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Constantinos Christou

Nicholas Mousoulides

Marios Pittalis

Demetra Pitta-Pantazi

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Constantinos Christou, Nicholas Mousoulides, Marios Pittalis & Demetra Pitta-Pantazi

University of Cyprus (Cyprus)

Abstract: In this study, we considered dynamic geometry software (DGS) as the tool that mediates students' strategies in solving and posing problems. The purpose of the present study was twofold. First, to understand the way in which students can solve problems in the setting of a dynamic geometry environment, and second, to investigate how DGS provides opportunities for posing new problems. Two mathematical problems were presented to six pre-service teachers with prior experience in dynamic geometry. Each student participated in two interview sessions which were video recorded. The results of the study showed that DGS, as a mediation tool, encouraged students to use in problem solving and posing the processes of modeling, conjecturing, experimenting and generalizing. Furthermore, we found that DGS can play a significant role in engendering problem solving and posing by bringing about surprise and cognitive conflict as students use the dragging and measuring facilities of the software.

1. Introduction

In an attempt to inform the development of better pedagogical models, this paper reports some of the findings from a study of the integration of dynamic geometry software (DGS) in mathematics classrooms. One of the distinguishing features of DGS is the facility to construct geometrical objects and specify relationships between them. Within the computer environment the geometrical objects created on the screen can be manipulated, moved and reshaped interactively with the use of the mouse. The tools, definitions, exploration techniques, and visual representations associated with dynamic geometry contribute to a learning environment fundamentally removed from its straightedge-and-compass counterpart (Laborde, 1998).

The focus of this paper is on students' problem solving and posing processes in the learning environment of dynamic geometry when they work on problem solving and posing. The paper also examines how the increasing use of DGS may give rise to new problems and new ways of introducing problem solving in a variety of contexts.

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The study is based on the theoretical premise that computers are being introduced in education not only because they do a *better* job but because but they do the job *differently* (Aviram, 1999; 2001). There are good reasons to support that DGS has the potential to help students improve their abilities to solve a variety of mathematical problems in novel ways and provide a powerful means of posing new problems by applying different heuristic approaches (Gomes & Vergnaud, 2004).

The first part of the paper presents the theoretical background of the study with special reference to computers as mediation tools, and to the learning mediated through employing DGS. The second part of the study provides support to the theoretical part by indicating that students in the era of computers can do mathematics differently (Aviram, 2001) by solving and extending problems in ways they could not do with paper and pencil, and by exploring and investigating different possible answers to a problem.

2. Theoretical Background

2.1 DGS as a mediation tool

In this study, we investigated pre-service teachers' abilities to construct geometrical objects and solve and pose problems in a computer-based environment, which served as a mediation tool (artifact). The types of artifacts are closely related to the knowledge that students construct (Artique, 2002), and are central to the processes by which students mathematize their activities. In addition, artifacts support students' mathematical development by anticipating how students might act with particular tools, and what they might learn as they do so (Cobb, 1997). Jones (1997) asserted that artifacts stand between the learners and the knowledge that students are intended to learn. This assumes that learning within a DGS environment involves the interaction between students and the software, as they submit their previous knowledge to revision, modification, completion or rejection in the process of acquiring new knowledge (Jones, 2000). This interaction is more clearly explained as the interaction between two systems (Brousseau, 1997; Jones, 1997). The first system refers to students who attempt to solve or pose a problem, and the second system refers to the environment, which offers opportunities to students to act and react. The environment also includes the tools that mediate students' actions and exists between the students and the world of mathematics (Artigue, 2002), and, most importantly, transforms the students' activities upon the world.

Gomes and Vergnaud (2004) considered DGS as an integral part of the didactical environment, performing a specific mediation of knowledge. In problem solving and posing, DGS makes possible for students to generate and use specific strategies (Hölzl, 1996). Hölzl, for example, identified two components of the epistemology of DGS: first the nature of the interface, and second the consequences on students' conceptualizations. Specifically, the structure of a particular interface is a key determinant of the characteristics of the knowledge evolved using it. Problem solving and posing, using DGS, involves the direct and indirect effects of the software's interface on students' procedures and understandings. In addition, DGS's interface provides students with the opportunity to use visual reasoning in mathematics and helps them, through the dragging facilities, to generalize problems and relationships (Sinclair, 2004).

The main issue, however, is whether the involvement of students in such learning settings may result in understandings that could not be achieved through traditional instruction (Artique, 2002), and whether DGS is actually used and transformed by students in visually confirming and negating conjectures and in developing a new perspective on solving and posing original problems (Meira, 1998; Sinclair, 2004).

