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AIMS AND SCOPE

The Montana Mathematics Enthusiast is an eclectic journal which focuses on mathematics content, mathematics education research, interdisciplinary issues and pedagogy. The articles appearing in the journal address issues related to mathematical thinking, teaching and learning at **all** levels. The secondary focus includes specific mathematics content and advances in that area, as well as broader political and social issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is also interested in research based articles as well as historical, philosophical and cross-cultural perspectives on mathematics content, its teaching and learning.

The journal is accessed from 90+ countries and its readers include students of mathematics, future and practicing teachers, university mathematicians, mathematics educators as well as those who pursue mathematics recreationally. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor preferably in APA style. The typical time period from submission to publication (including peer review) is 7-10 months. Please visit the journal website at http://www.montanamath.org/TMME

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A Global Intellectual Collage

Bharath Sriraman, Editor
The University of Montana

Welcome to another eclectic blockbuster issue of the *Montana Mathematics Enthusiast*. This issue features nine journal articles from all around the world with a wide variety of interesting topics for math enthusiasts, math education researchers, teachers and students. The geographic range of the authors and the sheer magnitude of perspectives presented by the articles both embodies the spirit of the journal and again testifies to the benefits of open access for the exchange of ideas across institutional and national boundaries.

Some statistics of the growth of the journal have been requested by contributors and readers which are now candidly presented. In the time period May 2005-06, 46 manuscripts were received of which 16 were published in the two issues (vol3, no.1 and vol3, no.2). The acceptance rate of manuscripts after peer review is roughly 33% and the average time period from submission to publication is 6-10 months. Contributors should note that proof based math articles and mathematics education articles using esoteric statistical techniques take much longer to be reviewed in comparison to other manuscripts because of the difficulty of finding referees competent in highly specialized areas of research. Prospective authors should consult the aims and scope of the journal to determine whether our journal is an appropriate outlet for their work.

The journal has recently received indexing in *Academic Search Complete*, a new EBSCO product, in addition to being listed in the approximately 300 library directories worldwide. Access statistics continue to be amazing. Interested readers can look at the journal statistics given at the end of this issue for details on places from which our journal is being accessed.

Another issue that is need of discussion and serious consideration is the fact that mathematics education research is of growing interest in regions in the world historically under-represented in numerous publications. We would like to cultivate this growing interest and take steps to remedy this unfortunate situation for these researchers. From many of the e-mails and manuscripts I have received for the journal, the two main issues that have come up are: (1) Providing language support to non-native English speakers on promising manuscripts; (2) Providing current research literature on topics being researched and literature on advances in methodologies for researching problems. To date we have been able to provide this type of support for authors from under-represented countries and will continue to do so. Our long term goal is to use the journal as a platform to bridge schisms between countries on the cutting edge of advances in mathematics education and those attempting to get there. I will take this opportunity to thank members of the editorial board who have been particularly sympathetic to this vision and given of their time to help potential authors.

This journal issue contains mathematics education research articles that employ very interesting frameworks and methodologies. In the article by Martina Janáčková and Jaroslav Janáček, the combinatorial thinking of a high school student in the Slovak Republic is analyzed via the use of four isomorphic problems and by then carefully delineating student strategies by the method of "atomic analysis" developed by Milan Hejný. The method of "atomic analysis" examines every

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nuanced detail in the written work of a student and allows researchers to broaden the range of possible strategies employed by students. In this paper, these researchers describe ten different strategies used by a student on these combinatorial problems. The paper by Giorgio Bagni (Italy) reflects on 50 years since the publication of Ludwig Wittgensteins landmark work *Bemerkungen über die Grundlagen der Mathematik.* Bagni writes on the creative power of language, and how the language itself is embedded into the rest of human activities. Readers interested in the foundations of mathematics will find Bagni's exposition of interest. Birch Fett (Montana) writes a historical expository paper on the golden ratio and its repeated occurrence in nature, in art, and in music. Fett also points out to various non-European civilizations which were aware of the golden ratio. This paper also provides some unusual modern connections of the golden ratio with day to day life. The next paper comes from Saudi Arabia, in which Balarabe Yushau examines the influence of blended e-learning on students' attitudes towards mathematics and computers. Yushau's paper reports on the results of two quantitative attitude batteries administered to a random sample of 70 students of the preparatory year program of King Fahd University of Petroleum & Minerals (KFUPM) in Dhahran. Catherine Kelly explores problem solving in elementary classrooms while focusing on how children use (perform tasks) manipulatives and/or tools in problem solving while working on mathematical tasks. Her paper should serve as a useful research based resource for teachers contemplating the "correct" use of manipulatives.

Eddie Gray (UK) and Demetra Pitta-Pantazi (Cyprus) contribute an in depth analysis of the relationship between 8-11 year old students' numerical achievement and their possible disposition towards the construction of particular frames of reference. The paper uses the characteristics of a variety of kinds of images to focus upon frames of reference and explores a possible relationship between children's verbal descriptions of concrete and abstract nouns and the different ways they respond to aspects of elementary arithmetic. This paper provides an exemplary application of Marvin Minsky's framework of frame system theory. The next paper by Om P. Ahuja (Ohio) compares U.S performance on international assessments like TIMSS and PISA to other countries and discusses issues related to the goal of creating world-class high quality mathematics education for all K-12 American students. In particular, the author chooses to apply the success story of Singapore to the U.S context. The editor is interested in reactions to this paper from those interested in weighing the pros and cons of such comparative analyses. Alex Friedlander and Tzippora Resnick (Israel) contribute a paper on the use of sunrise and sunset data with ninth graders and present numerous pedagogical possibilities with the use of such activities. The final paper by Iliada Elia (Cyprus) and Panayotis Spyrou (Greece) makes use of implicative statistical analyses to propose a triarchic conceptual-semiotic model of the concept of function, which describes students' thinking and understanding of the notion of function across different modes of representation. The instrument used by these researchers provides interesting recognition tasks.

The last few pages provide worldwide access details of the journal. On a concluding note, we are still on schedule to release a special monograph issue of the journal focused on mathematics education and social justice (vol.3,no.3) in December 2006. Thank you for your continued support and input. As usual commentaries, book reviews and volunteers for refereeing are always welcome.

A classification of strategies employed by high school students in isomorphic combinatorial problems

Martina Janáčková[1]
Jaroslav Janáček[2]

Abstract: *The aim of the paper is to discuss some aspects of the combinatorial thinking of high school students. We took one student - Jane and gave her 4 isomorphic testing problems. Then we tried to classify the different strategies the students took in their solutions.*

Keywords: combinatorics, high school students, strategies

1. Introduction

Combinatorics plays an important role in school mathematics. This theme has been recurrent in the mathematics education literature (Fischbein & Gazit, 1988; English, 2005; Lesh & Heger, 2001; Muter 1999; Sriraman & English, 2004), as well as numerous curricular documents worldwide (NCTM, 1991, 2000). Among the most influential is work of Kapur (1970) who called for incorporating enumerative combinatorics in the school curriculum. He elicited the following reasons to justify the teaching of elementary combinatorics in schools:

(1) The independence of combinatorics from Calculus facilitates the tailoring of suitable problems for different grades and usually very challenging problems can be discussed with pupils so that they discover the need for more "sophisticated" mathematics to be created.

(2) Combinatorics can be used to train pupils in enumeration, making conjectures, generalization and systematic thinking; it can help the development of many concepts, such as equivalence and order relations, function, sample, etc.

(3) Many applications in different fields can be presented.

All these reasons justify the interest in improving the teaching of the topic. Nevertheless, students' approaches to combinatorial problems are known for a high occurrence of mistakes (Batanero, Navarro-Pelayo & Godino, 1997; English, 1993, 1998, 1999). These studies suggest that teachers pay attention to the nature of mistakes made by students in combinatorial problems and facilitate students' overcoming these mistakes by providing alternative isomorphic problems. They therefore argue that the teachers' goal should include not only attending to students' mistakes but also helping students to arrive at correct solution. It has been argued that discerning the origin of mistakes can help the teacher to understand how to support students' further learning. To understand the thinking of students, it is important to answer following two questions:

(1) Which strategies[3] are chosen by the student?

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[3] "Strategies are goal-directed operations employed to facilitate task performance." (Bjorklund, 1990)

(2) Why did the student chose a particular strategy?

The second question is deep and beyond the scope of this paper. However our intention here is to deal with the first question and create research-based implications for future research, which will enable us to answer the second question.

2. Theoretical Background

The research reported in this paper is grounded within the extant literature on this topic. In the massive literature review conducted by Sriraman & English (2004) on the state of research in the domain of combinatorial reasoning , they noted the following:

Piaget and Inhelder (1951) viewed combinatorial thinking as an aspect of the stage of formal operations. They characterized combinatorial reasoning as the capacity to determine all the possible ways in which one could link a given set of base associations with each other. Batanero, Navarro-Pelayo, and Godino (1997) provided a simple and highly illustrative account of Piaget and Inhelder's thesis on combinatorial reasoning: Given a problem where a set of objects are required to be arranged in all possible ways, children at the pre-operational stage use random listing procedures, without having an explicit systematic strategy. At the concrete operational level, children use trial and error strategies and are capable of devising "empirical procedures with a few elements" Finally at the stage of formal operations "adolescents discover systematic procedures of combinatorial construction, although for permutations, it is necessary to wait until children are 15 years old" (Batanero et. al, 1997). (Sriraman, B & English, L., 2004, p. 183)

Although Piaget's studies provided powerful insights into the development of combinatorial understanding, the materials that were used and the accompanying instructions were too scientific and abstract (Carey, 1985) for children. This would likely have masked the participating children's abilities in the combinatorial domain. Later research, which employed child-appropriate materials and meaningful task contexts, indicated that young children are able to link items from discrete sets in a systematic manner to form all possible combinations of items (e.g., English, 1991; 1992).

In one such study (English, 1991), 50 children aged between 4.5 years and 9.8 years were individually administered a series of 7 novel tasks that involved the dressing of cardboard toy bears (placed on stands) in all possible different outfits, with an outfit comprising a colored top and a colored pair of pants (or same-colored tops and skirts with different numbers of buttons, for two of the tasks). The findings indicate that, given an appropriate context, children are able to produce independently a systematic procedure for forming m x n combinations prior to the stage of formal operations postulated by Piaget and Inhelder. (Sriraman, B & English, L., 2004, p. 185-186)

Maher and her colleagues (Maher & Martino, 1996a, 1996b, 1997; Maher & Speiser, 1997; Martino & Maher, 1999; Muter & Maher, 1998; Muter, 1999; Speiser, 1997) conducted a series of longitudinal studies lasting up to ten years in which teaching experiments were set up to investigate the growth of mathematical knowledge via the use of combinatorial problems. The fascinating aspect about these studies was that the researchers focused on a group of students and studied the evolution of their mathematical representations, reasoning, argumentation and methods of proof, starting from grade five through grade twelve. The researchers in these studies typically used two or more related problems that were conducive to the formation of isomorphic mathematical structures. It was found that the problem solving strategies of the group of students *who worked on these problems evolved as they worked through these problems, on and off from 1993 to 2000. The representations used by these students became more and more abstract. As fourth-graders, these students discovered properties of combinations with reference to the given problems. The properties of combinations, for these students, grew from very concrete images, such as towers and pizzas (Maher, 1993; Maher & Speiser, 1997; Maher & Martino, 1996a & b; Maher & Kiczek, 2000). However, as tenth-graders, they were able to link these concrete notions to abstract notions of combinations and binomial coefficients found in Pascal's triangle. These findings do not simply confirm the findings of Piaget but also reveal how the development of combinatorial reasoning can "accelerate" from grades four to ten. The Piagetian model spans an eleven-year time period, whereas the findings of the longitudinal studies conducted by Maher and her colleagues indicate that with appropriate instructional scaffolding, students' combinatorial thinking can evolve into sophisticated structures in only seven to eight years! It should be noted that this rapid development is dependent on the use of appropriate tasks in order to facilitate this development in a much shorter time span.* (Sriraman, B & English, L., 2004, p. 184)

Another important finding of these studies was that there was a relationship between "carefully monitoring students' constructions leading to a problem solution" and teacher questioning at appropriate stages of problem solving, which challenged the students to pursue general solutions (Martino & Maher, 1999, p.53). The findings reported by Maher and her colleagues validate the Piagetian notion of how combinatorial reasoning evolves in problems requiring a set of objects to be arranged in all possible ways. These studies revealed that students' strategies evolved from random listing strategies and other trial and error or "empirical procedures" (Davydov, 1996) as fourth-graders, to systematic counting strategies as tenth-graders. This compares with the findings of English (1991, 1992), except in her studies, cited earlier, the children developed sophisticated strategies across a set of tasks within the period of task administration. Increasing notational sophistication, a disposition to think abstractly, the ability to generalize and an affinity for constructing proofs characterized the evolving strategies of the students (Maher, 1993; Maher & Martino, 1996a, 1996b, 1997; Maher & Speiser, 1997; Martino & Maher, 1999; Muter & Maher, 1998; Maher & Kiczek, 2000; Speiser, 1997).

3. The Present Study

Given the precedence of types of problems effective for research on combinatorial thinking, we used isomorphic testing problems in this research. However our attempt was not a mere replication of previous research. The research reported in this paper systematizes and synthesizes perspectives on combinatorial thinking to create an effective instrument for the comprehensive classification of combinatorial strategies employed by high school students in a new geographic location (namely the Slovak Republic). This furthers the aim of the mathematics education community to create research-based knowledge generalizable to age groups across geographic locations.

4. Method

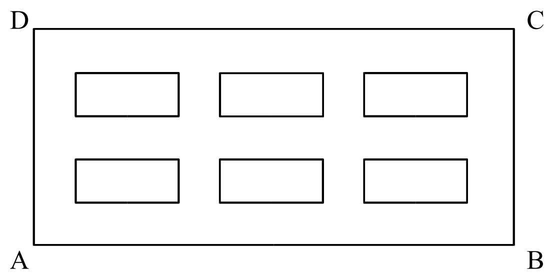
4.1. The Problems of the Study

Siegler (1977) defined the concept *isomorphic problem* (or *isomorphs*) as follows: "Isomorphs are problems that are formally identical but differ in their surface structure". If we expect that the solution of a problem is influenced by numerous parameters, it is necessary to keep all but one of the parameters invariant to establish the influence of the chosen parameter on the solution. In the Slovak Republic, high school students meet the phenomenon of isomorphism during the traditional teaching of combinatorics. They are often required to solve problems that are similar to standard "ground" types of the combinatorial problems.

As a starting point we used 4 isomorphic combinatorial problems:

The Town Problem

There are houses marked as rectangles on the figure. There are streets between them. By how many different ways can we get from the place A to the place C, if we move through the streets of the town only in the directions upwards and to the right?



The Ice Hockey Problem

An ice hockey match finished 2:3. What are the possible partial scores that could have led to the final score of this match? Find all different possibilities.

The Pigeonhole Problem

Write all possibilities in which 5 balls A, B, C, D, E can be placed into 2 pigeonholes u and v such that 2 balls are in the pigeonhole u and 3 balls are in the pigeonhole v.

The Line Problem

In how many ways is it possible to line up 3 ○ and 2 □?

Note that the problems in the instrument are robust because they yield the following isomorphic solutions. Each possibility which is a part of the solution of the preceding problems can be coded by the sequence of **0s** (three symbols) and **1s** (two symbols), where the symbol **0** means:

1. in the Town problem the move "to the right" (see forthcoming extract of the protocol)

2. in the Ice Hockey problem the goal scored by the opposing team

3. in the Pigeonhole problem the selection of the ball into the pigeonhole *v*

4. in the Line problem the symbol: ○

and the symbol **1** signifies:

1. in the Town problem the move "upwards" (see forthcoming extract of the protocol)

2. in the Ice Hockey problem the goal scored the "home" team

3. in the Pigeonhole problem the selection of the ball into the pigeonhole *u*

4. in the Line problem the symbol: □

Another important concept that emerged as will be revealed in the subsequent sections was that of “position”. By “position” we understand:

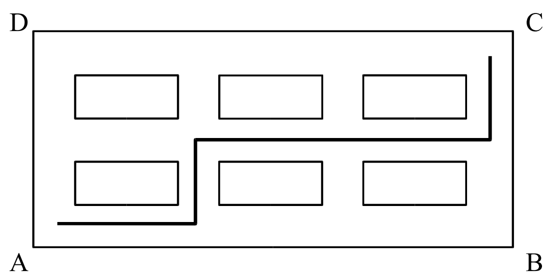
1. in the Town problem the serial number (or running count) of the move on the path from A to C

2. in the Ice Hockey problem the serial number of the goal

3. in the Pigeonhole problem the balls A, B, C, D, E (the ball A = 1st position, …)

4. in the Line problem the place in the line

The following example will help illustrate these nuances:



position	1.p.	2.p.	3.p.	4.p.	5.p.
path	**0**	**1**	**0**	**0**	**1**

4.2. Data Collection

These problems were assigned to 16-year-old Jane, a student in the 2nd year of high school. The student’s solution process was video recorded and later transcribed to create a time-stamped protocol. This protocol served as a basis for the detailed analysis that I will explain in the next section. As an illustration, I present a part of the solution protocol of the Town problem and Jane‘s solutions of the remaining problems.

Extract of the protocol

0.00 J. takes the red crayon and looks at the figure.

0.02 J. marks the path **00011** (a).

0.04 J. looks at the figure.

0.06 J. marks quickly in succession the paths **00110** (b), **00101** (c), **01010** (d).

0.12 J. looks at the figure.

0.14 J. marks quickly in succession the paths **01100** (e), **11000** (f), **10100** (g).

0.20 J. scratches her head and looks at the figure.

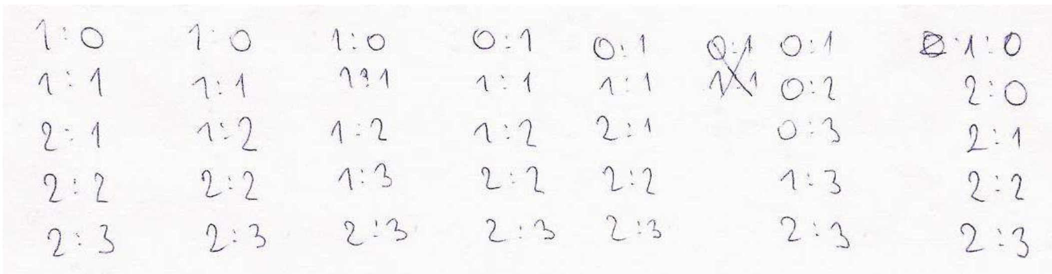
0.30 J. raises the crayon into the air and looks at the figure.

0.46 J. indicates by the crayon the paths **10001** (h), **10010** (i) in the air closely above the figure.

0.50 J. raises the crayon into the air, juggles with her hair and looks at the figure.

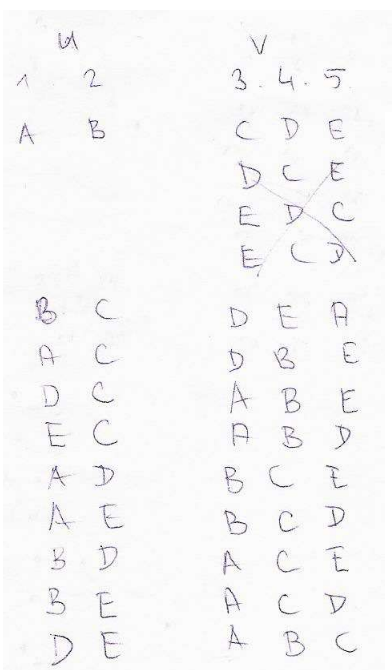
0.76 J.: "There are nine possibilities."

The solutions of the remaining problems



(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
10100	**10010**	**10001**	**01010**	**01100**	**01**	**00011**	**11000**

Figure 1. The Ice Hockey Problem. Each column depicts one of Jane's solutions for how the ice hockey match could have progressed. The chart beneath Jane's solution depicts each match's progress as a sequence of zeros and ones, using the coding explained previously



(a) **11000**

(b) **11**

(c) **01100**

(d) **10100**

(e) **00110**

(f) **00101**

(g) **10010**

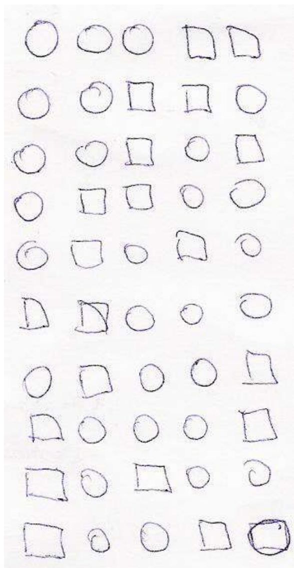
(h) **10001**

(i) **01010**

(j) **01001**

(k) **00011**

Figure 2. The Pigeonhole Problem



(a) **00011**
(b) **00110**
(c) **00101**
(d) **01100**
(e) **01010**
(f) **11000**
(g) **01001**
(h) **10001**
(i) **10100**
(j) **10010**

Figure 3. The Line Problem

4.3. Data Analysis

In order to compare the solutions of isomorphic problems presented in varying contexts with the goal of classifying strategies as well as understanding the influence of context on a particular strategy it was first necessary to identify and describe the strategies which the Jane used during the solution of a particular type of problem. To identify the strategies, we used the method of atomic analysis, introduced in work of Hejný (1992). This method consists of a thorough investigation of every detail – every "graphical atom" of the written work of a student. We examined nuanced details of Jane's solutions and characterized particular strategies that she followed when solving the problems. We then identified each strategy on the basis of the changes (i.e. permutations) occurring in the use of symbols within a solution. The next step consisted of describing these strategies. To make the description valid for each solution of the four problems, each permutation was converted to a sequence of zeros **0** and ones **1**. Each of the identified strategies was explained on the example of two associated succeeding permutations of three zeros and two ones. Then the strategy was generalized for any initial permutation.

We will explain a derivation of strategies on Jane's solutions of the Ice Hockey and Line problems.

Our basic assumption when analyzing students' solutions was that high school students create lists of possibilities in accordance to some guiding principle (i.e., not randomly). We also conjectured that the students would use the principle until they exhausted all the possibilities that it allowed them to identify.

When I compared associated running scores of matches (a) and (b) in the Ice Hockey problem, I found the only difference - in the third step: (a) *2:1* (b) *1:2*. If the team, that scored the third goal in (b), were identical to the team, that scored the third goal in (a), the two solutions for the scores would be identical. It looks like that Jane takes over the running scores from the last generated solution up to the point when the next step in the new solution must necessarily differ if the two solutions are not to be identical. If Jane used this principle systematically to guide her generation of different solutions, we should be able to identify it again in Jane's transition from the solution (b) to (c). Matches (b) and (c) differ in the fourth running score for the first time. It supports our hypothesis, because taking over the score *2:2* from (b) would result in generating a solution identical to (b).

Jane used this guiding principle in generating solutions of other problems as well. As an example, the ordering (c) in the problem "Line" generated in accordance to the same principle. Jane takes over the sequence of ○ and □ from the solution (b) up to the symbol in which the two sequences must differ in order to be different.

The following figure depicts the running scores of ice hockey matches (a), (b), (c) coded into the sequence of zeros and ones.

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**
	↓	↓			
(b)	**1**	**0**	**0**	**1**	**0**
	↓	↓	↓		
(c)	**1**	**0**	**0**	**0**	**1**

This coding transcends the context of the original ice hockey problem and can be interpreted in the context of any of the isomorphic problems used with Jane. For example, in the Line problem the previous coded sequences would represent following solutions:

□ ○ □ ○ ○

□ ○ ○ □ ○

□ ○ ○ ○ □

The above-mentioned guiding solution principle that Jane used in generating solution (b) in the ice hockey problem can then be described as follows: *If the change from **1** to **0** doesn't occur at the position 3, we would not arrive at the solution different from the preceding solution (a).*

This guiding principle can be applied to any initial sequence to generate a new solution. We will call this generalized principle a *Strategy of a constant beginning.* This strategy can be executed in the following way

(1) Start copying the initial sequence from the left

(2) Identify the "critical" position, that is, the right-most position on which the new sequence can no longer be identical to the initial sequence if the two sequences are not to be identical.

(3) Change the symbol on the "critical" position and finish the new sequence accordingly

5. Results

In Jane's solutions we discovered 11 strategies. By (x)/(y) we mean that the model for creation of the permutation (y) was the permutation (x). By (x-y) we will denote all the permutations (x), (x+1), ..., (y).

1. Strategy of exhausted subset

We look for a new strategy because we have exhausted the subset of permutations with a common feature that we could have enumerated using the preceding strategies. We present several such examples:

I. All the permutations with a common prefix of a certain length have been found (i.e. all the permutations that are identical up to a certain position). For example: In case of the permutations (a), (b), (c) in the figure 1. it is about all the permutations, that begin with the symbols **10**.

II. All the permutations whose progress up to a certain position is based on the regular alternation of **1s** and **0s** have been found. For example: In case of the permutations (a)-(e) in the figure 1. it is about all the permutations, where **1**s and **0**s alternate in the first two positions in any order (i.e. **01** or **10**). Although one of such permutations is missing (**01001**), the set is considered to be exhausted, since it is not possible to obtain the missing permutation using the strategies 3[(a)/(b)] 3[(b)/(c)] 1[(a-c)/(…)] 2[(a-c)/(d-f)] 5[(a)/(d)] 5 [(b)/(e)] 5[(c)/(f)].

III. All the permutations whose progress begins with a sequence of one of the symbols (either **1** or **0**) and continues with a sequence of the other symbol have been found. For example: figure 1. – the permutations (g), (h).

IV. All the permutations having the symbol **1** in a certain position have been found (see the strategy of a constant element). For example: In case of the permutations (c)-(f) in the figure 2. it is about all the permutations that have the symbol **1** it the 3rd position.

V. All the permutations that contain all possible arrangements of two symbols **1** in given positions have been found. For example: In case of the permutations (a), (c), (d) in the figure 2. it is about all the permutations that have symbols **1** located in any two of three positions (1, 2 and 3).

This strategy is present in the solutions of all problems.

2. Group strategy

A preceding subset of permutations (with two elements at least) is used as a model for creating new permutations using some of the presented strategies. We will refer to this subset as to a *model group*.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.		1.p.	2.p.	3.p.	4.p.	5.p.		1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**	(b)	**1**	**0**	**0**	**1**	**0**	(c)	**1**	**0**	**0**	**0**	**1**
	∇	∇	∇	∇			∇	∇	∇	∇							
(d)	**0**	**1**	**0**	**1**	**0**	(e)	**0**	**1**	**1**	**0**	**0**						

∇- the change to the other symbol

The permutations (d) and (e) have been created from the permutations (a) and (b) in mentioned order by the symmetry strategy. The permutation (c) has not been used as a model for creation of an additional permutation because it would be identical to (e).

This strategy has been used in co-operation with other strategies in the solutions of the problems „Town“ (figure 4.; (d-e)/(f-g), see extract of the protocol), „Ice Hockey“ (figure 1.; (a-c)/(d-f), see the symmetry strategy) and „Line“ (figure 3.; (b-c)/(d-e), see the parallelism strategy).

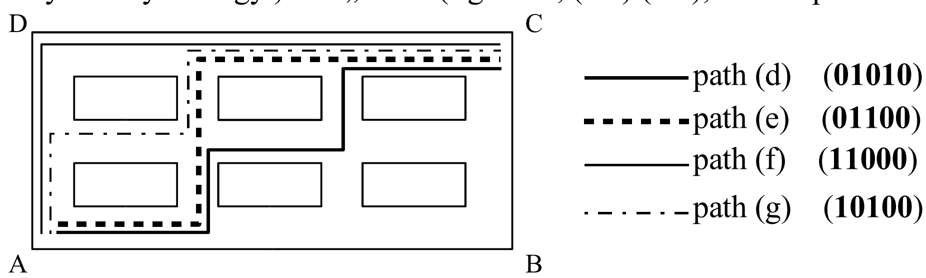


Figure 4. The Town Problem

3. Strategy of a constant beginning

The progress remains identical up to „the highest possible“ position (it is such a position that if the symbol in it is not changed, the entire permutation will have to be identical to the model).

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**
	↓	↓			
(b)	**1**	**0**	**0**	**1**	**0**
	↓	↓	↓		
(c)	**1**	**0**	**0**	**0**	**1**

If the symbol in the position 3 (b) or in the position 4 (c) is not changed from **1** to **0** , the symbols in the higher positions will also have to remain unchanged, leading to the same permutation.

This strategy is present in the solutions of the problems „Town“ (figure 5.; for example (a)/(b), (b)/(c)), „Ice Hockey“ (figure 1.; (a)/(b), (b)/(c)) and „Line“(figure 3.; for example (a)/(b), (b)/(c)).

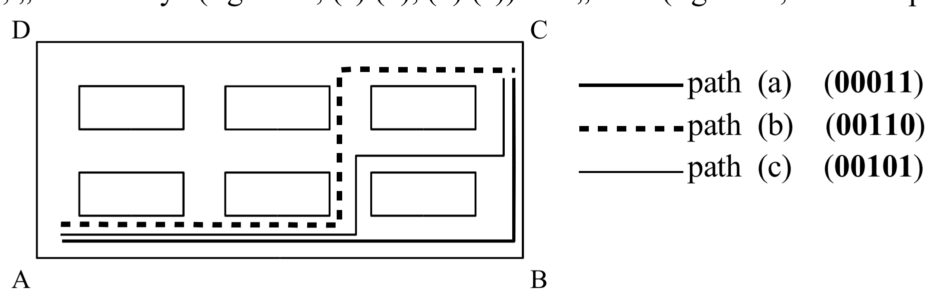


Figure 5. The Town Problem

4. Strategy of the same number of the permutations in groups

If a subset of permutations derived from a model group using the group strategy has less elements than the model group, other permutations are added to it to make the number of elements equal to the number of elements of the model group. These additional permutations are chosen so that all permutations in the resulting subset share a common feature that distinguishes them from the permutations of the model group.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**
(b)	**1**	**0**	**0**	**1**	**0**
(c)	**1**	**0**	**0**	**0**	**1**
(d)	**0**	**1**	**0**	**1**	**0**
(e)	**0**	**1**	**1**	**0**	**0**
(f)	**0**	**1**			
(g)	**0**	**0**	**0**	**1**	**1**

Because (c) cannot be used as a model for creation of (f) (see the group strategy example), the group of the permutations beginning with the sequence **01** is exhausted, but it has less elements (by one) than its model group (a-c). A random permutation beginning with the symbol **0**(just like (d) and (e)) is added to the group. The feature that distinguishes the new group from the model group is the symbol in the first position in this case.

This strategy is present in the solution of the problem „Ice Hockey"(figure 1.; (a-f)/(g)).

5. Strategy of symmetry

The symbols **1** are replaced with **0**s and the symbols **0** are replaced with **1**s in all positions up to the position where this kind of change is no longer possible because the exact number of **0**s and **1**s in the permutation is given. The remaining positions are filled with **0**s.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**
	∇	∇	∇	∇	
(b)	**0**	**1**	**0**	**1**	**0**

The symbol **0** in the 5th position cannot be replaced with the symbol **1**, because the two symbols **1** have already been used.

This strategy is present in the solution of the problem „Ice Hockey" in co-operation with the group strategy (figure 1.; (a)/(d), (b)/(e), (c)/(f)).

6. Strategy of parallelism

All symbols **0** move by one position to the right (we will denote this strategy as the strategy of parallelism **0**R), or to the left (strategy of parallelism **0**L), and the unoccupied positions are filled up with symbols **1**. If a symbol cannot be moved in the first step (because it is in the first or the last position respectively), it remains in its current position and only the other symbols move as described. We define the strategies of parallelism **1**R and **1**L for the movement of symbols **1** by analogy.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**0**	**1**	**0**	**1**	**0**
		↙		↙	
(b)	**1**	**0**	**1**	**0**	**0**
	↓		↙		
(c)	**1**	**1**	**0**	**0**	**0**

The symbols **1** move from the 2nd and the 4th position (a) to the 1st and the 3rd position (b). The remaining positions (2nd, 4th and 5th) are filled with with **0**s. Because the further movement of the symbol **1** in the first position to the left is not possible, it remains in its current position and the other symbol **1** moves from the 3rd position to the 2nd position.

This strategy is present in the solutions of the problems „Town" (figure 4.; (d)/(g), (e)/(f))) and „Line" (figure 3.; (b-c)/(d-e)) in co-operation with the group strategy.

7. Strategy of a constant element

One of the symbols **1** remains in its position, the other one takes a random position of the remaining ones. We shall think about this strategy only in the case when the subset of permutations having the symbol **1** in a certain position is exhausted in a continuous sequence of steps (see the strategy of exhausted subset, example IV). If we considered only two successive permutations regardless of the context, we would identify other strategies as well. For this reason we consider it necessary to introduce a requirement that, if a subset is exhausted (in the sense of the strategy of the exhausted subset, case IV), we will consider it to be exclusively according to the strategy of a constant element. As an exception, if there is a strategy leading to exhaustion of the same subset of permutations as the strategy of a constant element, we shall consider them both (or all of them if there are more such strategies).

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**0**	**1**	**1**	**0**	**0**
			↓		
(b)	**1**	**0**	**1**	**0**	**0**
			↓		
(c)	**0**	**0**	**1**	**1**	**0**
			↓		
(d)	**0**	**0**	**1**	**0**	**1**

One of the symbols **1** remains in the 3rd position, while the other one progressively occupies all remaining positions. It is evident from the sequence of the individual permutations that the strategy of a constant element is used exclusively, although we could identify also the strategy of constant beginning between permutations (c) and (d).

This strategy is present in the solutions of the problems „Town" (figure 6.; (e)-(h)), „Pigeonholes" (figure 2.; (d)/(e), (e)/(f)) and „Line" (figure 3.; (f)/(h), (h)/(i), (i)/(j)).

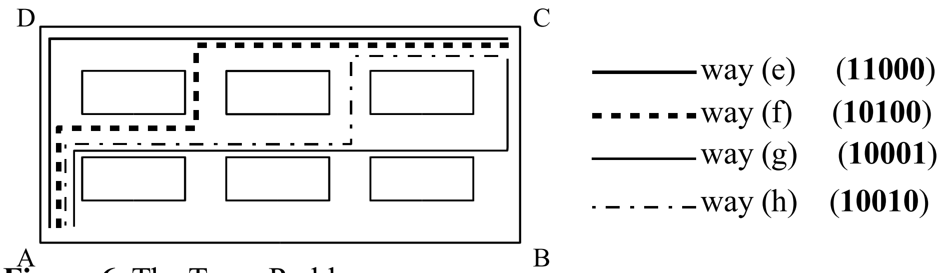


Figure 6. The Town Problem

8. Strategy of complement of all arrangements

A subset of permutations containing all except one possible arrangements of the two symbols **1** in given positions is completed with the missing permutation to form an exhausted subset in the sense of the strategy of exhausted subset, case V.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**1**	**0**	**0**	**0**
(b)	**0**	**1**	**1**	**0**	**0**
(c)	**1**	**0**	**1**	**0**	**0**

The permutations (a) and (b) represent two elements from the three element subset of the permutations that include all arrangements of two symbols **1** in the 1st, 2nd and the 3rd position. The missing permutation is added.

This strategy is present in the solutions of the problems „Town“ (figure 5.; (a-b)/(c)) „Pigeonholes“ (figure 2.; (a-c)/(d)) and „Line“ (figure 3.; (a-b)/(c)).

9. Strategy of the odometer[4]

The principle of the odometer is already mentioned in the papers by L. D. English (1993): „This pattern is so named because of its similarity to the odometer in a vehicle.“ We have modified the description of this principle to correspond to the definitions of our problems because our problems and those in the cited papers differ. One of the symbols **1** remains in the position *x* (called constant element) while the other one progressively occupies all remaining positions from the lowest one to the highest one, without repeating previously discovered permutations. After exhausting all possibilities, next position (x+1) is chosen for the constant element and the process repeats. The strategy ends when all possibilities for the choice of the constant element position are exhausted. If we considered only two successive permutations regardless of the context, we would identify other strategies as well. For this reason we consider it necessary to introduce a requirement that, if a subset is exhausted according to this strategy, we will consider it to be exclusively according to this strategy, and we will not take the other possible strategies into account.

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[4] We mean the distance counter in a vehicle.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**1**	**0**	**0**	**0**
(b)	**1**	**0**	**1**	**0**	**0**
(c)	**1**	**0**	**0**	**1**	**0**
(d)	**1**	**0**	**0**	**0**	**1**
(e)	**0**	**1**	**1**	**0**	**0**
(f)	**0**	**1**	**0**	**1**	**0**
(g)	**0**	**1**	**0**	**0**	**1**
(h)	**0**	**0**	**1**	**1**	**0**
(i)	**0**	**0**	**1**	**0**	**1**
(j)	**0**	**0**	**0**	**1**	**1**

The constant element remains in the 1st position, the second one of the symbols **1** progressively occupies the 2nd to 5th position. The constant element remains in the 2nd position, the second one of the symbols **1** progressively occupies the 3rd to 5th position. It cannot occupy the 1st position, because this permutation would be identical to (a). The strategy finishes by the occupation of the 4th position by the constant element, because there are no possible positions for the second symbol **1**, when the constant element occupies the 5th position, that would yield a new permutation.

This strategy is present in the solution of the problem „Pigeonholes" (figure 2.; (f)/(g), (g)/(h),(h)/(i), (i)/(j), (j)/(k)).

10. Strategy of rotation

The new permutation is created by rotating the preceding one by 180°.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**1**	**0**	**0**	**0**
(b)	**0**	**0**	**0**	**1**	**1**

A statement of the type:

Jane: „Here, when one (team) scored in a row." (The common description of both permutations (a) and (b) in the „Ice Hockey" problem) plays an important role in the identification of this strategy in a special case because if the order of the permutations was 00011, 11000, we could identify the used strategy as the strategy of symmetry. However, the statement indicates that it is not the progress of the model that is important for the respondent, but rather the fact that the result is to be a permutation rotated by 180°.

This strategy is present in the solution of the problem "Ice Hockey" (figure 1.; (g)/(h)).

11. Strategy of complement of the exhausted subset

While in the preceding steps the exhaustion of certain subset (the minimal number of the components is 5) of permutations with some common symbol occurred, the permutations are looked for with such common character which for it is valid:

Having exhausted a subset of permutations with a common feature (containing at least 5 elements) we look for a subset of permutations with another common feature such that:

1. the exhausted subset and the looked for subset of permutations are disjunct;
2. the exhausted subset and the looked for subset form a set of all permutations.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**0**	**0**	**0**	**1**	**1**
(b)	**0**	**0**	**1**	**1**	**0**
(c)	**0**	**0**	**1**	**0**	**1**
(d)	**0**	**1**	**0**	**1**	**0**
(e)	**0**	**1**	**1**	**0**	**0**
(f)	**1**	**1**	**0**	**0**	**0**
(g)	**1**	**0**	**1**	**0**	**0**
(h)	**1**	**0**	**0**	**0**	**1**
(i)	**1**	**0**	**0**	**1**	**0**

The permutations (a)- (g) form the exhausted subset of permutations with the difference of the positions occupied by the symbols **1** equal to 1 or 2. The permutations are looked for that have the difference of the positions occupied by the symbols **1** equal to 3 or 4.

This strategy is present in the solutions of the problems "Town" (see the extract of protocol (a-g)/ (h-i)) and "Line" (figure 3.; (a-e)/(f, h-j)).

Overview of strategies used in particular problems

permutation	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)
"Ice Hockey"	3.	3., 1.I	2., 5.	5.	5., 1 I	4.	1.II, 10., 1.III			
"Pigeonholes"			8., 1.V	7.	7., 1.IV	9.	9.	9.	9.	9.
"Line"	3., 1.III	3., 1.I or 8.	3. or 2., 6.	6,1.I or 3.	2., 6. or 11. or 3.	3., 1.I	3. or 7. or 11.	7. or 11.	3., 1.I or 7., 1.IV or 11.	
"Town"	3., 1.III	3. or 8.	1. I	2., 6.	2., 6.	6.	3. or 7. or 11	3. or 7. or 11.		

6. Conclusions

Event though the presented problems are isomorphic, students have used different strategies to solve them. This observation correlates to the observations of other authors (Tőrner, 1987; Bauersfeld, 1985; English, 1999; Hefendehl-Hebeker&Törner, 1984; Hesse, 1985; …). It would be interesting in a future research to find which aspects in the problem context influence the strategy selection and the completion of the solution.

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Numbers and Polynomials

50 years since the publication of Wittgenstein's *Bemerkungen über die grundlagen der mathematik* (1956): mathematical and educational reflections

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Abstract: *According to L. Wittgenstein, the meaning of a mathematical object is to be grounded upon its use. In this paper we consider Robinson theory Q, the subtheory of first-order Peano Arithmetic PA; some theorems and conjectures can be interpreted over one model of Q given by a universe of polynomials; with respect to nonconstant polynomials some proofs by elementary methods are given and compared with corresponding results in the standard model of PA. We conclude that the creative power of the language can be pointed out in how the language itself is embedded into the rest of human activities, and this is an important track to follow for researchers in mathematics education.*

Keywords: Peano Arithmetic; Robinson Arithmetic; Wittgenstein

1. Introduction

> Knowledge in mathematics: Here one has to keep on reminding oneself of the unimportance of the 'inner process' or 'state' and ask «Why should it be important?» What does it matter to me? What is interesting is how we *use* mathematical propositions.
>
> Ludwig Wittgenstein (1969, n. 38)

Although, from the ontological point of view, the "Platonic" conception in which mathematical objects exist independent of their representations cannot be stated uncritically, mathematical objects do not exist as real, concrete objects and mathematical knowledge can only be attained through representations. This fact leads us to consider the so-called *cognitive paradox of mathematical thought*, pointed out by R. Duval, who underlines that although mathematical learning is conceptual, any activity involving mathematical objects takes place only through semiotic representations (Duval, 1995). As a consequence, it is necessary to make a clear distinction between the mathematical object (if it exists) and its different semiotic representations (Otte, 2001, p. 33).

However the plurality of representations of a mathematical object can imply the recognition of a plurality of objects: how can we coordinate this diversity of objects taking into account that the professional mathematician sees a unique object? It would be possible, according to Ludwig Wittgenstein (1889-1951), to accept the grammatical intra-discursive nature of mathematical objects: the doctrine of "meaning as use" (Wittgenstein, 1953, n. 43) is connected to the key concept of "context embeddedness", where the term is understood not merely as the physical environment of a linguistic utterance, but is referred to a wider cultural context (McGinn , 1984; McDonough, 1989; Godino & Batanero, 1997). A philosophical problem facing epistemological realism after the "linguistic turn" can be summarized in the following question: how can the assumption that there is an independently existing world be compatible with the linguistic position according to which we cannot have unmediated access to reality? (Habermas, 1999).

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Let us consider a quotation from Wittgenstein's *Philosophical Investigations*, published posthumously in 1953 (*Philosophische Untersuchungen* in German):

"Perhaps you say: two can only be ostensively defined in this way: «This *number* is called 'two'». For the word «number» here shews what place in language, in grammar, we assign to the word. But this means that the word «number» must be explained before the ostensive definition can be understood. – The word «number» in the definition does indeed shew this place; does shew the post at which we station the word. (…) Whether the word «number» is necessary in the ostensive definition depends on whether without it the other person takes the definition otherwise than I wish. And that will depend on the circumstances under which it is given, and on the person I give it to. (…) So one might say: the ostensive definition explains the use –the meaning– of the word when the overall role of the word in language is clear" (Wittgenstein, 1953, nn. 29-30).

