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Abstract: *In a previous study that explored epistemological perspectives on solving problems with Computer Algebra Systems (CAS) we concluded that awareness of the special ways that the software utilizes symbols in algebraic manipulations and in implicit plotting should be encouraged (Zehavi, 2004). Such awareness is required for, and encouraged by treating geometry analytically with a symbolic-graphical system. In this paper we compare a traditional solution of a problem in analytic geometry with CAS-based solutions to the same problem. The discussion will focus on the role of reflective thinking, namely selection of techniques, monitoring of the solution process, insight, and conceptualization, play in the creation of instrumented techniques (Guin & Trouche, 1999). Teachers, who experienced learning activities from a resource e-book for teaching analytic geometry with CAS, contributed to the design of tasks and to the analysis of instrumented techniques.*

Introduction

Since 1996 a team at the Weizmann Institute of Science has been preparing CAS-based activities for junior high school, and for the senior high school. The activities complement the current syllabus aiming to broaden learning opportunities and to promote greater mathematical understanding. Research studies that accompany the development of the learning activities indicate that students' interaction with CAS and students' reflections are intertwined (Zehavi & Mann, 2003; Mann, Zehavi, & Halifa, 2003). We have recently developed a resource e-book for teaching Analytic Geometry, containing activities for students, and an extended teacher guide including annotated CAS files (we use *Derive*). Although symbolic-graphical technology is not allowed at this stage in the final exams, an increasing number of mathematics teachers incorporate this technology in their work. The activities were presented to in-service teachers in professional workshops as part of the formative development of the learning materials. The practicing of instrumented techniques led the teachers to extend the pedagogical scope of the activities. Here we discuss the epistemological value added to the pragmatic production of solutions by instrumented techniques, [see: Guin & Trouche (1999), Artigue (2002), and Lagrange (2005)]. We first analyze a traditional solution to the problem of finding the *director circle of an ellipse*. The analysis method we developed for this purpose links the cognitive and meta-cognitive levels, namely the execution of the solution and the reflective thinking. Then we analyze by the same method CAS-based solutions. Implications of the

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analysis to our understanding of the changes that computer algebra systems bring to mathematics education will appear in the concluding part.

The Analysis Method

The steps of the solution are analyzed in two levels: *execution* and *reflective thinking*. The basic components of the *execution* of problem solving in analytic geometry (or any other domain that requires modeling) are: constructing a mathematical *model* for the problem, *manipulations* within the model to obtain results, *interpretation* of the results in the contexts of the problem, and *representations* (graphical or symbolic) of the model or the manipulations or the interpretations. We use the term *reflective thinking* for the meta-cognitive level referring to four categories: selection of *techniques*, *monitoring* of the solution process, *insight* or ingenuity, and *conceptualization* (i.e. connecting concepts and meaning).

The reflective thinking components are inferred from the written 'execution' of the solution and from explanations given in textbooks. To make the reflective thinking more transparent we asked teachers and students to add annotations to their CAS worksheets and to discuss them verbally. The classification associated to solution steps, however, should be regarded as subjective.

A traditional solution

The problem is presented as a task: *"Find the locus of the points of intersection of perpendicular tangents to the ellipse defined by the equation* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$*"*. This task appears in traditional textbooks and is regarded as quite sophisticated for high school students. Therefore, some textbooks provide a solution to the problem (For example, Barry, 1963). The steps of the traditional solution of this problem are described in the following (Chart 1).

Step 1

<u>Reflective thinking: selecting technique</u>

The equation of a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = mx + \sqrt{a^2m^2+b^2}$ *or* $y = mx - \sqrt{a^2m^2+b^2}$.

A line parallel to the vertical axis is not considered in this equation.

Step 2

Execution: Modeling

A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through a point (p, q) if and only if $q = mp + \sqrt{a^2m^2+b^2}$ *or* $q = mp - \sqrt{a^2m^2+b^2}$

We look for values of *m* that satisfy the above condition.

Step 3

Reflective thinking: insight, selecting technique

In order to utilize Viète's formula the equation $q = mp \pm \sqrt{a^2m^2+b^2}$

should be "simplified" in a special way to get

a quadratic equation in the form $Am^2 + Bm + C = 0$.

Step 4

Execution > manipulations

…….. $(p^2-a^2)m^2 - 2pqm + q^2 - b^2 = 0$

Step 5

Reflective thinking: conceptualization

The product of the slopes of two orthogonal lines is -1.

Step 6

Execution: manipulations

Viète's formula states that $m_1 \cdot m_2 = \frac{q^2-b^2}{p^2-a^2}$. Thus we have $\frac{q^2-b^2}{p^2-a^2} = -1$.

