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THE MONTANA MATHEMATICS ENTHUSIAST

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AIMS AND SCOPE

The Montana Mathematics Enthusiast is an eclectic journal which focuses on mathematics content, mathematics education research, interdisciplinary issues and pedagogy. The articles appearing in the journal address issues related to mathematical thinking, teaching and learning at **all** levels. The secondary focus includes specific mathematics content and advances in that area, as well as broader political and social issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is also interested in research based articles as well as historical, philosophical and cross-cultural perspectives on mathematics content, its teaching and learning.

The journal is accessed from 45+ countries and its readers include students of mathematics, future and practicing teachers, university mathematicians, mathematics educators as well as those who pursue mathematics recreationally. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed.

Manuscripts are to be submitted in electronic format to the editor preferably in APA style. The typical time period from submission to publication (including peer review) is 7-10 months. Please visit the journal website at http://www.montanamath.org/TMME

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Editorial: Growth & Change

Bharath Sriraman, Editor
The University of Montana

2006 heralds the third year of *The Montana Mathematics Enthusiast*. The journal has undergone healthy mutations since its rebirth in April 2004. We now have in place since October 2005 an illustrious international editorial board and contributing editors with a very wide range of experience and expertise. The aims, scope and editorial information link on the journal website provides this information for the interested reader. The peer review process for papers submitted to the journal has also been smooth and timely, which has helped in attracting more submissions with quality control checks in place to maintain the scholarly status of the journal. *TMME* has also begun the process of acquiring indexing in well known research databases worldwide. The website statistics for *Vol2no2 (August 2005)* and *TMME* in general have been nothing short of staggering in terms of the places from which the journal was accessed. We have thus far been accessed from 91 different countries (!) and counting. A new statistical feature on the journal website allows readers to get a rolling glimpse of countries from which the journal is accessed based on the last 100 page loads. Sample statistics on journal access during the last five months is included at the end of this issue for the interested reader. The current issue: Volume 3, no1 is both wide in scope and dense with ideas, consisting of seven articles focused on topics within mathematics; mathematics and philosophy; mathematics education history; talent development and challenges for mathematically promising students. One underlying theme of many of the articles is ways in which mathematics can stimulate us, capture our imagination, and even excite us with its possibilities for teaching and learning from the elementary school level onto the professional levels. The geographic range of the authors attests to the benefits of open access for the wide dissemination of ideas without institutional and subscription restrictions.

The first paper by Joran Elias (Montana) provides an interesting application of Wu's method of proving geometric theorems algorithmically. The paper also serves as an accessible introduction to ideas from elementary algebraic geometry for those interested in this area of mathematics. Viktor Freiman (Canada) contributes a research based article based on a 7-year longitudinal study in K-6 classrooms in Eastern Canada on ways to boost mathematical talent in the early grades. The paper provides a glimpse at the sophisticated mathematical capacities of young children once a challenging situation captures their interest. Freiman also makes novel use of Krutetskii's findings on the mathematical abilities of school children and Guy Brousseau's theory of didactical situations to illustrate how his model of boosting mathematical talent works in the mixed-ability classroom setting. One question that has perplexed researchers is how and why natural mathematical talent gets stifled in the institutionalized school setting despite the best intentions of teachers and curricula. Some conjecture that this happens because of the non-recreational and non-realistic characteristic of mathematics in the school curriculum as a student progresses from kindergarten onto high school. The physicist, George Gamov (1904-1968), also took an interest in education as evidenced in his numerous writings accessible to "lay" persons. Gamov proposed the building blocks problem as a recreational problem to determine the center of gravity of blocks laid on top of another, staggered by a fixed number. The problem is intended to provoke mathematical thought and a solution not relying on any numerical formulas although solutions can involve the use of the harmonic series and the logarithmic function. Yutaka

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Nishiyama (Japan) writes that university students majoring in the sciences are unable to solve this problem although their Calculus background provides them with (context independent) knowledge of showing how harmonic series diverge. Nishiyama argues the need to provide context to mathematics if the goal is to get students at the tertiary level excited about mathematics. Steffen Iversen (Denmark) investigates authentic situations which allow for philosophical competencies to develop in the high school mathematics classroom, and presents a conceptual model for further developing interdisciplinary connections between mathematics and philosophy. Iversen's views are pragmatic in nature and warn us to be wary of implementing interdisciplinary reform prescribed by a governmental body (the Danish ministry of Education) without fully thinking of the didactical consequences, both positive and negative. The paper presents the results from a series of qualitative interviews of high school teachers on interdisciplinary activities which integrate mathematics and philosophy at the high school level. The paper presents a didactical model for integrating math and philosophy, which Steffen Iversen and Claus Michelsen are trying to expand to the sphere of physics and other subjects of natural sciences. The next three papers explore mathematics education history and talent development. Fulvia Furinghetti (Italy) sketches important elements of mathematics education history in Italy, starting with the contributions of influential mathematicians like Guiseppe Peano, Luigi Cremona, Federigo Enriques, onto the formation of the present research community of mathematics education researchers as a result of their experiences in the international community after WWII. In this article, the reader will identify current day tension between research, practice and policy. However Furinghetti's article stresses the possibility of a mutually supportive relationship between the mathematics community and researchers engaged in mathematics education research as well as the dependency of national policy and priorities with the political history of the country. In a similar vein, in the U.S context, Linda Sheffield (USA) contributes a paper on the history of mathematics education in the U.S with an emphasis on the changes needed in current policy to maintain a technological edge in today's world. Sheffield draws attention to the growing inequity in the United States in education and examines some consequences of "squandering" opportunities of nurturing talent in mathematics and science. The paper by Agnis Andžans, Inese Berzina & Dace Bonka (Latvia) examines the role of mathematical competitions in fostering mathematical talent at the secondary level. They write that contests are of great importance in Latvia and provide a classification of suitable problems which are algorithmic in nature, are accessible to younger students and take into account recent trends in mathematics.

On a concluding note, readers are informed that Volume 3 will consist of three issues (February 2006, August 2006 and December 2006) as opposed to the normal frequency of 2/year. The third issue (Vol3, no.3) scheduled to appear in December 2006 will be a special issue focused on social justice issues in mathematics education worldwide. Putting together a special issue on this topic is a non-trivial task and will be in no means exhaustive on the topic. We are aiming towards a multitude of worldwide perspectives on the subject. Therefore the journal would like to provide readers interested in contributing articles for the special issue to contact the Editor. We especially welcome classroom teachers working with vulnerable populations to contribute short articles on *reflective practice*. A network of experienced researchers is available to provide teachers support with the writing and the review process. We have received commitments from distinguished researchers in South Africa, Australia, U.S and Europe to contribute papers to this issue and are open to practitioner's perspectives on this issue.

Automated Geometric Theorem Proving: Wu's Method

Joran Elias
University of Montana

Abstract: *Wu's Method for proving geometric theorems is well known. We investigate the underlying algorithms involved, including the concepts of pseudodivision, Ritt's Principle and Ritt's Decomposition algorithm. A simple implementation for these algorithms in Maple is presented, which we then use to prove a few simple geometric theorems to illustrate the method.*

1 Introduction

This article will discuss algebraic methods in automatic geometric theorem proving, specifically Wu's Method. Proving geometric statements algorithmically is an area of research which has particular importance in the fields of robotics and artificial intelligence. While a computer implementing Wu's Method can hardly be said to be "thinking" geometrically in the same sense as a human might, it can lend a computer the ability to interact with its physical environment in a fairly sophisticated and independent manner (see the discussion of robotic arms in [4]).

In general, we will follow the subject as presented in [1]. First, we will discuss the translation of geometric statements to the realm of algebra. After considering some examples we will move on to record some basic algebraic results needed throughout the rest of the paper. Next, we motivate Wu's Method with a brief discussion of geometry theorem proving using Groebner basis techniques. Third, we introduce the details of Wu's Method including the concepts of pseudodivision, ascending chains and characteristic sets and Ritt's Decomposition Algorithm. Next, we illustrate how Wu's Method is used to prove geometric theorems. The last section consists of a very basic implementation of Wu's Method in Maple, and its application to several examples.

Here we briefly outline Wu's Method:

- Translate a geometric theorem into a system of algebraic equations, yielding a set of hypotheses equations $f_1, \ldots, f_r$ and a conclusion g (Section 2).
- Transform our system of hypothesis equations into a triangular form using pseudodivision (Section 4.1). By triangular form, we mean that the hypothesis equations can be written as:

$$\begin{aligned} f_1 &= f_1(u_1, \ldots, u_d, x_1) \\ f_2 &= f_2(u_1, \ldots, u_d, x_1, x_2) \\ &\vdots \\ f_r &= f_r(u_1, \ldots, u_d, x_1, \ldots, x_r) \end{aligned}$$

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and the variety $V(f_1, \ldots, f_r)$ contains the irreducible components of the original variety defined by the hypothesis equations (see Section 4.2 for details on this special triangular form).

- Perform successive pseudodivision (Section 4.1.1) on the transformed hypotheses in triangular form and the conclusion equation, yielding a final remainder. If this final remainder is zero, we will say that the conclusion g follows from the hypotheses $f_1, \ldots, f_r$.
- Examine the nondegenerate conditions that arose while triangulating the hypotheses (Section 5). In particular, we conclude that g follows from the hypotheses $f_1, \ldots, f_r$ given that the nondegenerate conditions hold. These conditions take the form $p \neq 0$ where p is a polynomial that arises naturally during our triangulation process.

2 Algebraic Formulation of Geometric Theorems

To illustrate the translation of geometric statements into a suitable system of algebraic equations, we consider a few examples. The simplest place to start is the theorem stating that the intersection of the diagonals of a parallelogram in the plane bisects the diagonals (this theorem is used repeatedly as an example in both [1] and [4]). The situation we have in mind is illustrated below.

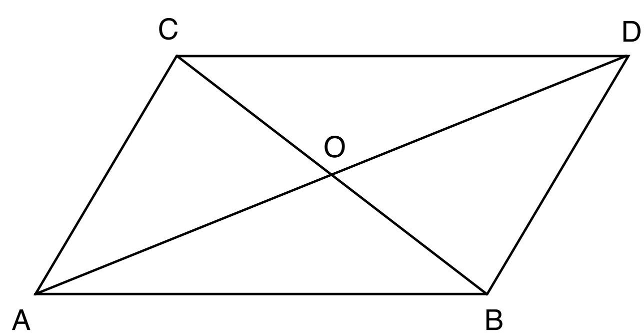


Figure 1: Parallelogram

Example 1 The basic idea is to place the figure above in the coordinate plane and then to interpret the hypotheses of the theorem as statements in coordinate, rather than Euclidean, geometry. So we begin by coordinatizing the parallelogram by placing the point A at the origin, so $A = (0, 0)$. Now we can say that the point B corresponds to $(u_1, 0)$, and that C corresponds to (u_2, u_3). The last vertex, D, is completely determined by the other three. We indicate this distinction in its coordinates by labeling D with the coordinates (x_1, x_2).

It will always be the case that some coordinates will depend upon our choices for other points. In other words, some points will be arbitrary while others

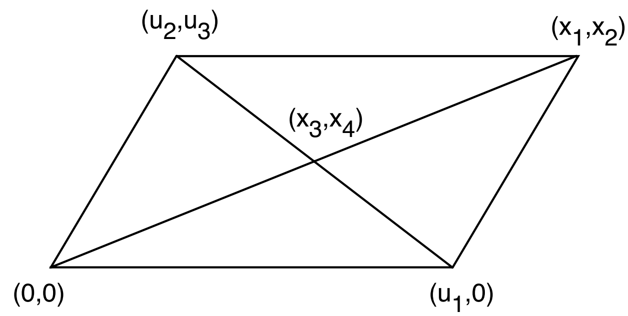


Figure 2: Coordinatized Parallelogram

will be completely determined. We will distinguish these points by using the u_i for arbitrary coordinates and the x_i for the completely determined points. Finally, the coordinates for the intersection of the diagonals, O, are also completely determined by the previous points so we let $O = (x_3, x_4)$.

The first hypothesis in our theorem is that $ABCD$ is a parallelogram. This can be restated as saying that both $\overline{AB} \| \overline{CD}$ and $\overline{AC} \| \overline{BD}$. We can translate these statements into equations by relating their slopes. For example, the slope of the line determined by the points A and B is the same as the slope of the line determined by C and D. After clearing denominators, this yields the equations:

$$\begin{aligned} x_2 - u_3 &= 0 \\ (x_1 - u_1)u_3 - x_2 u_2 &= 0 \end{aligned}$$

We label the polynomials on the left hand sides in the above equations h_1 and h_2. (The labels h_1, h_2 etc. will always refer to the *polynomials* in the equations we get upon translating our theorem. For brevity, we will not call attention to this distinction from now on. If we speak of assigning a label to an equation, we mean the polynomials as in above.) Now we must consider the assumption that O is indeed the intersection of the two diagonals. In other words, we mean that A, O, D and B, O, C are sets of collinear points. Again using the slope formula we get the equations:

$$\begin{aligned} x_4 x_1 - x_3 u_3 &= 0 \\ x_4(u_2 - u_1) - (x_3 - u_1)u_3 &= 0 \end{aligned}$$

Call these h_3 and h_4. Hence we have a system of four equations representing the hypotheses. A simple use of the distance formula gives us the following equations representing the conclusion of our theorem:

$$
\begin{aligned}
x_1^2 - 2x_1x_3 - 2x_4x_2 + x_2^2 &= 0 \\
2x_3u_1 - 2x_3u_2 - 2x_4u_3 - u_1^2 + u_2^2 + u_3^2 &= 0
\end{aligned}
$$

which we label g_1 and g_2. So the algebraic version of our theorem states that $g_1 = 0$ and $g_2 = 0$ should hold whenever $h_1 = 0, h_2 = 0, h_3 = 0, h_4 = 0$ also hold.

Note that our conclusion is represented by two equations, not just one. In general, our conclusion may involve several algebraic equations.

See Example 2 in Section A for a demonstration of the remaining steps in Wu's Method.

The following two examples are taken from exercises in [4].

Example 2 Another standard geometry theorem states that the altitudes of a triangle $\triangle ABC$ all meet in a single point, H, called the orthocenter (see Figure 3).

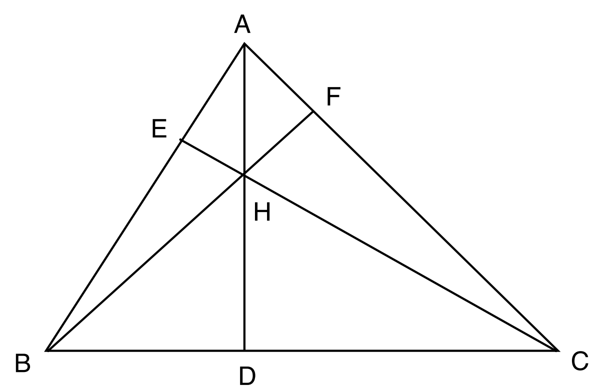


Figure 3: Orthocenter Diagram

First we construct the triangle in the coordinate plane by letting $A = (u_2, u_3)$, $B = (0, 0), C = (u_1, 0)$, as in Figure 3. Next we construct the altitudes. For example, if we let D be the point given by $(u_2, 0)$ then the line segment $\overline{AD}$ is the altitude from A. The other two altitudes require more work.

Let $E = (x_1, x_2)$ and $F = (x_3, x_4)$ be points such that $\overline{BF}, \overline{CE}$ are the altitudes from B, C respectively. This means that we must have B, E, A and C, F, A collinear. Also, we must have $\overline{CE} \perp \overline{AB}, \overline{BF} \perp \overline{AC}$. This yields the following four hypotheses:

$$
\begin{aligned}
x_2u_2 - x_1u_3 &= 0 \\
x_4(u_2 - u_1) - u_3(x_3 - u_1) &= 0 \\
x_2u_3 + u_2(x_1 - u_1) &= 0 \\
x_4u_3 + x_3(u_2 - u_1) &= 0
\end{aligned}
$$

labeling the polynomials as h_1, h_2, h_3 and h_4. Now, we want to conclude that all three altitudes meet at a single point. Hence we construct the following two additional points: $G = (u_2, x_5)$ and $H = (u_2, x_6)$. We intend that G should be the intersection of $\overline{AD}$ and $\overline{CE}$ while H should be the intersection of the line segments $\overline{AD}$ and $\overline{BF}$. Hence we need the additional hypotheses that G, E, C and H, B, F are collinear yielding the following two equations:

$$\begin{aligned} (x_2 - x_5)(x_1 - u_1) - x_2(x_1 - u_2) &= 0 \\ x_6 x_3 - x_4 u_2 &= 0 \end{aligned}$$

which we call h_5 and h_6. Finally, our conclusion becomes the assertion that the points G and H are in fact identical. Hence, we get the equation:

$$x_5 - x_6 = 0.$$

Call this polynomial g. We should mention here that the translation of geometric problems is in general much more difficult than establishing their validity algorithmically. For example, it should be clear from our examples that we could have performed these translations in slightly different ways. We frequently have a certain degree of latitude in translating geometry theorems. While this will typically not alter the validity of the conclusion (for an exception see Example 6 in Appendix A) some translations may be substantially easier to work with. For these reasons, a human is usually needed to perform the translation accurately.

A common difficulty that arises while translating theorems is that the typical statement of geometry theorems contains implicit assumptions that are easy to overlook. As an example of what can go wrong, consider the following example.

Example 3 Let $\triangle ABC$ be a triangle in the plane. Construct three points A', B', C' so that $\triangle ABC', \triangle AB'C, \triangle A'BC$ are equilateral triangles. This situation we have in mind is illustrated below (ignore imperfections in the figure).

A theorem of classical geometry states that the line segments $\overline{AA'}, \overline{BB'}, \overline{CC'}$ all meet at a single point, S, called the Steiner point.

If we translate the theorem directly as stated above, and then attempt to use the methods described below to prove the theorem, we will fail. The reason is that we tacitly assumed that the point A' should be on a specific side of the segment $\overline{AC}$ (and similarly for B', C'). We could have constructed the figure with the equilateral triangles "folded over" so that they overlapped the original triangle:

This construction is consistent with the theorem (again ignoring imperfections in the figure), but it is obviously not what we intended. Indeed, in this case the three lines in question do not meet in a single point S. If we reformulate the theorem in such a way that this alternate construction is excluded, then

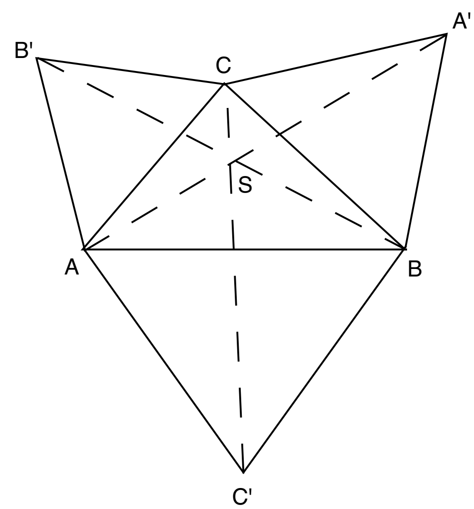


Figure 4: Steiner Point Theorem

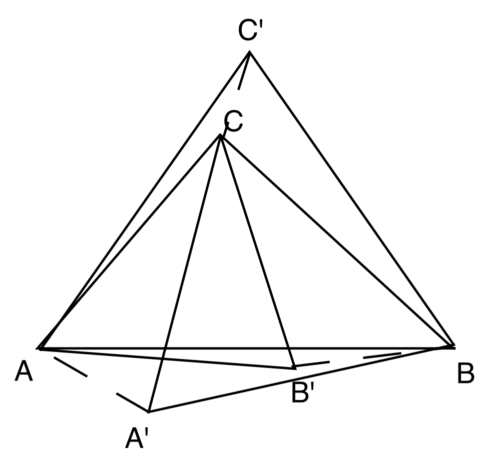


Figure 5: Incorrect Steiner Point Theorem

Wu's Method will be successful. Specifically, we could include the hypothesis that the distance from A to A' is equal to the sum of the distances from A to S and from S to A', which is easily translated using the distance formula.

Now that we've seen how to translate plane geometry theorems into systems of algebraic equations, in the next section we will summarize the algebraic results assumed for the rest of the article. Then we will specify what it means for an algebraic equation to "follow" from a system of additional algebraic equations (see Section 3.2).

3 Preliminaries

3.1 Algebraic Results

Here we set out the prerequisite notation and results from algebra that we will need in developing the notions underlying Wu's Method. In general we assume the reader is familiar with basic results involving rings, fields, ideals, prime and radical ideals, and algebraic and transcendental field extensions. If the reader is interested in proofs of these results, see [4], or any standard algebra text (e.g. [5]).

Let k be a field and denote by $k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k. Similarly, $k(x_1, \ldots, x_n)$ is the field of rational functions of k in n variables. We need the following theorem due to Hilbert,

Theorem 3.1 (Hilbert Basis Theorem)**.** *Every ideal I of $k[x_1, \ldots, x_n]$ is finitely generated, or equivalently, $k[x_1, \ldots, x_n]$ has no infinite strictly increasing sequences of ideals.*

In particular, given any ideal I in $k[x_1, \ldots, x_n]$, we can write $I = \langle f_1, \ldots, f_r \rangle$ where the f_i are a finite set of polynomials. We denote the radical of the ideal I by $\sqrt{I}$.

We say that a field F is an **extension** of the field k if k is a subfield of F. Let F be an extension of k and let α be an element of F. Then α is said to be **algebraic** over k if it is the root of some nonzero polynomial with coefficients in k. Otherwise, α is **transcendental**. Let $\alpha_1, \ldots, \alpha_r$ be elements of an extension F, of k. The subfield generated by $\alpha_1, \ldots, \alpha_r$ over k is denoted by $k(\alpha_1, \ldots, \alpha_r)$ (the respective subring is given by $k[\alpha_1, \ldots, \alpha_r]$). We need the following theorem.

Theorem 3.2. *Let F be an extension of the field k and let $\alpha \in F$. If α is algebraic over k then,*

(i) $k(\alpha) = k[\alpha]$

(ii) $k(\alpha) \cong k[x]/\langle f \rangle$ *where x is an indeterminate and f is an irreducible polynomial of degree $n \geq 1$ and $f(\alpha) = 0$.*

(iii) Every element of $k(\alpha)$ can be expressed uniquely in the form $c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha + c_0$, where $c_i \in k$.

We also need the ability to factor polynomials in our polynomial ring, and also in algebraic extensions, so we include the following theorems.

Theorem 3.3. *If D is a unique factorization domain, then so is the polynomial ring $D[x_1, \ldots, x_n]$. In particular, $k[x_1, \ldots, x_n]$ is a UFD.*

Theorem 3.4. *Let D be a UFD with quotient field k. Let α be in any extension of k that is algebraic over k. If there is an algorithm for factoring in D then,*

(i) there is an algorithm for factorization in the polynomial rings $D[x]$ and $k[x]$.

(ii) there is an algorithm for factorization in the polynomial ring $k(\alpha)[x]$.

This last theorem is certainly not trivial. For proofs see [9], or [10, Section 25]. Chou developed an algorithm in [2] for factoring polynomials over successive quadratic extensions over fields of rational functions that worked efficiently for most of the geometry theorems proved in [1].

We also need some basic results from affine algebraic geometry. Again, let F be an extension of the field k and let $k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k.

Definition 3.5. *Given a nonempty set of polynomials $S \subset k[x_1, \ldots, x_n]$, the* ***variety*** *$V(S)$ is defined to be the set of common zeroes of all the elements of S, i.e.*

$$V(S) = \{(a_1, \ldots, a_n) \in k^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in S\}$$

We can define varieties in terms of ideals as well. If I is the ideal generated by the polynomial set S in $k[x_1, \ldots, x_n]$ then $V(S) = V(I)$ and by the Hilbert Basis Theorem we can write, $V(I) = V(f_1, \ldots, f_r)$ where the ideal I is generated by the f_i. Hence, every algebraic variety is the set of common zeroes of a finite polynomial set.

We may also define an ideal using a nonempty subset U of k^n by letting

$$I(U) = \{f \mid f \in k[x_1, \ldots, x_n] \text{ and } f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in U\}$$

The following useful properties of V and I are easy to check: $S \subset I(V(S))$ and $U \subset V(I(U))$.

Proposition 3.6. *Let S_1 and S_2 be polynomial sets and S_1S_2 be the set of all products of an element of S_1 with an element of S_2. Then,* *(i)* $V(S_1 \cup S_2) = V(S_1) \cap V(S_2)$

(ii) $V(S_1 S_2) = V(S_1) \cup V(S_2)$

It is often possible to decompose varieties into unions of smaller varieties.

Definition 3.7. *A nonempty variety V is **irreducible** if whenever V is written in the form $V = V_1 \cup V_2$ where V_1, V_2 are varieties, then either $V = V_1$ or $V = V_2$.*

Definition 3.8. *Let V be a variety. A decomposition $V = V_1 \cup \cdots \cup V_s$, where each V_i is irreducible and $V_i \not\subset V_j$ for all $i \neq j$ is called a **minimal decomposition**.*

Note that the irreducibility of a variety depends on whether or not k is algebraically closed.

When k is algebraically closed we have the following convenient characterization of irreducible varieties,

Proposition 3.9. *Let V be a nonempty variety over an algebraically closed field k. Then V is irreducible if and only if $I(V)$ is a prime ideal. If k is not algebraically closed, the converse still holds.*

Theorem 3.10. *Let V be a variety. Then V has a minimal decomposition, $V = V_1 \cup \cdots \cup V_s$, and this decomposition is unique up to the order in which the V_i are written.*

Definition 3.11. *The **dimension of a prime ideal** P (also known as its co-height) is the transcendence degree of the quotient field of the integral domain $k[x_1, \ldots, x_n]/P$ over the field k. Equivalently, its dimension is the supremum of the lengths of chains of distinct prime ideals containing P. The dimension of an irreducible variety V is the dimension of its prime ideal $I(V)$. The dimension of a (reducible) variety V is the highest dimension of one of its components.*

The following definition is crucial in light of our distinction between dependent and independent variables when translating geometric theorems.

Definition 3.12. *Let V be an irreducible variety with $P = I(V)$ its prime ideal. Let U be a subset of the variables x_i in the ring $k[x_1, \ldots, x_n]$. The variables in U are said to be **algebraically independent** on V if P does not contain a nonzero polynomial involving only variables from U. Otherwise, the variables in U are said to be **algebraically dependent**.*

Definition 3.13. *A **generic zero** of an ideal $I \lhd k[x_1, \ldots, x_n]$ is a zero $\alpha = (a_1, \ldots, a_n)$ of I in an extension of k such that $f \in I$ if and only if $f(a_1, \ldots, a_n) = 0$.*

Theorem 3.14. *An ideal I has a generic zero α in some extension of k if and only if it is a proper prime ideal.*

Proof. First suppose that I has a generic zero α in some extension of k. Since $1 \notin I$, I is proper. Let f, g be polynomials such that $fg \in I$. Then $(fg)(\alpha) = f(\alpha)g(\alpha) = 0$, which implies that either $f(\alpha)$ or $g(\alpha)$ is zero. Hence either f or g must be in I, so I is prime.

Now suppose that I is a proper prime ideal. Let $R = k[x_1, \ldots, x_n]_I$ be the localization of $k[x_1, \ldots, x_n]$ at I, and consider the field R/I_I containing k. Let $\alpha = (\bar{x}_1, \ldots, \bar{x}_n)$ where $\bar{x}_i \in R/I_I$ is the canonical image of x_i. So α is the canonical image under the mappings:

$$x_i \mapsto \frac{x_i}{1} \mapsto \frac{x_i}{1} + I_I = \bar{x}_i$$

We claim that α is a generic zero of I. To see this, let $f \in I$. Then $f = \sum_J a_J x_J$ where each x_J is a product of the variables x_i and $a_J \in k$. Evaluating at α we get:

$$f(\alpha) = \sum_J a_J \bar{x}_J = \sum_J a_J x_J + I_I = 0$$

The last equality above holds since $\sum_J a_J x_J \in I \subset I_I$.

Now, for an arbitrary $g \in k[x_1, \ldots, x_n]$, suppose that $g(\alpha) = 0$. This implies (by the equalities above) that in fact $g \in I_I$. So $g = \sum_{i=1}^{r} \frac{h_i}{p_i} f_i$ where $p_i \notin I$ and $f_i \in I$. So we have that $p_1 \cdots p_r g \in I$, and since I is prime and $p_i \notin I$, we conclude that $g \in I$. $\square$

Corollary 3.15. *If $\alpha = (a_1, \ldots, a_n)$ is a generic zero of I, then $k[a_1, \ldots, a_n]$ is isomorphic to the quotient ring $k[x_1, \ldots, x_n]/I$ under the mapping $a_i \mapsto \tilde{x}_i$ where $\tilde{x}_i$ is the canonical image of x_i in $k[x_1, \ldots, x_n]/I$. Also, $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is a generic zero of I and the dimension of I is the transcendence degree of $a_1, \ldots, a_n$ over k.*

Proof. That the mapping described in the corollary is an isomorphism is easily checked. Suppose that $f(\tilde{x}_1, \ldots, \tilde{x}_n) = 0$ in $k[x_1, \ldots, x_n]/I$. By our isomorphism, we have that $f(a_1, \ldots, a_n) = 0$, and hence $f \in I$. Also, if $f \in I$ then $f(\tilde{x}_1, \ldots, \tilde{x}_n) = 0$. Hence $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is a generic zero of f. Finally, the dimension of I is just the transcendence degree of $\mathrm{Frac}(k[x_1, \ldots, x_n]/I) \cong k[\tilde{x}_1, \ldots, \tilde{x}_n]$ over k and our isomorphism shows that this is the same as the transcendence degree of $k(a_1, \ldots, a_n)$ over k. $\square$

Remark The best way to interpret this degree is the size of any maximally algebraically independent subset of $a_1, \ldots, a_n$.

For the following results, and henceforth, we assume that k is algebraically closed. There are two equivalent forms of Hilbert's Nullstellensatz and one important consequence (we present them as in [1]).

Theorem 3.16 (Hilbert's Weak Nullstellensatz)**.** *If I is a proper ideal in $k[x_1, \ldots, x_n]$ then $V(I) \neq \emptyset$.*

Theorem 3.17 (Hilbert's Strong Nullstellensatz)**.** *Given any ideal I in the polynomial ring $k[x_1, \ldots, x_n]$, we have that $I(V(I)) = \sqrt{I}$.*

Proposition 3.18. *If P is a proper prime ideal in $k[x_1, \ldots, x_n]$ then $V(P)$ is irreducible and $I(V(P)) = P$.*

3.2 Proving Translated Theorems

We have seen that we can translate a geometric theorem into a system of algebraic equations in the ring $k[u_1, \ldots, u_d, x_1, \ldots, x_r]$: $h_1, \ldots, h_r$ (the hypotheses) and $g_1, \ldots, g_s$ (the conclusions). From now on we will assume that our translation only yielded one conclusion ($s = 1$) since we can always consider each g_i individually. In what sense then does our conclusion, g, follow from the hypotheses, $h_1, \ldots, h_r$?

The basic idea is that we want g to be satisfied by every point that satisfies $h_1, \ldots, h_r$. In other words, we want every point in the variety defined by the hypotheses to satisfy g. Hence we start with the following definition.

Definition 3.19. *The conclusion g **follows strictly** from the hypotheses $h_1, \ldots, h_r$ if $g \in I(V) \subset k[u_1, \ldots, u_d, x_1, \ldots, x_r]$ where $V = V(h_1, \ldots, h_r)$.*

We will briefly investigate a straightforward attempt to use this definition which will serve to motivate both a revised definition and the practicality of Wu's Method. The techniques employed for this brief discussion rest upon Groebner Basis methods that we will not treat in any detail here. If the reader is unfamiliar with the concepts used below, see [4]. We use this approach simply because it allows us a direct way to motivate Definition 3.21.

In general, the field k may not be algebraically closed, so we cannot rely on computing $I(V)$ directly using Hilbert's Nullstellensatz. We can, however, use the following test.

Proposition 3.20. *If $g \in \sqrt{(h_1, \ldots, h_r)}$, then g follows strictly from $h_1, \ldots, h_r$.*

Proof. The hypothesis $g \in \sqrt{(h_1, \ldots, h_r)}$ means that $g^s \in \langle h_1, \ldots, h_r \rangle$ for some s. Hence $g^s = \sum_{i=1}^{n} A_i h_i$, where $A_i \in k[u_1, \ldots, u_d, x_1, \ldots, x_r]$. Then g^s must vanish whenever the h_i vanish, and hence g does as well. □

This test is useful because we have an algorithm for determining if $g \in \sqrt{(h_1, \ldots, h_r)}$.[1] Let us recall Example 1, and consider attempting to show that the first conclusion follows from our hypotheses. Hence we have the following hypotheses:

[1] Specifically, we have containment if and only if $\{1\}$ is the reduced Groebner basis for the ideal $\langle h_1, \ldots, h_r, 1 - yg \rangle \subset k[u_1, u_2, u_3, x_1, x_2, x_3, x_4, y]$. See Chapter 6 Section 4 in [4] for more details.

$$
\begin{aligned}
h_1 &= x_2 - u_3\\
h_2 &= (x_1 - u_1)u_3 - x_2u_2\\
h_3 &= x_4x_1 - x_3u_3\\
h_4 &= x_4(u_2 - u_1) - (x_3 - u_1)u_3.
\end{aligned}
$$

The conclusion we are interested in is

$$g_1 = x_1^2 - 2x_1x_3 - 2x_4x_2 + x_2^2.$$

To use Proposition 3.20 we compute a Groebner basis for the ideal, $\langle h_1, h_2, h_3, h_4, 1 - yg_1\rangle$ in the polynomial ring $\mathbb{R}[u_1, u_2, u_3, x_1, x_2, x_3, x_4, y]$. Unfortunately, we do not get the Groebner basis $\{1\}$ as we should. The cause of our problem lies in the variety defined by the hypotheses: $V(h_1, h_2, h_3, h_4)$. If one computes a Groebner basis for these four equations one sees [2] that this variety is actually reducible. In particular, after some calculation we see that the variety defined by our hypotheses actually has four components, $V = V' \cup U_1 \cup U_2 \cup U_3$ defined by:

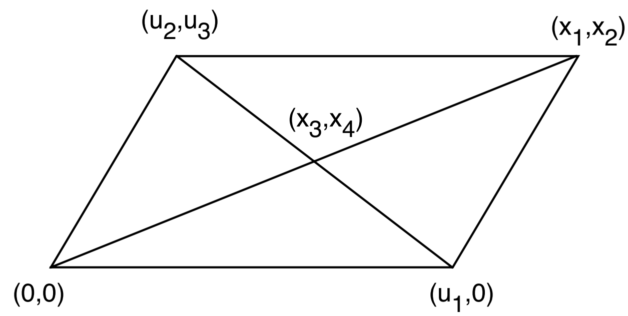
$$
\begin{aligned}
V' &= V\left(x_1 - u_1 - u_2, x_2 - u_3, x_3 - \frac{u_1 + u_2}{2}, x_4 - \frac{u_3}{2}\right)\\
U_1 &= V(x_2, x_4, u_3)\\
U_2 &= V(x_1, x_2, u_1 - u_2, u_3)\\
U_3 &= V(x_1 - u_2, x_2 - u_3, x_3u_3 - x_4u_2, u_1).
\end{aligned}
$$

Our original strategy revolved around showing that the conclusion, $g_1 = x_1^2 - 2x_1x_3 - 2x_4x_2 + x_2^2$, vanishes on the variety defined by our hypotheses. But this clearly cannot happen on some of the components above. Consider the U_i. Each has as one of its defining equations an expression that involves only the u_i. But now recall our construction of our theorem concerning the diagonals of a parallelogram

In our construction, the coordinates corresponding to the u_i were intended to be arbitrary. But in U_1 for example, we must have $u_3 = 0$. In this case, we won't have a genuine parallelogram. It now becomes clear that $u_3 = 0$ is a degenerate case of our diagram. Since each of the U_i contain equations that involve only the u_i, each U_i corresponds to degenerate cases of our theorem. If we repeated our approach using only the component V', then Proposition 3.20 will work as we intended.