So the meaning of a mathematical object (for instance of numbers) can be grounded upon its use. But can we always consider a particular use of the mathematical words (for instance with regard to Arithmetics) as completely natural? In this paper we shall consider two non-isomorphic models of an arithmetic theory in order to investigate the following issue: apart from representation registers employed, is this philosophical approach embodied into mathematics itself? More generally, what is the relationship between Mathematics and some crucial philosophical issues regarding the meaning? For instance, can we state that "there are infinitely many couples of primes p, q such that $q = p+2$" (the celebrated Twin Prime Conjecture) without possible misunderstandings?

Previous considerations are related to teaching undergraduate mathematics: as a matter of fact the theoretical analysis of what it means to understand a concept and how understanding can be constructed by the learner (Asiala & Al., 1996) requires an investigation of some fundamental epistemological and, more generally, philosophical issues. So, in our opinion, problems dealing with the meaning of a mathematical object and of its expression are relevant to mathematics education, both for teachers and for students. Of course our aim is not to provide complete answers to these fundamental questions; but a reflection upon the language (in particular, the mathematical language) can be based upon some considerations, for instance about Mathematical Logic and Arithmetics. So we are going to propose a contribution to the debate based upon the discussion of a mathematical example that will be introduced and discussed by elementary methods.

2. Arithmetic theories and models

Natural numbers, or counting numbers, are grounded on our common everyday experience and their perception and interpretation can be considered as a very important aspect of our common sense (nevertheless some basic remarks can be considered: Wittgenstein, 1969). But can we state that this interpretation is always totally clear? More precisely, can we state that the meaning attributed to the common arithmetical language is independent from interpretation? We shall try to reflect about these questions: in other words, we are going to investigate if Arithmetics itself can be interpreted according several (theoretical, mathematical) point of views.

In this paper we shall consider, by elementary methods, the set **N** of natural numbers and a particular set of polynomials. In order to present and clarify our choice, let us remember some well known considerations about Number Theories[1]. *Robinson Arithmetic* (introduced by Tarski, Mostowski and Robinson in 1953 and usually denoted by Q) is weaker than *Peano*

[1] For a theoretical study of weak Arithmetics see Macintyre, 1987 and the quoted references.

Arithmetic, denoted by *PA* (Mendelson, 1997, p. 128 and p. 187); *Q* can be obtained from *PA* if the induction:

$$\varphi(0)\wedge(\forall y)(\varphi(y)\rightarrow\varphi(s(y)))\rightarrow(\forall y)\varphi(y)$$

(in the language {+, ·, s, 0}, s is the successor function) is replaced by the axiom:

$$(\forall y)(y \neq 0\rightarrow(\exists z)(y = s(z)))$$

that is a theorem in *PA* and can by easily proved by induction.[2]

In order to show the importance of the (mathematical) interpretation of basic arithmetical objects, we shall consider some models of arithmetic theories, in particular models of *PA* and *Q* (Kaye, 1991), by elementary methods.

The set **N** of natural numbers with the addition and the multiplication is the standard model of *PA*, <**N**, +, ·, s, 0>; the existence of non-standard models of *PA* (models non-isomorphic to the standard model) was proved in 1934 by Skolem. While non-standard models of *PA* are not (educationally) simple to be proposed, it is interesting to present models of *Q* non-isomorphic to **N**: for instance, we shall denote by $Z^*[x]$ the set whose elements are 0 and all polynomials with integral coefficients whose leading coefficients are positive: $Z^*[x]$ with the addition and the multiplication is a model of *Q* (Mendelson, 1997, p. 188), <$Z^*[x]$, +, ·, s, 0>.

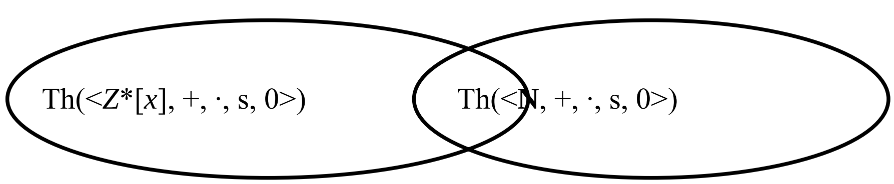
3. A comparison between <$Z^*[x]$, +, ·, s, 0> and <N, +, ·, s, 0>

First of all, let us underline some meaningful differences between the considered models: the paragraphs 3 and 4 will be devoted to this comparison, that will be relevant to the aim of our paper.

As a matter of fact, <$Z^*[x]$, +, ·, s, 0> is not a model of *PA*:

$$(\forall y)(\exists z)(z+z = y \vee z+z = y+1)$$

that can be proved by induction, is not in $Z^*[x]$ (every nonconstant polynomial of $Z^*[x]$, $B(x) = a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ whose coefficients a_n, a_{n-1}, $...+a_1$ aren't all even can be considered as a counterexample)[3]: so considered models <$Z^*[x]$, +, ·, s, 0> and <**N**, +, ·, s, 0> are not elementary equivalent. Let us underline that we shall find true propositions in <$Z^*[x]$, +, ·, s, 0> that are false with reference to <**N**, +, ·, s, 0> (this can be stated theoretically, too: if not, the models <**N**, +, ·, s, 0> and <$Z^*[x]$, +, ·, s, 0> would be equivalent, and this is absurd; see for instance: Chang & Keisler, 1973, p. 32). We shall summarize previous statements in the following picture, where Th(*M*) usually indicates the set of all sentences true in *M*:



[2] The role of the axiom schema of induction and of the phenomenon of incompleteness in *PA* and in subtheories are important fields of contemporary research; see Hájek & Pudlák, 1993, where fragments of *PA* resulting by restricting the induction schema to formulas belonging to a prescribed class are studied.

[3] Every proposition that can be proved in *Q* can be proved in *PA*, too; there are propositions that can be proved in *PA* and that cannot be proved in *Q*; of course a proposition that can be proved in *PA* cannot be confuted in *Q*. If any proposition can be proved in *PA* and can be confuted in *Q*, being *PA* an extension *Q*, then *PA* would be inconsistent.

Of course Th(<**N**, +, ·, s, 0>)∩Th(<$Z^*[x]$, +, ·, s, 0>) ≠ ∅; in fact, it includes the set of all sentences deducible from *Q*.

We noticed that an element of Th(<**N**, +, ·, s, 0>)–Th(<$Z^*[x]$, +, ·, s, 0>) is: $(\forall y)(\exists z)(z+z = y \vee z+z = y+1)$. Later, we shall present an element of Th(<$Z^*[x]$, +, ·, s, 0>)–Th(<**N**, +, ·, s, 0>), too.

4. Order in $Z^*[x]$

According to an axiom of *Q*, the order is defined in $Z^*[x]$ as follows:

$f(x) \le g(x)$ iff (def.) $g(x)-f(x) \in Z^*[x]$
$f(x) < g(x)$ iff (def.) $0 \ne g(x)-f(x) \in Z^*[x]$

We can state some basic properties: if $f(x)$, $g(x)$, $h(x)$ belong to $Z^*[x]$:

if $f(x) \le g(x)$ then $f(x)+h(x) \le g(x)+h(x)$
if $f(x) < g(x)$ then $f(x)\cdot h(x) < g(x)\cdot h(x)$
if $f(x) \le g(x)$ then $f(x)+h(x) \le g(x)+h(x)$
if $f(x) < g(x)$ then $f(x)\cdot h(x) < g(x)\cdot h(x)$ (being $h(x) \ne 0$)

If $f(x)$, $g(x)$, $h(x)$, $h(x)-f(x)$, $h(x)-g(x)$ belong to $Z^*[x]$:

if $f(x) \le g(x)$ then $h(x)-g(x) \le h(x)-f(x)$
if $f(x) < g(x)$ then $h(x)-g(x) < h(x)-f(x)$

As regards the minimum element of $Z^*[x]$, for every $f(x) \in Z^*[x]$: $0 \le f(x)$.

These properties hold in $Z^*[x]$ being provable in *Q*; moreover, the following results are trivial:

- If $f(x), g(x) \in Z^*[x]$, then either $f(x) \le g(x)$ or $g(x) \le f(x)$.
- If $f(x), g(x) \in Z^*[x]$ and $f(x) < g(x) \le f(x)+1$, then $g(x) = f(x)+1$.
- If $f(x) \in Z^*[x]$, $g(x)$ is a nonconstant element of $Z^*[x]$, $f(x) < g(x)$ and $g(x)-f(x)$ is nonconstant, for every *n*, *k* positive integers, it is $f(x)+n < g(x)-k$.

This last property is interesting: by that we present an infinity of couples of elements $f, g \in Z^*[x]$ such that $f<g$ and an infinity of couples of elements $n, k \in Z^*[x]$ such that $f+n<g-k$. Such property holds with reference to $Z^*[x]$, but it does not hold in **N**. So, have we found an element of Th(<$Z^*[x]$, +, ·, s, 0>)–Th(<**N**, +, ·, s, 0>)? The problem is that logical quantifiers are finitary, so we cannot use an infinity of existential quantifiers in the same sentence[4].

Let us now underline an interesting fact: any nonconstant $g(x) \in Z^*[x]$ could be considered as an "infinite" element; in fact, for every natural number *n* we can write $n<g(x)$ (proof is trivial). So in $Z^*[x]$ there are different "infinite" elements, for instance $x < x+1 < x^2 < x^2+1$ and so on.

For every $n \in \mathbf{N}$, $a \in \mathbf{Z}$ it is: $n<x+a$; so we have, in $Z^*[x]$:

$$0 < 1 < 2 < \ldots < x-2 < x-1 < x < x+1 < x+2 < \ldots$$

If by $[x]$ we mean … $x-2$, $x-1$, x, $x+1$, $x+2$ … (a "copy of **Z**"), let us write:

---

[4] Concerning natural numbers, if we want to express that the property P(*n*) holds for an infinity of *n*, we can write, for instance: $(\forall m)(\exists n)(m<n \wedge P(n))$, but a similar expression cannot be now used in $Z^*[x]$ in order to express our statement.

$Z^*[x] = \{\mathbf{N}, [x]\}$

where we state moreover that the copy of **Z** $[x]$ is "adjacent" to **N**. The following results are trivial:

- No $f(x) \in Z^*[x]$ whose degree is 1 and leading coefficient is greater than 1, or whose degree is greater than 1, is such that $n < f(x) < x+a$, $n \in \mathbf{N}$, $a \in \mathbf{Z}$.
- If the degree of $f(x) \in Z^*[x]$ is lower than the degree of $g(x) \in Z^*[x]$, then $f(x) < g(x)$.
- If the degree of $f(x) \in Z^*[x]$ is equal to the degree of $g(x) \in Z^*[x]$ and if the leading coefficient of $f(x)$ is lower than the leading coefficient of $g(x)$, then $f(x) < g(x)$.
- Let $f(x)$, $g(x)$ nonconstant elements of $Z^*[x]$, having the same degree and the same leading coefficient; let n be the maximum degree for which coefficients a_n of $f(x)$ and b_n of $g(x)$ are not equal; if $a_n < b_n$, then $f(x) < g(x)$.

We write $Z^*[x]$ in the following way, with reference to ordered "copies of **Z**":

$Z^*[x] = \{\mathbf{N}, [x], [2x], [3x], [4x] \ldots$
$\ldots [x^2-2x], [x^2-x], [x^2], [x^2+x], [x^2+2x] \ldots$
$\ldots [2x^2-2x], [2x^2-x], [2x^2], [2x^2+x], [2x^2+2x] \ldots [x^3] \ldots\}$

5. Prime elements belonging to $Z^*[x]$

Let us now turn back to the questions proposed in the first paragraph; so we shall consider some propositions in order to point out differences between what happens in **N** and in $Z^*[x]$.

It is easy to interpret constant non-negative polynomials and natural numbers (**N** and the subset of constant elements belonging to $Z^*[x]$ are isomorphic), so **N** is a submodel of $Z^*[x]$ (Chang & Keisler, 1973, p. 21): so every proposition with a single existential quantifier that is true in **N** is of course true in $Z^*[x]$ too, and every proposition with a single universal quantifier true in $Z^*[x]$ is true in **N**, too. These considerations will be important with reference to the rest of our paper.

We shall present some conjectures and frequently we shall consider "prime elements". Let us give the following definition: $p \in Z^*[x]$ is *prime* if it is different from 0 and from 1 and if there are not two elements belonging to $Z^*[x]$, both of them different from 1, whose product is p; so a polynomial is prime if and only if it is irreducible and primitive (i.e. the gcd of its coefficients is 1), too. So we can express $\Pr(y)$ ("y is prime") by:

$$y \neq 0 \wedge y \neq 1 \wedge (\neg(\exists a)(\exists b)(a \neq 1 \wedge b \neq 1 \wedge ab = y))$$

As regards a comparison between numbers and polynomials, some differences are immediately clear: for instance, in $Z^*[x]$ for every integer k the polynomial $x+k$ is prime, while if a natural number $n>2$ is prime, its successor is even so it is not prime. This remark is interesting: in fact, by writing:

$$(\exists y)(y \neq 2 \wedge \Pr(y) \wedge \Pr(y+1))$$

we have found an element of $\mathrm{Th}(<Z^*[x], +, \cdot, s, 0>) - \mathrm{Th}(<\mathbf{N}, +, \cdot, s, 0>)$.

It is trivial to show some arithmetic propositions in $Z^*[x]$ (as regards arithmetic conjectures, see: Guy, 1994). Let us consider the presence of primes in any arithmetic progression (according to a well known theorem proved in 1837 by Dirichlet, if $h>1$ and $a \neq 0$ are relatively prime then the progression: a, $a+h$, $a+2h$, $a+3h$, … includes infinitely many prime numbers: Ribenboim, 1995, p. 205). With respect to polynomials, it is easy to find arithmetic progressions entirely including prime elements; for instance, if h is any integer, $h \neq 0$, all polynomials of the progression x, $x+h$, $x+2h$, $x+3h$, … are prime. It follows, for instance, the version of the Twin Prime conjecture in $Z^*[x]$[5]: it is trivial to verify that there are infinitely many couples of prime elements ($P(x)$; $Q(x)$) belonging to $Z^*[x]$ such that $Q(x) = P(x)+2$ (e.g. $P(x) = x+k$, $Q(x) = x+k+2$, for every $k\in\mathbf{Z}$).

Another interesting remark is referred to prime elements that can be written as n^2+1: are they infinitely many? It is an open problem in **N** (2005). It is trivial to show that there are infinitely many elements $P(x) \in Z^*[x]$ such that $[P(x)]^2+1$ is a prime element of $Z^*[x]$ (e.g. $P(x) = x+k$ for every $k\in\mathbf{Z}$; it follows: $[P(x)]^2+1 = x^2+2kx+k^2+1$ that is prime, being primitive and irreducible: $\Delta(k) = -4 < 0$: Bagni, 2002). A general form of the last conjecture in **N** is the following: if a, b, c are relatively prime, a is positive, $a+b$ and c are not both even and b^2-4ac is not a square, then there are infinitely many primes an^2+bn+c (Hardy & Wright, 1979, p. 19). As regards $Z^*[x]$, it is trivial to prove that if a, b, c are relatively prime, a is positive, b^2-4ac is not a square, then there are infinitely many elements $P(x)\in Z^*[x]$ such that $a[P(x)]^2+bP(x)+c$ is a prime element of $Z^*[x]$ (once again, consider $P(x) = x+k$ for every $k\in\mathbf{Z}$).

It is interesting to consider in $Z^*[x]$ some results of the additive Number Theory. For instance, the well known Lagrange's theorem which states that every natural number is the sum of four squares (see for instance: Nathanson, 1996a, p. 37and 1996b) doesn't hold in $Z^*[x]$: there are elements of $Z^*[x]$ that cannot be expressed as a sum of square elements of $Z^*[x]$ (e.g. any polynomial of $Z^*[x]$ whose degree is 1 cannot be expressed as the sum of squares of $Z^*[x]$).

6. Two great problems: Catalan and Goldbach conjectures

Paragraphs 6 and 7 are devoted to some classical problems: we shall consider them with reference to both the models **N** and $Z^*[x]$.

Let us remember the Catalan conjecture in **N**, which asserts that 8 and 9 are the only consecutive powers (Nathanson, 2000, p. 186); equivalently, it states that the only solution of the equation $x^m-y^n = 1$, being x, y, m, n natural numbers greater than 1, is: $x = n = 3$, $y = m = 2$. To prove or disprove this conjecture was a great problem in Number Theory until its proof, announced by P. Mihailescu in 2002.[6]

Of course we shall not try to prove the Catalan conjecture in $Z^*[x]$ by elementary methods: such proof would imply a proof of the conjecture in **N**, too. However, it is possible to prove that $x^m-y^n = 1$, being $m\geq 2$, $n\geq 2$ natural numbers, has no solution x, y in nonconstant polynomials belonging to $\mathbf{C}(t)$ (Nathanson, 1974). So Catalan conjecture holds for nonconstant polynomials of $Z^*[x]$.

[5] From the formal point of view, let us underline once again that logical quantifiers are finitary, while the Twin Prime conjecture considers the existence of *infinitely* many couples of twin primes; so it must be expressed as follows: $(\forall n)(\exists p)[\Pr(p)\wedge\Pr(p+2)\wedge(p>n)]$ (where $\Pr(m)$ means "m is prime"). It is interesting to remember that we don't know if there are infinitely many twin primes (2005), but in 1919 Brun proved that the sum of the reciprocals of twin primes converges to 1.902160577783278… (it is the so-called Brun's constant).

[6] In 1999, M. Mignotte proved that eventual exceptions to Catalan conjecture (of course, if they exist) would be such that: $m > 7.15\cdot 10^{11}$, $n > 7.58\cdot 10^{16}$ (Peterson, 2000).

In order to consider the Goldbach conjecture in $Z^*[x]$, we must underline that it is a conjecture where there is a universal quantifier[7]: once again, we shall examine only nonconstant polynomials (Bagni, 2002). Let us prove the following result:

PROPOSITION 1. If the nonconstant polynomial $Q(x) \in Z^*[x]$ is not primitive, then there are two prime polynomials $Q_1(x) \in Z^*[x]$, $Q_2(x) \in Z^*[x]$ such that $Q(x) = Q_1(x)+Q_2(x)$.

PROOF. Let us consider a nonconstant and non-primitive polynomial belonging to $Z^*[x]$ (where pq is the gcd of its coefficients and p is prime):

$$Q(x) = pqa_n x^n + pqa_{n-1} x^{n-1} + \ldots + pqa_1 x + pqa_0$$

Let us consider the following polynomials belonging to $Z^*[x]$, being $t \in \mathbf{Z}$:

$$Q_1(x) = x^n + pqa_{n-1} x^{n-1} + \ldots + pqa_1 x - p(pt+1)$$
$$Q_2(x) = (pqa_n - 1)x^n + p(qa_0 + pt + 1)$$

whose sum is $Q(x)$ for every t. We shall show that it is possible to find t such that both polynomials $Q_1(x)$ and $Q_2(x)$ are prime.

For every t, $Q_1(x)$ is irreducible from Eisenstein criterion: the prime p divides its coefficients apart the leading one and p^2 doesn't divide $p(pt+1)$; moreover $Q_1(x)$ is primitive so it is prime.

If it is not $qa_0 \equiv -1 \pmod{p}$, then qa_0+pt+1 is not a multiple of p so Eisenstein criterion can be applied to $Q_2(x)$ too and $Q_2(x)$ is irreducible for every t. Let us show that $Q_2(x)$ is primitive for some t: $(qa_0+1)+pt$ is prime for infinitely many t from Dirichlet theorem (qa_0+1 and p are relatively prime) and t can be chosen such that qa_0+pt+1 is prime and greater than pqa_n-1.

If $qa_0 \equiv -1 \pmod{p}$, so $qa_0 = kp-1$ being k an integer, it is:

$$Q_2(x) = (pqa_n - 1)x^n + p^2(k+t)$$

There are infinitely many t such that $k+t$ is prime and greater than pqa_n-1: so we can find t such that $Q_2(x)$ is irreducible from Eisenstein criterion and primitive. ■

From this proposition (being $p = 2$) it follows that the Goldbach conjecture holds for nonconstant polynomials of $Z^*[x]$, where we call *even* a polynomial such that the gcd of its divisors is even.[8]

We can summarize previous statements in the following figure. With respect to $Z^*[x]$ (and to ordered "copies of **Z**"), we notice that Goldbach conjecture is empirically verified for an initial (finite) set of natural numbers; then its validity is not proved for infinitely many natural numbers; finally, it holds for all nonconstant polynomials of $Z^*[x]$.

$Z^*[x]$	**N**	$[x]$	$[2x]$	$[3x]$	...
Goldbach conjecture	*verified* ...	*it holds for nonconstant polynomials*			

[7] In Goldbach conjecture there is not only a universal quantifier: in fact, it states that for every even integer n greater than 2 there is a couple of primes (p, q) such that $p+q = n$: so there are two existential quantifiers, too: however if n is an integer, p and q are integers, too. As regard experimental verifications, in 1998 Richstein verified Goldbach conjecture up to $4 \cdot 10^{14}$.

[8] Concerning Goldbach conjecture let us indicate Weyl, 1942, Erdös, 1965, Wang, 1984.

7. The Last Fermat Theorem

Another interesting situation can be finally described with reference to the Fermat Last Theorem. It is trivial to extend the theorem from **N** to $Z^*[x]$: if there are three non-zero and nonconstant polynomials $A(x)$; $B(x)$; $C(x)$ belonging to $Z^*[x]$ and a natural number $n\geq 3$ such that $[A(x)]^n+[B(x)]^n = [C(x)]^n$, we can assign a value to x such that $A(x)$, $B(x)$, $C(x)$ are positive (the leading coefficients are positive) so the (proved) Last Fermat Theorem in **N** would not hold: absurd.

Concerning nonconstant elements of $Z^*[x]$, it is possible to prove the Fermat Last Theorem independently from its proof in **N**, too: it is possible to prove that the Fermat equation $a^n+b^n = c^n$ has no (noncostant) polynomial solutions if $n\geq 3$ (Greenleaf, 1969; such equation has solutions in polynomials for $n = 2$, for instance $a = (x^2-1)^2$; $b = (2x)^2$; $c = (x^2+1)^2$: Nathanson, 2000, p. 183).

$Z^*[x]$	**N**	$[x]$	$[2x]$	$[3x]$	...
Fermat, Catalan (Mihailescu)	*proved*	*it holds for nonconstant polynomials*			

8. Final reflections

> One cannot contrast *mathematical* certainty with the relative uncertainty of empirical propositions. For the mathematical proposition has been obtained by a series of actions that are in no way different from the actions of the rest of our lives, and are in the same degree liable to forgetfulness, oversight and illusion. Now can I prophesy that men will never throw over the present arithmetical propositions, never say that now at last they know how the matter stands? Yet would that justify a doubt on our part?
>
> Ludwig Wittgenstein (1969, nn. 651-652)

Let us briefly turn back to the question proposed in the first paragraph of this paper: can we really state that "there are infinitely many couples of primes p, q such that $q = p+2$" with no possible misunderstandings? Previous considerations show that many other similar questions can be proposed.

Of course the answer would require attention and care. By that we do not mean that the mathematical language is surely ambiguous: nevertheless different models of arithmetic theories, in particular some models of *PA* and *Q*, can be considered: and the study of the presented model of *Q* non-isomorphic to **N** points out that, for instance, the sentence "there are infinitely many couples of primes p, q such that $q = p+2$" depends on the particular context.[9]

So how can we "translate" a mathematical statement? What is the meaning to be attributed to the sentence (see paragraph 5) "there is a prime different from 2 such that its successor is prime"? Is it true or false? According to Quine, there are always different ways to distribute functions among words, and this cause the so-called "indeterminacy of translation" (Quine,

[9] The fundamental problem of the meaning can be considered with reference to mathematical theories, too: is it possible to discuss the meaning of *PA* and *Q*? Further research can be devoted to this issue.

1960). As previoulsy noted, this does not lead to state that mathematical words and concepts are meaningless: but can we always raise absolute question of “right” or “wrong” in translating (or interpreting) mathematical language? Different “theories of translation” can be based upon different abnalytical hypotheses (let us remember that, according to Quine, in order for a statement to be analytic, it must be true by definition). And Wittgenstein, too, pointed out that mathematical propositions describe neither abstract entities nor empirical reality (their a priori status is due to the fact that their role is a normative one: Glock, 1996): their certainty is obtained by operations grounded in our actual lives (Wittgenstein, 1969, nn. 651-652), so it depends upon particular facts, that is, upon contexts or situations: words (mathematical words, too) have meaning insofar as they are candidates for use within propositions that have meaning, and propositions are meaningful as used within a context (Morawetz, 1980, p. 59).

As previously pointed out, our aim is not to provide answers to these philosophical questions (Habermas, 1999); but in our opinion, aforementioned remarks can be useful from the educational point of view, too. As a matter of fact, Steinbring underlines that in classroom interactions the use of mathematical language is frequently acquired “by means of social participation, and not (…) according to strict rules” (Steinbring, 2002, p. 10), so we underline the primary importance of an adequate negotiation of meanings between teacher and pupils.[10] Moreover, let us underline that the problem of the meaning is relevant to all representative registers employed (Duval, 1995), being connected to the language itself; let us quote once again Wittgenstein:

> “Instead of producing something common to all that we call language, I am saying that these phenomena have no one thing in common which makes us use the same word for all, – but that they are related to one another in many different ways. And it is because of this relationship, or these relationships, that we call them all «language»” (Wittgenstein, 1953, n. 65).

But our language “did not emerge from some kind of ratiocination” (Wittgenstein, 1969, n. 475), and the origin of a “language game” (in the sense of: Wittgenstein, 1953) is a reaction. So, following Wittgenstein, we can conclude that language is not (just) a code, whose power can be mainly referred to its syntax; its creative power lies in how the language itself is embedded into the rest of human activities (Morawetz, 1980; Shotter, 1996), and the mathematical study of the models of a theory provides an example in order to underline the primary importance of the context. In our opinion, this is an interesting track to follow for researchers in mathematics education.

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[10] Of course this does not mean that if notation and terms are introduced correctly, no misconception will occur.

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An In-depth Investigation of the Divine Ratio

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Abstract**:** The interesting thing about mathematical concepts is that we can trace their development or discoveries throughout history. Most cultures of the ancient world had some form of mathematics, and these basic skills developed into what we now call modern mathematics. The divine ratio is similar in that it was used in many different sections of history. The divine ratio, sometimes called the golden ratio or golden section, has been found in very diverse areas. The mathematical concepts of the golden ration have been found throughout nature, in architecture, music as well as in art. Phi is an astonishing number because it has inspired thinkers in many disciplines, more-so than any other number has in the history of mathematics. This paper investigates how the golden ratio has influenced civilizations throughout history and has intrigued mathematicians and others by its prevalence.

Keywords: Egyptian mathematics; Fibonacci; Golden mean; Golden ratio; Greek mathematics; Indian mathematics; mathematical aesthetics; mathematics in nature

Introduction

Throughout this paper, the terms golden ratio, divine ratio, golden mean, golden section and Phi (φ) are interchangeably used. Wasler, (2001) defines the golden ratio as a line segment that is divided into the ratio of the larger segment being related to the smaller segment exactly as the whole segment is related to the larger segment. The divine ratio is the ratio of the larger segment, AB, of line AC to the smaller segment BC of the line AC.

This same definition was first given by Euclid of Alexandria around 300 B.C. He defined this proportion and called it "extreme and mean ratio" (Livio, 2002). Let us assume that the total length of line AC is x+1 units and the larger segment AB has a length of x. This would mean that the shorter segment BC would have a length of 1 unit. Now we can set up a proportion of AC/AB = AB/BC.

$$\frac{x+1}{x} = \frac{x}{1}$$

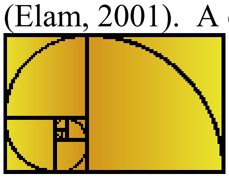
By cross multiplying it yields $x^2 - x - 1$. Using the quadratic formula, two solutions become apparent $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, and we only use the positive solution because we are in terms of a length. The positive solution is $(1+\sqrt{5})/2$. Phi is the only number that has the unique property that $\varphi*\varphi'=-1$where φ' is the negative solution to the quadratic $(1-\sqrt{5})/2$ (Huntley, 1970).

Additional Information on the Golden Ratio

In professional mathematical literature, the golden ratio is represented by the Greek letter tau. The symbol (τ) means "the cut" or "the section" in Greek. In the early twentieth century, American mathematician gave the golden ratio a new name. Mark Barr represented the Golden Ratio as phi (φ), which is the first Greek letter in the name of Phidias.[1] Barr chose to honor the great sculptor because many of Phidias's sculptors contained the Divine Ratio.

The golden ratio is a known irrational number. Irrational numbers have been around for sometime. Most historians believe that irrational numbers were discovered in the fifth century B.C. Pythagoreans knew about irrational numbers and believed that the existence of such numbers was due to a cosmic error (Livio, 2001).

The Golden section is aesthetically pleasing in nature. Phi represents some remarkable relationships between the proportions of patterns of living plants and animals. Contour spirals of shells, such as the chambered nautilus, reveal growth patterns that are related to the golden ratio. The nautilus shell has patterns that are logarithmic spirals[2] of the golden section. Each section is characterized by a spiral, and the new spiral is extremely close to the proportion of the golden section square larger than the previous. The growth patterns in nature approach the golden ration, and in some cases come very close to it, but never actually reach the exact proportion (Elam, 2001). A construction of the golden rectangle and logarithmic spiral can be seen below.



Logarithmic spirals can be found through-out nature. Ram horns and elephant tusks, although they do not lie in a plane, follow logarithmic spirals. Logarithmic spirals are also closely related to Golden Triangles[3]. Starting with a Golden Triangle ABC, the bisector of angle B meets AC at point D and is the golden cut of AC. With this bisection, triangle ABC has been cut into two isosceles triangles that have golden proportion (the ratio of their areas is φ:1. Continuing this process by bisecting angle C, point E is obtained. Again point E is the golden cut along line BD, thus constructing two more golden triangles. This process produces a series of gnomons[4] that will eventually converge to a limiting point O, which is the pole of a logarithmic spiral passing

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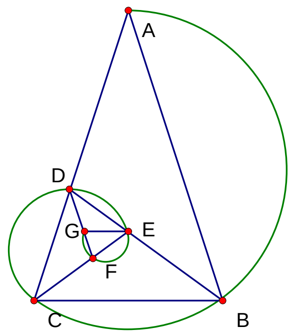
[1] Phidias was an Greek sculptor who lived between 490 and 430 B.C. His sculptors included "Athena Parthenos" which is located in Athens and "Zeus" which is located in the temple of Olympia.
Comment: astonishing is a strange word to use here…how about great?

[2] Logarithmic spirals have a unique property. Each increment in the length of the shell is accompanied by a proportional increase in its radius. This implies that the shape remains unchanged over time and growth. As a logarithmic spiral grows wider, the distance between its coils increases and it moves away from its original starting point (pole). It turns by equal angles and increases the distance from the pole by equal ratios.

[3] Golden Triangles are isosceles triangles that exhibit base angles of 72 degrees and an apex angle of 36 degrees. From the Pythagoreans and the construction of the pentagram (which has five equal-area golden triangles) it can be seen that the length of the longer side to that of the shorter side is in golden proportion.

[4] A gnomon is a portion of a figure which has been added to another figure so that the whole is of the same shape as the smaller figure.

successively and in the same order through the three vertices (…A,B,C,D…) of each of the series of the triangles (Huntley, 1970).



If we begin with GF and call it the unit length, then:

FE = 1φ
ED = 1φ + 1
DC = 2φ + 1
CB = 3φ + 2
BA = 4φ + 3

By bisecting the base angles of the successive gnomons, the lengths of these segments form a Fibonacci series, which we have already seen to converge to the Golden Ratio.

Pine cones and sunflowers are closely related to the Golden Ratio. Each seed in a pine cone is part of a spiral growth pattern that closely relates to φ. The seeds of pine cones grow along two intersecting spirals that move in opposite directions. Interestingly, each seed belongs to both spirals. Eight of the spirals move in the clockwise direction and the remaining thirteen move counter clockwise. As seen above, the numbers 8 and 13 are consecutive Fibonacci numbers which converge to the Golden Ratio. The proportion of 8:13 is 1:1.625. Sunflowers exhibit the same spiral patterns as seen in pine cones. Sunflowers have 21 clockwise spirals and 34 counter clockwise. The proportion of 21:34 is even closer to the Golden Ratio than that of pine cones; it is 1:1.619 (Elam, 2001).

The geometry of plant axis flexure is the result of orthotropic growth and the stress caused by a vertical weight distribution along the axis. A flexed plant axis is shown to conform to a portion of a logarithmic spiral. With numerous plants, this mode of curvature is the most prevalent condition of plants lacking or having secondary growth. Plants like sunflowers represent this growth pattern (Niklas and O'Rourke, 1982).

In 1907 the German mathematician G. van Iterson showed that the human eye would pick out patterns of winding spirals when successive points were packed tightly together. The points were separated by the Golden Angle which measures to 137.5 degrees. The familiar spirals that the human eye would pick out consisted of counter clockwise and clockwise patterns of consecutive Fibonacci numbers. Nature, specifically sunflowers, grows in the most efficient way[5] of sharing horizontal space, which is in proportion of the Golden Ratio. Most sunflowers

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[5] Mathematicians, Harold S. M. Coxeter and I. Adler, showed that buds of roses which were placed in union with spirals generated by the Golden Angle were the most efficient. For example, if the angle used was 360/*n* where *n* is an integer, the leaves would be aligned radially along *n* lines, thus leaving large spaces. Using the Golden Angle,

have a 21:34 ratio, but few have been reported with proportions of 89:55, 144:89 and 233:144 (Livio, 2001).

The Golden Ratio can be found in many examples throughout the world. Phi can be seen in many places; from the layout of seeds in an apple to Salvador Dali's painting "Sacrament of the Last Supper" (Livio, 2002). In the following sections, an in-depth look is taken on the occurrences of Phi in as well as the development of Phi throughout history.

The Golden Ratio and Fibonacci Numbers

Leonardo de Pisa, born around 1175 A.D., commonly known as Fibonacci[6] introduced the world to the rabbit problem. The rabbit problem asked to find the number of rabbits after *n* months, given that adult rabbits produce a pair of rabbits each month, offspring take one month to reach reproductive maturity, and that all the rabbits are immortal. This problem gave the mathematical world the series of Fibonacci numbers[7].

In 1202 A.D. Fibonacci wrote, *Liber Abaci*, which was a book based on the arithmetic and algebra that he had accumulated in his travels. This book was widely copied and introduced the Hindu-Arabic place-value decimal system and the use of Arabic numerals into Europe. Most of the problems in *Liber Abaci* were aimed at merchants and related to the price of goods, how to calculate profit on transactions, and how to convert between the various currencies in use in the Mediterranean countries. Fibonacci is most remembered for presenting the world with the "rabbit problem" which is located in the third section of *Liber Abaci*.

Looking at the ratio of successive Fibonacci numbers, an interesting value appears. As the n increases, the ratio of F_n/F_{n+1} approaches the golden ratio. The values (n=1…10) can be seen in the table below:

n	F(n)	F(n)/F(n-1)
1	0	
2	1	
3	1	1
4	2	2
5	3	1.5
6	5	1.666667
7	8	1.6
8	13	1.625
9	21	1.615385
10	34	1.619048

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which is an irrational multiple of 360 degrees, ensures that the do not line up in a specific radial direction and this leaves no space unfilled.

[6] Fibonacci is a shortened form of Filius Bonaccio (son of Bonaccio). Fibonaci was taught the Arabic system of numbers in the 13th century. He later published the book *Liber Abaci* (Book of Abacus). This book introduced the Arabic numbering system to Europe and gave Fibonacci everlasting fame as a mathematician. (Dunlap, 1997)

[7] Fibonacci Numbers are represented by the recursive relation $A_{n=2} = A_{n+1} + A_n$

The convergence of Fibonacci numbers to the Golden Ratio can be seen in the “rabbit problem”. A Scottish mathematician, in the early 1700's made the connection between the Golden Ratio and the rabbit problem. Robert Simson (1687-1768), noticed that consecutive terms of the solution to the rabbit problem converged to the Golden Ratio (Johnson, 1999). A geometric sequence can be constructed on the basis of the breeding rabbits. For example, let adult rabbits be represented by ‘A’ and their offspring represented by ‘b’. The arrangement of adults (A) and their offspring (b), can be written as AbAAbAbAAbAAbAbAAbAbA… The sequence of A’s and b’s may be extended indefinitely in a unique way because the rule for generating the next character is well defined. The ratio of adults to offspring rabbits in the limit of an infinite sequence is equal to the Golden Ratio (Dunlap, 1997).

$$\lim_{n\to\varphi} A/b = \varphi$$

The Golden Ratio in Ancient Greece

The Golden Ratio can be found throughout nature, which will be discussed below, but it can also be found in the history of the heavens. Plato (428-347 B.C.) prophesied the significance even before Euclid described it in *Elements*. Plato saw the world in terms of perfect geometric proportions and symmetry. His ideas were based on Platonic Solids.[8] He divided the heavens into four basic elements, earth, water, air, and fire. Each of these elements was assigned a Platonic Solid; a cube for earth, tetrahedron for fire, octahedron for air and an icosahedron for water. Using this foundation, Plato created a chemistry that is similar to modern day chemistry[9] (Livio, 2003).

The five Platonic solids are the only existing solids in which all of the faces are identical and equilateral and each vertex is convex. Interestingly, each of the solids can be circumscribed by a sphere with all of its vertices lying of the sphere. The tetrahedron consisted of four triangular faces, the cube with six square faces, the octahedron with eight triangular faces, the dodecahedron with twelve pentagonal faces and the icosahedron with twenty triangular faces (Livio, 2002).

Each face of the regular polyhedron is a regular polygon with n edges. It is known that the values of n are $\{n: 3 \le n < \infty\}$ with n being related to the interior angle α.[10] Each vertex of the three dimensional polygon is defined by the intersection of a number of faces, m, where $m \ge 3$. In order

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[8] The Platonic Solids Plato used consisted of five shapes. The first three; tetrahedron, octahedron and the icosahedron, were based on equilateral triangles. The remaining two; cube and dodecahedron were made from the square and regular pentagram.

[9] Plato's theory was much more than a symbolic association. He noted that the faces of the tetrahedron, cube, octahedron, and dodecahedron could be constructed out of two types of right angled triangles, the isosceles 45-90-45 and the 30-60-90 triangle. Plato explained that his chemical reactions could be described using these properties. For example, when water is heated by fire, it produces two particles of vapor (air) and one particle of fire, $\{water\} \to 2\{air\} + \{fire\}$. In Platonic chemistry, balancing the number of faces involved (in the Platonic solids that represent these elements) we get 20=2*8+4. The central idea is that particles in the universe and their interactions can be described by a mathematical' theory that possesses certain symmetries.

[10] In general a regular n-gon has n edges and interior angles given by the equation $\alpha=[1-(2/n)]*180$.

for a convex vertex to be formed, $m\alpha<360$ degrees. [11] There are only five combinations of integers that satisfy these equations and they correspond with the five Platonic Solids and are listed below[12] (Dunlap ,1997).

solid	n	m	e	f	v
tetrahedron	3	3	6	4	4
cube (hexahedron)	4	3	12	6	8
octahedron	3	4	12	8	6
dodecahedron	5	3	30	12	20
icosahedron	3	5	30	20	12

The Golden Ratio is of relevance to the geometry of figures with fivefold symmetry. The dodecahedron and the icosahedron are of particular interest. If either one of these Platonic Solids are constructed with an edge length of one unit, it is easy to see the important role the Golden Ratio play in their dimensions (Dunlap, 1997).

solid	surface area	volume
dodecahedron	$15\varphi/(3-\varphi)$	$5\varphi^3/(6-2\varphi)$
icosahedron	$5\sqrt{3}$	$5\varphi^5/6$

Plato and his foundations using Platonic Solids for the heavens may suggest that the Golden Ratio may have been known in ancient Greece. However, the full mathematical properties of Platonic Solids may not have been known in antiquity. Plato and his followers may have created and used Platonic Solids in the foundations of the universe based on sheer beauty.

Many authors researching ancient Greek mathematics are unsure if the works of Plato were influenced by Pythagoras and the Pythagoreans. Pythagoras[13] was born around 570 B.C. on the island of Samos. Pythagoras and the Pythagoreans are best known for their role in the development of mathematics and for the application of mathematics to the concept of order (Livio, 2002).

The Pythagoreans assigned special properties to odd and even numbers as well as individual numbers. The number one was considered the generator of all other numbers and geometrically, the generator of all dimensions. The number two was considered the first female number and the number of opinion and division. Geometrically, the number two was expressed by the line

[11] We have to place certain restrictions on the values of m. These reasons are if m=2 then an edge is formed, not a vertex. And if $m\alpha=360$ degrees, then the vertex is merely a point on a plane and if $m\alpha>360$ degrees then the faces overlap.

[12] The table above lists the characteristics of the five Platonic Solids. The quantities n and m are the number of edges per face and the number of faces per vertex. The quantities e, f, and v are the total number of edges, faces, and vertices for the respective solid.

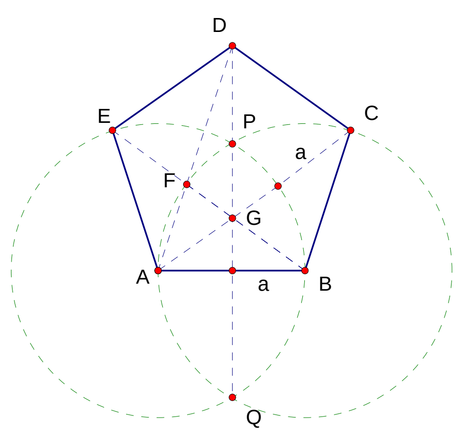
[13] Pythagoras emigrated to Croton in southern Italy sometime between 530 and 510. He studied Egyptian, and Babylonian mathematics, but both of these prove too applied for him. There are many different accounts of the Mathematician's life and death, but what is known for sure is that he was responsible for mathematics, and philosophy of life and religion.

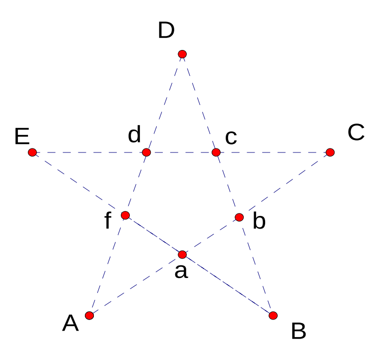
which has one dimension. The number three is considered by the Pythagoreans to be the first male number and the number of harmony because it combines the unity number (one) and the division number (two). The geometric expression of the number three was a triangle, where the area of the triangle has two dimensions. Justice and order was expressed in the number four. On the surface of the Earth, four directions provide orientation for humans to identify their coordinates in space. Four points, not in the same plane, form a tetrahedron. The number six is the first perfect number and considered the number of creation. It is the number of creation because it is the product of the first female number (two) and the first male number (three). Six is a perfect number because it is the sum of all the smaller numbers that divide into it. The first three perfect numbers are listed below (Livio, 2002).

