A Finite Integration Method for A Time-Dependent Heat Source Identification of Inverse Problem

Rama Lesmana, Areena Hazanee, Aniruth Phon-On, Jarunee Saelee
Department of Mathematics and Computer Science
Faculty of Science and Technology, Prince of Songkla University
Pattani, Thailand
Email : lesmanarama5@gmail.com, areena.h@psu.ac.th, aniruth.p@psu.ac.th, jarunee.sa@psu.ac.th.

ABSTRACT

We investigate an inverse problem of reconstructing a timewise-dependent source for the heat equation. The solution of this problem is uniquely solvable, yet unstable. The inverse source problem two unknowns is reformulated to be a new form of forward problem one unknown. Furthermore, we propose that the finite integration method combined with the backward finite difference method can be used to solve the reformulated heat equation. The Tikhonov regularization method is employed to stabilize the noisy data. The proposed algorithm is not only easy to use but also can give an accurate and stable solution. Numerical result is presented and discussed.

Keywords: Heat Equation, Inverse Problem, Tikhonov Regularization.

I. INTRODUCTION

Inverse source problem for the heat equation commonly appear in mathematical modeling to identify the unknown source function in polution source intensity, melting and freezing process. Recently, the inverse problem has been the point of interest by many authors, see [1,4,5,6,11]. The source can be determined as a function depending on both space and time for one dimension heat equation, in practical many researchers considered the heat source as a function of either space or time only. The identification of space-dependent heat source function can be seen in [2,7] whereas for the time-dependent heat source function can be seen in [13,14]. Furthermore, several numerical methods have been employed to seek out the time-dependent heat source function of the inverse problem [3,13,14].

In this present paper, we only focus on the identification for the time-dependent source function for the heat equation under the initial and the Neuman boundary conditions, together with the given observed data considered as the over-determination condition. Accordingly, we propose the finite integration method (FIM) with the ordinary linear approximation (OLA) to solve the inverse source problem. This proposed method is based on the trapezoidal rule which is numerical integration of using linear function to approximate the integral. The FIM (OLA) was first reconstructed and introduced by [8] and has been improving to be able to solve various kinds of the differential equations. Therefore, this method has been extensively used for dealing the direct problem with both ordinary and partial differential equations. For example, the problem of nonlocal elastic bar under static [8], fractional-order of PDE [12] and extended to two dimensional potential problem [9]. However, no author has been using the FIM (OLA) to solve the inverse heat source problem. To deal with the inverse problem, there are many methods/procedures for obtaining the numerical solution such as method of fundamental solution (MFS), boundary element method (BEM) and the direct numerical method. In [13] has used the direct numerical method which is a method about to reform the inverse problem into the direct problem by using differentiation and integration. Therefore, in this study, we propose to use the FIM (OLA) to solve the inverse problem of finding the time-dependent heat source function by the direct numerical method.

The paper is organized as follows. In Section 2, the problem is clearly stated. In Section 3, the use of the direct method is applied mathematically to seek the time-dependent source function. In Section 4, the FIM (OLA) is employed together with the backward difference in order to discretize the problem obtained from Section 3. In the Section 5, we use the Tikhonov regularization method to stabilize and approximate the noisy function. To ilustrate a clear overview and test the accuracy of the proposed method, in Section 6, a benchmark numerical example is provided. Section 7 ends this paper with the conclusion.

II. The Problem Statement

Let $D_T = C^2\left[0,1\right]\times C^1\left[0,T\right]$ be the solution domain with the final time $T>0$. We consider the inverse problem of finding the pair solution $\left(f(t),u(x,t)\right)\in C\left([0,T]\right)\times C^{2,1}\left(D_T\right)$ for the following the heat conduction equation,

$$u_t(x,t) = u_{xx}(x,t) + f(t), \quad 0<x<1, \quad 0<t<T, \qquad (1)$$

subject to the initial data and boundary conditions

$$u(x,t) = u_0(x), \qquad 0<x<1, \qquad (2)$$

$$u_x(0,t) = s(t), \qquad u_x(1,t) = r(t), \qquad 0<t<T. \qquad (3)$$

The additional condition is considered as

$$u(x_f,t) = g(t), \qquad 0\le x_f \le 1, \qquad (4)$$

In order to ensure the existence and uniqueness of the solution, the given functions $u_0(x), s(t)$ and $r(t)$ are assumed to satisfy the following compatibility conditions:

1) $s, r\in C[0,T], g\in C^1[0,T]$, and $u_0\in C^1[0,1]$.
2) $(u_0)_x(0) = s(0), \ (u_0)_x(1) = r(0), \ g(0) = u_0(x_f)$.