Now it should be clear that our goal is to develop a general method for establishing the validity of our conclusion only on those components of V that do not correspond

[2] In fact, we get $\{x_1x_4 + x_4u_1 - x_4u_2 - u_1u_3, x_1u_3 - u_1u_3 - u_2u_3, x_2 - u_3, x_3u_3 + x_4u_1 - x_4u_2 - u_1u_3, x_4u_1^2 - x_4u_1u_2 - \frac{1}{2}u_1^2u_3 + \frac{1}{2}u_1u_2u_3, x_4u_1u_3 - \frac{1}{2}u_1u_3^2\}$, which is reducible. Specifically, we can factor three of these equations.



to degenerate cases of our theorem. In other words, we are only interested in those components of V on which the u_i are algebraically independent. Let us revise Definition 3.19 accordingly.

Definition 3.21. *A conclusion g* ***follows generically*** *from the hypotheses $h_1, \ldots, h_r$ if $g \in I(V') \subset k[u_1, \ldots, u_d, x_1, \ldots, x_r]$ where V' is the union of those irreducible components of $V(h_1, \ldots, h_r)$ on which the u_i are algebraically independent.*

Now that we have a clearer definition to work with we can move on to discuss Wu's Method. The approach we used in Proposition 3.20 relied upon Groebner Basis techniques. While it is possible to design theorem provers around these techniques Wu's Method is tailored more specifically to the task and hence is often more computationally efficient (see [3],[6],[7]).

4 Wu's Method

4.1 Pseudodivision

The primary tool in Wu's Method is a variation on the division algorithm for multivariable polynomials (see [4] for a description) called pseudodivision. Let $f, g \in k[x_1, \ldots, x_n, y]$, with $g = a_p y^p + \cdots + a_0$ and $f = b_m y^m + \cdots + b_0$, where the a_i, b_j are polynomials in the $x_1, \ldots, x_n$. Then we have the following result.

Proposition 4.1. *Let f, g be as above and assume that $m \leq p$ and that $f \neq 0$. Then,*

(i) There is an equation

$$b_m^s g = qf + r$$

where $q, r \in k[x_1, \ldots, x_n, y]$, $s \geq 0$, and r is either the zero polynomial or its degree in y is less than m.

(ii) r is in the ideal $\langle f, g \rangle$ in the ring $k[x_1, \ldots, x_n, y]$.

Proof. First, we will use the notations $\deg(f, y)$ and $\textsc{lc}(f, y)$ to denote the degree of f in the variable y and the leading coefficient of f as a polynomial in y. We will establish the proposition using the following algorithm:

Input: f, g
Output: r, q

$r := g, q := 0$
While $r \neq 0$ and $\deg(r, y) \geq m$ Do
$r := b_m r - \textsc{lc}(r, y) f y^{\deg(r,y)-m}$
$q := b_m q + \textsc{lc}(r, y) y^{\deg(r,y)-m}$

We begin by using induction to show that the first part of (i) holds at each iteration of the above algorithm, or that after the i^{th} iteration we have $b_m^i g = q_i f + r_i$. For the base case, consider the situation after one time through the above algorithm. We get that

$$q_1 f + r_1 = a_p y^{p-m} f + b_m g - a_p y^{p-m} f = b_m g.$$

So indeed we have that $b_m g = q_1 f + r_1$. Now suppose that $b_m^i g = q_i f + r_i$ and consider what happens on interation $i + 1$. We get:

$$\begin{aligned}
q_{i+1} f + r_{i+1} &= \left(b_m q_i + \textsc{lc}(r_i, y) y^{\deg(r_i,y)-m}\right) f + \left(b_m r_i - \textsc{lc}(r_i, y) f y^{\deg(r_i,y)-m}\right) \\
&= b_m q_i f + b_m r_i \\
&= b_m (q_i f + r_i) \\
&= b_m^{i+1} g.
\end{aligned}$$

The assertion that either $r = 0$ or $\deg(r, y) < m$ follows from the While statement in the algorithm assuming that the algorithm terminates. Now we show that the algorithm terminates. The claim is that the degree of r_i in y is strictly decreasing with each iteration of the algorithm. To see this, consider r_{i+1}.

$$r_{i+1} = b_m r_i - \textsc{lc}(r_i, y) f y^{\deg(r_i,y)-m}.$$

Now, the highest y-degree term in both $b_m r_i$ and $\textsc{lc}(r_i, y) f y^{\deg(r_i,y)-m}$ are both of degree $\deg(r_i, y)$, and they have the same coefficient. Hence these terms cancel, meaning that the degree of r_{i+1} in y is strictly less than that of r_i. Hence the algorithm does terminate. Part (ii) follows trivially. ∎

The proof of Proposition 4.1 shows that if the variable x_i does not occur in f then $\deg(r, x_i)$ and $\deg(q, x_i)$ are less than or equal to $\deg(g, x_i)$.

Note that this algorithm outputs a unique q, r. However, if no restrictions (beyond being nonegative) are placed on the exponent s then there are not unique q, r such that $b_m^s g = qf + r$. In particular, q and r are unique if s is minimal(For a brief discussion of this, see Chapter 6 of [4]). For our purposes, it is enough that our algorithm outputs a unique q, r. Hence, we denote the remainder on pseudodivision (pseudoremainder) of f by g with respect to the variable y by $\text{prem}(f, g, y)$.

4.1.1 Successive Pseudodivision

The critical use of the pseudodivision algorithm comes in performing successive pseudodivision. Suppose that $f_1, \ldots, f_r$ are a set of hypothesis equations that are in triangular form, so that we can write them as:

$$\begin{aligned} f_1 &= f_1(u_1, \ldots, u_d, x_1) \\ f_2 &= f_2(u_1, \ldots, u_d, x_1, x_2) \\ &\vdots \\ f_r &= f_r(u_1, \ldots, u_d, x_1, \ldots, x_r). \end{aligned}$$

Let $g = g(u_1, \ldots, u_d, x_1, \ldots, x_r)$ be our conclusion equation. Performing successive pseudodivision simply involves the following: set $R_r = g$, $R_{r-1} = \text{prem}(R_r, f_r, x_r)$, $R_{r-2} = \text{prem}(R_{r-1}, f_{r-1}, x_{r-1}), \ldots$ etc. Continuing in this fashion, we get a final remainder $R_0 = \text{prem}(R_1, f_1, x_1)$. R_0 is called the final remainder upon successive pseudodivision of g by $f_1, \ldots, f_r$ and is denoted $\text{prem}(g, f_1, \ldots, f_r)$. We have the following result.

Proposition 4.2. *Suppose that the polynomials $f_1, \ldots, f_r$ are in triangular form and $g = g(u_1, \ldots, u_d, x_1, \ldots, x_r)$ is our conclusion. Let $R_0 = prem(g, f_1, \ldots, f_r)$ and let d_j be the leading coefficient of f_j as a polynomial in x_j. Then*

(i) There exist integers $s_1, \ldots, s_r \geq 0$ and polynomials $A_1, \ldots, A_r$ such that

$$d_1^{s_1} \cdots d_r^{s_r} g = A_1 f_1 + \cdots + A_r f_r + R_0$$

(ii) Either $R_0 = 0$ or $deg(R_0, x_i) < deg(f_i, x_i)$ for $i = 1, \ldots, r$.

Proof. To establish (i) and (ii) we use induction on r. If $r = 1$ then we are simply performing normal pseudodivision (see Proposition 4.1) and the result holds. Suppose that (i) and (ii) hold for $r - 1$, so that we have

$$d_1^{s_1} \cdots d_{r-1}^{s_{r-1}} R_{r-1} = A_1 f_1 + \cdots + A_{r-1} f_{r-1} + R_0$$

with $\deg(R_0, x_i) < \deg(f_i, x_i)$ for $i = 1, \ldots, r-1$. Now note that R_{r-1} can also be written $R_{r-1} = d_r^{s_r} g - A_r f_r$ and substitute this into the equation above. The result follows. □

Example For a simple illustration of this process consider the following system of equations in triangular form:

$$\begin{aligned} f_1 &= u_1x_1 - u_1u_3 \\ f_2 &= u_3x_2 - (u_2 - u_1)x_1 \\ f_3 &= (u_3x_2 - u_2x_1 - u_1u_3)x_3 + u_1u_3x_1 \\ f_4 &= u_3x_4 - u_2x_3 \end{aligned}$$

and let $g = 2u_2x_4 + 2u_3x_3 - u_3^2 - u_2^2$. Now if we perform successive pseudodivision on this system we get:

$$\begin{aligned} R_3 =& \text{prem}(g, f_4, x_4) = (2u_3^2 + 2u_2^2)x_3 - u_3^3 - u_2^2u_3 \\ R_2 =& \text{prem}(R_3, f_3, x_3) = (-u_3^4 - u_2^2u_3^2)x_1 + \\ & ((u_2 - 2u_1)u_3^3 + (u_2^3 - 2u_1u_2^2)u_3)x_1 + u_1u_3^4 + u_1u_2^2u_3^2 \\ R_1 =& \text{prem}(R_2, f_2, x_2) = (-u_1u_3^4 - u_1u_2^2u_3^2)x_1 + u_1u_3^5 \\ & + u_1u_2^2u_3^3 \\ R_0 =& \text{prem}(R_1, f_1, x_1) = 0. \end{aligned}$$

Since the final remainder upon successive pseudodivision is zero, we have shown that g follows from the hypothesis equations f_1, f_2, f_3, f_4.

Remark - We can still calculate $\text{prem}(g, f_1, \ldots, f_r)$ even if the f_i are not quite in triangular form. Specifically, as long as the leading variables in each f_i are distinct we can find $\text{prem}(g, f_1, \ldots, f_r)$ inductively by defining it to be $\text{prem}(\text{prem}(g, f_2, \ldots, f_r), f_1)$. The above remainder formula still holds. The reason for presenting successive pseudodivision in the context of a system in triangular form is that this will be the form our system will be in when actually performing Wu's Method (see the discussion of the Dimensionality Requirement following Definition 4.13).

4.2 Ascending Chains and Characteristic Sets

The next several sections focus on specifically how Wu's Method takes our hypothesis equations and transforms them into a triangular form. To do this we need to discuss the notions of ascending chains and characteristic sets. First we introduce some notation: all polynomials under consideration are in $k[x_1, \ldots, x_n]$ (here we temporarily abandon our distinction between the u_i and x_i to simplify our notation). We say that the **class** of a polynomial f, denoted class(f), is the smallest integer c such that $f \in k[x_1, \ldots, x_c]$. If $f \in k$ then class$(f) = 0$. We call x_c the **leading variable** of f, denoted $\text{LV}(f)$. Similarly, we say that $\text{LC}(f)$ is the **leading coefficient** of f as a polynomial in x_c. We will sometimes refer to this coefficient as the **initial** of f. Also, the degree of f in its leading variable is denoted $\text{LD}(f)$.

A polynomial g is **reduced** with respect to f if $\deg(g, x_c) < \deg(f, x_c)$ where class$(f) = c > 0$. In other words, $\text{prem}(g, f, x_c) = g$. Note that by our pseudo-division algorithm, $\text{prem}(g, f, x_c)$ is always reduced with respect to f. Also, for any finite set of polynomials, $f_1, f_2, \ldots, f_r$, we say that g is **reduced with respect** to $f_1, f_2, \ldots, f_r$ if $\deg(g, x_i) < \deg(f_i, x_i)$ for each $1 \leq i \leq r$ where x_i is the leading variable of each f_i.

The basic ideas introduced here are that ascending chains are polynomial sets that are close to being triangular, and characteristic sets will be defined to be "minimal" ascending chains in a sense to be explained below.

Definition 4.3. *Let $C = f_1, f_2, \ldots, f_r$ be a sequence of polynomials in $k[x_1, \ldots, x_n]$. It is a **quasi-ascending chain** if either*

(i) $r = 1$ and $f_1 \neq 0$ or,

(ii) $r > 1$ and $0 < class(f_1) < \cdots < class(f_r)$.

*We say that a quasi-ascending chain is an **ascending chain** if f_j is reduced with respect to f_i for all $i < j$.*

Note that in a quasi-ascending chain, f_j is automatically reduced with respect to f_i for all $i > j$. So in an ascending chain, f_j is reduced with respect to f_i for all $i \neq j$.

We will briefly illustrate this definition with a few examples.

Example The set $\{f_1 = y_1^5, f_2 = y_1^6 + y_2\}$ is not an ascending chain since the degree of f_2 in y_1 is greater than that in f_1 (it is still a quasi-ascending chain). However, the set $\{f_1 = y_1^2, f_2 = y_1 + y_2^3\}$ is an ascending chain.

Example If $f_1, \ldots, f_n$ is an ascending chain, then f_j is reduced with respect to f_i for all $i < j$. Specifically, this means that the variable x_i must appear with a lower degree in f_j than it does in f_i, for each $i < j$. In particular, this implies that the class variable of f_i appears to a lower degree in the initial of f_j. Hence, the initials of f_j are reduced with respect to f_i for $i < j$.

Example Additionally, if $f_1, \ldots, f_n$ is an ascending chain, then since the initials of the f_j are reduced with respect to all the previous elements of the ascending chain, then we must have that $\text{prem}(d_i, f_1, \ldots, f_n) \neq 0$ for $i = 1, \ldots, n$ (Here

d_i is the initial, or leading coefficient of f_i). This can be seen if we use the recursive definition of successive pseudodivision. Since d_i is reduced with respect to $f_1, \ldots, f_{i-1}$ we have that $\text{prem}(d_i, f_1, \ldots, f_{i-1}) = d_i$. And since d_i is clearly reduced with respect to the remaining polynomials in the ascending chain we get that $\text{prem}(d_i, f_1, \ldots, f_n) = d_i \neq 0$.

Now we define the following partial ordering on the ring $k[x_1, \ldots, x_n]$.

Definition 4.4. *Given $f, g \in k[x_1, \ldots, x_n]$ we say that $f < g$ (g is **higher**, or of **higher rank**) if either*

(i) class$(f) <$ class(g), or

(ii) class$(f) =$ class(g) and $\text{LD}(f) < \text{LD}(g)$.

*Polynomials f and g have the **same rank** if they are not comparable, i.e. if class$(f) =$ class(g) and $\text{LD}(f) = \text{LD}(g)$.*

Note that distinct polynomials may have the same rank.

Proposition 4.5. *The partial ordering $<$ defined above on $k[x_1, \ldots, x_n]$ is a well-ordering. In other words, under this ordering, every set has a (not necessarily unique) minimal element.*

Proof. Let $S \subseteq k[x_1, \ldots, x_n]$. If S contains an element of k, than this element is minimal. Otherwise, by the fact that the positive intergers are well-ordered, let S_1 be the subset of S consisting of polynomials of minimal class. Again, by the well-ordering of the positive integers, choose an element of S_1 of minimal leading degree. This is a minimal element of S. $\square$

Now we use this ordering to define a partial order on ascending chains,

Definition 4.6. *Let $C = f_1, \ldots, f_r$ and $C_1 = g_1, \ldots, g_m$ be ascending chains. We say that $\mathbf{C} < \mathbf{C_1}$ if either,*

(i) $\exists s \leq min(r, m)$ such that f_i, g_i are of the same rank for $i < s$ and $f_s < g_s$, or

(ii) $m < r$ and f_i and g_i are of the same rank for $i \leq m$.

Not surprisingly, this ordering is also a well-ordering,

Proposition 4.7. *Let Γ be a set of ascending chains. Then Γ has a minimal element with respect to our ordering $<$ on ascending chains.*

Proof. By our well-ordering on polynomials defined above, we can let Γ_1 be the subset of Γ consisting of ascending chains whose first polynomials are minimal among

all the first polynomials in all ascending chains in Γ. If all the ascending chains in Γ_1 have only one polynomial, than any of them are minimal. Otherwise, define Γ_2 similarly as above: take the subset of Γ_1 whose second polynomials are minimal among all second polynomials in the ascending chains in Γ_1.
Repeat this process at most m times where m is the size of the largest ascending chain in Γ. Any of the ascending chains in Γ_m are minimal. □

An obvious use for a well-ordering on ascending chains is that it allows us to pick out a minimal ascending chain. In this way we introduce the idea of a characteristic set.

Definition 4.8. *Let S be a nonempty set of polynomials in $k[x_1, \ldots, x_n]$. A minimal ascending chain among all ascending chains formed by polynomials in S is called a* ***characteristic set****.*

If $C = f_1, \ldots, f_r$ is a characteristic set, then we say that g is reduced with respect to C if for each $f \in C$ with $\text{class}(f) = i$, $\deg(g, x_i) < \deg(f, x_i)$ for all $i = 1, \ldots, r$.

We are particularly interested in the algorithmic contruction of characteristic sets. The following two results will help us show that characteristic sets can be found algorithmically.

Proposition 4.9. *Let $C = f_1, \ldots, f_r$ be a characteristic set of the polynomial set S with class$(f_1) > 0$. Let g be a nonzero polynomial that is reduced with respect to C. Then $S_1 = S \cup \{g\}$ has a characteristic set less than C.*

Proof. If $\text{class}(g) \leq \text{class}(f_1)$ then the set $\{g\}$ is a characteristic set strictly lower than C. This is true since $g < f_1$.

Now suppose that $\text{class}(g) > \text{class}(f_1)$, and let $j = \max\{i \,|\, \text{class}(f_i) < \text{class}(g)\}$. So f_j is the "biggest" element of C that is still lower than than g. Then we claim that the set $f_1, \ldots, f_j, g$ is an ascending chain lower than C.

It is an ascending chain since we have that $\text{class}(f_1) < \cdots < \text{class}(f_j) < \text{class}(g)$ and each polynomial is reduced with respect to the previous polynomials (g is reduced with respect to C). It is lower than C since the polynomials are of the same rank except for $g < f_{j+1}$. □

Proposition 4.10. *Let $C = f_1, \ldots, f_r$ be an ascending chain in the polynomial set S with class$(f_1) > 0$. Then C is a characteristic set of S if and only if S contains no nonzero polynomials reduced with respect to C.*

Proof. First, suppose that C is a characteristic set of S. If there were some g in S reduced with respect to C then by Proposition 4.9, we can find a smaller ascending chain, contradicting the minimality of C.

To prove the opposite direction, suppose that C is not a characteristic set of S, i.e. there is a $C_1 = g_1, \ldots, g_m$ that is strictly lower than C. Now we have the following two cases;

Case 1 There exists $s \leq \min(r, m)$ with f_i, g_i having the same rank for $i < s$ and $g_s < f_s$. Then g_s is reduced with respect to all the preceding f_i's since they are of the same rank as the corresponding g_i's and g_s is reduced with respect to the other f_i's since $g_s < f_i$ for $i \geq s$.

Case 2 $r < m$ and f_i, g_i are of the same rank for $i \leq r$. Then g_{r+1} is reduced with respect to C.

So in either case there exists an element of S reduced with respect to C. □

Now we can say something about the actual construction of characteristic sets.

Theorem 4.11. *Every nonempty polynomial set S has a characteristic set. When S is finite, there is an algorithm for constructing this characteristic set.*

Proof. This first statement follows from the well-ordering property proved above. Suppose that S is finite, and let f_1 be a polynomial of minimal rank in S. If $\text{class}(f_1) = 0$ then the set f_1 is a characteristic set, so suppose further that $\text{class}(f_1) > 0$.

We can construct the set

$$S_1 = \{g \in S \mid g \text{ is reduced w/respect to } f_1\}$$

by computing $\deg(g, \text{LV}(f_1))$ for every $g \in S$. If S_1 is empty then f_1 is a characteristic set. Otherwise, every polynomial in S_1 is of higher class than f_1. Now let f_2 be a polynomial of minimal rank in S_1 and let S_2 be the set

$$S_2 = \{g \in S_1 \mid g \text{ is reduced w/respect to } f_2\}$$

If S_2 is empty then $\{f_1, f_2\}$ is a characteristic set. Otherwise repeat this process. Since S was finite, this process must terminate, yielding a characteristic set $\{f_1, \ldots, f_r\}$. □

We end this section by noting a property of characteristic sets for polynomial ideals.

Proposition 4.12. *Let $C = f_1, \ldots, f_r$ be a characteristic set of the ideal $I \lhd k[x_1, \ldots, x_n]$.*

(i) If $g \in I$ then prem$(g, f_1, \ldots, f_r) = 0$

(ii) If I is a prime ideal, then prem$(g, f_1, \ldots, f_n) = 0 \Rightarrow g \in I$.

Proof. First recall that finding pseudoremainders in this situation is still possible even though C may not be in triangular form. See the Remark at the end of Section 4.1.1.

(i) Let $g \in I$. By the properties of pseudodivision, we see that prem$(g, f_1, \ldots, f_r)$ $\in I$ and is reduced with respect to C. But by Proposition 4.10, it must be zero, for otherwise C would not be a characteristic set.

(ii) Let I be a prime ideal, and suppose that prem$(g, f_1, \ldots, f_r) = 0$. Again, by the properties of pseudodivision, we get that

$$d_1^{s_1} \cdots d_r^{s_r} g = A_1 f_1 + \cdots + A_r f_r$$

where d_i is the initial (leading coefficient) of f_i. Note that the d_i are in fact nonzero and reduced with respect to C (see the examples following Definition 4.3), so by Proposition 4.10 $d_i \notin I$. Hence $g \in I$.

□

4.2.1 Irreducible Ascending Chains

Recall that our goal is to develop a method for "triangulating" our system of hypotheses in such a way that we can use successive pseudodivision and Definition 3.21 to establish our geometric result. Our introduction of the concepts of ascending chains and characteristic sets has taken us a long way in that direction. However, recall our attempt to prove the geometric theorem in Example 1 using a Groebner basis. We discovered that we ran into difficulties if the variety defined by the hypotheses was reducible. In particular, we saw that we could factor several of the equations in the Groebner bases for this variety, and that this yielded subvarieties corresponding to degenerate conditions of our theorem.

Since the "triangular form" we've been heading towards involves ascending chains, we might attempt to investigate the irreducibility of polynomials in ascending chains. This suggests the following definition.

Definition 4.13. *Let $C = f_1, \ldots, f_r$ be an ascending chain with no constants and with each $f_i \in k[x_1, \ldots, x_n]$. Rename the variables x_i in such a way that we can write:*

$$
\begin{aligned}
f_1 &= f_1(u_1, \ldots, u_d, x_1) \\
f_2 &= f_2(u_1, \ldots, u_d, x_1, x_2) \\
&\vdots \\
f_r &= f_r(u_1, \ldots, u_d, x_1, \ldots, x_r)
\end{aligned}
$$

*so that $n = d + r$. An **irreducible ascending chain** is an ascending chain in the form above such that each $f_i \in C$ is irreducible in the ring $k(u_1, \ldots, u_d)[x_1, \ldots, x_i]/\langle f_1, \ldots, f_{i-1}\rangle$.*

Example The ascending chain $f_1 = x_1^2 - u_1, f_2 = x_2^2 - 2x_1x_2 + u_1$ is reducible since f_2 is reducible over $F_1 = \mathbb{Q}(u_1)[x_1]/\langle f_1\rangle$. In particular, $f_2 = (x_2 - x_1)^2$ where $x_1^2 = u_1$.

Notice that at this point we have resumed the distinction in variables between the dependent and independent coordinates. In practice, any relabeling of variables is rarely necessary, since most properly translated geometric theorems are in this form already. However, occasionally we may translate a theorem and find that some dependent coordinate x_i actually does not appear in any of our hypothesis equations.[3] This is the only situation in which relabeling the variables may be necessary. As Chou notes (see [1, pages 52-53]) this often implies that something deeper is taking place in the theorem then previously thought. In particular, a hidden hypothesis is usually to blame, as in Example 3. Reformulating the problem with this in mind generally solves the problem. Chou actually excludes this from occurring by adding what he calls a Dimensionality Requirement, which demands that each dependent variable, x_i actually occur as the leading variable in f_i in our ascending chain. We will follow Chou and assume this as well.

Remark 4.14 (Dimensionality Requirement)**.** *In an ascending chain, each dependent variable x_i must actually appear as the leading variable in the polynomial f_i.*

Some other notes on the above definition are necessary. First, the ideals $\langle f_1\rangle, \langle f_1, f_2\rangle$, etc. are in fact ideals in the ring $k(u_1, \ldots, u_d)[x_1]$, etc. So we are allowing denominators in the u_i. Second, if we do have an irreducible ascending chain C as above, then the following sequence forms a tower of field extensions

[3] As an example, consider the triangle $\triangle ABC$ with medians $\overline{AD}, \overline{BE}, \overline{CF}$. Let $G = \overline{AD} \cap \overline{BE}$ and let $H = \overline{CF} \cap \overline{AD}$. Finally, let P be a point on the line $\overline{GH}$. If we translate these hypotheses (there are ten) we will find that the ascending chain we obtain does not include the variable x_{10}. In this case, the cause is the fact that $G = H$ is always true.

$$
\begin{aligned}
F_0 &= k(u_1, \ldots, u_d) \\
F_1 &= F_0[x_1]/\langle f_1 \rangle \\
F_2 &= F_1[x_2]/\langle f_2 \rangle \\
&\vdots \\
F_r &= F_{r-1}[x_r]/\langle f_r \rangle,
\end{aligned}
$$

and each $f_i \in C$ may be considered as a polynomial in x_i over the field F_{i-1}. We have the following result on irreducible ascending chains.

Theorem 4.15. *Let $C = f_1, \ldots, f_r$ be an irreducible ascending chain as in Definition 4.13 and let $g \in k[u_1, \ldots, u_d, x_1, \ldots, x_r]$ and $F_r = k(u_1, \ldots, u_d)[x_1, \ldots, x_r]/\langle f_1, \ldots, f_r \rangle$. Then the following statements are equivalent:*

(i) $prem(g, f_1, \ldots, f_r) = 0$

(ii) Let E be any extension of the field k. If $\mu = (\tilde{u}_1, \ldots, \tilde{u}_d, \tilde{x}_1, \ldots, \tilde{x}_r) \in E^{d+r}$ is in $V(f_1, \ldots, f_r)$ with $\tilde{u}_1, \ldots, \tilde{u}_d$ transcendental over k, then $\mu \in V(g)$.

(iii) Viewed as an element of F_r, g is zero. In other words, the canonical image of g in F_r is 0.

(iv) There exist finitely many nonzero polynomials $c_1, \ldots, c_s \in k[u_1, \ldots, u_d]$ such that $c_1 \cdots c_s g$ belongs to the ideal in $k[u_1, \ldots, u_d, x_1, \ldots, x_r]$ generated by $f_1, \ldots, f_r$.

First we must establish the following lemma.

Lemma 1. *Let $p = a_s x_m^s + \cdots + a_0$ be a polynomial with $1 \leq m \leq r$, $0 \leq s$, where the a_i are polynomials in $k[u_1, \ldots, u_d, x_1, \ldots, x_{m-1}]$, and suppose that p is reduced with respect to $f_1, \ldots f_r$. Then if μ from (ii) in Theorem 4.15 is a zero of p then p is in fact the zero polynomial.*

Proof. First note that the Lemma is trivial when $s = 0$, so we assume that $s \geq 1$. We use induction on m. Let $\tilde{p}$ denote the polynomial obtained upon substitution of μ. Suppose that $m = 1$. Then $p(\mu) = 0$ implies that

$$\tilde{p} = \tilde{a}_s \tilde{x}_1^s + \cdots + \tilde{a}_0 = 0$$

(recall that the a_i are polynomials as well, so we denote the substitution of μ in the a_i with a tilde). Since p is reduced with respect to f_1, we may assume that $s < \deg(f_1, x_1)$. Now recall the uniqueness of an algebraic expression in an extension of k (Theorem 3.2 (*iii*)). Specifically, if we evaluate f_1 only at the $\tilde{u}_i$'s, we get the polynomial $f_1(\tilde{u}_1, \ldots, \tilde{u}_d, x_1)$ which is irreducible in the ring $k(\tilde{u}_1, \ldots, \tilde{u}_d)[x_1]$ and has a root $\tilde{x}_1$. So we can consider $\tilde{p}$ above to be in the extension field of k given by

$$k(\tilde{u}_1, \ldots, \tilde{u}_d)(\tilde{x}_1) \cong k(\tilde{u}_1, \ldots, \tilde{u}_d)[x_1]/\langle f_1(\tilde{u}_1, \ldots, \tilde{u}_d, x_1)\rangle$$

Then by the uniqueness of an expression equal to zero in this extension, we must have $\tilde{a}_j = 0$. But the $\tilde{u}_1, \ldots, \tilde{u}_d$ were chosen to be transcendental over k, so the only way the $\tilde{a}_j$ could evaluate to zero is if each a_j is the zero polynomial. Hence p is the zero polynomial.

Now assume that the Lemma holds for $m-1$, and let $p(\mu) = 0$ where $p = a_s x_m^s + \cdots + a_0$. Then we get that

$$\tilde{p} = \tilde{a}_s \tilde{x}_m^s + \cdots + \tilde{a}_0 = 0$$

Again, since $s < \deg(f_m, x_m)$ we can use the unique representation of an algebraic expression in an extension (using a similar argument as above) to conclude that all $\tilde{a}_j = 0$. So μ is a zero of all the a_j. Now note that each a_j is in fact reduced with respect to $f_1, \ldots, f_r$, so we can use the induction hypothesis on each to conclude that each a_j is the zero polynomial. Hence, p is the zero polynomial, as desired. □

Now we can prove the theorem using this lemma.

Proof. $(ii) \Rightarrow (i)$ Let μ be as in (ii) (such a μ always exists, consider for example the canonical images of $u_1, \ldots, u_d, x_1, \ldots, x_r$ in F_r viewed as an extension of k), and suppose that $g(\mu) = 0$. Let $R = \text{prem}(g, f_1, \ldots, f_r)$ so that we have

$$d_1^{s_1} \cdots d_r^{s_r} g = A_1 f_1 + \cdots + A_r f_r + R.$$

Hence $R(\mu) = 0$ (recall that $f_i(\mu) = 0$ for all i since $\mu \in V(f_1, \ldots, f_r)$). But by Proposition 4.2(ii), R is reduced with respect to $f_1, \ldots, f_r$ so we may invoke the Lemma to conclude that $R = 0$.

$(i) \Rightarrow (ii)$ Now suppose that $\text{prem}(g, f_1, \ldots, f_r) = 0$, so upon pseudodivision we have the equation

$$d_1^{s_1} \cdots d_r^{s_r} g = A_1 f_1 + \cdots + A_r f_r.$$

Since $f_1, \ldots, f_r$ is an ascending chain, it has the property that $\text{prem}(d_k, f_1, \ldots, f_r) \neq 0$ (see Example 3 following Definition 4.3). But by the proof of $(ii) \Rightarrow (i)$ this implies that $d_k(\mu) \neq 0$, which in turn implies that $g(\mu) = 0$.

$(i) \Leftrightarrow (iii)$ (iii) is a particular case of (ii) using μ defined as the canonical images of the variables $u_1, \ldots, u_d, x_1 \ldots, x_r$ as noted above in $(ii) \Rightarrow (i)$, and so our previous arguments give us $(iii) \Leftrightarrow (i)$.

$(iv) \Rightarrow (i)$ Suppose, as in (iv), there exist finitely many nonzero polynomials $c_1, \ldots, c_s \in k[u_1, \ldots, u_d]$ such that $c_1 \cdots c_s g \in \langle f_1, \ldots, f_r \rangle$, the ideal generated by the f_i in the ring $k[u_1, \ldots, u_d, x_1, \ldots, x_r]$. Let μ be as in (ii). Then since the $\tilde{u}_i$ are transcendental over k and the c_i are nonzero we must have $c_i(\mu) \neq 0$. But $\mu \in V(f_1, \ldots, f_r)$, so we must have $g(\mu) = 0$. Hence, since $(ii) \Rightarrow (i)$, we can conclude that $\mathrm{prem}(g, f_1, \ldots, f_r) = 0$.

$(i) \Rightarrow (iv)$ Suppose that $\mathrm{prem}(g, f_1, \ldots, f_r) = 0$. In other words,

$$d_1^{s_1} \cdots d_r^{s_r} g = A_1 f_1 + \cdots + A_r f_r. \tag{1}$$

In the field $F_r = k(u_1, \ldots, u_d)[x_1, \ldots, x_r]/\langle f_1, \ldots, f_r \rangle$, we claim that $p = d_1^{s_1} \cdots d_r^{s_r}$ is not zero. If this were not the case, then we would have the formula

$$d_1^{s_1} \cdots d_r^{s_r} = Q_1 f_1 + \cdots + Q_r f_r$$

which implies that $\mathrm{prem}(d_r, f_1, \ldots, f_r) = 0$. But this contradicts the fact that the initial d_r is reduced with respect to f_r.

This means that p has an inverse in F_r, or in other words that there is a $q \in k(u_1, \ldots, u_d)[x_1, \ldots, x_r]$ such that $qp - 1 \in \langle f_1, \ldots, f_r \rangle$ (viewed as an ideal in the ring $k(u_1, \ldots, u_d)[x_1, \ldots, x_r]$). So we have

$$qp - 1 = Q_1 f_1 + \cdots + Q_r f_r$$

Clearing denominators yields

$$q_1 p - c = \bar{Q}_1 f_1 + \cdots + \bar{Q}_r f_r$$

Where c involves only the variables $u_1, \ldots, u_d$. Now if we multiply (1) by q_1 we get

$$\begin{aligned} q_1(A_1 f_1 + \cdots + A_r f_r) &= d_1^{s_1} \cdots d_r^{s_r} g q_1 \\ &= p g q_1 \\ &= (\bar{Q}_1 f_1 + \cdots + \bar{Q}_r f_r + c) g \end{aligned}$$

Upon rearranging the last equation we see that $gc \in \langle f_1, \ldots, f_r \rangle$ as an ideal in the ring $k[u_1, \ldots, u_d, x_1, \ldots, x_r]$. But c involves only the u_i, so we are done.

$\square$

The point $\mu = (\tilde{u}_1, \dots, \tilde{u}_d, \tilde{x}_1, \dots, \tilde{x}_r)$ discussed in (ii) of the previous theorem is of particular importance, so we give it a name: any point $\mu \in E$ that is in $V(f_1, \dots, f_r)$ with the $\tilde{u}_1, \dots, \tilde{u}_d$ transcendental over k we call a **generic point** of the ascending chain $f_1, \dots, f_r$ in an extension E of k. (Not to be confused with a *generic zero* discussed in Section 3.1.)

Proposition 4.16. *Let $f_1, \dots, f_r$ be an irreducible ascending chain and g any polynomial. If prem$(g, f_1, \dots, f_r) \neq 0$ then there are polynomials q, p with $p \neq 0$ such that $qg - p \in \langle f_1, \dots, f_r \rangle$ and $p \in k[u_1, \dots, u_d]$.*

Proof. If $\text{prem}(g, f_1, \dots, f_r) \neq 0$ then we have that

$$d_1^{s_1} \cdots d_r^{s_r} g = A_1 f_1 + \cdots + A_r f_r + R, \tag{2}$$

where $R \neq 0$. As in the proof that $(i) \Rightarrow (iv)$ in Theorem 4.15, we conclude that R has an inverse in the ring

$$F_r = k(u_1, \dots, u_d)[x_1, \dots, x_r]/\langle f_1, \dots, f_r \rangle$$

.

In other words we have that

$$Rq - 1 \in \langle f_1, \dots, f_r \rangle \subset k(u_1, \dots, u_d)[x_1, \dots, x_r]$$

for some (rational) polynomial q. Now if we clear denominators we get

$$R\tilde{q} - c = Q_1 f_1 + \cdots + Q_r f_r \tag{3}$$

Note that since only q had a denominator, R remains unchanged and $c \in k[u_1, \dots, u_d]$. Multiply equation (2) on both sides by $\tilde{q}$ to get

$$d_1^{s_1} \cdots d_r^{s_r} \tilde{q} g = \tilde{A}_1 f_1 + \cdots + \tilde{A}_r f_r + R\tilde{q}.$$

Now use equation (3) to rewrite $R\tilde{q}$ in the above equation, yielding

$$\bar{q} g - c = \bar{A}_1 f_1 + \cdots + \bar{A}_r f_r$$

which establishes the result. □

The following theorem gives us a method for transforming reducible ascending chains into irreducible ones while preserving most of our "triangular" properties.