2.2 Problem solving and posing

Problem posing, problem solving, and conjecturing are three important mathematical activities (National Council of Teachers of Mathematics (NCTM), 2000). In geometry, these activities involve some tasks that technology performs efficiently and well, such as computing and graphing. In this study, two problems were assigned to students, which show how the computing and graphing capabilities of DGS can be used in making conjectures, in problem solving and problem posing within geometry tasks. The importance and relevance of these mathematical activities is supported by "The Principles and Standards for School Mathematics Document" (NCTM, 2000). For example, this document states that instructional programs should provide opportunities for all students to "use visualization, spatial reasoning, and geometric modeling to solve problems" (p. 308). The document also calls for students to "formulate interesting problems based on a wide variety of situations, both within and outside mathematics" (p. 258). In addition, the document recommends that students should make and investigate mathematical conjectures and learn how to generalize and extend problems by posing follow-up questions. Moreover, Silver and Cai (1996) refer explicitly to problem posing by arguing that students should have some experience in recognizing and formulating their own problems. Most authors agree that problem posing is a term used to mean both the generation of new problems and the reformulation of given problems (Cai & Hwang, 2002; English, 2003; Silver & Cai, 1996).

3. The Present Study

In this study, based on the theoretical dimensions discussed in the previous section, we considered DGS as the tool that mediates students' strategies in solving and posing problems. We perceived the construct of mediation in two different ways. First, in a cognitive framework, we addressed the students' attempts that arose from the exploration of the possible extensions and solutions in the assigned problems, and second, in a social framework, we discussed the results or the solutions through the evidence provided by the software.

3.1 The Purpose of the Study

The purpose of the present study was twofold. First, to understand the way in which students can solve non-routine problems in the setting of DGS, and second, to investigate how DGS provides opportunities to students for posing new problems. Both purposes are explored under the assumption that DGS constitutes the artifact that helps us understand how computers can be used in education in doing mathematics in a different way. Thus, the research questions concern firstly the exploration of DGS as a tool that fosters the development of problem solving processes, and secondly the investigation of how DGS mediates the generation of new problems. More specifically, the questions of the present study are:

(a) In what ways does the DGS mediates students' problem solving processes in geometry problems?

(b) In what way the DGS environment provides opportunities for students to pose and solve their own geometry problems?

In order to meet the purposes of the study, two mathematical problems or situations were presented to students. These problems illustrate how students may be engaged in conjecturing, problem solving, and problem posing with the aid of DGS. The problems also illustrate the power of such environments to engage students at various levels of mathematical sophistication.

4. Method

4.1 The Problems of the Study

In order to answer the research questions formulated above, the following two problems, as were slightly adapted from Contreras (2003), were assigned to the students:

Problem 1. The authorities of four towns are planning to build an airport that will serve the needs of their citizens. Identify the optimal place for the location of the airport so that the needs of the four towns are served in a fair way.

Problem 2. What is the figure formed by the angle bisectors of the interior angles of a parallelogram?

The first problem is open-ended and purposefully not well defined. Thus, students had to provide a context for the problem in order to clarify the situation in which they would work. The second problem is a pure geometrical situation, which allows students to explore a seemingly standard problem, but in the solving process they may encounter surprising results. Both problems provided the opportunity to students to generate new problems by altering the situations or extending and reformulating the given problems in different ways (English, 1997, 2003; Silver & Cai, 1996).

4.2 The Participants

The participants of the study were six pre-service teachers with prior experience in dynamic geometry. All students attended a course on the integration of computers in elementary education. The course focused on mathematical applications in the teaching and learning of mathematics in grades 1-6, with special emphasis on the integration of dynamic geometry. Therefore, students had a basic understanding of Geometer's Sketchpad's drawings, menus, and construction features.

4.3 Data Collection and Procedures

The interviewees participated on a voluntary basis. Six students were interviewed while they were working on two non-routine problems. Three of the students worked on the airport problem and three on the bisectors problem. Each student participated in two interview sessions, which corresponded to the two aspects of the mediation construct, namely, the cognitive and social aspects. During the first session, students were asked to solve the problems. During the second session, we worked with the students and discussed not only their solutions but also possible ways of extending, posing and solving new problems.

The interviews were conducted in the mathematics laboratory, which was equipped with computers loaded with the Greek version of the Geometer's Sketchpad. A video recording of the sessions (as opposed to audio) was decided as the means of recording the interviews since we wished to capture not only the discussions but also the actions occurring on the computer screen as interviewees talked about their work. The setting was informal with students being able to analyze and build geometric constructions that they thought would help them solve the problems without any time constraints being set. The data was collected during unstructured interviews. One of the most important benefits of the unstructured interview approach has been described by Cobb (1986) as the process of "negotiating meaning". It gives the opportunity to the researchers to ask the subjects to clarify or explain their activities or comments.

Analysis of the data followed interpretative techniques (Miles & Huberman, 1994). Video records helped us identify the unique ways the software facilitated the students to solve the problems as well as the sequence of the cognitive processes and strategies used during the solution of the problems. The interviews helped to identify the ways in which students used the software in order to pose new problems and how these new problems were related to the original problems. Detailed analysis of all the data was then used to develop categories of problem posing and solving processes that could be checked against participants' own accounts of their work.

