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Iliada Elia

Panayotis Spyrou

HOW STUDENTS CONCEIVE FUNCTION: A TRIARCHIC CONCEPTUAL-SEMIOTIC MODEL OF THE UNDERSTANDING OF A COMPLEX CONCEPT

Iliada Elia & **Panayotis Spyrou**

Department of Education **Department of Mathematics**

University of Cyprus (Cyprus) **University of Athens (Greece)**

Abstract: This study explores conceptions for function amongst 164-second year students of the Department of Education at the University of Cyprus and their relationship with students' abilities in dealing with tasks involving different modes of representations of function. The test that was administered to the students included recognition tasks given in various representations and questions requesting definition and examples of function's applications in real life situations. Results have shown that students' definitions and examples of the notion are closely related to their ability to use different modes of representations of function. These three factors, i.e., definitions given by the students, functions considered by them as examples of application in real life situations, and different representations of functions, seemed to contribute in their own unique way to students' acquisition of this complex concept. Thus, support was provided for the use of a triarchic conceptual-semiotic model of the concept of function, which enables students' thinking and understanding of the notion to be analyzed and described across these three features.

Key words: function, representations, compartmentalization, concept definition, concept image, triarchic conceptual-semiotic model, similarity diagram, implicative method

1. INTRODUCTION

The concept of function is of fundamental importance in the learning of mathematics and has been a major focus of attention for the mathematics education research community over the past decades (e.g., Evangelidou, Spyrou, Elia and Gagatsis, 2004; Sfard, 1992; Sierpinska, 1992; Vinner and Dreyfus, 1989). Research related to functions has been directed towards various domains. We will focus on two strands of research that have a bearing on this study in order to clarify the basic goal of it. The first research domain refers to the concept image of function in the students' minds (e.g., Vinner and Dreyfus, 1989) and the second one concerns the different representations of the notion and the passage from one to another (e.g., Duval, 2002; Hitt, 1998).

This paper is an attempt to examine the relationship between students' concept definitions, examples of function and their ability to use and connect different representations of the notion, on the basis of two theoretical semiotic perspectives (Duval, 2002; Steinbring, 1997) having a central focus of attention on students' construction of meaning and understanding of mathematical concepts. This relationship is incorporated in a new triarchic conceptual-semiotic model which integrates three fundamental components of the understanding of the function concept: defining the concept; giving examples of the application of the concept in everyday life; identifying functions in different modes of representation and changing systems of representation.

1.1 Understanding of the concept of function

1.1.1 Concept image and concept definitions

Concept image and concept definitions are two terms that have been discussed extensively in the literature concerning students' conceptions of function (Vinner and Dreyfus, 1989; Tall and Vinner, 1981). Although formal definitions of mathematical concepts are introduced to high school or college students, students do not essentially use them when asked to identify or construct a mathematical object concerning or not this concept. They are frequently based on a concept image which refers to "the set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them" (Vinner and Dreyfus, 1989, p. 356). Thus, on the basis of a model of cognitive processes concerning the relation between the definition of the concept and concept image, different categories of students' definitions and concept images were identified in the study of Vinner and Dreyfus (1989).

1.1.2 Representations and the understanding of function

The understanding of functions does not appear to be easy, given the diversity of representations related to this concept (Hitt, 1998). Sierpinska (1992) indicated that students have difficulties in making the connections between different representations of the notion (formulas, graphs, diagrams, and word descriptions), in interpreting graphs and manipulating symbols related to functions. Some students' difficulties in the construction of concepts are linked to the restriction of representations when teaching. Mathematics instructors, at the secondary level, traditionally have focused their instruction on the use of algebraic representations of functions rather than the approach of them from the graphical point of view (Eisenberg and Dreyfus, 1991; Kaldrimidou and Iconomou, 1998). Markovits, Eylon and Bruckheimer (1986) observed that translation from graphical to algebraic form was more difficult than the reverse conversion and that the examples given by the students were limited in the graphical and algebraic form.

The findings of the above studies are related to the phenomenon of compartmentalization. The existence of compartmentalization reveals a cognitive difficulty that arises from the need to accomplish flexible and competent conversion back and forth between different kinds of mathematical representations of the same situation (Duval, 2002), which according to Arcavi (2003) is at the core of mathematical understanding. Gagatsis, Elia and Andreou (2003) found that 14-year-old students were not in a position to change systems of representation of the same mathematical content of functions in a coherent way, indicating that systems of representations remained compartmentalized and mathematical thinking was fragmentary.

1.2 Representations and mathematics learning: Two semiotic theories

The theoretical position that we are taking in our research is based on two semiotic perspectives. These also serve as a basis of the triarchic conceptual-semiotic model that we propose. The first basic idea we adopt in our framework deals with the importance of the diversity of semiotic representations and their transformation for the development of mathematical thought. According to Duval (1993, 2002) mathematical activity can be analyzed into two types of transformations of semiotic representations: treatments and conversions. Treatments are transformations of representations which take place within the same system where they have been formed. Conversions are transformations of representation that consist of changing a system of representation, without changing the objects being denoted. The conversion of representations is considered as a fundamental process for mathematical understanding (Duval, 2002, 2005).