Theorem 4.17. *Let $f_1, \dots, f_r$ be an ascending chain. Suppose that $f_1, \dots, f_{k-1}$ is irreducible, but that $f_1, \dots, f_k$ is reducible. Then there exist polynomials g, h in the ring $k[u_1, \dots, u_d, x_1, \dots, x_r]$ that are reduced with respect to $f_1, \dots, f_r$ and such that $class(g) = class(h) = class(f_k)$, and $gh \in \langle f_1, \dots, f_k \rangle$.*

Proof. Suppose that f_k is reducible in the ring $F_{k-1}[x_k]$. Then we can factor f_k viewed as a member of the one variable polynomial ring $F_{k-1}[x_k]$ (this is often the most difficult computational hurdle in Wu's Method; we need factorization over algebraic extensions).

Hence there exist polynomials $g'', h'' \in k(u_1, \dots, u_d)[x_1, \dots, x_k]$ of positive degree in x_k such that $f_k - g''h'' = 0$ in $F_{k-1}[x_k]$. Specifically, we get an equation

$$f_k - g''h'' = A_m x_k^m + \cdots + A_0 \tag{4}$$

where each A_i belongs to $k(u_1, \dots, u_d)[x_1, \dots, x_{k-1}]$ and is zero in

$$F_{k-1} = k(u_1, \dots, u_d)[x_1, \dots, x_{k-1}]/\langle f_1, \dots, f_{k-1} \rangle.$$

The equality in (4) still holds if we evaluate the right hand side at $x_k = 1$. Then clear denominators to get $Qf_k - g'h' = p$ where p is the resulting polynomial from the right hand side of (4) and p is a polynomial in $k[u_1, \dots, u_d, x_1, \dots, x_{k-1}]$ (since we evaluated at $x_k = 1$).

Now note that $p \equiv 0$ in the ring $F_{k-1}[x_k]$, so we can use $(iii) \Rightarrow (i)$ of Theorem 4.15 to conclude that $\mathrm{prem}(p, f_1, \dots, f_{k-1}) = 0$. Then a simple series of algebraic manipulations yields the following series of equations:

$$\begin{aligned}
d_1^{s_1} \cdots d_{k-1}^{s_{k-1}} p &= Q_1 f_1 + \cdots + Q_{k-1} f_{k-1} \\
d_1^{s_1} \cdots d_{k-1}^{s_{k-1}} (Qf_k - g'h') &= Q_1 f_1 + \cdots + Q_{k-1} f_{k-1} \\
-(d_1^{s_1} \cdots d_{k-1}^{s_{k-1}}) g'h' &= Q_1 f_1 + \cdots + Q_{k-1} f_{k-1} - \tilde{Q} f_k.
\end{aligned}$$

So we have that $(d_1^{s_1} \cdots d_{k-1}^{s_{k-1}}) g'h'$ is in the ideal $\langle f_1, \dots, f_k \rangle$. Let

$$\begin{aligned}
g &= \mathrm{prem}((d_1^{s_1} \cdots d_{k-1}^{s_{k-1}}) g', f_1, \dots, f_{k-1}) \\
h &= \mathrm{prem}(h', f_1, \dots, f_{k-1}).
\end{aligned}$$

It is an easy calculation using the remainder formula from pseudodivision to check that $gh \in \langle f_1, \dots, f_k \rangle$. Also, the properties of pseudoremainders ensure that g, h are both reduced with respect to $f_1, \dots, f_{k-1}$. We noted above that g'', h'' were both reduced with respect to f_k. This implies that g', h' are as well, and in turn that g, h are reduced with respect to f_k. Since both g, h were obtained from f_k by factoring and division, the highest variable appearing in each must be x_k, so they must be reduced with respect to $f_{k+1}, \dots, f_r$ since f_k was as well.

Finally, we need to check that $\text{class}(g) = \text{class}(h) = \text{class}(f_k)$. First, in the factorization of f_k, we must have that the class of both g'', h'' are the same as f_k. Second, when we rationalized the denominator, this contributed only u_i's, so the class of g', h' remained the same. Third, pseudodivision by $f_1, \dots, f_{k-1}$ won't effect the appearance of x_k, so the class of g, h will remain the same. Hence g, h have all the desired properties. $\square$

The usefulness of irreducible chains is illustrated by the following theorem.

Theorem 4.18. *Let $f_1, \dots, f_r$ be an irreducible ascending chain and let P be defined by*

$$P = \{g \mid g \in k[u_1, \dots, u_d, x_1, \dots, x_r] \text{ and } prem(g, f_1, \dots, f_r) = 0\}$$

Then the following assertions are true:

(i) P is a prime ideal with $f_1, \dots, f_r$ as a characteristic set.

(ii) A generic point of $f_1, \dots, f_r$ is a generic zero of P.

(iii) If k is algebraically closed, then a polynomial g vanishes on $V(P)$ if and only if $prem(g, f_1, \dots, f_r) = 0$.

(iv) For any field k, $dim(V(P)) \geq d$ (the number of independent variables, u_i) where $V(P) = \{x \in k^n \mid f(x) = 0 \;\forall\, f \in P\}$. If $prem(g, f_1, \dots, f_r) = 0$ then g vanishes on $V(P) \subseteq k^{d+r}$.

Proof. First recall a result from Section 3.1, Theorem 3.14. Let μ be a generic point of the irreducible ascending chain $f_1, \dots, f_r$. Then by $(i) \Leftrightarrow (ii)$ in Theorem 4.15 we have that $P = \{g \mid g(\mu) = 0\}$. This establishes (ii), and also easily implies that P is in fact an ideal. So μ is a generic zero of the ideal P which by Theorem 3.14 mentioned above implies that P is in fact a proper prime ideal.

In addition, since everything in P has remainder of zero when divided by $f_1, \dots, f_r$, we have that there are no nonzero polynomials in P that are reduced with respect to $f_1, \dots, f_r$. Hence by Proposition 4.10 we see that $f_1, \dots, f_r$ is in fact a characteristic set of P. This establishes (i).

If k is algebraically closed, then we have that $I(V(P)) = P$ by Theorem 3.17 and the fact that all prime ideals are radical. Hence a polynomial g vanishes on $V(P)$ if and only if $g \in P$, i.e. $\text{prem}(g, f_1, \ldots, f_r) = 0$. This establishes (iii).

If k is any field (not necessarily algebraically closed) then the dimension of $V(P)$ is the same as the dimension of its prime ideal $I(V(P))$. Note that $I(V(P)) \supset P$ so we have that $\dim I(V(P)) \geq \dim P$. Now, the dimension of the prime ideal P is the transcendence degree of the quotient field of $k[u_1, \ldots, u_d, x_1, \ldots, x_r]/P$ over k.

From the proof of $(ii) \Rightarrow (i)$ in Theorem 4.15 we know that the characteristic set $f_1, \ldots, f_r$ has a generic point μ, and by (ii) in the present theorem we see that μ is a generic zero of P. Then by Corollary 3.15 we see that

$$k[u_1, \ldots, u_d, x_1, \ldots, x_r]/P \cong k[\tilde{u}_1, \ldots, \tilde{u}_d, \tilde{x}_1, \ldots, \tilde{x}_r].$$

Since μ is a generic point, the $\tilde{u}_1, \ldots, \tilde{u}_d$ are algebraically independent over k. Hence the transcendence degree of the quotient field of $k[u_1, \ldots, u_d, x_1, \ldots, x_r]/P$ over k is at least d. Hence $\dim I(V(P)) \geq d$, so we have that $\dim V(P) \geq d$.

The remaining statement in (iv) follows easily from the fact that every prime ideal is radical and for any field we have $\sqrt{P} \subset I(V(P))$. $\square$

We should note here what happens with $V(P)$ if k is not algebraically closed. In particular, we cannot conclude that the variety $V(P)$ is irreducible. This is troublesome because we wish to use characteristic sets and their prime ideals P in order to find an irreducible decomposition of the original variety defined by the hypothesis equations. However, we do have the following statement.

Proposition 4.19. *In the situation above, if $V(P)$ is of dimension d, then it is irreducible.*

Proof. Suppose that k is not algebraically closed, and that $V(P)$ is of dimension d. Let $V_1 \subset V(P)$ be a component of dimension d. Then if we take the ideal of both sides we get that $I(V_1) \supset I(V(P)) \supset P$. Now $I(V_1)$ is a prime ideal with dimension d and it contains P. We also have that the dimension of P is d, since the u_i's are assumed to be algebraically independent over P.

Now recall that the dimension of a prime ideal is also defined as the supremum of the lengths of chains of distinct prime ideals that contain it.

Hence we claim that $P = I(V_1) = P_1$. Specifically, if $P \neq P_1$ then the dimension of P_1 would be strictly smaller than that of P.

We conclude that $V(P) = V(P_1) = V_1$, so $V(P)$ is irreducible. $\square$

We've seen that we can generate prime ideals from irreducible ascending chains. The following theorem allows us to move in the opposite direction.

Theorem 4.20. *Let P be a nontrivial prime ideal of $k[u_1, \ldots, u_d, x_1, \ldots, x_r]$, and let $f_1, \ldots, f_r$ be a characteristic set of P. Then $f_1, \ldots, f_r$ is irreducible.*

Proof. From Proposition 4.12 we know that $P = \{g \mid \text{prem}(g, f_1, \ldots, f_r) = 0\}$. Suppose, to get a contradiction, that $f_1, \ldots, f_r$ is reducible. Then there is a $k > 0$ such that $f_1, \ldots, f_{k-1}$ is irreducible but $f_1, \ldots, f_k$ is reducible. By Theorem 4.17 we can find polynomials g, h such that they are reduced with respect to $f_1, \ldots, f_r$ and $gh \in \langle f_1, \ldots, f_k \rangle \subset P$ (also, the degrees of g, h in x_k are positive).

Since g, h are both reduced with respect to $f_1, \ldots, f_r$, we have that $\text{prem}(g, f_1, \ldots, f_r)$ $\neq 0$ (the same is true of h as well). But this implies that neither are contained in P, while their product is in P. This contradicts P being a prime ideal. ☐

4.3 Ritt's Principle

Previously, our construction of characteristic sets always involved picking polynomials from the original polynomial set. Here we introduce a slight generalization, called an extended characteristic set, where the elements of the characteristic set are not necessarily in our original polynomial set, but they are in the ideal generated by our original polynomial set. The definition we have in mind is the following:

Definition 4.21. *Let $S = \{h_1, \ldots, h_m\}$ be a finite nonempty set of polynomials in the ring $k[x_1, \ldots, x_n]$, and let $I = \langle h_1, \ldots, h_m \rangle$. An **extended characteristic set** is an ascending chain C such that either*

(i) C consists of just an element of $k \cap I$, or

(ii) $C = \{f_1, \ldots, f_r\}$ with class$(f_1) > 0$ such that $f_i \in I$ and prem$(h_j, f_1, \ldots, f_r) = 0$ for all i, j.

Note the differences between this definition and our definition of characteristic sets. Before we only required that no element of S be reduced with respect to C, here we demand that the remainder actually is zero. Also, as noted above, here the elements of C may not come from S, although they will be in the ideal I.

We also note that every extended characteristic set of a polynomial set $S = \{h_1, \ldots, h_m\}$ is also a characteristic set of the ideal $I = \langle h_1, \ldots, h_m \rangle$.

Proposition 4.22. *Let $S = \{h_1, \ldots, h_r\}$ be a polynomial set in $k[x_1, \ldots, x_n]$, with extended characteristic set $C_e = f_1, \ldots, f_r$. Then C_e is also a characteristic set of the ideal $I = \langle h_1, \ldots, h_m \rangle$.*

Proof. If the extended characteristic set C_e consists of only an element from the field k, then I certainly doesn't contain any nonzero elements that are reduced with respect to C_e and so by Proposition 4.10, it is a characteristic set.

If C_e is not a trivial extended characteristic set, we proceed by contradiction. Suppose that $g \in I$ is a nonzero polynomial that is reduced with respect to C_e. Then we see that $\text{prem}(g, f_1, \ldots, f_r) = g$. We also know that $f_i \in I$ and that $\text{prem}(h_j, f_1, \ldots, f_r) = 0$ for all $i = 1, \ldots, r$ and all $j = 1, \ldots, m$. Hence for each $j = 1, \ldots, m$ we can write the equation

$$d_1^{s_{1j}} \cdots d_r^{s_{rj}} h_j = Q_{1j} f_1 + \cdots + Q_{rj} f_r. \tag{5}$$

Now let $\overline{s}_i = \max \{s_{ij} \mid j = 1, \ldots, m\}$. But from the fact that $g \in I$ we see that

$$g = A_1 h_1 + \cdots + A_m h_m$$

for some polynomials $A_1, \ldots, A_m$. Now multiply this equation on both sides by the polynomial $d_1^{\overline{s}_1} \cdots d_r^{\overline{s}_r}$, yielding

$$d_1^{\overline{s}_1} \cdots d_r^{\overline{s}_r} g = \overline{A}_1 \left(d_1^{s_{11}} \cdots d_r^{s_{r1}} h_1\right) + \cdots + \overline{A}_m \left(d_1^{s_{1m}} \cdots d_r^{s_{rm}} h_m\right)$$

Then by using the equations (for $j = 1, \ldots, m$) mentioned in (5) above we see that we can write g as

$$d_1^{\overline{s}_1} \cdots d_r^{\overline{s}_r} g = \overline{Q}_1 f_1 + \cdots + \overline{Q}_r f_r.$$

But this contradicts the fact g is reduced with respect to $f_1, \ldots, f_r$ noted above. Hence I must not contain any polynomials reduced with respect to C_e, and so C_e must be a characteristic set by Proposition 4.10. $\square$

Theorem 4.23 (Ritt's Principle)**.** *Let $S = \{h_1, \ldots, h_m\}$ be a finite, nonempty set of polynomials in $k[x_1, \ldots, x_n]$, and let $I = \langle h_1, \ldots, h_m \rangle$. There is an algorithm to find an extended characteristic set C of S.*

Proof. By Theorem 4.11 we can construct a characteristic set C_1 of the polynomial set $S = S_1$. If C_1 contains only a constant, then we have (i) in Definition 4.21. Otherwise we expand S_1 by adding all nonzero remainders of elements of S_1 on pseudodivision by $C_1 = f_1, \ldots, f_r$ to get a new polynomial set S_2. Specifically, we find $\text{prem}(h_j, f_1, \ldots, f_r)$ for all j. If the remainder is nonzero we include it in S_2. If $S_1 = S_2$ then we are in (ii) of Definition 4.21. Otherwise repeat this process on S_2, yielding the characteristic set C_2. By Proposition 4.9 we know that S_2 has a characteristic set that is strictly lower than C_1. Then the characteristic set found by our algorithm in Theorem 4.11 must be lower than C_1; i.e. we have that $C_1 > C_2$.

Repeating this process yields a sequence of polynomial sets

$$S_1 \subset S_2 \subset \cdots$$

and a corresponding decreasing sequence of characteristic sets

$$C_1 > C_2 \cdots$$

Since characteristic sets are well-ordered, this strictly decreasing chain must terminate, i.e. we must have that $S_k = S_{k+1}$ or C_k consisting of only a constant. We claim that in either case C_k has the properties in Definition 4.21. If C_k is only a constant, this is trivial.

By the construction of $C_k = f_1, \ldots, f_r$ we have that $\mathrm{prem}(h_j, f_1, \ldots, f_r) = 0$ for all j. It remains to show that $f_i \in I$ for all i. We use induction to show that for all i, both $S_i \subset I$ and $C_i \subset I$. The base case $(i = 1)$ is trivial. Now suppose that $C_i \subset I$ and $S_i \subset I$. To get the characteristic set C_{i+1} we add the nonzero remainders of elements of S_i upon pseudodivision by C_i. It is an easy consequence of the remainder formula for pseudodivision that this remainder also lies in I. This establishes the result. $\square$

We emphasize here that this algorithm produced an increasing sequence of sets and a corresponding decreasing sequence of characteristic sets. When the algorithm terminates, we have a final characteristic set, which we call C and a final polynomial set which we call S'.

We need the following property of extended characteristic sets.

Proposition 4.24. *Let $S = \{h_1, \ldots, h_n\}$, and suppose that $C = f_1, \ldots, f_r$ is an extended characteristic set of S (with no constants). Let d_j denote the initials (leading coefficients) of the f_j and let $S_j = S \cup \{d_j\}$. Finally let $P = \{g \mid prem(g, f_1, \ldots, f_r) = 0\}$. Then we have that*

(i) $V(f_1, \ldots, f_r) - (V(d_1) \cup \cdots \cup V(d_r)) \subset V(P) \subset V(S) \subset V(f_1, \ldots, f_r)$

(ii) $V(S) = V(P) \cup V(S_1) \cup \cdots \cup V(S_r)$

Proof. (i) Let $p \in V(f_1, \ldots, f_r) - (V(d_1) \cup \cdots \cup V(d_r))$. Then we have that $f_i(p) = 0$ but $d_i(p) \neq 0$ for all i. For any $g \in P$ we have by pseudodivision the following formula,

$$d_1^{s_1} \cdots d_r^{s_r} g = Q_1 f_1 + \cdots + Q_r f_r$$

and this forces us to conclude that $g(p) = 0$. So $p \in V(P)$. The same reasoning using the pseudoremainder property of extended characteristic sets shows that $p \in V(S)$.

(ii) First we claim that $V(S) \subset V(P) \cup (V(d_1) \cup \cdots \cup V(d_r))$. To see this note that using (i) we get:

$$\begin{aligned} V(S) \subset V(f_1, \ldots, f_r) &\Rightarrow V(S) - (V(d_1) \cup \cdots \cup V(d_r)) \subset V(f_1, \ldots, f_r) - \\ &\quad (V(d_1) \cup \cdots \cup V(d_r)) \\ &\Rightarrow V(S) - (V(d_1) \cup \cdots \cup V(d_r)) \subset V(P) \\ &\Rightarrow V(S) \subset V(P) \cup (V(d_1) \cup \cdots \cup V(d_r)) \,. \end{aligned}$$

Now suppose that $p \in V(S)$. Then by the claim above, p is contained in $V(P) \cup (V(d_1) \cup \cdots \cup V(d_r))$. Then $p \in V(P)$ or $p \in V(d_j)$ for some j. In either case, we have $V(S) \subset V(P) \cup V(S_1) \cup \cdots \cup V(S_r)$, since $V(S \cup \{d_j\}) = V(S_j)$.

Now suppose that $p \in V(P) \cup V(S_1) \cup \cdots \cup V(S_r)$. If $p \in V(S_j)$ for some j, then clearly $p \in V(S)$. And finally, if $p \in V(P)$, then we have $p \in V(S)$ by (i).

□

4.4 Ritt's Decomposition Algorithm

Now we are ready to present Ritt's full algorithm for completely decomposing varieties. Recall the situation presented in Section 2. We have a collection of hypotheses $h_1, \ldots, h_r$ and a conclusion equation g, all in the polynomial ring $k[u_1, \ldots, u_d, x_1, \ldots, x_r]$. Our method depends upon deciding if g vanishes on the irreducible components of the variety $V(h_1, \ldots, h_r)$ that do not correspond to degenerate cases of our theorem.

Theorem 4.25. *Let S be a finite nonempty polynomial set in the ring $k[x_1, \ldots, x_n]$. There is an algorithm to determine whether $\langle S \rangle = k[x_1, \ldots, x_n]$ or otherwise to decompose the variety,*

$$V(S) = V(P_1) \cup \cdots \cup V(P_s)$$

where each P_i is the prime ideal given by an irreducible characteristic set as in Theorem 4.18 (i).

Proof. Let D be a set of characteristic sets, which to begin our algorithm is empty. We can apply Theorem 4.23 to the polynomial set S to get an extended characteristic set C and also the corresponding polynomial set S' (the final polynomial set in the increasing sequence that arose in the algorithm in Ritt's Principle). Then we have the following cases:

Case 1 C consists of just a constant. In this case we conclude that $V(S)$ is empty and $\langle S\rangle = k[x_1, \ldots, x_n]$.

Case 2 $C = \{f_1, \ldots, f_r\}$ is an irreducible ascending chain. Let d_k be the initials of the f_k, and let $S_k = S' \cup \{d_k\}$. Then by (ii) of Proposition 4.24 we have that

$$V(S) = V(P_1) \cup V(S_1) \cup \cdots \cup V(S_r)$$

where P_1 is the prime ideal with characteristic set C, so that

$$P_1 = \{g \mid \mathrm{prem}(g, f_1, \ldots, f_r) = 0\}$$

by Theorem 4.18 (i). Then by Proposition 4.9 we know that each S_k has a characteristic set strictly lower than C.

Add the characteristic set C to D and repeat this algorithm on each S_k.

Case 3 $C = \{f_1, \ldots, f_r\}$ is a reducible ascending chain. Specifically, there is a $k > 0$ such that $f_1, \ldots, f_{k-1}$ is irreducible but $f_1, \ldots, f_k$ is reducible. In this case we use Theorem 4.17 (here we need to be able to factor polynomials over algebraic extensions) to conclude that there are polynomials g, h, both of the same class as f_k and reduced with respect to $f_1, \ldots, f_r$ such that

$$gh \in \langle f_1, \ldots, f_k\rangle$$

We claim that $V(S) = V(S') = V(S_1) \cup V(S_2)$ where $S_1 = S' \cup \{g\}$ and $S_2 = S' \cup \{h\}$.

Since $S \subset S'$ we have that $V(S) \supset V(S')$. To establish the opposite containment it suffices to show that $V(S_i) \subset V(S_{i+1})$ for all i in the algorithm outlined in Ritt's Principle (Theorem 4.23).

Let $p \in V(S_i)$, where S_i is a polynomial set in the increasing sequence generated in the algorithm for Ritt's Principle. The polynomials that are in S_{i+1} but not in S_i are all remainders given by the formula

$$d_{1_i}^{s_{1_i}} \cdots d_{m_i}^{s_{m_i}} g = Q_{i_1} h_{1_i} + \cdots + Q_{i_r} h_{m_i} + R$$

where $g \in S_i$ and $h_{1_i}, \cdots, h_{m_i}$ is the characteristic set of S_i. Now, $g(p) = 0$ and we also must have that $h_{j_i}(p) = 0$ for all j since $h_{j_i} \in S_i$ (by our construction of characteristic sets). But this forces $R(p) = 0$. Hence $V(S) \subset V(S')$ and we have the opposite containment.

Now note that $V(S_1) \cup V(S_2) \subset V(S')$ is trivial, so let $p \in V(S') = V(S)$. This means that $p \in V(f_1, \ldots, f_r)$. But we know that

$$gh \in \langle f_1, \ldots, f_k\rangle$$

so we must have either $g(p) = 0$ or $h(p) = 0$. Hence $p \in V(S_1) \cup V(S_2)$. This establishes the claim.

Now repeat this algorithm on S_1 and S_2.

The above algorithm only adds characteristic sets to D that are strictly lower than the previous ones. Hence this process must terminate in one of the following two cases:

(i) $D = \emptyset$. In this case $V(S) = \emptyset$ and $\langle S \rangle = k[x_1, \dots, x_n]$.

(ii) $D = \{C_1, \dots, C_s\}$ and $V(S) = V(P_1) \cup \dots \cup V(P_s)$ where each P_k is the prime ideal given by a characteristic set C_k.

$\square$

5 Using Wu's Method to Prove Theorems

Now we wish to use Ritt's Decomposition algorithm to describe a method for actually proving geometric theorems. To see how this is done, first recall from the end of Chapter 3 our definition of what it means for a conclusion g to follow generically from $h_1, \dots, h_r$:

Definition 5.1. *A conclusion g* ***follows generically*** *from the hypotheses $h_1, \dots, h_r$ if $g \in I(V') \subset k[u_1, \dots, u_d, x_1, \dots, x_r]$ where V' is the union of those irreducible components of $V(h_1, \dots, h_r)$ on which the u_i are algebraically independent.*

The idea is to first apply Ritt's Decomposition algorithm to our hypotheses. This will yield a collection of extended characteristic sets, $D = \{C_1, \dots, C_s\}$, which correspond to components of $V(h_1, \dots, h_r)$ defined by the prime ideals $P_1, \dots, P_s$. Note that if k is algebraically closed, we may conclude that these varieties are irreducible, but that if k is not algebraically closed we may not.

We wish to pick out those irreducible $V(P_i)$ on which the u_i's are algebraically independent. (In other words, we are looking for the P_i that do not contain any nonzero u-polynomials.) Identifying on which components the u_i are independent is simple: we pick those $V(P_i)$ such that the corresponding extended characteristic set C_i contains no polynomials involving only the u_i's.

To see that this is sufficient, consider some C_k that contains no polynomials only in the u_i. Suppose that some u-polynomial $g \in P_k$. This implies that $\text{prem}(g, C_k) = 0$, which is impossible, since g must be reduced with respect to C_k.

It is possible that of the components on which the u_i are algebraically independent, some have dimension higher than d. (Recall that in Theorem 4.18 (*iv*) we only proved that dim $V(P) \geq d$.) However, as Chou notes ([1] p. 47) this is very rare. He observes that among the 600 theorems proved by his implementation, none had any components that fit this description. Thus, we will treat this occurrence as a degenerate condition (as Chou does), and ignore these components with dimension greater than d.

The remaining components are all of dimension d, so by Proposition 4.19 we know that they are irreducible.

Recall that we have assumed that the C_k, which are irreducible ascending chains, all satisfy the Dimensionality Requirement (See Remark 4.14). In other words we are requiring that each of the x_i's actually appear as the leading variable of a polynomial in our ascending chain. If they do not, and some dependent variable x_i is missing, we should reexamine the translation of the problem.

Let $\text{prem}(g, C_k)$ denote successive pseudodivision of g by the elements of the characteristic set C_k. Now, by (iv) of Proposition 4.18, we know that if $\text{prem}(g, C_k) = 0$ then g vanishes on $V(P_k)$, the component of $V(h_1, \ldots, h_n)$ corresponding to C_k.

Hence to check the conditions in the definition above simply find $\text{prem}(g, C_k)$ for each C_k that does not contain a polynomial involving only the u_i. If in each case the pseudoremainder is zero, then g follows generally from $h_1, \ldots, h_n$.

This last comment omitted an important exception. When we find each pseudoremainder, we get an expression of the form

$$d_1^{s_1} \cdots d_r^{s_r} g = Q_1 f_1 + \cdots + Q_r f_r + R.$$

So in order to conclude that g does indeed vanish on the component of $V(h_1, \ldots, h_r)$ corresponding to this characteristic set we must additionally assume that each $d_j \neq 0$. These comprise our nondegenerate conditions for our geometric theorem. This discussion establishes the following result

Theorem 5.2. *Let $h_1, \ldots, h_r, g$ be as above and let $D = \{C_1, \ldots, C_s\}$ be just those extended characteristic sets obtained from Ritt's Decomposition algorithm on which the u_i are algebraically independent. Then if $prem(g, C_k) = 0$ for all k then g is generically true under the degenerate conditions $d_j \neq 0$, where the d_j are the initials of the polynomials in each C_k.*

It may be that we get a pseudoremainder of zero on some but not all of the components in the above theorem. In this case the formulation of the geometric theorem should be reexamined for errors or hidden hypotheses. However, if we get a nonzero remainder on every component in the above theorem, then we may safely conclude that g is generally false.

We note again that we have assumed throughout that our hypothesis (and hence all resulting characteristic sets) satisfy the Dimensionality Requirement (Remark 4.14), since a failure to meet this condition usually implies a need to reformulate the theorem.

Finally, as noted by Chou ([1, page 54]), it is very rare that Ritt's Decomposition algorithm will yield more than one characteristic set. Specifically, it is usually the case that the variety V' in Definition 5.1 above is actually irreducible.

A Implementing Wu's Method in Maple

We present in this appendix some very basic Maple code that performs the essential elements of Wu's Method. If the reader is interested, a more extensive implementation was created by Dongming Wang in the Maple package CharSet. For our purposes, we wish only to implement the basic parts of Ritt's Decomposition algorithm.

We begin with some very simple procedures that we will need as tools later on in Ritt's Algorithm. First we have a procedure that returns the class of a given polynomial.

```
class:= proc(p::polynom,depvars::list)
        local V,test,i;
        V:=indets(p);
        V:=V[];
        V:=[V];
        V:=sort(V);
        for i from 0 to nops(depvars)-1 do
              if member(depvars[nops(depvars)-i],V)
                       then RETURN(nops(depvars)-i);
              fi;
        od;
        RETURN(0);
end;
```

In general, our code requires the input of the dependent variables, i.e. the x_i. This is not a terribly restrictive requirement, since a human must typically translate the theorem. Next, recall that we discussed an ordering on polynomials using the notion of class. Hence we have a procedure that compares two polynomials and returns TRUE if the first is less than the second:

```
PolyCompare:= proc(f::polynom,g::polynom,depvars::list)
              if class(f,depvars) < class(g,depvars) then
                      RETURN(true);
              elif class(f,depvars)=class(g,depvars) then
                      i:=class(f,depvars);
                      if degree(f,depvars[i])< degree(g,depvars[i]) then
                               RETURN(true);
                      else RETURN(false);
                      fi;
              else RETURN(false);
```

```
              fi;
end;
```

Now in the algorithm described in Theorem 4.11 we have a sequence of polynomial sets from which we must select the least polynomial. Our next procedure performs this task on a polynomial set.

```
LeastPoly:=proc(S::list,depvars::list)
               if nops(S)=1 then
                      RETURN(S[1]);
               fi;
               i:=1;
               j:=1;
               counter:=1;
               IsLeastPoly:=false;
               while IsLeastPoly = false do
                      if i=jthen
                            j:=j+1;
                            counter:=counter+1;
                      elif PolyCompare(S[j],S[i],depvars)=true then
                            i:=j;
                            j:=1;
                            counter:=1;
                      else
                            counter:=counter+1;
                            j:=j+1;
                      fi;
                      if counter=nops(S)+1 then
                            IsLeastPoly:=true;
                      fi;
               od;
               RETURN(S[i]);
end;
```

The algorithm in Theorem 4.11 also requires that we decide whether one polynomial is reduced with respect to another. So we introduce a procedure that performs this simple task.

```
Reduced:=proc(f::polynom,g::polynom,depvars::list)
             gClass:=class(g,depvars);
             fDegree:=degree(f,depvars[gClass]);
             gDegree:=degree(g,depvars[gClass]);
             if fDegree< gDegree then
                       RETURN(true);
             else
```

```
                    RETURN(false);
          fi;
end;
```

Now we are ready to write a procedure that performs the algorithm in Theorem 4.11. Since our implementation is intended to be used on geometric theorems, we have ignored the possibility that our starting polynomial set may contain a constant. This in general will not occur in a properly translated theorem. The following procedure yields a characteristic set of a given polynomial set (as usual we require the input of the list of independent variables).

```
CharSet:=proc(S::list,depvars::list)
        C:=[];
        S1:=S;
        SCopy:=S;
        isCharSet:=false;
        while isCharSet=false do
              C:=[op(C),LeastPoly(SCopy,depvars)];
              S1:=[];
              for j from1 to nops(SCopy) do
                        isReduced:=true;
                        for i from1 to nops(C) do
                                  Check:=Reduced(SCopy[j],C[i],depvars);
                                  if Check=falsethen
                                               isReduced:=false;
                                  fi;
                        od;
              if isReduced=true then
                        S1:=[op(S1),SCopy[j]];
              fi;
              od;

              if nops(S1)=0 then
                        isCharSet:=true;
              fi;
              SCopy:=S1;
        od;
        RETURN(C);
end;
```

Before we present the code for producing an extended characteristic set, we need procedures that perform successive pseudodivision. For completeness, we include both a version that handles polynomials that are in triangular form and another that performs the recursively defined version mentioned at the end of Section 4.1.1. We call them SuccessivePrem and RecursivePrem respectively.

```
SuccessivePrem:=proc(g,L::list,depvars::list)
          local i,R;
          R:=g;
          for ifrom  0 to nops(L)-1 do
               R:=prem(R,L[nops(depvars)-i],depvars[nops(depvars)-i]);
          od;
end;
```

```
RecursivePrem:=proc(g::polynom,S::list,depvars::list)
   r:=g;
   for ifrom 0 to nops(S)-1 do
        r:=prem(r,S[nops(S)-i],depvars[eval(class(S[nops(S)-i],depvars))]);
   od;
   RETURN(r);
end;
```

Now we have the tools necessary to write a procedure that performs the algorithm described in Ritt's Principle. This procedure takes a set of polynomials (and the list of independent variables) and returns an extended characteristic set.

```
ExtCharSet:=proc(S::list,depvars::list)
                S1:=S;
                S2Unchanged:=false;
                S2:=[];

                while S2Unchanged=false do
                C1:=CharSet(S1,depvars);
                counter:=0;
                       for i from 1 to nops(S1) do
                             r:=RecursivePrem(S1[i],C1,depvars);
                             if member(r,C1) then
                             fi;
                             if r<>0then
                                     S2:=[op(S2),r];
                                     counter:=counter+1;
                             fi;
                       od;
                       if counter=0 then
                             S2Unchanged:=true;
                       fi;
                       S1:=[op(S1),op(S2)];
                od;
                RETURN(C1);
end;
```

Obviously, this procedure does not check if the resulting extended characteristic set is irreducible. This can certainly be done, as Maple has numerous tools for factoring polynomials. However, we were more interested in the implementation of algorithms for producing characteristic sets than in algorithms for factoring polynomials. Also, as we've noted before, in most cases in plane geometry the resulting extended characteristic set will indeed be irreducible. If this is not the case, the user can easily check each polynomial in the extended characteristic set for factorability, and then repeat the process on each resulting polynomial set using the code above.

We have tested our code on the following examples: (These examples were drawn from theorems proven mechanically by Chou's implementation in [1]. Our implementation differs somewhat from his, so the extended characteristic sets found in these examples may be different than in [1].) Also, in all of these examples, the characteristic set is irreducible.

Example 1 Let $ABCD$ be a square, with $\overline{CG}$ parallel to $\overline{BD}$. Construct a point E on $\overline{CG}$ such that $\overline{BE} \equiv \overline{BD}$. F is the intersection of $\overline{BE}$ and $\overline{DC}$. Then $\overline{DF} \equiv \overline{DE}$.

If we let $A = (0,0), B = (u_1, 0), C = (u_1, u_1), D = (0, u_1), E = (x_1, x_2)$ and $F = (x_3, u_1)$, then we can express the hypotheses as $h_1 = x_2^2 + x_1^2 - 2u_1x_1 - u_1^2, h_2 = -u_1x_2 - u_1x_1 + 2u_1^2, h_3 = -x_2x_3 + u_1x_2 + u_1x_1 - u_1^2$. The conclusion is given by $g = x_3^2 - x_2^2 + 2u_1x_2 - x_1^2 - u_1^2$. Using the code above, our calculations in Maple are as follows:

```
> S1:=[x2^2+x1^2-2*u1*x1-u1^2,-u1*x2-u1*x1+2*u1^2,
-x2*x3+u1*x2+u1*x1-u1^2]:
> g:=x3^2-x2^2+2*u1*x2-x1^2-u1^2:
> C:=ExtCharSet(S1,[x1,x2,x3]);
```

$$[2u1^2x1^2 - 6u1^3x1 + 3u1^4, -u1\,x2 - u1\,x1 + 2u1^2, -u1^3 - u1\,x1\,x3 + 2u1^2x3]$$

```
> SuccessivePrem(g,C,[x1,x2,x3]);
                                    0
```

Recall from our discussions above that this means we have proven this theorem under certain degenerate conditions. Specifically, these degenerate conditions are the leading coefficients of the polynomials in the extended characteristic set C. So the theorem is true under the conditions:

$$2u_1^2 \neq 0$$
$$-u_1 \neq 0$$
$$2u_1^2 - u_1x_1 \neq 0.$$

Under these restrictions, the above theorem holds. For example, the first condition requires that A and B are distinct points.

