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An In-depth Investigation of the Divine Ratio

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Abstract**:** The interesting thing about mathematical concepts is that we can trace their development or discoveries throughout history. Most cultures of the ancient world had some form of mathematics, and these basic skills developed into what we now call modern mathematics. The divine ratio is similar in that it was used in many different sections of history. The divine ratio, sometimes called the golden ratio or golden section, has been found in very diverse areas. The mathematical concepts of the golden ration have been found throughout nature, in architecture, music as well as in art. Phi is an astonishing number because it has inspired thinkers in many disciplines, more-so than any other number has in the history of mathematics. This paper investigates how the golden ratio has influenced civilizations throughout history and has intrigued mathematicians and others by its prevalence.

Keywords: Egyptian mathematics; Fibonacci; Golden mean; Golden ratio; Greek mathematics; Indian mathematics; mathematical aesthetics; mathematics in nature

Introduction

Throughout this paper, the terms golden ratio, divine ratio, golden mean, golden section and Phi (φ) are interchangeably used. Wasler, (2001) defines the golden ratio as a line segment that is divided into the ratio of the larger segment being related to the smaller segment exactly as the whole segment is related to the larger segment. The divine ratio is the ratio of the larger segment, AB, of line AC to the smaller segment BC of the line AC.

This same definition was first given by Euclid of Alexandria around 300 B.C. He defined this proportion and called it “extreme and mean ratio” (Livio, 2002). Let us assume that the total length of line AC is x+1 units and the larger segment AB has a length of x. This would mean that the shorter segment BC would have a length of 1 unit. Now we can set up a proportion of AC/AB = AB/BC.

$$\frac{x+1}{x} = \frac{x}{1}$$

By cross multiplying it yields $x^2 - x - 1$. Using the quadratic formula, two solutions become apparent (1+√5)/2 and (1-√5)/2, and we only use the positive solution because we are in terms of a length. The positive solution is (1+√5)/2. Phi is the only number that has the unique property that φ*φ'=-1where φ' is the negative solution to the quadratic (1-√5)/2 (Huntley, 1970).

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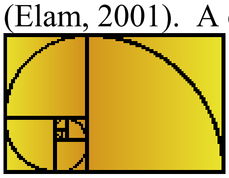
Additional Information on the Golden Ratio

In professional mathematical literature, the golden ratio is represented by the Greek letter tau. The symbol (τ) means "the cut" or "the section" in Greek. In the early twentieth century, American mathematician gave the golden ratio a new name. Mark Barr represented the Golden Ratio as phi (φ), which is the first Greek letter in the name of Phidias.[1] Barr chose to honor the great sculptor because many of Phidias's sculptors contained the Divine Ratio.

The golden ratio is a known irrational number. Irrational numbers have been around for sometime. Most historians believe that irrational numbers were discovered in the fifth century B.C. Pythagoreans knew about irrational numbers and believed that the existence of such numbers was due to a cosmic error (Livio, 2001).

The Golden section is aesthetically pleasing in nature. Phi represents some remarkable relationships between the proportions of patterns of living plants and animals. Contour spirals of shells, such as the chambered nautilus, reveal growth patterns that are related to the golden ratio. The nautilus shell has patterns that are logarithmic spirals[2] of the golden section. Each section is characterized by a spiral, and the new spiral is extremely close to the proportion of the golden section square larger than the previous. The growth patterns in nature approach the golden ration, and in some cases come very close to it, but never actually reach the exact proportion (Elam, 2001). A construction of the golden rectangle and logarithmic spiral can be seen below.



Logarithmic spirals can be found through-out nature. Ram horns and elephant tusks, although they do not lie in a plane, follow logarithmic spirals. Logarithmic spirals are also closely related to Golden Triangles[3]. Starting with a Golden Triangle ABC, the bisector of angle B meets AC at point D and is the golden cut of AC. With this bisection, triangle ABC has been cut into two isosceles triangles that have golden proportion (the ratio of their areas is φ:1. Continuing this process by bisecting angle C, point E is obtained. Again point E is the golden cut along line BD, thus constructing two more golden triangles. This process produces a series of gnomons[4] that will eventually converge to a limiting point O, which is the pole of a logarithmic spiral passing

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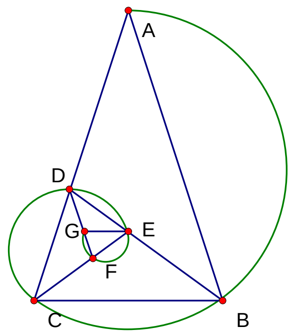
[1] Phidias was an Greek sculptor who lived between 490 and 430 B.C. His sculptors included "Athena Parthenos" which is located in Athens and "Zeus" which is located in the temple of Olympia.
Comment: astonishing is a strange word to use here…how about great?

[2] Logarithmic spirals have a unique property. Each increment in the length of the shell is accompanied by a proportional increase in its radius. This implies that the shape remains unchanged over time and growth. As a logarithmic spiral grows wider, the distance between its coils increases and it moves away from its original starting point (pole). It turns by equal angles and increases the distance from the pole by equal ratios.

[3] Golden Triangles are isosceles triangles that exhibit base angles of 72 degrees and an apex angle of 36 degrees. From the Pythagoreans and the construction of the pentagram (which has five equal-area golden triangles) it can be seen that the length of the longer side to that of the shorter side is in golden proportion.

[4] A gnomon is a portion of a figure which has been added to another figure so that the whole is of the same shape as the smaller figure.

successively and in the same order through the three vertices (…A,B,C,D…) of each of the series of the triangles (Huntley, 1970).



If we begin with GF and call it the unit length, then:

FE = 1φ
ED = 1φ + 1
DC = 2φ + 1
CB = 3φ + 2
BA = 4φ + 3

By bisecting the base angles of the successive gnomons, the lengths of these segments form a Fibonacci series, which we have already seen to converge to the Golden Ratio.