We also adopt Steinbring's (1997) idea that the meaning of a mathematical concept occurs in the interaction between sign/symbol systems and reference contexts or object domains. The triarchic conceptual-semiotic model of the understanding of function that is introduced in this study is constituted by the reference contexts and the signs of the notion. In particular, students' constructed definitions and examples of function correspond to the "reference contexts" that may change during the process of mathematical knowledge development. The systems of representation, such as graphs, symbolic expressions, arrow diagrams and verbal descriptions are considered as the "symbol systems" that are used for denoting and implementing the referential objects. Steinbring maintains that "the difference between the function of a symbol system and a structural reference context is essential for the generation of meaning in every new mathematical relationship" (1997, p. 78). Sierpinska (1992) considers the distinction of a function from the analytic tools used to describe its law as one of the essential conditions for understanding functions. Therefore, in this study students' constructions of definition and examples for the concept of function are distinguished from the transformation of representations.

As presented above, numerous studies have examined the role of representations on the understanding of function and students' concept image for it, separately. Taking into account Steinbring' s (1997) idea that the meaning of a mathematical concept occurs in the interaction between sign/symbol systems and reference contexts, we need to add to the mathematics education research community understanding of the way these two dimensions are interrelated as regards the concept of function.

In this paper we attempt to contribute to mathematics education research understanding with respect to the concept of function by investigating the relationship among the three aforementioned components that are constitutive of the meaning of function, i.e, D, E and R, and by interweaving them in a triarchic conceptual-semiotic model. "D" corresponds to the common definitions of the function concept given by a student; "E" signifies the set of mathematical or non-mathematical objects or relations considered by the student to be examples of the concept of function; and "R" designates the range of different representations of functions that the student deals with (R). We anticipate that this model will provide a coherent picture of students' construction of the meaning of function that is desirable for current approaches of instruction which aim at the development of the understanding of this concept. The potential power of the triarchic model will be verified by a statistical tool, namely, CHIC (Bodin, Coutourier and Gras, 2000), that has not been used previously in similar investigations.

In more specific terms the purpose of this paper is the following: First, to explore university students' conceptions of function on the basis of their concept definitions and examples of the notion; second, to examine students' performance to recognize functions in different forms of representation and transfer from one representation to another; third, to explore the relationship between their conceptions of function (D and E) and their ability to use different representations of the concept (R).

2. METHOD

2.1 Participants

The sample of the study consisted of 164 students who attended the course "Contemporary Mathematics" at the University of Cyprus. The questionnaire was completed by 154 second year students of the Department of Education and 10 four year students of the Department of

Mathematics and Statistics. The students come from diverse high school directions, which differ in the level and length of the mathematics courses that they involve. Nevertheless, all of the students who participated in this study had received a teaching on functions during the last three grades of high school. The content of this teaching is based mainly on a classical presentation of function: domain and range, derivatives, maximum and minimum and construction of graphs of first-, second-, third- and fourth-degree polynomial functions. It is noteworthy that the sample consists of future primary and secondary school teachers, who will in a way transfer, their mathematical thinking to their prospect students. The concept of function is not included in the curriculum of primary mathematics education in Cyprus, but other mathematical relations such as proportion or bijective types of correspondence are within the content areas that teachers are required to teach, similarly to the educational systems of other European countries. As for the secondary education, the concept of function is one of the basic topics that are included in the content of the mathematics curriculum in Cyprus and focuses on the "classical" topics of function, mentioned above.

2.2 Research instrument

A questionnaire (see Appendix) was administered a few weeks after the beginning of the course. It consisted of ten tasks, which were developed on the basis of the two types of transformation of semiotic representations proposed by Duval (2002): treatment and conversion. Yet, the tasks we developed differed from Duval's proposed activities in two ways: First, they included recognition whether mathematical relations in different modes of representation (verbal expressions, graphs, arrow diagrams and algebraic expressions) were functions or not, by applying the definition of the concept. Nevertheless, a general use of the processes of treatment and conversion was required for the solution of these tasks. Secondly, they involved conversions, which were employed either as complex coding activities or as point-to-point translations and were designed to correspond to school mathematics. For instance, a conversion could be accomplished by carrying out various kinds of treatment, such as calculations in the same notation system.

A variety of functions were used for the tasks of the questionnaire: linear, quadratic, discontinuous, piecewise and constant functions. Below we give a brief description of the questionnaire and the corresponding symbolization for the variables used for the analysis of the data: Question 1 (Q1A, Q1B, Q1C, Q1D), Question 4 (Q4A, Q4B, Q4C, Q4D, Q4E, Q4F), Question 6 (Q6A, Q6B, Q6C, Q6D, Q6E) and Question 7 (Q7A, Q7B, Q7C, Q7D) asked students to recognize functions in different modes of representation, i.e., verbal, algebraic, graphical and arrow diagrams, respectively, and to provide an explanation for their answer. Questions 2 (Q2), 3 (Q3) and 5 (Q5) required a conversion of a function from one representation to another. Question 8 (Q8) asked what a function is and Question 9 (Q9) requested two examples of functions from their application in real life situations.

2.3 Data Codification

Students' responses for the definition of function and its applications at the corresponding questions were grouped into particular categories to explore the relation of the different values of the former two dimensions of the triarchic model, i.e., D and E, to the latter one, i.e., R. Definitions in Question 8 were coded as follow:

D1: *Correct definition*. This group included the accurate set-theoretical definition.

D2: *An approximately correct definition.* This group involved answers with a correct reference to the relation between variables, but without defining the domain and range.

D3: *Definition of a special kind of function.* This group of answers made reference to a particular type of function (e.g., real, bijective, injective or continuous function).

D4: *Reference to an ambiguous relation*. Answers that made reference to a relation between variables or elements of sets, or a verbal or symbolic example were included in this group.