4. Results

4.1 The Airport Problem

The airport problem created a lot of discussion between the researchers and the individual students. The discussion revolved around (a) the meaning of the needs of the four towns and how one could interpret them, and (b) the meaning of the word "fair", which produced some disagreements concerning the population of each town. Following the discussions about the context and the meaning of the words involved in the problem, some of the students decided to consider the concept of "fairness" as "equidistance".

When the researcher asked students to solve the problem using the Sketchpad software, the three students who worked on that problem, modelled it, assuming that they should take into consideration the distance of the four towns from the airport. The students were not used to working with such investigations, since geometry textbooks dissuade students from making conjectures based on the limited evidence provided by a single shape. As a result, the students, trying out to find a solution to the airport problem, built "prototype conjectures" (see Hanna, 2000), which were based on common geometrical shapes such as rectangles, parallelograms or squares. Specifically, two of the students investigated the problem by assuming that the four towns were the vertices of a rectangle or a square. The following extract shows their attempts in finding a reasonable solution to the problem:

Student A: I assumed that the four towns are on the vertices of rectangle. Then I hypothesized that the best location for the airport should be in the centre of the rectangle.
Researchers: What do you mean by the "centre" of the rectangle?
Student A: Probably, this is the intersection of diagonals.
Researcher: Ok. How can you check your hypothesis?
Student A: (He points to the diagram on the computer screen). *I defined a point inside the rectangle and constructed the segments from the vertices to this point. I moved the point inside the rectangle* (See Figure 1). *In the meantime, I measured the length of each segment.*

In this case, student A found that the optimal location of the airport was the intersection of the diagonals of the shape, by dragging a point into the shape. He actually based his conjectures on measurements showing that each town is equidistant from the point of the intersection of the diagonals (see Figure 1).

At this point, when students were asked to generalize their findings to include all quadrilaterals, two of them intuitively answered that the point of the intersection of the diagonals should be the best location for the airport. However, they could not provide a reasonable justification, although they empirically tried other points inside an arbitrary quadrilateral. The dragging facilities of the software helped them to work inductively, i.e., from two or three examples they generalized to all quadrilaterals. Of course, they recognized the need for a formal proof in order to convince themselves and others about the truth of their generalization.

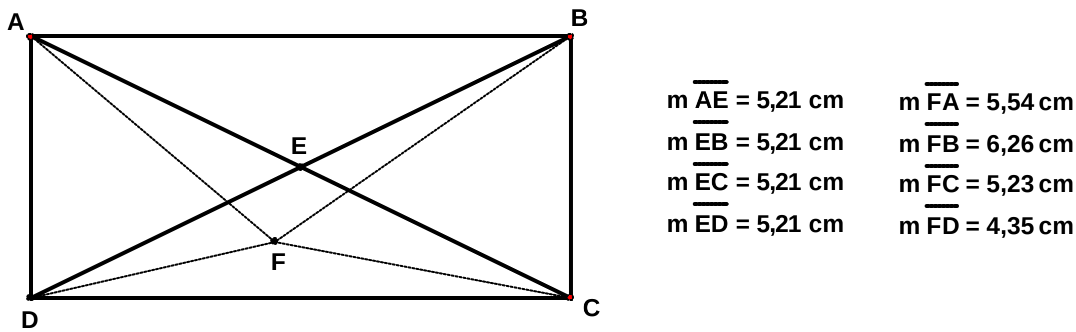


Fig. 1. Students' answers based on "prototype conjectures".

Student C perceived the problem in a quite different way. She conjectured that the four points representing the towns should be points on a circle. The following extract shows how DGS helped her to understand that her reasoning was true only under very specific circumstances.

Student C: The best location for the airport should be the centre of the circle, since the centre is equidistant from any point on the circumference of the circle. (She showed her work by drawing a circle and constructing four points on it, as shown in Figure 2a).

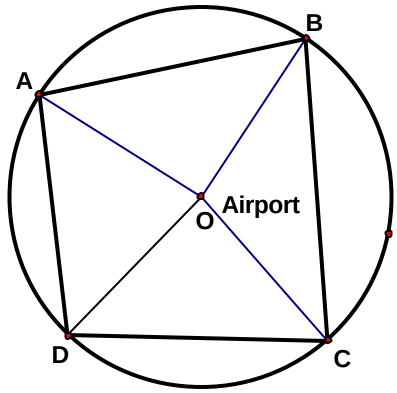
Researcher: Drag one of the vertices of your figure.