Example 2 We use the same situation as in Example 1 in Section 2. Then we get the following calculations in Maple:

```
> S1:=[x2-u3,(x1-u1)*u3-x2*u2,x4*x1-x3*u3,x4*(u2-u1)-(x3-u1)*u3]:
> g:=x1^2-2*x1*x3-2*x4*x2+x2^2:
> C:=ExtCharSet(S1,[x1,x2,x3,x4]);
```

$$[u3\,x1 - u1\,u3 - u3\,u2, x2 - u3, 2u1\,u3^2x3 - u1^2u3^2 - u3^2u2\,u1,\\ -u1^2u3^4 - u3^4u2\,u1 + 2u1^2u3^3x4 + 2u1\,u3^3x4\,u2]$$

```
> SuccessivePrem(g,C,[x1,x2,x3,x4]);
                                    0
```

The degenerate conditions are:

$$u_3 \neq 0$$
$$2u_1u_3^2 \neq 0$$
$$2u_1^2u_3^3 + 2u_1u_3^3u_2 \neq 0.$$

Example 3 Next we present Pascal's Theorem, as translated in [1]. Let O be a circle, and let A, B, C, D, E, F be points on O. Let $P = \overline{AB} \cap \overline{DF}, Q = \overline{BC} \cap \overline{FE}$ and $S = \overline{CD} \cap \overline{EA}$. Then the points P, Q and S are collinear. If we let $A = (0,0), O = (u_1, 0), B = (x_1, u_2), C = (x_2, u_3), D = (x_3, u_4), F = (x_4, u_5), E = (x_5, u_6), P = (x_7, x_6), Q = (x_9, x_8)$ and finally $S = (x_{11}, x_{10})$, then we get the following system of equations:

$$\begin{aligned}
h_1 &= x_1^2 - 2u_1x_1 + u_2^2 = 0\\
h_2 &= x_2^2 - 2u_1x_2 + u_3^2 = 0\\
h_3 &= x_3^2 - 2u_1x_3 + u_4^2 = 0\\
h_4 &= x_4^2 - 2u_1x_4 + u_5^2 = 0\\
h_5 &= x_5^2 - 2u_1x_5 + u_6^2 = 0\\
h_6 &= (u_5 - u_4)x_7 + (-x_4 + x_3)x_6 + u_4x_4 - u_5x_3 = 0\\
h_7 &= u_2x_7 - x_1x_6 = 0\\
h_8 &= (u_6 - u_5)x_9 + (-x_5 + x_4)x_8 + u_5x_5 - u_6x_4 = 0\\
h_9 &= (u_3 - u_2)x_9 + (-x_2 + x_1)x_8 + u_2x_2 - u_3x_1 = 0\\
h_{10} &= u_6x_{11} - x_5x_{10} = 0\\
h_{11} &= (u_4 - u_3)x_{11} + (-x_3 + x_2)x_{10} + u_3x_3 - u_4x_2 = 0\\
g &= (x_8 - x_6)x_{11} + (-x_9 + x_7)x_{10} + x_6x_9 - x_7x_8 = 0.
\end{aligned}$$

Then Maple gives us the following:

```
> S4:=[x1^2-2*u1*x1+u2^2,x2^2-2*u1*x2+u3^2,x3^2-2*u1*x3+u4^2,
x4^2-2*u1*x4+u5^2,x5^2-2*u1*x5+u6^2,(u5-u4)*x7+(-x4+x3)*x6+u4*x4-
u5*x3,u2*x7-x1*x6,(u6-u5)*x9+(-x5+x4)*x8+u5*x5-u6*x4,(u3-u2)*x9+
(-x2+x1)*x8+u2*x2-u3*x1,u6*x11-x5*x10,(u4-u3)*x11+(-x3+x2)*x10+
u3*x3-u4*x2]:
> g:=(x8-x6)*x11+(-x9+x7)*x10+x6*x9-x7*x8:
> C:=ExtCharSet(S4,[x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11]);
```

$$C = [x1^2-2u1\,x1+u2^2, x2^2-2u1\,x2+u3^2, x3^2-2x3\,u1+u4^2, x4^2-2u1\,x4+u5^2, x5^2-2u1\,x5+u6^2, -x1\,x6\,u5+x1\,x6\,u4+u2\,x6\,x4-u2\,x6\,x3-u2\,u4\,x4+u2\,u5\,x3, -x1\,u5^2x7+2x1\,u5\,x7\,u4-x1\,u5\,u4\,x4+x1\,u5^2x3-x1\,u4^2x7+x1\,u4^2x4-x1\,u4\,u5\,x3+u2\,x4\,x7\,u5-u2\,x4\,x7\,u4-u2\,x3\,x7\,u5+u2\,x3\,x7\,u4, -u6\,x8\,x2+u6\,x8\,x1+u6\,u2\,x2-u6\,u3\,x1+u5\,x8\,x2-u5\,x8\,x1-u5\,u2\,x2+u5\,u3\,x1+x8\,x5\,u3-x8\,x5\,u2-x8\,x4\,u3+x8\,x4\,u2-u5\,x5\,u3+u5\,x5\,u2+u6\,x4\,u3-u6\,x4\,u2, -x1\,u5^2x5+x1\,u5^2x9+u5^2x2\,x5-u5^2x2\,x9-u6^2x1\,x4+u6^2x1\,x9+u6^2x2\,x4-u6^2x2\,x9+x1\,u5\,u6\,x4+x5\,u3\,x9\,u6-x5\,u3\,x9\,u5-x5\,u2\,x9\,u6+x5\,u2\,x9\,u5-x4\,u3\,x9\,u6+x4\,u3\,x9\,u5+u2\,x4\,x9\,u6-u2\,x4\,x9\,u5+u6\,u2\,x2\,x5-u6\,u2\,x2\,x4-u6\,u3\,x1\,x5+u6\,u3\,x1\,x4-u5\,u2\,x2\,x5+u5\,u2\,x2\,x4+u5\,u3\,x1\,x5-u5\,u3\,x1\,x4+2u6\,x2\,x9\,u5-u6\,x2\,u5\,x5-2u6\,x1\,x9\,u5+u6\,x1\,u5\,x5-u5\,x2\,u6\,x4, -u6\,x10\,x3+u6\,x10\,x2+u6\,u3\,x3-u6\,u4\,x2+x5\,x10\,u4-x5\,x10\,u3, -u6^2x11\,x3+u6^2x11\,x2+u6\,x11\,x5\,u4-u6\,x11\,x5\,u3+x5\,u6\,u3\,x3-x5\,u6\,u4\,x2]$$

```
> SuccessivePrem(g,C,[x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11]);
                                 0
```

The degenerate conditions are:

$$\begin{aligned}
-x_1u_5+x_1u_4+u_2x_4-u_2x_3 &\neq 0\\
-x_1u_5^2+2x_1u_5u_4-x_1u_4^2+u_2x_4u_5-u_2x_4u_4-u_2x_3u_5+u_2x_3u_4 &\neq 0\\
-u_5x_1+u_5x_2-x_4u_3+x_5u_3-u_6x_1+x_4u_2 &\neq 0\\
x_1u_5^2+u_6x_4u_2-u_5^2x_2+\cdots-2u_6x_1u_5-u_6^2x_2 &\neq 0\\
-u_6x_3+u_6x_2+x_5u_4-x_5u_3 &\neq 0\\
u_6x_5u_4-x_5u_3u_6-u_6^2x_3+u_6^2x_2 &\neq 0.
\end{aligned}$$

Example 4 This example uses the same theorem as in Example 2 in Section 2 which stated that the altitudes of a triangle all meet in a single point (called the orthocenter). As we saw in that example our hypotheses and conclusion equations are given by:

$$
\begin{aligned}
h_1 &= x_2u_2 - x_1u_3 = 0\\
h_2 &= x_4(u_2-u_1) - u_3(x_3-u_1) = 0\\
h_3 &= x_2u_3 + u_2(x_1-u_1) = 0\\
h_4 &= x_4u_3 + x_3(u_2-u_1) = 0\\
h_5 &= (x_2-x_5)(x_1-u_1) - x_2(x_1-u_2) = 0\\
h_6 &= x_6x_3 - x_4u_2 = 0\\
g &= x_5 - x_6 = 0.
\end{aligned}
$$

When entered into Maple we get the following calculations:

```
> S1:=[x2*u2-x1*u3,x4*(u2-u1)-u3*(x3-u1),
x2*u3+u2*(x1u1),x4*u3+x3*(u2-u1),
(x2-x5)*(x1-u1)-x2*(x1-u2),x6*x3-x4*u2]:
> g:=x5-x6:
> C:=ExtCharSet(S1,[x1,x2,x3,x4,x5,x6]);
```

$$
\begin{aligned}
C = [&-\mathit{u2}^2\mathit{u1} + \mathit{u2}^2\mathit{x1} + \mathit{u3}^2\mathit{x1},\ \mathit{u2}^3\mathit{x2} + \mathit{u2}\,\mathit{x2}\,\mathit{u3}^2 - \mathit{u2}^2\mathit{u1}\,\mathit{u3},\ \mathit{x3}\,\mathit{u2}^2 -\\
&2\mathit{u2}\,\mathit{x3}\,\mathit{u1} + \mathit{x3}\,\mathit{u1}^2 + \mathit{u3}^2\mathit{x3} - \mathit{u3}^2\mathit{u1},\ \mathit{u2}^3\mathit{x4} - 3\mathit{u2}^2\mathit{u1}\,\mathit{x4} + \mathit{u2}^2\mathit{u1}\,\mathit{u3} + 3\mathit{u2}\,\mathit{u1}^2\mathit{x4} -\\
&2\mathit{u2}\,\mathit{u1}^2\mathit{u3} - \mathit{u1}^3\mathit{x4} + \mathit{u1}^3\mathit{u3} + \mathit{u3}^2\mathit{u2}\,\mathit{x4} - \mathit{u3}^2\mathit{u1}\,\mathit{x4},\ \mathit{u3}^2\mathit{u1}\,\mathit{x5}\,\mathit{u2} + \mathit{u3}\,\mathit{u1}\,\mathit{u2}^3 -\\
&\mathit{u3}\,\mathit{u1}^2\mathit{u2}^2,\ \mathit{u3}\,\mathit{u1}\,\mathit{u2}^3 - 2\mathit{u3}\,\mathit{u1}^2\mathit{u2}^2 + \mathit{u3}\,\mathit{u1}^3\mathit{u2} + \mathit{u2}\,\mathit{x6}\,\mathit{u1}\,\mathit{u3}^2 - \mathit{u3}^2\mathit{u1}^2\mathit{x6}]
\end{aligned}
$$

```
> SuccessivePrem(g,C,[x1,x2,x3,x4,x5,x6]);
                              0
```

The degenerate conditions are:

$$
\begin{aligned}
u_2^2 + u_3^2 &\neq 0\\
u_2^3 + u_2u_3^2 &\neq 0\\
u_2^2 - 2u_2u_1 + u_1^2 + u_3^2 &\neq 0\\
u_2^3 - 3u_2^2u_1 + 3u_2u_1^2 - u_1^3 + u_3^2u_2 - u_3^2u_1 &\neq 0\\
u_3^2u_1u_2 &\neq 0\\
u_2u_1u_3^2 - u_3^2u_1^2 &\neq 0.
\end{aligned}
$$

Example 5 Here we prove the well known theorem due to Pappus. Let A, B, C and A', B', C' be two sets of collinear points. Then let $P = \overline{AB'} \cap \overline{A'B}$, $Q = \overline{AC'} \cap \overline{A'C}$ and finally let $R = \overline{BC'} \cap \overline{B'C}$. Then the points P, Q, R are collinear.

For our translation, let $A = (0,0), B = (u_1, 0), C = (u_2, 0), A' = (u_3, u_4), B' = (u_5, u_6), C' = (u_7, x_1), P = (x_2, x_3), Q = (x_4, x_5), R = (x_6, x_7)$. Note that the point C' is partially dependent on our choices of A, B, A', B', so one of its coordinates is x_1. Then we have the following seven hypotheses and conclusion:

$$\begin{aligned}
h_1 &= (u_6 - u_4)(u_7 - u_3) - (x_1 - u_4)(u_5 - u_3) \\
h_2 &= x_3u_5 - u_6x_2 \\
h_3 &= u_4(x_2 - u_1) - x_3(u_3 - u_1) \\
h_4 &= x_5u_7 - x_1x_4 \\
h_5 &= x_5(u_3 - u_2) - u_4(x_4 - u_2) \\
h_6 &= x_7(u_7 - u_1) - x_1(x_6 - u_1) \\
h_7 &= u_6(x_6 - u_2) - x_7(u_5 - u_2) \\
g &= (x_5 - x_3)(x_6 - x_2) - (x_7 - x_3)(x_4 - x_2).
\end{aligned}$$

In Maple, this translation yields the following:

```
> pappus:=[(u6-u4)*(u7-u3)-(x1-u4)*(u5-u3),x3*u5-u6*x2,u4*(x2-u1)-
x3*(u3-u1),
x5*u7-x1*x4,x5*(u3-u2)-u4*(x4-u2),x7*(u7-u1)-x1*(x6-u1),u6*(x6-u2)-
x7*(u5-u2)]:
> c:=(x5-x3)*(x6-x2)-(x7-x3)*(x4-x2):
> C:=ExtCharSet(pappus,[x1,x2,x3,x4,x5,x6,x7]);
```

$$C = [(u6 - u4)(u7 - u3) - (x1 - u4)(u5 - u3)\ldots]$$

```
> SuccessivePrem(c,C,[x1,x2,x3,x4,x5,x6,x7]);
                                0
```

The (extremely large) set C has been omitted for space reason. Here the degenerate conditions are:

$$\begin{aligned}
-u_5 + u_3 &\neq 0 \\
u_5u_4 - u_6u_3 + u_6u_1 &\neq 0 \\
u_5^2u_4 - u_5u_6u_3 + u_5u_6u_1 &\neq 0 \\
-u_5u_7u_4 + u_6u_7u_3 - u_6u_7u_2 - u_6u_3^2 + u_6u_3u_2 + u_4u_7u_2 + u_4u_5u_3 - u_4u_5u_2 &\neq 0 \\
u_7^2u_4u_5^2 - u_5u_7^2u_6u_3 + \cdots + u_7^2u_6u_3^2 - u_3u_7^2u_6u_3 &\neq 0 \\
u_5u_6u_3 - u_6u_7u_3 + \cdots + u_4u_5u_2 - u_6u_3u_2 &\neq 0.
\end{aligned}$$

Example 6 Finally, we include an example in which we discovered an error in Chou's proof of Simson's Theorem as presented in [1]. Chou states the theorem as follows: Let D be a point on the circumscribed circle (with center O) of triangle ABC. From D draw three perpendiculars to the sides of the triangle,

$\overline{BC}, \overline{CA}, \overline{AB}$. Let E, F, G be the three feet respectively. Then E, F, G are collinear.

Chou translates the theorem as follows: let $A = (0, 0), B = (u_1, 0), C = (u_2, u_3), O = (x_2, x_1), D = (x_3, x_4), E = (x_5, x_4), F = (x_7, x_6), G = (x_3, 0)$. The hypotheses are:

$$
\begin{array}{ll}
h_1 = 2u_2x_2 + 2u_3x_1 - u_3^2 - u_2^2 = 0 & \overline{OA} \equiv \overline{OC} \\
h_2 = 2u_1x_2 - u_1^2 = 0 & \overline{OA} \equiv \overline{OB} \\
h_3 = -x_3^2 + 2x_2x_3 + 2u_4x_1 - u_4^2 = 0 & \overline{OA} \equiv \overline{OD} \\
h_4 = u_3x_5 + (-u_2 + u_1)x_4 - u_1u_3 = 0 & E, B, C \text{ collinear} \\
h_5 = (u_2 - u_1)x_5 + u_3x_4 + (-u_2 + u_1)x_3 - u_3u_4 & \overline{DE} \perp \overline{BC} \\
h_6 = u_3x_7 - u_2x_6 = 0 & F, A, C \text{ collinear} \\
h_7 = u_2x_7 + u_3x_6 - u_2x_3 - u_3u_4 = 0 & \overline{DF} \perp \overline{AC}
\end{array}
$$

and the conclusion is given by $g = x_4x_7 + (-x_5 + x_3)x_6 - x_3x_4 = 0$. Chou then triangulates these hypotheses yielding the irreducible ascending chain:

$$
\begin{aligned}
f_1 &= 4u_1u_3x_1 - 2u_1u_3^2 - 2u_1u_2^2 + 2u_1^2u_2 = 0 \\
f_2 &= 2u_1x_2 - u_1^2 = 0 \\
f_3 &= -x_3^2 + 2x_2x_3 + 2u_4x_1 - u_4^2 = 0 \\
f_4 &= (-u_3^2 - u_2^2 + 2u_1u_2 - u_1^2)x_4 + (u_2 - u_1)u_3x_3 + u_3^2u_4 + (-u_1u_2 + u_1^2)u_3 = 0 \\
f_5 &= u_3x_5 + (-u_2 + u_1)x_4 - u_1u_3 = 0 \\
f_6 &= (-u_3^2 - u_2^2)x_6 + u_2u_3x_3 + u_3^2u_4 = 0 \\
f_7 &= u_2x_7 + u_3x_6 - u_2x_3 - u_3u_4 = 0.
\end{aligned}
$$

However, it is easy to verify using Maple that successive pseudodivision on this set of equations does not yield a remainder of zero, as it should. We believe that Chou's error lies in his translation of the problem. His construction of the point $D = (x_3, u_4)$ is incorrect. If one constructs each point of the triangle ABC in succession, then we are left in a serious difficulty in constructing D. The coordinates for D are only partially restricted by our choices for the coordinates of A, B and C. In particular, we must have that x_3 doesn't force D to lie beyond our circle. Hence, x_3 cannot really be completely determined from the previous points.

Instead, we translated the theorem as follows: Let A, C, B, D be four point on a circle centered at O. From D draw three perpendiculars to the sides of the triangle ABC: $\overline{BC}, \overline{CA}, \overline{AB}$. Let E, F, G be the three feet respectively. Then E, F, G are collinear.

Our version is clearly equivalent, and yields the following translation: $A = (0, 0), O = (u_1, 0), B = (x_1, u_2), C = (x_2, u_3), D = (x_3, u_4), E = (x_4, x_5), F = (x_6, x_7), G = (x_8, x_9)$. This gives us the following nine hypotheses:

$$
\begin{array}{lr}
h_1 = u_1^2 - (x_1 - u_1)^2 - u_2^2 = 0 & \overline{AO} \equiv \overline{BO} \\
h_2 = u_1^2 - (x_2 - u_1)^2 - u_3^2 = 0 & \overline{AO} \equiv \overline{CO} \\
h_3 = u_1^2 - (x_3 - u_1)^2 - u_4^2 = 0 & \overline{AO} \equiv \overline{DO} \\
h_4 = -x_4u_2 + x_5x_1 = 0 & E, A, B \text{ collinear} \\
h_5 = x_6(u_3 - x_7) + x_7(x_6 - x_2) = 0 & A, F, C \text{ collinear} \\
h_6 = (x_1 - x_8)(x_9 - u_3) - (u_2 - x_9)(x_8 - x_2) = 0 & B, G, C \text{ collinear} \\
h_7 = -(x_3 - x_4)x_1 - (u_4 - x_5)u_2 = 0 & \overline{DE} \perp \overline{AB} \\
h_8 = (x_3 - x_6)x_2 + (u_4 - x_7)u_3 = 0 & \overline{DF} \perp \overline{AC} \\
h_9 = (x_3 - x_8)(x_1 - x_1) + (u_4 - x_9)(u_2 - u_3) = 0 & \overline{DG} \perp \overline{BC}
\end{array}
$$

and the conclusion is given by $g = (x_4 - x_6)(x_7 - x_9) - (x_5 - x_7)(x_6 - x_8) = 0$. Using these equations, we get the extended characteristic set:

```
> Simsons:=[u1^2-(x1-u1)^2-u2^2,u1^2-(x2-u1)^2-u3^2,u1^2-
(x3-u1)^2-u4^2,-x4*u2+x5*x1,x6*(u3-x7)+x7*(x6-x2),(x1-x8)*
(x9-u3)-(u2-x9)*(x8-x2),(x3-x4)*(-x1)+(u4-x5)*(-u2),(x3-x6)*x2
+(u4-x7)*u3,(x3-x8)*(x1-x2)+(u4-x9)*(u2-u3)]:
> SimsonConclusion:=(x4-x6)*(x7-x9)-(x5-x7)*(x6-x8):
> C:=ExtCharSet(Simsons,[x1,x2,x3,x4,x5,x6,x7,x8,x9]));
```

C=[$u1^2-(x1-u1)^2-u2^2, u1^2-(x2-u1)^2-u3^2, u1^2-(x3-u1)^2-u4^2, x1\,u2\,u4+$
$2x1\,u1\,x3-2x1\,u1\,x4-u2^2x3, -x1\,u2^2u4-2u2\,x1\,u1\,x3+u2^3x3+4u1^2x5\,x1-$
$2u1\,x5\,u2^2,$
$x2\,u3\,u4+2x2\,u1\,x3-2x2\,u1\,x6-u3^2x3, x2\,u3^2u4+2u3\,x2\,u1\,x3-u3^3x3-$
$4u1^2x7\,x2+2u1\,x7\,u3^2, -2x1\,x2\,x3+2x1\,x8\,x2+x1\,u2\,u4-x1\,u3\,u4-x2\,u2\,u4+$
$x2\,u3\,u4-x1\,u3\,u2+x1\,u3^2+2x8\,u3\,u2+u2^2x2-u2\,x2\,u3+2x2\,u1\,x3-$
$2x2\,u1\,x8-u3^2x3+2x1\,u1\,x3-2x1\,u1\,x8-u2^2x3, -4x2\,u1^2u2+2u2\,x1\,u1\,x3-$
$2u3\,x2\,u1\,x3-2x1\,x2\,x3\,u2+2x1\,x2\,x3\,u3-2x1\,u2\,u4\,u3+2x2\,u2\,u4\,u3+2x2\,u1\,x3\,u2-$
$2x1\,u1\,x3\,u3+2u3\,u2\,x1\,x9-2u3\,u2\,x9\,x2-2x2\,u1\,x1\,u3+$
$2x1\,u1\,u2\,x2+x1\,u2^2u4-u2^3x3+u3^3x3-x2\,u3^2u4-x1\,u3^3+u2^3x2+$
$x1\,u3^2u4-x2\,u2^2u4-x1\,u3\,u2^2-2x1\,u3^2u2+u2\,x2\,u3^2+2u2^2x2\,u3-u3^2x3\,u2+$
$u2^2x3\,u3+4x2\,u1^2x9+2u3^2x1\,x9+2u3^2u1\,u2-2u3^2u1\,x9-4x1\,u1^2x9+$
$4x1\,u1^2u3-2u2^2x9\,x2+2u2^2u1\,x9-2u2^2u1\,u3$]

And then under successive pseudodivision we get:

```
> SuccessivePrem(SimsonConclusion,C,[x1,x2,x3,x4,x5,x6,x7,x8,x9]);
                                     0
```

References

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Problems to discover and to boost mathematical talent in early grades: A Challenging Situations Approach

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Abstract: *Several studies of mathematical giftedness conducted in the past two decades reveal the importance of creation of learning and teaching environment favourable to the identification and nurturing mathematically talented students. Based on psychological, methodological and didactical models created by Krutetskii (1976), Shchedrovtiskii (1968), Brousseau (1997) and Sierpinska (1994), we have developed our challenging situation approach. During 7 years of field study in the elementary K-6 classroom, we collected sufficient amount of data that demonstrate how these challenging situations help to discover and to boost mathematical talent in very young children keeping and increasing their interest towards more advanced mathematics curriculum. In this article, we are going to present our model and illustrate how it works in the mixed-ability classroom. We will also discuss different roles that teachers and students might play in this kind of environment and how each side could benefit from it.*

1. Introduction

The biographers of famous mathematicians often refer to the evidence of a particular nature of their talent which can be detected already at a very young age. One can ask where this deep insight in mathematics comes from. How can teachers discover their talent and nurture it? And, as a result of this discovery, what kind of a classroom environment would be advantageous for these children? What can be done by teachers to help these children to realise their potential?

From their very early pre-school and school years, mathematically gifted children are active and curious in their learning, persistent and innovative in their efforts, flexible and fast in grasping complex and abstract mathematical concepts, and thus represent a unique human intellectual resource for our society, which we have no right to waste or to loose.

Numerous studies of mathematical giftedness conducted during past decades provide us with different lists of characteristics of gifted children and suggest various models of identification and fostering them in and beyond mathematics classroom.

Long time experimentation with schoolchildren and observations made by teachers allowed Krutetskii (1976) to construct a list of characteristics of mental activity have shown by mathematically gifted children in a comparatively early age :

- An ability to generalize mathematical material (an ability to discover the general in what is externally different or isolated)
- A flexibility of mental processes (an ability to switch rapidly from one operation to another, from one train of thought to another)

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- A striving to find the easiest, clearest, and most economical ways to solve problems
- An ability chiefly to remember generalized relations, reasoning schemas, and methods of solving type-problems
- Curtailment of the reasoning processes, a shortening of its individual links
- Formation of elementary forms of a particular 'mathematical' perception of the environment - as if many facts and phenomena were refracted through prism of mathematical relationships.

Miller (1990) mentions some other characteristics that may give important clues in discovering high mathematical talent:

- Awareness and curiosity about numerical information
- Quickness in learning, understanding and applying mathematical ideas
- High ability to think and work abstractly
- Ability to see mathematical patterns and relationships
- Ability to think and work abstractly in flexible, creative way
- Ability to transfer learning to new untaught mathematical situations

Another model focusing on giftedness as "intersection" of various factors has been developed by Renzulli (1977). By means of this model, Ridge and Renzulli (1981) define giftedness as an interaction among three basic clusters of human traits: above average general abilities, high levels of task commitment, and high levels of creativity. Upon their definition, gifted and talented children are those possessing or capable of developing this composite set of traits and applying them to any potentially valuable area of human performance.

In a similar way, Mingus and Grassl (1999) focused their study on students who display a combination of willingness to work hard, natural mathematical ability and / or creativity.
The authors consider **natural mathematical ability,** which might be represented by several characteristics discovered by Krutetskii (see above) as well as non-mathematical ones as **willingness to work hard** (that means being focused, committed, energetic, persistent, confident, and able to withstand stress and distraction) or **high creativity** (i.e. capacity of divergent thinking and of combining the experience and skills from seemingly disparate domains to synthesise new products or ideas). The authors labelled students possessing a high degree of mathematical ability, creativity, and willingness as "truly gifted".

Reflecting on our classroom observations of 4-5-year old children using educational software with some mathematical tasks we became interested in studying deeply mathematically precocious children

We noticed that some of them always choose more challenging activities, go through all the levels up to the highest ones, understand each activity almost without any explanation from the teacher, demonstrate very systematic approach to the problem, have very sharp selective memory of important facts, details, methods, they are very creative in their work with “open-ended” problems (such as creating puzzles and patterns), and often share their discoveries with their peers being very proud of themselves.
For example, working with counting tasks such as finding a domino piece with number of dots corresponding to a show number from 6 to 9, some children count all the dots on almost **every**

card using their fingers, others choose first one which contains **more** than 5 dots (for example, they may choose 8), then again, most of them count dots and if the result is not good, they jump randomly to another with similar number of dots. There is also a small group of children that try to spot a card with less than 8 dots on it. Finally, one child clicks immediately on card with 7 dots, saying "I know it's this one because 5 and 2 make 7".

Analyzing children's strategies, we could see their *different approach to numbers*. Some children see cards as pictures with objects to count and they use the same strategies as they were manipulative objects (like toys). Other children try to use a different, more complex approach – thinking globally (I see it's five here, I know that 7 is less than 8) and abstractly one (number as an abstract characteristic of a set of dots) along with using a number of shortcuts which helped them to increase efficiency of their mathematical work.

Our next example is a comparison task with two cards shown to the child: one with a certain number of dots arranged within an array 3x4 (12 dots as maximum) and another one with a number 1-12 written on it as a digit. The child has to decide *whether two cards present the same numbers or not.* For the most of 5 year old children, this is a relatively simple task but adding the time limit does make activity extremely challenging for children whose strategy of counting is limited by "finger pointing". The best winning strategy was found by children who used estimation (I know that I have much more dots here than number 3 on another side) and counting with eyes (without fingers). Some children were giving surprisingly deep comments like "I know this number of dots is 12 because I see 4 row of 3 dots which make 12" which demonstrate precocious insight into numbers and number relationships.

Some tasks give children an opportunity to create some patterns asking to construct a personage following certain pattern, or to create their own personage. This second option was seeing by many children more as an art activity, although our observation showed that some 4 year old children create personages upon more complex pattern of mathematical nature (like color, background, part of cloths). One activity presented a grid 6x6 with a set of different puzzles to reproduce (pictures are given as a model) or to create their own puzzle and many young children (4-5 year old) did it just as drawing another picture.

Again, we could notice few children building spontaneously more mathematically abstract tessellations using complex, sometimes symmetrical configurations of shapes which would be more expected from older children already familiar with geometric transformations like reflection or translation. Another activity presented a factory for making cookies with chocolate chips on them. One mode of this activity asks child to put a number of chips on a cookie corresponding to a randomly given number (1 to 10). Another mode prompts to create a cookie with an arbitrary chosen number of chips. Giving free choice to children it allowed us to observe some of them making cookies with consecutively chosen numbers from one to ten repeated in two rows. And even more, they were so fascinated with their result so they started to repeat the same pattern more and more without any visible fatigue, although it was a routine repetition of the same procedure. It seems that here we have an example of a mathematical creativity of a particular kind: seeing beauty of mathematical structure in the same repeating pattern.

Equalizing task is a complex task for very young children. For example, an activity of feeding rabbits with carrots shows some rabbits "waiting for a food", on another – an empty field in which a child has to put carrots keeping in mind that each rabbit would get one carrot. In fact, the child has to control two conditions at the same time to ensure that number of rabbits is equal to the number of carrots. Our observation shows that some children decide to arrange carrots in a certain geometric pattern (row, stair, or array) helping themselves to keep control of conditions showing thus more complex way of thinking.

Finally, working on ordering tasks like one of arranging 7 dolls "matreshka" in increasing or decreasing order by size, some children proceed rather by trial and error, others do it more systematically (looking at neighbors and switching if necessary). Few of them do it in a very systematical way: starting with putting a smallest/biggest one first, then going to the next smallest/biggest and so on. This strategy allows them to simplify the process of problem solving, and at the same time, shows their ability to apply more complex thinking.

Reflecting on these examples, one can ask: Why do these children demonstrate such unusual behavior at an early age? Is it simply related to the attractiveness of computer games on the screen, or does it reflect a much more complex structure of their mind? Our further study of these children’s strategies while solving “purely” mathematical tasks led us to believe that, indeed, the latter might be the case and that it is worth while searching for a specific structure of the mathematically able mind.

Our further questions were: How to identify “pure” mathematical components of the children's learning activity? What kind of cognitive structure enables a child to act like a mathematician? And, from the point of view of practicing teacher, we asked: How to organize children's mathematical activities so that they were motivated to act this way?

In the following section, we will analyze several theories that form our theoretical framework enabling us to analyze problems that help to boost mathematical talent in young children.

2. Theoretical background

Kulm (1990) remarks that since so much of school mathematics in the past has been focused on practised skills, the completion of a large number of exercises in a fixed time period has been accepted not only as a measure of mastery, but as an indication of giftedness and potential for doing advanced work. On the other hand, higher order thinking in mathematics is by very nature complex and multifaceted, requiring reflection, planning, and consideration of alternative strategies. Only the broadest limits on time for completion make sense on a test purposing to assess this type of thinking.

Burjan's (1991) recommendation to use

- Open-ended investigations and open-response problems rather than multiple-choice
- Problems allowing several different approaches

- Non-standard tasks rather than standard ones
- Tasks focusing on high-order-abilities rather than lower-level-skills
- Complex tasks requiring the use of several "pieces of mathematical knowledge" from different topics) rather than specific ones (based on one particular fact or technique)
- Knowledge-independent tasks rather than knowledge-based one

goes in the same direction.

Unfortunately, as it was mentioned by Greenes (1981), the bulk of our mathematics program is devoted to the development of computational skills and we tend to assess students' ability or capability based on successful performance of these computational algorithms (so called "good exercise doers") and have little opportunity to observe student's high order reasoning skills.

Sometimes, even a very banal math problem might deliver a clear message about distinguishing the gifted student from the good student. Greenes analyses a very simple word problem (given to 5th Grade children):

Mrs. Johnson travelled 360 km in 6 hours. How many kilometres did she travel each hour?

One bright student surprised the teacher by having difficulty to solve this easy problem. Finally, the teacher realised that the student has discovered that nothing was said about the same number of kilometres travelled each day. This example demonstrates the child's ability to detect ambiguities in the problem, which indicate him/her as mathematically gifted student.

That is why, in a later work Greenes (1997) insists on the importance of presenting situations in which students can demonstrate their talents: "One vehicle for both challenging students and encouraging them to reveal their talents is to use of rich problems and projects". Greenes mentions that such problems accomplish the following:

- Integrate the disciplines (application of concepts, skills, and strategies from the various sub-discipline of mathematics or from other content areas (including non-academic ones)
- Are open to interpretation or solution (open-beginning and open-ended problems)
- Require the formation of generalisations (recognition of common structures as basic to analogue reasoning)
- Demand the use of multiple reasoning methods (inductive, deductive, spatial, proportional, probabilistic, and analogue)
- Stimulate the formulation of extension questions
- Offer opportunities for firsthand inquiry (explore real-word problems, perform experiments and conduct investigations and surveys)
- Have social impact (well-being or safety of members of the community)
- Necessitate interaction with others

Many authors point at teacher's particular role in the process of identification of mathematically able children. Kennard (1998) affirms that the nature of the teacher's role is critical in terms of facilitating pupil’s exploration of challenging material. Hence, the identification of very able pupils becomes inextricably linked with both the provision of challenging material and forms of teacher-pupil interaction capable of revealing key mathematical abilities. The author votes for interactive and continuous model for providing identification through challenge which integrates the following strands:

- The interpretative framework
- The selection of appropriately challenging mathematical material
- The forms of interaction between teachers and pupils which provide opportunities for mathematical characteristics to be recognised and promoted
- The continuous provision of opportunities for mathematically able children to respond to challenging material

In Kennard's case study based on this model and Krutetskii's categories the identification was conducted by the so-called teacher-researcher in the classroom environment where the pupils are being taught as well as observed. The questioning approach was used in order to reveal aspects of pupils' mathematical approaches and understanding.

Ridge, Renzulli (1981) suggest three types of activities which are important for nurturing mathematical talents:

- General exploratory activities to stimulate interest in specific subject areas: experiences that would demonstrate various procedures in the professional or scientific world (through children's museums and science centres) in which students would get an opportunity to choose, explore, and experiment without the treat of having to prepare report or provide any sort of formal recapitulation.
- Group training activities to develop processes related to the areas of interest developed through general activities. The aim of these activities is to enable students to deal more effectively with content through the power of mind. Typical for these thinking and feeling processes are critical thinking, problem solving, reflective thinking, inquiry training, divergent thinking, sensitivity training, awareness development, and creative or productive thinking. Problem solving applies to
    1. The application of mathematics to the solution of problems in other fields
    2. The solution of puzzles or logically oriented problems
    3. The solution of problems requiring specific mathematical content and processes.
- Individual and small-group investigation of real problems. As giftedness becomes manifest as result of student's willingness to go engage in more complex, self-initiated investigative activities, the essence of this type of activities is that students become problem finders as well as problem solvers and that they investigate a real problem using methods of inquiry appropriate to the nature of the problem (p.231).

