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Problems to discover and to boost mathematical talent in early grades: A Challenging Situations Approach

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Abstract: *Several studies of mathematical giftedness conducted in the past two decades reveal the importance of creation of learning and teaching environment favourable to the identification and nurturing mathematically talented students. Based on psychological, methodological and didactical models created by Krutetskii (1976), Shchedrovtiskii (1968), Brousseau (1997) and Sierpinska (1994), we have developed our challenging situation approach. During 7 years of field study in the elementary K-6 classroom, we collected sufficient amount of data that demonstrate how these challenging situations help to discover and to boost mathematical talent in very young children keeping and increasing their interest towards more advanced mathematics curriculum. In this article, we are going to present our model and illustrate how it works in the mixed-ability classroom. We will also discuss different roles that teachers and students might play in this kind of environment and how each side could benefit from it.*

1. Introduction

The biographers of famous mathematicians often refer to the evidence of a particular nature of their talent which can be detected already at a very young age. One can ask where this deep insight in mathematics comes from. How can teachers discover their talent and nurture it? And, as a result of this discovery, what kind of a classroom environment would be advantageous for these children? What can be done by teachers to help these children to realise their potential?

From their very early pre-school and school years, mathematically gifted children are active and curious in their learning, persistent and innovative in their efforts, flexible and fast in grasping complex and abstract mathematical concepts, and thus represent a unique human intellectual resource for our society, which we have no right to waste or to loose.

Numerous studies of mathematical giftedness conducted during past decades provide us with different lists of characteristics of gifted children and suggest various models of identification and fostering them in and beyond mathematics classroom.

Long time experimentation with schoolchildren and observations made by teachers allowed Krutetskii (1976) to construct a list of characteristics of mental activity have shown by mathematically gifted children in a comparatively early age :

- An ability to generalize mathematical material (an ability to discover the general in what is externally different or isolated)
- A flexibility of mental processes (an ability to switch rapidly from one operation to another, from one train of thought to another)

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- A striving to find the easiest, clearest, and most economical ways to solve problems
- An ability chiefly to remember generalized relations, reasoning schemas, and methods of solving type-problems
- Curtailment of the reasoning processes, a shortening of its individual links
- Formation of elementary forms of a particular 'mathematical' perception of the environment - as if many facts and phenomena were refracted through prism of mathematical relationships.

Miller (1990) mentions some other characteristics that may give important clues in discovering high mathematical talent:

- Awareness and curiosity about numerical information
- Quickness in learning, understanding and applying mathematical ideas
- High ability to think and work abstractly
- Ability to see mathematical patterns and relationships
- Ability to think and work abstractly in flexible, creative way
- Ability to transfer learning to new untaught mathematical situations

Another model focusing on giftedness as "intersection" of various factors has been developed by Renzulli (1977). By means of this model, Ridge and Renzulli (1981) define giftedness as an interaction among three basic clusters of human traits: above average general abilities, high levels of task commitment, and high levels of creativity. Upon their definition, gifted and talented children are those possessing or capable of developing this composite set of traits and applying them to any potentially valuable area of human performance.

In a similar way, Mingus and Grassl (1999) focused their study on students who display a combination of willingness to work hard, natural mathematical ability and / or creativity.
The authors consider **natural mathematical ability,** which might be represented by several characteristics discovered by Krutetskii (see above) as well as non-mathematical ones as **willingness to work hard** (that means being focused, committed, energetic, persistent, confident, and able to withstand stress and distraction) or **high creativity** (i.e. capacity of divergent thinking and of combining the experience and skills from seemingly disparate domains to synthesise new products or ideas). The authors labelled students possessing a high degree of mathematical ability, creativity, and willingness as "truly gifted".

Reflecting on our classroom observations of 4-5-year old children using educational software with some mathematical tasks we became interested in studying deeply mathematically precocious children

We noticed that some of them always choose more challenging activities, go through all the levels up to the highest ones, understand each activity almost without any explanation from the teacher, demonstrate very systematic approach to the problem, have very sharp selective memory of important facts, details, methods, they are very creative in their work with “open-ended” problems (such as creating puzzles and patterns), and often share their discoveries with their peers being very proud of themselves.
For example, working with counting tasks such as finding a domino piece with number of dots corresponding to a show number from 6 to 9, some children count all the dots on almost **ever**y

card using their fingers, others choose first one which contains **more** than 5 dots (for example, they may choose 8), then again, most of them count dots and if the result is not good, they jump randomly to another with similar number of dots. There is also a small group of children that try to spot a card with less than 8 dots on it. Finally, one child clicks immediately on card with 7 dots, saying "I know it's this one because 5 and 2 make 7".

Analyzing children's strategies, we could see their *different approach to numbers*. Some children see cards as pictures with objects to count and they use the same strategies as they were manipulative objects (like toys). Other children try to use a different, more complex approach – thinking globally (I see it's five here, I know that 7 is less than 8) and abstractly one (number as an abstract characteristic of a set of dots) along with using a number of shortcuts which helped them to increase efficiency of their mathematical work.

Our next example is a comparison task with two cards shown to the child: one with a certain number of dots arranged within an array 3x4 (12 dots as maximum) and another one with a number 1-12 written on it as a digit. The child has to decide *whether two cards present the same numbers or not.* For the most of 5 year old children, this is a relatively simple task but adding the time limit does make activity extremely challenging for children whose strategy of counting is limited by "finger pointing". The best winning strategy was found by children who used estimation (I know that I have much more dots here than number 3 on another side) and counting with eyes (without fingers). Some children were giving surprisingly deep comments like "I know this number of dots is 12 because I see 4 row of 3 dots which make 12" which demonstrate precocious insight into numbers and number relationships.

Some tasks give children an opportunity to create some patterns asking to construct a personage following certain pattern, or to create their own personage. This second option was seeing by many children more as an art activity, although our observation showed that some 4 year old children create personages upon more complex pattern of mathematical nature (like color, background, part of cloths). One activity presented a grid 6x6 with a set of different puzzles to reproduce (pictures are given as a model) or to create their own puzzle and many young children (4-5 year old) did it just as drawing another picture.

Again, we could notice few children building spontaneously more mathematically abstract tessellations using complex, sometimes symmetrical configurations of shapes which would be more expected from older children already familiar with geometric transformations like reflection or translation. Another activity presented a factory for making cookies with chocolate chips on them. One mode of this activity asks child to put a number of chips on a cookie corresponding to a randomly given number (1 to 10). Another mode prompts to create a cookie with an arbitrary chosen number of chips. Giving free choice to children it allowed us to observe some of them making cookies with consecutively chosen numbers from one to ten repeated in two rows. And even more, they were so fascinated with their result so they started to repeat the same pattern more and more without any visible fatigue, although it was a routine repetition of the same procedure. It seems that here we have an example of a mathematical creativity of a particular kind: seeing beauty of mathematical structure in the same repeating pattern.