$6 = 1+2+3$
$28 = 1+2+4+7+14$
$496 = 1+2+4+8+16+31$

The number five deserves its own explanation. Five represents the union of the first female number and the first male number. This union suggests that five is the number of love and marriage. The main reason five is important to this discussion is because the Pythagoreans used the pentagram[14] as a symbol of their brotherhood (Livio, 2002).

The construction of the pentagon, using a compass and marked straight edge, leads to a pentagram. Given a line AB, use the compass to draw arcs of radius *a* about points A and B. Next construct the perpendicular bisector PQ of line AB. Using the straight edge plot two points that are *a* units apart and slide the straight edge so that it passes through point A, until one of the points falls on the arc of B. There are only two possible positions for these points, namely, C and F. Using the same directions, find points G and D, sliding the straight edge through point B until one of the points falls on the arc of A. The fifth vertex (E) can be found by the requirement that on line EGB, EG equals *a*. Using this construction of a pentagon, one can connect the vertices and build a pentagram (Herz-Fischler, 1987).

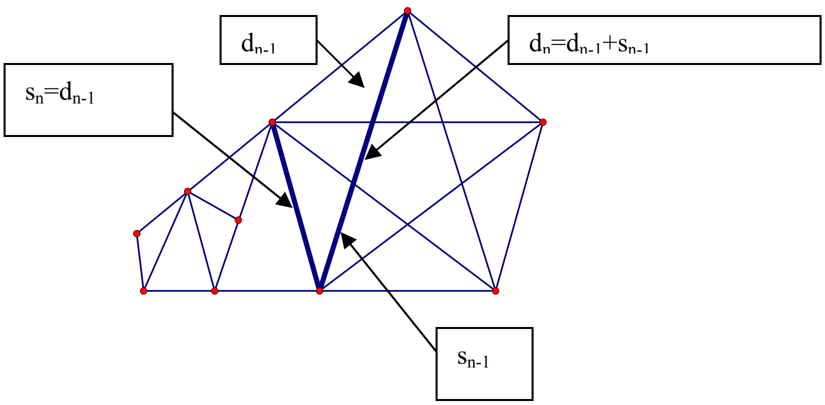




[14] The pentagram is closely related to the regular pentagon. If one is to connect all the vertices of the pentagon by diagonals, a pentagram is constructed. The diagonals of this pentagon form a smaller pentagram. This process can be continued to infinity, and every segment is smaller that its predecessor by a factor that is precisely equal to the Golden Ratio.

The pentagram is important to the discussion of the Golden Ratio because of its unique properties. The diagonals of a pentagon cut each other in the Golden Ratio and the larger of the two segments is equal to the side of the pentagon. The Pythagoreans choosing the pentagram as a symbol for brotherhood, and the given properties of the pentagram, suggests that the Pythagoreans were familiar with the Golden Number, but many historians are still under debate about this particular topic, due to inconclusive historical data (Herz-Fischler, 1987).

One theory, Heller (1958), suggests that the Pythagoreans used the pentagon to discover incommensurability and the division in extreme and mean ratio. Heller believes that the Pythagoreans discovered incommensurability through the observations of a series of pentagons when drawing diagonals.



The diagonal, d_{n-1}, becomes the side, s_n, of the next largest pentagon. The new diagonal d_n is the sum of the side and the diagonal, s_{n-1} and d_{n-1}, of the previous pentagon. With this information it is easy to see the recurrence relationships $s_n = d_{n-1}$; $d_n = d_{n-1} + s_{n-1}$. Using[15] $s_1 = 2$ and $d_1 = 3$, leads to the sequence of d_n:s_n ratios of 3/2, 5/3, 8/5, 13/8… which we have already seen to be successive Fibonacci numbers (Herz-Fischler, 1987). A formal proof of this can be found in The Golden Ratio: The Story of Phi the World's most Astonishing Number.

The Golden Mean in Ancient Egypt

Modern mathematicians have been trying to decide what civilizations used and understood the golden mean. Ancient Egypt, a civilization with profound mathematical accomplishments and astonishing monuments is under investigation for uses of the golden mean. Many interpretations of the golden mean use the properties of different geometrical figures. This may prove useless because it can produce an infinite chain of similar links. Math historians do need to focus on the ancient monuments and the mathematics of the respective time period. Ancient civilizations did not necessarily have the same numbering systems of modern times. This suggests that some things that work in modern numbering systems do not work in ancient systems (Rossi and Tout, 2002).

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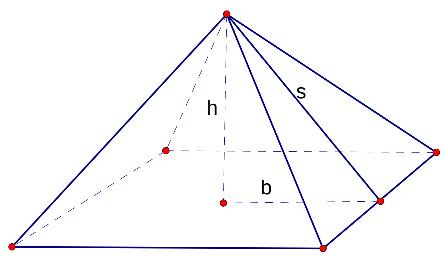
[15] Side and diagonal numbers of squares start off with the number one as the first number in the sequence. For pentagonal side and diagonal numbers, starting with one will lead to the degenerate case. Thus we have to start with the two as the first number in the sequence.

One theory about the use of the Golden Mean in ancient Egypt is that Egyptian architects designed the pyramids in a geometric way. Egyptian pyramids were based on geometrical processes of squares, rectangles and triangles. Of extreme importance was the process of the 8:5 triangles.[16] Egyptians used these triangles because the ratio of 8/5 was a good approximation of the Golden Mean. The theory continues to suggest that Egyptian architects gave their designs dimensions based on the corresponding numbers of the Fibonacci series. We have already seen that the ratio of corresponding Fibonacci numbers converges to the Golden Ratio (Rossi and Tout, 2002).

The Great Pyramid of Cheops, built before 2500 B.C., has been measured and many different dimensions are present. The majority of the dimensions are within one percent of 755.79 feet as the length of the base and 481.4 feet as the height. Some theories claim that the Great Pyramid of Cheops was designed so that the ratio of the slant height of the pyramid to half the length of the base would be the divine proportion (Markowsky, 1992).



In the above figure, h represents the height, b represents half the base, and s represents the slant height of the Great Pyramid of Cheops. Using 755.79 feet for the length of the base and 481.4 feet for the height, we can see that b=377.90 feet. Using the Pythagorean Theorem, $h^2 + b^2 = s^2$, we can find that s=612.01. This gives us a ratio of the slant height of the pyramid to half the length of the base as 612.01/377.90=1.62 which is very close to the Golden Mean (Markowsky, 1992). Another interesting feature of the Great Pyramid is that it has an apex angle of 63.43 degrees. This is very close to the apex angle of the Golden Rhombus[17] (63.435 degrees), which has dimensions derived for the Golden Ratio. The difference between the apex angle of the Great Pyramid and a Golden Rhombus is a mere 22 centimeters in the edge of the length of the pyramid base (Dunlap, 1997).

The question that needs to be answered is, was it possible for ancient Egyptians to construct a convergence of the Fibonacci numbers? Ancient Egyptians represented ratios as a sum of unit fractions. For example the fraction 3/5 would be represented as 1/2 + 1/10. As ratios continued to grow, many different representations become available. Take the ratio 13/21 for example. Egyptians could have represented this number in five different ways:

1. 1/2 + 1/10 + 1/56 + 1/840
2. 1/2 + 1/10 + 1/57 + 1/665
3. 1/2 + 1/10 + 1/60 + 1/420
4. 1/2 + 1/10 + 1/63 + 1/315
5. 1/2 + 1/10 + 1/65 + 1/273

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[16] The 8:5 triangle was an isosceles triangle in which the base was eight units and the height was five units.

[13] The Golden Rhombus is a two dimensional figure that has perpendicular diagonals which have a ratio of 1:φ.

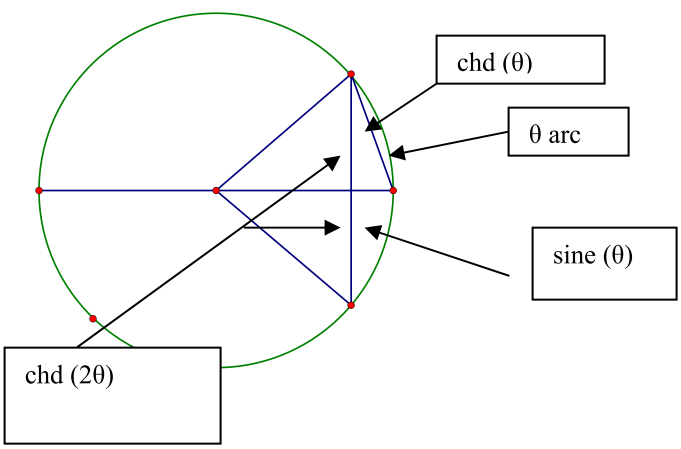
Egyptian scribes could have found a convergence of φ with their system of representing fractions. Adding to the previous sum of ratios a unit fraction whose denominator is given by the multiplication of the two previous denominators (in the ratio of Fibonacci numbers) yields the next value in the sum converging to the Golden Ratio. The sum of the ratios of the first few Fibonacci numbers converging to φ can be seen below (Rossi and Tout, 2002).

$1/2 = 1/2$
$3/5 = 1/2 + 1/10$
$8/13 = 1/2 + 1/10 + 1/65$
$21/34 = 1/2 + 1/10 + 1/65 + 1/442$
$55/89 = 1/2 + 1/10 + 1/65 + 1/442 + 1/3026$
$144/233 = 1/2 + 1/10 + 1/65 + 1/442 + 1/3026 + 1/20737$

The convergence above suggests that is was possible for ancient Egyptian scribes to evaluate the Golden Ratio. However, it seems unlikely that ancient Egyptians were aware of the Fibonacci numbers. Egyptian math is considered an applied math, no records have been found on the theory behind their mathematics. Only applications of Egyptian mathematics exist. This suggests that the Egyptians, although capable, did not recognize the golden ratio and it was a mere coincidence that the architecture of the pyramids is based on 8:5 triangles (Rossi and Tout, 2002).

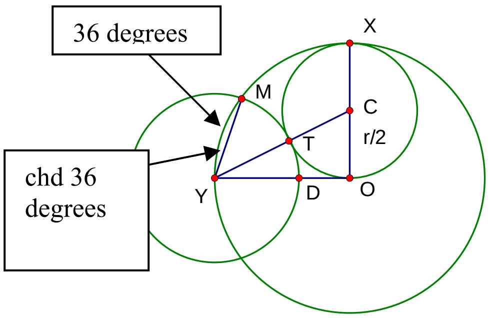
The Golden Ratio in Ancient India

The division in extreme and mean ratio appears in mathematical texts from India in connection with trigonometric functions. The Indian sine function is not the same as our modern day sine function. The Indian sine function can be defined as satisfying the relationship Sine (θ) = ½* chord (2θ). The circumference of the circle is divided into 360 degrees and then the radius of the circle is divided into 60 parts. With this, sine (30) = $a_6/2 = r/2 = 30$. And sine (18) = $a_{10}/2$ and sine (36) = $a_5/2$.



Bhaskara II (1114-1185) states without proof that Sine (18) = $(R(5r^2) - r)/4$. This is exactly the relationship sine (18) = $a_{10}/2$. Bhaskara, again without reason, tells to find the side of the pentagon inscribed in a circle, multiply the diameter by 70534/12000 (Amma, 1979). A proof of this statement is provided by Gupta (1976) and is provided below.

In a circle of radius r = OX = OY, let the arc YM = 36 degrees. Draw a semicircle OX about the midpoint C of OX and draw the arc MD about Y. Assume that the tow arcs meet at the single point T on line YC. Then Sine (18) = YM/2 = YT/2 = YC/2 –TC/2 = $(R(r^2 + (r/2)^2) - r/2)/2$ which is equivalent to Sine (18) = $(R(5r^2) - r)/4$.



The above proof and construction are considered incomplete because they do not explain why the arcs meet at point T. Gupta (1976) continues and completes the construction by: Think of Y and C as given points and draw the arc OTX of radius r/2. Draw arc MTD of radius YT. Thus, the circles are tangent at the point T on the line YTC connecting the centers.

How does this construction tie in with the discussion on Ancient Indians knowing the Golden Ratio? Concentrate on the triangle YOC and arcs DT and OT. With a close examination, it can be seen that OY is divided in extreme and mean ratio at D. In other words, YM = YD is the greater segment when OY is divided in extreme and mean ratio (Gupta, 1976).

Evidence of the Golden Ratio in the Arts

Countless illustrations of the proportions of the Golden Section are found in the works of humans. The Golden Section follows upon the basis of symmetry everywhere and the forms which are based upon the golden proportion are widely distributed. When speaking about the products of art and architecture, there is no equal symmetry, the artist or workmen unconsciously employ golden proportions. Irregular inequality and capricious division is aesthetically disagreeable, while golden proportions are pleasing to both hand an eye (Ackermann, 1895).

Many assertions claiming that the Golden Section was used in art are associated with the aesthetics of the proportion. When given an opportunity to choose the most visually pleasing rectangle, most people would choose rectangles with a close approximation of the Golden Rectangle. Although most humans cannot decipher between a rectangle with a ratio of 1.6 and a rectangle with ratio of 1.7, it suggests that humans do prefer rectangles in the range close to the Golden Rectangle (Markowsky, 1992).

Several decades after the Brotherhood of the Pythagoreans faded, the Golden Ratio continued to influence many artists and artisans. The Golden Ratio has influenced classical Greek architecture, notably the Parthenon in Athens. Inside the Parthenon stands a forty-foot-tall statue of the Greek Goddess Athena[18], which has also shown to have Golden proportions. Both the temple and the stature were designed by Phidias, who is the first artist known to use the Golden

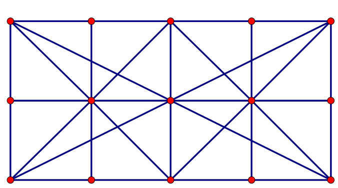
[18] Athena is the Greek goddess of wisdom, war, the arts, industry, justice and skill. Her father was Zeus and her mother was Metis, Zeus' first wife.

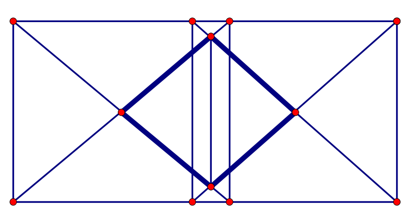
Ratio in his work. As said above, the symbol for the Golden Ratio is the first Greek letter phi, which also happens to be the first letter in Phidias's name (Johnson, 1999).

Ancient Greek scholar and architect Marcus Vitruvius Pollio, who is commonly known as Vitruvius, advised that "the architecture of temples should be based on the likeness of the perfectly proportioned human body where a harmony exists among all parts" (Elam, 2001).Vitruvius is credited with introducing the concept of a module to the architectural world. This concept was the same as the module of human proportions and became an important architectural idea. The Parthenon[19] in Athens is an example of this proportioning. The Parthenon can be inscribed by a Golden Rectangle (Elam, 2001).

When the triangular pediment was still intact[20], the Parthenon fit precisely into a Golden Rectangle. Another claim is that the height of the structure (from the top of the tympanum to the bottom of the pedestal) is divided into the Golden Ratio (Livio, 2001).Markowsky, (1992) has a contrasting view of the Parthenon. He believes that even though the Parthenon incorporates many geometric balances, its builders had no knowledge of the Golden Ratio. Depending on what sources are used, the dimensions of the Parthenon vary because the authors are measuring between different points. This implies that if the author is a Golden Ratio enthusiast they could choose which ever numbers give them the best approximation of φ.

Regardless whether or not the Parthenon's architecture was built accordingly to the Golden Ratio, it is still an amazing structure, and may get some of its beauty from the regular rhythms introduced by the repetition of the same column (Livio, 2001). Renaissance artists often used diagonals and other interior lines of rectangles to divide rectangular space proportionally. For example the main diagonals of a rectangle allow for division of the rectangle into halves, both vertically and horizontally. Continuing, the diagonals of the halves allow division into quarters. Another tactic used by Renaissance artists to construct they work was called rabatment. Rabatment is where the shorter sides of the picture rectangle are rotated onto the longer. The rotation creates vertical division and overlapping squares. If rabatment is applied to a Golden Rectangle, the diagonals of the two overlapping squares cut the diagonals of the rectangle in golden proportion (Brinkworth and Scott, 2001). A construction of division by diagonals is provided on the left and a construction by rabatment is provided on the right.





[19] The Parthenon, know as "the Virgin's place in Greek," in Athens was built in the fifth century B.C. and is one of the world's most famous structures. The Parthenon is a sacred temple to the cult of Athena Parthenos.
[20] On September 26, 1687, Venetian artillery directly hit the Parthenon. General Konigsmary said "How it dismayed His Excellency to destroy the beautiful temple which had existed for over three thousand years."

In the thirteenth century three artists' work contain close proportions to the Golden Rectangle. Italian painter and architect Giotto di Bondone (1267-1337) painted the "Ognissanti Madonna"[21] which is also known as "Madonna in Glory." Both the painting as a whole and the central figures in the painting can be inscribed by Golden Rectangles. Similarly, Sienese artist Duccio di Buoninsegna's (1255-1319) "Madonna Rucellai" and Florentine painter Cenni de Pepo's (1240-1302) "Santa Trinita Madonna" can be inscribed by Golden Rectangles. Both of these paintings are in the same room as the "Madonna in Glory". It is speculated that these three artists did not include the Golden Section into their paintings; rather they were driven by the unconscious aesthetic properties of the Golden Ratio. With respect to the time period, the three Madonnas were painted centuries before the publication of "The Divine Ratio" which brought the proportion into common knowledge (Livio, 2001).

Leonardo da Vinci inevitable comes into the discussion of the Divine Ratio and art. Five of his works have been speculated to host Golden Ratio properties: The unfinished canvas of "St. Jerome," the two version of "Madonna on the Rocks," the drawing of "a head of an old man," and the most famous of all, the "Mona Lisa"(Livio, 2001).

The two versions of "Madonna on the Rocks" have an interesting history. The first version, produced between 1483 and 1486, was done before da Vinci had any contact with Pacioli or his book "The Divine Ratio." The second version, which was completed around 1506, could have been influenced by Pacioli's book. Interestingly, both versions are very close to the Divine Ratio. In the first version, the dimensions are in proportion 1.64 and in the second version's dimensions are in proportion 1.58, both close estimates of φ (Livio, 2001).

Leonardo da Vinci's "head of an old man", is suggested to be a self-portrait which is overlaid with a square that is divided into rectangles. Some of these rectangles approximate Golden Rectangles but it is difficult to be absolutely sure. The rectangles are very roughly drawn and do not have square corners (Markowski, 1992). This suggests that depending on where one measures from, it is very possible to find some ratio that approximates the Golden Ratio. Leonardo da Vinci's "St. Jerome" has similar uncertainty. When overlaid with a Golden Rectangle, the left side of St. Jerome's body and his head are missed completely. The left side of the Golden Rectangle is tangent to a small fold of fabric and does not touch the body at all. Again, Leonardo was not introduced to Pacioli's book until thirteen years after the completion of "St. Jerome" (Markowski, 1992). His right arm also extends beyond the rectangle's side. The drawing of "a head of an old man,"[22] completed in 1490, is the closest demonstration that da Vinci used Golden Rectangles to determine dimensions in his paintings (Livio, 2001).

Human body proportions and facial features share similar mathematically proportioned relationships as other living organisms. The placement of facial features yields the classic proportions used by both the Romans and Greeks. Marcus Vitruvius Pollio described the height of a well proportioned man is equal to the length of his outstretched arms. The body and outstretched arms can be inscribed in a square, while the hands and feet are inscribed in a circle. With this system, the human body is divided into two parts at the naval. These parts are

[21] Bondone's painting "Madonna in Glory" is currently in the Uffizi Gallery in Florence. This painting features an enthroned Virgin with a child on her lap. Both Madonna and Child are surrounded by angles.

[22] The drawing of "a head of an old man" is currently in the Galleria dell' Accademia in Venice.

represented in the proportion of the Golden Rectangle. Classical statues from the fifth century such as Doryphoros the spear bearer and Zeus have the proportions suggested above (Elam, 2001).

The art described above deal with proportions of measurements. It should be noted that measurements, no matter how accurate, only provide reasonable estimates of the Golden Ratio (Fischler, 1981). The artist, painter or sculptor may or may not have been trying to conform to the proportion of the Golden Ratio. However close the approximations are, they could have been created with beauty in mind and with no intention to match the Golden Ratio[23].

Visually pleasing art is not the only form of art where the Golden Ratio can be found. Music and mathematics have been entwined since antiquity and it is not surprising that one accompanies the other[24]. The Golden Ratio is related to many forms of music. Many listeners, including people who are only casually acquainted with the music of Mozart (1756-1791), can pick up on the manifested form and balance the composer used when writing his music (Putz, 1995).

Mozart worked with mathematical figures throughout his life. In his early composing years, he took up the problem of composing minuets 'mechanically', by putting two-measure melodic fragments together in a specific order. By the age of nineteen, Mozart had composed his first sonata for piano[25]. Almost all of his sonatas were composed of two movements: 1) the Exposition in which the musical theme in introduced and the Development and Recapitulation in which the theme is developed and revisited (Newman, 1963). A visual representation of Mozart's sonata-form movement can be seen below.

Sonata-Form Movement

The first movement of the first sonata, K. 279, is 100 measures in length. It is divided so that the Development and Recapitulation section has a length of 62. It should be noted that the lengths of the movements are natural numbers because they measure counts. When reviewing the first movement of the first sonata, it can be seen that 100 cannot be divided any closer (using natural numbers) to the Golden Ratio than 38 and 62. This is true for the second sonata which has total length of 74 and is divided in 28 and 46. A table of some of Mozart's movements is listed below.

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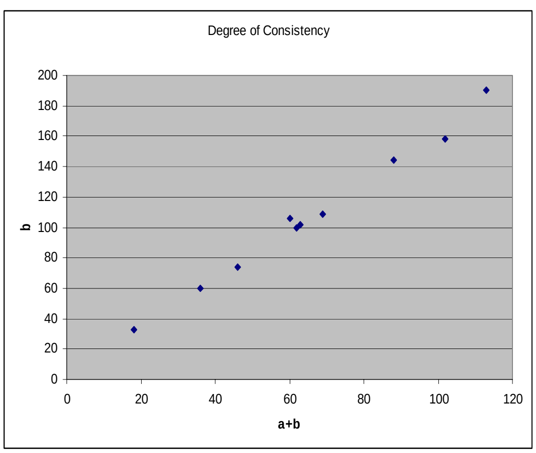
[23] Fischler (1981) gives a detailed description, complete with proofs of how certain data can be transformed to exhibit Golden Ratio characteristics.

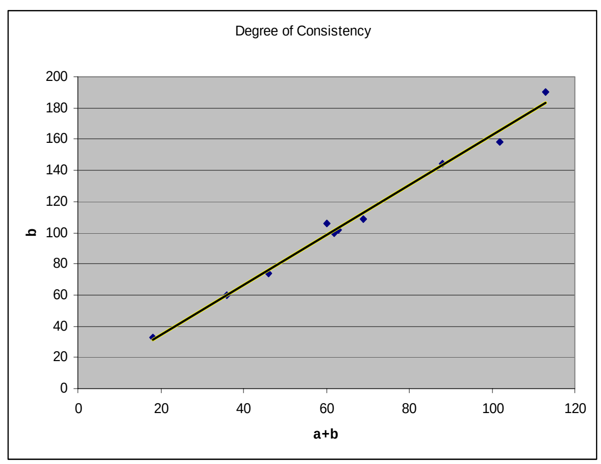
[24] When Mozart was learning arithmetic, he gave himself entirely to it. His sister recalls that he once covered the walls of the staircase and of all the rooms in their house with figures, then moved to the neighbors house as well (King, 1976).

[25] Mozart wrote 19 all together.

Piece and Movement	a	b	a+b
279, I	38	62	100
279, II	28	46	74
279, III	56	102	158
280, I	56	88	144
280, II	24	36	60
280, III	77	113	190
281, I	40	69	109
281, II	46	60	106
282, I	15	18	33
282, III	39	63	102

To evaluate the consistency of the ten proportions listed above, a scatter plot of b against a+b can be used. If a composer, Mozart in this case, is consistent with using the Golden Ratio in their works, the data should be linear and fall near the line y = φx. The graph on the left represents the degree of consistency by plotting the value of b with the values of a+b. The statistical analysis for the data shows an r^2 value of .994 which confirms an extremely high degree of linearity. The graph on the right shows the linear regression of the data (represented by the yellow line and the equation y = 1.59614205x + 2.733467326), and the line y = φx (black line) overlaid on the plot of the data. The statistical analysis of the data and the graphs below show that the data is linear and the points scarcely differ from the line y = φx. This is of impressive evidence that Mozart did partition sonata movements near the Golden Section (Putz, 1995).





If a movement is divided into the Golden Section, then both a/b and b/(a+b) should be near phi. Fischer (1981) provides a theorem and the following proof that b/(a+b) is always closer to φ than a/b is.

Theorem: $\left| \{b/(a+b)\} - \varphi \right| \leq \left| (a/b) - \varphi \right|$ where $0 \leq a \leq b$.

Proof: Let x = a/b. Then show that,

$$\left| \{1/(x)\} - \varphi \right| \leq \left| (x) - \varphi \right|$$

for all x $\in$ [0,1]. Let f(x) = 1/(x+1). By the Mean Value Theorem, for all x $\in$ [0,1] there is a z $\in$ (0,1) such that:

$$|f(x) - f(\varphi)| = |f'(z)|\,|x - \varphi|.$$

Now $f'(x) = -1/(x+1)^2$ satisfies

$$1/4 < |f'(x)| < 1$$

For x $\in$ (0,1). A simple calculation will show that φ is a fixed point of f, that is, that f(φ) = φ. So, for all x $\in$ [0,1],

$$|\{1/(x+1)\} - \varphi| \geq |(x) - \varphi|$$

with equality when x = φ. This theorem says that the ratio of consecutive terms of any Fibonacci-like sequence ($f_1 = a$, $f_2 = b$, $f_{n+2} = f_n + f_{n+1}$ with a and b not both zero) converges to φ.

Modern Implications of the Golden Ratio and Beauty

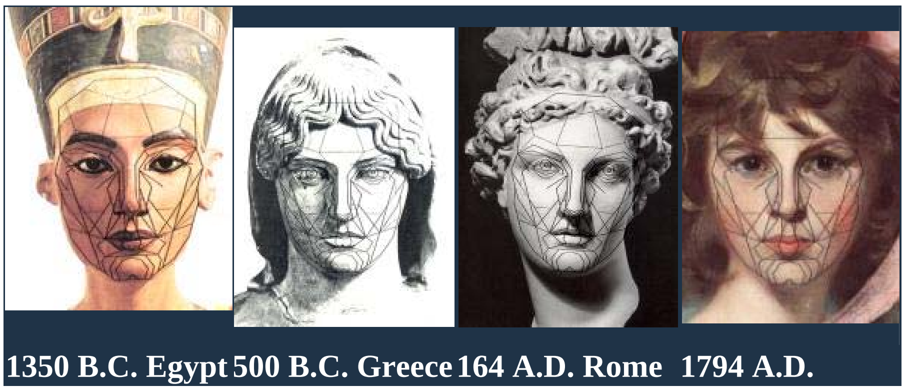
Beauty has been defined in many different ways since antiquity. A modern definition of beauty is "excelling in grace or form, charm or coloring, qualities which delight the eye and call forth that admiration of the human face in figure or other objects." Facial harmony can be activated through symmetry. Such symmetry exists when one side of the face is a mirror image of the other. The ideal face can be measured in symmetrical proportions. It should be noted that attractive faces are relatively symmetrical but not all symmetrical faces are considered beautiful (Adamson & Galli, 2003).

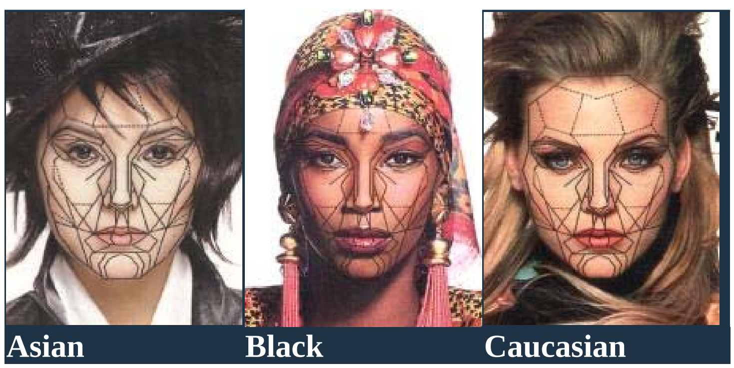
The Golden Ratio can also be found in human DNA structure[26] and has been found to be the only mathematical configuration that can duplicate itself ad infinitum without variance. It has been suggested that this represents a geometrically encoded instructional pattern in the brain that guides humans to recognize beauty.

The Golden Proportion can be found throughout a beautiful human face. The human head forms a Golden Rectangle with the eyes at the midpoint. The mouth and nose can each be placed at Golden Sections of the distance between the eyes and the bottom of the chin. With this information it is possible to construct a human face with dimensions exhibiting the Golden Ratio. This is exactly how some modern plastic surgeons are creating beauty. Dr. Stephen Marquardt created a Golden Decagon Mask, which is a two-dimensional visual perception of the face that has triangles with sides with ratios of 1:1.618. The Golden Decagon Mask is completed when

[26] DNA molecules are based on the Golden Ratio. A single DNA molecule measures 34 angstroms long by 21 angstroms wide for a full cycle of its double helix spiral. Both 34 and 21 are Fibonacci numbers which converge to the Golden Ratio. The double-stranded helix DNA molecule has two grooves in its spiral. The major groove measures 21 angstroms and the minor groove measure 13 angstroms, again, both are Fibonacci numbers. Another unique way that DNA is related to the Golden Number can be seen in a cross-sectional view of a DNA strand, which turns out to be a decagon. The golden properties of the decagon are discussed above.

forty-two secondary Golden Decagon matrices[27] are mathematically and geometrically positioned in the primary framework. The secondary matrices are geometrically locked on to the primary matrix by having at least two vertex radials, a vertex radial and an intersect of two vertex radials, or two intersects of vertex radials in common with the primary Golden Decagon matrix. These secondary Golden Decagon Matrices form the various features of the face (Marquardt, 2002). Below are some examples of how the Golden Ratio is perceived throughout history and through different cultures.





Regardless of how the human face seems to fit into a unique geometric figure, beauty will always be defined in more ways than one. Plastic surgeons may construct beautiful faces today to fit into a Golden Decagon, but this may not always be the case. The future may lead to a new definition of beauty based on other information than the golden ratio. But it does make you wonder if "beauty is in the phi of the beholder."

Concluding Thoughts

Phi could be the world's most astonishing number. It can be found in nature, throughout history, in art, music, and architecture. Many conflicting theories exist about the origins of phi (φ); however we cannot deny the principles that accompany it. Whether it is the mathematical

[27] The secondary Golden Decagon matrices are constructed exactly the same way as the primary Golden Decagon only smaller.

relationships that seem to form around the number or the sheer aesthetics of the proportion, we must be aware that φ is all around us and rightly called the Divine Ratio.

Acknowledgement
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The Effects of Blended E-Learning on Mathematics and Computer Attitudes in Pre-Calculus Algebra

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Abstract**:** This study examines the influence of blended e-learning on students' attitude towards mathematics and computers. A random sample of 70 students of the preparatory year program of King Fahd University of Petroleum & Minerals (KFUPM), Dhahran served as the sample of this study. Data were collected at the beginning (pre-program) and the end (post-program) of the semester using Aiken Mathematics Attitude Scale and Greessen and Loyd Computer Attitude Scale. The result indicates that the subjects have positive attitude towards mathematics and computer. However, analysis of variance shows no statistically significant change in students' attitudes towards mathematics and computer except for computer confidence and anxiety subscale.

Keywords: Affect; E-learning; Computer attitudes; Mathematics attitudes; Math anxiety

1. Introduction

Attitude has been defined as "a learned predisposition to respond positively or negatively to a specific object, situation, institution, or person" (Aiken 2000: 248). Therefore, attitude affects people in everything they do and in fact reflects what they are, and hence a determining factor of people's behavior. Also, it provides people with a framework within which to interpret the world and integrate new experiences (Galletta and Lederer, 1989). That is to say by understanding an individual's attitude towards something, one can predict with high precision the individual's overall pattern of behavior to the object (Ajzen and Fishbein, 1977). Some educators defined learning as a change in behavior. Since attitude is the determining factor of peoples' behavior, the issue then is critical in education. It is a common practice that if a new program is introduced part of the evaluation is to determine people's attitude toward the program. In most cases, positive attitudes are interpreted as an indicator that the program may succeed. Otherwise, there is a tendency of failure, and so, the attitude needs to be modified or possibly changed.

Mathematics as a subject has remained mysteriously difficult and unpopular for most students. This is despite the fact that no one is in doubt of its importance in almost all careers, especially in the science and technological fields. Many studies have found attitude to be one of the stumbling block for progress or otherwise in learning mathematics (Aiken, 1976). A strong correlation between mathematics achievement and attitude has been found in many studies, and that the two interrelates and affects each other in a reciprocal manner (Aiken, 1970 & 1985, Reyes 1984). Studies have shown that students that have positive attitude toward mathematics tend to do well in the subject, and students that have negative attitude toward mathematics tend to perform badly in the subject (Begle, 1979). As a result, intensive research has been done to determine students' attitude towards mathematics in relation to different variables. The aim was to identify the variables that can assist in developing positive attitude of students towards mathematics or at least reduce negative attitude. Most of these efforts are through modification/innovation of instructional approaches that make the subject attractive to the students. According to (Aiken 1976), most of these experimental methods of teaching mathematics have not been shown to be superior to traditional methods with respect to changes in attitude towards the mathematics. However, research has shown that technological aids such as calculators and computers have improvement effects on students' attitudes toward mathematics (Aiken, 1976; Collins, 1996).

On the other hand, given the pervasiveness of computers in all levels of educational system, it is likely that students will have developed some attitudes towards these machines. In a classroom setting, studies have shown that students often experience reactions towards computers either positively or negatively. This in turn either enhances or interferes with their development of effective learning (Geer, White & Barr, 1998). Furthermore, attitude towards computers has been found to influence not only the acceptance of computers in classroom, but also future behavior, such as using a computer as a professional tool or introducing computer applications into the classroom (see Al-Badr, 1992). A student with a negative attitude towards computers may not pay attention to anything to do with computers. Similarly, students that are computer enthusiasts may pay attention to any program that is computer based and this may influence their attitudes toward the subject. Studies have shown that computer attitudes are a strong predictor of performance and evaluation of a computer literacy courses (Batte, Fiske and Taylor, 1986). Some other studies have shown that the use of computer in education has the potential of changing students' attitudes positively towards mathematics and computers (Bangert, Kullik, & Kullik, 1983; Kulik, 1984; Ganguli, 1992 and Funkhouser, 1993).

For instance, Ganguli (1992) investigated the effect of using computers as a teaching aid in mathematics instruction on student attitudes toward mathematics. He used computers as a supplement to normal class instruction. The sample in the study consisted of 110 college students enrolled in four sections of an intermediate algebra class offered by the open-admissions undergraduate unit of a large Midwestern state university. The instruction focused on how to develop the concept of relationship between the shape of a graph and its function. The results indicated that the attitudes of the experimental group which was taught with computer aid were significantly changed in a positive direction whereas the control group that was taught without computer aid failed to show a similar result. Similarly, the results have shown that students in the microcomputer treatment group experienced a more positive self-concept in mathematics, more enjoyment of mathematics and more motivation to do mathematics than their counterparts in the control group. Furthermore, the two instructors who participated in the study both indicated that the computer-generated graphics led to more active classroom discussions in experimental sections and consequently created more rapport between the teacher and the students than in the control sections.

In a similar study conducted in Saudi Arabia, Al-Rami (1990) examined the students' attitude toward learning about and using computers and correlated their attitudes with their achievements in computer classes. One hundred and seventy two male students participated. Student attitudes were determined at the beginning and end of the semester using the Computer Attitude Scale (Loyd and Gressard, 1984). Academic achievement was based on end-of-semester scores. Findings indicate that students' attitudes toward computers were positive at all semester levels and almost the same at the beginning and end of the semester. Both pre-test and post-test attitude results were statistically significant in predicting achievement, with the post-test shown to have be more reliable in predicting achievement.

However, despite all these, very little is known on the effect of blended e-learning to students' attitudes towards mathematics and computers, especially among the pre-calculus algebra students. In line with this, this study seeks to explore this area of endeavor as it is a necessary step for successful blended learning.

2. What is blended E-Learning?

Education is one of the sectors that most benefited from the current technological advancement. With this development, time and space are no more barrier to education. As a matter of fact, the concept of distance learning has been revolutionized to what is now known as e-learning or Web-based learning programs. However, it has been observed that the first generation of e-learning programs focused on presenting physical classroom-based instructional content over the internet with very little attention given to the peculiar nature of this delivery program in comparison to the tradition classroom lesson (Singh, 2003). This observation has lead educators and researcher to realize that the two approaches are structurally different and so direct translation of traditional material to online will in no way yield a successful program. In addition, learning styles of each learner tend to be different, and hence, "a single mode of instructional delivery may not provide sufficient choices, engagement, social contact, relevance, and context needed to facilitate successful learning and performance" (Singh, 2003). An attempt to accommodate all these realized challenges is what lead to what come to be known as blended learning or blended e-learning. According to Singh,

> Blended learning mixes various event-based activities, including face-to-face classrooms, live e-learning, and self-paced learning. This often is a mix of traditional instructor-led training, synchronous online conferencing or training, asynchronous self-paced study (Singh, 2003).

Originally, blended learning according to Singh was often associated with simply linking traditional classroom training to e-learning activities; however, the term has now evolved to encompass a much richer set of learning strategies or dimensions. It is the combination of two or more of these dimensions that is today referred to as a blended learning. For a more detailed description of blended learning concept and it theoretical framework, one can see Badrul Khan (1997, 2001) and Sing (2003).

3. Methodology

A one semester experiment was designed to conduct the experiment and collect data for this study. Data were collected at the beginning (pre-program) and at the end (post-program) of the experimental semester. The pre-questionnaires was administered during the first week of the semester, and the post-questionnaire was given to the students in the last week of the semester. The software MINITAB was used to analyze the data collected.

3.1 Sample

The subjects of this study were 70 randomly selected students of the second pre-calculus course at the Prep Year Math program at King Fahd University of Petroleum & Minerals, Dhahran Saudi Arabia. These students are fresh from high school where the mode of teaching and the language of instruction are completely different. As a result of language switched from Arabic to English, all newly admitted students undergo a one year intensive English training. At the same time they are required to take two compulsory pre-calculus algebra courses (Math 001 and 002). The subjects of this study were the second pre-calculus course (Math 002) students at the preparatory year program at KFUPM.

3.2 Design of the experiment

Simplest form of blended learning was used in this experiment due to the fact that at the preparatory year, students are undergoing academic, social and environmental adjustment, and hence, need to be handled with sensitivity. Two modes of learning implemented during the experiment are the online and offline forms. The offline learning consisted of normal classroom lecture that was conducted three times in a week in a more or less traditional manner. The online learning consisted of a weekly computer lab session and availability of online learning resources in the intranet and internet available to the students. WebCT was used as a delivery mood of the online part of this course. Accounts were created for each student in the WebCT, and the following were some of the online activities of the students during the experiment.

- MATLAB manual was provided for the students online to enable them see an alternative way of solving problem using mathematical software. This was placed in WebCT platform for reference.
- Resources related to the material of the course were made available in the prep-year website for students' perusals. This includes solutions of the past exams, homework, and quizzes. The subjects were encouraged to use them.
- Some problems were posted online on weekly basis for the student to solve and submit online. These problems are usually none traditional, and in most cases students will require the uses of some software like MATLAB to solve.
- Self-Tests were provided online on regular basis. Students were expected to take these tests at their free time, and immediate feedback was provided to guide the student on his performance.
- Solutions of the exercises and exams are provided online.
- All announcements were sent online through WebCT
- Online discussion forum and e-mail communication was part of the program, and were vie WebCT.

3.3 Measurement

The Mathematics Attitude Scale by Aiken (2000) was used in this study for measuring students' attitude towards mathematics. Many different scales for measuring attitude towards mathematics and science are associated with Aiken. Three of these scales were reported in (Taylor, 1997). According to her, all the three scales "are characterized by their brevity, simplicity, and as such are useful instruments for both the teacher and the researcher" (p.125).

On the other hand, Computer Attitude Scale (CAS) developed by Loyd and Gressard (1984) was used to measure students' attitudes. According to Nash and Moroz (1997), the Scale is one such measure of attitude towards computers which has been used extensively with college students and professional educators. CAS consists of four separate subscales of different dimensions, these are: computer anxiety, which assesses the fear of computers; computer confidence, which assesses the confidence in the ability of dealing with computers; computer liking, which assesses the enjoyment of dealing with computers; and computer usefulness, which assesses the perception of the proliferation of computers on future jobs.

4. Results and Discussion

Table 1 presents the summary of the results in this study. The first column is the scales and their subscales. The second column gives the number of students that responded in the pre and post program. The third column is for the mean of the score of both the pre and post program, while the fifth column is for standard deviation of the two scores. The result of F-statistics is given in the sixth column, and the last column gives the significance or p-values. As can be noted from Table 1, only 50 students responded to the questionnaire during the pre-program data collection, and 65 responded in the post-program.

TABLE 1
Descriptive statistics and the summary of F and p-value of the comparison between Pre-program and Post-program data

Scale		N		Mean		Sta.Dev.		F	p
		Pre	Post	Pre	Post	Pre	Post		
Mathematics attitudes		50	65	55.22	51.51	12.57	10.29	3.03	0.08
Computer attitudes (total)		50	65	111.9	106.0	22.70	16.56	2.60	0.109
	Anxiety	50	65	28.64	26.19	6.30	5.09	5.35	0.023*
	Confidence	50	65	30.04	27.22	5.78	5.45	7.20	0.008*
	Liking	50	65	27.68	26.74	5.10	4.09	1.21	0.27
	Usefulness	50	65	29.66	28.49	5.49	4.96	1.43	0.24

*significant at p = 0.05

To be able to have a good interpretation of the data summary in Table 1, it is worth noting that in the computer attitude scale, the minimum point that one can obtain if he answer all the questions is 40, while the maximum is 160. Therefore, a student with a score of 100 or above is considered having positive attitudes. Similarly, in the mathematics attitude scale, student that answered all the questions gets a minimum score of 20, while the maximum is 80. Hence, a student with a score of 50 or above is considered having positive attitude toward mathematics. The mean of both mathematics and computer attitudes (total) in Table 1 indicated that the subjects have positive attitude towards mathematics and computer, and this was maintained in both pre-program and post program data. The subject appeared to have slightly higher mean in the pre-program data, but the difference is not statistically significant in almost all the items except for Computer Confidence (p = 0.008) and Anxiety (0.023). This finding is not inline with what was mostly reported in the literature; that the more computer use the less the anxiety and the more confidence (Or, 1997, Loyd and Gressard, 1984). It is difficult to trace the real reason for this slight change. However, a possible interpretation is that in addition to sudden change in the medium of instruction, the system in the preparatory year at KFUPM is more rigorous and higher in standard than what the students were used to in high schools. Consequently, at the end of each academic semester, a good number of students get exhausted and sometime frustrated. This frustration may result to some negative response. But this change is not peculiar to this experiment, rather general. Another possible reason might be due to the variation in the number of respondents in the pre and post data. As can be noted in Table 1, there are 15 people difference in the pre (50) and post (65) data. It is possible that most of those that did not respond in the first place are negatively oriented and so their responses in the post program bring down the mean. In any case, if the slight change of attitudes is due to the program, this was not noticed throughout the experiment. Student spent a lot of their time working and experimenting with various problems in the lab. Many students were fascinated with different type of graphs they were able generate using MATLAB, and the mathematics discussion in the webCT. The only complain we receive from some pocket of students was that they felt overworked compared to their other colleagues who were taking normal lecture in a traditional mood only, but they all agree that they have leant much more.