For his study, Krutetskii (1976) developed several sets of challenging mathematical problems and conducted interviews with each of chosen students offering an original way to study mathematical abilities within appropriate mathematical activity, which, taken in school instruction, consists of solving various kind of problems in the broad sense of word, including problems on proof, calculation, transformation, and construction. He analyses seven principles of a choice of mathematical problems suitable to discover mathematically able student:

1. The problems should represent about equally the different parts of school: mathematics-arithmetic, algebra, and geometry
2. Experimental problems should be of various degrees of difficulty
3. They ought to fulfil their direct purpose: solving them should help to clarify the structure of abilities
4. The problems should be oriented not so much toward a quantitative expression of the phenomenon being studied as toward revealing its qualitative features (process versus result)
5. We should try to choose problem the solving of which is primarily based on abilities, not on the knowledge, habits, or skills
6. The problems have to allow to determine how rapidly a pupil progressed in solving problems of a certain type, how well he achieved skill in solving these problems, and what were his maximum possibilities in this regard (instruction versus diagnostic)
7. The problems are supposed to allow some quintile analysis as well as qualitative one.

Analyzing different children's approaches to the problems, Krutetskii (1976) provides us with several key elements of mathematical ability showing how these challenging problems help us to recognize different children's approaches to mathematics. In a regular classroom, we often teach students direct methods of solving mathematical problems. Then, in order to test their knowledge, we give them the same kind of problem and expect them to (re-)produce the same solution.

This might lead to several paradoxes, such as Brousseau's (1997) paradox of devolution of situations when the teacher "is induced to tell the student how to solve the given problem or what answer to give, the student, having had neither to make a choice nor to try out any methods nor to modify her own knowledge or beliefs, will not give the expected evidence of the desired acquisition." Brousseau thus claims that everything the teacher undertakes in order to make the student produce the behaviors that she expects tends to deprive this student of the *necessary conditions for understanding and learning of the target notion*.

However, several studies point at the fact that, in order to access a higher level of knowledge or understanding, a person has to be able to proceed at once with an integration and reorganization (of previous knowledge). Sierpinska (1994) sees the need of "reorganizations" as one of the most serious problems in education. But we can not just tell the students "how to reorganize" their previous understanding, we can not tell them what to change and how to make shifts in focus or to generalize because we would have to do this in terms of knowledge they have not acquired yet.

Looking for new methodological approaches to teaching and learning, Shchedrovitskii (1968) gives striking examples of other paradoxes when we as educators want our children to master some kind of action by teaching it directly giving children tasks which are identical with this action. But classroom practice shows that the children not only do not learn actions that go beyond the tasks, they do not even learn the actions that we teach them within the tasks.

In our challenging situation model, we propose an active everyday use of open-ended mathematical activities that would engage children into a meaningful process of exploring, questioning, investigating, communicating and reflecting on mathematical structures and relationships. This model represents rather larger vision of mathematical giftedness that correlates with Sheffield's notion of a *mathematical promise* (Sheffield, 1999) and thus aims to give pleasure to more children to think and to act in a mathematically meaningful way.

3. Our study: general context

Our experiment reflects 7 years of classroom activities and observations with Grades K-6 children while teaching challenging mathematics courses. It has been conducted at one Montreal located private bilingual elementary school with French and English both taught as a first language. Along with a strong linguistic program (with a third language, Spanish or Italian), the school insists on offering enriched programs in all subjects including mathematics to all its students independently of their abilities and academic performance.

The school thus promotes education as a fundamental value by instilling the will to learn while developing the following intellectual aptitudes:

- being able to analyse and synthesize
- critical thinking
- art of learning

The mathematics curriculum is composed of a solid basic course whose level is almost a year ahead in comparison to the program of the Quebec's Ministry of Education (Programme de formation de l'école québécoise, 2001) and an enrichment (deeper exploration of difficult concepts and topics: logic, fractions, geometry, numbers as well as a strong emphasis on problem solving strategies). The active and intensive use of "Challenging mathematics" text-books (Lyons, Lyons) along with carefully chosen additional materials helps us create a learning environment in which the students participate in decisions about their learning in order to grow and progress at their own pace. Each child competes with himself (herself) and is encouraged to surpass himself (herself).

Since the school doesn't do any selection of students for the enriched mathematics courses, all its students (total of 238) participated in the experiment. With some of them, this author started to work at their age of 3-5, as a computer teacher. There were many students that we could observe during a long period of time (for example some of Grade 6 children in 2002-2003 were our students since Grade 1, some of them since the age of 3-5). During this period, some children had to leave the school, some of them joined the class later (in the same Grade 6, there were 2 students who started in our school in Grade 6). In terms of abilities, we can characterise our classroom as a mixed ability classroom with a significant variation in the level of achievement.

The enriched course aimed to foster children's logical reasoning and problem solving skills in all children. It is based on *challenging situations* presented in the 'Challenging Mathematics' textbook collection (Défi mathématique (Lyons, Lyons)) along with other different computer and printed resources (LOGO, Cabri, Game of Life, Internet, and so on) as well as situations created by the author. It included several topics earlier than in the regular curriculum; some topics were presented in more depth than in the regular curriculum; various topics which are not included in the regular curriculum. Such curriculum thus requires a mobilising of all the inner resources of the child: her motivation, hard mental work, curiosity, perseverance, thinking ability. Since all our students are exposed to this enriched curriculum, the differences between them become more evident.

In our further more detailed analysis, we will make explicit the role of the challenging situation itself showing that without the context of challenging situation, such opportunity for students and teachers would be lost.

4. Challenging tasks as powerful teaching and learning tools helping to discover and boost mathematical talent

The story of Gauss solving a routine problem of calculating the sum of the first hundred natural numbers is one of the well known examples of this kind. While all other children were desperately trying to add terms one by one, Gauss impressed the teacher by finding a quick and easy way to do it regrouping the terms in a special way (see, for example, *Dunham, 1990*).

But one can ask: what were the characteristics of the classroom situation, which allowed the gifted student to demonstrate his talent in mathematics?

The same story says that the teacher had chosen the task for its accessibility to all students (the task is routine) and the probably very long time that it would take the students to solve; he hoped to thus keep them all quiet and busy for a good while. What he didn't expect that one of the students would turn the routine task into a challenging one of finding a quick way of solving an otherwise tedious and long computational exercise. The situation was not planned to reveal a mathematical talent, yet it did so "spontaneously". The situation became a challenging one by chance.

In many similar cases, mathematical talent would not be identified. We could say that using routine drill tasks involving numerous standard algorithms is not, in general, offering a good opportunity to identify and nurture mathematical talents.

Sheffield (1999) calls such routine tasks "one dimensional". As an example, she cites a class of three-four graders reviewing addition of two-digit numbers with regrouping. Children are asked to complete a page of exercises such as: 57+45, 48+68, 59+37. As it usually happens with brighter and faster students, they finish all exercises before their classmates. So the teacher would "challenge" them with 3- or 4-digit additions. Although the calculations become longer and time consuming, the tasks themselves are not more complex or more mathematically interesting.

As a better didactical solution for these children, Sheffield suggests the use of meaningful tasks like one of *finding three consecutive integers with a sum of 162*:

> Students would continue to get the practice of adding two-digit numbers with regrouping, but they also would have the opportunity to make interesting discoveries. Students who are challenged to find the answer in as many ways as possible, to pose related questions, to investigate interesting patterns, to make and evaluate hypotheses about their

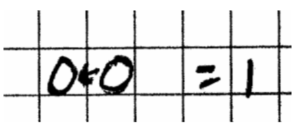
> observations, and to communicate their findings to their peers, teachers, and others will get plenty of practice adding two digit numbers, but they will also have the chance to do some real mathematics. (Sheffield, 1999: 47)

By giving children a challenging task we would expect them to make efforts in understanding a problem, to search for an efficient strategy of solving it, to find appropriate solutions and to make necessary generalizations.

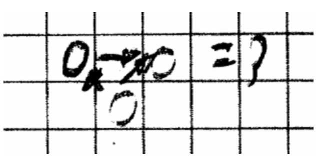
Following examples illustrate three very different approaches to the same problem of *finding a number of handshakes that we obtain when n people shake hands of each other* used by mathematically talented children.

Marc-Etienne (10) organized an experience with his classmates, considering systematically the cases n=2, n=3, etc. Then he made then necessary generalizations. Here is a transcript of his report in which we could observe several steps:

Step1: two circles connected with an arrow representing two people - one handshake. He wrote beside the picture '=1'.

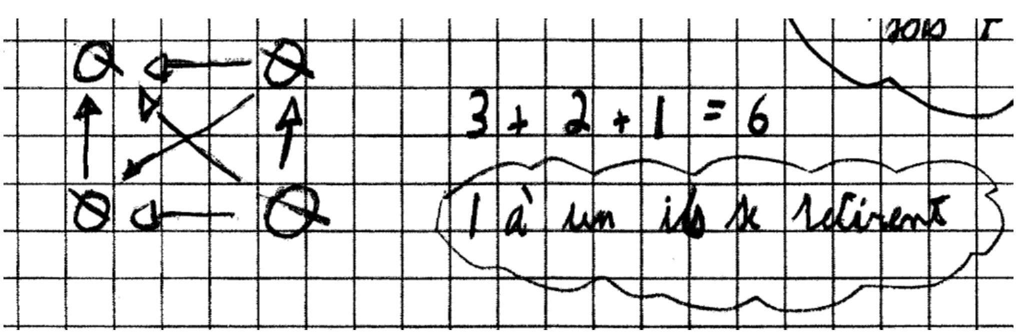


Step 2: three circles forming a triangle connected with three arrows representing three people - three handshakes. He wrote beside the picture '=3'

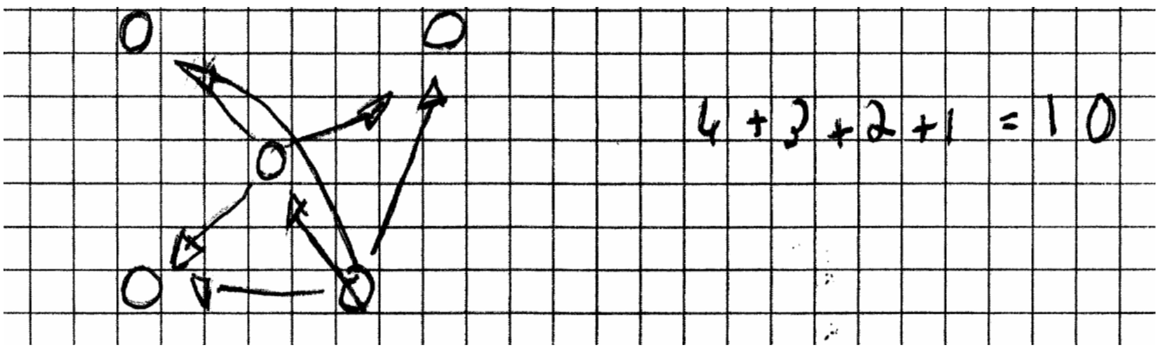


Step 3: four circles forming a square connected with six arrows representing four people - six handshakes. He wrote aside: '3+2+1=6' commenting:

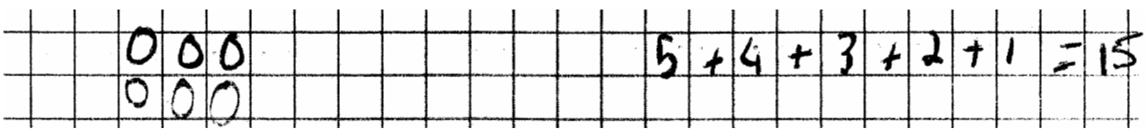
'1 after another, they leave'



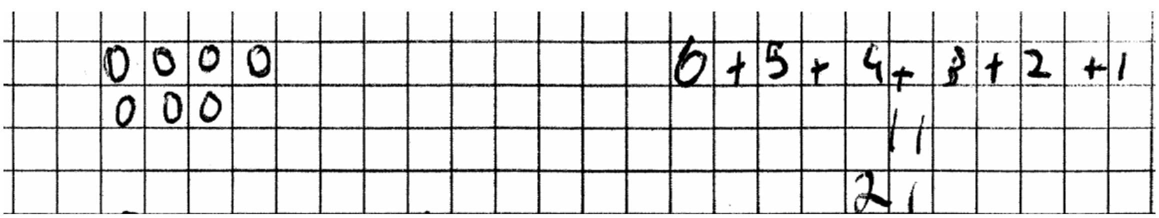
Step 4: Five circles form a 'domino-5-dots disposition' connected with only six arrows (some arrows are missing). However, he wrote '4+3+2+1=10' continuing the same pattern.



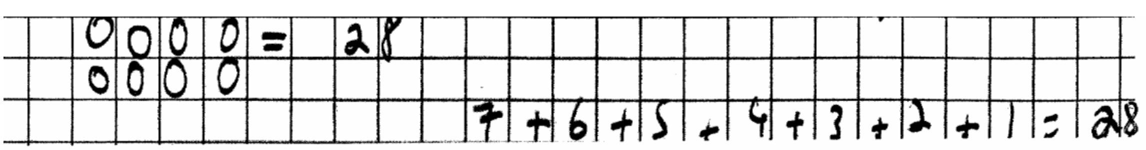
Step 5: Six circles disposed in two rows (by three), no arrows. He wrote '5+4+3+2+1=15'



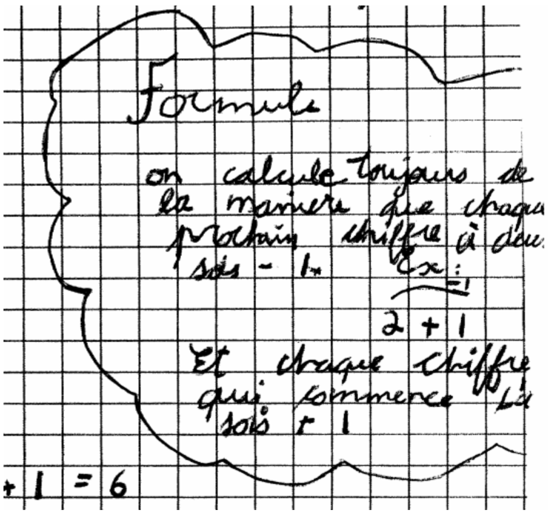
Step 6: Seven circles disposed in two rows (three+four), no arrows. He wrote '6+5+4+3+2+1=21'



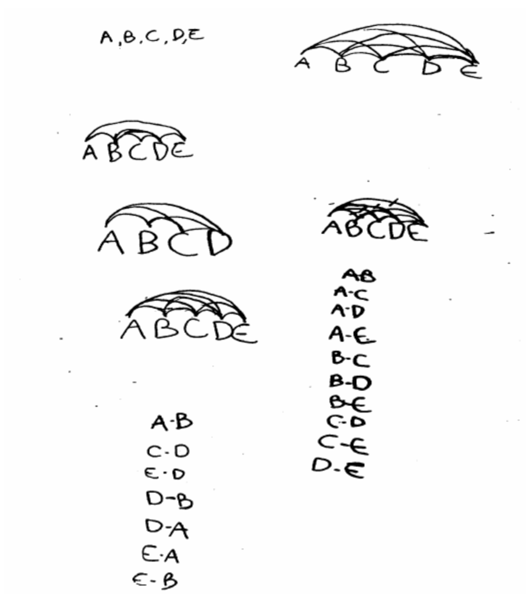
Step 7: Eight circles disposed in two rows (by four), no arrows. He wrote '7+6+5+4+3+2+1=28'



He concluded his generalization with following sentence that he called 'Formule': 'On calcule toujours de la manière que le prochain chiffre soit -1 '2+1' et que chaque chiffre qui précède soit +1'
(We calculate always in the same way in order that each next number in our sum would be 1 less. '2+1' and each previous number would be +1').



Charlotte (10) used a particular case of 5 people making diagrams doing systematic search for all possible combinations.



Christopher (10) was very short in his presentation writing just one single sentence:

1+2+3+4+5+7+8=36

He provided it with the oral explanation that if we have a group of people, each person has to shake hands to all people who came before, so with 2 people we would have 1 handshake, with 3 people - 2 more handshakes (1+2), and so on.

We can see that investigation of initial situation of 'handshakes' allowed children to explore the problem, to look for patterns and to make important mathematical generalizations. Moreover, with a simple boosting question what would be the number of handshakes with 101 people, the class arrived to the same problem that Gauss had to deal with but posed in different, challenging way. Thus, a further investigation might be provoked here in a more natural way.

And, from the point of view of practicing teacher, we can better understand *how to organize children's mathematical activities so that they were motivated to act this way.*

5. From challenging tasks to challenging curriculum: Example of kindergarten enrichment course

There are two basic approaches to design a mathematics curriculum for 5-6 year old children; one can be labelled as traditional and the other as innovative. The former is based on counting, ordering, classifying, introduces basic numbers, operations (addition and subtraction), relations (more, less, bigger, smaller, greater) and shapes. The latter puts more emphasis on learning while allowing children to play using manipulatives, colouring, arts and crafts, games with numbers and shapes. During the past decade, many creative teachers have been trying to use the best ideas from each of the two approaches also adding reasoning activities to the mathematics curriculum.

In our school, we used a traditional approach based on Quebec's *Passeport Mathématique* Grade 1 textbook along with a new French collection *Spirale (Maths CP2)* which represents the second, modern approach. However, even this combination doesn't provide our children with the material necessary for their mathematical development. There is still a gap between their level of ability and the requirements of the challenging curriculum that we use starting from Grade 1 (collection "Défi mathématique") and which is based on discovery, reasoning and understanding.

In order to fill the gap, we developed an enriched course offered to all kindergarten students (we have 30-35 children every year). The course was given on a weekly basis (1 hour a week). We base our teaching on the challenging situations approach, developing activities that stimulate mathematical questioning and investigations along with reflective thinking.

Each class starts with such questions as *What did we do last time? What problem did we have to solve? What was our way to deal with the problem? What strategies did we use?*, etc. This questioning aims to provoke reflection on the problems that children solved as well as on methods that they used. Without this reflection, rupture situation (in Shchedrovitskii's sense) would never arise, because a rupture is a break with previous knowledge, which needs to be brought to mind.

At the same time, we would ask questions that would indicate children's understanding of underlying mathematical concepts or methods that we aim to introduce (using appropriate vocabulary and/or symbolism).

During this initial discussion we usually try to bring in a new aspect which provides children with an opportunity to ask new questions, to look at the problem in a different way. Sometimes, we might ask them, simply, *what do they think we should do today?*

Thus we can pass to the new situation/new problem/new aspect of the old problem. We may do it by means of provoking questions, of interesting stories or introductory games. Following Shchedrovitskii and Brousseau, we try to avoid the teaching paradox by not providing children with direct description of the tasks or methods of solutions. We try also to keep their attention and motivate them.

After this introductory stage, children begin investigating a problem using different manipulatives: cubes, geometrical blocs, counters, etc. They work alone or in groups. During the phase of investigation, the role of the teacher becomes more modest: we give children certain autonomy to get familiar with the problem, to choose a necessary material, organise their work environment, and choose an appropriate strategy.

However, some work has to be done by the teacher to guide children through their actions. We have to make sure that the child understands the problem, the conditions that are given (rules of the game), the goal of the activity. As the child moves ahead, we shall verify his control of the situation: what she is doing now and what is the purpose of the action? (activating reflective action). We have to keep in mind that the exploration is used not only as a way to make the child do some actions but also and foremost as an introduction to mathematical concepts or methods.

Therefore, the teacher needs to be prepared to introduce the necessary mathematical vocabulary along with its mathematical meaning as well as mathematical methods of reasoning about the concepts and about the reasoning. In our experiment, we try to choose those mathematical aspects that are considered as difficult and are not normally included in the Kindergarten curriculum.

For example, when we want to introduce an activity with patterns, we would organise a game. We would start to make a line 'boy, girl, boy, girl,..' children find it easy and are happy to discover a pattern. Then we would start a new 'pattern' : 'boy, girl, boy, girl, boy, boy'. Many children would protest, saying that the pattern is wrong. But perhaps, some of them would try to look for different pattern, like "glasses, no glasses, glasses, no glasses, …".

As the game goes on, children get used to looking for familiar patterns. This is the time to challenge them more. For example, we may ask them, how many children would be in the line with the pattern 'boy, girl, boy, girl,…'. Since there were only 8 boys in the classroom, one child could make a hypothesis that it gives 8+8 children in the line. After such a line had been completed, teacher's silence could be broken by a child's voice - 'we can add one more child to the line – a girl in the beginning'.

The course is built of various challenging situations that we create in order to give children an opportunity to take a different look at mathematical activities that they usually do, to question their knowledge about mathematics trying to discover hidden links between different objects, to discover structures and relationships between data, learn to reason mathematically based on logical inference and at the same leave some space to children's mathematical creativity. We use different didactical variables in order to create obstacles making children re-organise their knowledge and create new means in order to overcome the obstacle. We were also asking our children to report on their investigations inviting them to communicate their discoveries by developing appropriate tools: diagrams, schemas, symbols, signs.

6. Guidelines for the design of challenging situations

We consider three kinds of challenging situations:

- open-ended problems and investigations
- routine work turned into a challenge by the teacher
- routine work turned into a challenge by a student

Let us consider these options in details:

6.1 Open-ended problems and investigations

As we look at the video protocol of interviews with 4-6 year old children conducted by Bednarz and Poirier (1987) within their study of number acquisition by young children, we see how the evidence of differences in organisation of mathematical work by very young children becomes explicit with the open character of given tasks.

The video presents children's work on different tasks related to the concept of number: counting, formation of collections, order, conservation, comparison. Each task that in a regular classroom might be seen as ordinary, was given by authors in a very original challenging, dynamic, and open-ended way.

The child was constantly invited to think about the process of her work (how did you do it?), to develop an efficient strategy, to re-organise, if necessary, her process, to co-ordinate her actions. Thus, the routine tasks became open-ended and a child was given an opportunity to become an organiser of her mathematical work.

In our experiment, we also tried to make problems more open than they were usually presented to the students.

For example, we can take a problem from one mathematical competition :

---->	**1**	**2**	**3**	
	4	**5**	**6**	
	7	**8**	**9**	**---->**

In this table, we enter by 1 and exit by 9.
One can only move horizontally or vertically, and it is impossible to step twice on one box. For example, moving through boxes 1-2-5-8-9, one gets a sum of 25. But not all the trajectories lead us to the number 25. Give all others 9 numbers.

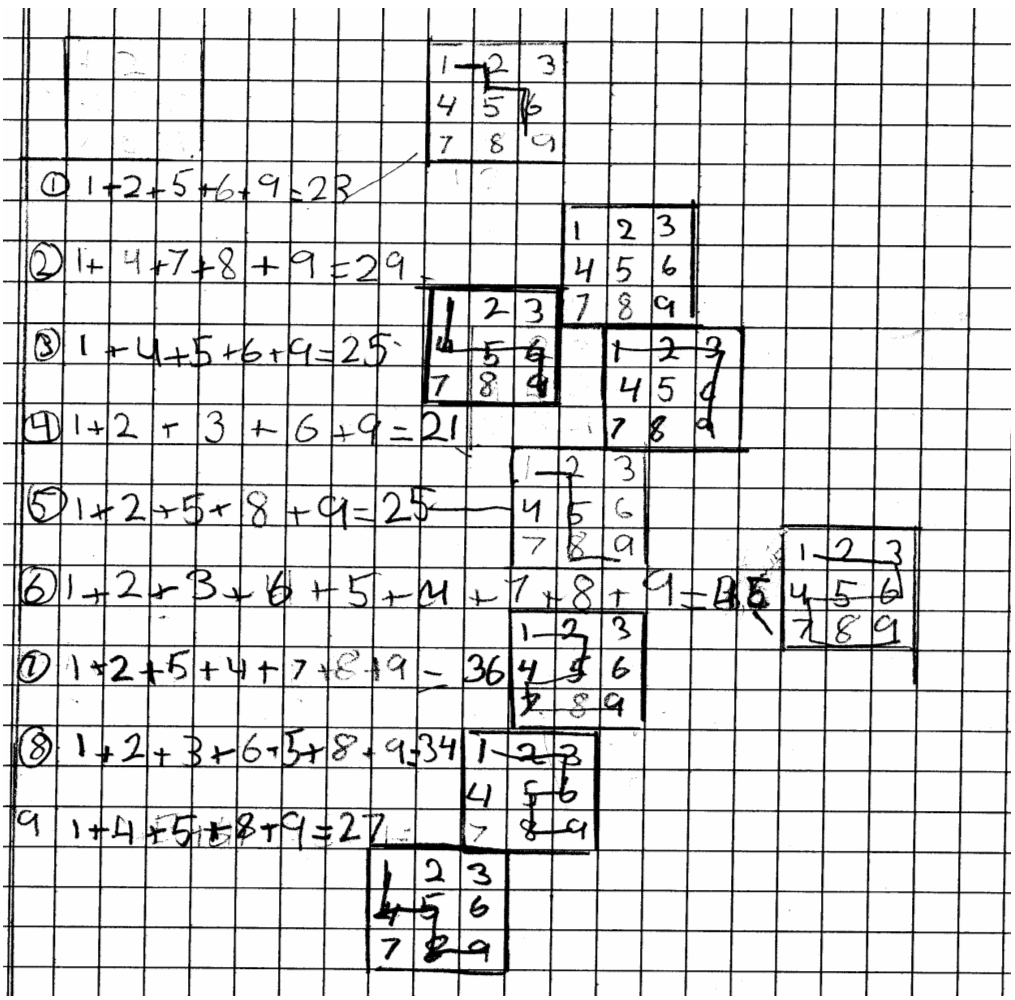
This problem was given to participants of the regional final of the Championnat International des Jeux Mathématiques et Logiques in 2000 for Grade 4-5 children (10-11 year old) **http://www.cijm.org/cijm.html**.

We found that this problem would become more challenging for children if posed in a different way (open-ended) :

Someone is going to visit a museum, which has 9 exhibition halls, arranged in a square 3x3. The number of paintings in each hall is written in the box. What are all the possible numbers of paintings that could be seen by this visitor who does not like to be in one hall twice ?

Not only do we hide the number of different ways, which makes this problem open, we give it to our Grade 1 students (6-7 year old). Every student had a task at his/her level (They will all be able to find at least a couple of solutions).

Following example illustrate the work of Chantal (6):



This example demonstrates how this open-ended situation helped the student to develop different abilities to organize systematic search and to keep tracks of her work

6.2 Routine work turned into a challenge by the teacher

The recent Quebec's school curriculum puts emphasis on the importance of mastering basic number facts (like multiplication tables). This routine task can be made more challenging in many different ways. For example, one day we wrote on the board the 9-table operations:
1 x 9 =
2 x 9 =
3 x 9 = , and so on.

Grade 3 children said immediately that it is a very easy table, because there is a well known regularity (writing down first digits of the product in order from 0 to 9 and the second ones down from 9 to 0, we obtain all the multiples of 9: 09, 18, 27, and so on). Among the answers one could find that 6 x 9 = 54.

So, the teacher comes to the board and writes 6 x 9 = 56 telling the story that when he was young, he had to memorise all answers, not just 'tricks', and he is sure that 6 x 9 = 56. The students are confused, but many of them started to think how to prove that their result (54 was the correct one).

Many of them went to the board to share their ideas as well as other ways to obtain a 9-table. As a result of the lesson, the 9-table has appeared a couple of times on the board, children said it many times aloud, so they could memorise it and at the same time do it in a meaningful way questioning and proving their methods and ways of reasoning.

6.3 Routine work turned into a challenge by a student

When Grade 4 children are asked to represent 1/8 of a rectangle, they find it an easy and routine task. That's why we were surprised by Christopher's way to divide a rectangle in 64 boxes (8 rows x 8 columns) and to colour 8 boxes randomly. He found that the task was not challenging enough and he wanted to make it more complicated.

6.4 Transformation of challenge within one situation

All three ways of creating of challenging situations are not isolated from one another. They can also be transformed one into another.

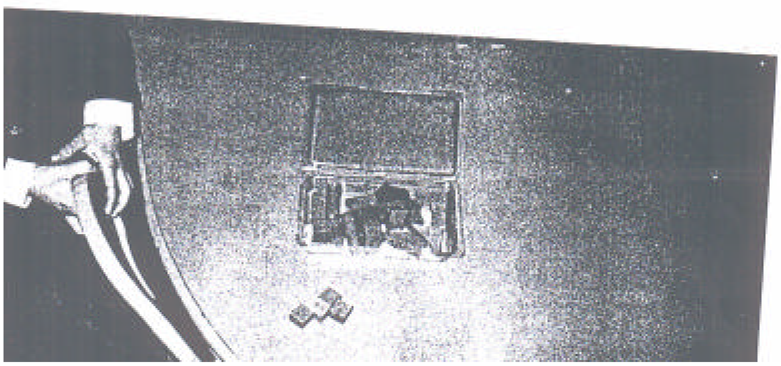
For example, a kindergarten class (5-6 year old) is working on an **open-ended** problem:
Amelie needs to build new houses for her farm animals. When one looks at the house from the sky, she sees that all of them have a roof in shape of a 'digit'. She has to build now a new house for her cows. What 'digit' do you suggest to use for the roof of this new house?

Children used blocs in form of different solids. The activity aimed to make them to explore different solids, to make different constructions with them. There are, basically, two ways of making constructions: three-dimensional or two-dimensional establishing thus different spatial relationships. For example, we may teach children to verify which shapes fit together recovering certain surface. The activity that we gave to our students didn't aim to teach any particular way of making constructions: many textbooks contain a lot of exercises asking children to reproduce one construction or another. Our situation was designed in order to help children get a certain 'spatial feeling' trying different way to layout blocs. The main challenge in it was to organise a mathematically meaningful investigation within an 'ill-defined' problem.

Some of them chose to imitate shapes of digits in the way we write them, others looked for different ways to create more 'economic' constructions taking care of geometric properties (like seeing if the blocs fit one to another) . Finally, there was a group of children who moved from the initially given situation of building a new 'house' and started to construct many digits 'writing' numbers (up to the "1000").

Soon, we could see that originally challenging and creative, the task became routine for many children. So, we decided to put some restrictions (new 'variables') that were sought as means to engage children in the investigation of a different problem in which we would be willing to construct house that has a "5"- shape and do this with a minimum of blocs. Thus, with the intervention of the teacher, a routine problem became a challenging one once again.

This method of a 'sudden' change of didactic variable (Brousseau, 1997) is important in our study of relationship between child's organisation of the problem-solving and mathematical giftedness because it provokes a reflection (what is new?) and re-organisation of the whole process of thinking and acting (what do I need to modify?) and thus gives students a chance to show their full potential.



In our experimental work with young children, we obtained a constant confirmation of the fruitfulness of such an approach, especially if one wants to identify and nurture gifted children. David (5 years old) was working on the 'minimum' task (see the picture above); he looked happy with his solution (4 blocks) but still in what looked like a 'state of alert'. At this moment, we began to discuss children's solutions. One group of children has presented a three-blocs solution. Suddenly, David started to change something in his configuration; he lost completely ‘fiveness’ of his shape while focusing on minimization task. But what is the most intriguing, is the rapid reactions of this child to the changing conditions (someone has found a better solution). This constant state of 'alert' is an important characteristics of giftedness which could be better activated in challenging situation that in the ordinary one.

This state of 'alert' leads them to constantly verify all the conditions going back and forth through the situation. Here is one more observation. Grade 4 students worked on their test. Answering a question of 'Is it true that if the sum a+c=8 then a and c are two different numbers?', Christopher hesitated a lot, saying however, that the numbers have to be different. As his work on the test went on, he had to solve a system of two equations with two variables: ab=16, a+b=8. He found easily a=4, b=4 as a solution then went back to his previous task and corrected his answer.

We could also observe another interesting phenomenon: challenging situation created by the teacher may initiate its further explorations by gifted students.

For example, doing the same activity with Grade 1 children (6-7 year old), we could state that it was seen as a routine problem by many of them and some of students lost completely their interest in it. Yet, we could still observe one girl looking for many different ways of building "5" using 4 blocs.

Not only she kept herself working on this problem, she came out with a new one: she started to look for possibilities to built a digit "4" with a minimum of blocks. Here, the problem was turned into a challenging one by the student.

6.5 The role of the teacher

In a challenging environment, the **role of the teacher** becomes crucial in all the stages:

- choice of a problem
- way of presenting it to the students
- organisation of student's work
- interpretation of results
- follow-up

One of the very important conditions of success of the challenging situations approach is the teacher's attitude. How should we, as teachers, control the student's work? Related to the learning paradox (described in the previous chapter), it is far from being obvious how to find a solution to this problem. On the one hand, every word and every gesture said by teacher can affect the whole challenge of situation in either a positive or a negative sense. On the other hand, the teacher has to have a full didactical control of the situation (otherwise a mathematical learning activity might become a sort of 'arts and crafts in mathematical wrapping').

Our experiment didn't provide us with clear recipes but rather with examples that can be open to further questioning and investigations. These examples allowed us to formulate teacher's approaches favourable for the challenging situation:

- ❑ Give a child an opportunity to think: being a flexible teacher
- ❑ Support of children's willingness to learn more about math
- ❑ Challenge students in informal situations: sense of humour
- ❑ Support children in their desire to go beyond pre-planed situations
- ❑ Giving hints without telling solutions
- ❑ Management of particular cases of mathematical giftedness
- ❑ Use of 'little tricks' as follows :

- While distributing manipulative material (blocs, cubes, etc.), we would give children time to touch it, to play with it, to get a feeling of it; sometimes it gives us important clues of children's organisations (how they put material, arrange it, order, classify, build different forms, etc.)
- When children finish their manipulation, we ask them to write a report. Sometimes it makes sense to give them time to break up their constructions. This opens the door to a variety of presentations (will the child reproduce his construction, add new details, draw a completely different pictures)

- When children are asked to communicate their results, it is important to motivate them to give detailed explanations. We often ask them to be 'mini-teachers' - to explain to somebody who doesn't understand the problem
- Children often ask us to teach them complicated things. Sometimes, a pedagogical effect can be bigger if the teacher makes them wait. Then, starting to teach it, children might become more motivated: *finally, we got it*!

6.6 The role of the student

The role of the students in a challenging situation differs significantly from those in the regular learning activity. They have to adapt to a new, open environment. They have no precise algorithm of actions, no clear instruction what to do. Therefore, they have an opportunity to:

- demonstrate different approaches to the problem
- act differently in different situations
- overcome obstacles, construct various means, discover new relationships
- work on mathematical problems based on structures and systems using properties and definitions, conjectures and proofs
- use of logical inference with fluency, control, rigour
- combine logic and creativity in problem solving
- invent new symbols and signs, use schemas and abstract drawings
- use reflective thinking
- ask mathematical questions, create new problems, investigate, use mathematics in non-mathematical situations, look around with 'mathematical eyes'

7. Conclusions and recommendations

There are a number of educational studies of mathematical giftedness. Various models of giftedness based on different characteristics of mathematically gifted students have been developed and implemented. Different programs of support provide gifted students with advanced curriculum and guidance of highly qualified professionals. Several mathematical contests, Olympiads, and competitions help in searching for mathematically gifted children and taking care of their development.

Yet, the problems of identification and nurturing of mathematical talent are far from being solved. Many children become bored, at a very early age, with the simplified curriculum, lose their interest in mathematics and waste their intellectual potential. Despite the ingenious testing system, some children never get admitted to special programs for gifted students. The regular school system is not equipped to help these children.

Our study aimed to contribute to filling this gap and providing elementary school (Grades K-6) teachers with methods of identification and fostering mathematically gifted children in the mixed ability classroom.

We have called our approach, the "challenging situations approach". The approach is theoretically grounded in Krutetskii's (1976) notion of mathematical ability, Shchedrovitskii's (1968) developmental model of reflective action, Bachelard's (1938) notion of epistemological obstacle, Sierpinska's (1994) distinction between theoretical and practical thinking in mathematics, and Brousseau's (1997) theory of didactic situations.

Following Krutetskii (1976), we have defined mathematical ability as a 'mathematical cast of mind', which represents a unique combination of psychological traits that enable young children to think in structures, to formalise, to generalise, to grasp relations between different concepts, structures, data and models and thus solve different mathematical problems more successfully than children of average or low ability.

