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Binary Search based Boundary Elimination Selection in Many-objective Evolutionary Optimization ☆

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Abstract

In many-objective optimization, the balance between convergence and diversity is hard to maintain, while the dominance resistant solutions (DRSs) could further harm the balance particularly in high-dimensional objective space. Thus, this paper proposes a novel selection strategy—boundary elimination selection based on binary search (called BESBS), trying to avoid the impact of DRSs during the optimization and achieve a good balance between convergence and diversity simultaneously. During the environmental selection, the binary search (BS) is used to adaptively adjust the ϵ value in the ϵ-dominance relationship and assist in detecting the well-distributed neighbors for the elite solutions. Then the ϵ value obtained by BS is used for serving the boundary elimination selection (BES) to guarantee the stability of the elite population. To improve the convergence, BES is mainly designed to select individuals approximating to the ideal point. By modifying the fitness of solutions and choosing solutions in terms of the shuffled sequence of objective axis, the DRSs will be eliminated during the selection. Thus, BESBS could achieve a good balance between convergence and diversity and avoid the impact from DRSs simultaneously. From

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a series of experiments with 35 instances, the experimental results have shown that BESBS is competitive against 8 state-of-art many-objective evolutionary algorithms.

Keywords: Many-objective optimization problems, boundary elimination selection, binary search

1. Introduction

In the real world, there are large numbers of multi- and many-objective (the number of objectives more than 3) optimization problems (MOPs) which are commonly non-linear and highly complicated, and their objectives usually conflict with each other. As a consequence, no single optimal solution exists but a set of trade-off Pareto solutions can be found to the decision maker (DM) for MOPs [1], [2].

In the past two decades, multi-objective evolutionary algorithms (MOEAs) and other population-based meta-heuristics have been demonstrated to be useful for solving MOPs, like NSGA-II [3], SPEA-II [4], MOEA/D [5], GrEA [6], to name but a few. Among various selection strategies, the Pareto dominance based sorting approaches are widely and effectively used in dealing with MOPs with 2 and 3 objectives [7], but their efficiency will seriously degrade with the increasing number of objectives. One reason for the deterioration is that most solutions tend to be non-Pareto dominant in a high-dimensional objective space [8]. Another reason is that the diversity maintaining mechanisms may lead to the population crowded with dominance resistant solutions (DRSs), therefore resulting in the deterioration of convergence and diversity [9], [10].

Thus, a number of techniques or ideas have been proposed to enhance the ability of MOEAs to obtain well-converged and well-distributed solutions in the many-objective optimization, which can be roughly classified into three categories [11], [12]. The first group is to modify the traditional Pareto dominance relationship, like enlarging the dominant areas of a solution, so as to improve the efficiency in distinguishing the relationship between the non-Pareto dominant solutions, such as ϵ-dominance [13], [14], angle-dominance [15], fuzzy dominance [16], [17], a grid dominance [6], preference order ranking [18], [19], and other ranking methods such as [20], [21].

The second category aims to select solutions based on indicators or metrics. Obviously, the selected solutions are highly qualified on demand of the indicators. There are three widely-used indicators based MOEAs such as IBEA [22], SMS-EMOA [23], and HypE [24]. IBEA predefines a goal of each solution to measure its contribution with ϵ indicator [25] or hypervolume (HV) indicator [26]. Both SMS-EMOA and HypE are based on HV estimation, where SMS-EMOA calculates the exact contribution to hypervolume indicator while HypE assigns a fitness based on approximations to the hypervolume indicator. The main advantage of the indicators based methods is not subject to the problems that the pareto-dominance based MOEAs encounter, but their computational cost is relatively expensive.

The third category is based on aggregation or preference information. Both MOEA/D [5] and MSOPS [27] need a set of uniform target vectors so as to convert the many-objective problem into a set of single problems. In MOEA/D, some neighbor-collaboration strategies between the sub-problems are introduced. MSOPS tries to find the best subproblem for a solution. In NSGA-III [28], the non-dominated solutions close to the reference points have high priority to be selected. On the basis of $\mu + \lambda$ framework of MOEA, RVEA [29] applies the reference vector guided selection and the reference vector adaptation to keep the balance between the convergence and diversity. There are some other preferences based methods like [30], [31], [32].