Student C: Oh! The centre of the circle (see Figure 2b) *is not always the best solution. In this case* (she points to Figure 2b) *there should be another point.*

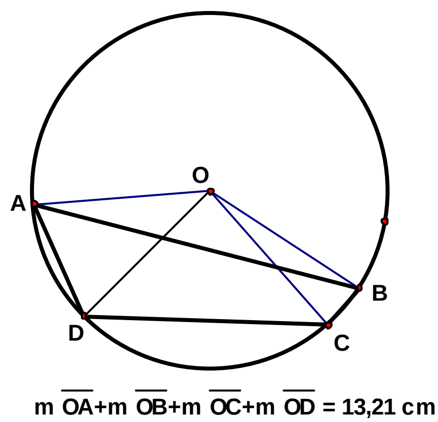
Researcher: Where should that be?

Student C: Probably it is inside the quadrilateral. … A point inside the quadrilateral could serve the towns in a more appropriate or economically better way.

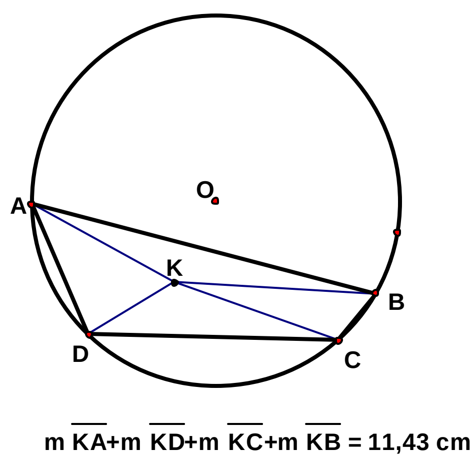
By constructing an arbitrary point inside the polygon (see point K in Figure 2c) and measuring the distances of that point from the four towns, she identified that her initial rule, i.e., the best location is the centre of the circle did not work. She then constructed a quadrilateral hypothesizing that the towns should be the vertices of that quadrilateral and with trial and error she tried to find the solution to the problem.



(a)



(b)



(c)

Fig. 2. Student’s C conjectures and processes for finding the solution of problem1

The students who individually worked on the airport problem could not consider all the possibilities and were not able to generalize their solutions. The day following the experiment, a meeting took place with the researchers and the students in order to present their solutions and find ways to extend the problem. This meeting lasted for an hour, and we realized that students, with the help of the software, had the enthusiasm to work further on the problem. We started the discussion with the work of student C. We prompted them to find the sum of the distances (a) from the centre (see Figure 2b), and (b) from a point inside the polygon (see Figure 2c). They realized that a point inside the quadrilateral would be a better location than the centre of the circle. This gave the opportunity to students construct the two diagonals of the quadrilateral and their point of intersection and label it P. In an attempt to add experimental evidence to support the conjecture, they moved point Q around other possible locations and observed that the point P seemed to be the optimal point (See Figure 3). They realized that the point of diagonals intersection P is the point for which the sum of its distances from each of the four points (i.e. the cities) is the smallest possible. The next step was to provide a mathematical proof of the conjecture so that it could become a mathematical theorem for the students (proofs are beyond the scope of the current paper).

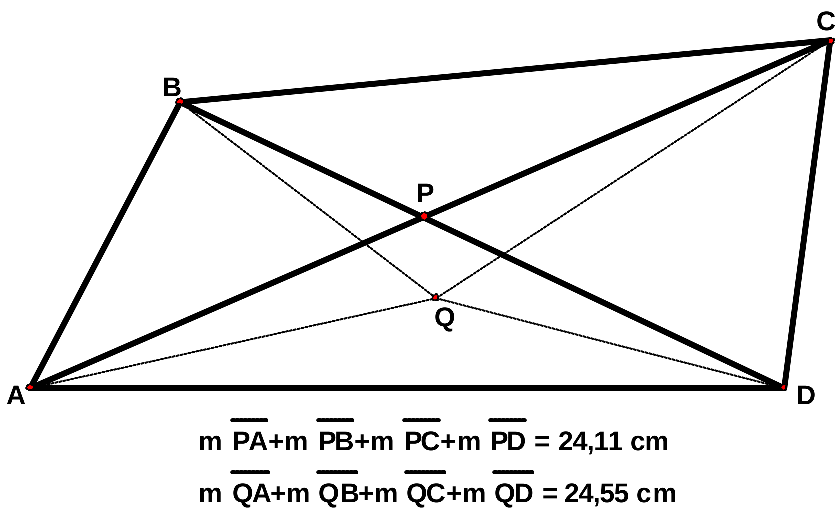
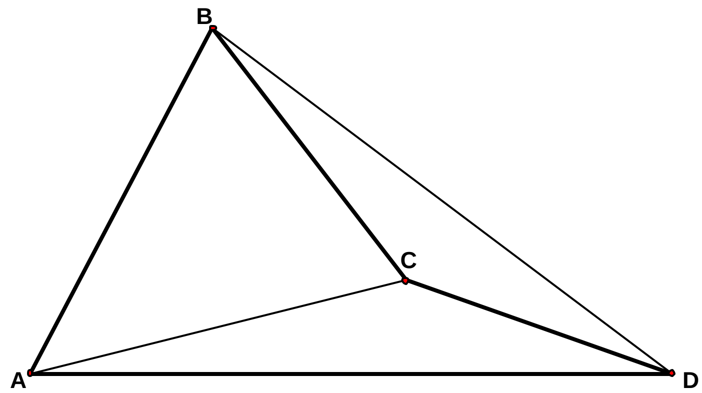


Fig. 3. P is the point where the sum of its distances to each of the four cities is the smallest possible sum.