One thing to note that although the inverse problem (1)-(4) under the above compatibility conditions is uniquely solvable, it is still ill-posed as the small errors in the input data leading to gain the large errors in the solution.

III. The Direct Method

In [13] Xiangtuan *et al*. have established a direct numerical method which is an algorithm for seeking the time-dependent and space-dependent heat source of the inverse problems. In this present study, we would like to apply the algorithm of this method to the time-wise inverse heat source problem as following explaination. The purpose of method is not to determine the source directly but rather to construct the forward problem as an access for obtaining the heat source eventualy. In order to employ the direct numerical method suggested by [13] for solving the inverse problem (1)-(4), we firstly take the derivative with respect to x over the heat equation (1), this yields

$$u_{tx}(x,t) = u_{xxx}(x,t). \qquad (5)$$

Let $w(x,t) = u_x(x,t)$ for $w(x,t)\in D_T$ and taking the integration with respect to x over $[x_f, x]$ gives

$$u(x,t) = \int_{x_f}^{x} w(y,t)dy + g(t). \qquad (6)$$

Since $w(x,t)\in D_T$, then the differential equation (5) and the initial and boundary conditions (2)-(3) become

$$w_t(x,t) = w_{xx}(x,t), \qquad (7)$$

with the reformed initial and boundary conditions

$$w(x,0) = (u_0)_x(x), \ \ w(0,t) = s(t), \ \ w(1,t) = r(t). \qquad (8)$$

Taking the integration with respect to x on $[x_f, x]$ over the equation (5) gives

$$\int_{x_f}^{x} u_{tx}(x,t)dx = \int_{x_f}^{x} u_{xxx}(x,t)dx. \qquad (9)$$

Then, we have

$$u_t(x,t) - u_t(x_f,t) = u_{xx}(x,t) - u_{xx}(x_f,t).$$

Consider the over-determination condition (4), i.e. $u(x_f,t) = g(t)$ and since $w_x(x_f,t) = u_{xx}(x_f,t)$, then we obtain

$$u_t(x,t) = u_{xx}(x,t) + g'(t) - w_x(x_f,t). \qquad (10)$$

Here, the above heat equation (10) is now written as the heat equation (1) with the source function defined as

$$f(t) = g'(t) - w_x(x_f,t). \qquad (11)$$

In general, the given data $g(t)$ normally consists some measurement errors unavoidably, we therefore use the

Graduate School and Faculty of Science and Technology, Prince of Songkla University, Pattani Thailand

Tikhonov regularization method to stabilize the noisy function denoted as $g^{\delta}(t)$. Now let us summarize the systematic step for determining $f(t)$ as follows

- Step 1. Solve numerically the following problem to get $w(x_f, t)$:

$$\begin{aligned} w_t(x,t) &= w_{xx}(x,t), && 0<x<1,\ 0<t<T, \\ w(x,0) &= (u_0)_x(x), && 0<x<1, \\ w(0,t) &= s(t),\ w(1,t)=r(t), && 0<x<1, \end{aligned} \tag{12}$$

- Step 2. Find $w_x(x_f, t)$ by using the central finite difference method.
- Step 3. Use the Tikhonov to stabilize noisy function $g(t)$.
- Step 4. Approximate the first-order derivative $g'(t)$ by the central finite difference method.
- Step 5. Compute $f(t)$ by $f(t) = g'(t) - w_x(x_f, t)$.

IV. THE USE OF FIM (OLA)

The FIM (OLA) is a renew numerical method for solving the differential equation (the direct problem) suggested by Li *et al* [8]. In this section, we propose to use the FIM (OLA), [12], to discretize the space-wise and employ the backward finite difference method to discretize the time-wise of the reformulated problem in step 1 of (12).