5. Conclusion

This study investigated the effect of blended e-learning on students' computer and mathematics attitudes. Student underwent a semester experiment of learning pre-calculus in both online and offline approach. Data were collected regarding students' attitudes towards mathematics and computer at the beginning and at the end of the program. Statistical methods were used to analyzed the data collected. The results did show any significant effect of the program in students attitudes toward mathematics and computers in all the items measured except for computer confidence and anxiety.

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Using Manipulatives in Mathematical Problem Solving: A Performance-Based Analysis

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Abstract: This article explores problem solving in elementary classrooms while focusing on how children use (perform tasks) manipulatives and/or tools in problem solving while working on mathematical tasks. Ways for teachers to assess children's learning through performance-based tool (manipulative) use will also be examined and suggested. Current research reveals that teachers need to teach and assess children's mathematical knowledge in ways that will allow them to show (perform) what they really understand. And, teachers must be able to see beyond obvious correct or incorrect answers into children's thinking processes by testing with "tests that allow students the opportunity to show what they know" (Van de Walle, 2003, p. 73).

Key words: Automacity; Classroom pedagogy; Manipulatives; Problem solving; Teacher practices..

1. Purpose and Introduction

In recent years and with the refinement of the *Principles and Standards for School Mathematics* (NCTM, 2000), it has become clear that standards-based mathematics teaching and learning is not only multi-faceted for the teacher, but also for the student. Acquisition of mathematical knowledge through problem solving and with manipulatives has long been considered to be time-consuming and labor intensive for many classroom teachers who are seemingly overwhelmed with high-stakes testing, published test scores, and unacceptable achievement scores on international measures (National Science Foundation, 2004). This has since encouraged educators to study ways in which teaching and learning occurs in the elementary classroom, particularly when a primary goal is to teach students fluency and flexibility with numbers and strategies for using those numbers (NCTM, 2000).

This article explores problem solving in elementary classrooms while focusing on how children use (perform tasks) manipulatives and/or tools in problem solving while working on mathematical tasks. Ways for teachers to assess children's learning through performance-based tool (manipulative) use will also be examined and suggested. The term, *manipulative*, will be defined as any tangible object, tool, model, or mechanism that may be used to clearly demonstrate a depth of understanding, while problem solving, about a specified mathematical topic or topics. *Performance-Based Assessment* implies that the measure encourages students to perform, create, and produce solutions while using contextualized problem solving and higher level thinking (Hatfield, Edwards, Bitter, & Morrow, 2005).

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Developing higher level thinking skills and fluency and flexibility with numbers in young students supports the idea for implementing manipulative-based problem solving in the classroom. While more traditional drill-and-practice, non-manipulative-based methods for teaching and learning mathematics might imply that understanding beyond automaticity does occur, it would seem that multiple and varied methods for teaching children mathematics should be explored.

Van de Walle (2004) describes automaticity as performing a task mindlessly and quickly (p.86), which implies that drill and repetition of isolated skills may not be as essential as it once was. There are certain mathematical skills that many believe must be committed to memory automatically (e.g. rote counting, basic facts, ordinal positioning, etc.); however, what a problem centered approach to mathematics teaching brings to the forefront is a connection to real life problem solving and connects with more students more of the time. “Developing flexible thinking strategies requires adequate opportunity with varied numbers and contexts…the development of different methods of thinking requires problem-based tasks” (Van de Walle, 2003, p. 87). If we want children to learn to think deeply and ponder real mathematics as well as to be able to use in depth thinking in real life scenarios, we must teach them and assess their knowledge in ways that will allow them to show us what they really understand about the tasks being tested. Additionally, teachers also need to be able to see beyond the correct or incorrect answers and garner a larger, more diversified view of each child’s mathematical understanding. These teacher-skills seem to develop quicker and more fully in classrooms where a problem-based approach is in place (Van de Walle, 2004).

However, growth in problem-based teaching techniques seems to evolve slowly. In analyzing how teachers teach mathematics, results from the TIMSS (Third International Math and Science Study) Video Study of Teaching revealed that of the two-thirds of teachers who felt that they were implementing real-world applications and group work with problem solving, only 19 percent actually implemented activities involving problem solving (Hiebert & Stigler, 2000). Additionally, the Program for International Student Assessment (PISA), which assessed student competency following mandatory education and looked at mathematical literacy and problem solving skills applied to real-life situations, reported mathematical skills scores for American 15 year olds that were below average, ranking them 24th out of 29 countries (NCTM, 2005). This has led mathematical experts to conclude that teachers are not teaching problem solving within and among real-world applications (NCTM, 2005) and that more work needs to done in strategizing ways in which to facilitate and guide teacher’s mathematics instruction.

2. Problem Solving in Elementary Mathematics

Lambdin (2003) describes problem solving as somewhat cyclical and interdependent with understanding. “Understanding enhances problem solving… learning through problem solving develops understanding” (as cited in Lester & Charles, Eds., NCTM, 2003, p. 7). As Lambdin (2003), Wilson, Fernandez, & Hadaway (1993) also imply that problem solving is cyclic in nature. When a student focuses on a problem, thinks s/he understands it, and devises a solution plan, a series of steps (processes) are initiated and revisited as the student’s thinking continues to evolve. Yet when implementing the plan of action, s/he discovers a misconnection in her/his understanding of the problem which requires revisiting the problem. Thus, it seems that problem solving is an iterative process and, if this is the case, all elementary, middle, and high school teachers might be well advised to include problem solving in the majority of their mathematics teaching. Van de Walle (2004) calls problem solving "a principle instructional strategy" (p.36) used to fully engage students in important mathematical learning; thus, it seems that problem solving might permeate not only almost every mathematical task, but also life in general.

The real question is: How do children use problem solving and what might their selection of tools, manipulatives, or materials for creating and showing their solutions look like? Reusser (2000) argues that, "children are active individuals who genuinely construct and modify their mathematical knowledge and skills through interacting with the physical environment, materials, teachers, and other children" (p.18). In reality, how many times are decisions made through problem solving right in the child's home at the breakfast table when five people choose to eat cereal for breakfast and only one cup of milk remains in the carton or when you get in your car to drive to work and suddenly find a flat tire? Aren't these everyday situations in nearly everyone's lives that imply some knowledge of problem solving skill? And, when needed, wouldn't most people solve either of these problems relatively efficiently? It seems that we do use problem solving on a regular basis, and, usually our problem solving tasks appear to be connected to some form of tool, tangible, or manipulative (tires, air pressure gauges, cereal types, sizes, and characteristics, and so on…). If this kind of problem solving were connected both directly and indirectly to problem solving with numbers and mathematics, might this become a natural conduit for a deepening sense of knowledge in mathematics? The goal here would be to guide elementary children to take the performance-based knowledge gained from choosing and using manipulative-based materials, as described previously, into the creation of models which are, subsequently, connected to numbers and algorithms.

Young children are immersed in problem solving as they experiment with and intuitively contemplate how things work, what makes it warm or cold outside, and what series of hand manipulations are required to tie the laces on a pair of shoes. If teachers and parents were to facilitate this type of experimentation with models, diagrams, and, finally numbers, it would seem that the fine line between everyday problem solving *with* tools and manipulatives and mathematical problem solving *through* tools and manipulatives would fade away. In other words, the mathematics would optimally gain fluency and flexibility and, at the same time, automaticity, with less traditional drill.

Realistically, sometimes adults assume far too much about the connections piece of the puzzle and hurriedly skip over a short lesson on the sequence of steps needed to tie a shoe (ordinal positioning) or what temperature, in numbers (degrees), really represents. For the latter, temperature is often reported to children as hot or cold, cloudy or sunny, windy or humid, rather than numerically with clear and accurate descriptions. Children are very able to decipher why 18º Fahrenheit is "cold" and 88º Fahrenheit is "hot", numerically and mathematically, when allowed to use problem solving in such situations with the guidance of the teacher. When the teacher uses accurate mathematical and meteorological language to describe the *difference* (70º) between the two temperatures as a part of the process, it most likely will become a functional part of the child's problem solving process and offer them more opportunities for developing real understanding.

3. The Role of the Teacher in Problem Solving

Innately, most teachers seem to teach problem solving as a series of steps and/or in linear fashion, while most students need not just the linear set of steps, but also a full array of ongoing, supported opportunities to indirectly develop and hone problem solving techniques. This in depth development of problem solving does not denote full understanding of the mathematical task at hand, nor does it imply that it is done in isolation, rather it is usually accomplished through engaging problems in which children connect new and previous information (Lambdin, 2003).

One of the foremost criteria for enhancing problem solving skill in children appears to involve being taught by teachers who are able to facilitate rather than direct the learning. Desoete, Roeyers, and Buysse (2001) studied the relationship between metacognition and problem solving in 80 third graders and found that a connection between these two variables occurred more significantly among above-average than novice students. This might imply that children who experience more metacognitive or introspective thinking opportunities may subsequently become stronger problem solvers, which might suggest that more interactive approaches to teaching could develop stronger thinkers and problem solvers. It is also important to ponder varied strategies for guiding teachers who are reluctant to embrace manipulatives and problem-based methods in their classrooms.

Part of what happens when teachers are reluctant to use perceived innovative, manipulative- and problem-based approaches relates back to perceptions of what mathematical problem solving and learning looked like in their own learning – most likely an abstract word problem or an algorithm written on the chalkboard. In reality, the latter technique is a relatively far-fetched approach to the problem solving that will occur everyday. This becomes the impetus for striving to apply realistic, everyday situations to the mathematical skill development in the classroom using a *situational context* or real-world context to teach and learn within, implying that the elementary mathematics teacher should observe, facilitate, and foster problem solving within and among academic disciplines to guide a natural and non-threatening approach to everyday problems. Such an approach also reduces the abstractness of the problem solving, moving it into a more concrete, reality-based setting. Concrete and less abstract situations may arise during reading groups where children will need to determine the number of pages read each day in order to complete a chapter book in a two-week period or how many children would need to be in each group if there were thirty-one children in all and each group needed five members. When children are encouraged to think like problem solvers on a regular, everyday basis, they will most likely become effective and confident mathematical problem solvers during mathematics instruction and use.

4. Strategies for Teaching and Assessing Problem Solving with Manipulatives

Good elementary teachers are masters at modeling appropriate strategies for children. They bring their own automaticity, or fluidity, to master instructional skill in mathematics when they not only use the numbers, algorithms, and processes needed to solve a specific problem, but also show, through interactive modeling, what the problem actually looks like. For many children, this is the moment of enlightenment in their mathematics learning because they are actually able to see and touch “the problem” while associating the “model” with the numbers. This is the essence of tool- and problem-based teaching and learning which, in turn, affords children multiple opportunities to construct mathematical knowledge while making reasonable connections to everyday tasks. This is the real bridge between the concrete and abstract.

Although formal testing of knowledge will continue to be evident in determining how much a person knows about a particular topic, it "need not be a collection of low-level skills exercises" (Van de Walle, 2003, p. 72). Assessment of mathematics learning should be cohesively connected to mathematics instruction, which is often done using models and/or manipulatives. Van de Walle (2004) argues that within a well-constructed test "much more information can be found than simply the number of correct or incorrect answers" (p.72). This is important since good instruction (prior to testing) should have included performance-based use of models, drawings, and other representational depictions through which students developed further relational understanding between and among mathematical concepts. Then, during testing, the same models and/or manipulatives should be included as a relevant piece of the assessment and not simply test fragments of learning in isolation (performance-based assessment) or "tests that allow students the opportunity to show what they know" (Van de Walle, 2003, p. 73).

Investigation of manipulative use in problem solving within a classroom setting often reveals teachers who are invested in meeting the diverse needs of all students (Equity Principle, *Principles and Standards for School Mathematics*, NCTM, 2004), but who are averse to using manipulatives for varied reasons.

First of all, teachers need to know when, why, and how to use manipulatives effectively in the classroom as well as opportunities to observe, first-hand, the impact of allowing learning through exploration with concrete objects. Constructivism has evolved from theorists such as Jean Piaget (1965) and Lev Vygotsky (1962). Piaget (1965) approached the construction of knowledge through questioning and building on children's answers while they constructed knowledge while Vygotsky (1962) felt that children could be guided to stronger mathematical understandings as they progressively analyzed complex skills on their own with the teacher nearby to scaffold or facilitate as needed.

Prior to discussing specific strategies, it is important to delineate some of the necessary benchmarks for effective manipulative use. First, it is essential for teachers to realize the impact of referring to manipulatives as *tools* to help students learn math more efficiently and effectively rather than as *toys or play things*. If manipulatives are referred to as "toys", students will see them as something to play with rather than as tools to work with to better understand mathematics. Second, manipulatives must be introduced in a detailed format with a set of behavior expectations held firmly in place for students to begin to develop a respectful knowledge-base about using manipulatives for math learning. Third, manipulatives need to be modeled often and directly by teachers in order to help students see their relevance and usefulness in problem solving and communicating mathematically. And, finally, manipulatives should be continuously included as a part of an exploratory workstation or work time once open explorations have been completed.

Teachers who consistently and effectively model manipulatives in front of all students will automatically offer all students a belief that using concrete objects to understand abstract concepts is acceptable and expected. Just as encouraging students to develop mental models, Venn diagrams, flowcharts, or matrices helps to broaden and enhance complex mathematics learning, so do manipulatives. If students feel that manipulatives are only used by those who are can't or don't understand or are less able, they will develop a negative attitude about manipulatives and an unwarranted stigma toward manipulatives will be launched.

5. Effectively Introducing and Implementing Manipulatives in Performance-Based Tasks

Preparing students to use concrete objects in mathematical exploration and problem solving is often overlooked, but is, truly one of the essential elements of successful implementation of a manipulative-based math program. Following these ten essential steps will help to establish a manipulative standard for the classroom (Author, 2002):

1. *Clearly Set and Maintain Behavior Standards for Manipulatives*

Students need to have a clearly established criteria for effectively handling and using manipulatives in the classroom. Without a clear set of expectations, students may misuse materials and teachers will become frustrated and disillusioned about manipulative use and, most likely, discontinue their use in the classroom. Rules for specific activities that incorporate manipulatives must be clearly articulated by the teacher, posted in the classroom, and re-affirmed consistently as needed during the manipulative lesson. In short, students need to be meaningfully guided to use and understand the purpose of the manipulative for the specific math task at hand, and then, it will gain relevance for them as mathematicians.

2. *Clearly State and Set the Purpose of the Manipulative Within the Mathematics Lesson*

If students know why the teacher has a certain expectation for a lesson, he/she will be much more likely to attend to the purpose of the task and handle the lesson manipulative correctly. It is important to remember that most math manipulatives are colorful, enticing, and closely resemble what most students have previously referred to as "toys". Since this is a natural association, it is of primary importance that teachers consciously facilitate understanding of the difference between math manipulative or tool and toy. If this is done carefully and effectively at the beginning of the academic year, students will be much less apt to misuse or mishandle mathematics manipulatives.

3. *Facilitate Cooperative and Partner Work to Enhance Mathematics Language Development*

The nature of manipulative use encourages interaction with not only objects but also with people since it usually involves an *action* on an object. Being able to learn and use mathematical language effectively helps to lay a strong foundation for conceptualizing and using abstract math skills in everyday life. It also helps students to develop and feel mathematical power as they become more able to articulate, both verbally and in written form, their math thinking processes. Using partner work with manipulatives to construct mathematical meaning also allows the more reticent math student a supported opportunity to explore strategies from both the observer and participant viewpoints.

4. *Allow Students an Introductory Timeframe for Free Exploration*

Once the purpose and behavior expectations have been established, students need to be given an opportunity to become familiar with the manipulative, discover its properties and limitations, and experiment with it in a variety of contexts. This, too, encourages cooperative work, language development, and risk-taking. Free exploration gives less active students individual opportunities to construct their own meaning and develop confidence in using the manipulative to solidify and enhance their math understanding.

5. *Model Manipulatives Clearly and Often*

Modeling on the overhead or in large or small group sessions will help students see how a particular manipulative can facilitate understanding. For example, when students are beginning to learn about measurement and non-standard or arbitrary units of measure, it is essential that they have a large variety of manipulatives (Cuisenaire rods, Unifix cubes, links, paper clips, pencils, etc.) with which to measure commonly used items (desks, chalk board ledges, window sills, etc.). In doing so, students will be developing real number sense about measurement. As a means for developing number sense, Marilyn Burns (1997) suggests that teachers include as much measurement as possible in their mathematics teaching as it is the foundation upon which strong mathematical understanding is built.

6. *Incorporate a Variety of Ways to Use Each Manipulative*

Offering students different ways to view the same problem will ensure that more of the students will gain a deeper and richer understanding of mathematics. In turn, they will further develop their own levels of fluency and flexibility with numbers as suggested by the new *Principles and Standards for School Mathematics* (2000). And, showing students how to use the same manipulative in a variety of ways will only tend to strengthen their everyday use and understanding of mathematics. For example, students may use pattern blocks initially to learn colors, shapes, or patterns and eventually create and decipher equal and unequal fractional parts.

7. *Support and Respect Manipulative Use by All Students*

The Equity Principle (NCTM, 2000) clearly states that high expectations and strong support for all students must be evident for excellence in mathematics education. Be sure to clearly and positively “set the stage” in your classroom for inclusion of all students in manipulative use. If you model and use manipulatives, mental models, and other tangible materials to problem-solve, your students will be much more apt to do the same. In turn, when teachers openly express less positive feelings about using manipulatives to solve problems, those who require tangible objects to reach success will be less likely to use them and, subsequently, less likely to gain a firm grasp of the math skill or concept in front of them.

8. *Make Manipulatives Available and Accessible*

In order to facilitate manipulative use at any grade level, the chosen and/or required manipulative must be stored in such a way that it will be physically reachable by all students, plentiful enough (in number) to allow each student to have access to a complete set, and labeled correctly with clear instructions as needed based on the intended purpose, which is not to say that creative manipulative use should not be sanctioned.

9. *Support Risk-Taking and Inventiveness in Both Students and Colleagues*

Teachers who model risk taking and are open to mistakes and re-thinking will enhance student’s abilities to move into uncharted territory. Supporting risk-taking and inventiveness in students leads them to explore unknowns and strive to reach unanswered questions as it facilitates open-mindedness and creativity. Students should be supported in seeking and using their own processes in problem-solving. Manipulatives are natural conduits for successful, interactive construction of knowledge through problem-solving.

10. *Establish a Performance-Based Assessment Process*

Since manipulative use is based on constructing or performing an action with a tangible object or set of objects, finding out what students know must also be based on active teacher observation and a set criteria of expected outcomes or, usually, a rubric-style assessment tool. Assessing hands-on inquiry can be a challenging task that implies a commitment of time and energy beyond paper-pencil measures of achievement. Authentically taking into account what a student knows and can do (perform) following a manipulative- and inquiry-based task requires keen observation skills and patience from the teacher. What the teacher actually *sees* the student do with a manipulative-based task is nearly as important as the mathematical thinking that the student can verbally organize and communicate coherently and clearly to teachers, peers, and parents. In assessing hands-on inquiry with manipulatives, the ten standards for school mathematics (NCTM, 2000) are actualized through verbal and non-verbal means. Developing rubric-style assessments for manipulative-based activities with students and colleagues helps to assure that the assessment actually measures what was taught and practiced, to bring strong student investment in the teaching-learning process, and develop real mathematical learning.

6. Conclusion

Early examples of the benefits of a manipulative-based mathematics program can be seen in kindergarten and primary classrooms where young children are using manipulatives, such as Algebocks, to learn algebraic concepts such as patterns and functions. In turn, bubbleology and materials, like Zometools, plastic Polydrons, and connected drinking straws, are helping very young children learn about the properties of angle, shape, and congruence in geometry. Realizing this, one might ask – "Would it be as easy to teach young children these more abstract concepts without manipulatives?" Probably not, and, in turn, isn't it plausible to surmise that middle and high school students who might struggle with math and discontinue math classes altogether after requirements are met, might also benefit from teacher encouragement through manipulative use? If manipulative use becomes an integral part of the academic structure for all students in mathematics classrooms, it may keep more students in higher-level math classes through college and beyond.

Students quickly pick up on what teachers verbal and body language tells them very quickly, particularly when it effects overall self-esteem and confidence. If teachers send messages to students that "only less able students need manipulatives" or "you don't need those anymore, do you?", students will be begin to devalue and stop using manipulatives and reduce their own chances for success in mathematics overall. Usually students will naturally choose tools that will best help them learn a particular concept, and, if manipulatives help to make mathematics "come to life", they should be encouraged.

Finally, with research on performance-based manipulative use seeming to be varied and limited, it would be beneficial for longitudinal study on the year-to-year transfer of children's problem solving abilities with manipulatives to be examined. Not only would it benefit teachers and parents, but also future employers and businesses.

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Frames of Reference and Achievement in Elementary Arithmetic

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Abstract: This paper considers the relationship between 8-11 years old students' numerical achievement and a possible disposition towards the construction of particular frames of reference. The paper uses the characteristics of a variety of *kinds* of images to focus upon frames of reference and explores a possible relationship between children's verbal descriptions of concrete and abstract nouns and the different ways they respond to aspects of elementary arithmetic. It seeks to establish whether or not children towards the extremes of arithmetical achievement (low and high achievement) have a disposition towards different kinds of frames of reference. The analysis suggests that at one extreme these may be largely descriptive, associated with episodes or specific characteristics that are derived from efforts to concretise the stimuli. At the other, these descriptive qualities are complimented by more relational characteristics that are indicative of greater flexibility in mathematical behaviour. The conclusions suggest that such differences may influence the interpretation that some children may make of the objects and actions that are the foundations of their numerical development and, as a consequence, this may affect the quality of the child's cognitive shift from concrete to abstract thought.

Key words: Frames of reference, arithmetic, mental representations

INTRODUCTION

In considering the different frames of reference that children project through their words this paper explores what may be a significant contributory factor in children's numerical development. It presents the possibility that the numerical growth may be influenced by a disposition towards articulating different frames of reference. We associate different kinds of frame with different kinds of a mental representation. The latter, seen as a mental reference that, divorced from the objects that give it place in the real world, is the product of imaging. Modality free, a mental representation may be seen as a presentation to the mind in the form of an idea or an image but unfortunately mental representations and mental processes are not directly observable. As a consequence, our guiding principle has not been to consider whether the form of a mental representation is verbal or visual but to recognise that human cognition requires different representational constructs to describe it. It is partly for this reason and partly because of the difficulties inherent in the study of mental representations that we identify the presentation made to the mind as a 'frame of reference'. Contextualised through an examination of selected literature on imagery, different frames of reference are identified and categorised from an analysis of what it is that a sample of children from the same elementary school choose to talk about when asked to respond to a variety of verbal stimuli.

Introduced by Minsky (1975) to indicate the way that an individual structures knowledge, frame of reference may be defined as:

> The context, viewpoint or set of presuppositions or of evaluative criteria with which a person's perceptions and thinking seem always to occur and which constrains selectively the course and outcomes of these activities (Bullock & Stromley, 1988, p. 334).

We conjecture that the notion may provide us with the means to utilise theories associated with different kinds of imagery to provide additional insight and understanding of cognitive development in arithmetic and in particular our understanding of a divergence in numerical thinking.

The perspective for our discussion is placed within the theoretical context that embraces the notion that mathematical symbolism is open to ambiguous interpretation. Learners' interpretation of this symbolism has suggested that there is a bifurcation in approach that on the one hand may lead to flexible thinking but on the other to a procedural cul-de-sac (Gray & Tall, 1994).

Why should this happen? To gain a partial answer to this question we seek to identify whether or not different frames of reference appear to be dominant within 8-11 years old children at the extremes of achievement in elementary arithmetic. Our task is to establish what ideas are central to the thoughts of a child when, in the absence of perceptual items, there is an invitation to articulate their perceptions of a selection of concrete and abstract nouns. Our fundamental thesis is that there is a relationship between what a child chooses to talk about and the ways s/he may think about elementary arithmetic.

THEORETICAL BACKGROUND

The conceived cognitive development of numerical concepts would appear to be underpinned by the encapsulation of actions with perceptual items (Piaget, 1965; Steffe, von Glaserfeld, Richards, & Cobb, 1983; Kamii, 1985; Gray & Tall, 1994). The reconstruction of well-rehearsed procedures continually involves a qualitative change that enables the concept of number to be conceived of as a construct that can be manipulated in the mind. Such a change suggests that there is no longer the need to transform the object-like essence of this mental object into an action associated with perceptual or figural items. Fortunate are those who recognise this. However, for others the meaning may remain at an enactive level whereby elementary arithmetic remains a matter of performing or re-presenting an action with real or imagined items (Steffe et al., 1983; Gray & Pitta, 1999). We suggest that these different perceptions may represent a spectrum of thinking that is typified by qualitative differences in the kinds of things children communicate when responding to a request to articulate what they have in mind when they think about a range of verbally presented concrete and abstract nouns.

Substantial interest in the cognitive development of mathematics has focused on the relationship between actions and entities (Dienes, 1960; Davies, 1984; Dubinsky, 1992; Sfard, 1992, Gray & Tall, 1994). The qualitative change associated with actions becoming objects of thought has been linked with theories accounting for the transformation of processes into concepts. These have helped to shift attention from *doing* mathematics to *conceptualizing* mathematics. Although consensus recognizes that reconstruction as a result of process/object encapsulation or procedural reification does occur, we are still a long way from being able to describe how this is done although there is evidence to suggest that we may confirm whether or not it has been done. For example, Gray and Tall (1994) suggest that an analysis of a child's interpretation and use of numerical symbolism can provide such evidence in the field of elementary arithmetic. On the one hand the evidence may point to the substantive use of a procedure such as count-all, whilst on the other it may be exemplified by the use of known fact to establish unknown ones. The essence of the differences in the level of sophistication that may be determined from such an observation is captured within the notion of the proceptual divide (Gray, Pinto, Pitta, & Tall, 1999). This suggests that on the one hand we may see a cognitive style more in tune with the flexibility that is associated with an appreciation that a numerical symbol can represent a process or a concept, whilst on the other, the cognitive style may be more dominantly associated with a procedural interpretation of the task in hand.

Though the theories mentioned above have intrinsic differences, they share common ground in their attempts to account for the cognitive reconstruction that underscores the development of conceptual thinking in mathematics. However, it is our conjecture that an individual's perception and interpretation of the original objects and the actions that may be associated with them will influence the quality of this development. Objects have different facets and therefore are subject to different interpretations. For example, counting starts with objects perceived in the external world which have properties of their own; they may be round or square, red or green or both round and red. However, these properties need to be ignored (or at least temporarily disregarded) if the counting process is to be encapsulated into a new entity — a *number* that is *named* and given a *symbol* — that may then be associated with new classifications and new relationships. This foundation can provide a basis for growing sophistication in the nature of the entities operated upon and this is manifest in an increasing detachment from immediate experience. Actions with physical objects become the basis for mental operations with the number symbols which then act as if they were objects.

Qualitatively Different Ways of Thinking

The qualitative differences in the mathematical thinking of novice and experts have been extensively discussed over the years and they still receive considerable interest (Kruteskii, 1976; Bransford, Brown & Cocking, 1999; Baroody & Dowker, 2003). Gray (1991) revealed that the qualitatively different ways that young children approached tasks in elementary arithmetic reflected the consequence of divergent ways of thinking. Some children remained at a procedural level, which, in terms of information processing, could make things very difficult for them. Others operated at a conceptual level that appeared to provide greater flexibility. Later re-evaluation of the children's interpretations of symbolism enabled these differing levels of sophistication to be placed within the context of the 'proceptual divide' (Gray & Tall, 1994). Hypothesized to be a cognitive difference between those children who processed information in a flexible way and those who invoked the use of procedures, this term was derived from the notion of procept, a mathematical symbol that ambiguously represents process and concept (Gray & Tall, 1994). It provides a basis for articulating the potential divergence between those who see mathematics as a sequence of procedures and those who are able to utilize the flexibility associated with process/concept duality. However, although it clearly focuses on interpretations of symbolism it may also have a relationship with other dichotomies in mathematical thinking for example, instrumental and relational (Skemp, 1977), procedural and conceptual (Hiebert & Lefevre, 1986) or operational and structural (Sfard, 1991). For those demonstrating instrumental, procedural or operational thinking it is possible to achieve success but they are arguably on a spiral leading to increasing difficulty in their personal construction of mathematical concepts. In contrast, it is suggested that those demonstrating relational, instrumental and conceptual thinking have a cognitive advantage since they appear more able to associate procedures that enable them to perform arithmetical operations with number concepts.

The existence of a proceptual divide not only seems to indicate that some give a different interpretation to an arithmetical activity but it also seems possible that they do not perceive the potential of the activity they engage with. Their perception may be closely linked with identifying properties of objects (Pitta-Pantazi, Gray & Christou, 2002) or on remembering particular actions with these objects. However, even when teaching programs are designed to shift the child's focus from processes to thinking strategies (Thornton, 1990; Howat, in preparation), children who may be entering the procedural cul-de–sac appear to resist a change in the apparent security offered by their use of well established procedures.

Pitta and Gray, (1997) suggest that children who have difficulty with elementary arithmetic often appear to have difficulty in isolating numerical symbolism from the perceptual and figural objects and actions that give it meaning. The assumption that all children will extract from their experience with particular representations that which will enable them to become "experts" in a particular aspect of mathematics has been questioned by Cobb, Yackel, and Wood (1992). Pitta and Gray (1997) suggest that children can focus on qualitatively different aspects of such representations which may have consequences for the quality of the object that eventually dominates the child's thought. Sáenz-Ludlow (2002) suggests that:

> In the classroom, mathematical concepts are constructed and comprehended through an intentional process of interpretation, guided by the teacher, of mathematical notations/representations. In such an interpreting process, mathematical notations could represent mathematical objects for the teacher but for the learner it will entail a recursive representational process before he comes to clearly see that mathematical object." (p.37)

A tendency towards a recursive representational process may be short lived amongst some learners but longer-term amongst others. Why is this so? One response to this question is that presented within this paper. Notwithstanding the variety of variants that may influence the acquisition of knowledge, for example social and cultural, the classroom environment, and the quality of pedagogy, this paper examines whether or not there may be a relationship between children's level and quality of achievement in elementary arithmetic and their frame of reference. Though the child's developmental increase is frequently self-evident within the elementary school, the psychological mechanisms and the components that support or undermine it are not fully identified or understood (Baroody & Dowker, 2003). If we could recognize these components and understand better these mechanisms and their associations we could probably increase our ability to describe the qualitative difference between those on different arms of the proceptual divide.

Different Kinds of Mental Representation

The notion that more than one type of mental representation exists has received extensive comments (Presmeg, 1986; DeBeni & Pazzaglia, 1995; Drake, 1996; Cifarelli, 1998). Presmeg's distinction between concrete pictorial imagery, which maintains a focus on irrelevant detail, and pattern imagery, which disregards such detail and focuses on relationships, is significant to our discussion in the emphasis it places on the difference between 'description' and 'relationship'.

Such a distinction was a theme alluded to by Drake (1996) who considered types of imagery at three levels of sophistication. Level 1, common to all respondents within her study, was characterized by the reporting of very concrete visual images in which respondents were either observers or participants. Such images, primarily visual and from the subject's known physical world, were regarded as a tool or a program to achieve a particular goal. A child who reports the use of imagined counting units such as fingers or counters might be seen to project imagery at this level. Subjects operating within level 2 identified concrete and highly pictorial images which could act as a symbol. Some of the number forms reported by Thomas, Mulligan, and Goldin, (1995) and Seron, Presenti, and Noël (1992) may fit this category. Such images were primarily visual but could come from other modalities. Images classified as Level 3 were abstract and formed from all modalities. Such images are only arbitrarily related to the real word and in essence are symbols.

The relationship between understanding and imagery suggests that concrete and memory images appear to dominate amongst instrumental thinkers whilst abstract imagery appears to predominate amongst relational thinkers (Brown & Presmeg, 1993). More particularly, the development of mathematical thinking can be seriously influenced by strong early attachments to particular dominant images (Pirie & Kieran, 1994). Such observation may be particularly relevant in a child's development of concepts within elementary arithmetic. Those who predominantly use procedures appear to display less inclination to filter out information (Gray & Pitta, 1997). Relational thinkers on the other hand appear to reject information or, to put it another way, they are more able to select the information that is most relevant to a particular situation. This would suggest that images that reflect different levels of sophistication might have their roots in qualitatively different types of abstraction and the formation of qualitatively different frames of reference. The individual's active mental process of making sense of data through direct involvement may in turn reflect a predisposition towards a particular type of interpretation that reflects a personal frame of reference.

Attributes associated with different types of personal involvement with concrete nouns have featured in the classification of different kinds of imagery explored by Cornoldi, De Beni and Pra Baldi (1988) and De Beni and Pazzaglia (1995). Cornoldi et al. suggest that images spontaneously evoked from a single verbal cue may be identified as *general*, *specific* or *autobiographical* with the frequency of occurrence appearing in decreasing proportions. General images represent a concept without any reference to a particular example or to specific characteristics of an item. Reference to a single well-defined example of the concept without reference to a specific episode is identified as specific images. Autobiographic images are seen to be special cases of the 'specific' category that is enlarged to include the involvement of the self-schema. These involve either the subject without a precise episodic reference or objects belonging to the subject.

De Beni and Pazzaglia (1995) questioned the meaning that may be given to the autobiographic image category. They suggested that there is a distinction between images that refer to a single episode in the subject's life (*episodic–autobiographic*) and those that actually involve the subject without a precise episodic reference (*autobiographic images*).

We see these terms helpful in our attempts to classify of the frames of reference identified from children's articulation of what came to mind when they were presented with a series of verbal stimuli. The notion that different frames of reference may be related to children's mathematical understanding is not only represented within the literature but it is also suggested by direct evidence drawn from children's mathematical behavior (Gray & Pitta, 1999). Therefore, it is possible that a child's tendency to respond to the stimuli in consistent ways and their ability to use the process/concept ambiguity of mathematical symbolism are probably linked. The quality of the cognitive shift, which supports an individual's recognition of the latter, is associated with the 'choice' that is associated with identifying the characteristics that form the essence of their mental representation of the activity.

The consequences of this "personal selection" process have been displayed during several of our encounters in school. Although the relative merits of choice, interpretation and enculturation may be difficult to determine, the outcome of making unsophisticated choices can lead to disquieting observations (Gray & Pitta, 1999). There is an obvious tension between the interpretation of numerical activity as the external manipulation of physical objects and the internal manipulation of mental objects (Gray & Pitta, 1997).

RESEARCH CONSIDERATIONS

A mathematical object may be seen as a theoretical construct but at issue is whether or not the frame of reference a child associates with such an object differs from that which the child associates with concrete objects identified from their conceptual labels. We will argue that consistency in the quality of the interpretation has implications for the individual's tendency to transform arithmetical processes into numerical concepts. Two questions guide the development:

- What frames of reference may be associated with children's discourse on a series of presented verbal stimuli?
- Do particular kinds dominate amongst children who reflect different levels of achievement in elementary arithmetic?

A phenomenological approach seemed be the best way to understand the frames of reference that the individuals associated with the concrete and abstract nouns they were invited to elaborate on. The work of De Beni and Pazzaglia (1995) that used high image evoking nouns reflects such an approach and it is one that we shall follow. However, since we were attempting to establish links between frames of reference associated with arithmetical concepts and frames of reference associated with image evoking nouns our item bank contained examples of both.

We apply a qualitative approach since we feel that it is the initial identification of the surface details and their possible relationship with other aspects of knowledge that is important but our attempt to investigate frames of reference is not infallible. Although the method follows a psychological perspective and involves both immediate and more prolonged reflection on the items presented, it is possible that our subjects may not be providing personally established frames. They may also be providing frames associated with descriptions, interpretations and beliefs that have been acquired through habituation or instrumental/rote learning. However, even these pieces of information have the potential to be informative if a discernible pattern can emerge in the quality of the responses but in an attempt to minimise such influences we consider children from within one school.

To establish the frames of reference associated with the presentation of given items we chose to concentrate on what it was children focus upon when they talked about objects of the real world and objects of the abstract world. Words (associated with introspection) have formed the backbone for much of the research into children's strategies in elementary arithmetic (Steffe, et al., 1983; Carpenter & Moser, 1982; Seigler & Jenkins, 1989; Gray & Tall, 1994). It is therefore an approach we use to examine the existence of different frames of reference.

Method

In designing a methodology that seeks to explore the relationship between arithmetical achievement and mental representations several different theoretical frameworks are interwoven:

- Accounting for differences in children's arithmetical behavior may be seen to have direct links with the notion of actions interiorized as concepts (Piaget, 1965), different forms of mathematical understanding (Skemp, 1976) and qualitatively different forms of mathematical thinking (Gray & Tall, 1994). The dimension presented within this study takes forward the latter by considering frames of reference associated with extreme levels of arithmetical achievement. The paper considers whether or not a disposition towards different particular frames of reference may contribute towards qualitatively different thinking about arithmetic.

- The way in which children use their knowledge inevitably falls within an information-processing paradigm. However, an investigation that seeks to discover how this knowledge is constructed and the way in which a disposition towards particular frames of reference may contribute to it, draws upon a constructivist philosophy,

* The development draws in the work of cognitive psychologists particularly that of De Beni and Pazzaglia (1994, 1995).

Our approach does not attempt to build a general model established from *common* cognitive processes but instead attempts to consider differences in behavior exhibited by children who *already display differences* because they represent different ends of a spectrum in numerical achievement.

Item Selection

Two phases guided the preparation of item clusters within the study. The first focused on the quality of the children's achievement in elementary arithmetic, and the second on the range of items selected to obtain insight into each child's general frame of reference.

The numerical items included a range of number combinations presented verbally — the children were invited to respond to these using mental methods only — and in written format to which children could respond using a written format. The intention of both was to establish achievement and to identify the strategies children applied to obtain solutions. The problems were derived from Gray (1991) and from a range of past Standard Assessment Tasks (SATs) published by the Schools Curriculum and Assessment Authority (SCAA, 1995. 1996, 1997). This item bank also formed an integral part of a comparative study examining the influence of curriculum change on children's achievement (Gray, Howat & Pitta, 2002)The range of verbally presented items included:

- Combinations to 20 such as: 3+2; 9–7; 7+6; 4+7; 17–13; 12–8
- Two digit combinations such as: 14+8; 29–6; 64–26; 27+62; 73–32, 45+57
- Three digit combinations such as: 188+267; 396–157.

The written combinations included:

- Combinations to 20 such as: 6+3; 3+5, 9–8; 15–8, 13–5, 9–2
- Two and three digit combinations presented in the style seen within particular SAT's

$$\begin{array}{r} 30 \\ \underline{+57} \end{array} \qquad \begin{array}{r} 47 \\ \underline{+15} \end{array} \qquad \begin{array}{r} 274 \\ \underline{+159} \end{array} \qquad 5 + 135 \qquad 11+696$$

$$\begin{array}{r} 21 \\ \underline{-15} \end{array} \qquad \begin{array}{r} 82 \\ \underline{-24} \end{array} \qquad \begin{array}{r} 293 \\ \underline{-185} \end{array} \qquad 438 - 21 \qquad 687 - 47$$

As each item was completed the child was asked to indicate what was happening in their head as it was being done. If necessary, a supplementary question or questions clarified a particular response. Thus for each child an overall level of achievement and the quality of this achievement could be established. Classifying the quality of the verbal responses to the number combinations to 20 followed those of Gray (1991) and thus the use of counting procedures (count-all and count-on together with analogous strategies for subtraction could be identified) and the use of known and/or derived facts was noted. Classifying approaches to combination in excess of 20 drew upon Oliver, Murray and Human (1990) and Gray (1994). Thus approaches were identified as sequence counting, demonstrated at its most sophisticated when children used accumulation or iterative strategies, or transformation strategies, which involved a rearrangement of the combination before the combination was attempted. Most responses to the written combinations were algorithmic but distinctions between left to right or right to left approaches were classified together with errors that occurred, for example, smallest from largest.

For the purposes of this paper, the items presented in standard paper and pencil form served the general purpose of providing a basis for establishing a child's level of achievement. In stating this we are not making any observation about the quality of thinking established from success or failure in making a response to these items. We recognise that if only the level of achievement is used as a criterion for analysis it is hard to distinguish between those who have simply demonstrated a high degree of procedural competence and those who have a sophisticated level of relational understanding. However, by associating this level of achievement with the quality of the understanding that may be determined from an analysis of the strategies used to resolve the verbally presented items it is possible to identify general differences.

Previous exploratory studies (Pitta & Gray, 1996; Pitta & Gray, 1997a) had indicated that what children chose to communicate when responding to invitations to consider a range of concrete and abstract nouns provided indications of differences in the quality of their communication. Thus two principles guided the selection of the words from which frames of reference could be classified.

Firstly, there should be nouns that denote objects that could be easily identified by the children in the real world and had shown their tendency to evoke strong mental representations (Paivio, Yuille & Rogers, 1969). The previous study had also indicated that such words could evoke different levels of communication from children. The list contained the nouns:

dog, table, dots, football, animal, furniture, ball

Secondly, there should be abstract nouns identified as conceptual labels representing mathematical ideas. Such labels could be associated with the outcome of a numerical process or the cognitive reconstruction of this process to form a new conceptual entity. They may evoke mental qualities associated with concreteness but they cannot be perceived directly by the senses. Little psychological evidence is available to establish how such labels may be determined within a framework of concreteness and associated imagery. Indeed a list of 925 words constructed with such qualities (Paivio et al., 1969) contained no such item. Here, then, we were 'in the dark'. However, drawing upon the experience of the previous studies and consulting teachers about their perception of children's experience with them the following list was constructed:

five, ninety-nine, half, three-quarters, nought point seven five, number fraction.

The sample and arithmetical achievement

Initially, 32 children within one English elementary school, eight within each year group from Y3 to Y6 were selected for the study. The eight students from each year group were subdivided into two groups of four, each group of four representing the extremes of 'high achievers' and 'low achievers' within each year. The terms 'high achiever' and 'low achiever' are used for simplicity and do not reflect any opinions on the underlying ability of the children, nor do they imply that any longer-term prognosis is being made concerning their eventual levels of achievement in mathematics. Selection depended on the outcome of school and national assessment associated with the level each child achieved in a range of Standard Assessment Tasks (see for example SCAA, 1998) taken at the age of 7+, the end of Y2, and with predictions for performance in an associated test to be taken during Y6. The later was derived from an ongoing cycle of school based tests within Y4, Y5 and Y6. Such a method of selection identified children within the lowest and highest quartiles of achievement within each of the four year groups that held children aged 8, 9, 10 and 11- years-old.

