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Abstract: Mathematical contests are of great importance for advanced education in Latvia today. Their content must be well-balanced and must correspond to the inner logic and recent trends of mathematics. A classification of algorithmic problems and characteristic examples are considered.
*Key words**:** Mathematical contests, algorithmic problems, method of interpretation.*

1. Introduction

Mathematical contests have become an essential part of middle and high school education in Latvia. They are the broadest national scale tests on advanced level. In the situation when the curricula of exact disciplines is reduced constantly in favor of social and humanitarian ones (considering this as “humanization” of education) math contests have not lost their high standards and are the most popular academic competitions in Latvian schools (e.g., the Open Latvian Mathematical Olympiad alone gathers more participants than competitions in all other disciplines together in Latvia).

In such a situation a great attention must be (and is) paid to the scientific content of contest problems. In accordance with the increasing role of the discrete branches of mathematics vs. continuous branches of it the proportion of combinatorial, number – theoretic etc. problems does not fall below 50% of the total number of them, being considerably higher in younger grades where the students have not yet accumulated enough knowledge to solve serious problems in algebra, geometry, calculus etc. Naturally, this leads to the fact that “Olympiad curricula” contains also many ideas and formal tools from computer science, which becomes the central discipline in today’s education. Without any doubt, the central concept of it is the concept of algorithm.

2. Main classes of algorithmic problems for contests

The problems of algorithmic nature mostly used in math competitions can roughly be classified as follows:

1. Games
    1.1. Games with symmetry
        1.1.1. Games with usual symmetry.
        1.1.2. Games with generalized symmetry.
    1.2. Model of the game.
        1.2.1. Model in the grid.
        1.2.2. Model in the graph.
    1.3. Games with prehistory.
    1.4. Indirect proofs on winning strategies.
    1.5. Invariant of the game.
    1.6. Probabilistic games.

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1.7. Continuous games, including games of search and ambush.

2. General combinatorial algorithms
   2.1. Inference of algorithms.
   2.2. Analysis of algorithms.
   2.3. Developing of algorithms.
      2.3.1. "Divide – and – conquer."
      2.3.2. Procedures.
      2.3.3. Inductive algorithms.
      2.3.4. Exhaustive search.
   2.4. Optimization of algorithms.
      2.4.1. Problems of searching and sorting.
      2.4.2. Algorithms for performing arithmetical operations.
      2.4.3. Algorithms in graphs.
      2.4.4. General methods of obtaining lower bounds.
   2.5. Proofs of the correctness of algorithms.
   2.6. Proofs of nonexistence of an algorithm.
      2.6.1. Uses of invariants and semi-invariants.
      2.6.2. Exhaustion.
      2.6.3. Modeling.
      2.6.4. General idea of a cycle.
   2.7. Nondeterministic algorithms.
   2.8. Probabilistic algorithms.
   2.9. Algorithms dealing with incomplete information.

Of course, not all of these types are suitable for junior students. Those we find appropriate for them are given in italics above. The above list shows also some shifts that have occurred during last decades. Until the 1960's the problems of geometric constructions were very popular; at present they have almost disappeared from "contest curricula". On the other hand, almost no examples of the type 1.4., 2.4.4., 2.5., 2.6.4., 2.7., 2.8, can be found in contests before 1970.

3. System of Math Contests for Junior Students in Latvia

There are two main classes of competitions, mainly in problem solving.

A. Mathematical Olympiads.

They are organized at three levels:

- school olympiads, often supported by universities; they are usually held in November,
- regional olympiads held in 39 different places in Latvia each year in February,
- Open math olympiad held each year in April. This competition is a very large one; more then 3000 participants arrive in Riga.

All these competitions are open to everybody who wants to participate.

Other present-way competitions are organised at schools, at summer camps etc.

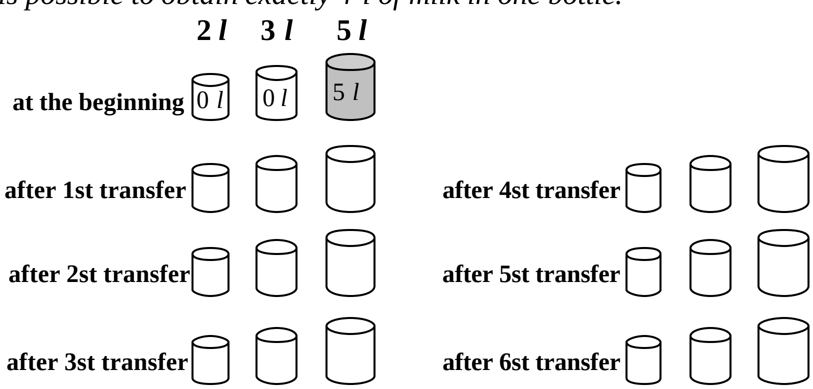
B. Corresponding contests.