D5: *Other answers*. This type of answers made reference to sets, but no reference to a relation, or reference to relation without reference to sets or elements of sets.

D6: *No answer*.

The following additional codes were given for the types of examples provided in Question 9: X1a: Example of a function with the use of discrete elements of sets; X1b: Example of a continuous function, usually, from physics; X2: Example of a one-to-one function; X3: Example presenting an ambiguous relation between elements of sets; X4: Example of an equation in verbal or symbolic form; X5: Example presenting an uncertain transformation of the real world; and X6: No example.

2.4 Data Analysis

2.4.α Qualitative Analysis.

The first part of the qualitative analysis is based on the explanations provided by the students when justifying their decision whether a relation represents a function or not. Next we present some indicative examples of the types of responses the students gave while trying to define and give examples of function.

2.4.b Quantitative Analysis

Primarily, the success percentages were accounted for the tasks of the test by using SPSS. A similarity diagram (Lerman, 1981) of students' responses at each item of the questionnaire was also constructed by using the statistical computer software CHIC (Classification Hiérarchique, Implicative et Cohésitive) (Bodin et al., 2000). The similarity diagram allows for the arrangement of the tasks into groups according to the homogeneity by which they were handled by the students. A similarity index is used to indicate the degree to which the variables of a group are similar to each other on the basis of students' answers. This aggregation may be indebted to the conceptual character of every group of variables. Unlike the range of the linear correlation coefficient (from -1 to +1), the similarity index is ranging from 0 to 1. As the similarity of a group gets stronger, the index gets closer to the value of 1. The similarity index corresponds to the length of the vertical segments that form each pair or group of variables. As these vertical segments get shorter, the similarity index approaches the value of 1. This means that the stronger the similarity relations (pairs or groups of variables), the shorter are their vertical segments.

It is worth noting that CHIC has been widely used for the processing of the data of several studies in the field of mathematics education in the last few years (e.g., Evangelidou et al., 2004; Gagatsis, Shiakalli and Panaoura, 2003; Gras and Totohasina, 1995).

3. RESULTS

3.1 Some indicative answers

An idea that was extensively observed among the students was that *a function must essentially contain two variables or unknowns* or that *the algebraic or graphical expression of a function must at any rate contain x and y*. The answers that the expressions (Q4A) 5x+3=0 and (Q4C) 4y+1=0 cannot define functions, were justified with "*x (or y) do not appear in the expression, therefore a function cannot be defined*". Moreover, the relation

(Q4D) $x^2+y^2=25$ was considered a function, since it included x and y. The same idea was apparent for the question requesting whether some Cartesian graphs have resulted from a function. Those graphs representing a straight line, parallel to the x- or the y- axis where not accounted as functions because "*x (or y) is constant and therefore it is not an unknown and a function must contain two unknowns*". Similarly some other students appeared to believe that *a function is an equation* and rejected (Q4A) and (Q4C) by explicitly saying that "*the expression … does not represent an equation, and therefore it cannot be a function*".

Another idea held by the students was that *a function is necessarily a bijective correspondence*. This was noticeable in the explanation given in (Q1D) asking whether the correspondence between every football game and the score achieved defines a function. Negative answers were justified with the fact that "*two football games may have the same score*". Also some students, while trying to explain their wrong decision that the algebraic expression in (Q4F) $f(x)=x$ for $x\geq 0$ and $f(x)=-x$ for $x<0$ does not represent a function, stated that "*two different values of x correspond to the same value of f(x) and therefore the expression is not a function*". Some other students used the same reasoning to reject the graph of the parabola in (Q6C).

According to some other students, *the variables should not come from a specific set, but should take random values*. This was expressed by those students who considered that the correspondences described verbally in Q1A (we correspond a girl to different friends of her with whom she dances at a party) and Q1D (we correspond every candidate with the post for which he applies for work in an organization) cannot define functions because "*the girl can only dance with a limited number of boys who attend the party*" and "*the candidates do not have a random a choice of jobs*".

The students were also very much distracted by the arrow diagrams, which, were presented in incompact frames, thus expressing the idea that *in a graph of a function domain and range should be compact sets*. Negative answers for the diagrams presented in Q8B and Q8C were that "*they do not represent functions because the correspondence starts or ends from a different set".*

A likewise idea was that *a graph of a function should be continuous* as the graph of a y=x, with domain the union of the intervals (-3,-1), (0,1), (2,3) was not considered as function with the same frequency as the other linear forms. Students justified their choice stating explicitly that "*the graph is not continuous, and therefore, cannot represent a function*".

In the question requiring the definition of function (Q9) the answers that gave an approximately correct definition (D2) were grouped together. Answers like "*Function is a relation between two variables so that one value of x (or the independent variable) corresponds to one value of y (or the dependent variable)*" were accounted in this group. Answers that referred to the accurate definition, but added some more conditions to it and as an outcome would give the definition of a specific type of function (like injective, bijective or real function), were coded as D3. Ambiguous answers like "*Function is an equation with two dependent variables*", "*Function is a relation in which an element x is linked with another element y*" or even "*Function is a mathematical relation connecting two quantities*" were coded as D4. As D5 we have coded answers, which made reference to sets, but did not mention relation, or made reference to relation but not to sets or elements of sets, such as "*Function is a relation*" or "*Function is a mathematical concept that is influenced by two variables*" or "*Function is the identification of parts of a set*".