Equalizing task is a complex task for very young children. For example, an activity of feeding rabbits with carrots shows some rabbits "waiting for a food", on another – an empty field in which a child has to put carrots keeping in mind that each rabbit would get one carrot. In fact, the child has to control two conditions at the same time to ensure that number of rabbits is equal to the number of carrots. Our observation shows that some children decide to arrange carrots in a certain geometric pattern (row, stair, or array) helping themselves to keep control of conditions showing thus more complex way of thinking.

Finally, working on ordering tasks like one of arranging 7 dolls "matreshka" in increasing or decreasing order by size, some children proceed rather by trial and error, others do it more systematically (looking at neighbors and switching if necessary). Few of them do it in a very systematical way: starting with putting a smallest/biggest one first, then going to the next smallest/biggest and so on. This strategy allows them to simplify the process of problem solving, and at the same time, shows their ability to apply more complex thinking.

Reflecting on these examples, one can ask: Why do these children demonstrate such unusual behavior at an early age? Is it simply related to the attractiveness of computer games on the screen, or does it reflect a much more complex structure of their mind? Our further study of these children’s strategies while solving “purely” mathematical tasks led us to believe that, indeed, the latter might be the case and that it is worth while searching for a specific structure of the mathematically able mind.

Our further questions were: How to identify “pure” mathematical components of the children's learning activity? What kind of cognitive structure enables a child to act like a mathematician? And, from the point of view of practicing teacher, we asked: How to organize children's mathematical activities so that they were motivated to act this way?

In the following section, we will analyze several theories that form our theoretical framework enabling us to analyze problems that help to boost mathematical talent in young children.

2. Theoretical background

Kulm (1990) remarks that since so much of school mathematics in the past has been focused on practised skills, the completion of a large number of exercises in a fixed time period has been accepted not only as a measure of mastery, but as an indication of giftedness and potential for doing advanced work. On the other hand, higher order thinking in mathematics is by very nature complex and multifaceted, requiring reflection, planning, and consideration of alternative strategies. Only the broadest limits on time for completion make sense on a test purposing to assess this type of thinking.

Burjan's (1991) recommendation to use

- Open-ended investigations and open-response problems rather than multiple-choice
- Problems allowing several different approaches

- Non-standard tasks rather than standard ones
- Tasks focusing on high-order-abilities rather than lower-level-skills
- Complex tasks requiring the use of several "pieces of mathematical knowledge" from different topics) rather than specific ones (based on one particular fact or technique)
- Knowledge-independent tasks rather than knowledge-based one

goes in the same direction.

Unfortunately, as it was mentioned by Greenes (1981), the bulk of our mathematics program is devoted to the development of computational skills and we tend to assess students' ability or capability based on successful performance of these computational algorithms (so called "good exercise doers") and have little opportunity to observe student's high order reasoning skills.

Sometimes, even a very banal math problem might deliver a clear message about distinguishing the gifted student from the good student. Greenes analyses a very simple word problem (given to 5th Grade children):

Mrs. Johnson travelled 360 km in 6 hours. How many kilometres did she travel each hour?

One bright student surprised the teacher by having difficulty to solve this easy problem. Finally, the teacher realised that the student has discovered that nothing was said about the same number of kilometres travelled each day. This example demonstrates the child's ability to detect ambiguities in the problem, which indicate him/her as mathematically gifted student.

That is why, in a later work Greenes (1997) insists on the importance of presenting situations in which students can demonstrate their talents: "One vehicle for both challenging students and encouraging them to reveal their talents is to use of rich problems and projects". Greenes mentions that such problems accomplish the following:

- Integrate the disciplines (application of concepts, skills, and strategies from the various sub-discipline of mathematics or from other content areas (including non-academic ones)
- Are open to interpretation or solution (open-beginning and open-ended problems)
- Require the formation of generalisations (recognition of common structures as basic to analogue reasoning)
- Demand the use of multiple reasoning methods (inductive, deductive, spatial, proportional, probabilistic, and analogue)
- Stimulate the formulation of extension questions
- Offer opportunities for firsthand inquiry (explore real-word problems, perform experiments and conduct investigations and surveys)
- Have social impact (well-being or safety of members of the community)
- Necessitate interaction with others

Many authors point at teacher's particular role in the process of identification of mathematically able children. Kennard (1998) affirms that the nature of the teacher's role is critical in terms of facilitating pupil's exploration of challenging material. Hence, the identification of very able pupils becomes inextricably linked with both the provision of challenging material and forms of teacher-pupil interaction capable of revealing key mathematical abilities. The author votes for interactive and continuous model for providing identification through challenge which integrates the following strands:

- The interpretative framework
- The selection of appropriately challenging mathematical material
- The forms of interaction between teachers and pupils which provide opportunities for mathematical characteristics to be recognised and promoted
- The continuous provision of opportunities for mathematically able children to respond to challenging material

In Kennard's case study based on this model and Krutetskii's categories the identification was conducted by the so-called teacher-researcher in the classroom environment where the pupils are being taught as well as observed. The questioning approach was used in order to reveal aspects of pupils' mathematical approaches and understanding.

Ridge, Renzulli (1981) suggest three types of activities which are important for nurturing mathematical talents:

- General exploratory activities to stimulate interest in specific subject areas: experiences that would demonstrate various procedures in the professional or scientific world (through children's museums and science centres) in which students would get an opportunity to choose, explore, and experiment without the treat of having to prepare report or provide any sort of formal recapitulation.
- Group training activities to develop processes related to the areas of interest developed through general activities. The aim of these activities is to enable students to deal more effectively with content through the power of mind. Typical for these thinking and feeling processes are critical thinking, problem solving, reflective thinking, inquiry training, divergent thinking, sensitivity training, awareness development, and creative or productive thinking. Problem solving applies to
    1. The application of mathematics to the solution of problems in other fields
    2. The solution of puzzles or logically oriented problems
    3. The solution of problems requiring specific mathematical content and processes.
- Individual and small-group investigation of real problems. As giftedness becomes manifest as result of student's willingness to go engage in more complex, self-initiated investigative activities, the essence of this type of activities is that students become problem finders as well as problem solvers and that they investigate a real problem using methods of inquiry appropriate to the nature of the problem (p.231).