Pine cones and sunflowers are closely related to the Golden Ratio. Each seed in a pine cone is part of a spiral growth pattern that closely relates to φ. The seeds of pine cones grow along two intersecting spirals that move in opposite directions. Interestingly, each seed belongs to both spirals. Eight of the spirals move in the clockwise direction and the remaining thirteen move counter clockwise. As seen above, the numbers 8 and 13 are consecutive Fibonacci numbers which converge to the Golden Ratio. The proportion of 8:13 is 1:1.625. Sunflowers exhibit the same spiral patterns as seen in pine cones. Sunflowers have 21 clockwise spirals and 34 counter clockwise. The proportion of 21:34 is even closer to the Golden Ratio than that of pine cones; it is 1:1.619 (Elam, 2001).

The geometry of plant axis flexure is the result of orthotropic growth and the stress caused by a vertical weight distribution along the axis. A flexed plant axis is shown to conform to a portion of a logarithmic spiral. With numerous plants, this mode of curvature is the most prevalent condition of plants lacking or having secondary growth. Plants like sunflowers represent this growth pattern (Niklas and O'Rourke, 1982).

In 1907 the German mathematician G. van Iterson showed that the human eye would pick out patterns of winding spirals when successive points were packed tightly together. The points were separated by the Golden Angle which measures to 137.5 degrees. The familiar spirals that the human eye would pick out consisted of counter clockwise and clockwise patterns of consecutive Fibonacci numbers. Nature, specifically sunflowers, grows in the most efficient way[5] of sharing horizontal space, which is in proportion of the Golden Ratio. Most sunflowers

[5] Mathematicians, Harold S. M. Coxeter and I. Adler, showed that buds of roses which were placed in union with spirals generated by the Golden Angle were the most efficient. For example, if the angle used was 360/*n* where *n* is an integer, the leaves would be aligned radially along *n* lines, thus leaving large spaces. Using the Golden Angle,

have a 21:34 ratio, but few have been reported with proportions of 89:55, 144:89 and 233:144 (Livio, 2001).

The Golden Ratio can be found in many examples throughout the world. Phi can be seen in many places; from the layout of seeds in an apple to Salvador Dali's painting "Sacrament of the Last Supper" (Livio, 2002). In the following sections, an in-depth look is taken on the occurrences of Phi in as well as the development of Phi throughout history.

The Golden Ratio and Fibonacci Numbers

Leonardo de Pisa, born around 1175 A.D., commonly known as Fibonacci[6] introduced the world to the rabbit problem. The rabbit problem asked to find the number of rabbits after *n* months, given that adult rabbits produce a pair of rabbits each month, offspring take one month to reach reproductive maturity, and that all the rabbits are immortal. This problem gave the mathematical world the series of Fibonacci numbers[7].

In 1202 A.D. Fibonacci wrote, *Liber Abaci*, which was a book based on the arithmetic and algebra that he had accumulated in his travels. This book was widely copied and introduced the Hindu-Arabic place-value decimal system and the use of Arabic numerals into Europe. Most of the problems in *Liber Abaci* were aimed at merchants and related to the price of goods, how to calculate profit on transactions, and how to convert between the various currencies in use in the Mediterranean countries. Fibonacci is most remembered for presenting the world with the "rabbit problem" which is located in the third section of *Liber Abaci*.

Looking at the ratio of successive Fibonacci numbers, an interesting value appears. As the n increases, the ratio of F_n/F_{n+1} approaches the golden ratio. The values (n=1…10) can be seen in the table below:

n	F(n)	F(n)/F(n-1)
1	0	
2	1	
3	1	1
4	2	2
5	3	1.5
6	5	1.666667
7	8	1.6
8	13	1.625
9	21	1.615385
10	34	1.619048

which is an irrational multiple of 360 degrees, ensures that the do not line up in a specific radial direction and this leaves no space unfilled.

[6] Fibonacci is a shortened form of Filius Bonaccio (son of Bonaccio). Fibonaci was taught the Arabic system of numbers in the 13th century. He later published the book *Liber Abaci* (Book of Abacus). This book introduced the Arabic numbering system to Europe and gave Fibonacci everlasting fame as a mathematician. (Dunlap, 1997)

[7] Fibonacci Numbers are represented by the recursive relation $A_{n=2} = A_{n+1} + A_n$

The convergence of Fibonacci numbers to the Golden Ratio can be seen in the "rabbit problem". A Scottish mathematician, in the early 1700's made the connection between the Golden Ratio and the rabbit problem. Robert Simson (1687-1768), noticed that consecutive terms of the solution to the rabbit problem converged to the Golden Ratio (Johnson, 1999). A geometric sequence can be constructed on the basis of the breeding rabbits. For example, let adult rabbits be represented by 'A' and their offspring represented by 'b'. The arrangement of adults (A) and their offspring (b), can be written as AbAAbAbAAbAAbAbAAbAbA… The sequence of A's and b's may be extended indefinitely in a unique way because the rule for generating the next character is well defined. The ratio of adults to offspring rabbits in the limit of an infinite sequence is equal to the Golden Ratio (Dunlap, 1997).

$$\lim_{n\to\varphi} A/b = \varphi$$

The Golden Ratio in Ancient Greece

The Golden Ratio can be found throughout nature, which will be discussed below, but it can also be found in the history of the heavens. Plato (428-347 B.C.) prophesied the significance even before Euclid described it in *Elements*. Plato saw the world in terms of perfect geometric proportions and symmetry. His ideas were based on Platonic Solids.[8] He divided the heavens into four basic elements, earth, water, air, and fire. Each of these elements was assigned a Platonic Solid; a cube for earth, tetrahedron for fire, octahedron for air and an icosahedron for water. Using this foundation, Plato created a chemistry that is similar to modern day chemistry[9] (Livio, 2003).

The five Platonic solids are the only existing solids in which all of the faces are identical and equilateral and each vertex is convex. Interestingly, each of the solids can be circumscribed by a sphere with all of its vertices lying of the sphere. The tetrahedron consisted of four triangular faces, the cube with six square faces, the octahedron with eight triangular faces, the dodecahedron with twelve pentagonal faces and the icosahedron with twenty triangular faces (Livio, 2002).