At a very early age, these children demonstrate high thinking potential in reasoning about mathematical concepts and systems of concepts along with the capacity to reason about their reasoning. From the outset, they are better prepared than other children for theoretical thinking, which is the foundation of pure mathematical thinking.

The critical point of our study was an understanding that a discovery and nurturing of theoretical thinking is not possible if children are working with routine arithmetical tasks, merely applying algorithms that had been provided by the teacher, telling her students what to do and how to do it.

The paradoxes of such classroom situations have been described by Brousseau (1997) in his Theory of Didactical Situations. Following Brousseau's theory, we bring a notion of challenging situation into our model of mathematical giftedness postulating that a gifted child will show her talent in mathematics only in specific situations when a real question has been asked and a real problem has been posed.

"Challenging situations" use open-ended problems and mathematical investigations. A challenging situation initiates the student's action of structuring a problem, and of searching for links between data and with her previous experience. Since a real challenge is possible only when the situation is new for the learner, the challenging situation must contain a rupture with what the student has previously learned, provoking the student to reflect on the insufficiency of the past knowledge and construct new means, new mechanisms of action adapted to the new conditions, activating her full intellectual potential.

Challenging situation in its very nature gives many growing up opportunities for mathematical talent by:

- ☐ providing the student with an opportunity to face an obstacle of a pure mathematical nature, the so called epistemological obstacle. In order to overcome it, the student will have to re-organise her mathematical knowledge, create new links, new structures following laws of logical inference. We claim that situations satisfying these conditions allow the teacher to identify and nurture mathematical giftedness among her students.
- ☐ presenting a problem, which goes above or beyond the average level of difficulty. The child is encouraged to surpass what is normally expected of children of her age, thus demonstrating her precocity, which is a sign of mathematical giftedness.
- ☐ helping to create a friendly environment in which a child compete with herself sharing her discoveries with other children and learning from others. Thus it gives

mathematically gifted children who are not high achievers to participate actively in class and to succeed.

Challenging situation cannot be created as an isolated learning task. It full developmental potential can be realised only within a system of teaching based on a challenging curriculum as a whole. This would allow creating a learning environment in which every child would be able to demonstrate her highest level of ability.

This is why, using a challenging situation model we are not only able to get gifted children involved in genuine mathematical activity but also help all children to increase their intellectual potential.

Finally, challenging situation has another opening for gifted children: they can always go further, go beyond situations, ask new questions, initiate their own investigations, be more creative in their mathematical work. This spontaneous mathematical reaction feeds back into the learning environment in a positive way and further enhances its potential for all children. We consider this feature of the approach as crucial from the point of view of mathematics education for all children.

Our study prompts different teaching approaches in mathematics. The teacher is no more re-translator of knowledge or instructor of methods of problem solving. In a challenging situation her role becomes more as moderators of discussions, listeners of student's ideas, student's guide through the discovery.
In helping students go through various obstacles, we shall encourage them to:

- **Organise** his/her mathematical work
- **Reason** mathematically
- **Control** several conditions (verification, adjustment, modification, reorganisation, awareness of contradictions, validation)
- **Choose/develop** efficient strategies/tools of problem solving
- **Reflect** on methods of mathematical work
- **Communicat**e his/her results in a "mathematical" way (oral/written form, use of symbols, giving valid explanations)

Thus, we will be able to **identify** gifted children who:

- ask spontaneously questions beyond given mathematical task
- look for patterns and relationships
- build links and mathematical structures
- search for a key (essential) of the problem
- produce original and deep ideas
- keep a problem situation under control
- pay attention to the details
- develop efficient strategies
- switch easily from one strategy to another, from one structure to another
- think critically
- persist in achieving goals

At the same time, we could nurture their curiosity, willingness to learn more about mathematics, provide them with an opportunity to go further in their mathematical learning, to create new structures, to pose new problems and thus **foster** the development of their mathematical abilities.

This approach is very demanding to the teaching. The teacher has to think constantly about challenging the students, look for different ways to stimulate children's work, demonstrate a high flexibility, ability to react spontaneously on changing conditions of the classroom situation, be ready to provoke students and to get provoked by students asking question which the teacher can not answer immediately.

A better understanding of how to help highly talented children to develop deeper mathematical thinking would lead to elaboration of efficient didactical approaches for all students. We shall agree with following general remark made by Young &Tyre (1992): "If we examine more closely what it is that makes prodigies, geniuses, gifted people, high achievers, champions and medallists, we may be better able to increase their number dramatically".

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Building Blocks Problem Related to Harmonic Series

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Abstract**:** *In this discussion I give an explanation of the divergence and convergence of infinite series through the building blocks problem and at the same time I touch on the fact that mathematics is not just about manipulating complicated numerical formulas but also a field in which logical ways of thought are learnt. I emphasize that in order to overcome the aversion of university students to mathematics, teachers must pour their energies into developing study materials taken from topics relevant to the students.*

Keywords: harmonic series, center of gravity, convergence and divergence, logarithmic functions, Mathematical education

1. Is it possible to stagger building blocks by more than the width of one block?

There has been a lot of publicity about how young people avoid and are "allergic" to mathematics. The goal of mathematics is not difficult numerical formulas but a mathematical way of looking at and thinking about things and I would like to present one example of this. Let us think about the building blocks problem in Figure **1**. There are a few building blocks stacked up, and the problem is whether or not it is possible to stack them in such a way that the positions of the bottom block and the top block are horizontally separated by more than the width of one block.

Most people asked this question would immediately answer that it is not possible. I wonder if this tendency to come to a conclusion before even attempting to think about whether something is possible or not is a reflection of the digital age. Sometimes it is possible, sometimes it is not possible, additionally sometimes we do not know. But they hate vague answers very much. This is not magic or a trick, I promise that a solution certainly exists. If a person is told to stack building blocks in a staggered way, he or she will stagger them uniformly. But if they are staggered uniformly they will fall down every time. I wonder if this tendency to stack the blocks uniformly is also a manifestation of digital thinking.

We will not obtain a solution immediately. Let us start by looking at the case of two blocks. It is intuitively obvious that the distance they can be staggered is 1/2 of the width of the blocks. So the problem is the third block. Let us hold the third block in our right hand and think about this problem. Most people would try to stack this block on top of the other two but then they always fall down. If your approach does not work it is important to abandon it, and you should search for an alternative approach. To be able to do this, it is necessary to change your way of thinking about the problem.

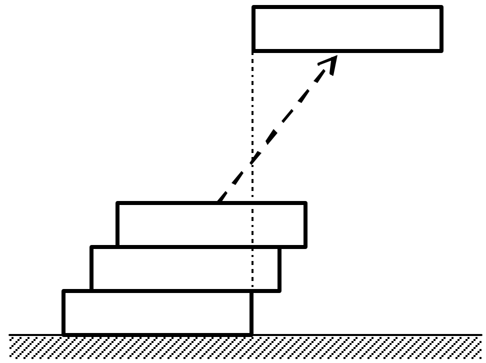

Figure **1**: Is it possible to stagger building blocks by more than the width of one block?

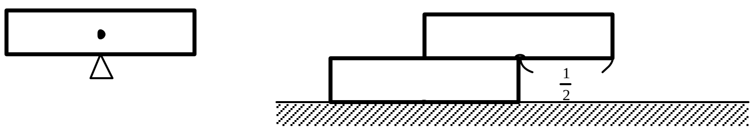

Figure **2**: 1/2 Stagger

2. Calculating the center of gravity

This is the problem of calculating the center of gravity. Rather than thinking about this problem with a pen and paper, it is surprisingly fast to use building blocks and look for the answer through trial and error.

Here I will give you a hint. Are building blocks best stacked on top of each other? You will probably be perplexed by this hint. That is because of the fixed preconception that it is because we stack them on top of each other that they are building blocks. But building blocks should not be stacked on top of each other; they should be slid under each other. If the third building block is placed at the bottom, and we gradually stagger the first two building blocks on top of the third building block while maintaining the relationship between the first two building blocks as it was, we find that we can stagger the top two blocks by 1/4. In the same way, the fourth block can be placed under the other three and staggered by 1/6, and the fifth block can be placed under the other four and staggered by 1/8. If we add 1/2, 1/4, 1/6 and 1/8 the sum is greater than 1. In other words, we have stacked the blocks in such a way that the position of the top block is horizontally separated from that of the bottom block by more than the width of one block.

While referring to Figures 2, 3, and 4, let us confirm the above approach as a center of gravity calculation using numerical formulas. First of all let us think about building block ? and building block ? . It is clear that we can only stagger them by 1/2 of the width of a block (Figure **2**).

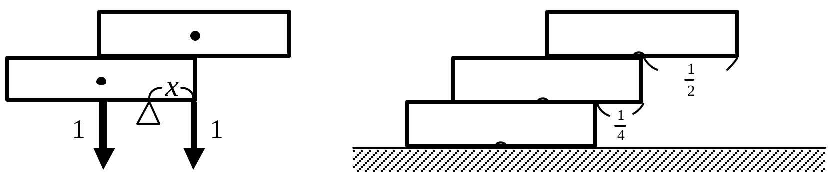

Figure **3**: 1/4 Stagger

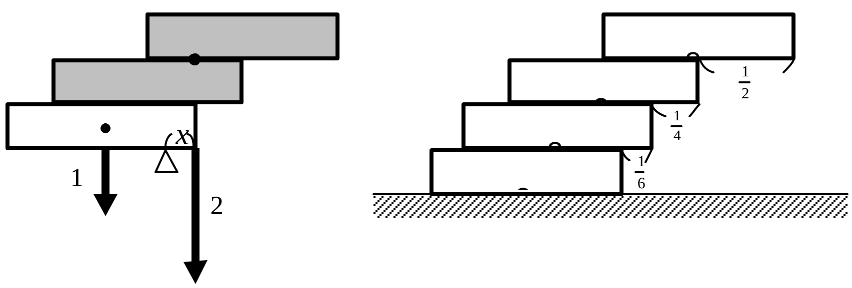

Figure **4**: 1/6 Stagger

Next we are going to put building block ? under the first two blocks so let us think about the center of gravity of building blocks ? and ? together (Figure **3**, left). Because building block ? can be staggered up to the center of gravity, I will obtain the moment, calling the stagger distance x. Moment is the product of weight and arm length so the moment of building block ? (rotated clockwise) is $1\times x$, the moment of building block ? (rotated anti-clockwise) is $1\times(\frac{1}{2}-x)$ and because these two values are equal,

$$1\times x = 1\times(\frac{1}{2}-x)$$

Solving this equation, we show that $x = 1/4$. In other words, the stagger distance for building block ? is 1/4 (Figure **3**, right).

Next we are going to place building block ? so let us think about the center of gravity of building blocks ? , ? and ? together. Let us obtain this center of gravity from the combination of the center of gravity of building blocks ? and ? together and the center of gravity of building block ? (Figure **4**, left). Taking building blocks ? and ? together gives a weight of 2. The moment of building blocks ? and ? (rotated clockwise) is $2\times x$, and the moment of building block ? (rotated anti-clockwise) is $1\times(\frac{1}{2}-x)$ and because these two values are equal,

$$2\times x = 1\times(\frac{1}{2}-x)$$

Solving this equation, we show that $x = 1/6$. In other words, the stagger distance for building block ? is 1/6 (Figure **4**, right).

Let us obtain the general result for the center of gravity of n building blocks. As this is determined by the center of gravity of $(n-1)$ building blocks plus the center of gravity of one building block, $(n-1)\times x = 1\times(\frac{1}{2}-x)$, therefore $x = \frac{1}{2n}$.

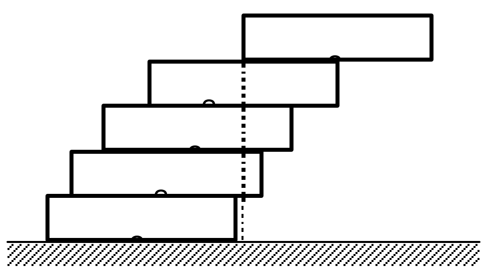

Figure **5**: 1-Block Stagger

Rearranging this equation we can see that if the stagger position is as follows

$$\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2n}, \cdots$$

then the building blocks can be stacked so that they will not fall down. When the progression produced by reciprocal numbers is an arithmetic progression, it is called a harmonic progression. For example, 1, 1/2, 1/3,? and 1, 1/3, 1/5, ? are harmonic progressions. Harmonic progressions are said to have been used in the study of harmonies theory by the Pythagorean School in ancient Greece and the name of harmonic progressions is derived from it. Harmonic series are the totals of harmonic progressions so we can also write:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}=\frac{1}{2}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})$$

So now let us calculate the value of this series.

$$\frac{1}{2}=0.5,\quad \frac{1}{4}=0.25,\quad \frac{1}{6}\approx 0.167,\quad \frac{1}{8}=0.125 \quad \text{therefore}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\approx 1.042>1$$

So we now know that if we have 5 building blocks we can stagger them by more than the width of one block (Figure **5**).

3. Convergence and divergence

In high school and university differential and integral calculus textbooks there are chapters on progressions and series. In those chapters the following exercise invariably appears:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}+\cdots\cdots$$

is divergent, and

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots+\frac{1}{n^2}+\cdots\cdots$$

is convergent.

When n goes to infinity, there are interesting exercises in which sometimes even if the general term of the progression converges to 0 the infinite series diverges. Convergence and divergence can be approximately known by performing integration as follows:

$$\sum\frac{1}{n}\approx\int\frac{dx}{x}=[\log x] \quad \text{therefore}$$

$$\sum\frac{1}{n^2}\approx\int\frac{dx}{x^2}=[-\frac{1}{x}].$$

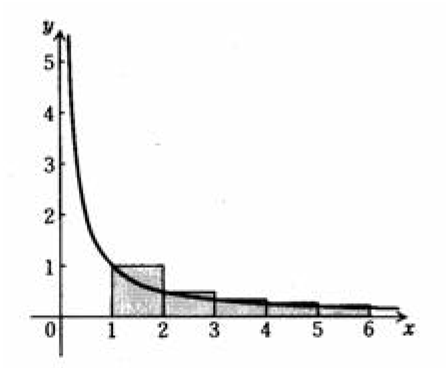

Figure **6**: $y = \frac{1}{x}$

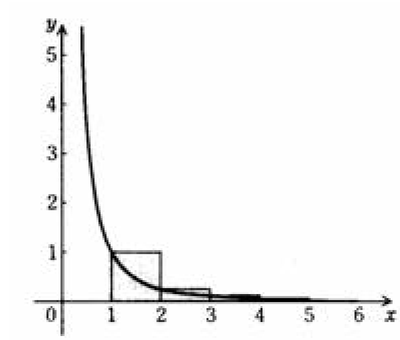

Figure **7**: $y = \frac{1}{x^2}$

The first of these two equations is in log order and diverges (Figure **6**), and the second of these two equations converges (Figure **7**). Generally, infinite series of the form $\sum \frac{1}{n^p}$ $(p > 0)$ diverge if $p \le 1$ and converge if $p > 1$. Furthermore, it is known that $\sum \frac{1}{n^2}$ converges to $\frac{p^2}{6}$. Furthermore, whether or not $\sum \frac{1}{n}$ converges is determined by the Cauchy convergence criteria for the progression.

The sum of the first n terms of the progression $a_1, a_2, \cdots, a_n, \cdots$ is defined as $S_n = a_1 + a_2 + \cdots + a_n$.

As for the necessary and sufficient condition for the series $\sum a_n$ to be convergent, if we make N sufficiently large compared to any given positive number e, for all n and m where $m > n > N$ it can be shown that:

$$| S_m - S_n | = | a_{n+1} + a_{n+2} + \cdots + a_m | < e .$$

Assuming that

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then no matter how big we make n,

$$\begin{aligned} | S_{2n} - S_n | &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \\ &> \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} \\ &= \frac{1}{2} \end{aligned}$$

So the *Cauchy* convergence criteria are not met. Therefore $\sum \frac{1}{n}$ is divergent. Let us look at this more closely. If we take the number of terms $2n$ as powers of 2 like this: 2, 4, 8,..., then

$$|S_2 - S_1| = \frac{1}{2}$$

$$|S_4 - S_2| = \frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$$

$$|S_8 - S_4| = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$$

$$|S_{2n} - S_1| = |S_{2n} - S_n| + \cdots + |S_8 - S_4| + |S_4 - S_2| + |S_2 - S_1|$$

$$> \frac{1}{2} + \cdots + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

So we can see that the series diverges.

4. Divergence in log n order

I have explained that the harmonic series $\sum \frac{1}{n}$ diverges to infinity but let us look closely at how quickly $\sum \frac{1}{2n}$ diverge. I used a personal computer to calculate the value of $\sum \frac{1}{2n}$, the total stagger distance. The results were as follows:

When $n = 4$ $\sum \frac{1}{2n} = 1.0417 > 1$

When $n = 31$ $\sum \frac{1}{2n} = 2.0136 > 2$

When $n = 227$ $\sum \frac{1}{2n} = 3.0022 > 3$

So the series does diverge to infinity but at an extremely slow speed.

If we now compare $\sum \frac{1}{n}$ with the integration of the function $y = \frac{1}{x}$ we can establish an inequality as follows:

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{dx}{x} < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$$

From the fact that $\int_1^n \frac{dx}{x} = [\log x]_1^n = \log n$, we can show that

$$\frac{1}{2}(\log n + \frac{1}{n}) < \sum \frac{1}{2n} < \frac{1}{2}(\log n + 1) .$$

So we know that when $n \to \infty$, $\sum \frac{1}{2n}$ diverges in $\frac{1}{2}\log n$ order.

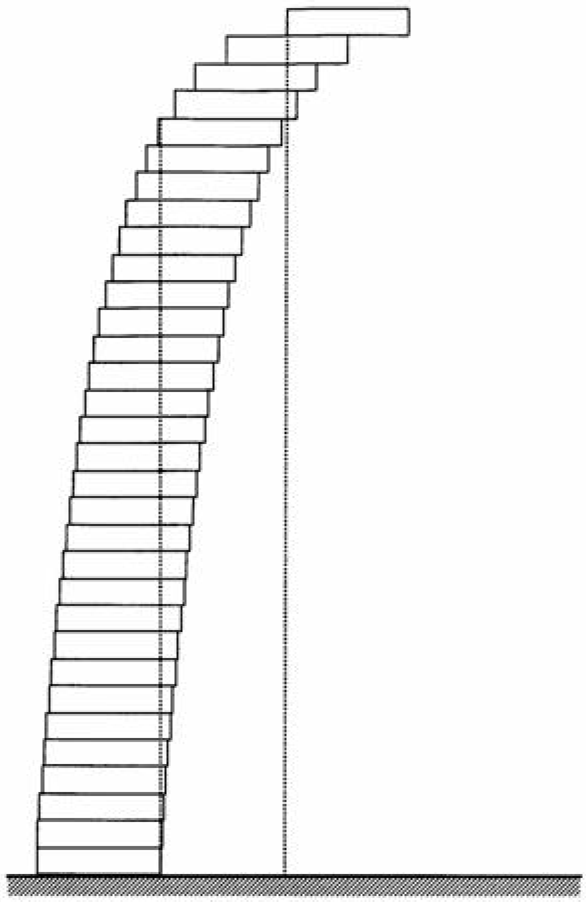

Figure **8**: 2-Block Stagger

Because one more extra building block is necessary at the bottom, the number of building blocks necessary is actually $n+1$. Only five building blocks (4+1) are sufficient to stagger the pile of building blocks by the width of one building block, but 32 building blocks (31+1) are necessary to stagger the pile by the width of two building blocks and 228 building blocks (227+1) are necessary to stagger the pile by the width of three building blocks. Figure **8** shows a stack of 32 building blocks but in practice it is impossible to stack up 32 blocks accurately staggered in this way. This is just a theoretical discussion. Figure **9** is a graph showing the function $y=\frac{1}{x}$ and $y=\log x$, the function resulting from the integration of $y=\frac{1}{x}$. If we rotate the log function 90 degrees clockwise and reverse it horizontally it becomes the building block stacking problem in Figure **8**. I will leave it to you to confirm this.

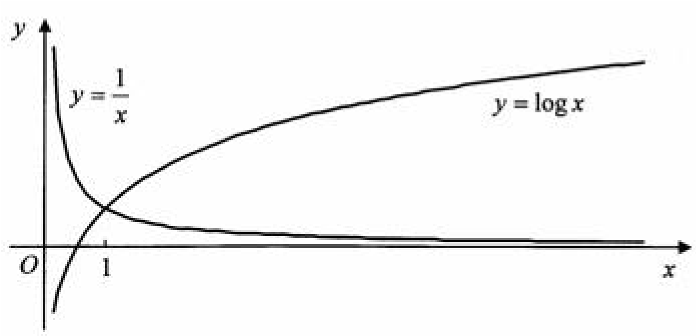

Figure **9**: Log function

That completes the proof. I have shown that the harmonic series $\sum \frac{1}{n}$ describes the solution to the building blocks problem. If we solve these kinds of problems, mathematics should be more enjoyable I think. Mathematics in high school and university progressively becomes more distant from reality and sometimes students come close to losing sight of them. At times like that the student must not forget to apply the problems to reality. The building blocks problem is the problem of the calculation of the center of gravity; it also involves harmonic series and is extremely mathematical. If we limit ourselves to just solving the problem, we do not need to use complicated numerical formulas. The important things are to employ logical ways of thought and to have the ability to change your way of thinking. Ironically, university students doing science subjects cannot solve this building blocks problem. They can prove with numerical formulas that harmonic series diverge to infinity, but they cannot solve the real world problem of the building blocks. This is a blind spot in modern education.

I leaned about the building blocks problem from a 1958 work by George Gamow. He was both a researcher and educator and it appears that he was of the opinion that the students will not get excited about mathematics if the teacher is not excited about it. If you would like to confirm the solution to the building blocks problem but you do not have any building blocks at hand, you could try doing it with ten volumes of an encyclopedia or ten video tapes.

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Modeling interdisciplinary activities involving mathematics and philosophy

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Abstract: *In this paper a didactical model is presented. The goal of the model is to work as a didactical tool, or conceptual frame, for developing, carrying through and evaluating interdisciplinary activities involving the subject of mathematics and philosophy in the high schools. Through the terms of Horizontal Intertwining, Vertical Structuring and Horizontal Propagation the model consists of three phases, each considering different aspects of the nature of interdisciplinary activities. The theoretical modelling is inspired by work which focuses on the students abilities to concept formation in expanded domains (Michelsen, 2001, 2005a, 2005b). Furthermore the theoretical description rest on a series of qualitative interviews with teachers from the Danish high school (grades 9-11) conducted recently. The special case of concrete interdisciplinary activities between mathematics and philosophy is also considered.*

1. Introduction

There is worldwide consensus that the society we live in today gets increasingly more and more complex. Earlier the problem was often to gather information, whereas the knowledge society of today is characterized by the fact that much information is easy accessible. The problem nowadays is therefore to survey and filter the great amount of accessible information rather than to gain access to it. Thus, the schools have to aim at producing students who are prepared to deal with such a great complexity of knowledge, that is, scientifically literate students (Gräber et al., 2001). In the educational system knowledge is still in a very large scale separated into distinct blocks by different subjects. This separation of knowledge has shown itself to be very efficient in producing and teaching new knowledge, but does not necessarily provide the students with the skills necessary to navigate through the constantly increasing amount of accessible information. Interdisciplinary activities between different subjects can help to develop a broader context of meaning or understanding for the student, and in this way contribute to the ongoing scholarly development and provide the student with the tools necessary to deal with complex problem solving waiting in the future.

In spite of the fact that many different subjects and areas often contain more and more mathematics-rich elements, mathematics, as a subject, mathematics remains quite isolated. The objective importance of mathematics from a social point of view exists side by side with its subjective irrelevance experienced by many people. Niss et al. (Niss, Jensen, Wedege, 1998) have characterized this as the *relevance-paradox* of mathematics. One reason for this could be found in the fact that mathematical knowledge is hard to transfer to new domains of knowledge by the student. Although the subject of mathematics in its very nature often is described as a tool, and therefore should be able to establish obvious connections to other contexts, such transfers of

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mathematical knowledge between different domains seldom occur (Hatano 1996, Michelsen 2001). Most of the time subjects have their own specific use of language and system of terminology and this can prevent the desirable transfer of mathematical knowledge to other contexts and domains.

The purpose of using interdisciplinary elements in the teaching of mathematics is, as concluded from the description above, 1) an attempt to broaden the students' curricular perspective and general view by removing the discrete lens that characterizes most schools' separation of knowledge into curriculums and present to the students a more real picture of the role and importance of mathematics in extra-mathematical contexts[1] 2) an attempt to help the students' abilities to transfer mathematical knowledge between different curricular domains.

To be able to work with interdisciplinary aspects in the teaching of mathematics one has to consider what connections the subject mathematics has to other subjects and areas of knowledge. In *the first International Symposium of mathematics and its Connections to the Arts and Sciences*[2] (Beckmann, Michelsen & Sriraman, 2005) such connections were discussed, and a sketch of a didactical model for interdisciplinary activities between mathematics and philosophy presented (Iversen, 2005).

Afterwards, the modeling of such activities involving mathematics has continued and the purpose of this paper is to present a didactical model, a conceptual frame for the planning, completion and evaluation of successfully interdisciplinary activities involving mathematics. The model will function as a tool to help develop activities that can facilitate a reasonable transfer of mathematical knowledge to other subjects and domains. The model is inspired by the work of Michelsen (2001, 2005a, 2005b) and is further developed through the special case of mathematics and philosophy and a section is therefore devoted to this specific topic. The section will also work as a demonstration of how the model should be understood and applied.

2. Theoretical Framework

Working with interdisciplinary activities implies a belief that there exist elements that is general and somewhat identical between the knowledge presented in different subjects. We assume that such an intersection of knowledge contains more elements the more related[3] the subjects are to one another, and at least *not-empty* (Dahland, 1998). There are different ways of trying to describe such assumed curricular intersections. In the development of the didactical model presented here a notion of *competencies* is used to identify and characterize the possible intersection of knowledge between mathematics and other subjects.

In the educational system of Denmark a huge step forward is taken with the completion of the *KOM-report* for mathematics (Niss et al., 2002). In this Niss lists eight mathematical competencies, valid for all steps of education, which is a meant to work as an overall frame for

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[1] This follows Sriraman (2004) who argued that students are used to viewing knowledge through the discrete lens of disjoint school subjects.

[2] The symposium took place 18-21 May 2005 in Schwäbish Gmünd, Germany. See Beckmann, A., Michelsen, C., & Sriraman, B (Eds.)., (2005). *Proceedings of the 1st International Symposium of Mathematics and its Connections to the Arts and Sciences.* The University of Education, Schwäbisch Gmünd, Germany, Franzbecker Verlag.

[3] Related should here be understood in a common way. The subject of mathematics is e.g. is supposed to be more *related* to physics than to English.

description of the education in mathematics in Denmark. The concept of a mathematical competence is here understood as some sort of *mathematical expertise*, and is more formally defined as *an insightful readiness to act appropriate in situations which contains a certain kind of mathematical challenges*.[4] The report has been a starting signal to similar competence descriptions of other subjects in the Danish educational system.

A description of mathematics by the means of competencies focuses more on the *purpose* of learning mathematics than to the specific curriculum. This description expresses a broader minded view on the teaching of mathematics than a normal curricular-dependent view. But Niss describes (Niss et al., 2002, p. 66) the eight mathematical competencies as strictly belonging to the sphere of mathematics thereby partly closing down the newly constructed bridge to other subject domains. Michelsen et al. (2005a) instead argues that some of the competences put forward by Niss et al (2002) are actually *interdisciplinary competences*, and mentions the modeling and representational competence as examples.

In this paper we will try to make use of the interdisciplinary potential inherent in a competence approach to mathematics on a theoretical didactical level suggested by Michelsen et al (2005a). A less bounded description of mathematical competences can then be substratum that enables an entanglement of mathematics with other subjects both on an educational theoretical level and on a practical level in the classrooms. It is here suggested that the notion of a mathematical competence should contain both a *narrow* and a *broad* dimension, by means of which such characterization of mathematical expertise in the student can both work as a description internally in mathematics and as a link to the rest of the world. As an example Niss (2002) mentions *the ability to reason mathematically* i.e. to be able to follow and judge mathematical argumentation, as one of the eight described competences. But the ability to be able to follow and judge a reasoning is far from restricted to the sphere of mathematics. It is the kind of expertise that is important to master in all the school's different subjects, and it could therefore be argued that some sort of *reasoning competence* is just as essential in physics or philosophy as it is in mathematics. Obviously arguments and reasoning often appear in different use of language and forms in different subjects, and therefore a reasoning competence is here suggested to be characterized by *the ability to follow and judge a reasoning in different curricular domains*, AND *being able to distinguish and characterize different types of arguments* thereby having the ability to go deeply into a certain subject and follow and judge a reasoning characteristic for this one subject.

Within mathematics valid arguments often have character of a proof, while arguments in other subjects, as e.g. philosophy or history, often are marked by less cogency and more contingent elements. In this context mastering the reasoning competence will be understood as the ability to distinguish different kinds of arguments but at the same time know why the different arguments work in different contexts, and to be able to dive into a specific argument, as e.g. a mathematical proof, and follow its string of reasoning.

This broad minded approach to the notion of competences should be understood as an attempt to, over time, change the educational practice which makes it possible that

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[4] My own translation from Danish (ibid.).

> *"Although critical thinking, problem solving and communication are real world skills that cut across the aforementioned disciplines students are led to believe that these skills are context dependent."(Sriraman, 2004, p.14).*

3. Interviewing high school teachers

During May 2005 a series of qualitative interviews were conducted. Six high school teachers were interviewed individually. The main purpose was to find out: *Which didactical (and practical) possibilities and obstacles exist for interdisciplinary activities between mathematics and other subjects (especially philosophy) in the Danish high school (grade 9-11)?*[5] The interviewees were teachers from different high schools in Denmark and varied both in age and seniority. They were chosen so that each one taught either mathematics or philosophy (or both) on a daily basis and moreover most of them had been engaged in relevant interdisciplinary activities. The hope was to be able to incorporate some of this real-life information into the development of the didactical model. In the following I will reproduce some of the, for this paper, relevant conclusions one can draw from the conducted interviews.[6]

Some of the interviewed teachers have conducted interdisciplinary activities between mathematics (or physics) and philosophy earlier on in their daily teaching. It has not been possible to find any writings about conducted activities between mathematics and philosophy in the Danish high school, but some of the interviewed teachers have been involved in documented activities involving physics and philosophy. Generally the experiences from these courses were positive

> *" It's easy for me to register that the students have been going through these activities (involving physics and philosophy) and other teachers can easily do so to. … They [the students] own more academically concepts than students usually have. They are really good at thinking different subjects together, and they also get very good at working together in little groups … I think they simply have a greater cultural and historical horizon." - Teacher 1*

The purpose of these activities involving physics and philosophy was primarily to strengthen the subject of physics. To embody the abstractness of physics as one of the interviewed teachers told me. This goal was in some sense achieved according to the teacher quoted above and the reports of evaluation carried out by the involved students and teachers afterwards. Besides the registered positive cognitive effects the students realized that physics can not be reduced to a mere collection of dead facts. Physics is a human activity that evolves and therefore argumentation actually do count. This shift in the students' perception of the subject physics from being a dusty collection of facts, to being relevant, is an experience that another of the involved teachers believe can be re-produced in the case of mathematics.

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[5] The fact that some of the asked questions particularly involved a reference to the Danish high schools(as opposed to any high schools) was because I wanted to find out which effect a forthcoming reform of the Danish high schools would have on the daily teaching practice. Most of questions asked involved only general educational components, and did not hold any particular reference to any Danish conditions.

[6] All the interviews were conducted in Danish, and the quotes given in the text is therefore my own translation. The text in the brackets is my insertions. They are there to give the right coherence in the teachers statements. The interviewed teachers are here given only a number, but all the quotes given in this paper are approved by the particularly teacher concerned.

> *"We can re-create the part about discovering in the case of mathematics … For example the students often only see the end-product when they see a proof for some mathematical relation. For them it's often a strange thing; How have "they" found out you are supposed to do like that? They ask themselves. The process from a proof starts to crystallize and right to the final version of the proof which needs polishing before it appears in a textbook, nice and rounded. That whole process one should try in teaching mathematics, I believe it would be very beneficial for the students." -Teacher 2*

Besides using philosophy as a tool to illustrate the world and methods of physics the teachers involved report how at the same time the activities created the perfect interdisciplinary context for developing central concepts from the philosophy of science. Ideas such as: *induction, empirically investigations* and *verification* were easy for the students to acquire and work with in this expanded domain. In this way the activities held the possibility that both involved subjects could engage in the work of developing the students' scientific literacy, but at the same time use the cross-curricular context to discover and develop relevant aspects specific to the different curriculums.

Others of the interviewed teachers had themselves planned and conducted interdisciplinary activities involving mathematics and philosophy. In both cases the activities had been carried out in relation to the daily teaching of mathematics, and both set of activities centered about *argumentation and proof in mathematics*. The purpose of the different activities varied slightly but fundamentally they both tried to illustrate characteristics of mathematical argumentation and how this often is worked out.

> *"When we speak about method, we did something about; When do you examine something and when do you actually construct a proof? And also, what is needed to construct a proof and what is the nature of a mathematical proof? These issues are very philosophical I think, and the activities were a great success for the students."- Teacher 3*

> *"We worked with paradoxes and reasoning and things like that … The overall theme was argumentation. It was a very good course, and the students were very fond of it." - Teacher 4*

The work with these topics in mathematics was carried through based on a wish to equip the students with some general tools, or concepts, which could function as some sort of cognitive scheme for their ongoing daily struggle for learning mathematics.

> *"A part of the teaching is about giving them [the students] a set of concepts which they can use to relate to what the are doing concretely. When they engage in a specific task in mathematics, they now have some concepts, some work habits,*

> *some patterns, some ways of thinking which they can use to throw light on what they are actually doing." - Teacher 3*

Mathematics propagates through a large and branching taxonomy of concepts and ideas. Several of the interviewed teachers pointed out that, cross-curricular activities between mathematics and a subject as philosophy should deal with concepts placed fairly high in the mathematical taxonomy used in the high school. To illustrate this point we can consider the relative position of two mathematical concepts in the taxonomy. Look for example at say the concept of *function* and a specific function as $f(x)=sin(x)$. Both entities can be considered as a concept that a student in the high school should become acquainted with at some point. The concept of function however will be placed highest of the two in a taxonomy of mathematical concepts, and we will therefore regard this as a *meta-concept* in comparison with $f(x)=sin(x)$. This way there also exists meta-concepts in comparison with the concept of function. The concept of *functional* is an example of a such, and the use of the name meta-concept will therefore always be relative.

For high school students the concept of *proof* will be regarded as a meta-concept most of the time and a direct investigation of this in the classroom by the students will often involve several problems. According to Dreyfuss (1999) most of the students on this educational level has a very restricted knowledge about what constitutes a mathematical proof. Also Hazzan and Zazkis (2005) point to the importance of trying to help the students acquire relevant mathematical meta-concepts as e.g. the proof.

According to Niss (1999) a major finding of research in mathematics education is students' alienation from proof and proving. Students' conceptions of the mathematical proof and those held by the mathematical community is separated by a huge gap. Niss concludes that

> *"Typically, at any level of mathematics education in which proof or proving are on the agenda, students experience great problems in understanding what a proof is (and is not) supposed to be, and what its purposes and functions are, as they have substantial problems in proving statements themselves, except in highly standardized situations." (Niss , 1999, p. 18).*

Instead the students' consider proofs and proving as strange rituals performed by professional mathematicians that are not really meant to be understood by ordinary human beings. The activities referred to above by the interviewed teachers are exactly concerned with these problems and shows how other subjects such as philosophy can be used in the struggles.