For his study, Krutetskii (1976) developed several sets of challenging mathematical problems and conducted interviews with each of chosen students offering an original way to study mathematical abilities within appropriate mathematical activity, which, taken in school instruction, consists of solving various kind of problems in the broad sense of word, including problems on proof, calculation, transformation, and construction. He analyses seven principles of a choice of mathematical problems suitable to discover mathematically able student:

1. The problems should represent about equally the different parts of school: mathematics-arithmetic, algebra, and geometry
2. Experimental problems should be of various degrees of difficulty
3. They ought to fulfil their direct purpose: solving them should help to clarify the structure of abilities
4. The problems should be oriented not so much toward a quantitative expression of the phenomenon being studied as toward revealing its qualitative features (process versus result)
5. We should try to choose problem the solving of which is primarily based on abilities, not on the knowledge, habits, or skills
6. The problems have to allow to determine how rapidly a pupil progressed in solving problems of a certain type, how well he achieved skill in solving these problems, and what were his maximum possibilities in this regard (instruction versus diagnostic)
7. The problems are supposed to allow some quintile analysis as well as qualitative one.

Analyzing different children's approaches to the problems, Krutetskii (1976) provides us with several key elements of mathematical ability showing how these challenging problems help us to recognize different children's approaches to mathematics. In a regular classroom, we often teach students direct methods of solving mathematical problems. Then, in order to test their knowledge, we give them the same kind of problem and expect them to (re-)produce the same solution.

This might lead to several paradoxes, such as Brousseau's (1997) paradox of devolution of situations when the teacher "is induced to tell the student how to solve the given problem or what answer to give, the student, having had neither to make a choice nor to try out any methods nor to modify her own knowledge or beliefs, will not give the expected evidence of the desired acquisition." Brousseau thus claims that everything the teacher undertakes in order to make the student produce the behaviors that she expects tends to deprive this student of the *necessary conditions for understanding and learning of the target notion*.

However, several studies point at the fact that, in order to access a higher level of knowledge or understanding, a person has to be able to proceed at once with an integration and reorganization (of previous knowledge). Sierpinska (1994) sees the need of "reorganizations" as one of the most serious problems in education. But we can not just tell the students "how to reorganize" their previous understanding, we can not tell them what to change and how to make shifts in focus or to generalize because we would have to do this in terms of knowledge they have not acquired yet.

Looking for new methodological approaches to teaching and learning, Shchedrovitskii (1968) gives striking examples of other paradoxes when we as educators want our children to master some kind of action by teaching it directly giving children tasks which are identical with this action. But classroom practice shows that the children not only do not learn actions that go beyond the tasks, they do not even learn the actions that we teach them within the tasks.

In our challenging situation model, we propose an active everyday use of open-ended mathematical activities that would engage children into a meaningful process of exploring, questioning, investigating, communicating and reflecting on mathematical structures and relationships. This model represents rather larger vision of mathematical giftedness that correlates with Sheffield’s notion of a *mathematical promise* (Sheffield, 1999) and thus aims to give pleasure to more children to think and to act in a mathematically meaningful way.

3. Our study: general context

Our experiment reflects 7 years of classroom activities and observations with Grades K-6 children while teaching challenging mathematics courses. It has been conducted at one Montreal located private bilingual elementary school with French and English both taught as a first language. Along with a strong linguistic program (with a third language, Spanish or Italian), the school insists on offering enriched programs in all subjects including mathematics to all its students independently of their abilities and academic performance.

The school thus promotes education as a fundamental value by instilling the will to learn while developing the following intellectual aptitudes:

- being able to analyse and synthesize
- critical thinking
- art of learning

The mathematics curriculum is composed of a solid basic course whose level is almost a year ahead in comparison to the program of the Quebec's Ministry of Education (Programme de formation de l'école québécoise, 2001) and an enrichment (deeper exploration of difficult concepts and topics: logic, fractions, geometry, numbers as well as a strong emphasis on problem solving strategies). The active and intensive use of "Challenging mathematics" text-books (Lyons, Lyons) along with carefully chosen additional materials helps us create a learning environment in which the students participate in decisions about their learning in order to grow and progress at their own pace. Each child competes with himself (herself) and is encouraged to surpass himself (herself).

Since the school doesn't do any selection of students for the enriched mathematics courses, all its students (total of 238) participated in the experiment. With some of them, this author started to work at their age of 3-5, as a computer teacher. There were many students that we could observe during a long period of time (for example some of Grade 6 children in 2002-2003 were our students since Grade 1, some of them since the age of 3-5). During this period, some children had to leave the school, some of them joined the class later (in the same Grade 6, there were 2 students who started in our school in Grade 6). In terms of abilities, we can characterise our classroom as a mixed ability classroom with a significant variation in the level of achievement.

The enriched course aimed to foster children's logical reasoning and problem solving skills in all children. It is based on *challenging situations* presented in the 'Challenging Mathematics' textbook collection (Défi mathématique (Lyons, Lyons)) along with other different computer and printed resources (LOGO, Cabri, Game of Life, Internet, and so on) as well as situations created by the author. It included several topics earlier than in the regular curriculum; some topics were presented in more depth than in the regular curriculum; various topics which are not included in the regular curriculum. Such curriculum thus requires a mobilising of all the inner resources of the child: her motivation, hard mental work, curiosity, perseverance, thinking ability. Since all our students are exposed to this enriched curriculum, the differences between them become more evident.

In our further more detailed analysis, we will make explicit the role of the challenging situation itself showing that without the context of challenging situation, such opportunity for students and teachers would be lost.

4. Challenging tasks as powerful teaching and learning tools helping to discover and boost mathematical talent

The story of Gauss solving a routine problem of calculating the sum of the first hundred natural numbers is one of the well known examples of this kind. While all other children were desperately trying to add terms one by one, Gauss impressed the teacher by finding a quick and easy way to do it regrouping the terms in a special way (see, for example, *Dunham, 1990*).

But one can ask: what were the characteristics of the classroom situation, which allowed the gifted student to demonstrate his talent in mathematics?

The same story says that the teacher had chosen the task for its accessibility to all students (the task is routine) and the probably very long time that it would take the students to solve; he hoped to thus keep them all quiet and busy for a good while. What he didn't expect that one of the students would turn the routine task into a challenging one of finding a quick way of solving an otherwise tedious and long computational exercise. The situation was not planned to reveal a mathematical talent, yet it did so "spontaneously". The situation became a challenging one by chance.

In many similar cases, mathematical talent would not be identified. We could say that using routine drill tasks involving numerous standard algorithms is not, in general, offering a good opportunity to identify and nurture mathematical talents.

Sheffield (1999) calls such routine tasks "one dimensional". As an example, she cites a class of three-four graders reviewing addition of two-digit numbers with regrouping. Children are asked to complete a page of exercises such as: 57+45, 48+68, 59+37. As it usually happens with brighter and faster students, they finish all exercises before their classmates. So the teacher would "challenge" them with 3- or 4-digit additions. Although the calculations become longer and time consuming, the tasks themselves are not more complex or more mathematically interesting.