Each face of the regular polyhedron is a regular polygon with n edges. It is known that the values of n are $\{n: 3 \leq n < \infty\}$ with n being related to the interior angle α.[10] Each vertex of the three dimensional polygon is defined by the intersection of a number of faces, m, where $m \geq 3$. In order

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[8] The Platonic Solids Plato used consisted of five shapes. The first three; tetrahedron, octahedron and the icosahedron, were based on equilateral triangles. The remaining two; cube and dodecahedron were made from the square and regular pentagram.

[9] Plato's theory was much more than a symbolic association. He noted that the faces of the tetrahedron, cube, octahedron, and dodecahedron could be constructed out of two types of right angled triangles, the isosceles 45-90-45 and the 30-60-90 triangle. Plato explained that his chemical reactions could be described using these properties. For example, when water is heated by fire, it produces two particles of vapor (air) and one particle of fire, $\{water\} \to 2\{air\} + \{fire\}$. In Platonic chemistry, balancing the number of faces involved (in the Platonic solids that represent these elements) we get 20=2*8+4. The central idea is that particles in the universe and their interactions can be described by a mathematical' theory that possesses certain symmetries.

[10] In general a regular n-gon has n edges and interior angles given by the equation $\alpha=[1-(2/n)]*180$.

for a convex vertex to be formed, $m\alpha<360$ degrees. [11] There are only five combinations of integers that satisfy these equations and they correspond with the five Platonic Solids and are listed below[12] (Dunlap ,1997).

solid	n	m	e	f	v
tetrahedron	3	3	6	4	4
cube (hexahedron)	4	3	12	6	8
octahedron	3	4	12	8	6
dodecahedron	5	3	30	12	20
icosahedron	3	5	30	20	12

The Golden Ratio is of relevance to the geometry of figures with fivefold symmetry. The dodecahedron and the icosahedron are of particular interest. If either one of these Platonic Solids are constructed with an edge length of one unit, it is easy to see the important role the Golden Ratio play in their dimensions (Dunlap, 1997).

solid	surface area	volume
dodecahedron	$15\varphi/(3-\varphi)$	$5\varphi^3/(6-2\varphi)$
icosahedron	$5\sqrt{3}$	$5\varphi^5/6$

Plato and his foundations using Platonic Solids for the heavens may suggest that the Golden Ratio may have been known in ancient Greece. However, the full mathematical properties of Platonic Solids may not have been known in antiquity. Plato and his followers may have created and used Platonic Solids in the foundations of the universe based on sheer beauty.

Many authors researching ancient Greek mathematics are unsure if the works of Plato were influenced by Pythagoras and the Pythagoreans. Pythagoras[13] was born around 570 B.C. on the island of Samos. Pythagoras and the Pythagoreans are best known for their role in the development of mathematics and for the application of mathematics to the concept of order (Livio, 2002).

The Pythagoreans assigned special properties to odd and even numbers as well as individual numbers. The number one was considered the generator of all other numbers and geometrically, the generator of all dimensions. The number two was considered the first female number and the number of opinion and division. Geometrically, the number two was expressed by the line

[11] We have to place certain restrictions on the values of m. These reasons are if m=2 then an edge is formed, not a vertex. And if $m\alpha=360$ degrees, then the vertex is merely a point on a plane and if $m\alpha>360$ degrees then the faces overlap.

[12] The table above lists the characteristics of the five Platonic Solids. The quantities n and m are the number of edges per face and the number of faces per vertex. The quantities e, f, and v are the total number of edges, faces, and vertices for the respective solid.

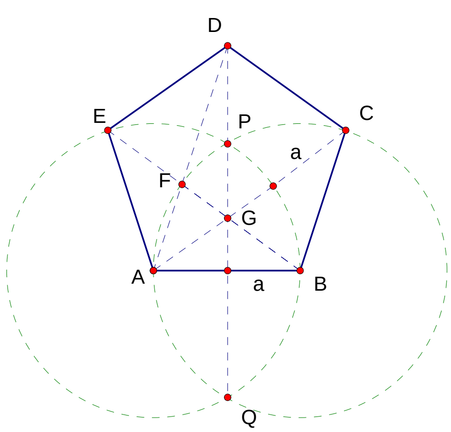
[13] Pythagoras emigrated to Croton in southern Italy sometime between 530 and 510. He studied Egyptian, and Babylonian mathematics, but both of these prove too applied for him. There are many different accounts of the Mathematician's life and death, but what is known for sure is that he was responsible for mathematics, and philosophy of life and religion.

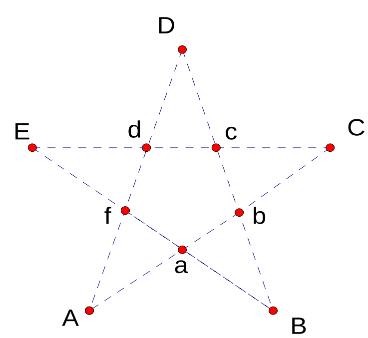
which has one dimension. The number three is considered by the Pythagoreans to be the first male number and the number of harmony because it combines the unity number (one) and the division number (two). The geometric expression of the number three was a triangle, where the area of the triangle has two dimensions. Justice and order was expressed in the number four. On the surface of the Earth, four directions provide orientation for humans to identify their coordinates in space. Four points, not in the same plane, form a tetrahedron. The number six is the first perfect number and considered the number of creation. It is the number of creation because it is the product of the first female number (two) and the first male number (three). Six is a perfect number because it is the sum of all the smaller numbers that divide into it. The first three perfect numbers are listed below (Livio, 2002).

6 = 1+2+3
28 = 1+2+4+7+14
496 = 1+2+4+8+16+31

The number five deserves its own explanation. Five represents the union of the first female number and the first male number. This union suggests that five is the number of love and marriage. The main reason five is important to this discussion is because the Pythagoreans used the pentagram[14] as a symbol of their brotherhood (Livio, 2002).