However, even though 32 children participated in the subsequent interview process, for purposes of reliability and efficiency it was eventually decided to narrow this range to 16 children, two at the extreme of achievement within each year group (identified as either "high" or "low" within Table 1). This sample was established by considering the children with the highest and lowest levels of achievement in the numerical items presented during this study.

Table 1: Numerical achievement of the sample within the study (%)

	Year 3		Year 4		Year 5		Year 6	
Mean Age	7years 8 months		8 Years 8 months		9 Years 7 months		10 years 8 months	
Class Average	58%		71%		70%		81%	
Selected Children	High	Low	High	Low	High	Low	High	Low
Child 1	87	20	97	30	90	37	100	50
Child 2	77	27	100	47	90	63	97	60

The individual level of achievement of each child that contributed to the study is seen in Table 1. Marks are given as a percentage and an indication of the class average is also provided.

Qualitative differences between the two groups were identified through the analysis of the strategies that they used to obtain solutions. With only one exception, a child from Y3, all high achievers responded to the verbally presented combinations to twenty by either knowing solutions or deriving them. There was very little evidence of knowing or deriving amongst the low achievers. The dominant solution processes were either count-on or count-back. However, across the full sample the success rate for this section of the numerical component was high, only one child having a recorded error.

Only one high achiever gave an incorrect solution to the verbally presented addition and subtraction two digit combinations. This occurred because of a counting error. In contrast no low achiever gave correct responses to more than five of the six items. Errors amongst low achievers became particularly common when they attempted to establish differences that required some form of transformation of the numbers. At least one low achieving child within each year group gave no correct solutions to the subtraction combinations. However, proficiency in sequencing, particularly in tens, enabled all except one to give correct solutions to the addition combinations.

These differences can be accounted for by the different strategies that were applied. The younger low achievers continued to apply counting procedures and their efforts were frequently accompanied by the support of perceptual items such as fingers. As indicated, this form of procedure caused great difficulties with subtraction. The older low achievers displayed a tendency to ‘chunk’ 10s and apply a sequence illustrative of the use of an accumulation strategy that involved incrementing in ones and/or 10s. Only in four separate instances did high achievers apply a similar approach. Decrementing in ones was the only approach used by a Y3 child to resolve the subtraction combinations whilst three others also demonstrated the extensive use of accumulation strategies. The dominant approach of high achievers was to use a transformation strategy that usually involved rounding up or down to the nearest ten and, if necessary, applying a corrective element.

When the two groups attempted to obtain answers to the verbally presented three digit combinations, the differences that had emerged in resolving the two digit combinations signaled an even sharper division. The strategies used by the high achievers for the two digit combinations were successfully transferred to the three digit combinations although standard algorithms dominated resolution of the visually present items. However, there were differences between their occurrence in subtraction (four out of five of the instances) and in addition (one out of three instances). None of the low achievers successfully solved any of the verbally presented 3 digit subtraction combinations.

Overall, the sample selected not only reflected differences in achievement but also confirmed the existence of qualitative differences in thinking (Gray & Tall, 1994). Low achievers emphasised the routine application of counting procedures either sequencing in ones or in tens. Frequently such processes led to errors or an inability to generalise to the next stage of difficulty, hence their well below the class average mark. High achievers demonstrated flexibility that suggested that they had an element of choice available to them. At its most sophisticated, such choice was reflected within their ability to derive a solution using an alternative range of knowledge, transform given numbers to more manageable numbers and apply routine procedures if required. Everything else failing they could always count.

Presenting the numeric and non-numeric verbal items

Each verbal cue was presented with the following instructions:

- First Response: *What is the first thing that comes to mind when you hear the word...?*

- 30 second Response: *Talk for 30 seconds about what comes in your mind when you hear the word…*

The development of a two part questioning process, a 'first response' and a 30 second response, gave the interviewee an initial opportunity to provide a reference and then further opportunity to enrich the first response with greater detail provided from a network of other relationships. Drake (1996) summarises the issues associated with this type of item delivery:

> "The generation of an image promotes the development of a trace in the brain that integrates the separate components. Thus, accessing a part of the information encoded in the memory prompts the retrieval of all the other pieces of information contained in the image" (Drake, 1996, p.7)

Two interviewers carried out the video recorded interviews, sometimes together but more often independently. Joint interviews were common at the start of each phase of questions so that an agreed procedure could be identified, evaluated and, if necessary, modified. It was also felt that this approach would contribute to the strength of discussion associated with the classification of responses.

Classifying Responses

An exploratory study (Pitta & Gray, 1996, 1997b) had indicated that the verbal expansion that came from the associations that children derived from concrete and abstract nouns could be categorised using a phenomenological approach and therefore this approach was used in the current study. However, repeated analysis of the data indicated that the classifications that were arising had striking similarities to those of De Beni and Pazzaglia (1995). Therefore, a modified version of De Beni's and Pazzaglia's quadripartite classifications of general, specific, contextual and autobiographic formed a basis for the frames of reference to be identified. In the section below we explain some of the modifications that were carried out and the reasons for doing so.

Applying the ‘contextualised’ category as identified by De Beni and Pazzaglia (1995) would suggest that some frames of reference could have distinctive and relational characteristics. However, such a classification did not satisfy clear distinctions observed in the quality of the responses of some of the subjects within the current study. The notion of contextualised as used by De Beni and Pazzaglia could describe a scene or a sequence of scenes. From the analysis of the children’s words a description of a scene or sequence of scenes could be best described as episodic and such descriptions were most often narrated in continuous speech. Thus the notion of episodic is used, but though there is no suggestion that a frame of reference identified as such is associated with the retrieval of a specific scene from the *remembered* past, it is associated with reference to a scene. In contrast however, characteristics that may be identified as ‘autobiographic-episodic’ are identified as of the outcome of autobiographic episodic memory and denote a specific scene that has occurred in an individual’s past life. However, there were other responses that implied the existence of a context but their structure was fragmentary; more a collection of disconnected, seemingly arbitrary, general statements which, though they originated from a particular stimulus, appeared to digress in a coherent way. Thus we included the notion of ‘multi-faceted’. Here the term is used to identify the fact that the presented conceptual label acts as a stimulus for associated yet diverging responses. Taken individually they may not appear to share a clear relationship but as a whole they are sourced from the same stimulus. In a sense we may identify multi-faceted as ‘multi general’.

The influence of a similar divergence that stemmed from presentation of the numerical words led to the identification of a set of equivalencies in that there could be different ways of saying the same thing. The proceptual responses showed evidence of an understanding of the mathematical concept and procedures associated with the numerical item in question. For example the item “half” could have evoked the responses, “fifty per cent”, “naught point five”, ”two quarters”, “one divided into two equal parts” etc. Frames of reference that demonstrated such an occurrence of were identified as “proceptual”.

As a result of this analysis the classifications of the frames of reference identified from the children’s responses were identified as “general”, “episodic”, “specific”, “episodic-autobiographic”, “multi-faceted” and “proceptual”. Each had the particular properties that are identified below.

A general fame of reference was identified when the subjects description did not specify any characteristics of the noun and nor did it suggest that the subject was talking about a specific item. For example, to the item ‘table’, the response “*A piece of furniture*” (Y4+. ‘table’)[1] was classified as ‘general’.

In the case of numerical items, the most frequent general response was a reference to the mathematical symbol or a very general comment, for example, ‘fraction’ was identified as “*Part of*” (Y6+, ‘fraction’).

[1] When using direct quotes from the children we shall give an abbreviation to indicate the group and the age of the child. Y4+ indicates a child within the fourth year of schooling (nine-year-old) within the above average group. Y5- would indicate a child within the fifth year of school (ten-year-old) from the below average group. Words such as “table” will indicate the item responded to.

Utilising De Beni's and Pazzaglia's prescription, a specific frame of reference was characterized through a clear example (or multiple examples) illustrative of the presented stimuli. Note that no reference was made to the specific context where one can meet this illustrative example. For example, a low achiever's response to the word 'animal' included "*... a cheetah is one, a rabbit is one, a dog is one a cat, a Labrador, Dalmatian, owl, eagle, buzzard, etc.*" (Y3–, 'animal'). In the numerical context, children's specific responses most often arose when they were asked about "number" or "fraction", for example, "*like one, two, three, four, five, six, and ten are numbers*" (Y4–, 'number') or the word 'fraction' exemplified by "*A half*" (Y6+, 'fraction').

Included within this category was also an extended description of what the item looked like or the addition of a self-schema. For example, when a child was asked about number five she volunteered that *"It's like a circle on the bottom."* (Y3-, 5), whilst another responded to the word 'ball' by saying *"I see my ball with my name on it"* (Y3-, 'ball'). The addition of such autobiographic detail in this category followed Kosslyn's (1994) notion that the inclusion of such detail is a special case of exemplar (specific) which has been enlarged by the addition of the self-schema,

Classification of an episodic frame of reference was identified when the item was associated with an 'episode' (a scene or sequence of scenes) that occurred in a specific context. The classification does not refer to elements and neither is it associated with a specific event in the child's past life. One aspect of the qualities of this classification was as an active scene narrated or described in full detail, for example:

> "Boys can kick it around and sometimes it can get lost over the field."(Y3–, 'ball')

> "Number five. I think of a row of numbers and light shines on number five. A light goes along and stops over the number five." (Y5–, 'number')

A productive statement that was common to the general concept identified the notion of a multi-faceted frame of reference. It was not a description of a sequential event that had a clear beginning and end but it was identified as a collection of statements that seemed to have the potential to produce new ideas. Though they had a 'general' quality, the statements diverged to produce different ideas related to the item.

> "Keeps you fit. An exciting game. Millions of fans. Important in every nation. Children and adults play it. Different types of football and balls." (Y4+, 'football')

> "... maths and writing. Seven you could be doing some adding or times and the number seven might come up. Seven is also played in sport on the back of a shirt, has one digit, phone number." (Y5+, 'seven').

In the context of this study, the definition of 'autobiographic-episodic' has been taken directly from De Beni and Pazzaglia. Their definition allowed for the "occurrence of a single episode in the subject's life connected to the concept" (p. 1361). Therefore, examples such as the following were classified as 'autobiographic-episodic':

> "My friend wasn't good at fractions and she had to take extra work home."(Y4+, 'fraction')

> "We have recently done reflections and they had lots of halves in them. We had to put our mirror down the side and see the rest of it. I saw lots of those." (Y4+, 'half')

The inclusion of a proceptual mental representation is additional to classifications identified by De Beni and Pazzaglia (1995) and it is unique to the numerical items. References to mathematical relationships, processes, concepts, manipulations of mathematical symbols and/or indications of equivalence were classified as such. For example,

> "Can also be a decimal and like 0.75. Not a whole. Add one third of it and you get a whole" (Y6+, "three quarters"),

> "3 parts out of 4, fraction, 0.75, more than half." (Y6+, 'three quarters)

RESULTS

Comparative Case Study: Children of the same age.

To give a flavor of the differences between the children we will consider two children of the same age who are at extremes in their level of achievement in the numerical component. Sonia is a low achiever aged nine who, although she was able to obtain correct responses to all of the combinations up to 20, she achieved an overall score 30% in the numerical component.

To obtain solutions she used a counting process in 25 out of the 29 combinations that she tried. She attempted a variety of approaches that included count-on (for example solving 7+6, 14+.8), count-all (3+2) and count back (12-8). However, where differences were relatively small count-back caused temporary problems:

> Now I have lost track. 12,11,10,9,8,7,6,5,4,4. This is the easy way [proceeding to reapply count back on her fingers]. (12–8, verbal,)

As numbers became larger and differences greater, for example in 17–13 and 29–6, these difficulties were not overcome. Sonia extensively used her fingers to support her counting when dealing with the verbally presented items but there was evidence of the use of verbal counting by allowing number words to stand in for countable items when she dealt with the visually presented items.

Sonia's overall approach to the combinations to 20 illustrated that she had constructed an abstract sense of number (Steffe et al, 1983). This allowed her to make fairly extensive use of count-on. However, there was a distinction in the nature of the unit she used to support her counting. Although there was evidence of verbal counting, Sonia generally approached every combination by re-presenting each verbal or visual representation of a number with figural objects that could be counted. Each combination appeared to suggest that a counting sequence needed to be performed. The same counting episode re-occurred for each sum with the only difference being the numbers that the individual would begin and end his/her counting. Thus to put it more clearly, the counting procedures invoked the application of a counting scene, associated with a specific context and the presented combination. Within each episode we may also see specificity in that numbers were specifically associated with fingers. However, attempts to generalise the episodes and the specific nature of the unit used to support these episodes failed as the combinations required the use of larger numbers. Her approach did suggest that she had a frame of reference associated with number combinations that was largely episodic and specific.

Malcolm, from the same class, was also nine years of age. Within the numerical component he obtained 100% and the strategies applied to the orally presented number combination were identified as known facts (all combinations to ten), derived facts (extensively used with combinations to 20) or transformation strategies.

The differences in responses that Malcolm and Sonia gave to the range of words may be seen in Figure 1.

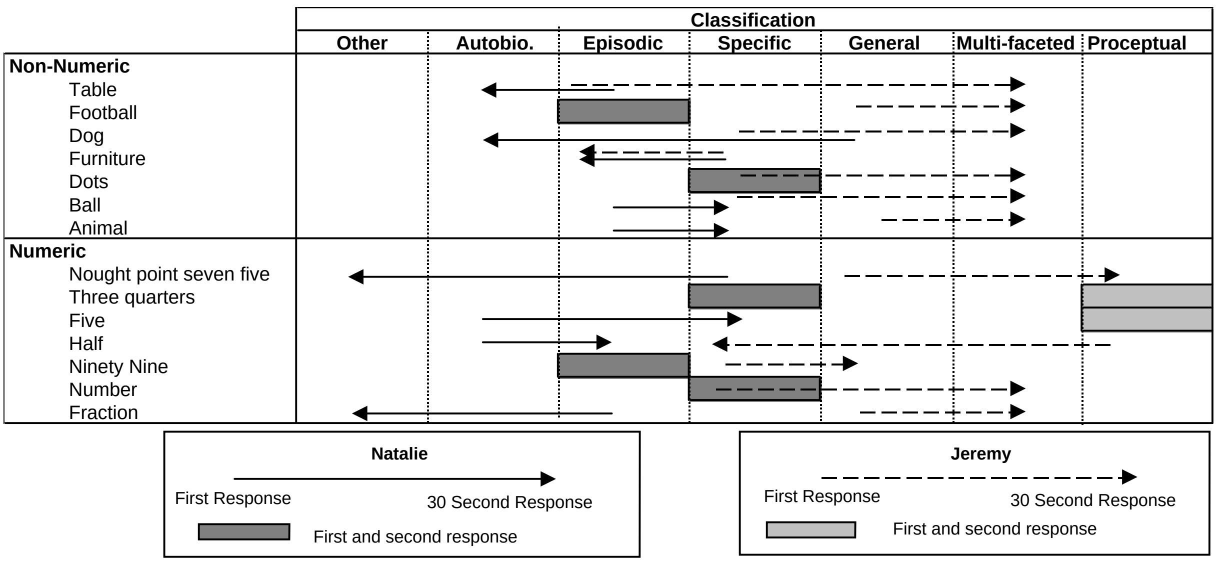

Figure1: First and second responses to the verbally presented items: Natalie and Jeremy

The order of the classifications within the figure has partly been guided by the literature (De Beni & Pazzaglia, 1995 in particular), the influence of the 30 second response and our analysis of the frequency of each classification. The articulation of a frame of reference identified as specific or episodic may start from the formation of a general frame of reference. The longer time span provided by the 30 seconds offers an opportunity for this frame to be enriched with more detail or with a network of relationships. Thus within Figure 1 we ascribe a somewhat pivotal role to the classification of responses identified as general. Those frames of reference identified from the specification of particular examples (specific) or particular episodes (episodic), and those more associated with relationships (multi-faceted) or relationship and equivalences (proceptual) are then arrayed to either side.

We also follow De Beni and Pazzaglia by taking note of their view and through analogy conjecture that episodic-autobiographic frame of reference have a different generation process:

> "[It is] not an enrichment of the general image, but a different process from the beginning. Given the verbal cue, a search takes place among the biographic memories associated with the cue leading to the choice of the one considered to be most representative"
> (DeBeni & Pazzaglia, 1995, pp.1361-1362)

It is for this reason that it is included in a position between the classification of 'other' responses — either the child's non-recognition of the item or an irrelevant comment that does not include an implicit reference to the item — and the classification 'episodic'. The relative positions of frames of reference identified as specific and episodic is solely due to the frequency with which each one occurs. Multi-faceted frames take precedence over proceptual ones not only because of their frequency but also because they may be identified from responses to either numeric or non-numeric items.

It can be seen from Figure 1 that Sonia's responses took little account of whether or not items were numerical or non-numerical. Not withstanding the items she did not respond to (nought point seven five, three quarters and fraction) her responses displayed qualities that were either general, specific or episodic.

Sonia's most common 30 second responses were episodic:

> "A ball is round and you kick it, throw it, roll it, hit it." (Ball, 30 seconds response)

> "I love dogs. I have been asking for one since I was born"(Dog, 30 seconds response`)

> "Its an age or a number… when it is your birthday… you are four and the next year you are five"(Five, 30 seconds response,)

Some responses identified as episodic indicated Sonia's general difficulty with the arithmetical component.

> "It's a sort of… sort of… sort of work. I've done it in class. It is difficult to tell all about it."
> (Three quarters, 30 seconds response,)

Such a response was in contrast to that identified as proceptual given by Malcolm:

> "It's a fraction. If you have a half of a half you need three of them to make it. Its not a whole — it's a quarter less than a whole." (Three quarters, 30 seconds response,)

Malcolm's initial responses were largely general or specific:

"A piece of furniture"	(Table, general, initial response)
"Fraction"	(Half, general, initial response)
"Football"	(Ball, specific, initial response)

During the 30 seconds Malcolm expanded his response and projected multi-faceted mental representations for half of the non-numeric items and proceptual mental representations for numerical items for example:

> "Some are older than others and they different things, rabbits jump, goldfish swim, tigers run. Some animals are vegetarians, some are predators and some are cannibals. They all live in different climates and in different habitats."
>
> (Animal, generic, 30 seconds response)
>
> "It's an odd number. Its factors are 1 and 5. There are five dots on a dice and five fingers on a hand" (Five, proceptual, 30 seconds response)

For him, naught point seven five had meaning established from the separate features of the symbolism:

> "Its Three quarters… I represented the naught by one hundred and took the seventy-five because seventy-five is three quarters of 100". (Five, 30 seconds response)

Summing up the responses to the numerical and non-numerical items we could argue that frequently Sonia's initial responses to the presented items were associated with a particular item, for example initially "furniture" was associated with the chair she was sitting on and this was then elaborated to talk about what it was made of and what could be done with it "You can put books on it on it". The frames of reference that Sonia associated with the items presented to her were linked with a specific item or framed within a particular episode.

On the other hand, Malcolm projected mainly general and specific initial responses. Given the opportunity to expand his thoughts through the thirty seconds response enabled Malcolm to demonstrate the way each non-numerical concept could represent a core idea which could be associated with related ideas and the way numerical items could be linked to a set of equivalences or processes.

The differences in Malcolm and Sonia's mathematical achievement and the quality of this achievement, as identified within the presented numerical items, was reflected in differences in the quality of the frame of reference that they apply to the presented concrete and abstract nouns. What we see within these two the tendency for Sonia to report on the items in largely an episodic and/or specific way. Although the evidence suggests that Malcolm also demonstrates these characteristics, he does so with less frequency. Given the opportunity to expand his thoughts he raised them to a more sophisticated level of thinking in that he identified associations and relationships which would seem to signal that the presented item could be a "core" concept that has the potential to lead him towards other concepts.

Identifying trends in frames of reference.

Figure 2 represents the distribution of the sixteen children's responses to the numerical and non-numerical nouns. The order of construction is similar to that of Figure 1—the classification of general has the other classifications arranged above and below it.

Looking at a whole group can mask some of the qualities that we may see with individuals, however, as Figure 2 indicates, some general patterns emerge from the analysis of the full sample responding to all of the items.

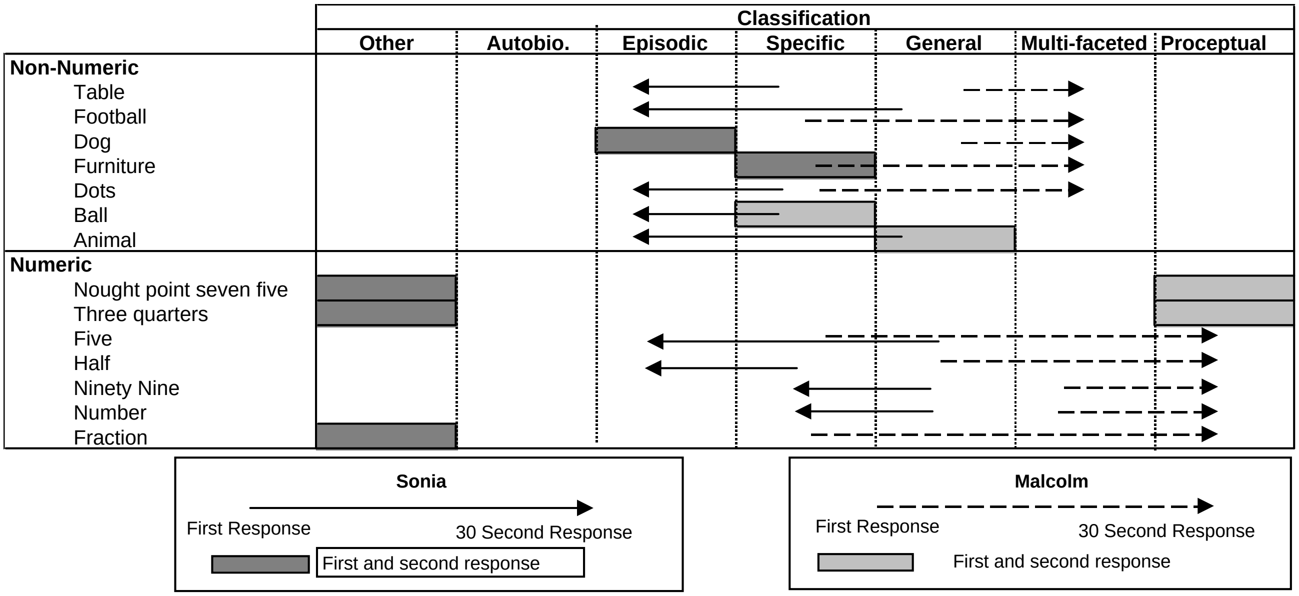

Figure 2: First and second responses to the verbally presented items: Sonia and Malcolm

It can be seen that low achievers provided a lower proportion of multi-faceted and proceptual responses than the high achievers. It is also evident that episodic and specific responses are common within both groups, the former to a much lesser extent amongst the high achievers than the latter. There are also indications that the general quality of the high achievers responses change between the initial response and the thirty seconds response.

Though Figure 2 provides us with an overall sense that some differences exist, a closer examination of the separated numerical and non-numerical nouns begins to present a clearer picture of these differences as shown within Figure 3 and 4. Interestingly the first responses of the two groups to the non-numerical items have a strong similarity particularly in the frequency of occurrence of episodic responses.

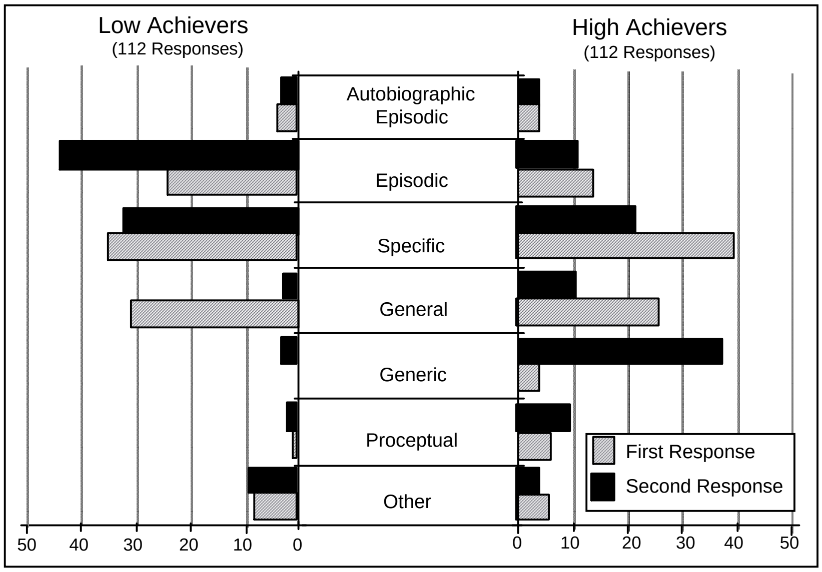

Figure 3: Distribution of frames of reference over the first and 30 second responses of high and low achievers.

Almost one half of each group's first responses are classified as specific but as we consider the second responses we see that the frequency of the high achievers episodic, specific and general responses are reduced by at least a half, whilst multi-faceted responses increase ten fold. Low achievers on the other hand maintain the frequency of specific responses and almost double the frequency of episodic responses.

Looking at Figure 4 we may see that all of the classifications are represented in different proportions by each of the two groups of children. What is noticeable is the frequency associated with the classification 'other'.

As indicated earlier, the younger children were not overtly familiar with decimal representations but also several of the low achievers did not know what to say about 'three-quarters' or 'fraction'. The general picture that emerged from the consideration of the non-numerical items also emerges here. General, specific and episodic classifications dominate the low achievers' responses. Also noticeable is that the frequency of the latter doubles on the second response.

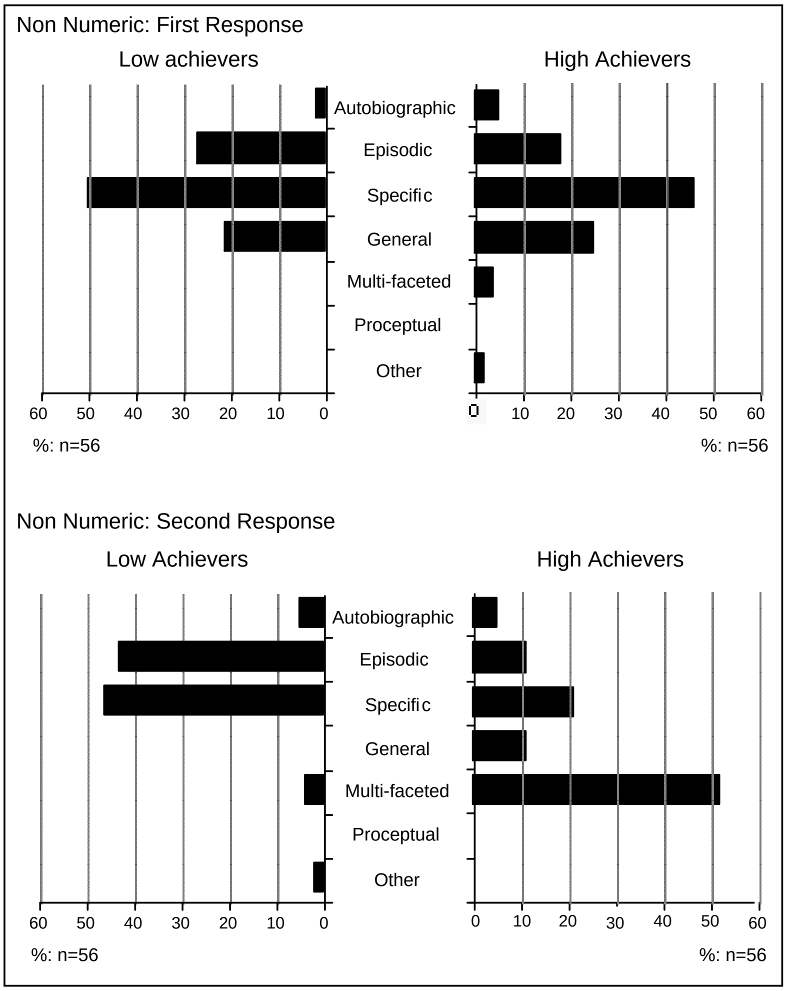

Figure 4: A comparison between the first and second responses of the high and low achievers to the non-numeric items.

Specific and episodic considerations also applied when we considered the strategies Sonia applied to obtain solutions to the number combinations.

Though specific and general responses account for almost two thirds of the high achievers' first responses their frequency declines to account for a third of the 30 seconds responses whilst multi faceted and proceptual responses increase almost three fold.

SOME GENERAL CONSIDERATIONS

Within both the numeric and the non-numeric sections children provided a one word 'general' frame of reference during the 30 seconds response. For example:

"Decimal." (Y4+, 'naught point seven five')

"Game." (Y6-, 'football')

However, given the opportunity to expand this frame, some, particularly the low achievers, frequently used the opportunity to either build around one idea or to provide multi-examples to give the sense of an item.

> "... a cheetah is one, a rabbit is one, a dog is one, a cat, a Labrador, a Dalmatian, owl, eagle, buzzard, etc." (Y4-, specific, 'animal')

> "There are thirty three worlds or countries, thirty three bins, thirty three doors, thirty three houses and some people are thirty three." (Y4-, specific, Thirty-three)

It may be seen from the examples above that this child provided responses to the non-numeric items and the numeric items which were qualitatively similar.

The additional time allowed for 30 seconds response also gave some, again particularly the low achievers, an opportunity to build very vivid stories around the words.

> "You sit at a table - tuck in your chair and you can have dinner at Christmas or a nice party and you can have all nice food and a table-cloth on it."(Y3-, episodic, 'table')

An appraisal of Figure 3 and 4 can suggest some general conclusions. There is a strong similarity between the low and high achievers in presenting a first response to non-numeric items. However, the 30 seconds response provided the low achievers with an opportunity to expand on their initial thoughts by providing qualitatively similar examples, adding further description to the idea originally presented or through associating the item with an episode from past experience. The thirty seconds response was an opportunity for high achievers to associate the initial idea with other related ideas. However, we should not lose sight of the fact that amongst the high achievers approximately one third of the 30 second responses were episodic or descriptive. These characteristics were not the feature of any one particular age group and nor of any individual but were demonstrated within the responses of almost every child together with responses that were general and multi-faceted. This suggests that the frames of reference of the high achievers may have multi-faceted characteristics whereas those of the low achievers are more descriptive in that they focus on either expanding or describing a specific item or upon episodes associated with that specific.

The frequency of an episodic frame of reference amongst low achievers in the thirty seconds numeric phase contrasts sharply with the diversity of frames of reference of high achievers. A frame of reference for numeric items dominated by episodic characteristics could explain the low achievers' tendency to place an over-reliance on counting.

Multi-faceted characteristics with less emphasis on the episodic were a feature of Malcolm responses, but it can also be seen as an overall feature of the response of those who were identified as high achievers. With the exceptions of one Y3 child and one Y5 child all of these provided evidence of a frame of reference that could be identified as proceptual. In contrast only one low achiever provided a response that could do the same.

The distribution of different frames of reference displayed by the two groups of children suggests that there are qualitative differences between them:

- The frames of reference of ‘low achievers’ are largely general, specific and episodic, the latter two characterised by descriptive attributes.

- ‘High achievers’ project a frame of reference which is multi-faceted and/or proceptual but they possess ‘general’ and ‘specific’ characteristics that resonate with the concrete or abstract nature of the item under discussion.

In essence, we see that the frame of reference of the low achievers does not show substantive change. No matter whether faced with numerical or non-numerical items, they consider them in the same way. Their ‘expansion’ of an initial general characteristic is epitomized by specific and/or ‘episodic’ frames. High achievers, on the other hand, appear more able to go either way. Whilst they can ‘expand’ a general frame in a specific or episodic way, they can also supply the multifaceted and proceptual qualities that indicate that their frames of reference are relational.

An examination of the general strategies used by the high and low achievers to solve the number combination to 20 illustrates the general differences that may be observed between the children. Table 2 provides a collective summary of the use of a variety of strategies that the children used to solve these problems.

Table 2: Comparison of the percentage frequency of the strategies used by high and low achieving children to solve the visually and verbally presented arithmetical combinations.

		Known Facts	Derived Fac	Count-on	Count-all	Count-back	Count-up
Verbal n=56	High Achievers	68	27	5	0	0	0
	Low Achievers	11	11	55	5	18	0
Visual n=80	High Achievers	80	11	4	0	0	5
	Low Achievers	28	10	46	3	13	0
Overall Percentage	High Achievers	75	18	3	0	0	4
	Low Achievers	21	10	50	4	15	0

Table 2 shows that the high achievers can recall the solutions to the majority of the combinations and that those they cannot recall they derive from combinations they do know. In contrast, low achievers, particularly the younger ones, rely extensively upon counting and interestingly none apply a count-up approach to resolve subtraction combinations. Where they do use counting subtraction is seen as the inverse of addition, count-back is used extensively. The evidence of the use of derived facts is fragmentary and largely restricted to one Y6 child who made extensive use of combinations that made ten, to obtain solutions to, for example, 4+7 (7+3+1) and 7+6 (7+3+3). Without this contribution the overall percentage of derived facts used by low achievers would be halved.

Overall then, with the noted exception, we see that the low achievers relied extensively upon counting episodes supported by the use of a specific counting unit — fingers. In contrast the high achievers largely recalled number facts to 20. However, drawing immediate conclusions from such an approach is difficult. We are not easily able to identify whether or not the fact is the outcome of rote learned knowledge or the basis for relational thinking. The evidence to provide an answer to this question only comes when we consider the way the knowledge may be used to identify the solution to combinations that are not known or the way it may be used to solve more difficult combination. We suggest that such evidence came from the children's efforts to solve the two and three digit combinations, particularly those that were presented verbally since many of the visually presented combination were solved using a standard algorithm.

The major distinction between the high achievers and the low achievers when attempting the two and three digit combinations presented verbally was the continued emphasis on counting that remained amongst the low achievers. However, such were the difficulties associated with this approach that children within Y3 and Y4 were not required to attempt more than two combinations. Sonia relied on counting. She attempted to count back 6 from 29 and managed to reach 27 but then made a simple guess to give answer of 18. She seemed to recognise the difficulties that her reliance on counting and the use of fingers as a counting unit were causing. For example, when she tried 27+62 she indicated that she "didn't know how to do it on fingers because I had to go up to six tens".

Overall counting brought very little success for the children. A Y5 child attempted an accumulation strategy for 27+62 using his fingers to count in ones from 62. He gave the answer as 94. Only when children were relatively proficient with such a strategy or provided evidence that they were attempting a transformation strategy, did there seem to be a chance of success. However even here there were problems since frequently the children's efforts to sort out one aspect of the combination led them into some confusion in what they were trying to do with another part. A Y6 child indicated that the solution to 73–32 was 47:

> Thirty away from seventy is 60, 50, 40. Then I just took two out of three to make 47.

Whereas collectively the low achievers only provided nine correct solutions to the verbally presented combinations high achievers provide forty eight (out of a possible fifty six). Counting was applied in only two instance; these by a Y3 child attempting 29–3 and 27+62. In both instances the correct solution was give. Transformation strategies accompanying the use of derived facts in almost 70% of instances. Again however, particularly with the three digit combinations, there were instances where children actually forgot what they were doing.

CONCLUSIONS

Other theories that may be closely related to the findings within this study fall into two groups: those associated with cognitive development from the standpoint of mathematics education and those associated with psychological research into the use of mental representations. Within the former should be placed theories associated with notions of interiorisation, encapsulation, reification and the notion of the proceptual divide. In the latter we must consider those associated with imagery and different kinds of mental representation.

It is generally accepted that the development of early arithmetic evolves from interaction with the environment, new knowledge been constructed by the learner through active methods. It was Piaget's belief that reflective abstraction was the key to the process through which the actions associated with the active methods were projected to thought. This requires the ability to concentrate the mind and give careful thought to an object, an action and an idea. This may require filtering out or temporarily ignoring irrelevancies so that ideas may be separated from their context. Of course, the ability to filter or ignore would suggest that superfluous properties are recognized. It may be that some children are not easily able to recognize those properties and actions that are important and compare them to those that are unimportant.

Abstraction involves the construction of relationships between and amongst objects and reflection upon the interrelationships of the actions on them. The results presented here suggest that individuals have the potential to interpret quite differently the objects that are acted upon. We would suggest that high achievers seem more capable of looking at the objects, mentally put aside their general and specific characteristics to look through them and place an emphasis on their more intrinsic qualities and their relationship with other objects. Such a disposition, which temporarily subsumes the descriptive and focuses on more relational characteristics, would seem to better support the construction of number concepts.

In any context that involves an action on objects, the individual has the possibility of attending to different aspects of the situation. Indeed this is a theme that Cobb, Yackel and Wood (1992) see as one of the great problems in learning mathematics, particularly if learning and teaching are approached from a representational context. We suggest that in their search for substance and meaning based upon descriptive aspects characterised by specificity and episodes, children who have difficulty with elementary arithmetic are disadvantaged right at the start of their mathematical development. However, it may be a disadvantage that does not make itself apparent in the earlier stages of numerical development. The use of counting may illustrate this point. The greater majority of young children count (and so do many adults). For a young child counting can be seen as part of a stage in concept development. However, an older child's extensive reliance on counting may be the result of necessity. At issue is whether or not we may be able to distinguish which outcome is probable.

Frames of reference that are dominated by specificity and episodic activity may be more easily associated with empirical and pseudo-empirical abstraction. The former is more strongly associated with geometric development. The latter, may lead to a form of procedural competence that can bring success within a well-rehearsed area of content. However, it can also be associated with procedural misinterpretations.

This study is suggesting that individuals internalize different things that are manifest as a result of their different frames of reference. It follows that in some instances, active methods, though they are universally recognised as having the potential to lead to the encapsulation of numerical processes, may prove to be a potential source of longer term difficulty for some. For these individuals it may not be easy to abstract the results of actions because it may be difficult to step out of the frames of reference that are more strongly associated with specificity and episodic characteristics. It may be that a tendency towards frames of reference that are dominated by such characteristics may make the very notion of encapsulation difficult if not impossible. To encapsulate a numerical process into a numerical object there must be a recognition that an object, detached from the 'real' objects and the associated actions can exist and that it can be used as basis for more sophisticated actions.

Though high achievers recognize that the actions exist, can describe their purpose, and name them, their focus can be wider. They appear able to temporarily put aside fundamental actions to consider the more abstract qualities of objects interiorised from a co-ordination of these actions. They can think in terms of encapsulated process or reified procedures. However, they may not be just compressing and squeeze everything down but they may be actively taking out some components. It was 'actively taking out' and then attempting to enhance mathematical relationships that guided a small teaching experiment with a child whose mental representations were analogues of real objects (Pitta & Gray, 1997). Such an approach needs to be considered further if children are not to establish frames of reference in elementary arithmetic that may be the source of later difficulty.

Qualitative differences in the interpretation of numerical symbolism have led to the notion of the proceptual divide (Gray & Tall, 1994). The evidence within this study suggests that a closer look at the frames of reference that influence a learner's development may provide a clearer picture of the causes of this divide. The disposition of the low achievers within this study to articulate the descriptive aspects of a range of items would seem to be indicative of the tendency to rely upon procedures that are fundamentally episodic.

By its very nature the study seeks to identify common forms of behaviour exhibited by children at extremes of arithmetical achievement. This is not to be interpreted as suggesting that all children at these extremes will behave in the ways presented. In supporting the hypotheses in the study we do not wish to lose sight of the individual and neither do we wish to loose sight of children with the wide range of achievements that have not been directly investigated. Translating what may seem to be the extreme cases reported within this paper across the full range of the children that we meet is not easy. It is evident that within any spectrum it is the extreme cases that may make the most interesting ones. However, these extremes may also indicate that within the very large group that lie between the extremes there is the strong possibility of meeting less clear cut differences.

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WORLD-CLASS HIGH QUALITY MATHEMATICS EDUCATION FOR ALL K-12 AMERICAN STUDENTS

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Dedicated to Professor Evelyn M. Silvia, University of California, Davis
(February 8, 1948 – January 21, 2005)

Abstract. In September 1989, the United States' Governors Conference in Charlottesville, Virginia set an ambitious goal by declaring that "By the year 2000, United States students will be first in the world in mathematics and science achievements". However, recent results of the 'Programme for International Student Achievement' and 'Trends in International Mathematics and Science Study' indicate that the United States students' achievements in mathematics are far below world class standards. This paper seeks to discuss issues in an international context related to the goal of creating world-class high quality mathematics education for all K-12 American students. In particular, the author also shares his reflections and depicts lessons from Singapore's success story in mathematics education.

Key words: Mathematics Achievement; International comparisons; PISA; Singapore Educational System; Standards; TIMSS

1. INTRODUCTION

The United States (U.S.) is viewed as a global leader in many aspects, including finance, medical research, higher education, sports, and scientific and technological advancements. And yet, according to 'Programme for International Student Achievement' (PISA) and 'Trends in International Mathematics and Science study' (TIMSS), the U.S. is still very far from being world class in K-12 mathematics and science education (Lane, 1996; Kaiser, et al., 1999; Ahuja, 2003; Braswell, et al., 2004; Gonzales, et al., 2004; Martin, et al., 2004; PISA Website; TIMSS Website).

In September 1989, the U.S. Governors Conference in Charlottesville Virginia set up an ambitious goal by declaring that "*By the year 2000, United States students will be first in the world in mathematics and science achievements*". That same year, the U.S. Department of Education announced a set of eight goals, and its fifth goal was that "*U.S. students will be first in the world in mathematics and science achievements*" (U.S. Department of Education, 1989). However, this ambitious goal is yet to be achieved.

There are several reasons for why the U.S. ought to have world-class school mathematics education. The Glenn (2000) commission observed four important and enduring reasons to achieve competency in mathematics and science: (i) the demands of our changing economy and workplace, (ii) our democracy's continuing need for a highly educated citizenry, (iii) the vital links of mathematics and science to the nation's national security interests, and (iv) the deeper value of mathematical and scientific knowledge. The commission recommended that all students must improve their performance in mathematics and science if they are to succeed in today's world and if the U.S. is to stay competitive in the integrated global economy. In fact, the commission believed that "*competency in mathematics, both in numerical manipulation and understanding its conceptual foundations, enhances a person's ability to handle the more ambitious and qualitative relationships that dominate our day-to-day decision making*". A powerful mathematics education system would also help in: (i) strengthening democracy by creating an informed adult population, (ii) empowering individuals and enabling them to develop toward their potential, and (iii) providing a sound basis for continuing national prosperity (Schmidt, et al., 1998). These three reasons involve political, personal and social, and economic goals.

The need for providing excellent mathematics education has increased in the global village of the 21st century. Rising global competition, workplace opportunities and challenges require better-educated workers who are adept at reasoning, problem solving, analyzing, and making sense of things. Having a deep understanding of mathematics is vital for achieving these skills. The ability to approach problems logically, to apply reasoning to decision making, and to understand how things work are exactly the kinds of skills that should be developed through meaningful mathematics and science education. Today's students must therefore master high-level mathematical concepts and complex approaches to solving problems to be prepared for college and careers of the 21st century, as well as the demands of everyday life (CBMS, 2000).