Step 7

Execution: interpretation, representation

The standard form of a Cartesian equation for the locus of points whose coordinates (p, q) verify the equation $p^2+q^2=a^2+b^2$ is $x^2+y^2=a^2+b^2$.

Chart 1: Steps of a traditional solution

Only a few high school students can come up with such a solution that requires good mastering of the mathematical meaning of symbols and a global view of the task. We dare to say that one should almost know the solution before actually working on it: the analysis indicates that conceptualization and insight are prior to the execution steps.

CAS-based solution

The task was presented to the teachers in a workshop. In order to get a visual product, the task involved a specific numerical example, $\frac{x^2}{9}+\frac{y^2}{4}=1$. In Chart 2 we present an example of a CAS-based solution using *Derive*'s notation.

Step 1

Execution: Modeling 1

The equation of a line (not parallel to the vertical axis) that passes through (p, q) is $y = mx - mp + q$. By substitution we get an equation for the *x* values of the intersection points of the ellipse and the line.

$$\frac{x^2}{9} + \frac{(m\cdot x - m\cdot p + q)^2}{4} = 1$$

Step 2

Reflective thinking: Selecting technique

$$x^2\cdot(9\cdot m^2 + 4) + 18\cdot m\cdot x\cdot(q - m\cdot p) + 9\cdot(m^2\cdot p^2 - 2\cdot m\cdot p\cdot q + q^2 - 4) = 0$$

Step 3

Execution: modeling 2

$$(18\cdot m\cdot(q - m\cdot p))^2 - 4\cdot(9\cdot m^2 + 4)\cdot 9\cdot(m^2\cdot p^2 - 2\cdot m\cdot p\cdot q + q^2 - 4) = 0$$

We look for values of *m* that satisfy the above condition, i.e. Discriminant = 0.

Step 4

Execution: manipulations

$$m = \frac{\sqrt{(4\cdot p^2 + 9\cdot(q^2 - 4))} - p\cdot q}{9 - p^2} \vee m = \frac{\sqrt{(4\cdot p^2 + 9\cdot(q^2 - 4))} + p\cdot q}{p^2 - 9}$$

Step 5

Reflective thinking: conceptualization

The product of the slopes of two orthogonal lines is -1, thus

$$\frac{\sqrt{(4\cdot p^2+9\cdot(q^2-4))}-p\cdot q}{9-p^2}\cdot\frac{\sqrt{(4\cdot p^2+9\cdot(q^2-4))}+p\cdot q}{p^2-9}=-1$$

Step 6

Reflective thinking: monitoring

Plot the equation in Step 5

Where do the "holes" come from? (see later)

Is this a circle? Why?

Let' simplify the equation.

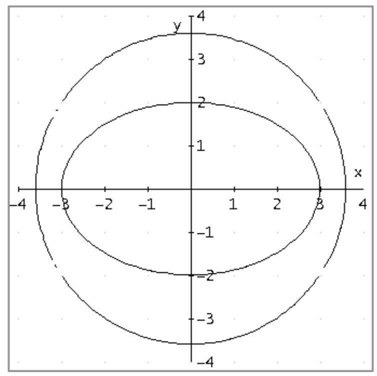

Figure1. Director circle with "holes"

Step 7

Execution: manipulations

Simplify the equation in Step 5,

$$\frac{q^2-4}{p^2-9}=-1$$

$$p^2+q^2=13$$ and plot.

Step 8

Execution: interpretation, symbolic representation

The standard form of a Cartesian equation for the locus of points whose coordinates (p, q) verify the equation $p^2+q^2=13$ is $x^2+y^2=13$.

Chart 2: Steps of a CAS-based solution

In contrast to the traditional solution which began with prior reflection, the CAS solution started with writing a "simple" equation for finding the intersection points of a line with slope *m* that passes through a point (p, q) and the given ellipse. Selecting a familiar technique for simplifying the equation led to the well known model ($\Delta = 0$) and utilizing the symbolic mechanism of the software to obtain two algebraic solutions for *m*. Translating the necessary and sufficient condition (if and only if) for lines to be perpendicular into an equation (Step 5) gave a strange result that called for monitoring. In Step 6 the teachers used the software to plot the graph of this equation. Various reactions were heard: Where do the "holes" come from? Our error? Bug of the implicit plotting? Is this a circle? Why? Let's simplify the equation: $\frac{q^2-4}{p^2-9}=-1$.