The construction of the pentagon, using a compass and marked straight edge, leads to a pentagram. Given a line AB, use the compass to draw arcs of radius *a* about points A and B. Next construct the perpendicular bisector PQ of line AB. Using the straight edge plot two points that are *a* units apart and slide the straight edge so that it passes through point A, until one of the points falls on the arc of B. There are only two possible positions for these points, namely, C and F. Using the same directions, find points G and D, sliding the straight edge through point B until one of the points falls on the arc of A. The fifth vertex (E) can be found by the requirement that on line EGB, EG equals *a*. Using this construction of a pentagon, one can connect the vertices and build a pentagram (Herz-Fischler, 1987).

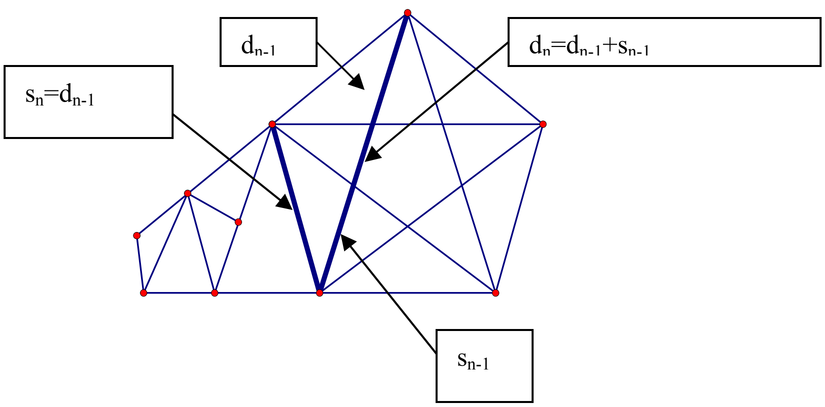




[14] The pentagram is closely related to the regular pentagon. If one is to connect all the vertices of the pentagon by diagonals, a pentagram is constructed. The diagonals of this pentagon form a smaller pentagram. This process can be continued to infinity, and every segment is smaller that its predecessor by a factor that is precisely equal to the Golden Ratio.

The pentagram is important to the discussion of the Golden Ratio because of its unique properties. The diagonals of a pentagon cut each other in the Golden Ratio and the larger of the two segments is equal to the side of the pentagon. The Pythagoreans choosing the pentagram as a symbol for brotherhood, and the given properties of the pentagram, suggests that the Pythagoreans were familiar with the Golden Number, but many historians are still under debate about this particular topic, due to inconclusive historical data (Herz-Fischler, 1987).

One theory, Heller (1958), suggests that the Pythagoreans used the pentagon to discover incommensurability and the division in extreme and mean ratio. Heller believes that the Pythagoreans discovered incommensurability through the observations of a series of pentagons when drawing diagonals.



The diagonal, d_{n-1}, becomes the side, s_n, of the next largest pentagon. The new diagonal d_n is the sum of the side and the diagonal, s_{n-1} and d_{n-1}, of the previous pentagon. With this information it is easy to see the recurrence relationships $s_n = d_{n-1}$; $d_n = d_{n-1} + s_{n-1}$. Using[15] $s_1 = 2$ and $d_1 = 3$, leads to the sequence of d_n:s_n ratios of 3/2, 5/3, 8/5, 13/8… which we have already seen to be successive Fibonacci numbers (Herz-Fischler, 1987). A formal proof of this can be found in The Golden Ratio: The Story of Phi the World's most Astonishing Number.

The Golden Mean in Ancient Egypt

Modern mathematicians have been trying to decide what civilizations used and understood the golden mean. Ancient Egypt, a civilization with profound mathematical accomplishments and astonishing monuments is under investigation for uses of the golden mean. Many interpretations of the golden mean use the properties of different geometrical figures. This may prove useless because it can produce an infinite chain of similar links. Math historians do need to focus on the ancient monuments and the mathematics of the respective time period. Ancient civilizations did not necessarily have the same numbering systems of modern times. This suggests that some things that work in modern numbering systems do not work in ancient systems (Rossi and Tout, 2002).

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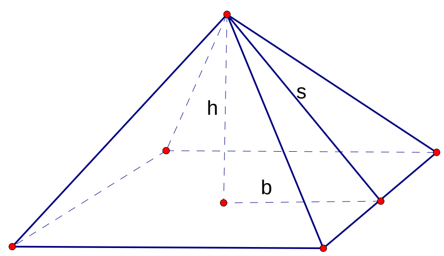
[15] Side and diagonal numbers of squares start off with the number one as the first number in the sequence. For pentagonal side and diagonal numbers, starting with one will lead to the degenerate case. Thus we have to start with the two as the first number in the sequence.

One theory about the use of the Golden Mean in ancient Egypt is that Egyptian architects designed the pyramids in a geometric way. Egyptian pyramids were based on geometrical processes of squares, rectangles and triangles. Of extreme importance was the process of the 8:5 triangles.[16] Egyptians used these triangles because the ratio of 8/5 was a good approximation of the Golden Mean. The theory continues to suggest that Egyptian architects gave their designs dimensions based on the corresponding numbers of the Fibonacci series. We have already seen that the ratio of corresponding Fibonacci numbers converges to the Golden Ratio (Rossi and Tout, 2002).

The Great Pyramid of Cheops, built before 2500 B.C., has been measured and many different dimensions are present. The majority of the dimensions are within one percent of 755.79 feet as the length of the base and 481.4 feet as the height. Some theories claim that the Great Pyramid of Cheops was designed so that the ratio of the slant height of the pyramid to half the length of the base would be the divine proportion (Markowsky, 1992).



In the above figure, h represents the height, b represents half the base, and s represents the slant height of the Great Pyramid of Cheops. Using 755.79 feet for the length of the base and 481.4 feet for the height, we can see that b=377.90 feet. Using the Pythagorean Theorem, $h^2 + b^2 = s^2$, we can find that s=612.01. This gives us a ratio of the slant height of the pyramid to half the length of the base as 612.01/377.90=1.62 which is very close to the Golden Mean (Markowsky, 1992). Another interesting feature of the Great Pyramid is that it has an apex angle of 63.43 degrees. This is very close to the apex angle of the Golden Rhombus[17] (63.435 degrees), which has dimensions derived for the Golden Ratio. The difference between the apex angle of the Great Pyramid and a Golden Rhombus is a mere 22 centimeters in the edge of the length of the pyramid base (Dunlap, 1997).