A. Finite difference method

This subsection is devoted to describe the numerical method for approximating the timewise first order derivative of the unknown function $w(x,t)$ with respect to t. Let $w_j = w(x,t_j)$ for $t_j = j\Delta t$, $j \in \{0,1,2,\cdots,M\}$ and $\Delta t = \frac{T}{M}$. Consider the uniform grid partitions

$$w_0 < \cdots < w_j < \cdots < w_M,\ j\in\{0\ ,\ 1,\ 2,\cdots,\ M\},$$

we can approximate the first order derivative of unknown function $w(x,t)$ by using the backward FDM which can be expressed as

$$w_t(x,t) = \frac{w(x,t_j) - w(x,t_{j-1})}{\Delta t}. \tag{13}$$

B. Finite integration method

For dealing with the FIM, we start with approximating a definite integral of a smooth function from a to b, $\int_a^b w_j(x)dx$, by using the trapezoidal rule as the following formula

$$\int_a^b w_j(x)dx = \frac{\Delta x}{2}\left[w_j(x_0) + 2w_j(x_1) + \cdots + 2w_j(x_{N-1}) + w_j(x_N)\right]$$

where $\Delta x = \frac{b-a}{N}$ and $x_i = a + i\Delta x$ for $i \in \{0,1,2,\cdots,N\}$. Define the (single-layer) definite integration function as

$$W^{(1)}(x_k) = \int_a^{x_k} w_j(x)dx \approx \sum_{i=0}^{k} a_{ki}^{(1)} w_j(x_i),$$

Where $a_{01}^{(1)} = 0$ and $a_{ki}^{(1)} = \Delta x \begin{cases} \frac{1}{2}, & i = 0, k, \\ 1 & i = 1,2,\cdots,\mathrm{k}-1. \end{cases}$

And also the matrix form of integration is expressed as follow:

$$\underline{W}_j^{(1)} = A^{(1)}\underline{w}_j,$$

where

$$\underline{W}_j^{(1)} = \left[\int_a^{x_0} w_j(x)dx, \int_a^{x_1} w_j(x)dx, \cdots, \int_a^{x_N} w_j(x)dx,\right]^T,$$

$$\underline{w}_j = \left[w_j(x_0), w_j(x_1), \cdots, w_j(x_N)\right]^T,$$

$$A^{(1)} = (\Delta x)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1 & 1 & 1/2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1/2 & 1 & 1 & 1 & 1 & 1/2 \end{bmatrix}_{(N+1)\times(N+1)}.$$

We then consider a double-layer definite integral as

$$W^{(2)}(x_k) = \int_a^{x_k}\int_a^{y_1} w_j(y)\,dy\,dy_1 = \sum_{i=0}^{k}\sum_{j=0}^{i} a_{ki}^{(1)} a_{ij}^{(1)} w_j(x_i).$$

Here, we reform the double-layer integral above as

$$W^{(2)}(x_k) = \sum_{i=0}^{k} a_{ki}^{(2)} w_j(x_i, t)$$

where

$$a_{01}^{(2)} = 0 \text{ and } a_{ki}^{(2)} = \Delta x \begin{cases} \dfrac{1+2(k-2)}{4}, & i = 0, \\ k-i, & i = 1, 2, \cdots, k-1, \\ \dfrac{1}{4}, & i = k. \end{cases}$$

Again, we can write this in matrix form as

$$\underline{W}_j^{(2)} = A^{(2)} \underline{w}_j,$$

where

$$\underline{W}_j^{(2)} = \left[\int_a^{x_0}\int_a^{y_1} w_j(y)\,dy\,dy_1, \int_a^{x_1}\int_a^{y_1} w_j(y)\,dy\,dy_1, \cdots, \int_a^{x_N}\int_a^{y_1} w_j(y)\,dy\,dy_1 \right]^T,$$

$$\underline{w}_j = \left[w_j(x_0), w_j(x_1), \cdots, w_j(x_N) \right]^T,$$

$$A^{(2)} = (\Delta x)^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 3/4 & 1 & 1/4 & 0 & 0 & 0 \\ 5/4 & 2 & 1 & 1/4 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1+2(N-1)}{4} & N-1 & N-2 & \cdots & 1 & 1/4 \end{bmatrix}_{(N+1)\times(N+1)}$$