In the question requiring examples of functions from their applications in real life (Q10) the variety of responses was even greater. The correct examples of a function were of two kinds

(X1a and X1b). Examples of the first kind (X1a), which made use of sets with discrete elements, were: "*Each person corresponds to the size of his shoes*", "*Each student corresponds to his/her mark at the test*". As (X1b) type of examples we have grouped examples of a continuous-linear function mainly from physics, such as "*The height of trees is a function of time*", "*Atmospheric pressure is a function of altitude*". The examples presenting a bijective function were coded separately as X2. Such answers were "*Every citizen has his own identity number*", "*Every graduate has his own different degree*" and "*Every country corresponds to its own unique name*". As X3 we coded the examples presenting a relation between elements or variables but without clarification of the uniqueness in function. Such answers were as follows: "*There is a relation between students and their books*", "*The prices of vegetables depend on the production*", "*We correspond the marks of girls in a classroom to those of boys*". Examples presenting an equation instead of a function were coded as X4: "*There are 2x boys and 3y girls in a classroom and all the children are 60. If the boys are 15 we can calculate the number of girls*", "*Kostas has x number of toffees and Giannis has double that number. How many toffees do the two friends have?*". The last category X5 included answers which were ambiguous, but furthermore did not define any variables or sets, just an uncertain transformation of the real world. Such answers were "*Health depends from smoking*", "*Success in a test depends on the hours of studying*", "*In the relation of children and parents, the children are the dependent variable and parents the independent variable*".

3.2 Success percentages

In this section we will only refer to the results that show the strongest trends among the students. Higher success scores (91%) were achieved in Question 4B (Q4B), which presented the algebraic form of a linear function (a well known figure from high school mathematics), while lower success rates (8%) were attained in one of the conversion problems (Q2). Only thirteen of the students succeeded in constructing the algebraic formula of the characteristic function of a set that was given verbally (Q2), probably because the change of system of representation was not a simple coding activity or transparent conversion (Duval, 2002), and required a global interpretation guided by the understanding of the qualitative variables and their relation. A large percentage (70%) of the students did not consider that the graph of a y=x, with domain the union of the intervals (-3,-1), (0,1), (2,3) in Question 6E was a function. Most of them justified their choice stating explicitly that "*the graph is not continuous, and therefore, cannot represent a function*". This kind of behaviour reveals students' idea that a graph of a function must be connected or "continuous". The majority of the students (62%) answered correctly to another conversion (verbal–algebraic) problem (Q3), which involved a function changing the initial prices to the sales prices of a shop, probably because this is a real life problem, concerns a linear function and involves a term-by-term conversion.

Low success percentages were observed in Q4C (26%) and Q4D (37%), as students did not think that the algebraic form in (Q4C) 4y+1=0 represented a function, while they considered that the formula of the circle in (Q4D) $x^2+y^2=25$, did. As deduced from their explanations, these responses were a consequence of the idea that a function must essentially contain two variables or unknowns. The same was true for the graphs in Question 6. For instance, students did not think that the graph of the straight-line, parallel to the horizontal axis (y=0), i.e., y=4/3 (Q6B) could have resulted from a function. Difficulties caused by the constant function were identified also by Markovits et al. (1986) among students of 14-15 years of age. Table 1 presents the success percentages in the last two questions requesting the definition of the function concept and examples of function from real life.

Table 1: *Percentages for the definition and example categories*

(Q9) Definition of function	Frequency N=164	Percentage %
D1: Correct definition	13	8
D2: An approximately correct definition	13	8
D3: Definition of a special kind of function	5	3
D4: Reference to an ambiguous relation	73	45
D5: Other answers	24	15
D6: No answer	36	22
Total	164	100%
(Q10) Example		
X1a: A function (using discrete elements of sets)	11	7
X1b: A continuous function usually from physics	8	5
X2: A one-to-one function	28	17
X3: An ambiguous relation between elements of sets	29	18
X4: An equation	8	5
X5: An uncertain transformation of the real world	27	16
X6: No example	53	32
Total	164	100%

It is apparent that the majority of the students (45%) did not give a correct definition, but made reference to an ambiguous relation between variables without establishing the uniqueness. In addition, 29% of the students gave a correct example of function with the majority of them 17% referring to a one-to-one function (X2). The largest percentage of the students (32%) could not find any example of function.

Table 2 presents the results of the cross tabs analysis, which was used to investigate students' achievement in each representational type of tasks and each conversion task in relation to their ideas for the definition and examples of function. The aforementioned categories for the given definitions and examples of function were grouped in such a way, so that the percentages refer to the students who gave acceptable definitions (D1-D3) or incorrect definitions (D2-D6), and acceptable examples (X1a, 1b, X2) or inappropriate examples (X3-X6).

Table 2: *Percentages of students from each definition and example group of categories who responded successfully to the tasks*

	Type of task	D1-D3 %	D2-D6 %	X1a, 1b, X2 %	X3-X6 %
Recognition tasks	Verbal expressions	68	47	64	45
	Algebraic expressions	77	56	70	56
	Cartesian Graphs	65	36	53	38
	Arrow diagrams	61	36	51	37
Conversion tasks	Problem 2	26	4	17	4
	Problem 3	84	56	81	54
	Problem 5	84	50	70	51
		N=31	N= 133	N=47	N=153

The results of the cross tabs analysis reveal that students who gave acceptable definitions or examples for function achieved higher levels of success in the recognition and conversion tasks in the diverse systems of representations of function, relative to the students who did not give correct definitions or examples. Moreover, a common phenomenon for all the groups of students irrespective of the correctness of their definitions or examples was the order of success in the recognition tasks with respect to their mode of representation. Most students succeeded in algebraic notation of functions, fewer in the verbal description of the tasks and the smallest percentages succeeded in the Cartesian graphs and the arrow diagrams. Nevertheless, a number of students exhibited inconsistent behavior in providing correct definitions or examples and using efficiently the various representations of function in the recognition and conversion tasks. For instance, a significant number of students, who did not give accurate definitions or examples, succeeded in most of the recognition and conversion tasks, while a number of students who gave acceptable definitions or examples were not in a position to handle different modes of representations in these tasks. This incoherent behavior constitutes an indication for the necessity of the consideration of the three dimensions of the triarchic model, that is D, E and R, for examining the understanding of function.