The question that needs to be answered is, was it possible for ancient Egyptians to construct a convergence of the Fibonacci numbers? Ancient Egyptians represented ratios as a sum of unit fractions. For example the fraction 3/5 would be represented as 1/2 + 1/10. As ratios continued to grow, many different representations become available. Take the ratio 13/21 for example. Egyptians could have represented this number in five different ways:

1. 1/2 + 1/10 + 1/56 + 1/840
2. 1/2 + 1/10 + 1/57 + 1/665
3. 1/2 + 1/10 + 1/60 + 1/420
4. 1/2 + 1/10 + 1/63 + 1/315
5. 1/2 + 1/10 + 1/65 + 1/273

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[16] The 8:5 triangle was an isosceles triangle in which the base was eight units and the height was five units.

[13] The Golden Rhombus is a two dimensional figure that has perpendicular diagonals which have a ratio of 1:φ.

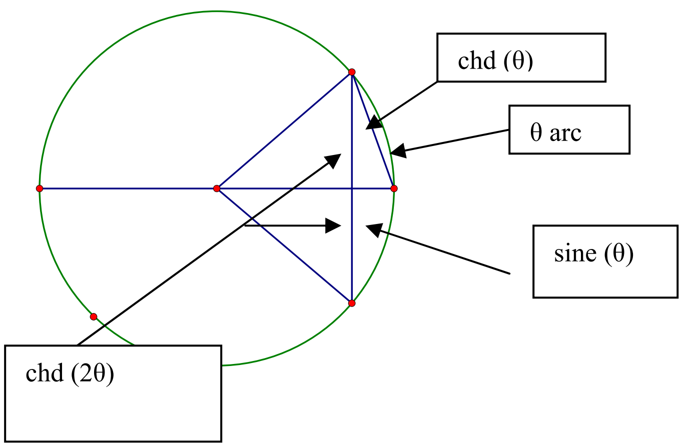
Egyptian scribes could have found a convergence of φ with their system of representing fractions. Adding to the previous sum of ratios a unit fraction whose denominator is given by the multiplication of the two previous denominators (in the ratio of Fibonacci numbers) yields the next value in the sum converging to the Golden Ratio. The sum of the ratios of the first few Fibonacci numbers converging to φ can be seen below (Rossi and Tout, 2002).

$1/2 = 1/2$
$3/5 = 1/2 + 1/10$
$8/13 = 1/2 + 1/10 + 1/65$
$21/34 = 1/2 + 1/10 + 1/65 + 1/442$
$55/89 = 1/2 + 1/10 + 1/65 + 1/442 + 1/3026$
$144/233 = 1/2 + 1/10 + 1/65 + 1/442 + 1/3026 + 1/20737$

The convergence above suggests that is was possible for ancient Egyptian scribes to evaluate the Golden Ratio. However, it seems unlikely that ancient Egyptians were aware of the Fibonacci numbers. Egyptian math is considered an applied math, no records have been found on the theory behind their mathematics. Only applications of Egyptian mathematics exist. This suggests that the Egyptians, although capable, did not recognize the golden ratio and it was a mere coincidence that the architecture of the pyramids is based on 8:5 triangles (Rossi and Tout, 2002).

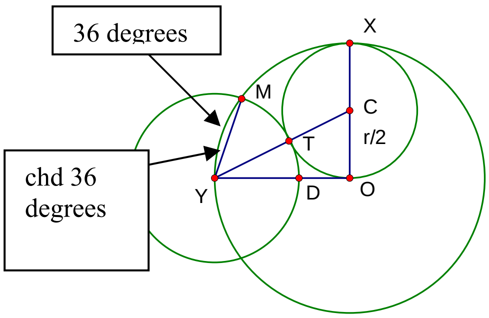
The Golden Ratio in Ancient India

The division in extreme and mean ratio appears in mathematical texts from India in connection with trigonometric functions. The Indian sine function is not the same as our modern day sine function. The Indian sine function can be defined as satisfying the relationship Sine (θ) = ½* chord (2θ). The circumference of the circle is divided into 360 degrees and then the radius of the circle is divided into 60 parts. With this, sine (30) = $a_6/2 = r/2 = 30$. And sine (18) = $a_{10}/2$ and sine (36) = $a_5/2$.



Bhaskara II (1114-1185) states without proof that Sine (18) = $(R(5r^2) – r)/4$. This is exactly the relationship sine (18) = $a_{10}/2$. Bhaskara, again without reason, tells to find the side of the pentagon inscribed in a circle, multiply the diameter by 70534/12000 (Amma, 1979). A proof of this statement is provided by Gupta (1976) and is provided below.

In a circle of radius r = OX = OY, let the arc YM = 36 degrees. Draw a semicircle OX about the midpoint C of OX and draw the arc MD about Y. Assume that the tow arcs meet at the single point T on line YC. Then Sine (18) = YM/2 = YT/2 = YC/2 –TC/2 = (R(r^2 + $(r/2)^2$) – r/2)/2 which is equivalent to Sine (18) = (R($5r^2$) – r)/4.



The above proof and construction are considered incomplete because they do not explain why the arcs meet at point T. Gupta (1976) continues and completes the construction by: Think of Y and C as given points and draw the arc OTX of radius r/2. Draw arc MTD of radius YT. Thus, the circles are tangent at the point T on the line YTC connecting the centers.

How does this construction tie in with the discussion on Ancient Indians knowing the Golden Ratio? Concentrate on the triangle YOC and arcs DT and OT. With a close examination, it can be seen that OY is divided in extreme and mean ratio at D. In other words, YM = YD is the greater segment when OY is divided in extreme and mean ratio (Gupta, 1976).

Evidence of the Golden Ratio in the Arts

Countless illustrations of the proportions of the Golden Section are found in the works of humans. The Golden Section follows upon the basis of symmetry everywhere and the forms which are based upon the golden proportion are widely distributed. When speaking about the products of art and architecture, there is no equal symmetry, the artist or workmen unconsciously employ golden proportions. Irregular inequality and capricious division is aesthetically disagreeable, while golden proportions are pleasing to both hand an eye (Ackermann, 1895).