There are also large numbers of many-objective optimization algorithms not belonging to the above categories like [9], [33]. In [9], the diversity management operator (DMO) does not maintain the diversity until the solutions converge into the Pareto optimal front. In [33], a shift-based density estimation strategy is proposed to penalize poorly converged solutions by assigning them a high density value. Furthermore, some researchers attempted to address many-objective problems by applying a reduced set of objectives [34], [35], [36].

Although the above studies have achieved great progress and provided some inspirations to deal with many-objective problems, more efforts are needed. Methods such as [22], [23] have to specify different parameters subject to varying problems. Also, optimizing two important features (convergence and diversity) separately may hard maintain the balance between diversity and convergence [3], [4]. In addition, how to avoid the negative influence of DRSs in the optimization is a challenge.

To deal with the above problems, this paper proposed a new selection strategy (boundary elimination selection) based on the binary search, in order to avoid the impact of DRSs during the optimization and achieve a good balance between convergence and diversity simultaneously. Following the framework of NSGA-II, some changes have taken place in the environmental selection. Firstly, the binary search (BS) is designed to adaptively adjust the ϵ value (in the ϵ-dominance relationship) for the selection of well-distributed elite solutions, and the value will be used in the boundary elimination selection (BES) to keep the stability of the elite population during the detection stage. Then, by modifying the fitness of solutions, the BES is designed to select the well-approximated solutions according to the shuffled indexes of coordinate axis in turns. During the selection, the DRSs will be eliminated by the adjacent selected solutions close to different coordinate axis. Thus, the advantage of the proposed algorithm is to keep the balance between convergence and diversity and avoid the impact of DRSs simultaneously.

The remainder of this paper is organized as follows. In Section II, the background is presented and discussed. Section III presents the related work and motivations in this paper. Section IV is devoted to the description of the proposed algorithm. Section V presents the algorithm settings, test functions, and performance metrics used for performance comparison. The experimental results and discussions are given in Section VI. Finally, Section VII provides some concluding remarks along with further investigation.

2. Background

In this section, some basic concepts and definitions in multi-objective optimization will be introduced; then ϵ-dominance relationship used in our algorithm will presented in detail.

2.1. Multi-objective optimization

A multi-objective optimization problem (MOP) can be stated as follows:

$$\begin{aligned} \text{minimize } F(p) = (f_1(p), f_2(p), \cdots, f_m(p))^T \\ \text{subject to } p \in \Omega. \end{aligned} \tag{1}$$

where p is a solution vector in Ω (the decision or variable space), and $F : \Omega \rightarrow \mathbb{R}^m$ consists m real-valued objective functions. $\mathbb{R}^m$ is the objective space.

Definition (Pareto dominance relationship): Let $p^1, p^2 \in \Omega$, $p^1 \prec p^2$ (p^1 dominates p^2), iff:

$$\begin{aligned} \forall i \in (1, 2, \cdots, m), f_i(p^1) \leq f_i(p^2) \\ \exists j \in (1, 2, \cdots, m), f_j(p^1) < f_j(p^2) \end{aligned} \tag{2}$$

where the optimal solution set in Ω is denoted to be the Pareto-optimal set (PoS), and the corresponding solution set in $\mathbb{R}^m$ is referred to as Pareto-optimal front (PoF) as shown in Fig.1.

In Fig.1, individuals p^1, p^2, p^3 in objective space, $p^1 \prec p^3$ and $p^1 \prec p^2$ because $f_1(p^1) < f_1(p^3) \wedge f_2(p^1) < f_2(p^3)$, $f_1(p^1) < f_1(p^2) \wedge f_2(p^1) < f_2(p^2)$. But p^3 and p^2 are non-dominated to each other because $f_1(p^3) < f_1(p^2)$ but $f_2(p^2) < f_2(p^3)$.