The interviewed teachers generally believed that interdisciplinary activities involving mathematics were very relevant for the students. Focusing on the special case of mathematics and philosophy some of teachers suggested that relevant activities could take as a starting point the purpose of illuminating the structure of mathematics, its fields of study and its characteristic form of argumentation. It comes as no surprise that the examples mentioned here are of a very general character. Engaging in interdisciplinary activities should hold the possibility of gaining something for all the involved subjects, and this would indeed be a very difficult premise to fulfill for *both* mathematics and philosophy if the activities centered about the quadratic equation and Socrates' famous Defence. Both are examples of a far to narrow approach to interdisciplinary activities determined too much by curricular considerations.

In spite of a general optimism shared by the interviewed teachers towards integrating the teaching of mathematics with other subjects, several of them also point to a number of difficulties with the subject of mathematics that must be overcome if the interdisciplinary activities should be rewarding.

The subject of mathematics is regarded as a subject that holds great technical difficulties for the students. According to the interviewed teachers exciting problems and topics in mathematics often demands a severe amount of preparation from the students before they can engage with the activities thereby losing the immediate interest that is so important for the learning process (Mitchell, 1993). Other subjects, e.g. philosophy, is for most students easier to engage in and this often leads to a shift in the students attention away from the mathematical content of the chosen topic. For that reason the development of successful interdisciplinary activities involving mathematics needs the development of a working culture among teachers and students where it is respected that a subject as mathematics can be hard accessible and show this problem extra attention in the classroom.

Most of the interviewed teachers highlighted the fact, that in many cases interdisciplinary activities end up bringing in the mathematics teacher to simply help the students read of some values on a prefabricated curve or similar. Here the actual mathematical content is far from challenging or relevant for the students (or the teacher). To avoid this situation one of the interviewed teachers point out that

> *"There's an interaction between the other subject [than mathematics],, the way it asks its questions and the areas of mathematics you can point out and work with. Sometimes mathematics and the other subject actually pose the same kinds of questions but they each give different kinds of answers. … The problems that the activities are meant to center on must have double-relevance, and that means that they should have relevance both in the reality to which they belong and also in mathematics. As a thought I think that is very correct because often they [the other teachers] say; Yes, this topic is really interesting could the mathematics teacher please come in here and help reading of the curve! I answer: No, no that's not really interdisciplinary activities." - Teacher 4*

The subject domains involved in the activities must in some sense meet and use each other properly. Subjects are not actually co-operating when the co-operation is reduced to a parasitic process where one of the subjects *de facto* is not gaining anything as described in the above quote.

4. Modeling interdisciplinary activities involving mathematics and philosophy

The purpose of developing a didactical model for interdisciplinary activities involving mathematics and philosophy is, as mentioned earlier on, multiple. The model should function as a link between educational theory and the daily teaching practice in mathematics, both in the development of new activities, the carrying through of already planned ones and the evaluation of completed activities. The model gets inspiration from the work of Michelsen (2001, 2005a, 2005b), and a former version was presented at MACAS 1 and described in Iversen (2005). The didactical model consists of three phases - *the horizontal intertwining*, *the vertical structuring* and *the horizontal propagation*. Freudenthal (1991) introduced the idea of two different types of mathematization in an educational context – horizontal and vertical mathematization. In the horizontal mathematization students develop mathematical tools that help them organize and work with mathematical problems situated in real-life situations. The process of reorganizing the mathematical system itself Freudenthal designates vertical mathematization. Also Harel & Kaput (1991) sees a distinction between horizontal and vertical growth of mathematical knowledge. They associate the term horizontal growth with the translation of mathematical ideas between extra-mathematical situations (and models of these) and across other representation systems. By vertical growth is understood the construction of new mathematical conceptual systems.

5. The Horizontal Intertwining

As mentioned by some of the interviewed teachers interdisciplinary activities involving mathematics very often end up as fictitious constructs without much relevant mathematical content. In the first phase of a cross-curricular collaboration the attention should be centered on the importance of obtaining a real intertwining of the involved subjects. Such a curricular intertwining involves considerations about which fields of study, problems and methods in mathematics and the other subjects involved that have potentiality as interdisciplinary elements. Such elements must not originate from oversimplified lingual similarities among the subjects, but instead from considerations about how these elements can be used later in the continued learning of e.g. mathematics. This kind of intertwining of the subjects' core subject matter the students will often experience as "the meeting of different subjects", and the term of *horizontal* refers therefore to the students pre-understanding of the chosen curricular element as belonging to both mathematics and another involved subject, but not necessarily as an subject-exceeding element. Often the students do not consider ideas to be related because of their logically connection, but because they are being used together in the same kind of problem solving situations (Lesh & Doerr, 2003; Lesh & Sriraman, 2005). Michelsen et al. (2005a) suggest the term horizontal linking to describe the process of identifying contexts across mathematics and other subjects of science that are suitable for integrated modeling courses. I will here suggest the notion of *horizontal intertwining* to describe a related process of identifying and characterizing interdisciplinary problems and context suitable for integrating the subjects of mathematics and philosophy, thereby emphasizing the broader scope the integration of mathematics with a subject not from the natural sciences demands.

The interdisciplinary activities should be chosen so they set up non-routine problems, which in order to be solved properly, need the involvement of all the involved subjects. A competence approach to the subject of mathematics contains a possibility to identify such relevant subject-exceeding elements, because this approach focuses on what the students master after going through the courses, and not on concrete curricula. As argued in the theoretical section of this paper such an approach demands a broadminded view on the notion of competencies to be able to work as an educational tool.

A horizontal intertwining of the subjects designates a weaving together of the involved subjects' core subject matter by identifying non-routine problems and contexts suitable for integrating mathematics and philosophy. In order to be able to do this one needs to clarify what constitutes such core subject matters. Furthermore, such a weaving together of subjects demands a clarification of the overall purpose of the activities. The purpose must have relevance for both mathematics and the other subjects involved in order to be justified. In practice it can span a wide field of areas; from helping the cognitive growth of the individual student (e.g. in relation to concept formation), trying to strengthen the motivation for the involved subjects or even trying to create a unified view of knowledge and science in the students.

6. The Vertical Structuring

A reasonable intertwining of the involved subjects facilitates the possibility that the student can identify with the cross-curricular aspects of the chosen problems, and thereby engage meaningfully in the activities. A clarification of the overall purpose with the activities will from the beginning help the teacher to follow the students' cognitive development along the activities. Such observations will often involve that the mathematics teacher abandons the usual authoritarian role and take on a more *guide-like* function instead.[7] From a combination of the involved subjects' core subject matter the student should under suitable guidance and activity go through a cognitive development – a so called vertical structuring - that will root the cross-curricular phenomenon concerned conceptually. It is crucial for a successful interdisciplinary engagement that the involved phenomena are central for the further learning of mathematics. If the purpose of the activities is the formation of new mathematical concepts the vertical structuring could be described as the construction of a new mathematical *concept image* (in the sense of Tall and Vinner, 1981). More theories describe how the formation of a new concept image in the student involves a qualitative change in the students perception of the specific concept. The change of perception is registered as a cognitive shift between perceiving the mathematical concept as an activity (or a process) and viewing the concept as an entity in itself i.e. a kind of structure or object (Dubinsky 1991, Sfard 1991, Tall 1997, 2001).

In activities where the over-all purpose is to equip the students with a greater curricular perspective and overview we can describe the vertical structuring as the cognitive development of a new cross-curricular platform in the student, whereto new knowledge later can be attached to and grow from.

7. The Horizontal Propagation

A successful vertical structuring should be evaluated in a greater perspective. The development of new significant concepts and connections based on interdisciplinary elements should be further developed in the different curricular domains of mathematics and philosophy. According to Lesh & Doerr (2003) the real challenge of the teacher is not only to introduce new ideas and concepts but also to create situations where the students need to express their current ways of thinking so this can be further tested and revised in directions of stronger development. In the case of mathematics the student should be allowed to use the newly learned knowledge in different mathematical activities and thereby apply, test and approve the specific mathematical concepts in question for the purpose of developing a more firm and generalized mathematical structure in the end. This is only possible if the original purpose with the activities is aimed at such a propagation of the new knowledge in other contexts. In other words the vertical structuring should be followed up by a horizontal propagation of the newly acquired structures in

[7] For a more developed description of this shift in the teachers role in the classroom, see e.g. Gravemeijer (1997).

the students and thereby this knowledge can find its use in both mathematics and other involved subjects.

In this way the cross-curricular elements can work as a new basis, or context, for the student which can use it in the continued learning of mathematics furthermore in the development of new interdisciplinary connections between subjects thereby being able to overcome the crucial problems of *transfer* mentioned in the theoretical section of this paper. This is the true gain of such interdisciplinary activities.

After the carrying through of a longer cross-curricular course one of the interviewed teachers describes an example of what could be characterized as a horizontal propagation as follows

> *"I see the acquired competencies applied in many different places. They [the students] simply travel faster over the learning-ground. One can say that they fundamentally have a greater prerequisite for both conceptual entities and in working contexts." - Teacher 1*

8. Designing relevant activities involving mathematics and philosophy

After sketching the different components that make up the didactical model it should be illustrated how it can be used in the development of relevant interdisciplinary activities between mathematics and philosophy. Here we consider the special case of proof and proving in mathematics and philosophy.

First we need to identify relevant non-routine problems, topics or phenomena which can function as curricular-exceeding elements between the two subjects and thereby establish a reasonable horizontal intertwining. We can use a competence approach to the curriculums of mathematics and philosophy respectively, hereby focusing on what cognitive qualities the two subjects aim at developing in the students. Common to the two subjects is a (seemingly endless) search for logically healthy arguments and conclusions and the ability to follow and judge such kind of reasoning therefore belongs to the core subject matter in both mathematics and philosophy. In planning the activities we can therefore reasonably focus on developing some sort of *reasoning competence* as mentioned earlier. This involves an ability to compare and differentiate the different kinds of argumentation used by the two subjects, but also the ability to dive into specific arguments from each subject and be able to follow and judge such specific reasoning.

In all of the school's different subjects the students' ability to argue clearly and reason reasonably plays an important role, and a development of this capacity is a key area in both mathematics and philosophy. Philosophy is in fact often characterized as a subject that tries to generate and develop the students' ability to understand and use forms of argumentation and knowledge that cut across the school's different disciplines and dimensions.

Mathematical reasoning takes many forms but is in its clearest form crystallized as actual proofs. The power to give a definite proof for a certain conjecture is characteristic for the subject of mathematics and the students' knowledge about the meta-concept of *proof* is, as argued earlier in this paper, therefore central in the teaching activities in the high school. In philosophy the idea of proof also plays a key role. Earlier on, philosophers tried to transfer the mathematical (in some sense Euclidean) idea of proof to actual philosophical arguments. The most famous philosophical "proofs" are the proofs of the existence of God. These were put forward by e.g. Anselm of Canterbury and Thomas Aquinas, who both believed that giving a formal proof of the existence of God was actually possible. The high school teachers who took part in the interviews also highlighted argumentation and the concept of proof as phenomena that could transcend the gap between the subjects of mathematics and philosophy and thereby overcome the problem of transferring mathematical knowledge to other contexts and domains.

To sum up we have, starting from a wish to advance the students' ability to argue and reason within mathematics and philosophy identified *the concept of proof* as a concrete topic suitable for a curricular intertwining of the two involved subjects.

The activities originate from a study of which role the actual proving of statements and conjectures holds within the two subjects. What constitutes a proof? At what point can we say we actually have proven something? And what kind of knowledge does a proof give us? Is it true? Is it unchangeable? [8] In practice one could use simple proofs, easy for the students to master mathematically, such as small proofs from the classical *Elements* by Euclid himself (Euclid, 2002). E.g. using the proof that *the sum of the angles in a (Euclidean) triangle* is equal to the sum of two right angles or the proof of *the Pythagorean theorem*. Then comparing these to actual proofs of philosophical character e.g. a modern version of *Anselms Ontological proof* of the existence of God. It is important that the students subsequently are placed in different situations where they themselves are forced to work out small proofs thereby experiencing the process of trying to argue for a conjecture. This will enable the students to apply, test and further develop their understanding of the concept of proof. An understanding that (hopefully) in time will evolve further and be a useful tool for the students.

Through an experimenting approach, as described above, to the idea of proof a vertical structuring of the meta-concept *proof* should be developed. At the same time focus is on the students' ability to separate different kinds of argumentation. Most of the interviewed teachers agree that this would be of significant importance in the students' continued engagement with both mathematics and philosophy.

A vertical structuring of the concept of proof subsequently work as a structure which must be applied, re-valued and tested further in the daily teaching practice that follows within both subjects. Hereby obtaining a horizontal propagation of the newly acquired knowledge which results in a greater basis or context for the further learning and understanding of both mathematics and philosophy.

The Danish Ministry of Education has recently published an Education Manual for the high schools. The manual focuses on interdisciplinary activities and a large part is devoted to paradigmatic examples of concrete activities. In this manual I've contributed to more fully describe activities between mathematics and philosophy as the one sketched above.[9]

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[8] All questions Niss (1999) emphasized as extremely difficult for students to answer properly.
[9] The manual can be found at http:us.uvm.dk/gymnasie/vejl/?menuid=15 (unfortunately only in Danish).

9. Conclusion

In the paper a didactical model which should function as a concept frame for the development, completion and evaluation of interdisciplinary activities involving mathematics and philosophy, was presented. The model consists of three phases that these activities involve; *The horizontal intertwining*, *the vertical structuring* and *the horizontal propagation*. Although the model is presented as linear, the process of going through the different phases is in some sense to be understood as an iterative process that can be run through several times by each student.

In the description of the first phase it was argued that it is of great importance that the actual mathematical content in interdisciplinary activities is not reduced to simple instrumental activities. Instead one should seek to identify and characterize interdisciplinary phenomena and contexts which can facilitate a proper intertwining of the different subjects involved by setting up relevant non-routine problems which need the involvement of both mathematics and philosophy to be answered. This can be enabled by a competence-approach as to what constitute mathematical skills. Such an approach is broader than the usual curriculum-approach to mathematics which often works as a drag to the development of successful interdisciplinary activities.

The model's second phase describes how the students' engagement in the planned activities should facilitate a vertical structuring which leads to the development of new conceptual systems, objects or contexts in the student. This can appear as a formation of new mathematical concept images, by which the interdisciplinary phenomenon considered, conceptually is anchored. This can work as a further basis in the students' continued learning of both mathematics and philosophy.

Finally the third phase focuses on how ongoing activities involving the newly acquired constructions are the overall purpose with all interdisciplinary activities. Furthermore it is argued that the cross-curricular phenomenon should be applicable in the daily teaching practice through a horizontal propagation of the considered phenomenon in both mathematics and the other subjects involved.

An anchoring of the model in the daily teaching practice was sought through a series of qualitative interviews of Danish high school teachers. Furthermore the model was illustrated through a design of a concrete interdisciplinary activity between mathematics and philosophy, and it was thereby argued how the model can be used to develop concrete interdisciplinary activities between these two subjects. The sketched activities take the concept of proofs and proving as a starting point and centers themselves around argumentation and reasoning in both mathematics and philosophy.

As the modeling of such activities is still (and perhaps always) a work-in-progress the presented model is somewhat tentative in its nature. The model originates from a wish to develop a concept frame for interdisciplinary activities between mathematics and philosophy, and found inspiration in the work of Michelsen (2001, 2005a, 2005b) which centers about interdisciplinary activities between mathematics and physics. A further perspective is to continue the work of developing concrete teaching activities, as well as trying to adapt and evaluate the model's strengths and weaknesses as a didactical tool to integrating the subjects of mathematics and philosophy. The author, therefore, invites all interested readers to further test and revise the model as well as concrete realizations and afterwards sharing experiences which hopefully will lead to the improvement of the didactical model as a result.

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Not out of the blue: Historical roots of mathematics education in Italy

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Abstract. *In this note I outline some elements of the history of mathematics education in Italy. Initially the chief characters were mathematicians who played a role in designing curricula and in editing textbooks. The development of the Italian community of mathematics educators towards the present day trend in research was fostered by participation in international activities after the Second World War. I also identify some elements of continuity with the past to stress the influence of some mathematicians in the development of present research.*

Key words: history of mathematical instruction; Italy; research in mathematics education; mathematicians.

1. The Past

Though the development of mathematics education as a discipline is affected by many factors I deem that the national policy of the system of instruction plays a major role. In turn this policy is strictly linked to the history of the country and to the academic world (in the case of mathematics the world of mathematical research.) As an example I briefly outline the events that preceded the birth of the Italian community of mathematics education research to catch a glimpse of links between the present situation and the past.

Italy became a unified country in 1861, before which it was composed of little states which had different systems of instruction or no system at all. To create a national system was one of the main concerns of the new government. It is remarkable that the concern about public instruction was already present before the unification, as evidenced by the proceedings of the annual meetings of scientists held from 1839 to 1847 in the future Italian territory. These scientists called themselves "Italian" before Italy existed as a political entity and planned the survey of the situation of the instruction in the Italian territory. It is said that the motto chosen for the proceedings of their meeting in 1846 was *"The educator and not the weapon will be in the future the arbiter of world's destiny"* and that this sentence was ink-cancelled (deleted) by order of the governor in almost all the already printed copies. Strong ideals were present in the scientific community: in particular, some important mathematicians participated personally in the independence wars and, when the process of unification was achieved, were involved in political activities (also as members of the parliament) concerning instruction. The evolution of the political situation in the following century changed the initial ideal position. The motto now proposed to school children in the 1930s (Fascist period) was *"Book and musket"*.

In the pioneering period after the unification the Italian community the relation of mathematics school teachers with professional mathematicians was sometimes difficult, as evidenced by the well known episode of the controversy around the teaching of elementary geometry. This episode parallels an analogous episode which happened in England and shows how similar situations may lead to different outputs in different countries. These are the facts. In Italy before the unification there was no tradition in mathematics education and textbooks were mainly imported from abroad. The first significant act of the new born nation was to publish an Italian mathematics textbook for secondary school. This happened in 1868 and the book was the edition of Euclid's *Elements* edited by two outstanding mathematicians (Enrico Betti and Francesco Brioschi). The Ministry of Education proposed it as a textbook to be adopted in Italian schools. The content was good from the mathematical point of view, but not suitable for secondary students. Teachers and mathematicians with some feeling of what mathematics education *should be* expressed a strong disappointment against the use of this book as a school text. A hot controversy was hosted in one of the two journals of mathematical research existing in Italy in those times (*Giornale di Matematiche*) from 1868 to 1871, see (Furinghetti and Somaglia, 2005). In one side of the duel there were the two editors of the *Elements* and Luigi Cremona, an important mathematician author of the official national programs for mathematics, in the other side there was a second rank mathematician who was caring for the pedagogical point of view and of school teachers' opinions. At those times the ordinary teachers had no voices, since professional journals did not exist, nor associations of mathematics teachers.

In England for many years the admission examinations to Cambridge, London and Oxford universities were based on rote exercises of Euclidean geometry. Many people were complaining about that, among them outstanding mathematicians such as Augustus De Morgan and James Sylvester. Books based on new syllabi were produced from 1868 onwards. In 1871 the A.I.G.T. (*Association for the Improvement of Geometrical Teaching*) was founded; it was the mother of the *Mathematical Association* founded in 1894. John Perry's address on 'The teaching of mathematics' delivered to the new 'Education' section of the British Association (1901) opened new perspectives to this problem: the educators were pushed to hear the voices of those students who would not become mathematicians and needed of a kind of mathematical education close to the requirements of the changing society. Perry's ideas were clearly expressed in the article 'The teaching of mathematics' (*Nature*, 1900, 317-320), see Howson (1982, pp. 147-148):

> *The young applier of physics, the engineer, needs a teaching of mathematics which will make his mathematical knowledge part of his mental machinery, which he shall use […] readily and certainly […]*
>
> *[This] method is one which may be adopted in every school in the country, and adopted even with the one or two boys in a thousand who are likely to become able mathematicians.*

In Italy things evolved in a different way. The academic power of mathematicians choked the timid attempts of rebellion to the use of the *Elements*. A sentence in the mathematics programs issued after Italian unification epitomises the official attitude towards mathematics in school: "mathematics is a gymnastic of the mind." This view was not unanimously accepted (especially by school teachers) and ironic references to this expression are present in papers appeared in the following years.

Many factors affected the different evolution in the two countries. Not only authors such as Herbart influenced the view of certain mathematics educators in England, but also the different level of industrialisation which called for a different role of education in society. This latter fact is evidenced by Godfrey's passage as reported in (Howson, 1982, p. 158):

> *In England we have a ruling class whose interests are sporting, athletic and literary. They do not know, or if they know do not realise, that this western civilisation on which they are parasitic is based on applied mathematics. This defect will lead to difficulties, it is curable and the place for curing it is school.*

A relevant factor in the different developments was mathematicians' attitude about rigour. In Italy at the beginning of the twentieth century the concept of rigour was shifted from the Euclidean rigour to Hilbert's and Peano's rigour, but still remained the main concern of university professors when discussing mathematics teaching in school. This strong concern is epitomised by the important report on the various types of rigour in textbooks at the first big international meeting of I.C.M.I. in Milan (see Castelnuovo, 1911).

In the meanwhile teachers were growing up professionally. In 1874 the first Italian journal devoted to mathematics teaching was founded. After its death a journal was founded, which was the cradle of the Italian association of mathematics teachers born in 1895 (*Mathesis*). These journals were concerned with discussing details of mathematical subjects taught in school rather than on pedagogical issues. In principle the association of mathematics teachers should have been the right place to discuss educational issues, but this did not happen. Most energies were devoted to decide if university professors could be admitted as members. The association had various deaths and resurrections until it acquired a rather stable status in 1921 under the chair of Federigo Enriques, one of the greatest Italian mathematicians of the twentieth century. He was a researcher in algebraic geometry, and also author of textbooks and books for teachers translated into foreign languages. The first half of twentieth century was dominated by this relevant personage, who had to face events important for the Italian system of instruction, such as the reform promoted by the philosopher Giovanni Gentile. Unfortunately, in accordance with the idealistic philosophical theory of Gentile, scientific culture (including mathematics) was relegated to a second rank position. Other Italian mathematicians were contributing to the discussion on mathematics teaching and had contacts with the international milieu of I.C.M.I.. Besides Enriques, Guido Castelnuovo and Gino Loria were among the nine persons awarded by I.C.M.I. with the special acknowledgement for their work in the field of mathematics instruction at the world Congress of mathematicians in Oslo (1936).

We see that, as it happened in the pioneering period of the nineteenth century, the chief characters in mathematics education of the first half of twentieth century were mainly university mathematicians. In summarising their attitudes towards mathematics teaching we may say that Enriques and Loria were interested in the dynamic of mathematics (its history, the psychology of the great mathematicians, the relationship of mathematics with painting, music,…). As a historian Loria was a pioneer in facing the problem of the use of history in mathematics teaching, especially in teacher education. Castelnuovo stressed the importance of modelling and application of mathematics; already at the beginning of the twentieth century he proposed the introduction of probability in mathematical programs. A singular position was that of Giuseppe Peano, who tried to apply directly the object of his research (logic) to school practice. According to him the language of logic, which is clear and not ambiguous, should make mathematical knowledge accessible to all students. Peano's project was utopian, but his enthusiasm and good willingness attracted secondary teachers who collaborated with him. His environment constitutes an early example of a mixed group of university professors and school teachers working on didactic problems.

2. The Present

The international panorama has changed since the period I have considered before. In the period after the second world war, we saw international efforts of important initiatives, which slowly lead the community of mathematics educators to become a community of researchers in the new discipline of mathematics education, (see Bishop, 1992; Dreyfus and Paola, 2004; Freudenthal, 1968-1969; Kaufman, B.A. and Steiner, 1968-1969; Niss, 1999; Sierpinska and Kilpatrick, 1998). The wrench with the past was marked by the creation of the journal *Educational Studies in Mathematics* in 1968, which initially gathered the contributions of mathematics teachers and university mathematicians. This was the time of the birth of the ICME conferences. In this international movement Italy was represented by few persons. One of them, the secondary teacher Emma Castelnuovo, daughter of Guido, was member of the first editorial board of *Educational Studies in Mathematics*. The impact inside the country of what was happening abroad was confined to a few groups of researchers in some Italian universities. Some good projects for renewing the mathematics teaching were carried out under the guidance of mathematicians, who were interested in mathematics teaching. Until ICME 5 in Berkeley (1984) the Italian participants to ICME conferences were very few. As a consequence also the involvement in the activities of the affiliated Study Group (HPM[1] and PME) created in 1976 was very poor. Initially the conferences of the commission for improving the mathematics teaching CIEAEM were the main bridge of Italians with the international community. The sudden increasing of the number of Italian participants at ICME 6 (1988 in Budapest) may be taken as a mark in the internationalisation of our community.

Important aspects of the development of mathematics education research in Italy until the 1990s are outlined in (Arzarello and Bartolini, 1998). Moreover, since ICME 6 (Québec, 1992) the national community of mathematics educators has issued special books containing summaries of papers authored by Italian researchers and surveys of the Italian streams of research.

[1] HPM: History and Pedagogy of Mathematics Group; PME: International Group of the Psychology of Mathematics Education.

I feel that the Italian community has developed its own identity and independence from the mother-community of mathematicians, nevertheless I observe remarkable elements of continuity. Firstly, though our attitude towards rigour has strongly changed, still the interest for the approach to proof in secondary school is central in our research as for all the stages (exploring, conjecturing, proving) and for all mediators (paper and pencil, computer, mathematical instruments, language), (see Boero, 2002). Secondly, in Italy many groups of research are characterised by close collaboration of teachers and researchers in planning and carrying out educational studies. This contributes to make the relation between theory and practice less problematic than in other countries. Our research has always in mind the classroom. Unfortunately the position of teachers as researchers is also not officially acknowledged by the Ministry of Education and the involvement of teachers is voluntary and without official rewards. In conclusion, as chair of the HPM Study Group in the years 2000-2004, I can not forget the historical flavour present in many Italian works, which is a direct heritage of Enriques's and Loria's style of approaching mathematics education problems.

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Mathematically Promising Students from the Space Age to the Information Age

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1. The Space Age

On October 4, 1957, with the launch of Sputnik 1 by the Soviet Union, the world entered the Space Age and the United States became quite concerned that the Soviet Union had a head start in the space race. A year later, realizing that the support of gifted and talented mathematics and science students was critical to national security, the United States federal government passed the National Defense Education Act (NDEA), providing aid to education in the United States at all levels, primarily to stimulate the advancement of education in science, mathematics, and modern foreign languages. Also, during this time, "new math" was introduced with an emphasis on more abstract concepts and unifying ideas. One of the most unique of the projects developed during that time, the *Comprehensive School Mathematics Program* (CSMP) from McREL, Mid-continent Research for Education and Learning, continues to be available online at http://ceure.buffalostate.edu/~csmp/. Although never fully implemented as intended, some of the "new math" projects along with the NDEA contributed to the dominance of the United States in science and technology in the latter part of the twentieth century as they inspired thousands of students to enjoy mathematical investigations and to pursue degrees in mathematics, science and technology.

On July 16, 1969, the Apollo 11 launched from the Kennedy Space Center and on July 20, 1969, Commander Neil Armstrong became the first man on the moon and said the historic words, "One small step for man, one giant leap for mankind." The sixth and final manned moon landing occurred in December 1972, and the United States declared victory in the space race. For fifteen years, Americans had supported gifted and talented students interested in learning mathematics and science, especially as related to space technology, but what has happened since that time?

2. The Growth of Technology

Partially in reaction to the "new math", the 1970s saw a strong "back-to-basics" movement with an emphasis on basic skills such as computation. In 1980, the National Council of Teachers of Mathematics (NCTM) published *An Agenda for Action* noting that the most important basic skill was problem solving. The following statement, from this same report pointed to the growing recognition of the importance of the development of gifted mathematics students.

> The student most neglected, in terms of realizing full potential, is the gifted student of mathematics. Outstanding mathematical ability is a precious societal resource, sorely needed to maintain leadership in a technological world.
> NCTM, 1980, p. 18

In 1983, The National Commission on Excellence in Education warned in its report, *A Nation at Risk*, that the skills and knowledge of the U.S. workforce would have to improve dramatically in

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order for the nation to remain internationally competitive. In 1989, the first President Bush convened an Education Summit with the nation's Governors and adopted six National Education Goals. The fifth goal was: "U.S. students will be first in the world in mathematics and science achievement by the year 2000." In spite of the public acknowledgement of the importance of students with high-level skills in mathematics and science, little has been done in the past 25 years to support our most promising students.

3. The Information Age

In 1993, Richard Riley, the U. S. Secretary of Education, in the introduction to *National Excellence: A Case for Developing America's Talent*, stated, "All of our students, including the most able, can learn more than we now expect. But it will take a major national commitment for this to occur." (Ross, 1993, p. iii) The report goes on to point to a "quiet crisis in educating talented students" with the following statement.

> The United States is squandering one of its most precious resources – the gifts, talents, and high interests of many of its students.
>
> Ross, 1993, p.1

The year after this report came out, the NCTM appointed a Task Force on Mathematically Promising Students to analyze this issue specifically for mathematics. The Task Force agreed that a major national commitment was needed to turn around this quiet crisis for mathematically promising students who were defined as "those who have the potential to become the leaders and problem solvers of the future". The Task Force called for a strategy that seeks to greatly increase the numbers and levels of mathematically promising students by maximizing their ability, motivation, beliefs, and experiences/opportunities. The report pointed out that these four factors are all variables that could and should be increased with proper support and encouragement. Noting research on brain functioning that demonstrates that significant changes in the brain are due to experiences, the report called on administrators, teachers, parents and students themselves to make sure that all students have the opportunity to experience the joy of solving challenging mathematical problems on a regular basis and that high-level mathematics courses are available to all students regardless of where they go to school. Recognizing that the culture in the United States often works against students' desire to excel in science, technology and mathematics, the report also noted the importance of students' realizing that excellence in mathematics is not only possible, but also leads to careers in fulfilling and intriguing areas. (Sheffield, et al, 1995) The recent popularity of the television series *Numb3rs* goes a long way toward supporting this goal, but much more is needed.

4. The Twenty-First Century

By 2000, it was evident that the United States was a long way from the goal of being first in the world in math and science. The Trends in International Mathematics and Science Study (TIMSS) in 1995 and the repeat of the study in 1999 and 2003 showed that not only were we not first, but top students in the United States were not at the same level as top students in other countries. In1995, 9% of U. S. fourth graders and 39% of Singapore fourth graders scored above the 90^{th} percentile on the mathematics portion of the TIMSS test. That year, 5% of U. S. eighth graders and 45% of Singapore eighth graders scored above the 90^{th} percentile on the TIMSS mathematics test. By 2003, 40% of the eighth grade students in Singapore, 38% of eighth graders in Taiwan, and 7% of U. S. eighth graders scored at the most advanced level. Although this was an improvement for students in the United States, it was still far behind other developed countries.

Similar results were found by the Program for International Student Assessment (PISA). In 2003, U.S. performance in mathematics literacy and problem solving was lower than the average performance for most OECD (Organization for Economic Co-operation and Development) countries. Even the highest U.S. achievers (those in the top 10 percent in the United States) were outperformed on average by their OECD counterparts. (National Center for Education Statistics, 2003)

The *No Child Left Behind Act of 2001* had as a major purpose that all students reach proficiency on challenging state standards and assessments, closing the achievement gap between high and low-achieving students. But what happens to students for whom moving toward proficiency is moving backwards when there is a goal to close the achievement gap between high and low-performing students?

In a study of the effects of teachers and schools on student learning, William Sanders and his staff at the Tennessee Value-Added Assessment System put in this way:

> "Student achievement level was the second most important predictor of student learning. The higher the achievement level, the less growth a student was likely to have." DeLacy, 2004, p. 40

Certainly one way to close the achievement gap between high and low-performing students is to slow down the learning of high-performing students, but is that a goal that we can afford?

> The United States is losing its edge in innovation and is watching the erosion of its capacity to create new scientific and technological breakthroughs. …If America is to sustain its international competitiveness, its national security and the quality of life of its citizens, then it must move quickly to achieve significant improvements in the participation of all students in mathematics and science.
> Business-Higher Education Forum, 2005, p. 1, 3

In 2005, the Annual Conference of the National Association of Gifted Children (NAGC) featured a special strand on Mathematics and Science with a keynote address by Jim Rubillo, the Executive Director of the National Council of Teachers of Mathematics and Gerry Wheeler, the Executive Director of the National Science Teachers Association, and NAGC appointed a Math/Science Task Force to continue this work. If the United States is to maintain leadership in this technological world, it is critical that we collaborate to take immediate drastic action to recognize, support, create and develop the mathematical promise in large numbers of students and their teachers - male and female; black and white; preschool through graduate school; rich and poor; rural and urban. As we approach the fiftieth anniversary of Sputnik and the National Defense Education Act, let's join together to inspire a new generation of students to excel in these areas critical to the welfare of our country and indeed of the entire world.

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Algorithmic Problems in Junior Contests in Latvia

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Abstract: Mathematical contests are of great importance for advanced education in Latvia today. Their content must be well-balanced and must correspond to the inner logic and recent trends of mathematics. A classification of algorithmic problems and characteristic examples are considered.
Key words: Mathematical contests, algorithmic problems, method of interpretation.

1. Introduction

Mathematical contests have become an essential part of middle and high school education in Latvia. They are the broadest national scale tests on advanced level. In the situation when the curricula of exact disciplines is reduced constantly in favor of social and humanitarian ones (considering this as "humanization" of education) math contests have not lost their high standards and are the most popular academic competitions in Latvian schools (e.g., the Open Latvian Mathematical Olympiad alone gathers more participants than competitions in all other disciplines together in Latvia).

In such a situation a great attention must be (and is) paid to the scientific content of contest problems. In accordance with the increasing role of the discrete branches of mathematics vs. continuous branches of it the proportion of combinatorial, number – theoretic etc. problems does not fall below 50% of the total number of them, being considerably higher in younger grades where the students have not yet accumulated enough knowledge to solve serious problems in algebra, geometry, calculus etc. Naturally, this leads to the fact that "Olympiad curricula" contains also many ideas and formal tools from computer science, which becomes the central discipline in today's education. Without any doubt, the central concept of it is the concept of algorithm.

2. Main classes of algorithmic problems for contests

The problems of algorithmic nature mostly used in math competitions can roughly be classified as follows:

1. Games
    - 1.1. Games with symmetry
        - 1.1.1. Games with usual symmetry.
        - 1.1.2. Games with generalized symmetry.
    - 1.2. Model of the game.
        - 1.2.1. Model in the grid.
        - 1.2.2. Model in the graph.
    - 1.3. Games with prehistory.
    - 1.4. Indirect proofs on winning strategies.
    - 1.5. Invariant of the game.
    - 1.6. Probabilistic games.

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1.7. Continuous games, including games of search and ambush.
2. General combinatorial algorithms
   2.1. Inference of algorithms.
   2.2. Analysis of algorithms.
   2.3. Developing of algorithms.
      2.3.1. "Divide – and – conquer."
      2.3.2. Procedures.
      2.3.3. Inductive algorithms.
      2.3.4. Exhaustive search.
   2.4. Optimization of algorithms.
      2.4.1. Problems of searching and sorting.
      2.4.2. Algorithms for performing arithmetical operations.
      2.4.3. Algorithms in graphs.
      2.4.4. General methods of obtaining lower bounds.
   2.5. Proofs of the correctness of algorithms.
   2.6. Proofs of nonexistence of an algorithm.
      2.6.1. Uses of invariants and semi-invariants.
      2.6.2. Exhaustion.
      2.6.3. Modeling.
      2.6.4. General idea of a cycle.
   2.7. Nondeterministic algorithms.
   2.8. Probabilistic algorithms.
   2.9. Algorithms dealing with incomplete information.

Of course, not all of these types are suitable for junior students. Those we find appropriate for them are given in italics above. The above list shows also some shifts that have occurred during last decades. Until the 1960's the problems of geometric constructions were very popular; at present they have almost disappeared from "contest curricula". On the other hand, almost no examples of the type 1.4., 2.4.4., 2.5., 2.6.4., 2.7., 2.8, can be found in contests before 1970.

3. System of Math Contests for Junior Students in Latvia

There are two main classes of competitions, mainly in problem solving.