One thing to note that $A^{(2)} = A^{(1)}A^{(1)}$. Therefore if we denote $A = A^{(1)}$, we can get $A^2 = A^{(2)}$. To use the FIM (OLA) for solving the differentiation equation as in problem (12) we can perform by taking the integration with respect x twice over the heat equation and combining with (13). Therefore, the PDE in (12) becomes

$$A^2 \underline{w}_j - \Delta t \underline{w}_j = A^2 \underline{w}_{j-1} + c_0 \underline{x} + c_1 \underline{i},$$

in discrete sense. Where c_0 and c_1 are integral constants, $\underline{x} = \left[x_0, x_1, \ldots, x_N\right]^T$ and $\underline{i} = \left[1, 1, \ldots, 1\right]^T$. For more detail on how to solve the system, we will describe in the section of numerical example.

V. THE TIKHONOV REGULARIZATION

As along the previous section, the equation (11) holds the first-order derivative $g'(t)$. Since the measured data $g(t)$ is normally obtained from the experiment and there exist measurement errors unavoidably. We denote g^{δ} as a noisy observed data. This brings us to involve an ill-posed problem of the first-order numerical differentiation.

Hence, we should tackle the first-order numerical differentiation stable approximation method. In this section, we wish to employ the Tikhonov regularization method to stabilize the noisy data g^{δ} by the following Tikhonov functional,

$$H_{\alpha}(g_{\alpha}^{\delta}) = \left\| g_{\alpha}^{\delta}(t) - g^{\delta}(t) \right\|^2 + \alpha \left\| \frac{d^2 g_{\alpha}^{\delta}}{dt^2}(t) \right\|^2, \tag{14}$$

where g_{α}^{δ} is the selected data obtained from the minimation (14) with the appropriate regularization parameter α. Evantualy in the numerical process, we can obtain a stabilize data $g_{\alpha}^{\delta}(t)$ by minimizing the functional (14) as

$$\underline{g}_{\alpha}^{\delta} = \left(I + \alpha R^T R \right)^{-1} \left(\underline{g}^{\delta} \right), \tag{15}$$

where R is the regularization matrix to be used as the second order derivative as

$$R_2 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}_{(N+1)\times(N+1)}.$$

Here, we are using the scaling technique to avoid the large value by omitting the step size term $\frac{1}{(\Delta t)^2}$. Thus stabilized data g_α^δ is now used to approximate the first-order derivative $\left(g_\alpha^\delta\right)'$ by using the finite difference method as following formula:

For $i = 0$,

$$\left(g_\alpha^\delta\right)'(t_i) = \frac{g_\alpha^\delta(t_{i+1}) - g_\alpha^\delta(t_i)}{\Delta t},$$

for $i = 1, 2, \ldots, M-1$,

$$\left(g_\alpha^\delta\right)'(t_i) = \frac{g_\alpha^\delta(t_{i+1}) - g_\alpha^\delta(t_{i-1})}{2\Delta t},$$

For $i = M$,

$$\left(g_\alpha^\delta\right)'(t_i) = \frac{g_\alpha^\delta(t_i) - g_\alpha^\delta(t_{i-1})}{\Delta t},$$

Therefore, the source function $f(t)$ can be calculated as the formula in step 5 of (12) as $f(t) = (g_\alpha^\delta)'(t) - w_x(x_f, t).$

VI. Numerical Example

In this section, we present a benchmark test example to illustrate the accuracy of the method presented in the previous section. In order to review the accuracy of the numerical result, we introduce the root mean square error (RMSE) defined as

$$\text{RMSE}(f(t)) = \sqrt{\frac{T}{M+1}\sum_{i=0}^{M}\left(f_{\text{exact}}(t_i) - f_{\text{numerical}}(t_i)\right)^2}.$$