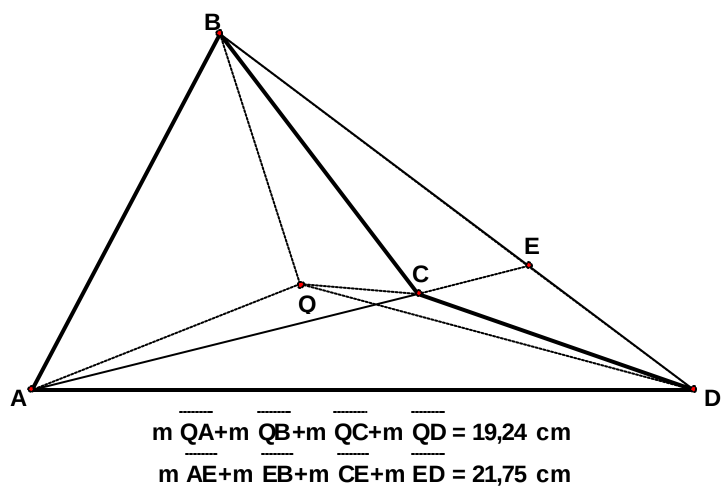
The discussion up to this point seemed to satisfy the students who concluded that the intersection of the diagonals would be the optimal point and the optimal solution of the problem. However, the prompt of the researcher led to further investigations of the problem by considering general or special cases and posing other follow-up problems as shown in the following extracts.

Researcher: Draw a quadrilateral and drag it in such a way as to transform it to a non –convex quadrilateral (see Figure 4a, which illustrates the drawing of a non-convex quadrilateral ABCD as constructed by student C). *What do you observe?*
Student C: The diagonals AC and BD do not intersect "insight" the figure.
Researcher: Does it mean that your previous conclusion is not correct?
Student C: I don't know.
Student A: We can find their point of intersection by extending AC. This may be the optimal point (point E in Figure 4b).
(At this point of the discussion students used the dragging and measure facilities of DGS to examine their hypothesis).
Student B: No, the point of diagonals' intersection is not the correct answer. (Student B points to her diagram). *I constructed a point into the figure and I measured the distances from it.*
Researcher: What are you looking for?
Student B: The best location. I mean, I am trying to find the point from which the sum of the distances is the smallest one.
Researcher: How can you find it?
Student B: (She explains her reasoning using the diagram on the screen). *I moved this point and I found that the total distance from any point inside the figure is always smaller than the distance from any point outside the figure.* (The student showed her work to the group).
Researchers: Ok. What is your answer to the problem?
Student A: We have to try by dragging the point. (All students worked by dragging a point inside the figure).
Student B: It seems that as the point reaches C, the total distance gets smaller.
Student A: Yes, it should be vertex C. Is it correct?

The environment of DGS provided students the opportunity to investigate the location of the new optimal point. Again, by moving point Q, they obtained the tentative location of the new optimal point as shown in Figure 4b. The above extract also addressed the conflicts that usually arise from the exploration of the possible extensions in the assigned problems. Surprisingly, students discovered that the optimal point coincided with point C. In other words, the optimal point seems to be the vertex of the reflex interior angle. Students concluded that the location of the optimal point depends on the type of quadrilateral (convex or non-convex). It should be noted that the dragging capabilities of DGS allowed them to discover that the optimal point of a quadrilateral is not always the point of intersection of the diagonals because such point does not always exist. Finally, students extended the previous conjecture or problem not only to non-convex quadrilaterals but also to the case where three or more points are collinear or to the cases where there are fewer or more points.



(a)



(b)

Fig. 4. The extension of problem 1 to a non-convex quadrilateral.

To answer the first question of the study, i.e., to find out the solution strategies employed by students in the DGS environment, we need first to summarize and interpret the students' work during the solution of the airport problem. In fact, all students used the strategies of modeling, conjecturing, experimenting, and generalizing. Specifically, all students first modelled the problem by representing it in different ways (the cities as points on the circumference of a circle, as vertices of quadrilaterals, etc). Second, they hypothesized the solution of the problem based on how they perceived and modelled the problem and tried to verify their conjectures by dragging and measuring. Finally, they tried to generalize their solutions in their attempts to provide a solution to the problem at hand. However, students seemed to over-generalize their solutions based on certain cases and failed to extend the problem to all possible situations. The latter was achieved during the discussion among the students and the researcher, where the emphasis was to extend the problem and help students to pose new problems.