3.3 Results based on the similarity diagram

The similarity diagram shown in Figure 1 provides a general structure of students' responses at the tasks of the test. It can be observed that there is a connection between four small groups Gr1, Gr2, Gr3, Gr4 that comprise the bigger cluster A. From these groups, the "strongest" is Gr2 formed by the variables D1, X2 and D2 that present a considerably strong similarity (0,99999). That means that students who gave a correct (D1) or an approximately correct definition (D2) in Question 8, gave an example of a one-to-one function (X2) in Question 9. This group is completed with the answers in Question 2 (Q2), which concerns the conversion from a verbal representation of a piecewise function to the algebraic form. It is indicated that a non-transparent conversion of representations was accomplished mostly by the students who achieved a conceptual understanding of function. This strong group is linked to Gr3 which involves the variables Q6B, Q6D and Q6E, representing the correct recognition of some non-conventional cases of relations in the form of Cartesian graphs. Within the same group, these variables are associated with the answers to the four parts of Task 7 (Q7A, Q7B, Q7C, Q7D). Task 7 concerns the recognition of functions presented in the form of arrow diagrams.

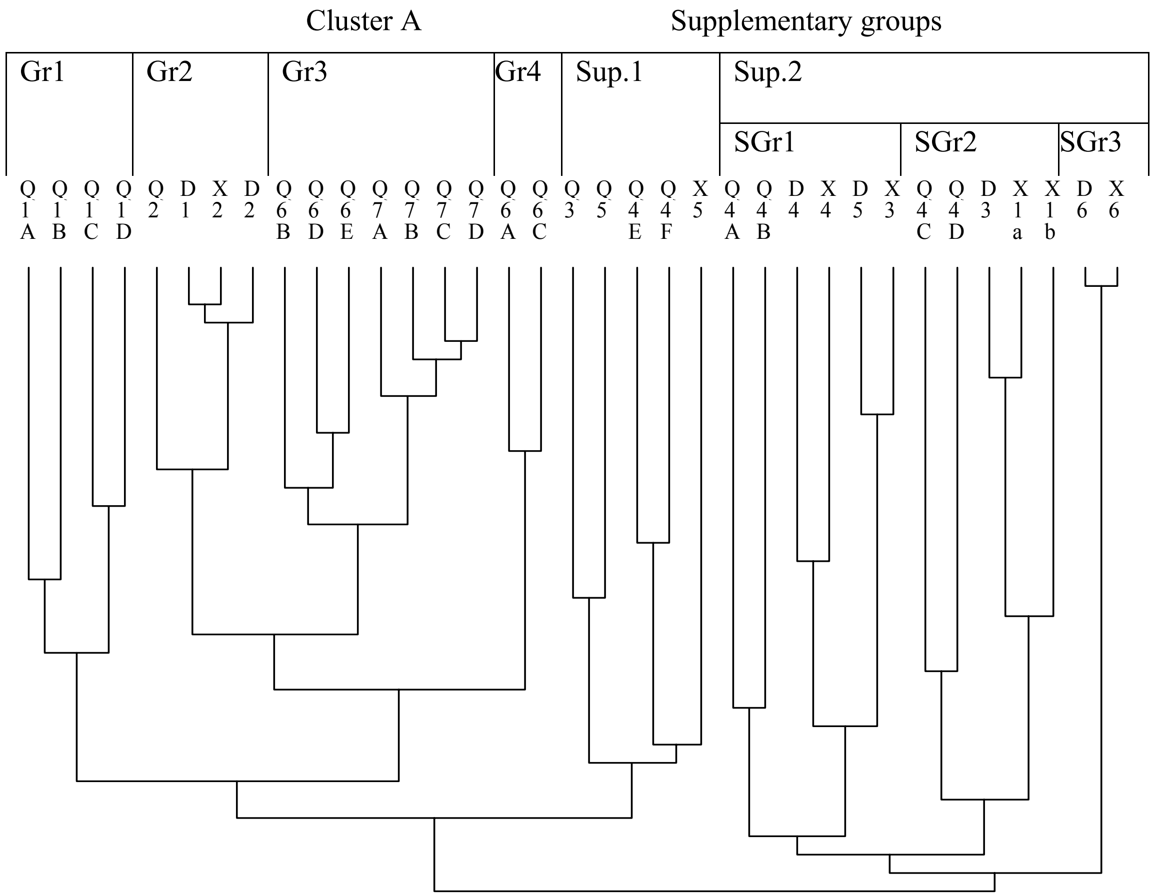


Figure 1: *Similarity diagram of students' responses to the tasks of the questionnaire*