Standard algebraic manipulations and interpretation yield the representation in the form of equation of the circle $x^2+y^2=13$. The circle and the given ellipse have the same center. In the general case, i.e. for an ellipse given by the canonical equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the radius of circle which is obtained by a working session as above is equal to $\sqrt{a^2+b^2}$. This circle is called the *directo*r circle (or *orthoptic* circle, or *Monge* circle) of the ellipse given by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The surprising holes around the four points (3, 2), (3, -2), (-3, -2), (-3, 2) are explained algebraically by the denominator in the equation; the graphical interpretation draws our attention to the exceptional tangents (to the ellipse) that are parallel to the x-y axes. The instrumented scheme that the teachers implemented has an epistemic value: The problem is that we work in a neighborhood of a singular point of the equation whose graph has been plotted (the singularity is caused by what we did at the beginning: we did not consider lines parallel to the y-axis). This is a general problem for computerized drawing of curves (see Dana-Picard, 2005)

After the surprising phenomenon of the holes has been understood, another question appeared: how can one see from Equation (1) that it would actually simplify to equation (2)? In the nominators of Equation (1) we can see the pattern (A – B)(A +B). At this stage the teachers became interested in investigating the expression under the square sign. Plotting the inequality $4p^2+9(q^2-4)\geq 0$ added more insight to the solution process: we see the outside of the given ellipse; since the expression under the square sign appear in the solution for the slope *m* of the tangent, the solution of the inequality shows, in fact, that it is impossible to draw a real tangent to the ellipse through a point within the circle.

The teachers suggested adding pragmatic value to the above exploration, namely, to produce pairs of perpendicular tangents to the ellipse given by the equation $\frac{x^2}{9}+\frac{y^2}{4}=1$ (see Figure 2). Some of them claimed that this should be stated initially as the goal of the task, so that the efforts in identifying the geometric locus of points of intersection of such pairs of tangents would be the means to achieve the goal. Others argued against such a pragmatic goal and preferred to consider the animation of pairs of tangents as an implementation of the result. The instrumented technique needed for this task involves the use of a slider bar to view in a dynamic way pairs of tangents that intersect in a point T= $(p,\sqrt{13-p^2})$ on the director circle whose equation is $x^2+y^2=13$. We substitute $\sqrt{13-p^2}$ for *q* in one of the expression for *m* in Step 4, and write the equation of two perpendicular tangents through T.

$$m1 := \frac{\sqrt{(81 - 5\cdot p^2)} - p\cdot\sqrt{(13 - p^2)}}{9 - p^2}$$

$$\left[y = m1\cdot x + \sqrt{(13 - p^2)} - m1\cdot p,\ y = -\frac{1}{m1}\cdot x + \sqrt{(13 - p^2)} + \frac{1}{m1}\cdot p\right]$$

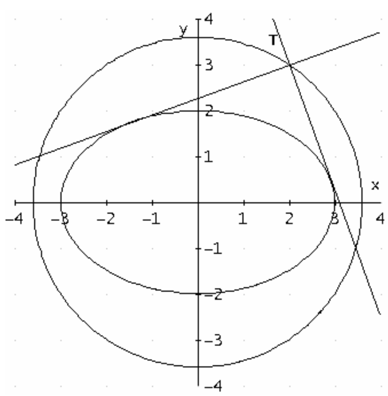

Figure 2. 'Animation' of perpendicular tangents

The teachers agreed that visualizing the tangents should be an integral part of the activity because it can provides feedback and control to student's actions. Not less important is the satisfaction feeling in obtaining a nice product.

Changes that computer algebra systems bring to mathematics education

Based on the example we described in this paper, and other similar examples we attempt to identify changes that CAS brings to the mathematical environment of teachers and students.

In a traditional solution one must have a full blown strategy from the beginning in order to solve the problem, and to master sophisticated methods of manipulations (e.g. Viète's formulas) to carry out the strategy. In a CAS solution one can start the solution process by using the symbolic power of the software to perform familiar manipulations and then obtain representations of the results. Having some result and being free from technical work one can gradually consolidate a solution strategy.

One implication of the above is that some topics of the core traditional curriculum may become obsolete. Viète's formulas and other algebraic ingenuities have been taught to facilitate manipulations by hand, but one can do without them when using software that was designed to perform the manipulations. These human culture developments should be appreciated and recognized, but not necessarily in the core mathematics curriculum. Instead we should develop strategies that develop awareness to pragmatic and epistemic values of instrumented techniques.

Our analysis indicates that in traditional solutions conceptualization and insight are prior to the execution steps, while in CAS solution the reflection steps (conceptualization, insight, monitoring, and selecting techniques) are inseparable from the execution steps.

A consequent implication is that advanced problems that have been traditional reserved for those few gifted with mathematical intuition, can now be accessed effectively by a greater population with appropriate instruction by the teachers.