In this example, we consider the inverse problem (1)-(4), with $T = 1$, the input data are given as

$$u(x,t) = u_0(x) = x^2 \quad \text{for } x \in [0,1]$$

$$u_x(0,t) = 0 = s(t), \text{ and } u_x(1,t) = 2 = l(t), \quad t \in [0,1]$$

and the additional condition is given by

$$u\left(\frac{1}{2}, t\right) = g(t) = \frac{1}{4} + 2t + \sin(2\pi t), \quad t \in [0,1].$$

The number of discretization of space x and time t are $N = 10$ and $M = 30$, respectively. We investigate the solution $f(t)$ along the noisy data $g^\delta(t)$, contamined as

$$\underline{g}^\delta = \underline{g} + random('Normal', 0, \sigma, 1, M),$$

where the $random('Normal', 0, \sigma, 1, M)$ is a command in MATLAB generating randomly the variable from normal distribution with zero mean and standard deviation σ which is taken to be $\sigma = p \times \max_{0 \le t \le T}|g(t)| = 2.25p$, and p is the percentage of the error. A regularization formula holds regularization parameter α, basically, a regularization parameter $\alpha > 0$ controls the neighborhood properties of the auxiliary problem. Larger values of α indicates higher stability of the approximate solution but this makes the auxiliary problem being far from the original one. While values of α near zero expresses the auxiliary problem close to the original one, but this leads to become still unstable as $\alpha \to 0$. Hence, the suitable regularization parameter has to be chosen carefully with consideration between the conflicting purpose of stability and approximating, [15]. Actually, there are many methods to choose the regularization parameter α such as the discrepancy principle criterion, the generalized cross-validation (GCV) or the L-curve method. Nevertheless in this study, the regularization parameter α is chosen according to the trial and error. This means that we consider the error in each value cases and then select the regularization parameter which yields the smallest error. In order to illustrute the accuracy of the method, the analytical solution of this inverse problem is given as

$$f(t) = 2\pi\cos(2\pi t), \quad t \in [0,1].$$

For the forward problem (1)-(3), $f(t)$ is known function, we have tried to solve the forward problem by using the FIM (OLA) together with the FDM, we first discretize the first derivative of u with respect to t by FDM, then take the integration twice over its discretized equation and yield

$$A^2\underline{u}_j - \Delta t\underline{u}_j = A^2\underline{u}_{j-1} + \frac{\Delta t}{2}f_j\underline{x}^2 + c_0\underline{x} + c_1\underline{i},$$

where c_0 and c_1 are integral constants, $\underline{x} = [x_0, x_1, \ldots, x_N]^T$ and $\underline{i} = [1, 1, \ldots, 1]^T$. Although we do not present the temperature result $u(x,t)$ graphicaly yet, we can even know how good the method for solving the forward problem as its mean average error is less than 1%.

The powerful method extends to the inverse problem. Firstly, we consider the case of exact data, i.e. no noise is added to the additional condition. The analytical and numerical solutions of $f(t)$ are displayed in Figure 1. This can be clearly seen that the proposed method in this study can capture the heat source term $f(t)$ in very good agreement with $\text{RMSE} = 0.0324$ as shown in table 1.

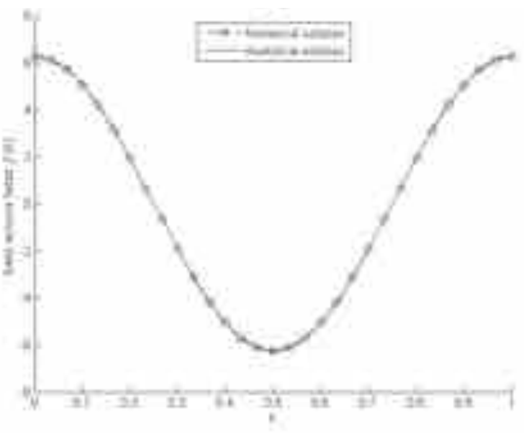


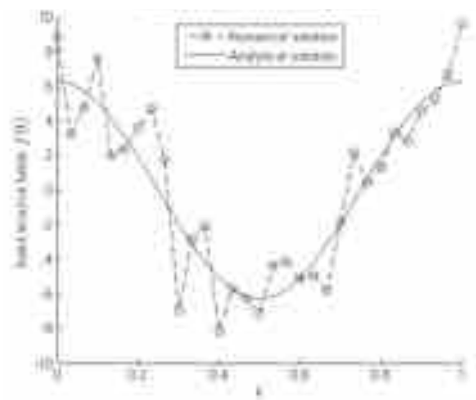
Figure 12. The analytical and numerical result of $f(t)$ for the exact data.