The two groups Gr2 and Gr3 are connected with Gr4 that includes the answers to the other two parts of Question 6 (recognition of function given in graphical form). The groups Gr2-Gr3-Gr4 connect with Gr1 that includes the answers to Task 1; that is the recognition of functions represented in verbal form. Conclusively the connection of groups Gr1-Gr2-Gr3-Gr4 creates a cluster of students' responses, which entail a conceptual approach to function. Finally this whole cluster A (Gr1-Gr2-Gr3-Gr4) connects with two of the conversion tasks of the questionnaire (Q3 and Q5) and the responses to the tasks Q4E and Q4F requiring the recognition of functions in algebraic form. These are linked with the group that gave an example presenting an uncertain transformation of the world (X5). This is the first "supplement" (Sup.1) of cluster A. The second supplement (Sup.2) is embodied by three similarity groups, which are connected to each other. The first group of Sup.2 (SGr1) involves the definition and example variables D4-X4 and D5-X3, which illustrate a vagueness or limited idea of the definition and the examples of function. These variables connect with answers to questions Q4A and Q4B, which have a linear algebraic character. The second group of Sup.2 (SGr2) is formed by the variables D3, X1a and X1b. This means that students, who provided a definition of a special kind of function (D3), gave an example of function with the use of discrete elements of sets (X1a) or an example of a continuous function (X1b). These variables are connected with Q4c and Q4d, which are treated in a way that shows the conception that symbols “x” and “y” must always appear in the algebraic form of a function. The third group of Sup.2 (SGr3), which is the strongest one in the whole similarity diagram, is characterized by the most doubtful idea about the notion of function, since it includes D6 and X6 (i.e., those students that did not attempt to give any definition or example of function), and is not linked directly to the use of any representation of function.

Within the similarity diagram, one can also observe the formation of groups or subgroups of variables of students’ responses in recognition tasks involving the same mode of representation of functions, i.e., in verbal form (Gr1), in an arrow diagram or in graphical form (Gr3) and in algebraic form (supplementary groups). The particular observation reveals the consistency by which students dealt with tasks in the same representational format, but with different mathematical relations. However, lack of direct connections between variables of similar content but different representational format indicate that some students may be able to identify a function in a particular mode of representation (e.g., algebraic form), but not necessarily in another mode of representation (e.g., graph). This inconsistent behavior among different modes of representation is an indication of the existence of compartmentalization.

The structure of the connections established in the similarity diagram seems to offer support to the triarchic model, proposed here. Closely connected pairs or terns of definitions and examples which are generated due to their common accurate or inaccurate features, are associated with students’ responses in tasks involving particular types of representation. Different aspects of students’ image for the function concept (e.g., conceptual understanding or ambiguous ideas) are indicated by the formation of different similarity groups, each incorporating the distinct but interrelated factors of the triarchic model: D, E and R.

4. DISCUSSION

4.1 Students’ main ideas for the concept of function

Conclusively the results of the study have revealed some of the ideas that university students had about function. Such an idea is the identification of “function” by a large percentage of students with the narrow concept of one-to-one function. This finding is in accord with the results of previous studies indicating that one-valuedness is a dominating criterion that students use for deciding whether a given correspondence is a function or not (Vinner and Dreyfus, 1989). This idea is also associated with the process of enumeration, which involves one-to-one correspondence as a matter of routine for the students. Another idea was that function is an analytic relation between two variables (as it worked historically, initially with Bernoulli’s definition, and more clearly with Euler’s). A number of students have even stated this explicitly in their justifications when attempting to identify functions among other algebraic relations. Moreover, students’ dominating idea that a graph of a function must be connected or “continuous” caused difficulties in recognition and conversion tasks involving disconnectedness of a function’s graph.

4.2 Ability to use diverse representations of function

One of the main goals of the present study was to examine students’ performance in recognition and conversion tasks involving different modes of representation of function. Higher success rates were observed in the tasks which involved algebraic representations, relative to the tasks involving verbal and graphic representations (either Cartesian graphs or arrow diagrams). This finding can be attributed to the fact that mathematics instruction in schools focuses on the use of algebraic representations of functions, thus hindering the approach of function in other representational modes (e.g., Kaldrimidou and Iconomou, 1998).

In addition, students responded in tasks involving the same type of representation in a consistent and coherent manner. Nevertheless, they approached in a distinct way the different forms of representation of functions, providing support to the existence of the compartmentalization phenomenon (Gagatsis et al., 2003). Students probably considered the different systems of representation as different and autonomous mathematical objects and not as distinct means of representing the same concept (Duval, 1993). This was apparent also from students' failure in a conversion task of representations that was not transparent. Since a concept is not acquired when some components of mathematical thought are compartmentalized, teaching needs to accomplish the breach of compartmentalization, i.e., de-compartmentalization and coordination among different types of representations. One way to achieve this is by giving students the opportunity to engage in conversions of representation that can be congruent or not in different directions (Duval, 2002).

4.3 The connection of students' concept definitions and examples with the use of different representations of function

Findings showed that strong similarity connections exist between the definitions and the examples given by the students for function and their abilities to handle different modes of representation of the concept in recognition and conversion tasks. This indicates that concept definitions, examples and ability to handle different representations are not independent entities, but are interrelated in students' thought processes. The group of students, who accomplished a conceptual understanding of function involved strong connections with representations in the form of arrow diagrams, Cartesian graphs and verbal description, and had a higher level of success when dealing with most of the representations of the concept and a non transparent conversion. The group of students who had ambiguous or limited ideas for the function concept was exemplified by the answers of the students who kept coherently mostly the connection with the idea of linear function and seemed to be competent at handling more efficiently the algebraic form of representations than any other mode and the simple (term-by-term) conversions. Some students' incompetence in giving a definition and an example for function was not related to the use of any representation of the concept. These findings are in line with the view of a number of researchers that students' errors may be a result of deficient use of representations or a lack of coordination between representations (e.g., Greeno and Hall, 1997; Smith, diSessa and Rochelle, 1993).