Many assertions claiming that the Golden Section was used in art are associated with the aesthetics of the proportion. When given an opportunity to choose the most visually pleasing rectangle, most people would choose rectangles with a close approximation of the Golden Rectangle. Although most humans cannot decipher between a rectangle with a ratio of 1.6 and a rectangle with ratio of 1.7, it suggests that humans do prefer rectangles in the range close to the Golden Rectangle (Markowsky, 1992).

Several decades after the Brotherhood of the Pythagoreans faded, the Golden Ratio continued to influence many artists and artisans. The Golden Ratio has influenced classical Greek architecture, notably the Parthenon in Athens. Inside the Parthenon stands a forty-foot-tall statue of the Greek Goddess Athena[18], which has also shown to have Golden proportions. Both the temple and the stature were designed by Phidias, who is the first artist known to use the Golden

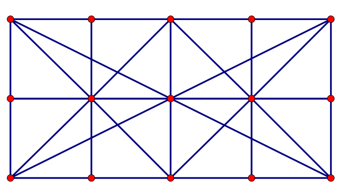
[18] Athena is the Greek goddess of wisdom, war, the arts, industry, justice and skill. Her father was Zeus and her mother was Metis, Zeus' first wife.

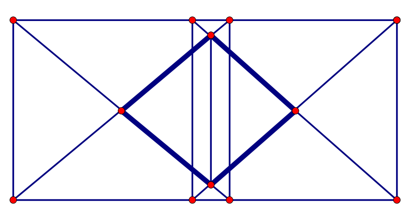
Ratio in his work. As said above, the symbol for the Golden Ratio is the first Greek letter phi, which also happens to be the first letter in Phidias's name (Johnson, 1999).

Ancient Greek scholar and architect Marcus Vitruvius Pollio, who is commonly known as Vitruvius, advised that "the architecture of temples should be based on the likeness of the perfectly proportioned human body where a harmony exists among all parts" (Elam, 2001).Vitruvius is credited with introducing the concept of a module to the architectural world. This concept was the same as the module of human proportions and became an important architectural idea. The Parthenon[19] in Athens is an example of this proportioning. The Parthenon can be inscribed by a Golden Rectangle (Elam, 2001).

When the triangular pediment was still intact[20], the Parthenon fit precisely into a Golden Rectangle. Another claim is that the height of the structure (from the top of the tympanum to the bottom of the pedestal) is divided into the Golden Ratio (Livio, 2001).Markowsky, (1992) has a contrasting view of the Parthenon. He believes that even though the Parthenon incorporates many geometric balances, its builders had no knowledge of the Golden Ratio. Depending on what sources are used, the dimensions of the Parthenon vary because the authors are measuring between different points. This implies that if the author is a Golden Ratio enthusiast they could choose which ever numbers give them the best approximation of φ.

Regardless whether or not the Parthenon's architecture was built accordingly to the Golden Ratio, it is still an amazing structure, and may get some of its beauty from the regular rhythms introduced by the repetition of the same column (Livio, 2001). Renaissance artists often used diagonals and other interior lines of rectangles to divide rectangular space proportionally. For example the main diagonals of a rectangle allow for division of the rectangle into halves, both vertically and horizontally. Continuing, the diagonals of the halves allow division into quarters. Another tactic used by Renaissance artists to construct they work was called rabatment. Rabatment is where the shorter sides of the picture rectangle are rotated onto the longer. The rotation creates vertical division and overlapping squares. If rabatment is applied to a Golden Rectangle, the diagonals of the two overlapping squares cut the diagonals of the rectangle in golden proportion (Brinkworth and Scott, 2001). A construction of division by diagonals is provided on the left and a construction by rabatment is provided on the right.





[19] The Parthenon, know as "the Virgin's place in Greek," in Athens was built in the fifth century B.C. and is one of the world's most famous structures. The Parthenon is a sacred temple to the cult of Athena Parthenos.
[20] On September 26, 1687, Venetian artillery directly hit the Parthenon. General Konigsmary said "How it dismayed His Excellency to destroy the beautiful temple which had existed for over three thousand years."

In the thirteenth century three artists' work contain close proportions to the Golden Rectangle. Italian painter and architect Giotto di Bondone (1267-1337) painted the "Ognissanti Madonna"[21] which is also known as "Madonna in Glory." Both the painting as a whole and the central figures in the painting can be inscribed by Golden Rectangles. Similarly, Sienese artist Duccio di Buoninsegna's (1255-1319) "Madonna Rucellai" and Florentine painter Cenni de Pepo's (1240-1302) "Santa Trinita Madonna" can be inscribed by Golden Rectangles. Both of these paintings are in the same room as the "Madonna in Glory". It is speculated that these three artists did not include the Golden Section into their paintings; rather they were driven by the unconscious aesthetic properties of the Golden Ratio. With respect to the time period, the three Madonnas were painted centuries before the publication of "The Divine Ratio" which brought the proportion into common knowledge (Livio, 2001).

Leonardo da Vinci inevitable comes into the discussion of the Divine Ratio and art. Five of his works have been speculated to host Golden Ratio properties: The unfinished canvas of "St. Jerome," the two version of "Madonna on the Rocks," the drawing of "a head of an old man," and the most famous of all, the "Mona Lisa"(Livio, 2001).

The two versions of "Madonna on the Rocks" have an interesting history. The first version, produced between 1483 and 1486, was done before da Vinci had any contact with Pacioli or his book "The Divine Ratio." The second version, which was completed around 1506, could have been influenced by Pacioli's book. Interestingly, both versions are very close to the Divine Ratio. In the first version, the dimensions are in proportion 1.64 and in the second version's dimensions are in proportion 1.58, both close estimates of φ (Livio, 2001).