2.2. ϵ-dominance relationship

Definition (ϵ-dominance relationship)[13]: Let individual $p^1, p^2 \in \mathbb{R}^m$. Then p^1 is said to ϵ-dominate p^2 for $\epsilon > 0$, denoted as $p^1 \prec_\epsilon p^2$, if:

$$\forall i \in \{1, 2, \cdots, m\} : (1 - \epsilon) f_i(p^1) \leq f_i(p^2) \tag{3}$$

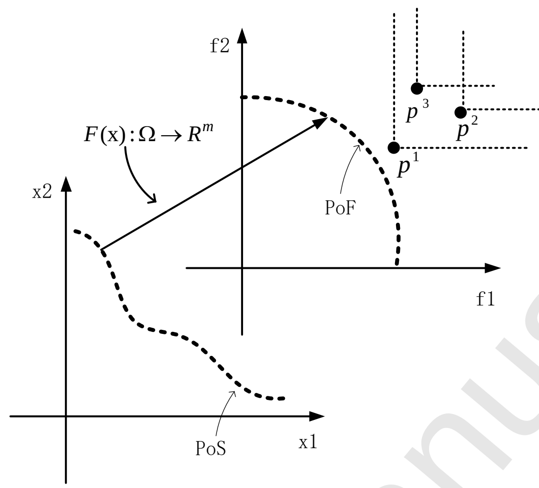


Figure 1: Illustration of the relationship between the PoS and PoF, and the Pareto dominance relationship.

Definition (ϵ-approximate Pareto set):Let $\mathbb{S} \subset \mathbb{R}^m$ be a set of vectors and $\epsilon > 0$. Then a set $\mathbb{S}_\epsilon$ is called an ϵ-approximate Pareto set of $\mathbb{S}$, if any vector $p \in \mathbb{S}$ is ϵ-dominated by at least one vector $q \in \mathbb{S}_\epsilon$, i.e.

$$\forall p \in \mathbb{S} : \exists q \in \mathbb{S}_\epsilon, st.q \prec_\epsilon p \tag{4}$$

The set of all ϵ-approximate Pareto sets of $\mathbb{S}$ is denoted as $P_\epsilon(\mathbb{S})$.

In Fig.2, $p^1, p^2, p^3 \in P$ and $P \subset \mathbb{R}^m$. The corresponding $p^1_\epsilon, p^2_\epsilon, p^3_\epsilon \in P_\epsilon$ and $P_\epsilon \subset \mathbb{R}^m$, where $\overrightarrow{p^j_\epsilon} = (1-\epsilon)\overrightarrow{p^j}$, and $j \in \{1, 2, 3\}$. According to Pareto dominance relationship, $p^1 \prec p^2$, p^1 and p^3 are non-dominated solutions. However, according to ϵ-dominance relationship, $p^1 \prec_\epsilon p^2$ and $p^1 \prec_\epsilon p^3$ because $p^1_\epsilon \prec p^2$ and $p^1_\epsilon \prec p^3$.

3. Related work and motivation

3.1. Related work

In this paper, more emphases are put on dealing with many-objective optimization problems. Two key issues are concerned during the optimization,

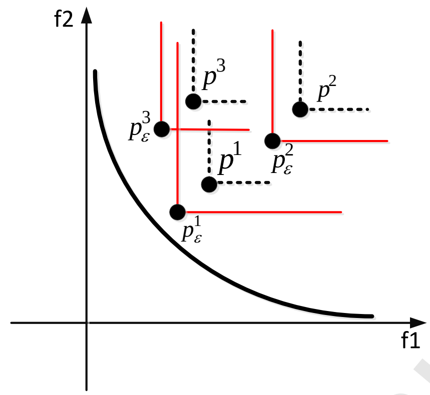


Figure 2: Illustration of ϵ-dominance relationship.

namely, convergence and diversity. On the one hand, the convergence is highlighted to evaluate how much the obtained solutions are close to the Pareto optimal front. On the other hand, the diversity is also important to provide more choices to the decision maker, which is to evaluate the solutions distributing to the whole Pareto optimal front.