The fact that using and representing functions in a diversity of representations are strongly related to the appropriate meaning of function and its applications has pedagogical implications. The understanding of function may be enhanced by designing didactic activities that are not restricted in certain types of representation, but involve recognition and transformation activities of the notion in various representations (Sierpinka, 1992; Duval, 2002; Even, 1998; Hitt, 1998). Furthermore, assessment tools of students' learning of function need to include tasks carried out in various semiotic representations. This study' s findings revealed that succeeding in transformation or recognition tasks in particular systems of representation was not indicative of students' understanding of function. For example, a significant percentage of students (from 28% to 60%), who gave an incorrect definition of function, were in a position to identify the concept in certain forms of representation (mainly the algebraic one).

The above example indicates that despite the close similarity relations between students' images of function and their ability to handle different representations of it, discrepancies between them were relatively frequent. Students' definitions or examples did not always have a predictive role in how students would apply the concept in various forms of representation. Hence, all three factors of mathematical thought examined in this study, D, E and R were found to describe in their own unique way different aspects of students' acquisition of the complex concept of function. It is not sufficient to make general inferences such as "students have an understanding of the concept of function" in the sense that they are reasonably successful in giving a definition of the concept or providing examples or even recognizing functions in different forms of representation, separately. The use of the triarchic conceptual-semiotic model of understanding of the function concept is, thus, validated. Adequate understanding of the concept may be indicated by approximately correct definition and examples, and flexibility in dealing with multiple representations in recognition and conversion tasks of function. Limited and ambiguous aspects of the function concept may be revealed by students' deficits in dealing with at least one of the three dimensions: D, E or R.

The above remarks have direct implications for teaching and assessment. One must remember that in order to teach functions to a group similar to the sample of this study, it is important to include the three different dimensions of studying function in his/her instruction and assessment: D, E and R. To employ effectively the triarchic model it is also important for the teachers to have in mind and make appropriate use of the connection among its components. By using the triarchic model in students' assessment, teachers can identify in which of the three domains students have difficulties as regards the understanding of function. On the basis of the assessment results, teaching must develop mathematical understanding in a way that it builds on students' constructed knowledge and abilities. In other words, strong emphasis should be given on the domains that are less familiar or known in some aspects and on their connection to the domains or aspects of a domain that students are more capable at. For example, students who are able to give an appropriate definition and examples of function applications, can be helped to elaborate their knowledge at first by using a familiar representation system and a diversity of other representations to represent their definition and examples; next, by recognizing whether a given mathematical relation in different systems of representation is a function or not in terms of their definition, by identifying the same types of function in various representations and carrying out a conversion of a function from one system of representation to another in different directions. These didactical implications are in line with Steinbring' s (1997) idea that mathematical meaning is developed in the interplay between a reference context and sign systems of the mathematical concept in question. Nevertheless, further research is needed to investigate at a practical level the effectiveness of such didactical processes for teaching the complex concept of function addressing prospective teachers.

4.4 Can we succeed de-compartmentalization? Implications of an on-going research

In an attempt to accomplish de-compartmentalization an experimental study was designed by Gagatsis, Spyrou, Evangelidou and Elia (2004) that constitutes the second stage of the research reported in the present paper. The researchers developed two experimental programs for teaching functions to university students, based on two different perspectives. The students who participated in the experimental study were divided into two groups. Each group received a different experimental program. Students of Experimental Group 1 were exposed to Experimental Program 1 and students of Experimental Group 2 received Experimental Program 2. Next, students of Experimental Group 1 were compared with students of Experimental Group 2. To compare the two groups two tests (a pre-test, before instruction and a post-test, after instruction) similar to each other and also similar to the test that was used in the present study, were designed to investigate students' understanding of functions.

The two experimental programs, conducted by two different university professors (Professors A and B), approached the teaching of the notion of function from two different perspectives. Experimental Program 1 started by providing a revision of some of the functions that were already known to the students from school mathematics, physics and economics. Different types of functions were presented next, starting from the simple ones and proceeding to the more complicated ones. The program ended by giving the set-theoretical definition of a function.

Experimental Program 2 encouraged the interplay between different modes of representation of a function in a systematic way. The instruction that was developed by Professor B on functions was based on two dimensions. The first dimension involved the intuitive approach and the definition of function. The second dimension emphasized the various representations of function, and the different conversions between them.

In the light of the above, an essential epistemological difference can be identified between the two experimental programs: Experimental Program 1 involved an instruction of a classic nature and widely used at the university level. On the contrary Experimental Program 2 was based on a continuous interplay between different representations of various functions.

The preliminary results of the new study provided evidence for the appearance of the phenomenon of compartmentalization in the similarity diagrams of the answers of the students of Experimental Group 1, before and after instruction, especially in using the graphical representations and arrow diagrams. On the contrary, the compartmentalization that was evident in the similarity diagram involving the responses of students of Experimental Group 2 before instruction disappeared in the corresponding similarity diagram after instruction. Similarity connections indicated students' consistency in recognizing functions in different modes of representation. In other words, success was independent from the mode of representation of the mathematical relation. This finding revealed that Experimental Program 2 was successful in developing students' abilities to use flexibly various modes of representation of functions and thus accomplished the breach of compartmentalization in their performance. The research towards the direction, described briefly above, continues so as to provide explanations for the success of Experimental Program 2 and to determine those features of the intervention that were particularly effective in accomplishing de-compartmentalization. The results of such an attempt may help educators at a university level to place stronger emphasis on certain dimensions of the notion of function and techniques of teaching functions, so that students can be helped to construct a solid and deeper understanding of the particular construct.