Leonardo da Vinci's "head of an old man", is suggested to be a self-portrait which is overlaid with a square that is divided into rectangles. Some of these rectangles approximate Golden Rectangles but it is difficult to be absolutely sure. The rectangles are very roughly drawn and do not have square corners (Markowski, 1992). This suggests that depending on where one measures from, it is very possible to find some ratio that approximates the Golden Ratio. Leonardo da Vinci's "St. Jerome" has similar uncertainty. When overlaid with a Golden Rectangle, the left side of St. Jerome's body and his head are missed completely. The left side of the Golden Rectangle is tangent to a small fold of fabric and does not touch the body at all. Again, Leonardo was not introduced to Pacioli's book until thirteen years after the completion of "St. Jerome" (Markowski, 1992). His right arm also extends beyond the rectangle's side. The drawing of "a head of an old man,"[22] completed in 1490, is the closest demonstration that da Vinci used Golden Rectangles to determine dimensions in his paintings (Livio, 2001).

Human body proportions and facial features share similar mathematically proportioned relationships as other living organisms. The placement of facial features yields the classic proportions used by both the Romans and Greeks. Marcus Vitruvius Pollio described the height of a well proportioned man is equal to the length of his outstretched arms. The body and outstretched arms can be inscribed in a square, while the hands and feet are inscribed in a circle. With this system, the human body is divided into two parts at the naval. These parts are

[21] Bondone's painting "Madonna in Glory" is currently in the Uffizi Gallery in Florence. This painting features an enthroned Virgin with a child on her lap. Both Madonna and Child are surrounded by angles.

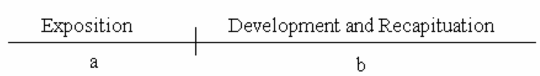
[22] The drawing of "a head of an old man" is currently in the Galleria dell' Accademia in Venice.

represented in the proportion of the Golden Rectangle. Classical statues from the fifth century such as Doryphoros the spear bearer and Zeus have the proportions suggested above (Elam, 2001).

The art described above deal with proportions of measurements. It should be noted that measurements, no matter how accurate, only provide reasonable estimates of the Golden Ratio (Fischler, 1981). The artist, painter or sculptor may or may not have been trying to conform to the proportion of the Golden Ratio. However close the approximations are, they could have been created with beauty in mind and with no intention to match the Golden Ratio[23].

Visually pleasing art is not the only form of art where the Golden Ratio can be found. Music and mathematics have been entwined since antiquity and it is not surprising that one accompanies the other[24]. The Golden Ratio is related to many forms of music. Many listeners, including people who are only casually acquainted with the music of Mozart (1756-1791), can pick up on the manifested form and balance the composer used when writing his music (Putz, 1995).

Mozart worked with mathematical figures throughout his life. In his early composing years, he took up the problem of composing minuets 'mechanically', by putting two-measure melodic fragments together in a specific order. By the age of nineteen, Mozart had composed his first sonata for piano[25]. Almost all of his sonatas were composed of two movements: 1) the Exposition in which the musical theme in introduced and the Development and Recapitulation in which the theme is developed and revisited (Newman, 1963). A visual representation of Mozart's sonata-form movement can be seen below.



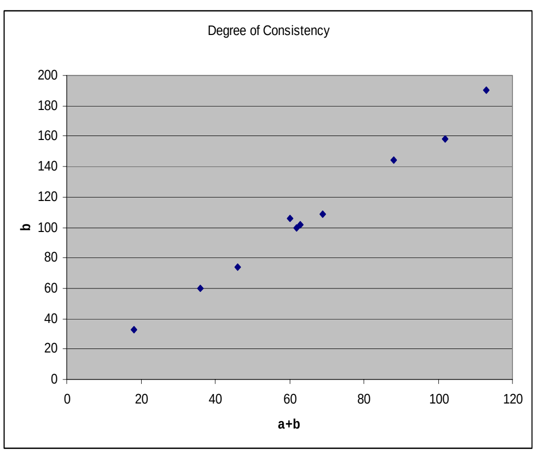
Sonata-Form Movement

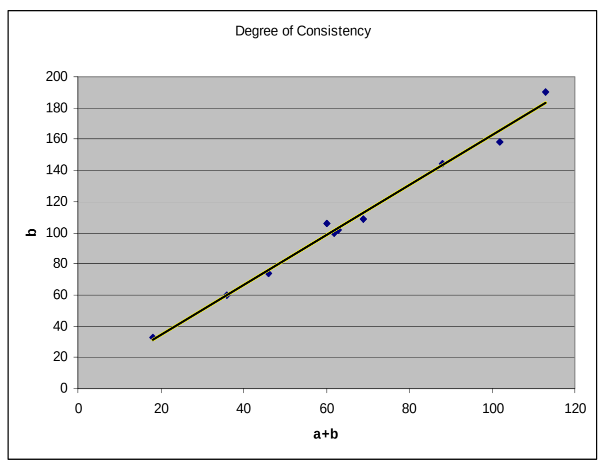
The first movement of the first sonata, K. 279, is 100 measures in length. It is divided so that the Development and Recapitulation section has a length of 62. It should be noted that the lengths of the movements are natural numbers because they measure counts. When reviewing the first movement of the first sonata, it can be seen that 100 cannot be divided any closer (using natural numbers) to the Golden Ratio than 38 and 62. This is true for the second sonata which has total length of 74 and is divided in 28 and 46. A table of some of Mozart's movements is listed below.

[23] Fischler (1981) gives a detailed description, complete with proofs of how certain data can be transformed to exhibit Golden Ratio characteristics.
[24] When Mozart was learning arithmetic, he gave himself entirely to it. His sister recalls that he once covered the walls of the staircase and of all the rooms in their house with figures, then moved to the neighbors house as well (King, 1976).
[25] Mozart wrote 19 all together.

Piece and Movement	a	b	a+b
279, I	38	62	100
279, II	28	46	74
279, III	56	102	158
280, I	56	88	144
280, II	24	36	60
280, III	77	113	190
281, I	40	69	109
281, II	46	60	106
282, I	15	18	33
282, III	39	63	102

To evaluate the consistency of the ten proportions listed above, a scatter plot of b against a+b can be used. If a composer, Mozart in this case, is consistent with using the Golden Ratio in their works, the data should be linear and fall near the line y = φx. The graph on the left represents the degree of consistency by plotting the value of b with the values of a+b. The statistical analysis for the data shows an r^2 value of .994 which confirms an extremely high degree of linearity. The graph on the right shows the linear regression of the data (represented by the yellow line and the equation y = 1.59614205x + 2.733467326), and the line y = φx (black line) overlaid on the plot of the data. The statistical analysis of the data and the graphs below show that the data is linear and the points scarcely differ from the line y = φx. This is of impressive evidence that Mozart did partition sonata movements near the Golden Section (Putz, 1995).