As a better didactical solution for these children, Sheffield suggests the use of meaningful tasks like one of *finding three consecutive integers with a sum of 162*:

> Students would continue to get the practice of adding two-digit numbers with regrouping, but they also would have the opportunity to make interesting discoveries. Students who are challenged to find the answer in as many ways as possible, to pose related questions, to investigate interesting patterns, to make and evaluate hypotheses about their

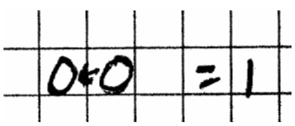
> observations, and to communicate their findings to their peers, teachers, and others will get plenty of practice adding two digit numbers, but they will also have the chance to do some real mathematics. (Sheffield, 1999: 47)

By giving children a challenging task we would expect them to make efforts in understanding a problem, to search for an efficient strategy of solving it, to find appropriate solutions and to make necessary generalizations.

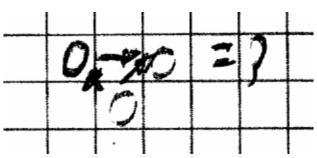
Following examples illustrate three very different approaches to the same problem of *finding a number of handshakes that we obtain when n people shake hands of each other* used by mathematically talented children.

Marc-Etienne (10) organized an experience with his classmates, considering systematically the cases n=2, n=3, etc. Then he made then necessary generalizations. Here is a transcript of his report in which we could observe several steps:

Step1: two circles connected with an arrow representing two people - one handshake. He wrote beside the picture '=1'.

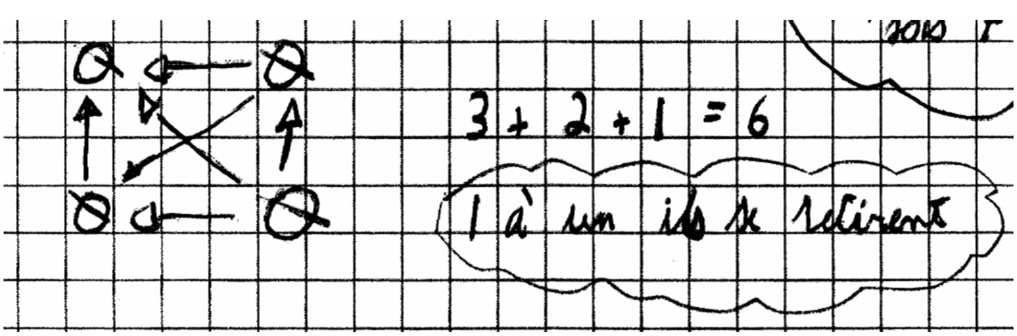


Step 2: three circles forming a triangle connected with three arrows representing three people - three handshakes. He wrote beside the picture '=3'

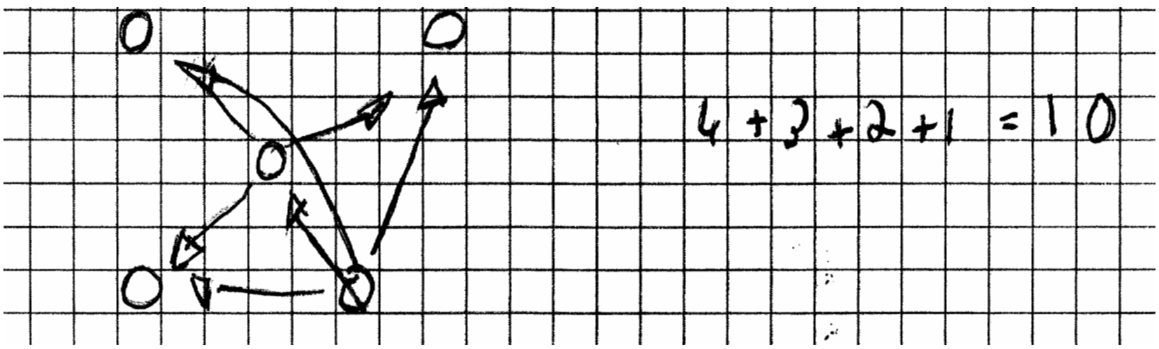


Step 3: four circles forming a square connected with six arrows representing four people - six handshakes. He wrote aside: '3+2+1=6' commenting:

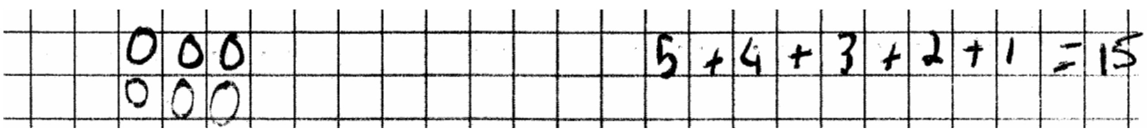
'1 after another, they leave'



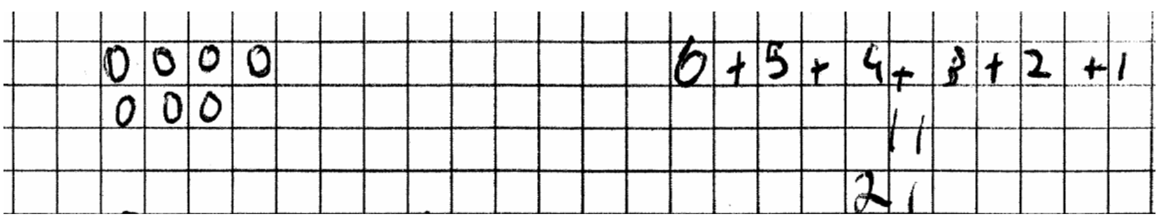
Step 4: Five circles form a 'domino-5-dots disposition' connected with only six arrows (some arrows are missing). However, he wrote '4+3+2+1=10' continuing the same pattern.



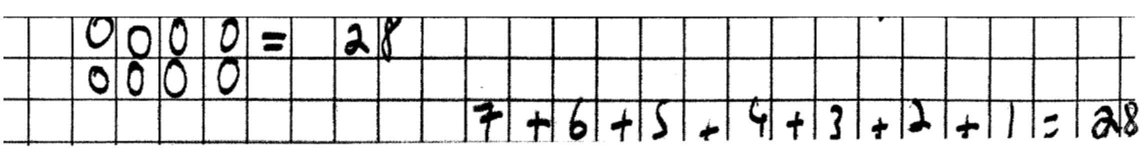
Step 5: Six circles disposed in two rows (by three), no arrows. He wrote '5+4+3+2+1=15'



Step 6: Seven circles disposed in two rows (three+four), no arrows. He wrote '6+5+4+3+2+1=21'