4.2 The Problem with the Bisectors of the interior angles of a parallelogram

To answer the second question of the study (i.e., to find ways in which DGS provides opportunities for students to pose and solve their own geometry problems), students were asked to solve the bisectors problem, which is a common geometrical problem found in most geometry textbooks. The software helped students to construct the parallelogram as well as the bisectors of the interior angles. Figure 5a shows students' construction of the angle bisectors of the interior angles of a parallelogram, and the following extracts show the way in which the DGS helped them to pose and solve new problems.

Student D: The figure formed by the bisectors of the interior angles seems to be a rectangle. (He dragged one of its vertices, and verified his answer).
Researcher: Is that always true? (Students tried to transform the parallelogram to other shapes such as rectangles, squares and rhombuses).
Student E: This is not always true. If you drag the parallelogram until it becomes a rhombus, the interior figure disappears. (See Figure 5b)
Student D: It didn't disappear. It became a point.
Student F: … If the original parallelogram becomes a rectangle, the figure is a square. (She verified it by dragging one of the vertices of the original rectangle). (See Figure 6).
Researcher: Try to extend the problem to other quadrilaterals.
Student D: This is an isosceles trapezium.
Student F: The angle bisectors of the interior angles of an isosceles trapezoid form a kite with two right angles. (See Figure 7).
Researcher: What about the figure formed in a non-isosceles trapezium?
Student D: It may be a kite. (Student D constructed the appropriate shape). *Yes, it's a kite without two right angles.*

The above constructions could certainly be reached without the computer, and the students could also prove the conjecture. However, without the use of the dynamic software students would not be able to add experimental evidence to their conjectures as they did by dragging any of the flexible points of the parallelogram and notice, as previously conjectured, that the figure might be a rectangle. An important finding lies on the fact that by dragging one of the flexible points of the parallelogram until it becomes a rhombus, the students observed that the figure no longer formed a rectangle but a point (See Figure 5b). This revealed to them that their first conjecture does not always hold and led them to consider a point as a degenerated rectangle! Again, it was evident that the dragging capabilities of DGS allowed individuals to consider extreme cases of a geometric configuration, cases that textbook authors fail to consider.

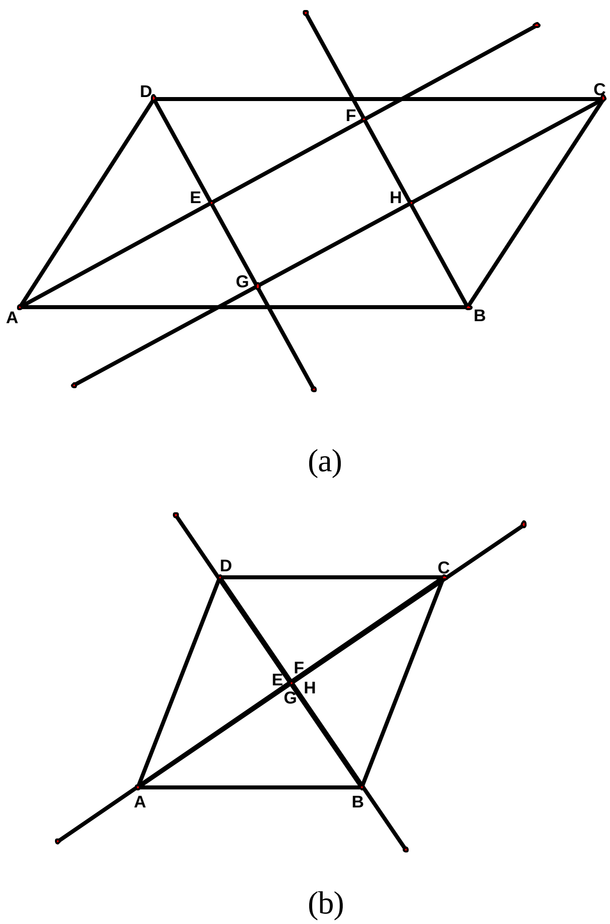


Fig. 5. The figure formed by the bisectors of a parallelogram and a rhombus.

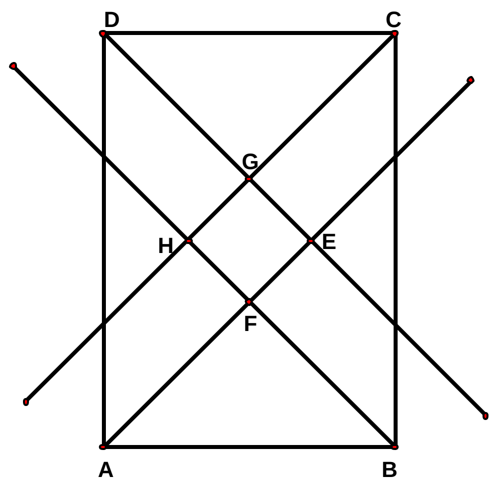


Fig. 6. The figure formed by the bisectors of a rectangle.