If a movement is divided into the Golden Section, then both a/b and b/(a+b) should be near phi. Fischer (1981) provides a theorem and the following proof that b/(a+b) is always closer to φ than a/b is.

Theorem: $\left| \{b/(a+b)\} - \varphi \right| \leq \left| (a/b) - \varphi \right|$ where $0 \leq a \leq b$.

Proof: Let x = a/b. Then show that,

$$\left| \{1/(x)\} - \varphi \right| \leq \left| (x) - \varphi \right|$$

for all x Є [0,1]. Let f(x) = 1/(x+1). By the Mean Value Theorem, for all x Є [0,1] there is a z Є (0,1) such that:

$$\left| f(x) - f(\varphi) \right| = \left| f'(z) \right| \left| x - \varphi \right|.$$

Now $f'(x) = -1/(x+1)^2$ satisfies

$$\tfrac{1}{4} < \left| f'(x) \right| < 1$$

For x Є (0,1). A simple calculation will show that φ is a fixed point of f, that is, that f(φ) = φ. So, for all x Є [0,1],

$$\left| \{1/(x+1)\} - \varphi \right| \geq \left| (x) - \varphi \right|$$

with equality when x = φ. This theorem says that the ratio of consecutive terms of any Fibonacci-like sequence ($f_1 = a$, $f_2 = b$, $f_{n+2} = f_n + f_{n+1}$ with a and b not both zero) converges to φ.

Modern Implications of the Golden Ratio and Beauty

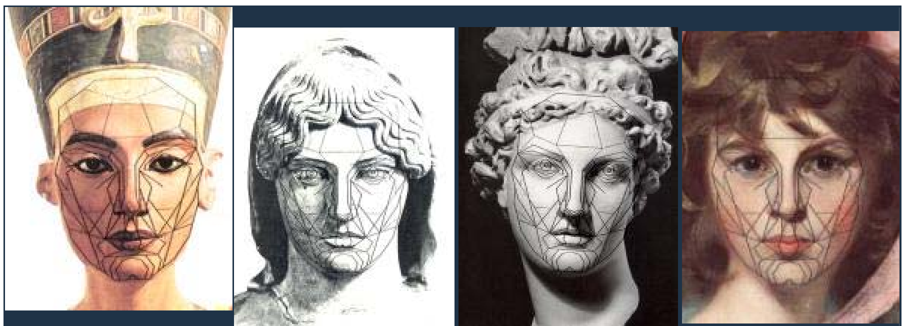
Beauty has been defined in many different ways since antiquity. A modern definition of beauty is "excelling in grace or form, charm or coloring, qualities which delight the eye and call forth that admiration of the human face in figure or other objects." Facial harmony can be activated through symmetry. Such symmetry exists when one side of the face is a mirror image of the other. The ideal face can be measured in symmetrical proportions. It should be noted that attractive faces are relatively symmetrical but not all symmetrical faces are considered beautiful (Adamson & Galli, 2003).

The Golden Ratio can also be found in human DNA structure[26] and has been found to be the only mathematical configuration that can duplicate itself ad infinitum without variance. It has been suggested that this represents a geometrically encoded instructional pattern in the brain that guides humans to recognize beauty.

The Golden Proportion can be found throughout a beautiful human face. The human head forms a Golden Rectangle with the eyes at the midpoint. The mouth and nose can each be placed at Golden Sections of the distance between the eyes and the bottom of the chin. With this information it is possible to construct a human face with dimensions exhibiting the Golden Ratio. This is exactly how some modern plastic surgeons are creating beauty. Dr. Stephen Marquardt created a Golden Decagon Mask, which is a two-dimensional visual perception of the face that has triangles with sides with ratios of 1:1.618. The Golden Decagon Mask is completed when

[26] DNA molecules are based on the Golden Ratio. A single DNA molecule measures 34 angstroms long by 21 angstroms wide for a full cycle of its double helix spiral. Both 34 and 21 are Fibonacci numbers which converge to the Golden Ratio. The double-stranded helix DNA molecule has two grooves in its spiral. The major groove measures 21 angstroms and the minor groove measure 13 angstroms, again, both are Fibonacci numbers. Another unique way that DNA is related to the Golden Number can be seen in a cross-sectional view of a DNA strand, which turns out to be a decagon. The golden properties of the decagon are discussed above.

forty-two secondary Golden Decagon matrices [27] are mathematically and geometrically positioned in the primary framework. The secondary matrices are geometrically locked on to the primary matrix by having at least two vertex radials, a vertex radial and an intersect of two vertex radials, or two intersects of vertex radials in common with the primary Golden Decagon matrix. These secondary Golden Decagon Matrices form the various features of the face (Marquardt, 2002). Below are some examples of how the Golden Ratio is perceived throughout history and through different cultures.



1350 B.C. Egypt 500 B.C. Greece 164 A.D. Rome 1794 A.D.

Asian Black Caucasian

Regardless of how the human face seems to fit into a unique geometric figure, beauty will always be defined in more ways than one. Plastic surgeons may construct beautiful faces today to fit into a Golden Decagon, but this may not always be the case. The future may lead to a new definition of beauty based on other information than the golden ratio. But it does make you wonder if "beauty is in the phi of the beholder."

Concluding Thoughts

Phi could be the world's most astonishing number. It can be found in nature, throughout history, in art, music, and architecture. Many conflicting theories exist about the origins of phi (φ); however we cannot deny the principles that accompany it. Whether it is the mathematical

[27] The secondary Golden Decagon matrices are constructed exactly the same way as the primary Golden Decagon only smaller.

relationships that seem to form around the number or the sheer aesthetics of the proportion, we must be aware that φ is all around us and rightly called the Divine Ratio.

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