There are many students who need more than some 4-5 hours (usually allowed during math olympiads) to go deep enough into the problem. For such children a system of correspondence contests has been developed:

- “Club of Professor Littledigit” (CPL) for students up to the 9th Grade. There are 6 rounds each year, each containing 6 relatively easy and 6 harder problems. Problems are published in the newspaper “Latvijas Avize” (having the largest circulation in Latvia), and on the INTERNET.
- “Contest of young mathematicians”(CYM) for students up to 7th Grade, originally developed for weaker students than the participants of CPL, especially in Latgale, the eastern region of Latvia. The problems are published in regional newspapers and on the INTERNET, and today it has become popular all over Latvia.
- “So much or… how much?” (SMHM) contest for the students up to 4th Grade, organized jointly with colleagues from Lithuania and Belorussia. The problems are published in Internet. At the end of the school year an international correspondence competition between the students of three countries is organized.

4. Characteristic examples

Example 1 (SMHM). *There is a bottle of volume 5l, full of milk. There also 2 empty bottles of volumes 2l and 3l correspondingly. The milk can be continuosly transferred from one bottle to another one until either the first bottle is empty or the second bottle is full. Show how it is possible to obtain exactly 4 l of milk in one bottle.*



Answer: it is enough to use ________ transfers.

The task can be accomplished within 3 transfers. This is a typical representative of the class 2.3.4 (see above). The problem appeared to be relatively easy.

Comment 1. If similar problem was proposed for the students of higher grades, possibly it should include also the question about the minimality of the number of transfers.

Comment 2. It is worth attention that a form for providing a solution is included in the text of it. Our experience shows that teaching how to write down the solution is not less important than teaching how to find it, and it needs a constant effort. As SMHM contest is the very first for many students, they should be given examples of correctly formulated solutions.

Example 2 (CYM). *There are 2005 points marked on the circumference; 999 of them are red while the other are green. Each of obtained 2005 arcs is marked with an integer:*

a) if both endpoints of the arc are red, it is marked with "-1",

b) if both endpoints are green, it is marked with "1",

c) if both endpoints are different, the arc is marked with "0".

Find the sum of all integers with which the arcs are marked.

Solution. Mark each red point with "-1" and each green point with "1". It is easy to see that the sum of all these new marks equals the requested sum.

Comment. This is a problem from the class 2.3.2; the idea of a basis is used, though indirectly, in the solution. It is a common praxis in Latvia to construct the problems in such a way that simple appearances of far-reaching ideas can be encomposed in the solution.

Example 3 (CPL, easy part). *There are 8 coins in the row. By one move we can interchange two neighboring coins. We must achieve the situation that each coin has "visited" both the left end and the right end of the row. Prove that 33 moves are not enough.*

Solution. Let the distance between neighboring coins be 1 unit. During one move the distance of 2 units is covered in common. The coin initially occupying the first place must cover the distance 7, the coin initially occupying the second place must cover the distance 8, etc. The sum of all distances that must be covered is 2 (7+8+9+10) = $2\cdot 34$. So at least 34 moves are needed.

Comment 1. The problem appeared to be a hard one. Most solutions tried to analyse the "worst case" not argumenting why it is really the worst one, what is the typical situation in solutions of the problems of class 2.4.1.

Comment 2. The real minimal sufficient number of moves is 40. That was a problem for the hard part of the contest.

Example 4 (CPL, easy part.). *There are 100 first-graders in a row, all facing the teacher standing in front of them. After the command "Turn to the right!" some of them turned to the right, while the other turned to the left. After that after each second each two pupils who stood face to face with each other turn around. Prove that the movement will stop after at most 99 seconds.*

Solution. It is easy to understand that the development of the process depends only on the fact into which direction the pupils occupying correspondingly the 1^{st}, 2^{nd}, …, 100^{th} place are looking at each moment, but not on the fact **which** particular pupil occupies the 1^{st}, 2^{nd}, …, 100^{th} place. Let's consider another similar process in which the pupils don't turn around but step forward interchanging their places. There is an isomorphism between the two processes in the sense that for each *i*, $1=i=100$, the pupil on the *i*-th place in the first process is looking to the right iff so does the pupil on the *i*-th place in the second process. On the other hand it is clear that no pupil can make more than 99 steps, so the conclusion follows.