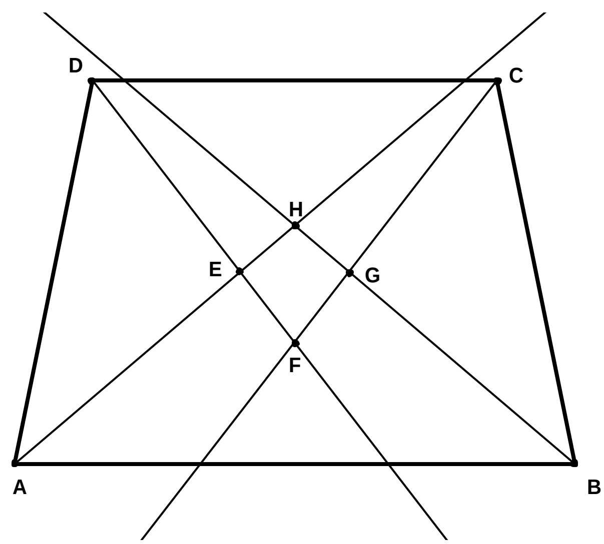


Fig. 7. The figure formed by the bisectors of an isosceles trapezium.

These activities led students to engage in problem posing by experimenting, generalizing, specializing, and extending the problem through the modification of the conditions of the given problem. A special case of the problem was to start with a rectangle instead of a parallelogram as done by student F. Figure 6 suggests that the figure formed by the angle bisectors of the interior angles of a rectangle is possibly a square. Students also considered general cases. They conjectured from Figure 7 and by dragging one of the flexible points that the figure formed by the angle bisectors of the interior angles of an isosceles trapezoid is a kite with two right angles.

Another interesting extension to this problem was posed during the discussion, when one of the students suggested that it would be interesting to find out the figure, which can be formed by joining the mid-points of the figures constructed by the bisectors (Figure 8). This, of course, led to different conjectures based on the solution of the original bisectors problem. For example, one student conjectured that the new shape would be similar to the shape formed by the bisectors, while others generalized the theorem of the mid-points of a quadrilateral, predicting that in all cases, with the exception of squares and rhombuses, the shape would be a parallelogram. Figure 9 shows most of the constructions proposed by students in extending the original problem.

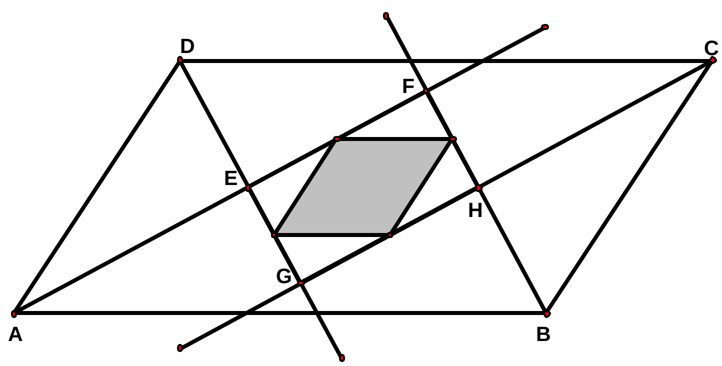


Fig. 8. The figure formed by the mid points of the segments defined by the bisectors of the parallelogram.