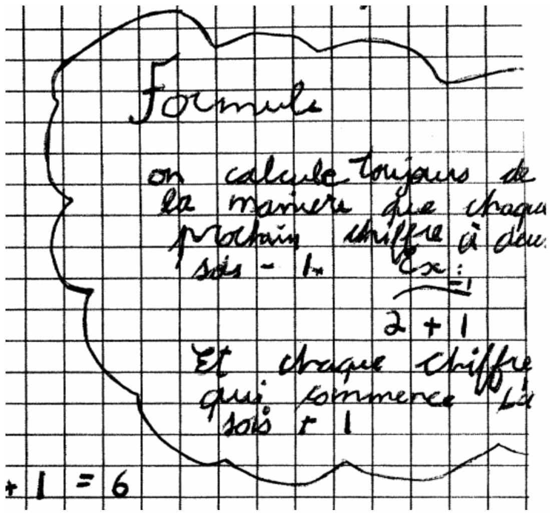


Step 7: Eight circles disposed in two rows (by four), no arrows. He wrote '7+6+5+4+3+2+1=28'

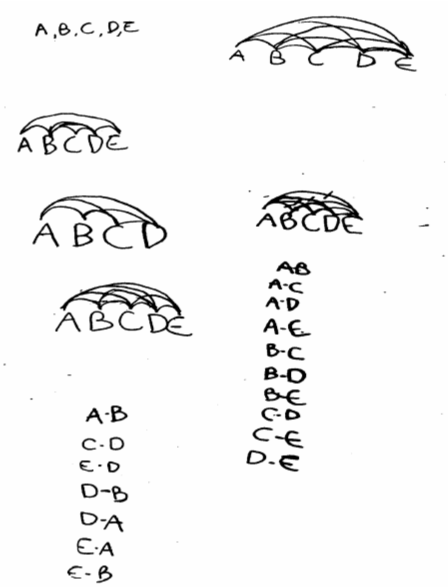


He concluded his generalization with following sentence that he called 'Formule': 'On calcule toujours de la manière que le prochain chiffre soit -1 '2+1' et que chaque chiffre qui précède soit +1'

(We calculate always in the same way in order that each next number in our sum would be 1 less. '2+1' and each previous number would be +1').



Charlotte (10) used a particular case of 5 people making diagrams doing systematic search for all possible combinations.



Christopher (10) was very short in his presentation writing just one single sentence:

1+2+3+4+5+7+8=36

He provided it with the oral explanation that if we have a group of people, each person has to shake hands to all people who came before, so with 2 people we would have 1 handshake, with 3 people - 2 more handshakes (1+2), and so on.

We can see that investigation of initial situation of ‘handshakes’ allowed children to explore the problem, to look for patterns and to make important mathematical generalizations. Moreover, with a simple boosting question what would be the number of handshakes with 101 people, the class arrived to the same problem that Gauss had to deal with but posed in different, challenging way. Thus, a further investigation might be provoked here in a more natural way.

And, from the point of view of practicing teacher, we can better understand *how to organize children's mathematical activities so that they were motivated to act this way.*

5. From challenging tasks to challenging curriculum: Example of kindergarten enrichment course

There are two basic approaches to design a mathematics curriculum for 5-6 year old children; one can be labelled as traditional and the other as innovative. The former is based on counting, ordering, classifying, introduces basic numbers, operations (addition and subtraction), relations (more, less, bigger, smaller, greater) and shapes. The latter puts more emphasis on learning while allowing children to play using manipulatives, colouring, arts and crafts, games with numbers and shapes. During the past decade, many creative teachers have been trying to use the best ideas from each of the two approaches also adding reasoning activities to the mathematics curriculum.

In our school, we used a traditional approach based on Quebec's *Passeport Mathématique* Grade 1 textbook along with a new French collection *Spirale (Maths CP2)* which represents the second, modern approach. However, even this combination doesn't provide our children with the material necessary for their mathematical development. There is still a gap between their level of ability and the requirements of the challenging curriculum that we use starting from Grade 1 (collection "Défi mathématique") and which is based on discovery, reasoning and understanding.

In order to fill the gap, we developed an enriched course offered to all kindergarten students (we have 30-35 children every year). The course was given on a weekly basis (1 hour a week). We base our teaching on the challenging situations approach, developing activities that stimulate mathematical questioning and investigations along with reflective thinking.

Each class starts with such questions as *What did we do last time? What problem did we have to solve? What was our way to deal with the problem? What strategies did we use?*, etc. This questioning aims to provoke reflection on the problems that children solved as well as on methods that they used. Without this reflection, rupture situation (in Shchedrovitskii's sense) would never arise, because a rupture is a break with previous knowledge, which needs to be brought to mind.

At the same time, we would ask questions that would indicate children's understanding of underlying mathematical concepts or methods that we aim to introduce (using appropriate vocabulary and/or symbolism).

During this initial discussion we usually try to bring in a new aspect which provides children with an opportunity to ask new questions, to look at the problem in a different way. Sometimes, we might ask them, simply, *what do they think we should do today?*

Thus we can pass to the new situation/new problem/new aspect of the old problem. We may do it by means of provoking questions, of interesting stories or introductory games. Following Shchedrovitskii and Brousseau, we try to avoid the teaching paradox by not providing children with direct description of the tasks or methods of solutions. We try also to keep their attention and motivate them.

After this introductory stage, children begin investigating a problem using different manipulatives: cubes, geometrical blocs, counters, etc. They work alone or in groups. During the phase of investigation, the role of the teacher becomes more modest: we give children certain autonomy to get familiar with the problem, to choose a necessary material, organise their work environment, and choose an appropriate strategy.

However, some work has to be done by the teacher to guide children through their actions. We have to make sure that the child understands the problem, the conditions that are given (rules of the game), the goal of the activity. As the child moves ahead, we shall verify his control of the situation: what she is doing now and what is the purpose of the action? (activating reflective action). We have to keep in mind that the exploration is used not only as a way to make the child do some actions but also and foremost as an introduction to mathematical concepts or methods.

Therefore, the teacher needs to be prepared to introduce the necessary mathematical vocabulary along with its mathematical meaning as well as mathematical methods of reasoning about the concepts and about the reasoning. In our experiment, we try to choose those mathematical aspects that are considered as difficult and are not normally included in the Kindergarten curriculum.

For example, when we want to introduce an activity with patterns, we would organise a game. We would start to make a line 'boy, girl, boy, girl,..' children find it easy and are happy to discover a pattern. Then we would start a new 'pattern' : 'boy, girl, boy, girl, boy, boy'. Many children would protest, saying that the pattern is wrong. But perhaps, some of them would try to look for different pattern, like "glasses, no glasses, glasses, no glasses, …".

As the game goes on, children get used to looking for familiar patterns. This is the time to challenge them more. For example, we may ask them, how many children would be in the line with the pattern 'boy, girl, boy, girl,…'. Since there were only 8 boys in the classroom, one child could make a hypothesis that it gives 8+8 children in the line. After such a line had been completed, teacher's silence could be broken by a child's voice - 'we can add one more child to the line – a girl in the beginning'.

