giving a foundation to arithmetic?' Yes, it did. But then, who said that it had to be useful for this purpose?» (Wittgenstein, 1956, V, § 13).

So Frege's logic is quite useless for giving a foundation to arithmetic: but if we consider it from an educational perspective we must reconsider our judgement. As a matter of fact the contradiction could have very different consequences and a very different influence upon language games: firstly, it is simply possible that no one derives the contradiction itself; secondly, Russell's paradox can be considered by our students (looking for teacher's approval: Wright, 1980) as a strange, bizarre statement, and never used in order to derive arbitrary propositions: «We shall see the contradiction in a quite different light if we look at its occurrence and its consequences as it were anthropologically – and when we look at it with a mathematician's exasperation. That is to say, we shall look at it differently, if we try merely to *describe* how the contradiction influences language-games, and if we look at it from the point of view of the mathematical law-giver» (Wittgenstein, 1956, II, § 87).

We can suggest, following R. Rorty, that the major point is related to «the attempt to model knowledge of perception and to treat 'knowledge of' grounding 'knowledge that'» (Rorty, 1979, p. 316); more precisely, we could say that according to a new perspective our perception of Russell's paradox will not be based just upon *knowledge of* the paradox itself (and its potentially destructive use): «our certainty will be a matter of conversation between persons, rather than a matter of interactions with nonhuman reality» (Rorty, 1979, p. 318). And, as previously noticed, this conversation does not lead speakers to the derivation of the contradiction.

3. Concluding remarks

Previous considerations suggest us to turn briefly to a more general issue. According to Luis Radford's perspective, knowledge is linked to activities of individuals and this is essentially related to cultural institutions (Radford, 1997), so knowledge is built in a wide social context. Moreover Radford states: «Drawing from the epistemologists Wartofsky and Ilyenkov, I have suggested that knowledge is the product of a specific type of human activity – namely *thinking*. And thinking is a mode of social *praxis*, a form of reflection of the world in accordance to conceptual, ethical, aesthetic and other cultural conceptual categories» (D'Amore, Radford & Bagni, 2006).

This connection knowledge-social *praxis* is a crucial point, from the educational point of view, too, and several issues ought to be considered: for instance, what do we mean by "pupils' minds"? More generally, can we still consider our mind as a "mirror of nature" (following Rorty, 1979), and make reference to our "inner representations" uncritically? R. Rorty underlines the crucial importance of "the community as source of epistemic authority" (Rorty, 1979, p. 380), and states: «We need to turn outward rather then inward, toward the social context of justification rather than to the relations between inner representations» (Rorty, 1979, p. 424).

Let us turn back to our previous remarks. We can state that surely the presence of a contradiction in a logical theory, and particularly its possible destructive use, is a potentially dangerous trap: but in the examined case it is a well-marked one. Of course it is important to keep such a danger in mind, and this is, in a certain sense, what happens in didactical practice: «The pernicious thing is not, to produce a contradiction in the region in which neither the consistent nor the contradictory proposition has any kind of work to accomplish; no, what *is* pernicious is: not to know how one reached the place where contradiction no longer does any harm» (Wittgenstein, 1956, III, § 60).

We previously noticed that a mathematical proposition expresses a grammatical rule and is necessarily connected to a decision to be taken into a community. Now, in a community of practice (e.g. in the classroom or, more generally, in the educational "custom", defined as a set of compulsory practices induced by habit, often implicitly: Balacheff, 1988, p. 21) we can decide to follow a particular rule that forbids the derivation of arbitrary sentences from a contradiction (or, according to Kripke's perspective, we say someone is obeying the considered rule when he/she does not actually derive arbitrary sentences from the contradiction); so, by that, we really "reached the place where contradiction no longer does any harm". Now we must know "how" we reached this place. For instance, our decision must be framed within a social and cultural context, whose features can have a relevant role (let us notice that Wittgenstein himself explicitly states that a conceptual system must be considered with reference to a particular context: Wittgenstein, 1953, XII). Since, in Radford's words, «knowledge is the result of thinking, and thinking is a cognitive social *praxis*» (D'Amore, Radford & Bagni, 2006), our choice can be considered as a form of real and effective knowledge. According to Habermas (1999), the rationality itself has three different roots, closely related the one to the others: the predicative structure of knowledge at an institutional level, the teleological structure of the action and the communicative structure of the discourse. We can state that the rule that forbids the derivation of arbitrary sentences from contradiction is mainly related to the second root.

We previously underlined that a *community view* is fundamental in Wittgenstein's approach. So any "ingenuous set theory" would risk a future change, in the community of practice, of the aforementioned rule that (nowadays) forbids the derivation of arbitrary sentences from a contradiction, with fatal consequences for our theory: «'Up to now a good angel has preserved us from going *this* way'. Well, what more do you want? One might say, I believe: a good angel will always be necessary, whatever you do» (Wittgenstein, 1956, V, § 13).

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