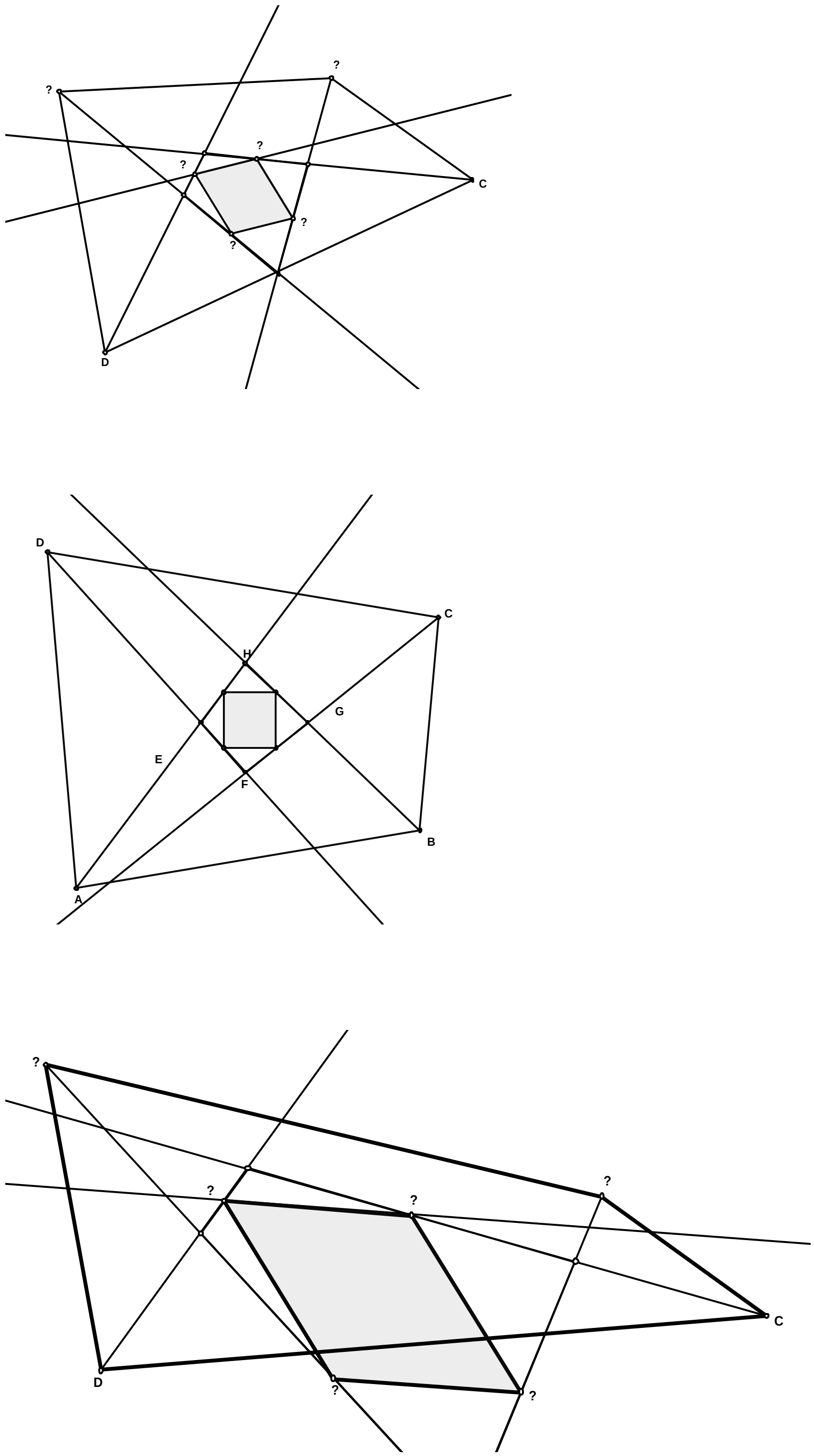


Fig. 9. The figure formed by the mid points of the segments defined by the bisectors of the quadrilateral.

5. Conclusions

The growing and nearly universal availability of technological tools facilitates teachers in teaching and improving the mathematical experiences of students. This paper focused on the use of a DGS in problem solving, inquiry, and exploration in mathematics. We provided some ideas on how students can use the tools of DGS to solve and pose mathematical problems. The paper also addressed the ways in which DGS may be associated with new problems that do not usually appear in the traditional geometry textbooks, and new ways of introducing problem solving and posing in a variety of contexts. Two examples were provided to show how the computing, graphing, and dragging capabilities of dynamic geometry software can enable students to explore and make mathematical conjectures, solve problems, and pose related problems.

In the two examples, DGS acted as a mediation tool (Artigue, 2000; Jones, 2000) in the implementation of an inquiry approach to teaching and learning mathematics as recommended in current mathematics education documents (NCTM, 2000). Specifically, this study showed that DGS can play a significant role in engendering problem solving and posing. First, the new information students obtained through dragging and measuring helped them understand the problems, and added challenge to the exploration of the possible answers to a problem. It was shown that dragging is an important tool for problem solving and posing, and measuring is an important tool for checking the correctness of students' conjectures.

Second, DGS as a mediation tool, encouraged students to use in problem solving and posing the processes of modeling, conjecturing, experimenting and generalizing. Through modeling, students constructed accurate images of the problems, which helped them to visually explore the problems and reflect on them. The meaning students extracted from the constructed images enabled them to explore at a perceptual level and to make conjectures about the possible solutions to the problems. Through experimentation, students also visually confirmed or negated their conjectures, and thus proceeded to suggest possible solutions or extensions to the assigned problems. The results of this study also show that in the DGS environment the problem solving processes involve the generation of new problems, supporting the relationship between problem solving strategy use and the tendency to pose extension problems (Cai & Hwang, 2002).

Third, DGS provided a context in which we can do mathematics in a different way (Aviram, 2001). The data of the study showed that DGS environment can bring about cognitive conflict and/or surprise, as it appeared mainly in the airport problem with the vertex quadrilateral. Since a particular paper and pencil figure usually displays a general case, it is difficult for students to appreciate the significance of special cases. However, students using the DGS are very likely to drag a figure past a special case, and thus more likely to stop on a special case and be faced with the consequences.

Finally, on a practical level, the present study of DGS learning can benefit teachers, and curriculum developers. Teachers faced with limited time and crowded computer labs may use research results to identify fruitful ideas in the language and construction actions of their students. In addition, curriculum developers may find inspiration for new activities aimed at the needs of dynamic geometry learners.

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