One can argue that "*the American K-12 system is failing to provide the mathematics and science skills necessary for kids to compete in the 21st century workforce, and the U.S. higher education system cannot produce enough scientists and engineers to support the growth of the high-tech industry that is so crucial to economic prosperity*" (AeA, 2004). The U.S. Secretary of Education Rod Paige admitted: "…*Unfortunately, we are average across the board compared to other industrialized nations. In the global economy, these countries are our competitors - average is not good enough for American kids*" (Paige, 2001). As stated in the Wall Street Journal on October 7, 2004: "*American companies don't simply go to foreign countries for inexpensive labor; they are increasingly going abroad to find skills that aren't available, or plentiful, in their own backyard*".

In 2002, the U.S. Secretary of Education Rod Paige signed a six-page memorandum of understanding with Singapore Minister of Education Teo Chee Hean in which the U.S. and Singapore agreed to help each other improve mathematics and science education (Hoff, 2002). Mr. Paige said in his statement about the agreement that "*Singapore's students score among the highest in the world in mathematics and science and there is much we can learn about its system of education which leads to such high achievement*" (Hoff, 2002). It may also be argued that "*Singapore's elementary mathematics teachers, like other elements of Singapore's mathematics education system, are superior in overall quality when compared to their U.S. counterparts*" (Ginsburg, et al., 2004, p.118). More than one hundred school districts in the U.S. have been experimenting in their schools with several features of Singapore's mathematics curriculum, teaching methods, and text books (Viadero, 2000; Hoff J. D., 2002). Some of such major ongoing pilot projects are Montgomery County Public Schools (Maryland), Baltimore City Public Schools (Maryland), Peterson (New Jersey), and North Middlesex (Massachusetts).

An awareness of Singapore's success story will help us to share its cultural and educational practices and traditions. Singapore, meaning 'lion city', a highly developed and successful free market economy enjoys a remarkably open and corruption-free environment, stable prices, and a per capita GDP equal to that of the Big 4 West European countries. Like the U.S., Singapore too is multiethnic, multi religious, multilingual, and a democratic nation. It is a small island - 26 miles in length and 14 miles in breadth - and has a total land area of about 228 square miles. It was discovered in 1819 by the British, and became a part of Malaysia in 1963. Two years later, Singapore became an independent city-state-nation. Singapore became developed and prosperous in the last few decades. Singapore's Gross Domestic Product rose from $300 in 1970 to $27,800 in 2004. Its economy depends heavily on exports, particularly in electronics and manufacturing. Its current population is about 4.4 million with a population density of 18,261 per square mile. It is ethnically diverse with 77% Chinese, 14% Malays, 8% Indians and 1% of other ethnic groups. The general literacy rate is 93.7%. All children in Singapore are required to study two languages: English and their mother tongue. It has four national languages, though English is the primary language of instruction for all subjects except mother tongue.

This paper seeks to discuss issues in an international context related to the goal of creating world-class high quality K-12 mathematics education in the U.S. Section 2 illustrates the international perspective of American schools' mathematics education. Section 3 compares mathematics education in the U.S. and Singapore. Section 4 opens with observations and reflections of the author and goes on to look at the features that make Singapore number one in mathematics education in the world. The last section explores how the U.S. can achieve its goal of creating a world-class mathematics education system in which its students will be top in the world in mathematics achievements.

2. U.S. SCHOOL MATHEMATICS FROM AN INTERNATIONAL PERSPECTIVE

An international comparison of the U.S. students' achievements in mathematics is conducted by 'Programme for International Student Assessment' (PISA) and 'Trends in International Mathematics and Science Study' (TIMSS). It may be noted that originally TIMSS stood for 'Third International Mathematics and Science Study'. These international studies allow the U.S. to compare its students' performance in mathematics to that of other countries. PISA uses an age-based sample and tells us about the mathematical literacy of 15-year-olds, while TIMSS uses grade-based samples and reports on curricular achievements in mathematics. In 2003, the number of countries which participated in PISA and TIMSS were 41 and 46 respectively. Incidentally, India and China (mainland) did not participate in these international studies. While Singapore participates in TIMSS, it does not participate in PISA.

PISA is an international educational research study conducted every three years by the 'Organization for Economic Cooperation and Development' (OECD) (PISA Website). Table 1 show that its participants in 2003 included all 29 of the OECD countries and 10 non-OECD countries. But, the international average of 500 reported was based only on these 29 OECD countries. Its emphasis is on the 15-year old students' ability to apply a range of knowledge and skills in mathematics to a variety of problems with real-life contexts. In fact, its goal is to answer the question "what knowledge and skills do students have at age 15?" taking into account schooling and other factors that may influence their performance. Results of PISA 2003 in Table 1 show that on the mathematics section, the U.S. ranked 24th out of the 29 member nations of OECD, dropping below Poland, Hungary, and Spain in the three years since the previous assessment. For further details, see Lemke (2004) and the PISA Website.

Table 1: *Average Combined Mathematics Literacy Scores in PISA* 2003

Rank	OECD countries	Score	Rank	OECD countries	Score
1	Finland	544	**24**	**USA**	**483**
2	Korea	542	25	Portugal	466
3	Netherlands	538	26	Italy	466
4	Japan	534	27	Greece	445
5	Canada	532	28	Turkey	423
6	Belgium	529	29	Mexico	385
7	Switzerland	527			
8	Australia	524		OECD COUNTRIES AVERAGE: 500	
9	New Zealand	523		**NON-OECD COUNTRIES**	
10	Czech Republic	516	**1**	**Hong Kong, China**	**550**
11	Iceland	515	**2**	**Liechtenstein**	**536**
12	Denmark	514	**3**	**Macao-China**	**527**
13	France	511	**4**	**Latvia**	**483**
14	Sweden	509	5	Russian Federation	468
15	Austria	506	6	Serbia & Montenegro	437
16	Germany	503	7	Uruguay	422
17	Ireland	503	8	Thailand	417
18	Stovak Republic	498	9	Indonesia	360
19	Norway	495	10	Tunisia	359
20	Luxembourg	493			
21	Poland	490		SOURCE: PISA Website	
22	Hungry	490			
23	Spain	485			

TIMSS is an international educational research study conducted every four years, comprising over half a million students across 5 continents and 46 countries. TIMSS 1995, 1999 and 2003 were projects of the International Study Center at Boston College. TIMSS International averages are generally based on 46 countries including 13 industrialized countries. It analyzes background information on the aims of school mathematics and their curricula, the delivery of instruction including textbooks and classroom practices, the students' attitudes and their mathematical achievements, the amount of parental support, and the qualification and training of teachers, among other information (TIMSS Website). TIMSS also reviews video-taped lessons prepared from classrooms in industrialized nations including the U.S., Germany, and Japan (NCES, 2000). All TIMSS data are based on the performance of both public and private schools in the U.S. and other participating countries. Achievement test scores on TIMMS studies typically range between 200 and 800, out of a possible of 1000. TIMSS identifies four international benchmark levels: Low (reaching 400 pts); Intermediate (reaching 475 pts); High (reaching 550 pts); and Advanced (reaching 625 pts). These benchmark levels describe what students know and can do in mathematics (Gonzales, et al., 2004; TIMSS Website).

The results of TIMSS consecutively in 1995, 1999, and 2003 indicate that the U.S. students' achievements in mathematics are not world class. The disheartening results of these studies have led to major stories in the Wall Street Journal, the New York Times, U.S. Times, and other leading national newspapers. The Wall Street Journal, for example, painted this picture with the headlines "*Economic Time Bomb: U.S. Teens Are Among Worst at Math*" on December 7, 2004.

Table 2 shows that in 1995 and 2003, there was no change in the TIMSS results in the U.S. fourth-grade students' average score of 518. The U.S. rank, however, went down from 6th to 12th in these eight years. Note that there was no TIMSS for fourth-graders in 1999. For further details, see Gonzales, et al., (2004) and Martin, et al., (2004).

Table 2: *TIMSS Av Scale Scores of Fourth-Graders in U.S. vs. Other Countries*

Country (International Average)	1995 (496)	2003 (495)		Country (International Average)	1995 (496)	2003 (495)
Singapore	*590 (1)*	*594 (1)*		**United States**	**518 (6)**	**518 (12)**
Hong Kong SAR	*557 (3)*	*575(2)*		Cyprus	475	510
Japan	*567 (2)*	*565 (3)*		Moldova	-	504
Chinese Taipei	-	564(4)		Italy	-	503
Belgium-Flemish	-	551 (5)		Australia	495	499
Netherlands	*549 (4)*	*540 (6)*		New Zealand	469	493
Latvia- LSS	*499*	*536 (7)*		Scotland	493	490
Lithuania	-	*534 (8)*		Slovenia	462	479
Russian Federation	-	532 (9)		Armenia	-	456
England	*484*	531 (10)		Norway	476	451
Hungry	*521 (5)*	529 (11)		Iran	387	389
United States	**518 (6)**	**518 (12)**		Philippines	-	358

SOURCE: TIMSS Website

It is evident from Table 3 that the U.S. eighth-graders in TIMSS 2003 improved in their average mathematics performance over the eight-year period between 1995 and 2003. In 1995, U.S. eighth-graders ranked 17th with an average score of 492. In 2003, their rank improved to 15th with an average score of 504 far exceeding the international average of 467. For further details, see Gonzales, et al., (2004) and Martin, et al., (2004).

Table 3: *TIMSS Average Scale Scores of Eighth-Graders in U.S. vs. Other Countries*

Country (International Average)	1995 (519) (Rank)	1999 (521) (Rank)	2003 (467) (Rank)		Country (International Average)	1995 (519)	1999 (521)	2003 (467)
Singapore	609 (1)	604 (1)	605 (1)		Scotland	493	-	498
Korea, Republic of	581	587 (2)	589 (2)		Israel	-	466	496
Hong Kong	569	582 (4)	586 (3)		New Zealand	501	491	494
Chinese Taipei	-	585 (3)	585 (4)		Slovenia	494	-	493
Japan	581	579 (5)	570 (5)		Italy	-	479	484
Belgium –Flemish	550	558 (6)	537 (6)		**Bulgaria**	**527**	**511 (15)**	476
Netherlands	529	540 (7)	536 (7)		Romania	474	472	475
Estonia	-	-	531 (8)		Norway	498	-	461
Hungary	527	532 (9)	529 (9)		Moldova	-	469	460
Malaysia	-	519 (14)	508 (10)		Cyprus	468	476	459
Russian Federation	524	526 (11)	508 (11)		Macedonia	-	447	435
Slovak Republic	534	534 (8)	508 (12)		Jordan	-	428	424
Latvia-LSS	488	505 (16)	505 (13)		Iran	418	422	411
Australia	509	525 (12)	505 (14)		Indonesia	-	403	411
United States	**492 (17)**	**502 (17)**	**504 (15)**		Tunisia	-	448	410
Canada	**521**	**531 (10)**	-		Chile	-	392	387
Czech	**546**	**520 (13)**	-		Philippines	-	345	378
Lithuania	472	482	502		South Africa	-	275	264
Sweden	540	-	499					

SOURCE: TIMSS Website

Several studies, videos and books which analyze the U.S. results in their international setting discuss in detail the possible reasons for this sub-optimal performance and its consequences for U.S. mathematics education (Lane, 1996; Schmidt, et al., 1998; Kaiser, et al. 1999; NCES, 2000; Hodges, et al., 2001; Ginsburg, et al., 2004). These studies conclude that there is no single coherent vision which dominates how students practice mathematics in the U.S. These studies also suggest that the reasons for the low mathematics performance amongst American students include: curricula that lack focus and intellectual challenges; a lack of coherence across the topics in mathematics frameworks; mathematics textbooks with low standards and lack of focus; lack of competent mathematics teachers; a lack of motivation and positive attitude amongst students; and a lack of parental support.

3. COMPARISON OF MATHEMATICS EDUCATION IN THE U.S. AND SINGAPORE

A comparison of the TIMSS results in the U.S. and Singapore reveals a significant difference in the performance of students in these two nations. In fact, their performance differences lie in deep-rooted complex aspects of teaching, learning processes (such as thinking skills and heuristics), certain special skills (such as estimation, approximation, mental calculation, communication, arithmetic and algebraic manipulation), mathematics curricula, textbooks, student attitudes, culture, and parental support.

3.1 Students' Performance Differences

Table 4 illustrates that about 35% of the American fourth-graders in 2003 reached the "High" benchmark level as compared to 73% of their Singaporean counterparts. Moreover, during the period from 1995 to 2003, the U.S. fourth-graders' average score reaching the "High" benchmark fell down by 2% (from 37% to 35%), while their counterparts in Singapore went up by 3% (from 70% to 73%). On the other hand, the U.S. fourth-graders' average score reaching the "Advanced" benchmark also dropped by 2% (from 9% in 1995 to 7% in 2003); while their Singapore's counterparts remained at 38%.

Table 4: *Percentage of Fourth-Graders Reaching International Benchmarks*

	Low (400)		Intermediate (475)		High (550)		Advanced (625)	
	1995	2003	1995	2003	1995	2003	1995	2003
United States	92 %	93 %	71 %	72 %	37 %	35 %	9 %	7 %
Singapore	96 %	97 %	89 %	91 %	70 %	73 %	38 %	38 %
International Av	85 %	88 %	63 %	69 %	33 %	36 %	10 %	10 %

SOURCE: TIMSS Website

Table 5 reveals that in 2003 about one-fourth (29%) of 8th graders in the U.S. reached the "High" benchmark as compared to about three-fourth (77%) of their counterparts in Singapore. This table further demonstrates that in TIMSS 2003, only about 7% of 8th graders in the U.S. reached the "Advanced" benchmark compared to about 44% of their counterparts in Singapore. It is a matter of concern that about 10% of U.S. eighth graders in 2003 could not reach the "Low" benchmark, that is, these students do not have basic mathematical knowledge such as the understanding of whole numbers and how to do simple computations with them.

A comparison of Table 4 and Table 5 demonstrate that as 4th graders in 1995 progressed to 8th grade in 1999, the mathematics achievements of American students fell but the achievements of their Singaporean counterparts went up. This fact is further illustrated in Figure 1 for students reaching "high" benchmark. In fact, if this trend continues the U.S. universities might find it increasingly hard to get sufficient number of American students for their challenging programs in mathematics and science.

Table 5: *Percentage of Eighth-Graders Reaching International Benchmarks*

	Low (400)			Intermediate (475)			High (550)			Advanced (625)		
	1995	1999	2003	1995	1999	2003	1995	1999	2003	1995	1999	2003
United States	86%	87%	90%	61%	62%	64%	26%	30%	29%	4%	7%	7%
Singapore	100%	99%	99%	98%	94%	93%	84%	77%	77%	40%	42%	44%
International Av	89%	80%	80%	69%	57%	56%	37%	31%	28%	11%	10%	8%

SOURCE: TIMSS Website

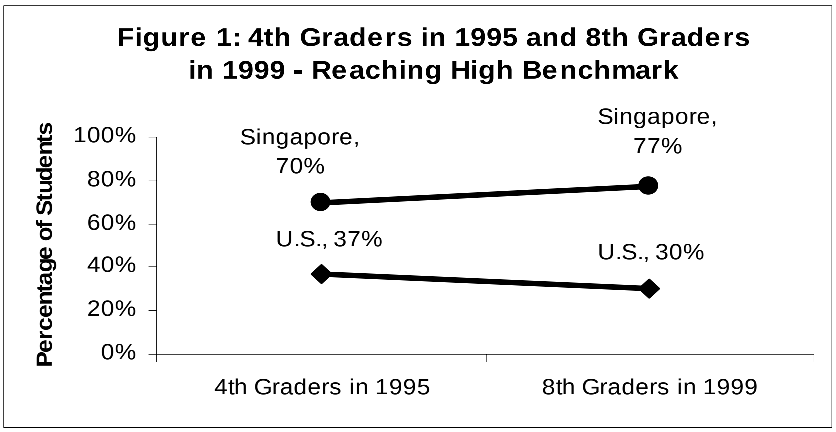


Table 6: *Average Scale Score Achievement by Gender on TIMSS 2003*

	Grade 4		Grade 8		Who is better in math abilities?
	Girls	Boys	Girls	Boys	
United States	502	507	514	522	Boys better
Singapore	611	601	599	590	Girls better
International Av	467	466	495	496	Almost same

SOURCE: TIMSS Website

We discover from Table 6 that there are interesting differences in mathematical abilities amongst boys and girls in the U.S. and Singapore. In the U.S., the mathematical abilities of boys are significantly better than that of girls. In Singapore, however, the girls outperform boys in mathematics.

Table 7 identifies that the U.S. students' mathematics achievements on TIMSS are significantly lower in all five content areas listed in Column 1. In particular, Table 7 also exhibits that a majority of the American students are weak in Measurement, Geometry and Algebra. Although the American students' average in these content areas has significantly improved in the last five years, it is still close to the international average except 'Data' analysis. On the other hand, Singapore students' average is about 25% higher than the international average.

Table 7: *Eighth-graders: % Correct Items on TIMSS*

	United States		Singapore		International Average	
Content area	1999	2003	1999	2003	1999	2003
Number	50 %	54 %	80 %	78%	50 %	48 %
Measurement	40 %	42 %	76 %	74%	44 %	42 %
Geometry	44 %	45 %	73 %	71%	51 %	50 %
Data	68 %	72 %	81 %	79%	62 %	62 %
Algebra	47 %	50 %	69 %	69%	50 %	51 %
Average	50 %	51 %	76 %	74%	51%	51%

SOURCE: TIMSS Website

Tables 4 to 7 and Figure 1 also reveal that there are significant differences in student performances in these two nations. Recent studies indicate that the treatment of a particular topic in any content area (listed in Column 1, Table 7) in an American public school may be insufficiently extended, treated in insufficient depth, inadequately consolidated, or assessed without due attention to content validity (Macnab, 2000; NCES, 2000; Ginsburg, et al. 2005). These studies also show that there are significant gaps between the intended curriculum (set by the school district), the implemented curriculum (taught by the teacher), and the achieved curriculum (learned by students).

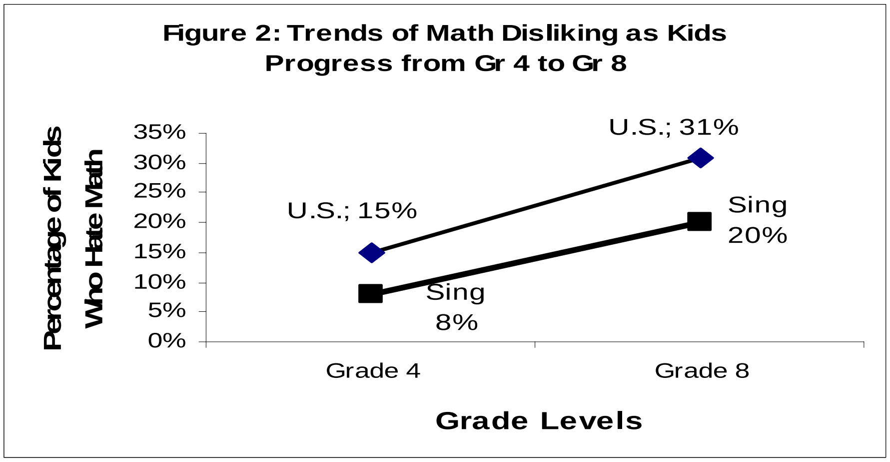
3.2 Effect of Mathematics Anxiety on Students' Performance

Several research studies have shown that the dislike of, or anxiety towards mathematics has a negative effect on mathematics performance. Table 8 reveals that many students develop an increasing dislike towards mathematics as they progress from grade 4 to grade 8. This increasing trend becomes more evident from Figure 2 which shows trends when 4^{th} grades in 1995 progressed to 8^{th} grades in 1999. Although this trend is universal, Singaporean students' dislike for mathematics is much lower than their counterparts in the U.S. These results suggest that in order to develop a positive attitude towards learning mathematics, children need to be shown from an early age that mathematics can be fun.

Table 8: *Students who dislike Mathematics on TIMSS*

	1995		1999		2003	
	Gr 4	Gr 8	Gr 4	Gr 8	Gr 4	Gr 8
United States	15%	30%	No study	31%	20%	40%
Singapore	8%	22%	No study	20%	15%	25%
International average	16%	37%	No study	31%	22%	40%

Source: TIMSS Website



3.3 Comparison of Mathematics Frameworks/ Standards

Table 9 provides highlights of mathematics frameworks in the U.S. and Singapore. The key difference is that while the U.S. has no official national mathematics framework, Singapore has a grade-by-grade national framework, which focuses on and emphasizes high-level cognitive processes. Each State in the U.S. and even many of the cities and school districts within each state, have their own mathematics framework or curriculum. While some states such as Virginia and California do have focused and grade-specific curricula, Ginsburg et al. (2005) observed that the 'National Council of Teachers of Mathematics' (NCTM) framework (NCTM, 2000), while emphasizing higher order 21st century skills in a visionary way, lacks the logical mathematical structure of Singapore's framework. They also discovered that the NCTM framework identifies content only within broad grade bands (e.g., K-2, 3-5) and only in general terms, thus providing inadequate content guidance to educators. Nevertheless, because of the 'No Child Left Behind' (NCLB) Act (U.S. Congress, 2002), grade-by-grade assessments are now required and many states are shifting to grade-by-grade content standards.

The essential finding of the K-12 'State of the State Mathematics Standards 2005' (Klein et al., 2005) confirms that an overwhelming majority of states in the U.S. today have inadequate mathematics standards. The national average grade is a "high D". Only six states earned the "honors" grades of 'A 'or 'B'. California, Indiana, and Massachusetts received an 'A' grade and are considered to have first-class mathematics standards, worthy of emulation. Alabama, Georgia, and New Mexico received a 'B' grade, while 15 states received Cs, 18 received Ds, and 11 states received Fs. For the complete report, see Klein et al. (2005).

Table 9: *Mathematics Frameworks*

	United States	Singapore
1.	Regional and local, mostly unfocused, repetitive, low-level knowledge and skills.	Federal, focused, non-repetitive (except for slow learners), emphasize on high level cognitive processes.
2.	NCTM framework (not a national framework) identifies content only within broad grade bands and only in general terms; some of the states provide grade-by-grade, focused and specific framework.	Framework covers grade-to-grade, specific and challenging following a spiral organization.

SOURCE: TIMSS Website

3.4 Supports for Slower Mathematics Students

Table 10 highlights the differences in support for slower mathematics students in the U.S. and in Singapore. Unlike Singapore, most state and NCTM curricula in the U.S. do not provide any alternative framework for slower mathematics students. Certain U.S. school districts such as 'San Jose Unified' and 'Los Angeles Unified' in California have created continuation schools, which may be viewed as models for alternative education. However, in most of the states as observed in the TIMSS data, slow mathematics students in the U.S. are often tracked into slower and watered down mathematics courses where they are generally not taught the required mathematics materials. In fact, it was revealed by Schmidt (1998) that in over three-fourths of the American schools, eighth graders often take different mathematics courses – regular or 'general' math, remedial math, enriched math, prealgebra and algebra.

Table 10: *Support for the Slower Mathematics Students*

	United States	Singapore
1.	No alternative math frameworks; unofficial tracking often exists but the students rarely learn all the required mathematics content.	Alternative math framework at a slower pace and with some repetition; students generally learn all the required math content.
2.	Teacher aids for extra help.	Specially trained teachers for extra help.
3.	NCLB addresses the needs of failing schools, but not of failing students.	Ministry of Education addresses the needs of all schools and students.
4.	NCLB judges schools based on a measure of "Adequate Yearly Progress" (ADY).	Ministry of Education rewards schools based on a measure of each school's value-added contribution to student achievement.

SOURCES: Ginsburg, et al. 2005; TIMSS Website

3.5 Comparison of Mathematics Textbooks

A world-class K-8 mathematics textbook should provide rich mathematical content that is aligned with the required standards and framework. It should have extensive problem sets that include routine and non-routine problems, and should use sound pedagogical approaches (McKnight 1987; Tyson, 1989; AAAS, 2000). In view of these criteria, Singapore mathematics texts are considered world-class. Singaporean textbooks provide a deep understanding of mathematical concepts and skills and use a spiral approach without unnecessary repetition. These texts also use picture representations or real life/practical examples to explain abstract concepts. On the other hand, American mathematics textbooks are generally too long, and written to cater for about 15,000 mathematics curricula in the nation (Ginsburg et al, 2005). Table 11 summarizes a comparison of most of the traditional mathematics textbooks used in these two nations.

Table 11: *Mathematics Textbooks*

	United States	Singapore
1.	Built for enhancing mechanical ability to apply mathematics concepts.	Built for deep understanding of mathematics concepts.
2.	Simple routine problems.	Multi-step challenging problems.
3.	Beautiful colored pictures, but lack of visual representations to guide conceptual understanding.	Illustrations range from a concrete to pictorial to abstract approach.
4.	Lack of focus since they are written for several state frameworks.	Focused and follows national framework.
5.	Spiral approach, but with a lot of repetition.	Spiral approach, with a little repetition.
6.	More topics, a lot of review but less depth (700 pages average).	Less topics, more coverage (between 250- 300 pages average).

SOURCES: TIMSS Website; Tyson-Bernstein, 1988, Tyson, et al. 1989; Ginsburg, et al. 2005

3.6 Teachers' Qualifications and Professional Development

"*About one-third of practicing mathematics teachers in the U.S. has neither a major nor a minor in mathematics during their undergraduate degree. These teachers typically teach 26% of the country's mathematics students*" (Ingersoll, 1999). It was further revealed by Glenn (2000, p19) that "*more than 12% of all new teachers in the U.S. enter the classroom without any formal training; another 14% start work without meeting the teaching standards of their states*". Moreover, I've also observed in my classes that the majority of preservice teachers for early childhood and middle grade programs come in with negative attitudes towards teaching and learning mathematics. Although they may have rote knowledge of arithmetic, many preservice teachers do not have the deep conceptual understanding that they need for teaching mathematics in these grades. On the other hand, although elementary school teachers in Singapore typically have considerably less college education than their U.S. counterparts, most of them are competent and qualified to teach mathematics (Ginsburg, et. al., 2005).

Although many states have induction polices, the overall support for new teachers in the U.S. is fragmented due to a wide variation in legislation, policy, and type of support available (Wang, et al., 2003). Only about a half of the new teachers in the U.S. receive formal teacher induction programs in their first year of teaching (Choy, et al., 1998). On the other hand, Singapore provides all new teachers with induction support during their first year (MOE, 1999), which includes reduced teaching load, mentoring, and on-the-job training during the first year.

The Singapore education system has four essential elements in its continuing professional development program: (i) An annual target of 100 hours of professional development for each teacher, (ii) A modular approach to teacher training in order to upgrade each teacher's skills in various topics to varying depths, (iii) Online teaching to supplement face-to-face instruction, and (iv) Formal recognition of teachers for courses taken (MOE, 1999; Hean, 2000). In contrast, a typical professional development program in the U.S. is not as high in quality and consists of no more than a day on a specific content area (Parsad, et al., 2001). Nevertheless, there are several states and school systems in the U.S. that are providing excellent professional development programs. An example worth mentioning is that of Connecticut (Sykes, et al., 2004).

3.7 Teaching Practices and Teacher Training

Table 12 shows that there are major differences in teacher training programs, teaching strategies and teaching practices in these two nations. Singapore's basic philosophy about a teacher's knowledge can be best described by a famous Chinese proverb, "*A teacher needs to have a bucket of water before he is able to give students a bowl of water*." Mathematics teachers therefore need a solid foundation in mathematics, with knowledge that is much deeper than what they are expected to teach.

Table 12: *Teaching Practices*

	Issue	United States	Singapore
1.	Class Time	Teachers seldom have time to go in depth because of too many topics.	Teachers have time to go into more depth.
2.	Subject knowledge of mathematics teachers	Most teachers are trained in education.	Most teachers have a strong background in mathematics.
3.	Teacher's goal	Teach students math skills.	Help students to understand mathematics concepts and skills.
4.	Using research	Only a few teachers use latest research ideas.	Most teachers widely use latest research ideas.
5.	Support and sharing	Unfocused and generally work in isolation.	Focused sessions and a lot of sharing of strategies and lesson plans.

SOURCES: Ginsburg, et al., 2005; Ahuja, 2003; TIMSS Website

4. REFLECTIONS ON SINGAPORE'S SUCCESS STORY

Let us now look at some of the factors which make Singapore the topmost in K-12 mathematics education.

In the Second International Science study (SISS) in 1983-84, the performance of Singapore's 4th graders (13th among 15 nations) and 8th graders (13th among 18 nations) in mathematics was quite unsatisfactory. In 1990, the Ministry of Education of Singapore enhanced the mathematics curriculum to one that emphasized on process skills (e.g. thinking skills, heuristics), attitude development, and streaming based on student ability (MOE, 1996).

While in Singapore, I witnessed the rise of the educational system to one that was becoming superior to those of most other nations. I witnessed their unfaltering belief that a strong educational system provides the means to stay ahead of competing nations. Confucian beliefs about the role of effort and ability in achievements are developed in the kids by their parents and schools right from childhood. Good manners, good attitude, a neat appearance, and perfect attendance are emphasized in all schools. When students fail or get a poor grade, they generally attribute it to their lack of efforts. This failure creates determination in many students to work harder and to pay more attention to their academics.

Some of the key features contributing to Singapore's success include: students' high educational aspirations and positive attitudes toward mathematics, world-class facilities in all schools, safe school environments, alternative mathematics framework and special assistance for slow learners, gifted education program, excellent textbooks, and competent and dedicated mathematics teachers. One of the most important features of the Singaporean educational system is that of streaming. The basic philosophy of streaming is reflected in the Singapore Ministry of Education's mission statement: "*Every child must be encouraged to progress through the education system as far as his ability allows. Advancement must always depend on performance and merit to ensure equal opportunity for all*" (MOE, 2003). On the basis of their abilities, students from grade 5 are placed in two or three different streams. But, slow learners study the same mathematics topics as the other two streams over a longer period of time and with extra assistance from teachers. There is an alternative mathematics framework with mathematics textbooks for each grade in the slower stream and well-trained specialist mathematics teachers. Furthermore, there are several opportunities for students with varying abilities to attend night tuition classes organized by various associations and private companies.

Features contributing to Singapore's success include: a lighter workload for new teachers; the mentoring of new teachers by more experienced teachers; common teachers' rooms with individual desks to work at; well-informed and well-structured guides, worksheets and lesson plans; a lot of cooperation and sharing among teachers within schools, neighborhoods and at the national level; and the availability of manipulative, software and computers. Most Singaporean teachers make an effort to attend meetings, workshops, and conferences during the year. They share and hear about the successes and failures of other teachers. Thus there is a lot of professional interaction. Teachers generally incorporate a variety of methods in teaching mathematics such as assigning theme-based projects and using diagrams and models. Most teachers in elementary and middle grade schools make mathematics interesting by manipulating objects using various software (such as Graph Club, Graphmatica, TesselMania, and Geometer's Sketchpad), and incorporating a variety of research-based teaching techniques in their lessons. Most teachers also focus on active problem solving during class time.

Key features of Singapore's educational system include: common national examinations at the end of grade 6, grade 10, and grade 12; a broad-based school ranking system to keep up standard and competition (MOE, 2004); a national curriculum; textbooks and pedagogical guides; and a lot of investment in education. Most importantly, the taught mathematics curriculum in schools is generally more than the intended mathematics curriculum (set at national level).

The National Institute of Education (NIE) of the Nanyang Technological University (NTU) in Singapore is a world-class teacher training institute where academics and educators work together to help the Ministry of Education in improving mathematics education in Singapore. The important thing about NIE's Mathematics department is that the mathematicians and mathematics educators work together: they belong to the same department, they have the same goals and ideals, and they work together on students' programs. Moreover, almost all professors of NIE, whether in pedagogy or content areas, supervise student teaching in schools. NIE has world-class professors, salary scales, facilities, academic programs, and excellent conceptual frameworks. Special features of NIE's teacher training programs include: full salary, fee waiver, and full service benefits for student-teachers throughout their training period. However, they are required to sign a bond to teach for the next 3 to 4 years. All prospective teachers for elementary schools are trained in six main areas of study: Educational Studies, Curriculum Studies, Curriculum Content, Academic Subjects, Practicum, and Language Enrichment and Academic Discourse. For more information, one may refer to (NIE Website).
Most Singaporean school students (about 80% as per TIMSS Website) have the desire to work hard in mathematics. They spend more than one hour a day on mathematics outside school time (TIMSS Website).

5. CREATING WORLD CLASS HIGH QUALITY K-12 MATHEMATICS EDUCATION IN U.S.

How do we succeed in creating the world-class mathematics education in the U.S. that was proposed by the Governors Conference in September 1989 and set as the fifth goal by the U.S. Department of Education (U.S. Department of Education, 1989)? Surely, the U.S. or any other country cannot merely emulate the practices of Singapore or other top performing nations such as Japan, Korea, Hong Kong SAR, and Belgium. Each country's education practices go hand in hand with its culture and society. That being said, the U.S. and its educational agencies do need to reconsider their own practices and find alternative ways of applying the knowledge of top performing nations in light of their own society and context. Singapore's success story tells us that if any nation has the will and determination, such a goal is not difficult to achieve.

The U.S. has the resources to create a world-class mathematics education system. The NCLB Act clearly shows federal commitment toward improving mathematics education in the nation. Such a commitment is also evident from what President George W. Bush delivered in his '2006 State of the Union Address' (Bush, 2006); the following are the relevant excerpts from his address:

> *"…To keep America competitive, one commitment is necessary above all: we must continue to lead the world in human talent and creativity. …We need to encourage children to take more mathematics and science, and to make sure those courses are rigorous enough to compete with other nations. We've made good start in the early grades with the No Child Left Behind Act, which is raising standards and lifting test scores across our country. Tonight I propose to train 70,000 high school teachers to lead advanced-placement courses in math and science, bring 30,000 math and science professionals to teach in classrooms, and give early help to students who struggle with math, so they have a better chance at good, high-wage jobs…"* (Bush, 2006).

The U.S. has thousands of towns and cities, many of them smaller or about the same size as Singapore. Most of these towns and cities are free to design their own programs, although there is already some level of federal or state control over school districts because of the NCLB Act. If each school district or local government starts by raising its children to be the best in the world in mathematics, the whole nation can become number one in the world in mathematics education. In the words of President G. Bush, "…*If we insure that America's children succeed in life, they will ensure that America succeeds in the world…"* (Bush, 2006). Moreover, this is an achievable goal, since it was found by researchers (Snipes, et. al., 2002) that smaller school districts could reap significant academic benefits by ensuring that students learn high levels of uniform content, as students do in Singapore. We have some examples of school agencies in the U.S. which have created world-class mathematics education systems that are very similar to Singapore's educational system. Examples include Naperville School District #203 (IL), First in the World Consort (IL), and Montgomery County (MD). All these school districts achieved above the international average in TIMSS with scores comparable to any of the industrialized or G8 nations (TIMSS Website).

5.1 Create a World-Class Mathematics Framework and Curriculum

A world-class mathematics framework always provides a clear message for teachers, students, and parents. For example, the Introduction to New Mathematics Content for California Public Schools states:

> "*These standards are based on the premise that all students are capable of learning rigorous mathematics and learning it well, and all are capable of learning far more than is currently expected. Proficiency in mathematics is not an innate characteristic; it is achieved through persistence, effort, and practice on the part of students and rigorous and effective instruction on the part of the teachers*" (CDE, 1999).

The essence of a worthwhile vision for mathematics education should be the development of mathematical ability and expertise. Mastery of mathematical ideas, concepts, processes, and the ability to put them into practice, is what society quite reasonably expects of mathematics education (Macnab, 2000). In order to make students flexible and competent problem solvers so as to meet the demand of the 21st century competitive world, any mathematics framework should be centered on mathematical problem solving. Such a framework should place an emphasis on all those concepts, computational skills, and thinking processes which are needed for a child to become a flexible and competent mathematical problem solver. Although several state standards and NCTM standard include many great ideas, there is a need to consider integrating state or NCTM-based mathematics standards with grade-by-grade concepts (such as numerical, geometrical, algebraic and statistical), processes (e.g. thinking skills and heuristics) and core mathematics skills (e.g. estimation and approximation, mental calculation, communication, arithmetic and algebraic manipulation). Such a framework should also emphasize developing attitudes such as appreciation, interest, confidence, and perseverance, as well as metacognition, i.e. monitoring one's own thinking.

TIMSS data suggests that most U.S. mathematics curricula need to place greater emphasis on areas such as: attitudes towards learning mathematics, problem solving and high-level thinking skills, measurement, estimation and mental mathematics, geometric shapes, perimeter, area and volume, congruence, similarity, vectors, geometric transformations, and three-dimensional geometry. Most importantly, evidences reviewed in previous sections suggest that any mathematics curriculum should also include topics to be studied, the depth at a particular stage, the balance and relationship between knowledge, understanding and investigation, and the expected achievement levels at various ages. Also, see (Macnab, 2000). Additionally, there are the following ten suggestions for states and school districts to consider:

1. Design a mathematics curriculum which ensures that teachers teach to mastery so that there is no need for re-teaching (except for a short review) of the same content in the subsequent grade levels.
2. Consider the spiral approach but without excessive repetition in successive years.
3. Focus on a few topics at each grade level, teach those topics in greater depth, and help students master challenging mathematics.
4. Focus on mathematics content, mental mathematics skills, estimation, and multi-step problem solving.
5. Take arithmetic instruction seriously in the elementary grades and ensure that it is mastered before a student proceeds to high school (Klein, et al. 2005).
6. Emphasize reasoning and mathematical problem solving at every level of mathematics teaching.
7. Use technology as a tool that is a means to the end, rather than an end in itself.
8. Have integrated/cohesive system to minimize gaps between (i) Intended Curriculum (set by state, city, or school district), (ii) Implemented curriculum (by the respective school district or school), (iii) Taught curriculum (by the teachers), and (iv) Attained Curriculum (assessed by external examinations).
9. Develop an official 'Alternate Mathematics Framework' for slow learners. Such a framework will ensure that these students are taught all the required material in less depth but at a slower pace within an extended period of time. Moreover, research shows that children with mathematical learning disabilities do much better in more structural learning environments (Darch, 1984, Miller, et al., 1997).
10. Develop special mathematics programs for high-ability and gifted students in mathematics jointly with the universities.

5.2 Produce World-Class Mathematics Textbooks

A world-class mathematics curriculum cannot create world-class mathematics education unless there are world-class mathematics textbooks to supplement it. Generally, textbooks in the U.S.

are written with an aim to serve all categories of students in thousands of school districts. The school districts or states may consider asking the writers of grade K-12 mathematics textbooks to follow certain guidelines, such as:

1. Textbooks should build a deep understanding of mathematical concepts, contain concrete illustrations (wherever necessary), provide multi-step problems, use sound pedagogical approaches, and also contain mathematically rich problem-based examples and exercises and should have challenging mathematics assessments.
2. Simplify and reduce the size of textbooks considerably by reducing the number of unrelated topics and avoiding repetitive materials from previous grade levels.
3. Textbooks should have strong content development and use the model approach in explaining concepts, wherever possible.
4. There should be special mathematics textbooks and workbooks for slow learners, based on an alternate mathematics framework.

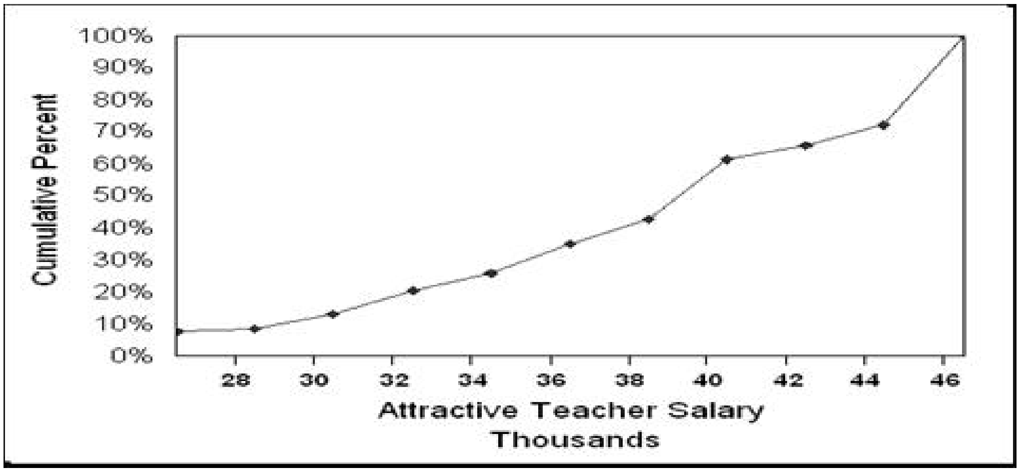
5.3 Recognize and Remove Barriers which Prevent Female Students from Learning Mathematics

As observed in Table 6, male students in the U.S. do better in mathematics than the female students. On the other hand, female students in Singapore are better in mathematics achievements than their male counterparts (Table 6). There is, in fact, a great deal of evidence to suggest that gender differences in mathematics achievement are not biologically or genetically based. It has been suggested that the decline of female achievements is the result of a strong pattern of socialization to mathematics success or failure rather than to gender differences in innate ability (Callahan et al., 1984). Since the goal of mathematics education is to promote students' mathematics achievement, and since gender equity is, in general, a societal goal, it is crucial to recognize and remove the barriers that prevent females from learning mathematics. Most importantly, we should (i) change parents' and teachers' attitudes towards female learning styles, and (ii) bring concrete changes in teaching methods and curricula, for example cooperative learning that promotes collegiality between male and female students (see Schwartz, et al.,1992).

5.4 Make Mathematics Teaching A World-Class and Attractive Profession

The Glenn Commission (Glenn, 2000) observed that American society "*frequently refuses to recognize the professional status of teachers, ranking them below doctors, lawyers, and clergy*". Gerstner (2004) also pointed out that "*the only way to ensure that we remain a world economic power is by elevating our public schools, particularly the teachers who lead them, to the higher tiers of American society*". In the same article, he further suggested that "*Elevating public schools to that level means that we should consider making the teaching profession world-class and attractive by offering salaries and benefits according to the qualifications and experience comparable to other professionals*".

Figure 3: *Cumulative Percentage of Talented College Sophomore and Junior Respondents Who Would Consider A Teaching Career* (SOURCE: *Milanowski, 2003)*



Milanowski (2003) through a research survey and Figure 3 discovered that *"mathematics, science, and technology majors see K-12 teaching as a low paid field, and that many would consider it if it paid substantially more than their current occupational choice"*. In addition to attracting, rewarding, and retaining high quality mathematics teachers, federal and state governments ought to consider having specialized mathematics teachers from Grade 4 onwards in all the public schools.