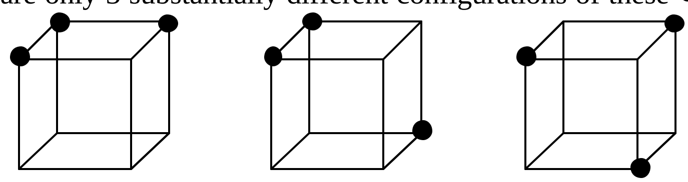
Comment 1. This is a typical problem of the class 2.2., using the method of interpretations (see [1]).

Comment 2. The problem comes from the theory of cellular automata. It is still another illustration of the great impact the theoretical computer science has made on math contests during last 30-40 years.

Example 5 (summer school math contest).

There are 3 convex polygons drawn inside the unit square: A, B, C. The contours of each two of them intersect each other at exactly two points and have no other common points; all 6 points of intersection are different. Two players X and Y play the following game. At first X chooses one of the polygon and paints either the inner or the outer region of *it; then Y does the same with one of the remaining polygons, and X paints the inner or the outer region of the only remaining polygon. Prove: X can ensure that the area of the three times painted region is* $\leq \frac{1}{6}$.

Solution. In an obvious way, represent the inner and outer regions of polygons by the faces of the cube. Then the 8 parts into which the square is dissected are represented by the vertices of the cube. Write the area of each part into the corresponding vertice; then the sum of all written numbers is 1. Mark the vertices with numbers $\leq \frac{1}{6}$ as ●; there are at least three ● in the cube. There are only 3 substantially different configurations of these ●:



Now, the move in the game is to choose one face of the cube and delete the opposite face from further consideration. It's almost obvious that the first player can ensure: the intersection of three chosen faces is marked with ●.

Comment. The problem appeared to be very hard. It is an example of class 1.2. with non-traditional application of the method of interpretations.

More examples can be found in [2].

5. On sources of algorithmic problems

The main, and, we hope, everlasting source of algorithmic problems for math contests is the current scientific research. For example, all rich area of "coin – weighing problems" has originated from the investigations in sorting algorithms. New types of problems arise in connection with non-traditional (from the students' point of view) types of algorithms.

Example 6 (Latvian summer competition). *There are 4 equally looking coins; all of them have different masses. We can use a pan balance without counterfeits. Develop an algorithm which uses a pan balance twice and find the heaviest coin with the probability* $\frac{3}{4}$.

Solution. At first, using any generator of random numbers (for example, throwing the fair coin twice), decide which coin will be called "read"; other coins will be called "blue". After that find the heaviest blue coin deterministically within two weighings in a standard way. Announce this coin the heaviest among all four.

Clearly there is a probability $\frac{3}{4}$ that the heaviest coin (among all four) will be blue. Then it will be announced the heaviest, QED.

Comment. This problem demonstrates the advantage of "clever" probabilistic algorithm over both deterministic algorithms and pure guessing. It can be easily proved that the task can not be completed deterministically. Of course, simple guessing gives the correct answer only with a probability $\frac{1}{4}$.

Example 7 (R.Freivalds). *There are 14 equally looking coins. The experts have established that 7 of them are exact and 7 of them are false. The court knows only that all exact coins have equal masses, all false coins have equal masses and an exact coin is heavier than the false one. How can expert demonstrate to the court which coins are exact and which are false using only 3 weighings on a pan balance without counterfeits?*

Solution. At first expert places one exact coin on the left pan and one false coin on the other. The court becomes aware "who is who" of these coins. The expert adds two exact coins to the false one and two false coins to the exact one – and the court again becomes aware "who is who". Then the expert gathers 3 "proved exact" coins on one pan and adds 4 "unproved false" coins to them; other 7 coins are placed on the other pan. It's not hard to understand that all should be clear to the court after this.

Comment. This problem has great educational value; it demonstrates to the student that a proof itself can be **principially** simpler than a process of establishing it. Really, an easy generalization shows that n exact coins can be separated from n false ones using $[\log_2 n]+1$ **demonstrations**; on the other hand, information theory lover bound shows that at least $n \cdot \log_3 2$ weighings are necessary to **establish** which n coins are the exact ones.

Other possible variations are to introduce the possibility of unreliable information, to consider parallel processes, to deal with more powerful/ more restricted identifying devices than yes/no questions or their equivalents, etc. All these are topics of serious investigations in computer science, but yet have not found an adequate reflection in math contests.

6. Concluding remarks

Many investigations have stressed the great educational value of discrete and combinatorial problems, e. g., [3]. Algorithmic problems are special among them. They develop the analytical and constructive skills of children and provide the possibilities of interdisciplinary education. They are always welcome by the students and often can be reformulated so that become suitable for independent investigations of them. Their connections with general reasoning methods make them a valuable educational tool.

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