TABLE IV. THE VALUE OF λ AND RMSE FOR $f(t)$ AND $g(t)$

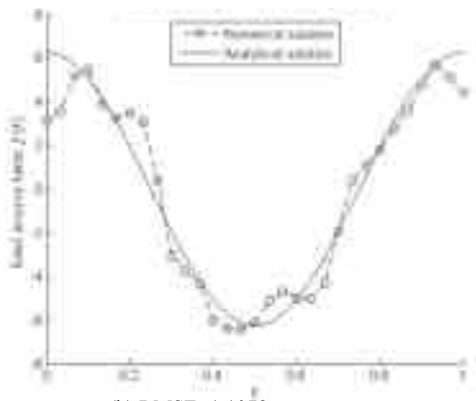
p	λ	RMSE of $g(t)$	RMSE of $f(t)$
0%	$\lambda = 0$	0	0.0324
3%	$\lambda = 0$	0.093977	1.9807
3%	$\lambda = 0.6$	0.062605	1.1078
3%	$\lambda = 3.1$	0.066401	1.3482

In the case of noisy data, as we have mentioned earlier, we add noise to the over-determination condition (4) with 3% noisy input. Then now the specific temperature is purtubed as g^{δ}. Figure 2(a) displays the numerical result of $f(t)$ obtained by using the algorithm introduced in Section 3 with $p = 3\%$ noisy input and with no regularzation, i.e. $\lambda = 0$. This can be seen that the numerical solution is inaccurate unstable since a 3% small pertubution causes significant error in the solution. In order to retrieve this issues, we then employ the Tikhonov regularization method that we have mentioned in the Section 4. By the trial and error of selecting the regularization parameter among 10^{-6} to 1, we found that $\lambda = 0.6$ is the most suitable regularization parameter for this problem.

Figure 2 illustrates the numerical results obtained when applying the second-order Tikhonov regularization. From Figure 2(b) we can observe that the numerical results are alleviated, compare to Figure 2(a). In addition, the smoothest result for this example can be obtained when setting $\lambda = 3.1$ and that is shown in Figure 3. This can be seen that the interior point of numerical solution, i.e. $t \in [0.1, 0.9]$ approximately, is more accurate and stable, whereas the starting and end point on $t \in \{0, 1\}$ are getting far away from the exact one.



(a) RMSE=1.9807, $p = 3\%$, $\lambda = 0$



(b) RMSE=1.1078, $p = 3\%$, $\lambda = 0.6$

Figure 13. The analytical and numerical result of $f(t)$ with $\lambda = \{0, 0.6\}$

The inaccuracy at both starting and end points is frequently found elsewhere when using stabilizing technique such as the Tikhonov regularization method. Accordingly, this is obviously seen that the FIM (OLA) and Tikhonov regularization can be used to deal with the inverse problem. In [13] Xiangtuan *et. al.* combined the direct numerical method with the finite difference method for solving this kind of problems. The method really works well with appropriate step length but this has one drawback: This algorithm always needs requirement of step length i.e. $\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$. This can be noted that with $N = 10$ and $M = 30$, which is not satisfied the above requirement. As a long to the requirement of step lenght, to set up $N = 10$ we need to put $M = 300$ which is a large number of time discretization and it also makes long computational time. Then we can conclude here that the FIM is a success method to deal with time-dependent inverse heat source, without any requirement of step lenght.