The course is built of various challenging situations that we create in order to give children an opportunity to take a different look at mathematical activities that they usually do, to question their knowledge about mathematics trying to discover hidden links between different objects, to discover structures and relationships between data, learn to reason mathematically based on logical inference and at the same leave some space to children's mathematical creativity. We use different didactical variables in order to create obstacles making children re-organise their knowledge and create new means in order to overcome the obstacle. We were also asking our children to report on their investigations inviting them to communicate their discoveries by developing appropriate tools: diagrams, schemas, symbols, signs.

6. Guidelines for the design of challenging situations

We consider three kinds of challenging situations:

- open-ended problems and investigations
- routine work turned into a challenge by the teacher
- routine work turned into a challenge by a student

Let us consider these options in details:

6.1 Open-ended problems and investigations

As we look at the video protocol of interviews with 4-6 year old children conducted by Bednarz and Poirier (1987) within their study of number acquisition by young children, we see how the evidence of differences in organisation of mathematical work by very young children becomes explicit with the open character of given tasks.

The video presents children's work on different tasks related to the concept of number: counting, formation of collections, order, conservation, comparison. Each task that in a regular classroom might be seen as ordinary, was given by authors in a very original challenging, dynamic, and open-ended way.

The child was constantly invited to think about the process of her work (how did you do it?), to develop an efficient strategy, to re-organise, if necessary, her process, to co-ordinate her actions. Thus, the routine tasks became open-ended and a child was given an opportunity to become an organiser of her mathematical work.

In our experiment, we also tried to make problems more open than they were usually presented to the students.

For example, we can take a problem from one mathematical competition :

---->	**1**	**2**	**3**	
	4	**5**	**6**	
	7	**8**	**9**	**---->**

In this table, we enter by 1 and exit by 9.
One can only move horizontally or vertically, and it is impossible to step twice on one box. For example, moving through boxes 1-2-5-8-9, one gets a sum of 25. But not all the trajectories lead us to the number 25. Give all others 9 numbers.

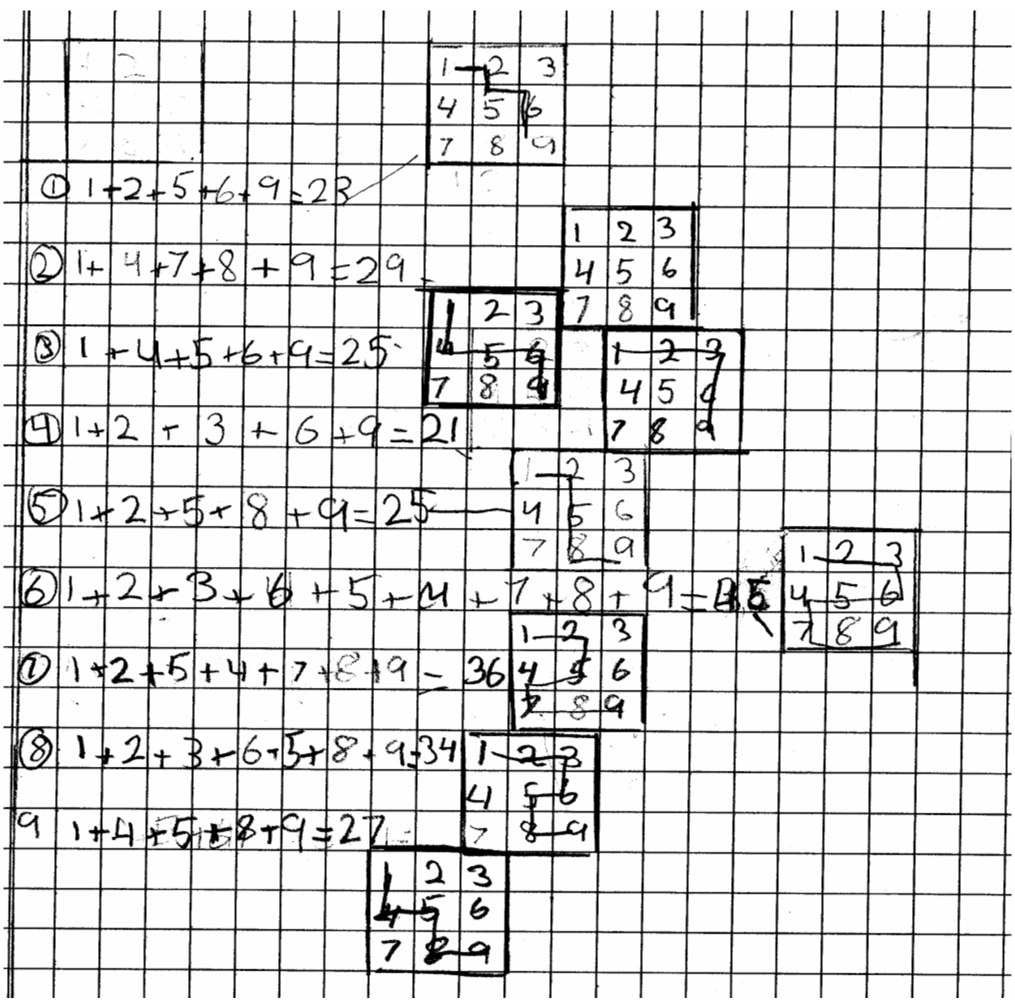
This problem was given to participants of the regional final of the Championnat International des Jeux Mathématiques et Logiques in 2000 for Grade 4-5 children (10-11 year old) **http://www.cijm.org/cijm.html**.

We found that this problem would become more challenging for children if posed in a different way (open-ended) :

Someone is going to visit a museum, which has 9 exhibition halls, arranged in a square 3x3. The number of paintings in each hall is written in the box. What are all the possible numbers of paintings that could be seen by this visitor who does not like to be in one hall twice ?

Not only do we hide the number of different ways, which makes this problem open, we give it to our Grade 1 students (6-7 year old). Every student had a task at his/her level (They will all be able to find at least a couple of solutions).

Following example illustrate the work of Chantal (6):



This example demonstrates how this open-ended situation helped the student to develop different abilities to organize systematic search and to keep tracks of her work

6.2 Routine work turned into a challenge by the teacher

The recent Quebec's school curriculum puts emphasis on the importance of mastering basic number facts (like multiplication tables). This routine task can be made more challenging in many different ways. For example, one day we wrote on the board the 9-table operations:
1 x 9 =
2 x 9 =
3 x 9 = , and so on.

Grade 3 children said immediately that it is a very easy table, because there is a well known regularity (writing down first digits of the product in order from 0 to 9 and the second ones down from 9 to 0, we obtain all the multiples of 9: 09, 18, 27, and so on). Among the answers one could find that 6 x 9 = 54.

So, the teacher comes to the board and writes 6 x 9 = 56 telling the story that when he was young, he had to memorise all answers, not just 'tricks', and he is sure that 6 x 9 = 56. The students are confused, but many of them started to think how to prove that their result (54 was the correct one).