5.5 Provide World-Class Professional Development for All Mathematics Teachers

In order to create a world class mathematics education system, it is necessary to provide ongoing world-class professional development programs for all mathematics teachers. Incidentally, the federal NCLB Accountability Act requires that all in-service teachers who teach mathematics should be highly competent on content-based mathematics. In this regard, consider the following suggestions:

1. Instead of the current system of short-term workshops, develop professional development programs which may offer credit or non-credit courses as a part of continuing education. States may follow Connecticut's style of a comprehensive 'professional model' for training teachers (Sykes et al., 2004). Alternatively, those teachers who are not highly qualified to teach mathematics should be encouraged to take up content based coursework in mathematics during weekends and long vacations.
2. School districts need to provide frequent in-class support from expert mathematics teachers and time and opportunities for mathematics teachers to meet together on a regular basis to discuss their model lessons; to discuss teaching strategies; to analyze and evaluate their teaching; to talk about their students' learning; and to discuss the use of new technologies in teaching mathematics.
3. Universities should offer specially designed content-based course work or programs in mathematics during summer vacations, with financial help from federal and state governments. Some of the universities have recently been progressing in this direction.

4. Encourage and reward in-service teachers who spend a considerable amount of time and effort, and show good improvement in professional training or take up content-based university courses in mathematics.
5. Provide and encourage mathematics teachers to take special training or courses for teaching mathematics to slow learners.

5.6 Provide World-Class Mathematics Education for Prospective Teachers

"*High quality teaching requires that teachers have a deep knowledge of subject matters*" (Glenn, 2000, p22). It is further observed that teachers' cognitive ability, content knowledge, and professional training are important in teacher quality and student achievement (Whitehurst, 2002). This fact is supported by NCLB Act which requires that only highly qualified teachers deliver instructions (U.S. Congress, 2002).

There is evidence of a vicious cycle in which too many prospective teachers enter college with an insufficient understanding of school mathematics, with little college instruction focused on the mathematics they will teach, and with inadequate preparation to enter their classrooms to teach mathematics to the following generation of students (Ball, 1999; CBMS, 2001). This is why a number of mathematicians and mathematics education researchers have recognized the special nature of the mathematical knowledge needed for K-12 teaching and its implication for the mathematical preparation of teachers (Ma, 1999; CBMS, 2001). In this regard, the following suggestions are offered:

1. Attract talented mathematics students for prospective education majors by offering them scholarships and/or full fee waiver.
2. Attract talented unemployed graduates in engineering, computer science, and business by offering them scholarships and by providing them one-year post-graduate teacher training programs.
3. Provide high quality mathematics education courses that develop a deep understanding of the mathematics content they will teach.

5.7 Promote Cooperation Between Mathematicians and Mathematics Educators

"Most mathematicians and mathematics educators in the U.S. live in different worlds: they have different cultures, different standards of rigor, and different languages to talk about mathematics learning" (Cuoco, 2003). Indeed, a Google search on "math war" will turn up thousands of newspaper articles and websites arguing for or against various approaches to teaching mathematics (Jackson, 1997; Cuoco, 2003, p781). Mathematicians and mathematics educators, instead of pursuing "math wars" as seen in the 21st century, should commit to shared accountability and responsibility to provide world-class mathematics education to prospective teachers.

5.8 Promote Parental Involvement in Creating World-Class Mathematics Education

Parental involvement can greatly help a school in providing world-class mathematics education. Every school should promote partnerships that will enhance parental involvement and participation in promoting the social, emotional, academic growth, and in particular, mathematics education of children. Elementary school years are important for fostering constructive learning habits that are reinforced in the home and are essential to life as an adult

(Duke 1986). It is therefore imperative that parents and, in fact, all other adults who play an important role in their children's home lives, help their children in developing self-discipline, perseverance, and a positive attitude towards learning and hard work. They should also help their children appreciate and value that mathematics is an important subject for their high school and college education, and also for many careers. In this regard, the parents or guardians of K-12 students should also be encouraged to:

1. Take an active interest in their children's mathematics education,
2. Make sure that their children do their mathematics homework consistently and prepare well in advance for their mathematics tests, projects, and examinations,
3. Ensure that schools do a good job in teaching mathematics, and
4. Understand and value the importance of mathematics in their children's education.

5.9 Other Suggestions for Federal and State Governments

It is still not too late to make all possible efforts to make the U.S. number one in school mathematics education in the world. In this regard, federal and state governments may also consider the following ten suggestions:

1. Consider improving the NCLB Accountability Act by holding schools, teachers, students, and parents accountable for their students' performances.
2. Replace the authors of weak standard documents with people who thoroughly understand mathematics, including university professors from mathematics departments (Klein, et al, 2005). In this regard, California is making headway by encouraging and involving university mathematicians in writing mathematics standards.
3. Consider borrowing a complete set of high quality mathematics standard from a top-scoring state (Klein, et al, 2005).
4. Reward schools that demonstrate students' academic growth measured by the difference in the students' entry level abilities and their abilities upon graduation. For example, see recently revised broad-based school ranking system in Singapore (MOE 2004).
5. Ensure world-class facilities in every classroom, such as a PC, projector, Internet, manipulative, concrete objects and others.
6. Ensure that weaker students in mathematics are given more time and extra help, after school hours, in small groups by specially trained and competent mathematics teachers.
7. Re-educate parents, administrators and the general public to understand and support reforms in mathematics education.
8. Popularize mathematics by making it fun and implementing programs such as 'Mathracy', 'Numeracy', 'Compute to your kids', 'Have fun in math with your kids' etc.
9. Provide all possible help to fix up financial crises in poor school districts that prevent them from advancing mathematics to worldwide standards.
10. Consider ways to invest heavily on education for at least the next ten years.

6. CONCLUSION

The United States is world-class in many areas. It is critical for the nation to become world-class in K-12 mathematics education as well. The consequences of supporting this vision include great economic prosperity and an overall higher quality of life for all Americans. Some states and school districts have been striving hard to achieve world-class mathematics education in their schools, for example, California, Indiana, and Massachusetts which received 'A' grades for having first-rate mathematics standards (Klein, et al., 2005). The United States can look forward to a top-notch K-12 mathematics program, and possibly succeed in achieving the goals set forth by NCLB if all the states in the country successfully develop world-class standards in mathematics education, align all other key educational policies (such as. salaries, teacher preparation and development, accountability, textbooks, graduation requirements etc) with those standards, and if their schools and teachers succeed in instructing students in the skills and content specified in those standards (also see Klein et al., 2005). Based on Glenn commission (Glenn, 2000), our motto should be "World-Class High Quality Mathematics Education for All K-12 American students, Without Any Delay! And –Without Any Excuses!"

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Sunrise… Sunset…

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Sunrise, sunset,
Sunrise, sunset,
Swiftly flow the days,
Seedlings turn overnight to sunflowers,
Blossoming even as we gaze.
(Fiddler on the Roof, Lyrics S. Harnick)

Introduction

Sunrise and sunset are obviously a part of our everyday life. Our connection to the movement of celestial bodies, including the Sun, has been somewhat diminished by modern life, in contrast to the practical interest, fascination or even awe of earlier civilizations. Mathematics and astronomy have a long history of cooperative work, but due to their complexity and technical difficulty, usually those who participated in this endeavor were at the advanced studies level, or occupied high religious positions.

However, today's technological tools can overcome many of these obstacles, and they have the potential to make mathematical–astronomical investigations possible at relatively early stages of education. In our opinion, currently our challenge is to raise student interest and to design appropriate investigative activities in this field. Beyond its social and historical importance, the investigation of the Sun's "movement" (actually, position) in our terrestrial world has many mathematical benefits.

In the following sections we will describe the *Sunrise – Sunset* activity, present some observed classroom reactions to this task, and finally discuss the learning potential of such an activity.

Sunrise – Sunset

The data for the *Sunrise-Sunset* activity were prepared in advance by the teacher, and presented to the students in the format of two spreadsheet tables. The tables contain the local time (in our case, Tel Aviv) for the sunrise and sunset at the first day of each of 24 consecutive months, starting January 1st. The dates were numbered from 1 to 24, and the time was presented both in standard form (figure 1a) and in decimal notation (figure 1b).

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	A	B	C	D	E	F	G	H
1		Month	Sunrise	Sunset		Month	Sunrise	Sunset
2		1	6:41	16:48		1	6.68	16.80
3		2	6:35	17:15		2	6.58	17.25
4		3	6:10	17:38		3	6.17	17.63
5		4	5:30	18:00		4	5.50	18.00
6		5	4:56	18:20		5	4.93	18.33
7		6	4:35	18:41		6	4.58	18.68
8		7	4:37	18:51		7	4.62	18.85
9		8	4:55	18:39		8	4.92	18.65
10		9	5:15	18:07		9	5.25	18.12
11		10	5:33	17:27		10	5.55	17.45
12		11	5:56	16:52		11	5.93	16.87
13		12	6:22	16:38		12	6.37	16.63
14		13	6:41	16:48		13	6.68	16.80
15		14	6:35	17:15		14	6.58	17.25
16		15	6:10	17:38		15	6.17	17.63
17		16	5:30	18:00		16	5.50	18.00
18		17	4:56	18:20		17	4.93	18.33
19		18	4:35	18:41		18	4.58	18.68
20		19	4:37	18:51		19	4.62	18.85
21		20	4:55	18:39		20	4.92	18.65
22		21	5:15	18:07		21	5.25	18.12
23		22	5:33	17:27		22	5.55	17.45
24		23	5:56	16:52		23	5.93	16.87
25		24	6:22	16:38		24	6.37	16.63

Figure 1. Spreadsheet timetables of sunrise and sunset (in Tel Aviv) in standard form (a) and in decimal notation (b).

The work of the students, a class of ninth graders, was observed during two class periods. Some student reactions are presented below (in italics).

1. Getting acquainted and making predictions.

First, the students understood the context and the data of the problem under investigation, made predictions about the shape of the graphs that show the change in time for the sun's rising and setting during two consecutive years, and became involved in the activity. This part consists of two steps:

- Class discussion and comparison (advantages, disadvantages) of the standard and decimal notation in the two tables (figure 1).
- Sketching (i.e. drawing in an unscaled coordinate system) two graphs, in order to show the general change in sunrise and sunset time during two consecutive years.

The students worked in pairs, and then looked and compared their graphs with those of their colleagues. Most of them sketched cyclic graphs (figure 2), and one pair of students sketched two identical graphs – a separate graph for each year.

Students S1 and S2 sketched a graph of straight lines (see figure 2A), and compared their work with the curved graph of their neighbors, S3 and S4 (figure 2B).

S1: I think they are right.
S2: Our [graph] is the same. This is only a sketch.
S1: Theirs is rounded – and this is better.
S2: It's the same – it's only a sketch.
S1: It's rounded…It's better.
S2: What do you mean, "rounded"? What is the difference between these two [graphs]?
S3 (Interrupts her work with her peer in Pair B and points to the neighborhood of the graph's maximum): Ours is slower here and yours drops sharply.
S4: It's the period between going up and going down… We have more points, and in yours it looks as if there is only one point.
S2: ("Giving up"): You want us to round it?
S1: Yes. It's more correct.
S2: Then let's round it off.
S1: They made them this way [showing that the graphs of pair B are parallel].
S2: Ours is OK [the sunrise graph starts by decreasing – in contrast with the phase difference of the sunset graph, which starts by increasing].

Some students drew the sunrise and the sunset graphs as intersecting curves (figure 2C). When asked by the teacher about the meaning of the intersection points, these students changed the position of their graphs.

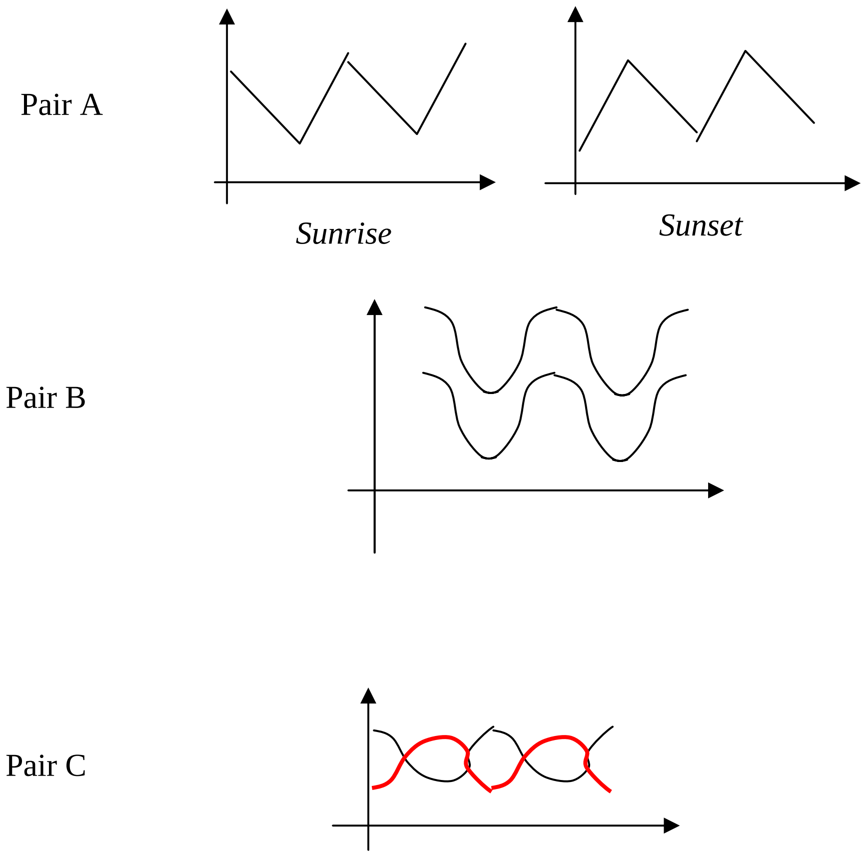

Figure 2. Students' predictions of shape and position of *Sunrise-Sunset* graphs.

- Reflecting on predictions. The issue of shape and position of the two graphs was discussed first in groups, and then in a forum involving the whole class.

2. *Analyzing data and drawing conclusions.*

In this part, the students produced graphical representations of the data, and analyzed their mathematical, astronomical, and daily-life meaning. This part consisted of several steps.

- Using spreadsheets to construct graphs of the sunrise and sunset times, and comparing the hand-sketched predictions and the spreadsheet graphs. Figure 3 shows the spreadsheet graphs of the sunrise and sunset times located in one coordinate system.

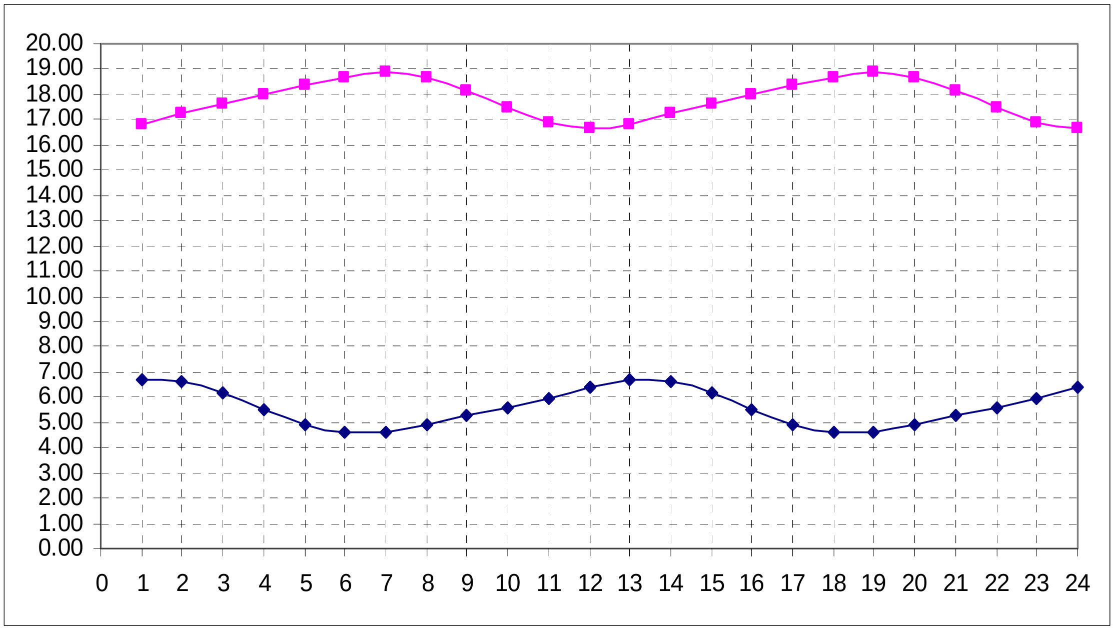

Figure 3. *Sunrise-Sunset* spreadsheet graphs.

An "Aha!" effect could be felt in the class, whenever the spreadsheet graphs of several students turned out to be different from their preliminary sketches.

- Investigating the data (patterns, extremes, cycles, axes of reflection, and rate of change) and making everyday-life interpretations of the given numbers and obtained graphs.

  Some students "talked" in a mathematical language, whereas others used context-bound expressions. Some examples:

  The functions are "monotonous" [intuitively meaning cyclic].

  The functions are opposite one another [intuitively meaning there is a phase difference, and the maximum of one graph corresponds to the minimum of the other].

  The sketch repeats itself.

  The sketch goes up and down.

  The sunset [graph] goes up, as the sunrise [graph] goes down.

  Every twelve months, the sunrise and the sunset occur at the same hours.

- Investigating patterns of daylight hours: How can we find the length of the daylight from the *Sunrise-Sunset* timetable? From the *Sunrise-Sunset* graphs? Construct a daylight column in the spreadsheet table, and produce a corresponding graph. Analyze the change in daylight hours (look again for patterns, extremes, cycles, axes of reflection, and rate of change).

  Students S1 and S2 (pair A) used the formula of the difference between sunrise and sunset (decimal) times to find the length of daylight in their spreadsheet table, and were surprised to receive negative numbers. They corrected their formula, and then wondered whether the "straight" difference provides the desired answer, or whether their formula should be corrected by plus or minus one. Subtracting whole hours and *checking by finger counting helped them decide that the "straight" difference gives the desired answer.*

Most students drew vertical segments between the two graphs as a graphical method of finding the length of daylight (figure 4). The students investigated patterns of daylight by using either the numerical data (the spreadsheet column) or a graphical representation (the vertical distances between the graphs).

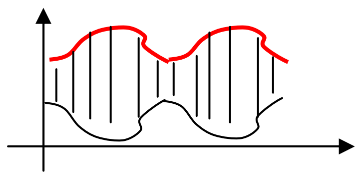

Figure 4. Using the *Sunrise-Sunset* graphs to look for patterns of variation of daylight.

3. Reflecting on the activity and on its mathematical implications.

During the summary in class, the students were asked to reflect on their investigation, and to consider the following issues:

- To what extent are the obtained graphs accurate – i.e. how well do they describe the real situation?

  A student raised the issue of the spreadsheet graphs' lack of symmetry (see figure 3). Most of the class had the intuitive feeling that the graphs should be symmetric. The teacher asked about the meaning of a symmetrical graph (sunrise and sunset times going up and down for equal periods), but the prevailing opinion was still that "the computer is probably right". The teacher indicated that the graphs could be different, if they were based on a more detailed (for example, a daily) timetable, or on monthly dates that are closer to the graphs' real extremes (the 21st or the 22nd of each month).
- If we separate the graphs from their context of sunrise-sunset time, and view them as mathematical "creatures", how can we characterize them?
- How would the *Sunrise-Sunset* graphs look in other places on the globe?
- How would the graphs look if the hours were not adjusted for daylight savings time?
- On what occasions can the *Sunrise-Sunset* data and conclusions be useful?

Conclusion

Finally, we would like to reflect on whether activities like *Sunrise-Sunset* provide satisfactory answers to the following questions.

- ***How can pattern recognition be used to develop students' problem-solving abilities?***

  The *Sunrise-Sunset* activity and similar investigations in other domains frequently require the following steps:

- define a research problem
- collect and organize the data
- predict the results
- analyze data / find patterns / look for solutions
- draw conclusions / compare the findings and predictions / reflect on the process.

This paradigm of work is common to both scientific inquiry and mathematical problem solving.

- ***How can patterns generated through spreadsheets enhance students' understanding of mathematics?***

Spreadsheets play an important role throughout the *Sunrise-Sunset* activity:

- They provide a quick, effective, and accurate passage from an extended numerical table to the corresponding graphs.
- They provide opportunities to make predictions based on raw data, subsequently comparing them to the results of a more extensive analysis.
- They emphasize global aspects and patterns of complex phenomena, based on a large quantity of local data.
- They provide a variety of representations (numerical, graphical, and algebraic), and allow the use of these representations, according to the task at hand.
- They help to perform calculations almost instantaneously, and as a result, they enable students to develop and employ higher-level skills – such as defining new variables, creating and using algebraic formulas, generalizing patterns, monitoring results, and drawing conclusions.

- ***How can the investigation of patterns lead to a better understanding of such concepts of algebra as variable, rate of change of a function and the slope of a graph?***

When students investigated the *Sunrise-Sunset* problem, we observed spontaneous references to the rate of change (especially with regard to the timetable) and the graphs' increase or decrease of slope. The observed class of ninth graders had not yet encountered a formal definition of a slope or of a linear function. The context allowed and even encouraged students to discuss and use these concepts with regard to natural phenomena, such as change in sunrise, sunset, and daylight time, or to mathematical phenomena, such as the steepness, curvature, and symmetry of a graph.

- ***How can explorations of patterns in a student's earlier experiences be used to develop more sophisticated topics?***

We tried to show here that our exploratory activity has the potential to provide an informal link in the continuous process of learning mathematical concepts related to the properties and patterns of linear and trigonometric functions.

- ***Where can pattern recognition in other disciplines be connected with familiar mathematical content?***

Similar activities of pattern recognition can relate to a variety of domains, such as architecture, plants, animals, physics, and geography. In these activities, the data can be represented and analyzed in various ways, and interesting patterns can be observed and explained. Finally, mathematical and context-based conclusions are drawn, and additional issues are perhaps raised. In our case, we observed students intuitively linking the familiar sunrise and sunset phenomenon to important mathematical concepts, later to be formalized in function analysis and trigonometry.

HOW STUDENTS CONCEIVE FUNCTION: A TRIARCHIC CONCEPTUAL-SEMIOTIC MODEL OF THE UNDERSTANDING OF A COMPLEX CONCEPT

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Abstract: This study explores conceptions for function amongst 164-second year students of the Department of Education at the University of Cyprus and their relationship with students' abilities in dealing with tasks involving different modes of representations of function. The test that was administered to the students included recognition tasks given in various representations and questions requesting definition and examples of function's applications in real life situations. Results have shown that students' definitions and examples of the notion are closely related to their ability to use different modes of representations of function. These three factors, i.e., definitions given by the students, functions considered by them as examples of application in real life situations, and different representations of functions, seemed to contribute in their own unique way to students' acquisition of this complex concept. Thus, support was provided for the use of a triarchic conceptual-semiotic model of the concept of function, which enables students' thinking and understanding of the notion to be analyzed and described across these three features.

Key words: function, representations, compartmentalization, concept definition, concept image, triarchic conceptual-semiotic model, similarity diagram, implicative method

1. INTRODUCTION

The concept of function is of fundamental importance in the learning of mathematics and has been a major focus of attention for the mathematics education research community over the past decades (e.g., Evangelidou, Spyrou, Elia and Gagatsis, 2004; Sfard, 1992; Sierpinska, 1992; Vinner and Dreyfus, 1989). Research related to functions has been directed towards various domains. We will focus on two strands of research that have a bearing on this study in order to clarify the basic goal of it. The first research domain refers to the concept image of function in the students' minds (e.g., Vinner and Dreyfus, 1989) and the second one concerns the different representations of the notion and the passage from one to another (e.g., Duval, 2002; Hitt, 1998).

This paper is an attempt to examine the relationship between students' concept definitions, examples of function and their ability to use and connect different representations of the notion, on the basis of two theoretical semiotic perspectives (Duval, 2002; Steinbring, 1997) having a central focus of attention on students' construction of meaning and understanding of mathematical concepts. This relationship is incorporated in a new triarchic conceptual-semiotic model which integrates three fundamental components of the understanding of the function concept: defining the concept; giving examples of the application of the concept in everyday life; identifying functions in different modes of representation and changing systems of representation.

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1.1 Understanding of the concept of function

1.1.1 Concept image and concept definitions

Concept image and concept definitions are two terms that have been discussed extensively in the literature concerning students' conceptions of function (Vinner and Dreyfus, 1989; Tall and Vinner, 1981). Although formal definitions of mathematical concepts are introduced to high school or college students, students do not essentially use them when asked to identify or construct a mathematical object concerning or not this concept. They are frequently based on a concept image which refers to "the set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them" (Vinner and Dreyfus, 1989, p. 356). Thus, on the basis of a model of cognitive processes concerning the relation between the definition of the concept and concept image, different categories of students' definitions and concept images were identified in the study of Vinner and Dreyfus (1989).

1.1.2 Representations and the understanding of function

The understanding of functions does not appear to be easy, given the diversity of representations related to this concept (Hitt, 1998). Sierpinska (1992) indicated that students have difficulties in making the connections between different representations of the notion (formulas, graphs, diagrams, and word descriptions), in interpreting graphs and manipulating symbols related to functions. Some students' difficulties in the construction of concepts are linked to the restriction of representations when teaching. Mathematics instructors, at the secondary level, traditionally have focused their instruction on the use of algebraic representations of functions rather than the approach of them from the graphical point of view (Eisenberg and Dreyfus, 1991; Kaldrimidou and Iconomou, 1998). Markovits, Eylon and Bruckheimer (1986) observed that translation from graphical to algebraic form was more difficult than the reverse conversion and that the examples given by the students were limited in the graphical and algebraic form.

The findings of the above studies are related to the phenomenon of compartmentalization. The existence of compartmentalization reveals a cognitive difficulty that arises from the need to accomplish flexible and competent conversion back and forth between different kinds of mathematical representations of the same situation (Duval, 2002), which according to Arcavi (2003) is at the core of mathematical understanding. Gagatsis, Elia and Andreou (2003) found that 14-year-old students were not in a position to change systems of representation of the same mathematical content of functions in a coherent way, indicating that systems of representations remained compartmentalized and mathematical thinking was fragmentary.

1.2 Representations and mathematics learning: Two semiotic theories

The theoretical position that we are taking in our research is based on two semiotic perspectives. These also serve as a basis of the triarchic conceptual-semiotic model that we propose. The first basic idea we adopt in our framework deals with the importance of the diversity of semiotic representations and their transformation for the development of mathematical thought. According to Duval (1993, 2002) mathematical activity can be analyzed into two types of transformations of semiotic representations: treatments and conversions. Treatments are transformations of representations which take place within the same system where they have been formed. Conversions are transformations of representation that consist of changing a system of representation, without changing the objects being denoted. The conversion of representations is considered as a fundamental process for mathematical understanding (Duval, 2002, 2005).

We also adopt Steinbring's (1997) idea that the meaning of a mathematical concept occurs in the interaction between sign/symbol systems and reference contexts or object domains. The triarchic conceptual-semiotic model of the understanding of function that is introduced in this study is constituted by the reference contexts and the signs of the notion. In particular, students' constructed definitions and examples of function correspond to the "reference contexts" that may change during the process of mathematical knowledge development. The systems of representation, such as graphs, symbolic expressions, arrow diagrams and verbal descriptions are considered as the "symbol systems" that are used for denoting and implementing the referential objects. Steinbring maintains that "the difference between the function of a symbol system and a structural reference context is essential for the generation of meaning in every new mathematical relationship" (1997, p. 78). Sierpinska (1992) considers the distinction of a function from the analytic tools used to describe its law as one of the essential conditions for understanding functions. Therefore, in this study students' constructions of definition and examples for the concept of function are distinguished from the transformation of representations.

As presented above, numerous studies have examined the role of representations on the understanding of function and students' concept image for it, separately. Taking into account Steinbring' s (1997) idea that the meaning of a mathematical concept occurs in the interaction between sign/symbol systems and reference contexts, we need to add to the mathematics education research community understanding of the way these two dimensions are interrelated as regards the concept of function.

In this paper we attempt to contribute to mathematics education research understanding with respect to the concept of function by investigating the relationship among the three aforementioned components that are constitutive of the meaning of function, i.e, D, E and R, and by interweaving them in a triarchic conceptual-semiotic model. "D" corresponds to the common definitions of the function concept given by a student; "E" signifies the set of mathematical or non-mathematical objects or relations considered by the student to be examples of the concept of function; and "R" designates the range of different representations of functions that the student deals with (R). We anticipate that this model will provide a coherent picture of students' construction of the meaning of function that is desirable for current approaches of instruction which aim at the development of the understanding of this concept. The potential power of the triarchic model will be verified by a statistical tool, namely, CHIC (Bodin, Coutourier and Gras, 2000), that has not been used previously in similar investigations.

In more specific terms the purpose of this paper is the following: First, to explore university students' conceptions of function on the basis of their concept definitions and examples of the notion; second, to examine students' performance to recognize functions in different forms of representation and transfer from one representation to another; third, to explore the relationship between their conceptions of function (D and E) and their ability to use different representations of the concept (R).

2. METHOD

2.1 Participants

The sample of the study consisted of 164 students who attended the course "Contemporary Mathematics" at the University of Cyprus. The questionnaire was completed by 154 second year students of the Department of Education and 10 four year students of the Department of

Mathematics and Statistics. The students come from diverse high school directions, which differ in the level and length of the mathematics courses that they involve. Nevertheless, all of the students who participated in this study had received a teaching on functions during the last three grades of high school. The content of this teaching is based mainly on a classical presentation of function: domain and range, derivatives, maximum and minimum and construction of graphs of first-, second-, third- and fourth-degree polynomial functions. It is noteworthy that the sample consists of future primary and secondary school teachers, who will in a way transfer, their mathematical thinking to their prospect students. The concept of function is not included in the curriculum of primary mathematics education in Cyprus, but other mathematical relations such as proportion or bijective types of correspondence are within the content areas that teachers are required to teach, similarly to the educational systems of other European countries. As for the secondary education, the concept of function is one of the basic topics that are included in the content of the mathematics curriculum in Cyprus and focuses on the "classical" topics of function, mentioned above.

2.2 Research instrument

A questionnaire (see Appendix) was administered a few weeks after the beginning of the course. It consisted of ten tasks, which were developed on the basis of the two types of transformation of semiotic representations proposed by Duval (2002): treatment and conversion. Yet, the tasks we developed differed from Duval's proposed activities in two ways: First, they included recognition whether mathematical relations in different modes of representation (verbal expressions, graphs, arrow diagrams and algebraic expressions) were functions or not, by applying the definition of the concept. Nevertheless, a general use of the processes of treatment and conversion was required for the solution of these tasks. Secondly, they involved conversions, which were employed either as complex coding activities or as point-to-point translations and were designed to correspond to school mathematics. For instance, a conversion could be accomplished by carrying out various kinds of treatment, such as calculations in the same notation system.

A variety of functions were used for the tasks of the questionnaire: linear, quadratic, discontinuous, piecewise and constant functions. Below we give a brief description of the questionnaire and the corresponding symbolization for the variables used for the analysis of the data: Question 1 (Q1A, Q1B, Q1C, Q1D), Question 4 (Q4A, Q4B, Q4C, Q4D, Q4E, Q4F), Question 6 (Q6A, Q6B, Q6C, Q6D, Q6E) and Question 7 (Q7A, Q7B, Q7C, Q7D) asked students to recognize functions in different modes of representation, i.e., verbal, algebraic, graphical and arrow diagrams, respectively, and to provide an explanation for their answer. Questions 2 (Q2), 3 (Q3) and 5 (Q5) required a conversion of a function from one representation to another. Question 8 (Q8) asked what a function is and Question 9 (Q9) requested two examples of functions from their application in real life situations.

2.3 Data Codification

Students' responses for the definition of function and its applications at the corresponding questions were grouped into particular categories to explore the relation of the different values of the former two dimensions of the triarchic model, i.e., D and E, to the latter one, i.e., R. Definitions in Question 8 were coded as follow:

D1: *Correct definition*. This group included the accurate set-theoretical definition.

D2: *An approximately correct definition.* This group involved answers with a correct reference to the relation between variables, but without defining the domain and range.

D3: *Definition of a special kind of function.* This group of answers made reference to a particular type of function (e.g., real, bijective, injective or continuous function).

D4: *Reference to an ambiguous relation*. Answers that made reference to a relation between variables or elements of sets, or a verbal or symbolic example were included in this group.

D5: *Other answers*. This type of answers made reference to sets, but no reference to a relation, or reference to relation without reference to sets or elements of sets.

D6: *No answer*.

The following additional codes were given for the types of examples provided in Question 9: X1a: Example of a function with the use of discrete elements of sets; X1b: Example of a continuous function, usually, from physics; X2: Example of a one-to-one function; X3: Example presenting an ambiguous relation between elements of sets; X4: Example of an equation in verbal or symbolic form; X5: Example presenting an uncertain transformation of the real world; and X6: No example.

2.4 Data Analysis

2.4.α Qualitative Analysis.

The first part of the qualitative analysis is based on the explanations provided by the students when justifying their decision whether a relation represents a function or not. Next we present some indicative examples of the types of responses the students gave while trying to define and give examples of function.

2.4.b Quantitative Analysis

Primarily, the success percentages were accounted for the tasks of the test by using SPSS. A similarity diagram (Lerman, 1981) of students' responses at each item of the questionnaire was also constructed by using the statistical computer software CHIC (Classification Hiérarchique, Implicative et Cohésitive) (Bodin et al., 2000). The similarity diagram allows for the arrangement of the tasks into groups according to the homogeneity by which they were handled by the students. A similarity index is used to indicate the degree to which the variables of a group are similar to each other on the basis of students' answers. This aggregation may be indebted to the conceptual character of every group of variables. Unlike the range of the linear correlation coefficient (from -1 to +1), the similarity index is ranging from 0 to 1. As the similarity of a group gets stronger, the index gets closer to the value of 1. The similarity index corresponds to the length of the vertical segments that form each pair or group of variables. As these vertical segments get shorter, the similarity index approaches the value of 1. This means that the stronger the similarity relations (pairs or groups of variables), the shorter are their vertical segments.

It is worth noting that CHIC has been widely used for the processing of the data of several studies in the field of mathematics education in the last few years (e.g., Evangelidou et al., 2004; Gagatsis, Shiakalli and Panaoura, 2003; Gras and Totohasina, 1995).

3. RESULTS

3.1 Some indicative answers

An idea that was extensively observed among the students was that *a function must essentially contain two variables or unknowns* or that *the algebraic or graphical expression of a function must at any rate contain x and y*. The answers that the expressions (Q4A) 5x+3=0 and (Q4C) 4y+1=0 cannot define functions, were justified with "*x (or y) do not appear in the expression, therefore a function cannot be defined*". Moreover, the relation

(Q4D) $x^2+y^2=25$ was considered a function, since it included x and y. The same idea was apparent for the question requesting whether some Cartesian graphs have resulted from a function. Those graphs representing a straight line, parallel to the x- or the y- axis where not accounted as functions because "*x (or y) is constant and therefore it is not an unknown and a function must contain two unknowns*". Similarly some other students appeared to believe that *a function is an equation* and rejected (Q4A) and (Q4C) by explicitly saying that "*the expression … does not represent an equation, and therefore it cannot be a function*".

Another idea held by the students was that *a function is necessarily a bijective correspondence*. This was noticeable in the explanation given in (Q1D) asking whether the correspondence between every football game and the score achieved defines a function. Negative answers were justified with the fact that "*two football games may have the same score*". Also some students, while trying to explain their wrong decision that the algebraic expression in (Q4F) $f(x)=x$ for $x\geq 0$ and $f(x)=-x$ for $x<0$ does not represent a function, stated that "*two different values of x correspond to the same value of f(x) and therefore the expression is not a function*". Some other students used the same reasoning to reject the graph of the parabola in (Q6C).

According to some other students, *the variables should not come from a specific set, but should take random values*. This was expressed by those students who considered that the correspondences described verbally in Q1A (we correspond a girl to different friends of her with whom she dances at a party) and Q1D (we correspond every candidate with the post for which he applies for work in an organization) cannot define functions because "*the girl can only dance with a limited number of boys who attend the party*" and "*the candidates do not have a random a choice of jobs*".

The students were also very much distracted by the arrow diagrams, which, were presented in incompact frames, thus expressing the idea that *in a graph of a function domain and range should be compact sets*. Negative answers for the diagrams presented in Q8B and Q8C were that "*they do not represent functions because the correspondence starts or ends from a different set".*

A likewise idea was that *a graph of a function should be continuous* as the graph of a y=x, with domain the union of the intervals (-3,-1), (0,1), (2,3) was not considered as function with the same frequency as the other linear forms. Students justified their choice stating explicitly that "*the graph is not continuous, and therefore, cannot represent a function*".

In the question requiring the definition of function (Q9) the answers that gave an approximately correct definition (D2) were grouped together. Answers like "*Function is a relation between two variables so that one value of x (or the independent variable) corresponds to one value of y (or the dependent variable)*" were accounted in this group. Answers that referred to the accurate definition, but added some more conditions to it and as an outcome would give the definition of a specific type of function (like injective, bijective or real function), were coded as D3. Ambiguous answers like "*Function is an equation with two dependent variables*", "*Function is a relation in which an element x is linked with another element y*" or even "*Function is a mathematical relation connecting two quantities*" were coded as D4. As D5 we have coded answers, which made reference to sets, but did not mention relation, or made reference to relation but not to sets or elements of sets, such as "*Function is a relation*" or "*Function is a mathematical concept that is influenced by two variables*" or "*Function is the identification of parts of a set*".

In the question requiring examples of functions from their applications in real life (Q10) the variety of responses was even greater. The correct examples of a function were of two kinds

(X1a and X1b). Examples of the first kind (X1a), which made use of sets with discrete elements, were: "*Each person corresponds to the size of his shoes*", "*Each student corresponds to his/her mark at the test*". As (X1b) type of examples we have grouped examples of a continuous-linear function mainly from physics, such as "*The height of trees is a function of time*", "*Atmospheric pressure is a function of altitude*". The examples presenting a bijective function were coded separately as X2. Such answers were "*Every citizen has his own identity number*", "*Every graduate has his own different degree*" and "*Every country corresponds to its own unique name*". As X3 we coded the examples presenting a relation between elements or variables but without clarification of the uniqueness in function. Such answers were as follows: "*There is a relation between students and their books*", "*The prices of vegetables depend on the production*", "*We correspond the marks of girls in a classroom to those of boys*". Examples presenting an equation instead of a function were coded as X4: "*There are 2x boys and 3y girls in a classroom and all the children are 60. If the boys are 15 we can calculate the number of girls*", "*Kostas has x number of toffees and Giannis has double that number. How many toffees do the two friends have?*". The last category X5 included answers which were ambiguous, but furthermore did not define any variables or sets, just an uncertain transformation of the real world. Such answers were "*Health depends from smoking*", "*Success in a test depends on the hours of studying*", "*In the relation of children and parents, the children are the dependent variable and parents the independent variable*".

3.2 Success percentages

In this section we will only refer to the results that show the strongest trends among the students. Higher success scores (91%) were achieved in Question 4B (Q4B), which presented the algebraic form of a linear function (a well known figure from high school mathematics), while lower success rates (8%) were attained in one of the conversion problems (Q2). Only thirteen of the students succeeded in constructing the algebraic formula of the characteristic function of a set that was given verbally (Q2), probably because the change of system of representation was not a simple coding activity or transparent conversion (Duval, 2002), and required a global interpretation guided by the understanding of the qualitative variables and their relation. A large percentage (70%) of the students did not consider that the graph of a y=x, with domain the union of the intervals (-3,-1), (0,1), (2,3) in Question 6E was a function. Most of them justified their choice stating explicitly that "*the graph is not continuous, and therefore, cannot represent a function*". This kind of behaviour reveals students' idea that a graph of a function must be connected or "continuous". The majority of the students (62%) answered correctly to another conversion (verbal–algebraic) problem (Q3), which involved a function changing the initial prices to the sales prices of a shop, probably because this is a real life problem, concerns a linear function and involves a term-by-term conversion.

Low success percentages were observed in Q4C (26%) and Q4D (37%), as students did not think that the algebraic form in (Q4C) 4y+1=0 represented a function, while they considered that the formula of the circle in (Q4D) $x^2+y^2=25$, did. As deduced from their explanations, these responses were a consequence of the idea that a function must essentially contain two variables or unknowns. The same was true for the graphs in Question 6. For instance, students did not think that the graph of the straight-line, parallel to the horizontal axis (y=0), i.e., y=4/3 (Q6B) could have resulted from a function. Difficulties caused by the constant function were identified also by Markovits et al. (1986) among students of 14-15 years of age. Table 1 presents the success percentages in the last two questions requesting the definition of the function concept and examples of function from real life.

Table 1: *Percentages for the definition and example categories*

(Q9) Definition of function	Frequency N=164	Percentage %
D1: Correct definition	13	8
D2: An approximately correct definition	13	8
D3: Definition of a special kind of function	5	3
D4: Reference to an ambiguous relation	73	45
D5: Other answers	24	15
D6: No answer	36	22
Total	164	100%
(Q10) Example		
X1a: A function (using discrete elements of sets)	11	7
X1b: A continuous function usually from physics	8	5
X2: A one-to-one function	28	17
X3: An ambiguous relation between elements of sets	29	18
X4: An equation	8	5
X5: An uncertain transformation of the real world	27	16
X6: No example	53	32
Total	164	100%

It is apparent that the majority of the students (45%) did not give a correct definition, but made reference to an ambiguous relation between variables without establishing the uniqueness. In addition, 29% of the students gave a correct example of function with the majority of them 17% referring to a one-to-one function (X2). The largest percentage of the students (32%) could not find any example of function.

Table 2 presents the results of the cross tabs analysis, which was used to investigate students' achievement in each representational type of tasks and each conversion task in relation to their ideas for the definition and examples of function. The aforementioned categories for the given definitions and examples of function were grouped in such a way, so that the percentages refer to the students who gave acceptable definitions (D1-D3) or incorrect definitions (D2-D6), and acceptable examples (X1a, 1b, X2) or inappropriate examples (X3-X6).

Table 2: *Percentages of students from each definition and example group of categories who responded successfully to the tasks*

	Type of task	D1-D3 %	D2-D6 %	X1a, 1b, X2 %	X3-X6 %
Recognition tasks	Verbal expressions	68	47	64	45
	Algebraic expressions	77	56	70	56
	Cartesian Graphs	65	36	53	38
	Arrow diagrams	61	36	51	37
Conversion tasks	Problem 2	26	4	17	4
	Problem 3	84	56	81	54
	Problem 5	84	50	70	51
		N=31	N= 133	N=47	N=153

The results of the cross tabs analysis reveal that students who gave acceptable definitions or examples for function achieved higher levels of success in the recognition and conversion tasks in the diverse systems of representations of function, relative to the students who did not give correct definitions or examples. Moreover, a common phenomenon for all the groups of students irrespective of the correctness of their definitions or examples was the order of success in the recognition tasks with respect to their mode of representation. Most students succeeded in algebraic notation of functions, fewer in the verbal description of the tasks and the smallest percentages succeeded in the Cartesian graphs and the arrow diagrams. Nevertheless, a number of students exhibited inconsistent behavior in providing correct definitions or examples and using efficiently the various representations of function in the recognition and conversion tasks. For instance, a significant number of students, who did not give accurate definitions or examples, succeeded in most of the recognition and conversion tasks, while a number of students who gave acceptable definitions or examples were not in a position to handle different modes of representations in these tasks. This incoherent behavior constitutes an indication for the necessity of the consideration of the three dimensions of the triarchic model, that is D, E and R, for examining the understanding of function.