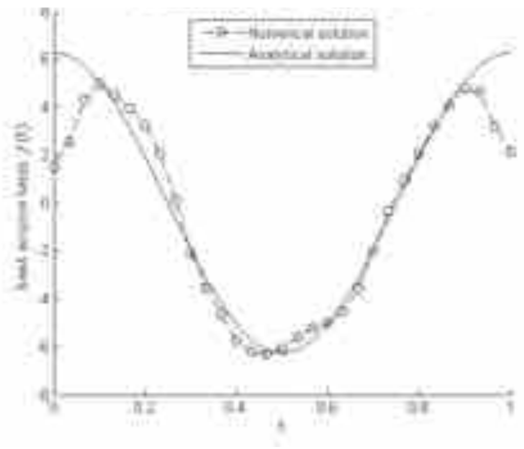


Figure 14. The analytical and numerical result of $f(t)$ with $\lambda = 3.1$

VII. Conclusion

The inverse problem of finding the time-dependent source function has been discussed. The inverse heat source problem, with two unknown, has been transformed to be a forward problem, with an unknown by employing the direct numerical method suggested by [13]. The numerical discretization of the forward problem was based on the finite integration method combined with the backward finite difference. Since to obtain the unknown source function $f(t)$ holds the first-order derivative $g(t)$ which is an observed data containing measurement errors. Furthermore, the Tikhonov regularization method has been employed together with the trial and error for selecting the suitable regularization parameter. The numerical result was found to be accurate and stable at interior points, yet not at both starting and end points.

Acknowledgment

Our massive gratitude goes to Faculty of Science and Technology, Prince of Songkla University for supporting the study with SAT-ASEAN Scholarship for international students and Graduate School, Prince of Songkla University for providing research funding.

References

[1] A. Farcas and D. Lesnic. The boundary-element method for the determining of a heat source dependent on one variable. Journal of Engineering Mathematics, 2006, pp. 375-388.

[2] A. Hasanov. Simultaneous determination of source terms in a linear parabolic problem from the final overdetermination: Weak solution approach.. Journal of Mathematical Analysis and Applications, 2007, pp. 766-779.

[3] A. Hazanee., M. I. Ismailov, D. Lesnic and N. B. Kerimov. An inverse time-dependent source problem for the heat equation. Applied Numerical Mathematics, 2013, pp. 13-33.

[4] A. Hazanee, D. Lesnic, M. I. Ismailov and N. B. Kerimov. An inverse time-dependent source problem for the heat equation with a non-clasical boundry condition. Applied Mathematical Modelling, 2015, pp. 6258-6272.

[5] M. I. Ismailov, F. Kanca, D. Lesnic. Determination of time-dependent heat source under nonlocal boundary and integral overdetermination conditions. Applied Mathematics and Computation, 2011, pp. 4138-4146.

[6] N. I. Ivanchov. On the determination of unknown source in the heat equation with nonlocal boundary conditions. Ukranian Mathematics Journal, 1995, pp. 1647-1652.

[7] T. Johansson and D. Lesnic. A variational method for identifying a spacewise-dependent heat source. Journal of Applied Mathematics, 2007, pp. 748-760.

[8] M. Li, Y. C. Hon, T. Korakianitis and P. H. Wen. Finite integration method for nonlocal elastic bar under static and dynamic loads. Engineering Analysis withBoundary Elements, 2013, pp. 842-849.

[9] M. Li, Y. C. Hon, Z. L. Tian, C. S. Chen and P. H. Wen. Improved finite integration method for partial differential equations. Engineering Analysis with Boundary Elements, 2016, pp. 230-236.

[10] S. Lu and Y. Wang. First and second order numerical differentiation with Tikhonov regularization. Numerical Mathematics, 2006, pp. 354-367.

[11] I. Malyshev. An inverse source problem for heat equation. Journal of Mathematical Analysis and Application, 1989, pp. 206-218.

[12] P. H. Wen, Y. C. Hon, M. Li. and T. Korakianitis. Finite integration method for partial differential equations. Applied Mathematical Modelling, 2013, pp. 100092-10106.

[13] X. Xiangtuan, Y. Yaomei and W. Junxia. A direct numerical method for solving inverse heat source problems. Journal of Physics: Conference Series, 2010, pp. 1-9.

[14] L. Yan. C. L. Fu and F. L. Yang. The method of fundamental solution for the inverse heat source problem. Engineering Analysis with Boundary Elements, 2008, pp. 216-222.

[15] T. Schuster, B. Kaltenbacher, B. Hofmann and K. S. Kazimierski. Regularization Methods In Banach Spaces. Berlin: Gruyter, 2012, pp.56–57.