Many of them went to the board to share their ideas as well as other ways to obtain a 9-table. As a result of the lesson, the 9-table has appeared a couple of times on the board, children said it many times aloud, so they could memorise it and at the same time do it in a meaningful way questioning and proving their methods and ways of reasoning.

6.3 Routine work turned into a challenge by a student

When Grade 4 children are asked to represent 1/8 of a rectangle, they find it an easy and routine task. That's why we were surprised by Christopher's way to divide a rectangle in 64 boxes (8 rows x 8 columns) and to colour 8 boxes randomly. He found that the task was not challenging enough and he wanted to make it more complicated.

6.4 Transformation of challenge within one situation

All three ways of creating of challenging situations are not isolated from one another. They can also be transformed one into another.

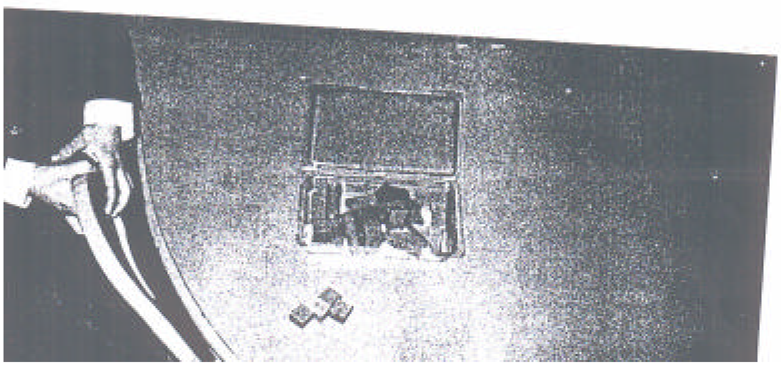
For example, a kindergarten class (5-6 year old) is working on an **open-ended** problem:
Amelie needs to build new houses for her farm animals. When one looks at the house from the sky, she sees that all of them have a roof in shape of a 'digit'. She has to build now a new house for her cows. What 'digit' do you suggest to use for the roof of this new house?

Children used blocs in form of different solids. The activity aimed to make them to explore different solids, to make different constructions with them. There are, basically, two ways of making constructions: three-dimensional or two-dimensional establishing thus different spatial relationships. For example, we may teach children to verify which shapes fit together recovering certain surface. The activity that we gave to our students didn't aim to teach any particular way of making constructions: many textbooks contain a lot of exercises asking children to reproduce one construction or another. Our situation was designed in order to help children get a certain 'spatial feeling' trying different way to layout blocs. The main challenge in it was to organise a mathematically meaningful investigation within an 'ill-defined' problem.

Some of them chose to imitate shapes of digits in the way we write them, others looked for different ways to create more 'economic' constructions taking care of geometric properties (like seeing if the blocs fit one to another) . Finally, there was a group of children who moved from the initially given situation of building a new 'house' and started to construct many digits 'writing' numbers (up to the "1000").

Soon, we could see that originally challenging and creative, the task became routine for many children. So, we decided to put some restrictions (new 'variables') that were sought as means to engage children in the investigation of a different problem in which we would be willing to construct house that has a "5"- shape and do this with a minimum of blocs. Thus, with the intervention of the teacher, a routine problem became a challenging one once again.

This method of a 'sudden' change of didactic variable (Brousseau, 1997) is important in our study of relationship between child's organisation of the problem-solving and mathematical giftedness because it provokes a reflection (what is new?) and re-organisation of the whole process of thinking and acting (what do I need to modify?) and thus gives students a chance to show their full potential.



In our experimental work with young children, we obtained a constant confirmation of the fruitfulness of such an approach, especially if one wants to identify and nurture gifted children. David (5 years old) was working on the 'minimum' task (see the picture above); he looked happy with his solution (4 blocks) but still in what looked like a 'state of alert'. At this moment, we began to discuss children's solutions. One group of children has presented a three-blocs solution. Suddenly, David started to change something in his configuration; he lost completely 'fiveness' of his shape while focusing on minimization task. But what is the most intriguing, is the rapid reactions of this child to the changing conditions (someone has found a better solution). This constant state of 'alert' is an important characteristics of giftedness which could be better activated in challenging situation that in the ordinary one.

This state of 'alert' leads them to constantly verify all the conditions going back and forth through the situation. Here is one more observation. Grade 4 students worked on their test. Answering a question of 'Is it true that if the sum a+c=8 then a and c are two different numbers?', Christopher hesitated a lot, saying however, that the numbers have to be different. As his work on the test went on, he had to solve a system of two equations with two variables: ab=16, a+b=8. He found easily a=4, b=4 as a solution then went back to his previous task and corrected his answer.

We could also observe another interesting phenomenon: challenging situation created by the teacher may initiate its further explorations by gifted students.

For example, doing the same activity with Grade 1 children (6-7 year old), we could state that it was seen as a routine problem by many of them and some of students lost completely their interest in it. Yet, we could still observe one girl looking for many different ways of building "5" using 4 blocs.

Not only she kept herself working on this problem, she came out with a new one: she started to look for possibilities to built a digit "4" with a minimum of blocks. Here, the problem was turned into a challenging one by the student.

6.5 The role of the teacher

In a challenging environment, the **role of the teacher** becomes crucial in all the stages:

- choice of a problem
- way of presenting it to the students
- organisation of student's work
- interpretation of results
- follow-up

One of the very important conditions of success of the challenging situations approach is the teacher's attitude. How should we, as teachers, control the student's work? Related to the learning paradox (described in the previous chapter), it is far from being obvious how to find a solution to this problem. On the one hand, every word and every gesture said by teacher can affect the whole challenge of situation in either a positive or a negative sense. On the other hand, the teacher has to have a full didactical control of the situation (otherwise a mathematical learning activity might become a sort of 'arts and crafts in mathematical wrapping').

Our experiment didn't provide us with clear recipes but rather with examples that can be open to further questioning and investigations. These examples allowed us to formulate teacher's approaches favourable for the challenging situation:

- ❑ Give a child an opportunity to think: being a flexible teacher
- ❑ Support of children's willingness to learn more about math
- ❑ Challenge students in informal situations: sense of humour
- ❑ Support children in their desire to go beyond pre-planed situations
- ❑ Giving hints without telling solutions
- ❑ Management of particular cases of mathematical giftedness
- ❑ Use of 'little tricks' as follows :

- While distributing manipulative material (blocs, cubes, etc.), we would give children time to touch it, to play with it, to get a feeling of it; sometimes it gives us important clues of children's organisations (how they put material, arrange it, order, classify, build different forms, etc.)
- When children finish their manipulation, we ask them to write a report. Sometimes it makes sense to give them time to break up their constructions. This opens the door to a variety of presentations (will the child reproduce his construction, add new details, draw a completely different pictures)

- When children are asked to communicate their results, it is important to motivate them to give detailed explanations. We often ask them to be 'mini-teachers' - to explain to somebody who doesn't understand the problem
- Children often ask us to teach them complicated things. Sometimes, a pedagogical effect can be bigger if the teacher makes them wait. Then, starting to teach it, children might become more motivated: *finally, we got it*!