A. Mathematical Olympiads.

They are organized at three levels:

- school olympiads, often supported by universities; they are usually held in November,
- regional olympiads held in 39 different places in Latvia each year in February,
- Open math olympiad held each year in April. This competition is a very large one; more then 3000 participants arrive in Riga.

All these competitions are open to everybody who wants to participate.
Other present-way competitions are organised at schools, at summer camps etc.

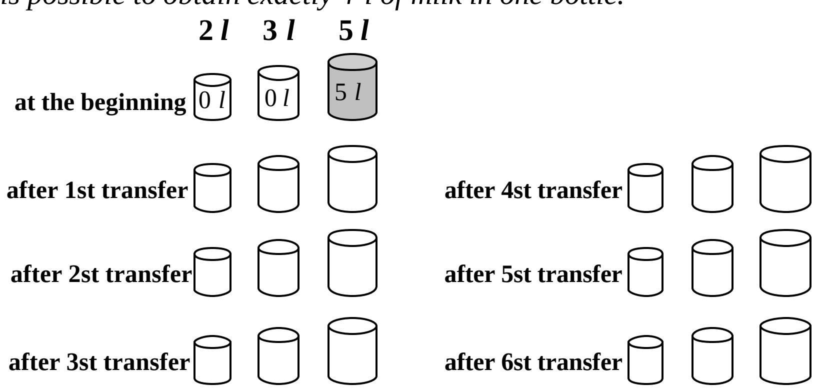
B. Corresponding contests.

There are many students who need more than some 4-5 hours (usually allowed during math olympiads) to go deep enough into the problem. For such children a system of correspondence contests has been developed:

- "Club of Professor Littledigit" (CPL) for students up to the 9^{th} Grade. There are 6 rounds each year, each containing 6 relatively easy and 6 harder problems. Problems are published in the newspaper "Latvijas Avize" (having the largest circulation in Latvia), and on the INTERNET.
- "Contest of young mathematicians"(CYM) for students up to 7^{th} Grade, originally developed for weaker students than the participants of CPL, especially in Latgale, the eastern region of Latvia. The problems are published in regional newspapers and on the INTERNET, and today it has become popular all over Latvia.
- "So much or… how much?" (SMHM) contest for the students up to 4^{th} Grade, organized jointly with colleagues from Lithuania and Belorussia. The problems are published in Internet. At the end of the school year an international correspondence competition between the students of three countries is organized.

4. Characteristic examples

Example 1 (SMHM). *There is a bottle of volume 5l, full of milk. There also 2 empty bottles of volumes 2l and 3l correspondingly. The milk can be continuosly transferred from one bottle to another one until either the first bottle is empty or the second bottle is full. Show how it is possible to obtain exactly 4 l of milk in one bottle.*



Answer: it is enough to use ________ transfers.

The task can be accomplished within 3 transfers. This is a typical representative of the class 2.3.4 (see above). The problem appeared to be relatively easy.

Comment 1. If similar problem was proposed for the students of higher grades, possibly it should include also the question about the minimality of the number of transfers.

Comment 2. It is worth attention that a form for providing a solution is included in the text of it. Our experience shows that teaching how to write down the solution is not less important than teaching how to find it, and it needs a constant effort. As SMHM contest is the very first for many students, they should be given examples of correctly formulated solutions.

Example 2 (CYM). *There are 2005 points marked on the circumference; 999 of them are red while the other are green. Each of obtained 2005 arcs is marked with an integer:*

a) if both endpoints of the arc are red, it is marked with "-1",

b) if both endpoints are green, it is marked with "1",

c) if both endpoints are different, the arc is marked with "0".

Find the sum of all integers with which the arcs are marked.

Solution. Mark each red point with "-1" and each green point with "1". It is easy to see that the sum of all these new marks equals the requested sum.

Comment. This is a problem from the class 2.3.2; the idea of a basis is used, though indirectly, in the solution. It is a common praxis in Latvia to construct the problems in such a way that simple appearances of far-reaching ideas can be encomposed in the solution.

Example 3 (CPL, easy part). *There are 8 coins in the row. By one move we can interchange two neighboring coins. We must achieve the situation that each coin has "visited" both the left end and the right end of the row. Prove that 33 moves are not enough.*

Solution. Let the distance between neighboring coins be 1 unit. During one move the distance of 2 units is covered in common. The coin initially occupying the first place must cover the distance 7, the coin initially occupying the second place must cover the distance 8, etc. The sum of all distances that must be covered is 2 (7+8+9+10) = $2 \cdot 34$. So at least 34 moves are needed.

Comment 1. The problem appeared to be a hard one. Most solutions tried to analyse the "worst case" not argumenting why it is really the worst one, what is the typical situation in solutions of the problems of class 2.4.1.

Comment 2. The real minimal sufficient number of moves is 40. That was a problem for the hard part of the contest.

Example 4 (CPL, easy part.). *There are 100 first-graders in a row, all facing the teacher standing in front of them. After the command "Turn to the right!" some of them turned to the right, while the other turned to the left. After that after each second each two pupils who stood face to face with each other turn around. Prove that the movement will stop after at most 99 seconds.*

Solution. It is easy to understand that the development of the process depends only on the fact into which direction the pupils occupying correspondingly the 1^{st}, 2^{nd}, …, 100^{th} place are looking at each moment, but not on the fact **which** particular pupil occupies the 1^{st}, 2^{nd}, …, 100^{th} place. Let's consider another similar process in which the pupils don't turn around but step forward interchanging their places. There is an isomorphism between the two processes in the sense that for each i, $1=i=100$, the pupil on the i-th place in the first process is looking to the right iff so does the pupil on the i-th place in the second process. On the other hand it is clear that no pupil can make more than 99 steps, so the conclusion follows.

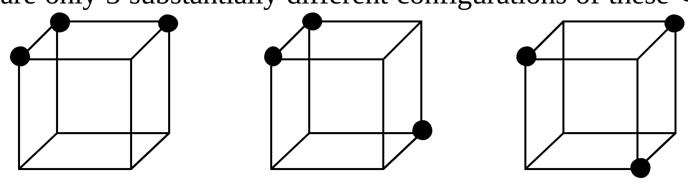
Comment 1. This is a typical problem of the class 2.2., using the method of interpretations (see [1]).

Comment 2. The problem comes from the theory of cellular automata. It is still another illustration of the great impact the theoretical computer science has made on math contests during last 30-40 years.

Example 5 (summer school math contest).

There are 3 convex polygons drawn inside the unit square: A, B, C. The contours of each two of them intersect each other at exactly two points and have no other common points; all 6 points of intersection are different. Two players X and Y play the following game. At first X chooses one of the polygon and paints either the inner or the outer region of *it; then Y does the same with one of the remaining polygons, and X paints the inner or the outer region of the only remaining polygon. Prove: X can ensure that the area of the three times painted region is* $\leq \frac{1}{6}$.

Solution. In an obvious way, represent the inner and outer regions of polygons by the faces of the cube. Then the 8 parts into which the square is dissected are represented by the vertices of the cube. Write the area of each part into the corresponding vertice; then the sum of all written numbers is 1. Mark the vertices with numbers $\leq \frac{1}{6}$ as ●; there are at least three ● in the cube. There are only 3 substantially different configurations of these ●:



Now, the move in the game is to choose one face of the cube and delete the opposite face from further consideration. It's almost obvious that the first player can ensure: the intersection of three chosen faces is marked with ●.

Comment. The problem appeared to be very hard. It is an example of class 1.2. with non-traditional application of the method of interpretations.

More examples can be found in [2].

5. On sources of algorithmic problems

The main, and, we hope, everlasting source of algorithmic problems for math contests is the current scientific research. For example, all rich area of "coin – weighing problems" has originated from the investigations in sorting algorithms. New types of problems arise in connection with non-traditional (from the students' point of view) types of algorithms.

Example 6 (Latvian summer competition). *There are 4 equally looking coins; all of them have different masses. We can use a pan balance without counterfeits. Develop an algorithm which uses a pan balance twice and find the heaviest coin with the probability* $\frac{3}{4}$.

Solution. At first, using any generator of random numbers (for example, throwing the fair coin twice), decide which coin will be called "read"; other coins will be called "blue". After that find the heaviest blue coin deterministically within two weighings in a standard way. Announce this coin the heaviest among all four.

Clearly there is a probability $\frac{3}{4}$ that the heaviest coin (among all four) will be blue. Then it will be announced the heaviest, QED.

Comment. This problem demonstrates the advantage of "clever" probabilistic algorithm over both deterministic algorithms and pure guessing. It can be easily proved that the task can not be completed deterministically. Of course, simple guessing gives the correct answer only with a probability $\frac{1}{4}$.

Example 7 (R.Freivalds). *There are 14 equally looking coins. The experts have established that 7 of them are exact and 7 of them are false. The court knows only that all exact coins have equal masses, all false coins have equal masses and an exact coin is heavier than the false one. How can expert demonstrate to the court which coins are exact and which are false using only 3 weighings on a pan balance without counterfeits?*

Solution. At first expert places one exact coin on the left pan and one false coin on the other. The court becomes aware "who is who" of these coins. The expert adds two exact coins to the false one and two false coins to the exact one – and the court again becomes aware "who is who". Then the expert gathers 3 "proved exact" coins on one pan and adds 4 "unproved false" coins to them; other 7 coins are placed on the other pan. It's not hard to understand that all should be clear to the court after this.

Comment. This problem has great educational value; it demonstrates to the student that a proof itself can be **principially** simpler than a process of establishing it. Really, an easy generalization shows that n exact coins can be separated from n false ones using $[\log_2 n]+1$ **demonstrations**; on the other hand, information theory lover bound shows that at least $n \cdot \log_3 2$ weighings are necessary to **establish** which n coins are the exact ones.

Other possible variations are to introduce the possibility of unreliable information, to consider parallel processes, to deal with more powerful/ more restricted identifying devices than yes/no questions or their equivalents, etc. All these are topics of serious investigations in computer science, but yet have not found an adequate reflection in math contests.

6. Concluding remarks

Many investigations have stressed the great educational value of discrete and combinatorial problems, e. g., [3]. Algorithmic problems are special among them. They develop the analytical and constructive skills of children and provide the possibilities of interdisciplinary education. They are always welcome by the students and often can be reformulated so that become suitable for independent investigations of them. Their connections with general reasoning methods make them a valuable educational tool.

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Linda Sheffield (USA) is Regents Professor of Mathematics Education and Gifted Education at Northern Kentucky University, and is internationally recognized for her work on developing and challenging students from the pre-kindergarten through the university level. She is past president of the School Science and Mathematics Association (SSMA), was chair of the Task Force on Promising Students for the National Council of Teachers of Mathematics (NCTM), and is chair of the Math/Science Task Force of the National Association for Gifted Children. She was also editor of the NCTM book *Developing Mathematically Promising Students*. Her other books include *Extending the Challenge in Mathematics* for teachers looking for ideas to amplify their students' mathematical power and co-authoring *Awesome Math Problems for Creative Thinking*, a series of problem solving books for children; the PreK–2 NCTM *Navigations* series; *Mentoring Mathematical Minds*, a series of mathematics units for talented elementary students; and a math methods book for elementary and middle school teachers. She has conducted seminars for teachers and students across the United States and as far away as Korea, Bulgaria, Denmark, Spain, Germany, England, Sicily, Japan, Australia, China, and Hungary with an emphasis on helping students develop their talents and abilities to the fullest extent possible.

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Agnis Andžans (Latvia) is Professor of mathematics at the University of Latvia, Corresponding Member of Latvian Academy of Sciences. His main research interests are theory of automata, new methods of advanced teaching of mathematics, the "informatization" of education. He has published 6 monographs, approximately. 90 research papers and approximately 120 teaching aids.

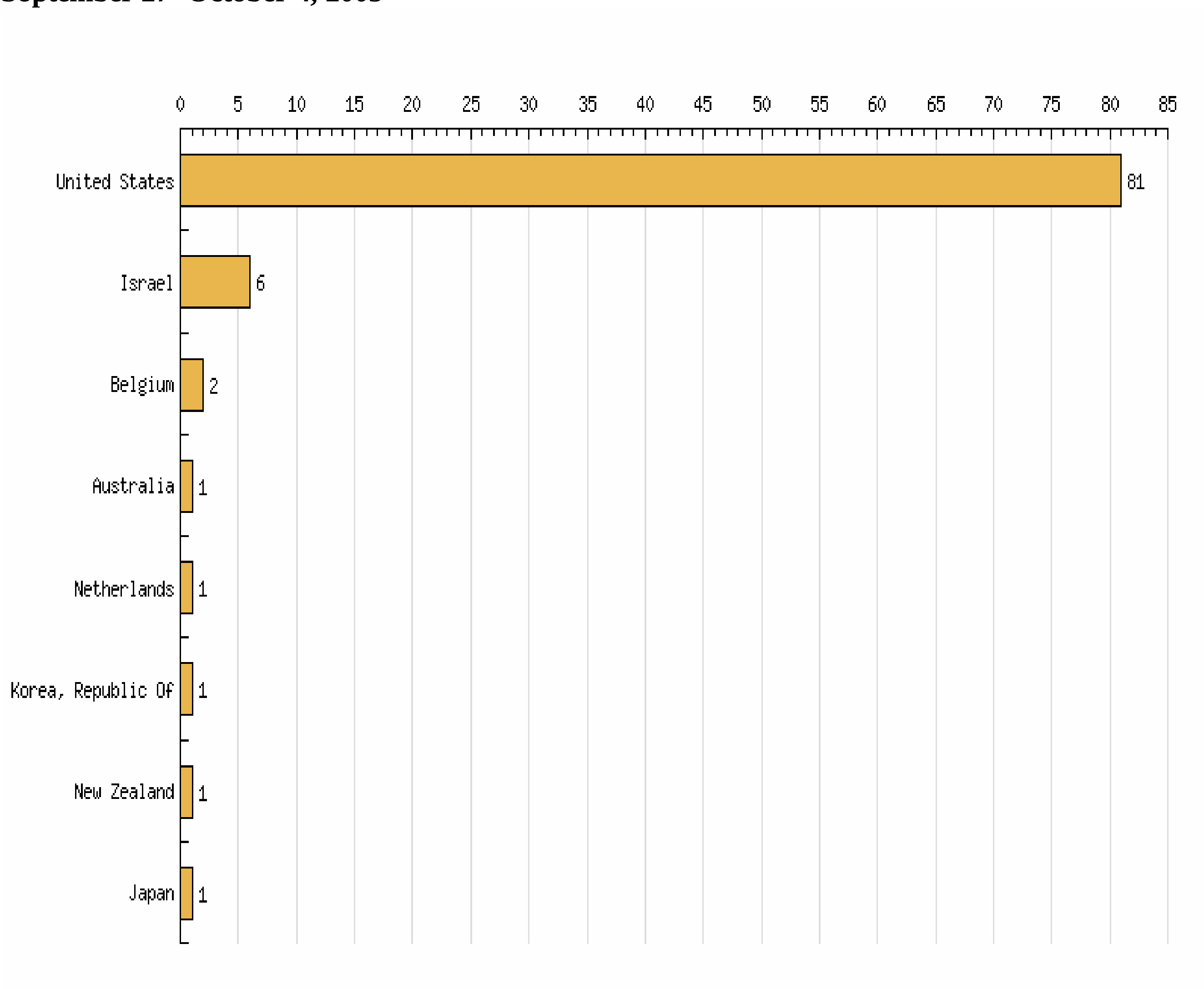
Inese Berzina (Latvia) is 3rd year mathematics bachelor student at the University of Latvia, Faculty of Physics and mathematics, and deputy educational director at A. Liepas Correspondence Mathematics School which is a centre of advanced mathematical educational system in Latvia. Her research area and main professional activities are connected with correspondence mathematics contests for junior students as well as with the curricula and teaching process in correspondence for gifted high school students, particularly those organized via internet. She is also active in arranging Olympiads, summer schools and regional mathematical clubs, mainly in the rural area of Latvia. She is the head of the team checking the papers of the multi-stage "Contest of young mathematicians", involving ~ 150 participants from all regions of Latvia. Her non-professional interests include choir singing, traveling, swimming etc.

Dace Bonka (Latvia) is a PhD student at the University of Latvia, lecturer at the University of Latvia and an educational director at A.Liepas Correspondence Mathematics School. Already at school Dace Bonka was inspired from math olympiads. She has been a member of organizing committee and jury of Latvian mathematics Olympiad already for 10 years. Her research area and main activities are correspondence mathematics contests for junior students in Latvia. She is initiator and leader of the contest "So much or… how much?" (SMHM) for the students up to 4th Grade in Latvia and co-leader of Junior International Math Olympiad – the last round of contest SMHM, organized jointly with colleagues from Lithuania and other neighboring countries. She is also the author of problem set for "Contest of young mathematicians" (CYM) for students up to 7th Grade. She has published 11 research papers and 5 teaching aids for junior and high school students.

A ROLLING GLIMPSE OF TMME'S WORLDWIDE CIRCULATION

The following bar graphs give a rolling snapshot of the worldwide access of *The Montana Mathematics Enthusiast,* based on samples of last 100 page loads on randomly chosen weeks from September 2005 – January 2006.

September 27- October 4, 2005



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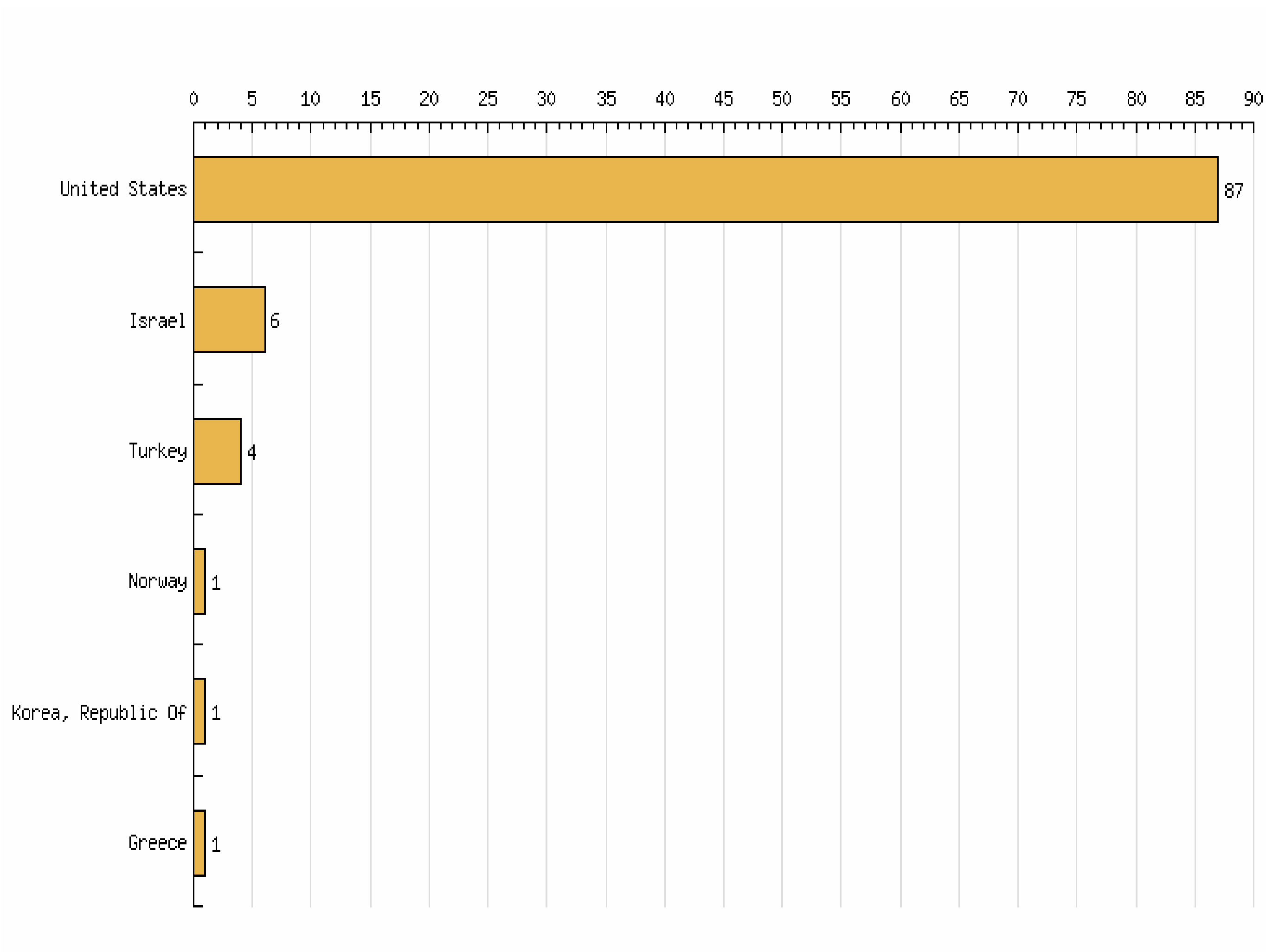
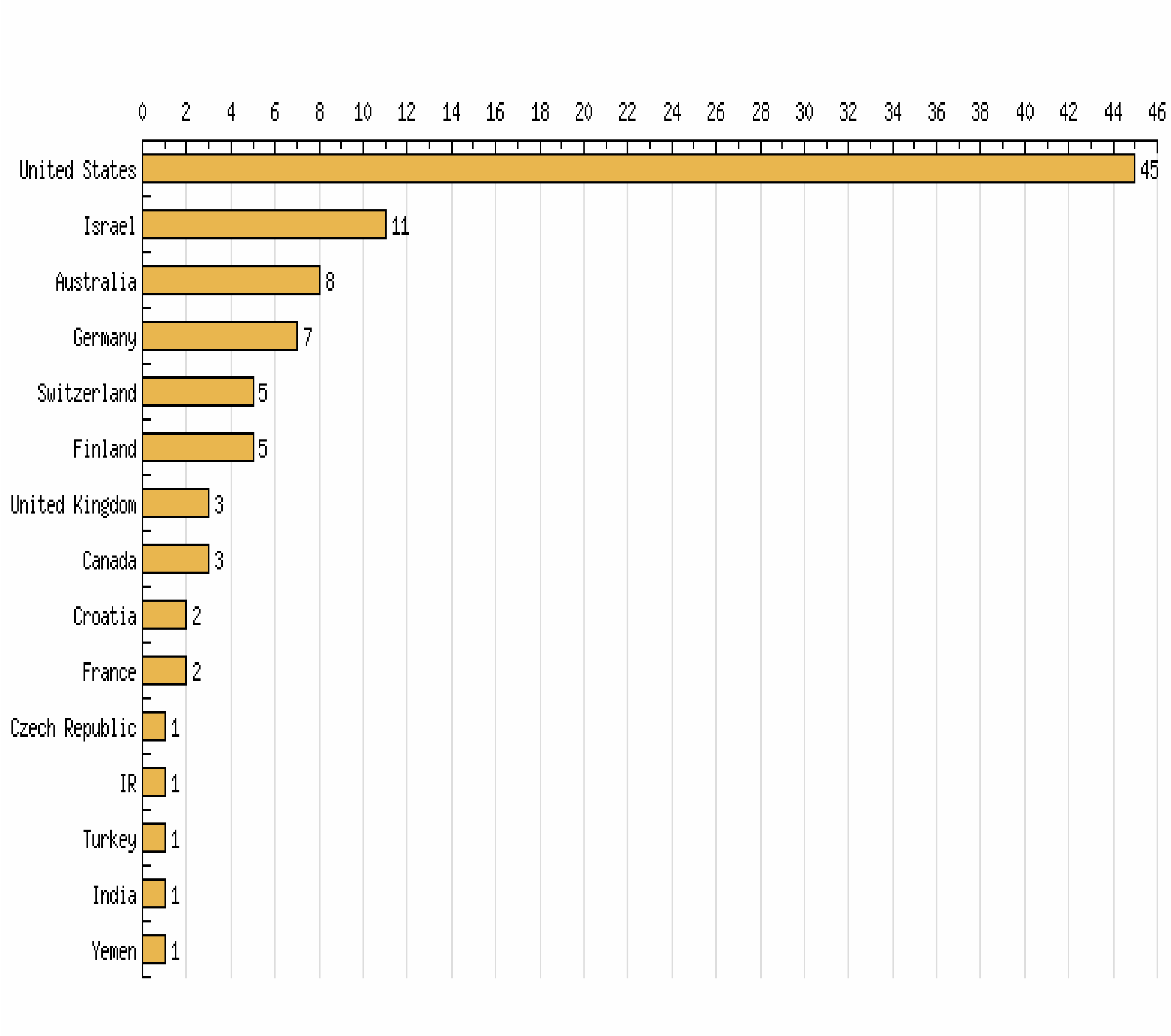
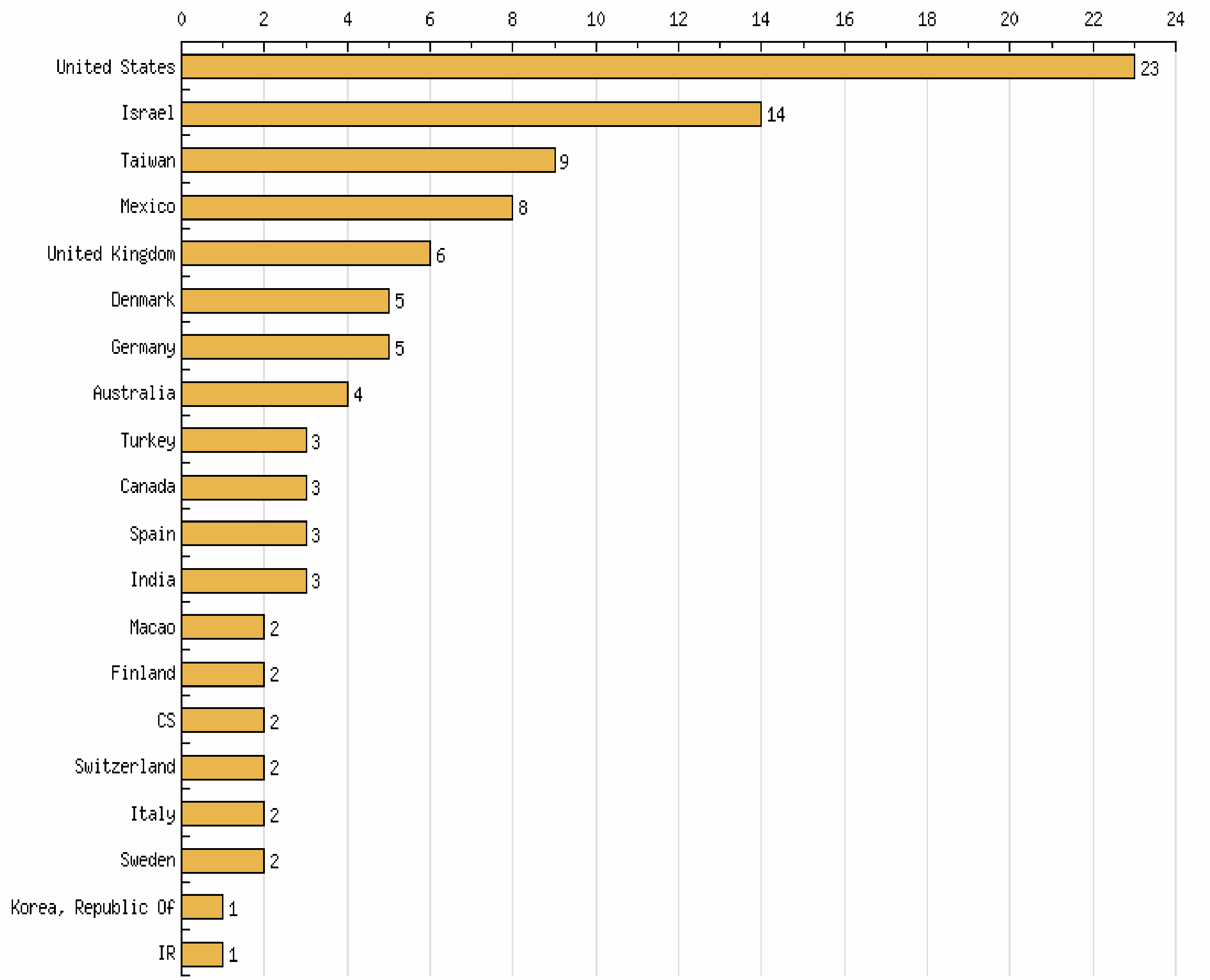
October 5- October 11, 2005



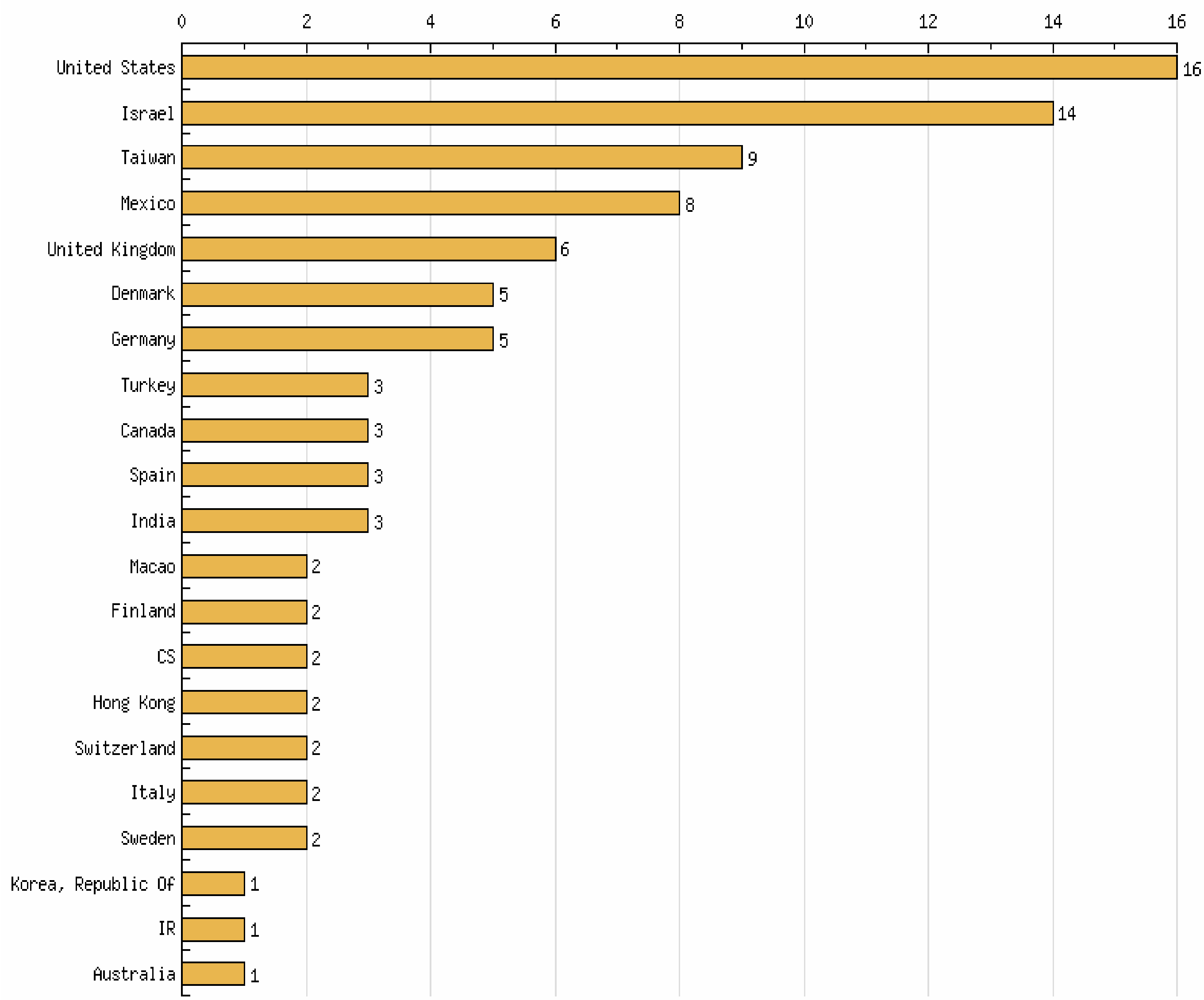
December 18 – December 25, 2005.



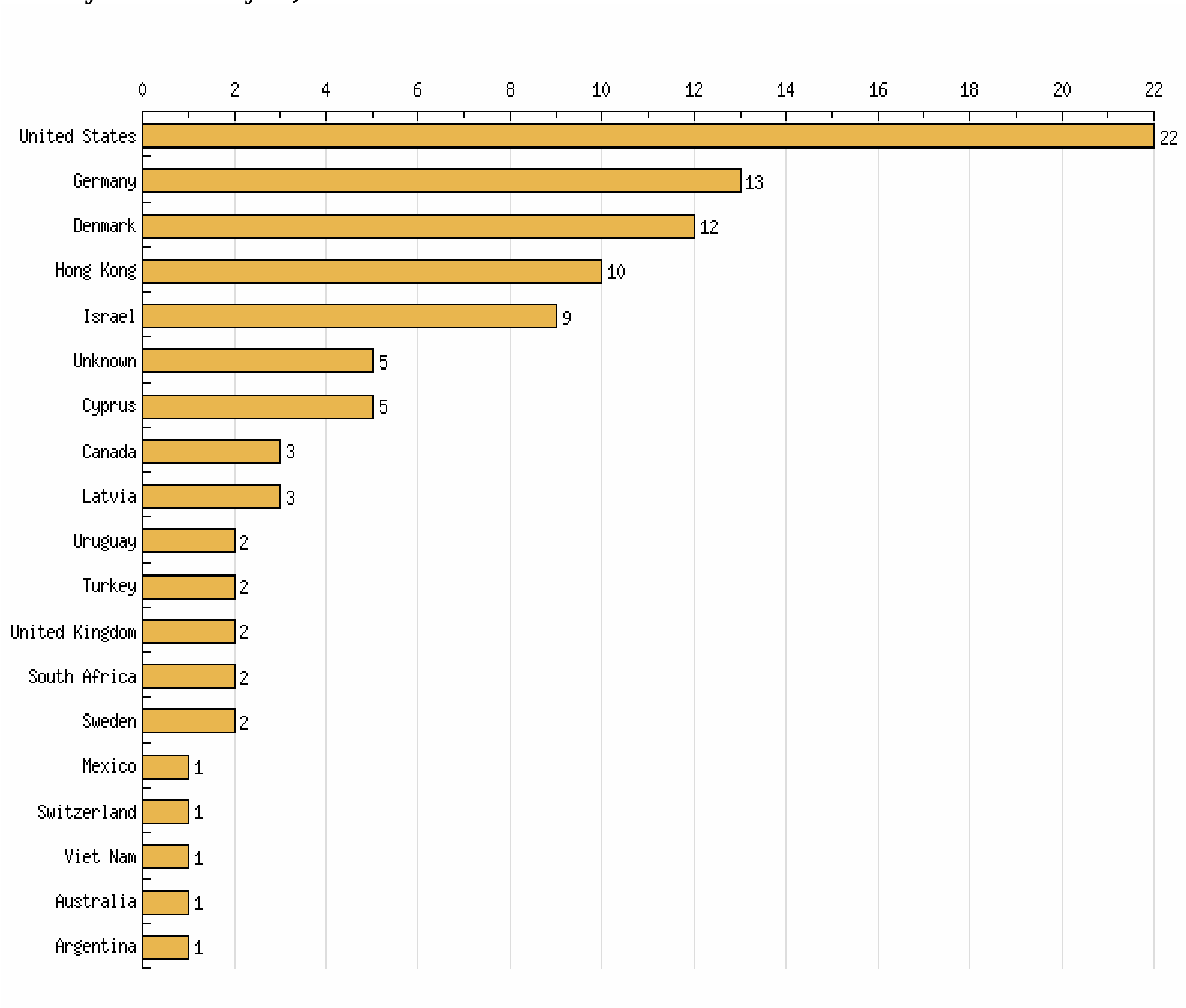
December 25, 2005 - January 01, 2006



December 27, 2005 – January 03, 2006



January 12 – January 19, 2006



January 19 – January 25, 2006