Appendix: The tasks of the questionnaire

1. Explain whether we define a function when we:

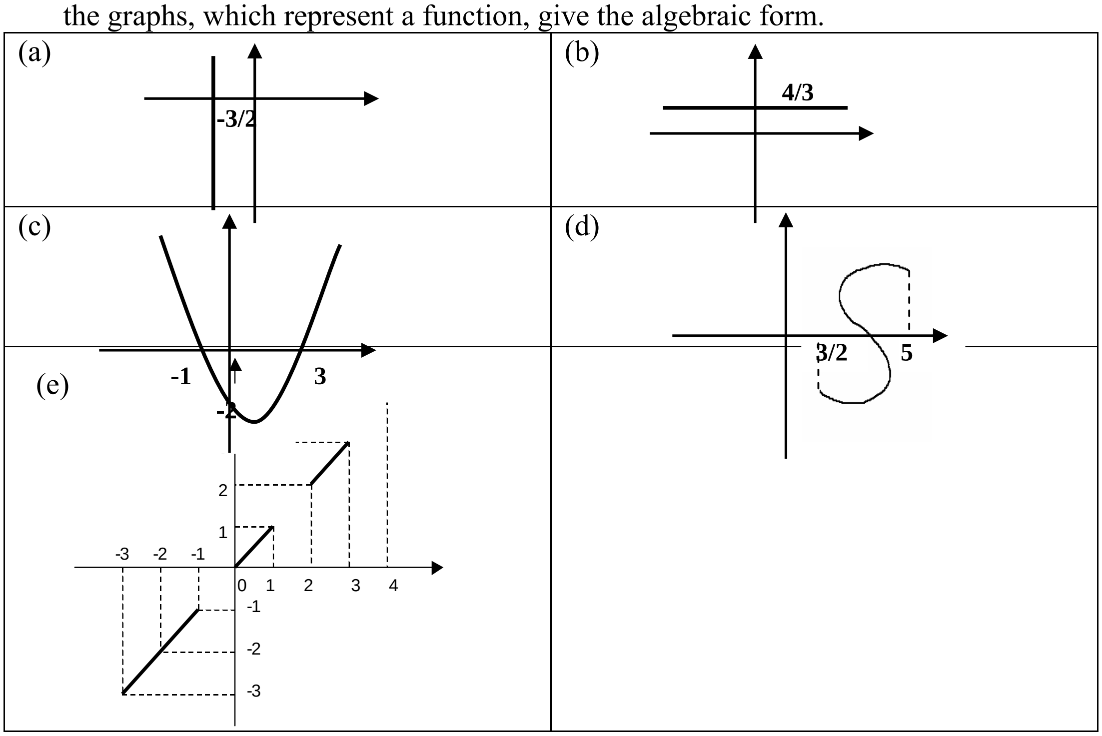
(a) correspond a girl with different friends of hers (George, Homer, Jason, Thanasis, etc.) with whom she will probably dance at a party.
(b) correspond every football game to the score achieved.
(c) at the university entrance examinations correspond every script to the couple of marks given by the first and the second examiner.
(d) correspond every candidate with the posts for which she applies for work in an organisation (candidates may apply for more than one post).

2. At the entrance examinations there are two types of candidates: successful and unsuccessful. Let A stand for the set of successful candidates and B stand for the unsuccessful candidates. Using symbols 1 and 0, construct a function, which describes this situation, and give the algebraic form.
3. Find the algebraic formula of the function that converts the initial prices of a shop that makes sales 20% in every item, to the new prices that emerge.
4. Examine whether the following symbolic expressions may define functions and justify your answer. For the expressions that define a function, indicate the symbol, which you consider as the independent variable.

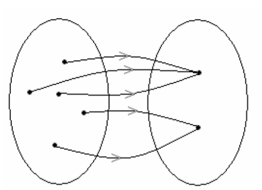
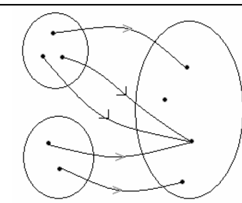
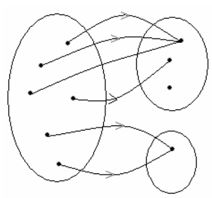
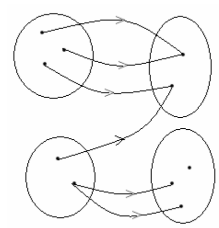
(a) $5x+3=0$ Yes / No, Explanation:
(b) $2x+y=0$ Yes / No, Explanation:
(c) $4y+1=0$ Yes / No, Explanation:
(d) $x^2+y^2=25$ Yes / No, Explanation:
(e) $x^3-y=0$ Yes / No, Explanation:
(f) $f(x) \begin{cases} x, & x\geq 0 \\ -x, & x<0, \end{cases}$ Yes / No, Explanation:

5. Draw the graph for one of the expressions of question 4, which you consider as a function.

6. Examine whether the following graphs represent a function and justify your answer. For the graphs, which represent a function, give the algebraic form.



7. Examine which of the following correspondences presented in the form of Venn diagrams are functions. Justify your answer.

(a) 	(b) 
(c) 	(d) 

8. According to you what is a function?
9. Give two simple examples from the applications of functions in everyday life.

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