6.6 The role of the student

The role of the students in a challenging situation differs significantly from those in the regular learning activity. They have to adapt to a new, open environment. They have no precise algorithm of actions, no clear instruction what to do. Therefore, they have an opportunity to:

- demonstrate different approaches to the problem
- act differently in different situations
- overcome obstacles, construct various means, discover new relationships
- work on mathematical problems based on structures and systems using properties and definitions, conjectures and proofs
- use of logical inference with fluency, control, rigour
- combine logic and creativity in problem solving
- invent new symbols and signs, use schemas and abstract drawings
- use reflective thinking
- ask mathematical questions, create new problems, investigate, use mathematics in non-mathematical situations, look around with 'mathematical eyes'

7. Conclusions and recommendations

There are a number of educational studies of mathematical giftedness. Various models of giftedness based on different characteristics of mathematically gifted students have been developed and implemented. Different programs of support provide gifted students with advanced curriculum and guidance of highly qualified professionals. Several mathematical contests, Olympiads, and competitions help in searching for mathematically gifted children and taking care of their development.

Yet, the problems of identification and nurturing of mathematical talent are far from being solved. Many children become bored, at a very early age, with the simplified curriculum, lose their interest in mathematics and waste their intellectual potential. Despite the ingenious testing system, some children never get admitted to special programs for gifted students. The regular school system is not equipped to help these children.

Our study aimed to contribute to filling this gap and providing elementary school (Grades K-6) teachers with methods of identification and fostering mathematically gifted children in the mixed ability classroom.

We have called our approach, the "challenging situations approach". The approach is theoretically grounded in Krutetskii's (1976) notion of mathematical ability, Shchedrovitskii's (1968) developmental model of reflective action, Bachelard's (1938) notion of epistemological obstacle, Sierpinska's (1994) distinction between theoretical and practical thinking in mathematics, and Brousseau's (1997) theory of didactic situations.

Following Krutetskii (1976), we have defined mathematical ability as a 'mathematical cast of mind', which represents a unique combination of psychological traits that enable young children to think in structures, to formalise, to generalise, to grasp relations between different concepts, structures, data and models and thus solve different mathematical problems more successfully than children of average or low ability.

At a very early age, these children demonstrate high thinking potential in reasoning about mathematical concepts and systems of concepts along with the capacity to reason about their reasoning. From the outset, they are better prepared than other children for theoretical thinking, which is the foundation of pure mathematical thinking.

The critical point of our study was an understanding that a discovery and nurturing of theoretical thinking is not possible if children are working with routine arithmetical tasks, merely applying algorithms that had been provided by the teacher, telling her students what to do and how to do it.

The paradoxes of such classroom situations have been described by Brousseau (1997) in his Theory of Didactical Situations. Following Brousseau's theory, we bring a notion of challenging situation into our model of mathematical giftedness postulating that a gifted child will show her talent in mathematics only in specific situations when a real question has been asked and a real problem has been posed.

"Challenging situations" use open-ended problems and mathematical investigations. A challenging situation initiates the student's action of structuring a problem, and of searching for links between data and with her previous experience. Since a real challenge is possible only when the situation is new for the learner, the challenging situation must contain a rupture with what the student has previously learned, provoking the student to reflect on the insufficiency of the past knowledge and construct new means, new mechanisms of action adapted to the new conditions, activating her full intellectual potential.

Challenging situation in its very nature gives many growing up opportunities for mathematical talent by:

- providing the student with an opportunity to face an obstacle of a pure mathematical nature, the so called epistemological obstacle. In order to overcome it, the student will have to re-organise her mathematical knowledge, create new links, new structures following laws of logical inference. We claim that situations satisfying these conditions allow the teacher to identify and nurture mathematical giftedness among her students.
- presenting a problem, which goes above or beyond the average level of difficulty. The child is encouraged to surpass what is normally expected of children of her age, thus demonstrating her precocity, which is a sign of mathematical giftedness.
- helping to create a friendly environment in which a child compete with herself sharing her discoveries with other children and learning from others. Thus it gives

mathematically gifted children who are not high achievers to participate actively in class and to succeed.

Challenging situation cannot be created as an isolated learning task. It full developmental potential can be realised only within a system of teaching based on a challenging curriculum as a whole. This would allow creating a learning environment in which every child would be able to demonstrate her highest level of ability.

This is why, using a challenging situation model we are not only able to get gifted children involved in genuine mathematical activity but also help all children to increase their intellectual potential.

Finally, challenging situation has another opening for gifted children: they can always go further, go beyond situations, ask new questions, initiate their own investigations, be more creative in their mathematical work. This spontaneous mathematical reaction feeds back into the learning environment in a positive way and further enhances its potential for all children. We consider this feature of the approach as crucial from the point of view of mathematics education for all children.

Our study prompts different teaching approaches in mathematics. The teacher is no more re-translator of knowledge or instructor of methods of problem solving. In a challenging situation her role becomes more as moderators of discussions, listeners of student's ideas, student's guide through the discovery.
In helping students go through various obstacles, we shall encourage them to:

- **Organise** his/her mathematical work
- **Reason** mathematically
- **Control** several conditions (verification, adjustment, modification, reorganisation, awareness of contradictions, validation)
- **Choose/develop** efficient strategies/tools of problem solving
- **Reflect** on methods of mathematical work
- **Communicat**e his/her results in a "mathematical" way (oral/written form, use of symbols, giving valid explanations)

Thus, we will be able to **identify** gifted children who:

- ask spontaneously questions beyond given mathematical task
- look for patterns and relationships
- build links and mathematical structures
- search for a key (essential) of the problem
- produce original and deep ideas
- keep a problem situation under control
- pay attention to the details
- develop efficient strategies
- switch easily from one strategy to another, from one structure to another
- think critically
- persist in achieving goals

At the same time, we could nurture their curiosity, willingness to learn more about mathematics, provide them with an opportunity to go further in their mathematical learning, to create new structures, to pose new problems and thus **foster** the development of their mathematical abilities.

This approach is very demanding to the teaching. The teacher has to think constantly about challenging the students, look for different ways to stimulate children's work, demonstrate a high flexibility, ability to react spontaneously on changing conditions of the classroom situation, be ready to provoke students and to get provoked by students asking question which the teacher can not answer immediately.

A better understanding of how to help highly talented children to develop deeper mathematical thinking would lead to elaboration of efficient didactical approaches for all students. We shall agree with following general remark made by Young &Tyre (1992): "If we examine more closely what it is that makes prodigies, geniuses, gifted people, high achievers, champions and medallists, we may be better able to increase their number dramatically".

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