The role of the teacher who teaches with modern technology is very complex, including aspects of the technology, of mathematics, and of didactics. Thus the structure of a computer based activity should initially be made clear to the teacher at a global level. To be able to guide effectively students in using the various instrumented techniques, teachers first need to review the relevant mathematical methods; they also need some experience and exposure to learning events that have the potential to intertwine execution and reflection. But most importantly, they should be partners in the task-design process. (This actually happened in one of our workshop that introduced the director circle of an ellipse.)

After finding the director circle of the ellipse the teacher usually explored loci of points of intersection of perpendicular tangents to an hyperbola and to a parabola, identifying the differences between the three cases. In one workshop some teachers were interested in finding the locus of the intersection point of tangent to an ellipse having an angle of 45^o between them. In the case of 90^o we had the simple equation $m_1 \cdot m_2 = -1$.

Here we have the equation $|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}| = 1$.

Plotting this implicit equation for the ellipse we used before gives a graphical representation of the locus (Figure 3).

A more traditional symbolic representation can be obtained by algebraic manipulations.

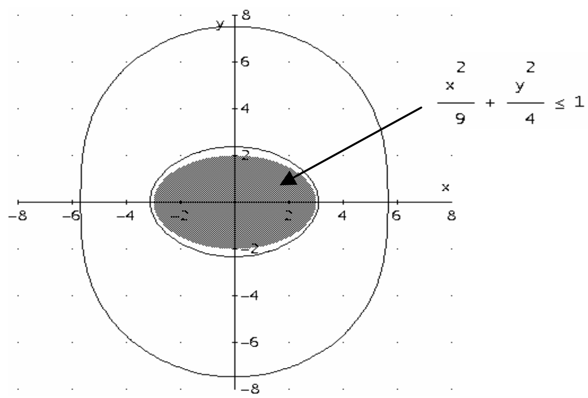

Figure 3. Seeing the ellipse in $45^o/135^o$

Now the questions come quick and fast: what about other angles (Figures 4, 5)? What about hyperbola, parabola?

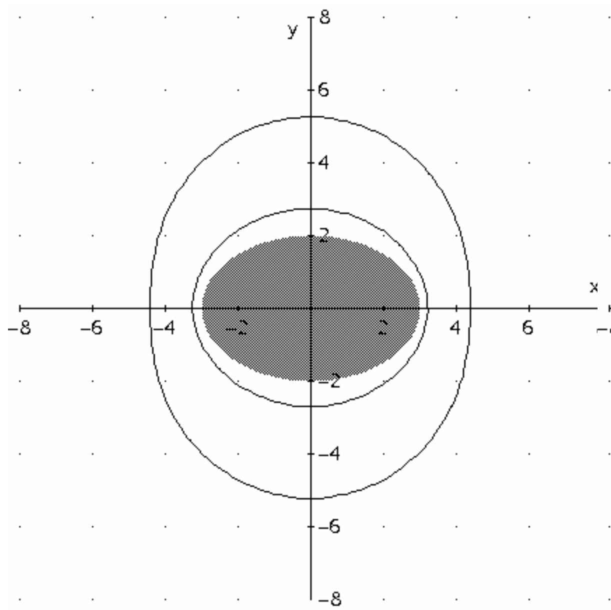

Figure 4. $|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}| = 2$

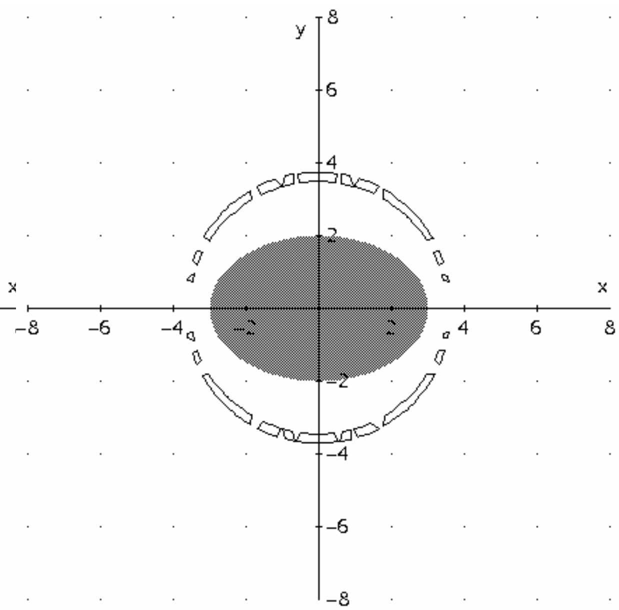

Figure 5. $|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}| = 20$

In this problem, as in the one we presented in detail, the implicit plotting plays an important role in making algebraic manipulation by the software and conceptual insight of the users inseparable. We invite the interested readers to explore the problem (with CAS, of course) and design a didactic sequence of tasks that suits their educational goals and the needs of their students.

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