3.3 Results based on the similarity diagram

The similarity diagram shown in Figure 1 provides a general structure of students' responses at the tasks of the test. It can be observed that there is a connection between four small groups Gr1, Gr2, Gr3, Gr4 that comprise the bigger cluster A. From these groups, the "strongest" is Gr2 formed by the variables D1, X2 and D2 that present a considerably strong similarity (0,99999). That means that students who gave a correct (D1) or an approximately correct definition (D2) in Question 8, gave an example of a one-to-one function (X2) in Question 9. This group is completed with the answers in Question 2 (Q2), which concerns the conversion from a verbal representation of a piecewise function to the algebraic form. It is indicated that a non-transparent conversion of representations was accomplished mostly by the students who achieved a conceptual understanding of function. This strong group is linked to Gr3 which involves the variables Q6B, Q6D and Q6E, representing the correct recognition of some non-conventional cases of relations in the form of Cartesian graphs. Within the same group, these variables are associated with the answers to the four parts of Task 7 (Q7A, Q7B, Q7C, Q7D). Task 7 concerns the recognition of functions presented in the form of arrow diagrams.

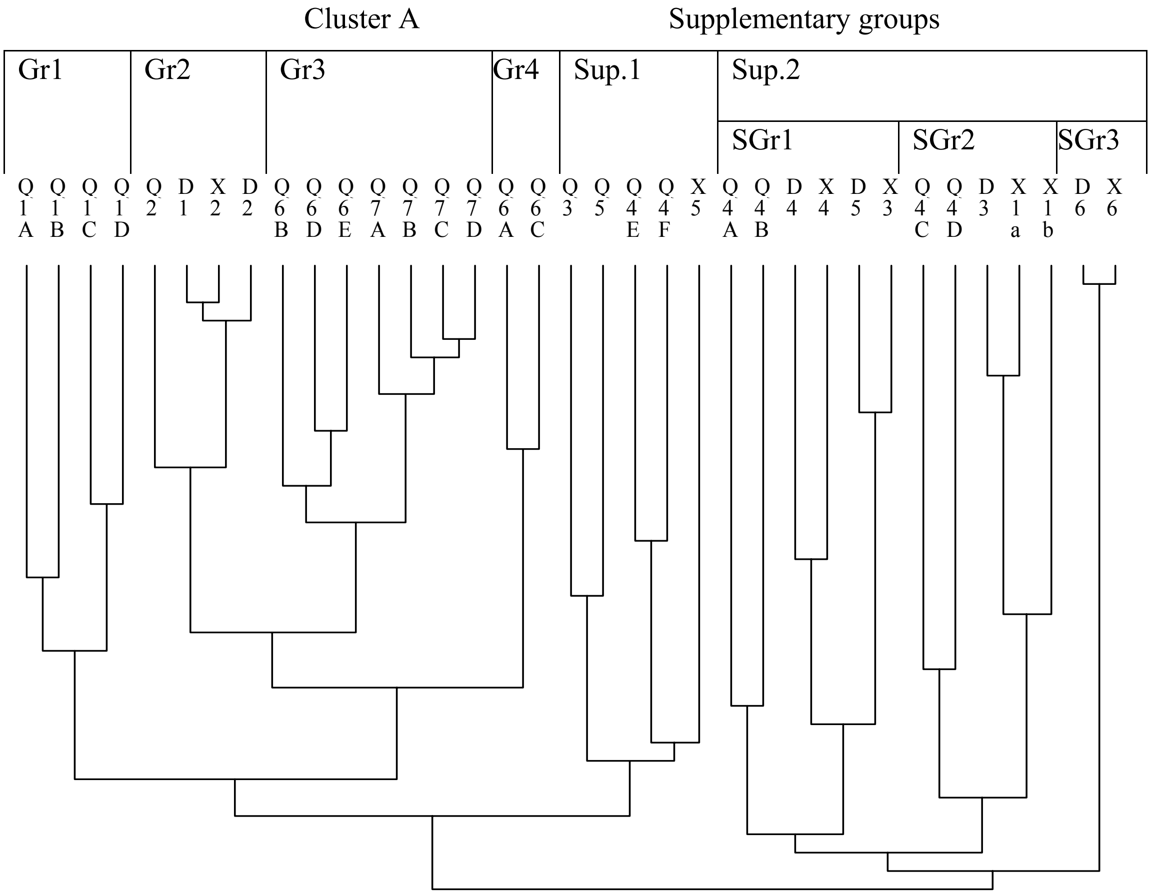


Figure 1: *Similarity diagram of students' responses to the tasks of the questionnaire*

The two groups Gr2 and Gr3 are connected with Gr4 that includes the answers to the other two parts of Question 6 (recognition of function given in graphical form). The groups Gr2-Gr3-Gr4 connect with Gr1 that includes the answers to Task 1; that is the recognition of functions represented in verbal form. Conclusively the connection of groups Gr1-Gr2-Gr3-Gr4 creates a cluster of students' responses, which entail a conceptual approach to function. Finally this whole cluster A (Gr1-Gr2-Gr3-Gr4) connects with two of the conversion tasks of the questionnaire (Q3 and Q5) and the responses to the tasks Q4E and Q4F requiring the recognition of functions in algebraic form. These are linked with the group that gave an example presenting an uncertain transformation of the world (X5). This is the first "supplement" (Sup.1) of cluster A. The second supplement (Sup.2) is embodied by three similarity groups, which are connected to each other. The first group of Sup.2 (SGr1) involves the definition and example variables D4-X4 and D5-X3, which illustrate a vagueness or limited idea of the definition and the examples of function. These variables connect with answers to questions Q4A and Q4B, which have a linear algebraic character. The second group of Sup.2 (SGr2) is formed by the variables D3, X1a and X1b. This means that students, who provided a definition of a special kind of function (D3), gave an example of function with the use of discrete elements of sets (X1a) or an example of a continuous function (X1b). These variables are connected with Q4c and Q4d, which are treated in a way that shows the conception that symbols “x” and “y” must always appear in the algebraic form of a function. The third group of Sup.2 (SGr3), which is the strongest one in the whole similarity diagram, is characterized by the most doubtful idea about the notion of function, since it includes D6 and X6 (i.e., those students that did not attempt to give any definition or example of function), and is not linked directly to the use of any representation of function.

Within the similarity diagram, one can also observe the formation of groups or subgroups of variables of students’ responses in recognition tasks involving the same mode of representation of functions, i.e., in verbal form (Gr1), in an arrow diagram or in graphical form (Gr3) and in algebraic form (supplementary groups). The particular observation reveals the consistency by which students dealt with tasks in the same representational format, but with different mathematical relations. However, lack of direct connections between variables of similar content but different representational format indicate that some students may be able to identify a function in a particular mode of representation (e.g., algebraic form), but not necessarily in another mode of representation (e.g., graph). This inconsistent behavior among different modes of representation is an indication of the existence of compartmentalization.

The structure of the connections established in the similarity diagram seems to offer support to the triarchic model, proposed here. Closely connected pairs or terns of definitions and examples which are generated due to their common accurate or inaccurate features, are associated with students’ responses in tasks involving particular types of representation. Different aspects of students’ image for the function concept (e.g., conceptual understanding or ambiguous ideas) are indicated by the formation of different similarity groups, each incorporating the distinct but interrelated factors of the triarchic model: D, E and R.

4. DISCUSSION

4.1 Students’ main ideas for the concept of function

Conclusively the results of the study have revealed some of the ideas that university students had about function. Such an idea is the identification of “function” by a large percentage of students with the narrow concept of one-to-one function. This finding is in accord with the results of previous studies indicating that one-valuedness is a dominating criterion that students use for deciding whether a given correspondence is a function or not (Vinner and Dreyfus, 1989). This idea is also associated with the process of enumeration, which involves one-to-one correspondence as a matter of routine for the students. Another idea was that function is an analytic relation between two variables (as it worked historically, initially with Bernoulli’s definition, and more clearly with Euler’s). A number of students have even stated this explicitly in their justifications when attempting to identify functions among other algebraic relations. Moreover, students’ dominating idea that a graph of a function must be connected or “continuous” caused difficulties in recognition and conversion tasks involving disconnectedness of a function’s graph.

4.2 Ability to use diverse representations of function

One of the main goals of the present study was to examine students’ performance in recognition and conversion tasks involving different modes of representation of function. Higher success rates were observed in the tasks which involved algebraic representations, relative to the tasks involving verbal and graphic representations (either Cartesian graphs or arrow diagrams). This finding can be attributed to the fact that mathematics instruction in schools focuses on the use of algebraic representations of functions, thus hindering the approach of function in other representational modes (e.g., Kaldrimidou and Iconomou, 1998).

In addition, students responded in tasks involving the same type of representation in a consistent and coherent manner. Nevertheless, they approached in a distinct way the different forms of representation of functions, providing support to the existence of the compartmentalization phenomenon (Gagatsis et al., 2003). Students probably considered the different systems of representation as different and autonomous mathematical objects and not as distinct means of representing the same concept (Duval, 1993). This was apparent also from students' failure in a conversion task of representations that was not transparent. Since a concept is not acquired when some components of mathematical thought are compartmentalized, teaching needs to accomplish the breach of compartmentalization, i.e., de-compartmentalization and coordination among different types of representations. One way to achieve this is by giving students the opportunity to engage in conversions of representation that can be congruent or not in different directions (Duval, 2002).

4.3 The connection of students' concept definitions and examples with the use of different representations of function

Findings showed that strong similarity connections exist between the definitions and the examples given by the students for function and their abilities to handle different modes of representation of the concept in recognition and conversion tasks. This indicates that concept definitions, examples and ability to handle different representations are not independent entities, but are interrelated in students' thought processes. The group of students, who accomplished a conceptual understanding of function involved strong connections with representations in the form of arrow diagrams, Cartesian graphs and verbal description, and had a higher level of success when dealing with most of the representations of the concept and a non transparent conversion. The group of students who had ambiguous or limited ideas for the function concept was exemplified by the answers of the students who kept coherently mostly the connection with the idea of linear function and seemed to be competent at handling more efficiently the algebraic form of representations than any other mode and the simple (term-by-term) conversions. Some students' incompetence in giving a definition and an example for function was not related to the use of any representation of the concept. These findings are in line with the view of a number of researchers that students' errors may be a result of deficient use of representations or a lack of coordination between representations (e.g., Greeno and Hall, 1997; Smith, diSessa and Rochelle, 1993).

The fact that using and representing functions in a diversity of representations are strongly related to the appropriate meaning of function and its applications has pedagogical implications. The understanding of function may be enhanced by designing didactic activities that are not restricted in certain types of representation, but involve recognition and transformation activities of the notion in various representations (Sierpinka, 1992; Duval, 2002; Even, 1998; Hitt, 1998). Furthermore, assessment tools of students' learning of function need to include tasks carried out in various semiotic representations. This study' s findings revealed that succeeding in transformation or recognition tasks in particular systems of representation was not indicative of students' understanding of function. For example, a significant percentage of students (from 28% to 60%), who gave an incorrect definition of function, were in a position to identify the concept in certain forms of representation (mainly the algebraic one).

The above example indicates that despite the close similarity relations between students' images of function and their ability to handle different representations of it, discrepancies between them were relatively frequent. Students' definitions or examples did not always have a predictive role in how students would apply the concept in various forms of representation. Hence, all three factors of mathematical thought examined in this study, D, E and R were found to describe in their own unique way different aspects of students' acquisition of the complex concept of function. It is not sufficient to make general inferences such as "students have an understanding of the concept of function" in the sense that they are reasonably successful in giving a definition of the concept or providing examples or even recognizing functions in different forms of representation, separately. The use of the triarchic conceptual-semiotic model of understanding of the function concept is, thus, validated. Adequate understanding of the concept may be indicated by approximately correct definition and examples, and flexibility in dealing with multiple representations in recognition and conversion tasks of function. Limited and ambiguous aspects of the function concept may be revealed by students' deficits in dealing with at least one of the three dimensions: D, E or R.

The above remarks have direct implications for teaching and assessment. One must remember that in order to teach functions to a group similar to the sample of this study, it is important to include the three different dimensions of studying function in his/her instruction and assessment: D, E and R. To employ effectively the triarchic model it is also important for the teachers to have in mind and make appropriate use of the connection among its components. By using the triarchic model in students' assessment, teachers can identify in which of the three domains students have difficulties as regards the understanding of function. On the basis of the assessment results, teaching must develop mathematical understanding in a way that it builds on students' constructed knowledge and abilities. In other words, strong emphasis should be given on the domains that are less familiar or known in some aspects and on their connection to the domains or aspects of a domain that students are more capable at. For example, students who are able to give an appropriate definition and examples of function applications, can be helped to elaborate their knowledge at first by using a familiar representation system and a diversity of other representations to represent their definition and examples; next, by recognizing whether a given mathematical relation in different systems of representation is a function or not in terms of their definition, by identifying the same types of function in various representations and carrying out a conversion of a function from one system of representation to another in different directions. These didactical implications are in line with Steinbring' s (1997) idea that mathematical meaning is developed in the interplay between a reference context and sign systems of the mathematical concept in question. Nevertheless, further research is needed to investigate at a practical level the effectiveness of such didactical processes for teaching the complex concept of function addressing prospective teachers.

4.4 Can we succeed de-compartmentalization? Implications of an on-going research

In an attempt to accomplish de-compartmentalization an experimental study was designed by Gagatsis, Spyrou, Evangelidou and Elia (2004) that constitutes the second stage of the research reported in the present paper. The researchers developed two experimental programs for teaching functions to university students, based on two different perspectives. The students who participated in the experimental study were divided into two groups. Each group received a different experimental program. Students of Experimental Group 1 were exposed to Experimental Program 1 and students of Experimental Group 2 received Experimental Program 2. Next, students of Experimental Group 1 were compared with students of Experimental Group 2. To compare the two groups two tests (a pre-test, before instruction and a post-test, after instruction) similar to each other and also similar to the test that was used in the present study, were designed to investigate students' understanding of functions.

The two experimental programs, conducted by two different university professors (Professors A and B), approached the teaching of the notion of function from two different perspectives. Experimental Program 1 started by providing a revision of some of the functions that were already known to the students from school mathematics, physics and economics. Different types of functions were presented next, starting from the simple ones and proceeding to the more complicated ones. The program ended by giving the set-theoretical definition of a function.

Experimental Program 2 encouraged the interplay between different modes of representation of a function in a systematic way. The instruction that was developed by Professor B on functions was based on two dimensions. The first dimension involved the intuitive approach and the definition of function. The second dimension emphasized the various representations of function, and the different conversions between them.

In the light of the above, an essential epistemological difference can be identified between the two experimental programs: Experimental Program 1 involved an instruction of a classic nature and widely used at the university level. On the contrary Experimental Program 2 was based on a continuous interplay between different representations of various functions.

The preliminary results of the new study provided evidence for the appearance of the phenomenon of compartmentalization in the similarity diagrams of the answers of the students of Experimental Group 1, before and after instruction, especially in using the graphical representations and arrow diagrams. On the contrary, the compartmentalization that was evident in the similarity diagram involving the responses of students of Experimental Group 2 before instruction disappeared in the corresponding similarity diagram after instruction. Similarity connections indicated students' consistency in recognizing functions in different modes of representation. In other words, success was independent from the mode of representation of the mathematical relation. This finding revealed that Experimental Program 2 was successful in developing students' abilities to use flexibly various modes of representation of functions and thus accomplished the breach of compartmentalization in their performance. The research towards the direction, described briefly above, continues so as to provide explanations for the success of Experimental Program 2 and to determine those features of the intervention that were particularly effective in accomplishing de-compartmentalization. The results of such an attempt may help educators at a university level to place stronger emphasis on certain dimensions of the notion of function and techniques of teaching functions, so that students can be helped to construct a solid and deeper understanding of the particular construct.

Appendix: The tasks of the questionnaire

1. Explain whether we define a function when we:

(a) correspond a girl with different friends of hers (George, Homer, Jason, Thanasis, etc.) with whom she will probably dance at a party.
(b) correspond every football game to the score achieved.
(c) at the university entrance examinations correspond every script to the couple of marks given by the first and the second examiner.
(d) correspond every candidate with the posts for which she applies for work in an organisation (candidates may apply for more than one post).

2. At the entrance examinations there are two types of candidates: successful and unsuccessful. Let A stand for the set of successful candidates and B stand for the unsuccessful candidates. Using symbols 1 and 0, construct a function, which describes this situation, and give the algebraic form.
3. Find the algebraic formula of the function that converts the initial prices of a shop that makes sales 20% in every item, to the new prices that emerge.
4. Examine whether the following symbolic expressions may define functions and justify your answer. For the expressions that define a function, indicate the symbol, which you consider as the independent variable.

(a) $5x+3=0$ Yes / No, Explanation:

(b) $2x+y=0$ Yes / No, Explanation:

(c) $4y+1=0$ Yes / No, Explanation:

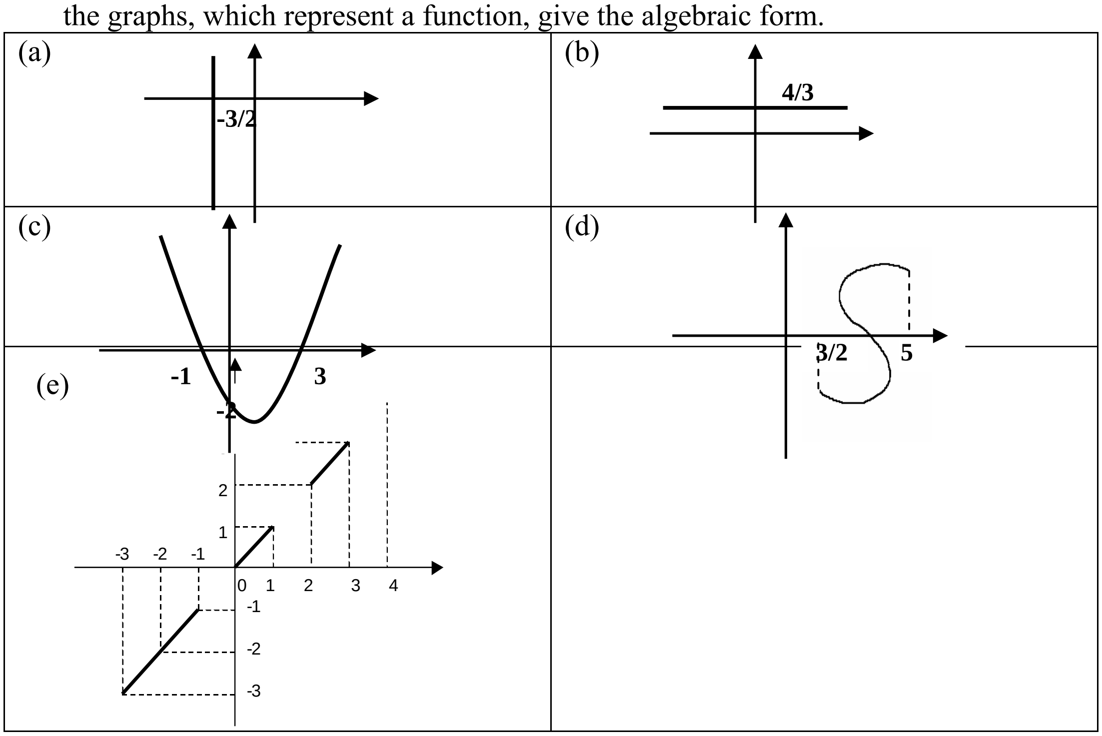
(d) $x^2+y^2=25$ Yes / No, Explanation:

(e) $x^3-y=0$ Yes / No, Explanation:

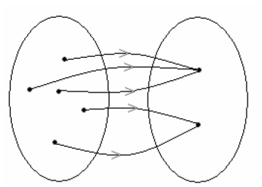
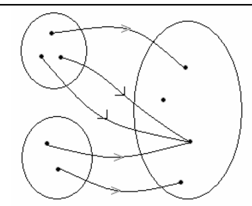
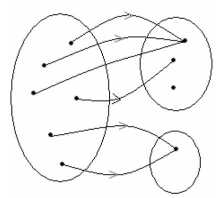
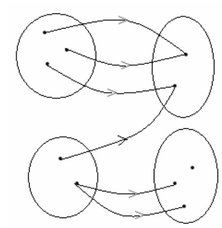
(f) $f(x) \begin{cases} x, & x\geq 0 \\ -x, & x<0, \end{cases}$ Yes / No, Explanation:

5. Draw the graph for one of the expressions of question 4, which you consider as a function.

6. Examine whether the following graphs represent a function and justify your answer. For the graphs, which represent a function, give the algebraic form.



7. Examine which of the following correspondences presented in the form of Venn diagrams are functions. Justify your answer.

(a) 	(b) 
(c) 	(d) 

8. According to you what is a function?
9. Give two simple examples from the applications of functions in everyday life.

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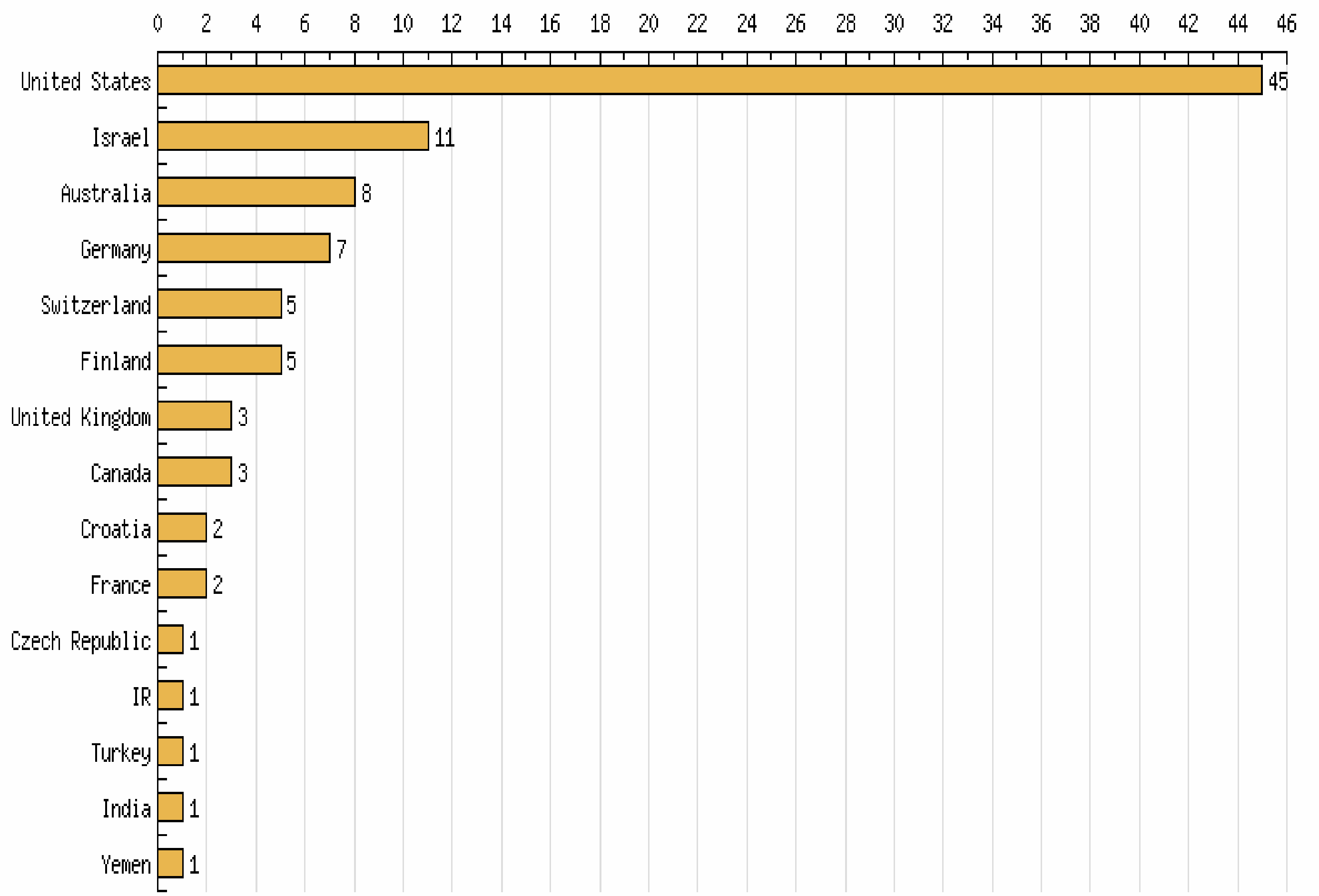

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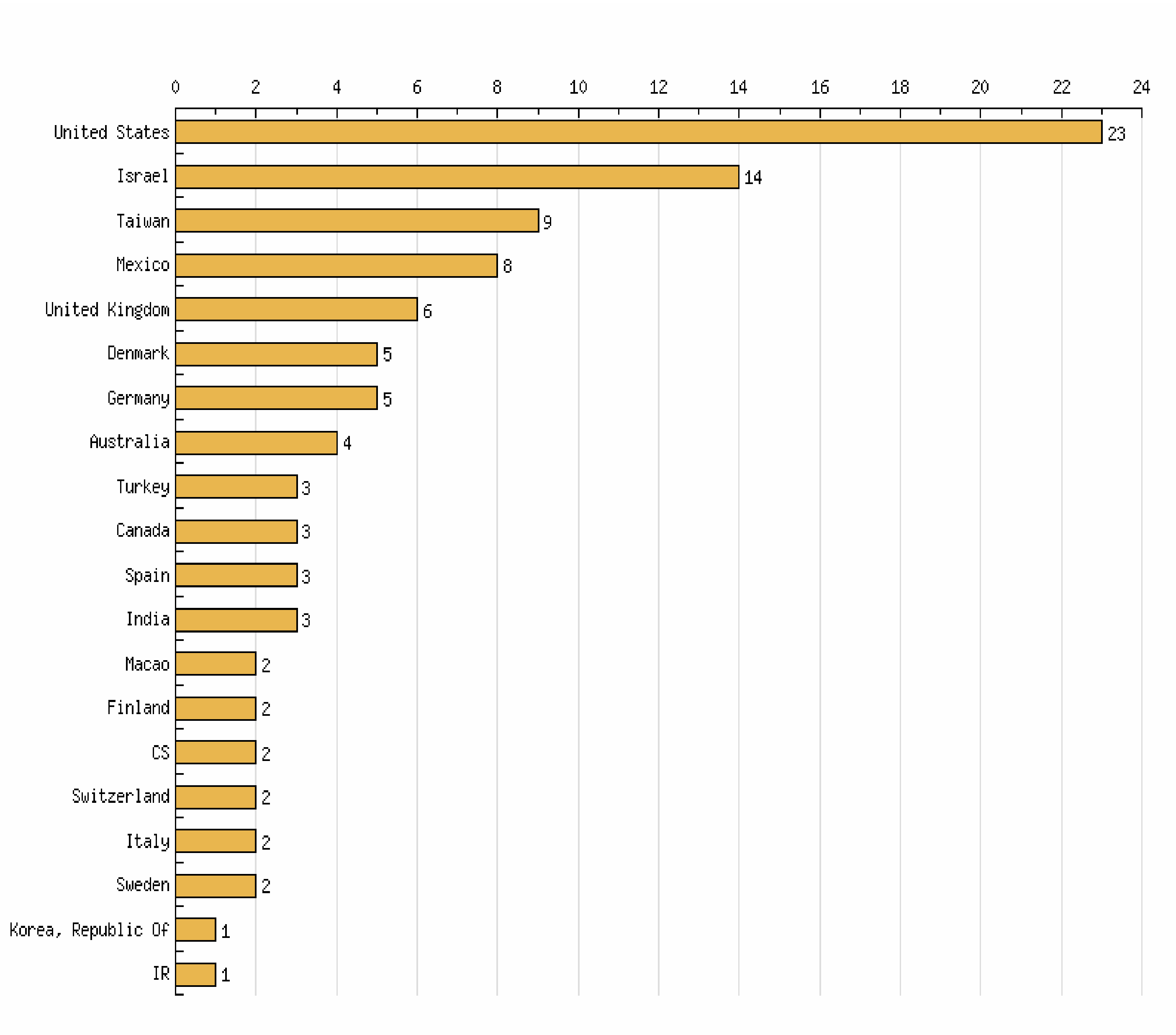

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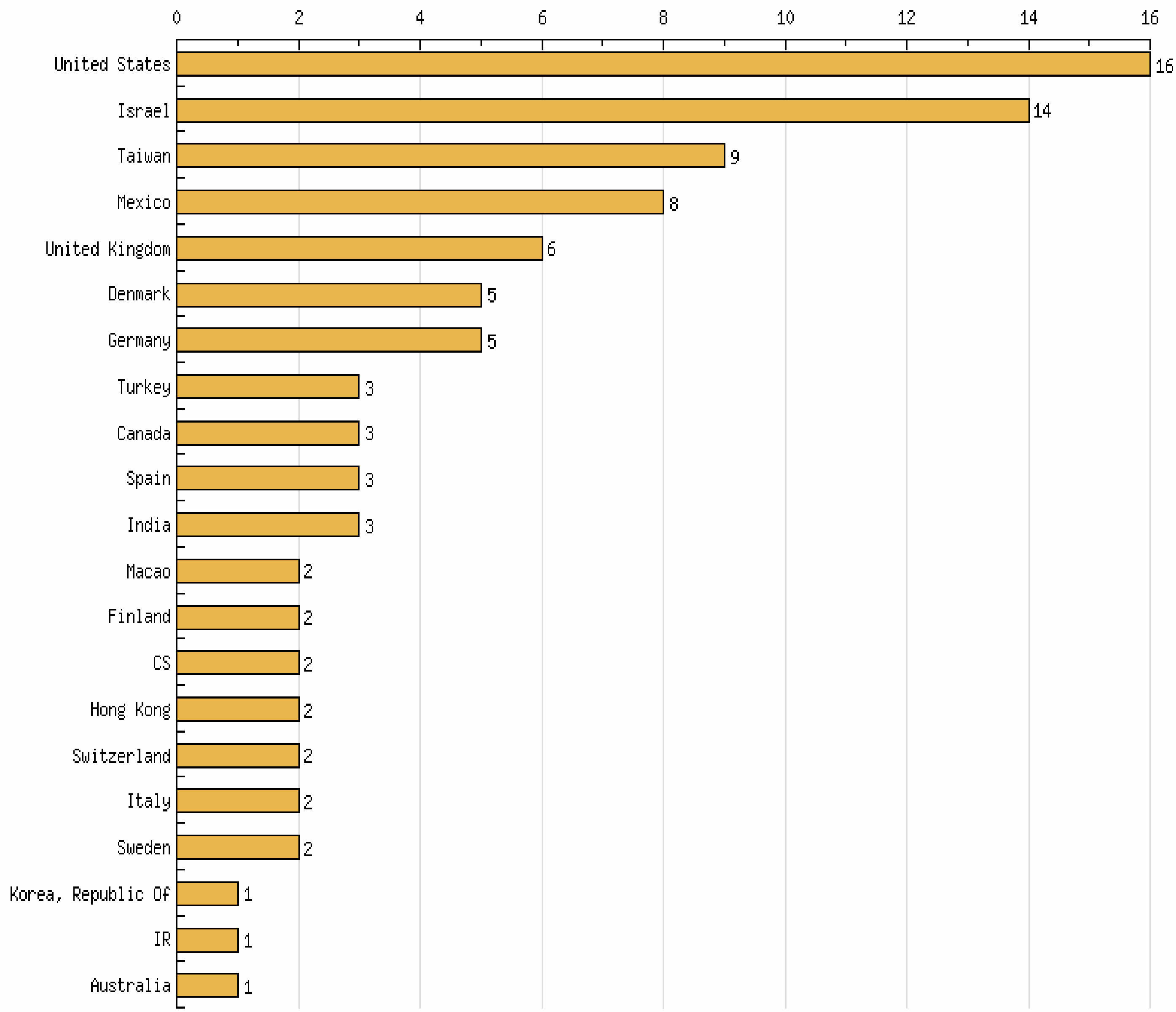

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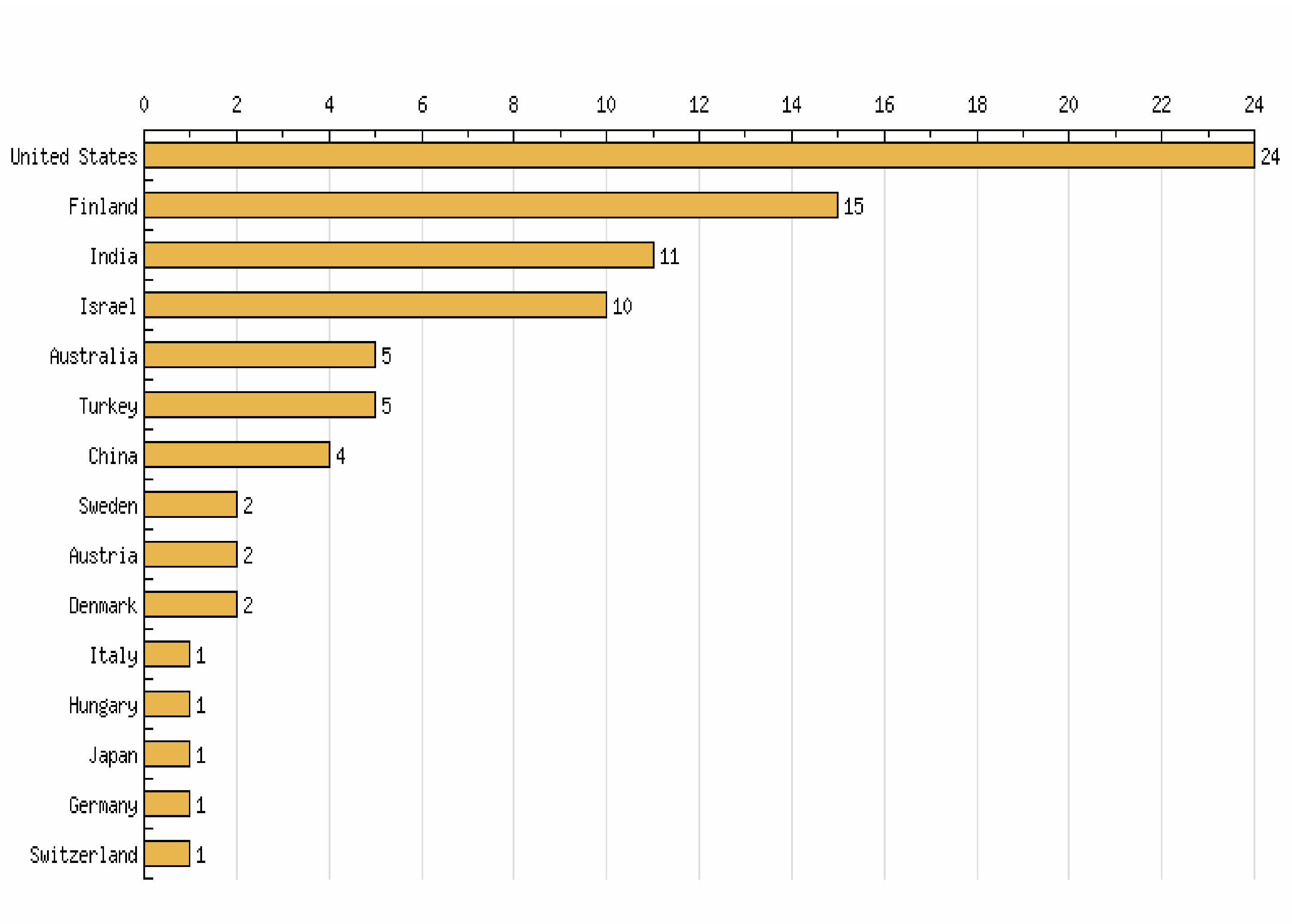

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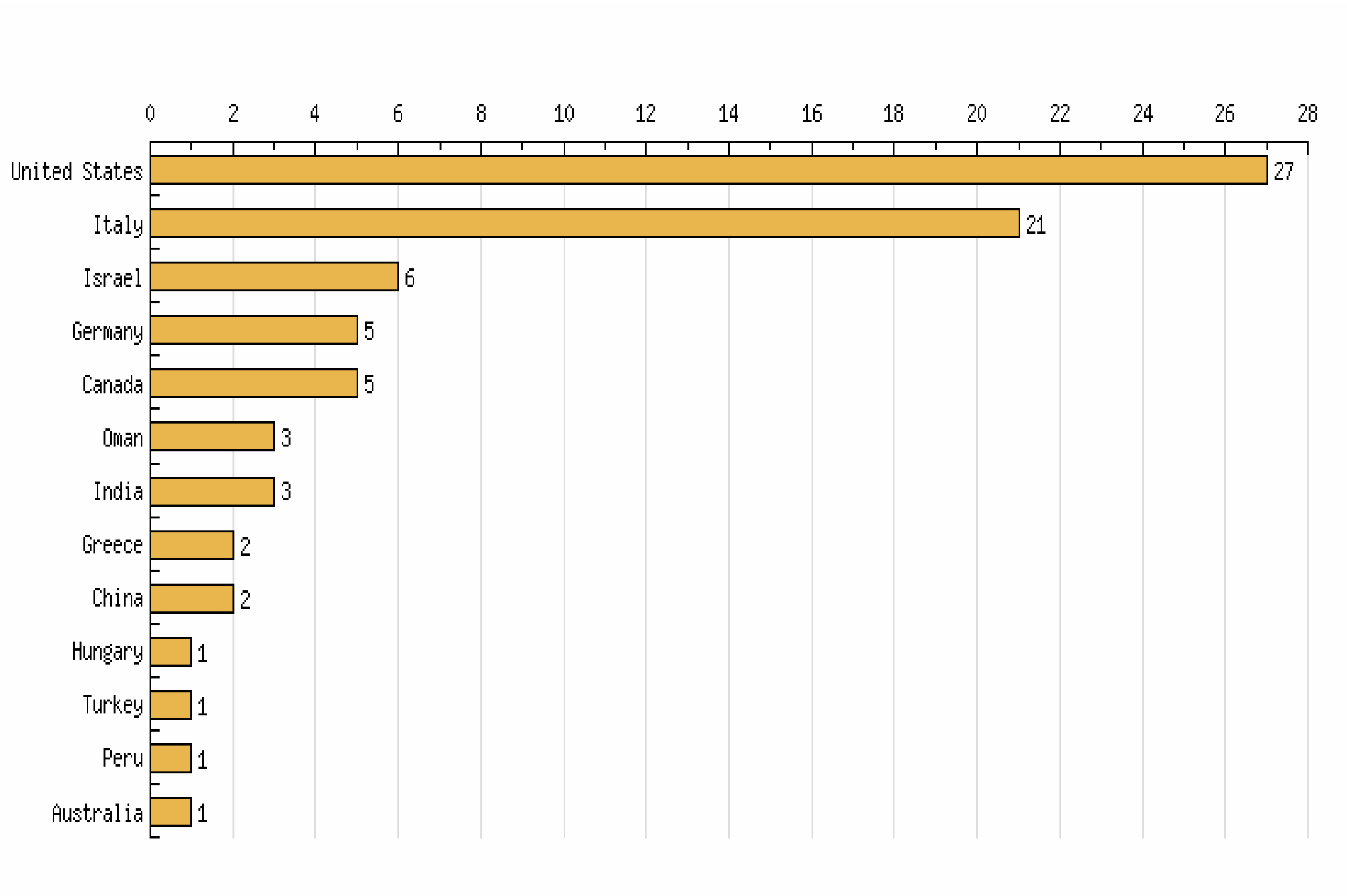

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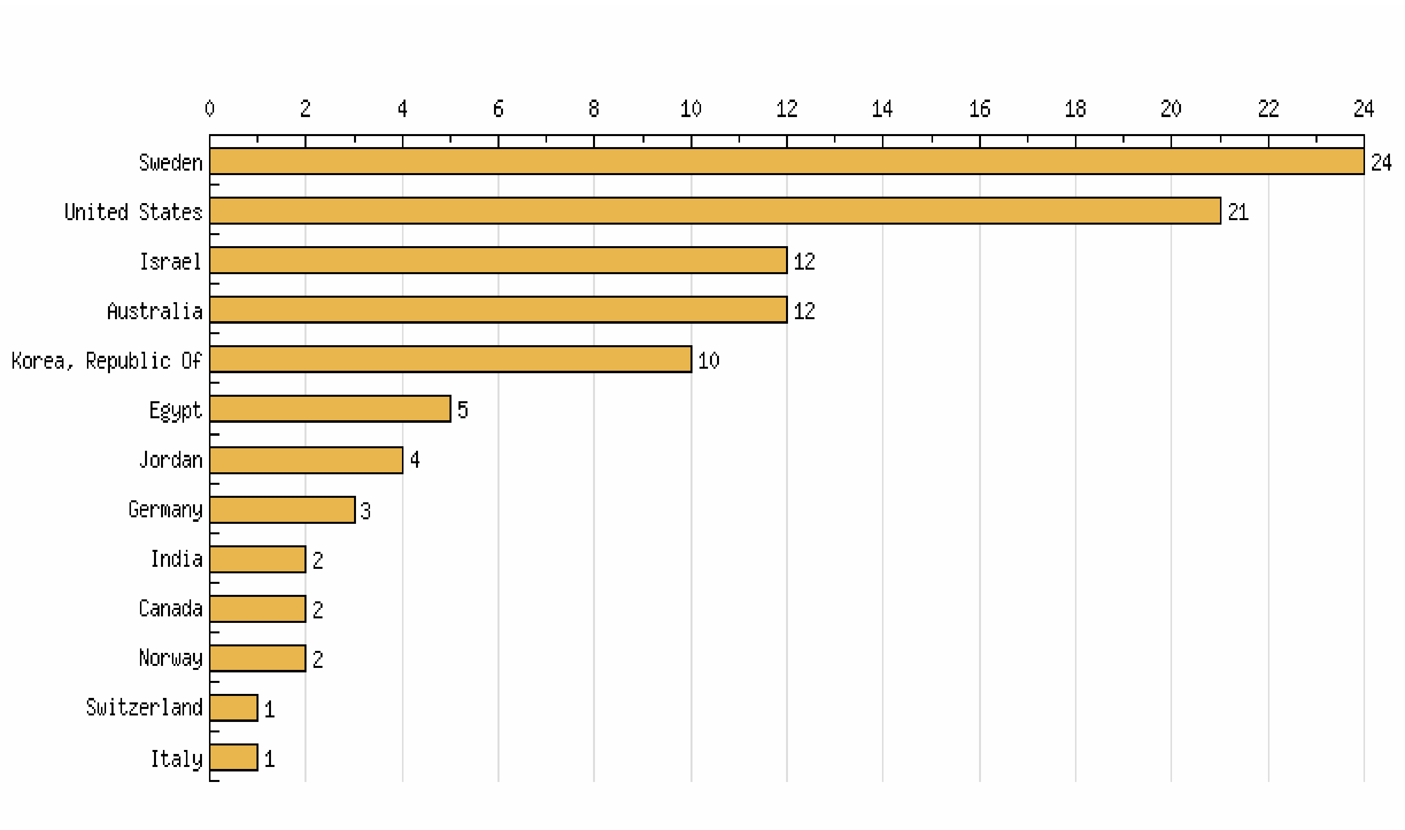

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