As mentioned in Section 1, different sorts of strategies are proposed to address these two issues, where the strategy to modify the dominance relationship is interesting to be investigated, especially the ϵ-dominance relationship [13]. There are many researches aiming at the modification of the ϵ-dominance relationship, so as to balance diversity and convergence during the optimization. [37] modifies the granularity of the hypergrids especially on the horizontal or vertical regions of the Pareto front, so as to retain more solutions in the blank areas in terms of the ϵ-dominance relationship. In other words, during the optimization, the algorithm considers different ϵ-dominance regions depending on the geometrical characteristics of the Pareto-optimal front, so as to change the density of the hypergrids in different regions. The shortage is hard to maintain the stability of the algorithm when the landscape of PoF is disconnected or mixed with convex and concave regions. In GDE-MOEA [38], the I_{GD}-selection is used as a convergence strategy and ϵ-selection mechanism as diversity strategy to explore the searching space. The ϵ value keeps decreasing until the specified number of solutions are obtained from the elite solution set, in terms of $\epsilon_j = (f_j^{max} - f_j^{min})/k$ by increasing the integer value of k. [39] is motivated similarly to the [37] to adaptively adjust the ϵ value according to the geometrical characteristics of the Pareto-optimal front. The shortage is to estimate the p value of the curve family associated to the Pareto-optimal front. There are also some other measures to adjust the ϵ value in the optimization like [40], [41].

Thus, the modification of the ϵ-dominance relationship in different researches play different roles in the optimization. It can be used in the exploration of the searching space, or maintenance of the distribution or diversity of the solutions, or enhancement of the convergence. And this paper also tries to adaptively adjusting the ϵ value, but the aim of this paper is to improve the ϵ-dominance relationship to sort the solutions during the environmental selection for the preparation of the boundary elimination selection, which could enhance the convergence, and meanwhile maintain the diversity to some extent.

3.2. Motivation

As illustrated in [42], [43], following the basic framework of NSGA-II [3], enhancing the diversity-promoting selection mechanisms has been identified as highly influential and useful to the optimization especially in dealing with many-objective problems [6], [28], etc. On the contrary, the diversity-promoting selection mechanisms also can do harm to the optimization with two reasons. First, obtaining a good diversity is relatively easy especially in a high-dimensional space, but the best diversity could easily favor these solutions with poor proximity. Second, back to the first selection applying the Pareto dominance to enhance the convergence, if the number of nondominant solutions is large, then the balance between convergence and diversity solely relies on the second selection, namely the diversity-promoting selection mechanisms. At the moment, the optimization process can come to standstill especially if there are some Pareto resistant solutions in the population.

Due to the above issues, this paper proposes a novel selection strategy—boundary elimination selection based on binary search (called BESBS), trying to avoid the impact of DRSs during the optimization and achieve good balance simultaneously. Following the basic framework of NSGA-II, during the environmental selection, the binary search (BS) is firstly used to assist in detecting well-distributed neighbors of elite solutions by adaptively adjust the ϵ in the ϵ-dominance relationship [13], [14], which serves for the boundary elimination selection (BES) by providing a suitable value so as to guarantee the stability of the size of the elite population. BES is mainly designed to select individuals approximating to the ideal point. By modifying the fitness of solutions and choosing solutions in terms of the shuffled sequence of objective axis, the DRSs will be eliminated in the selection. Thus, it could achieve a good balance between convergence and diversity and avoid the impact from DRSs simultaneously.

Thus, the aim of this paper is to propose a new idea to deal with multi- and many-objective problems that consider to retain the convergence information when promote the diversity during the second selection, thereby obtaining a good balance between convergence and diversity.

4. Proposed algorithm

In this section, we firstly present the overall framework of the proposed algorithm based on boundary elimination selection (BES) and binary search (BS). Next, we will describe the strategies of BES and BS in detail. Finally, we will analyze the computational complexity of the proposed algorithm.

4.1. Overall framework of the proposed algorithm

Algorithm 1 presents the overall framework of BESBS. In this frame, we adopt $\mu + \lambda$ strategy in common with NSGA-II [3]. Firstly, BESBS randomly generates the initial population P. Then the offspring population Q is obtained from P through evolutionary operations: mating selection (like 2-tournament

Algorithm 1 : Overall Framework of BESBS

Input: Population Size: n, Terminate Condition: $\mathcal{T}$

Output: Population: $P = \{x_1, x_2, \cdots, x_n\}$

$P = RandomInitiate\,(n)$
$Evaluation\,(P)$
while $\neg\mathcal{T}$ **do**
  $Q = MatingSelection\,(P)$
  $Q = Crossover\,(Q)$
  $Q = Mutation\,(Q)$
  $Evaluation\,(Q)$
  $R = P \cup Q$
  $P = EnvironmentalSelection\,(R, n)$
end while

selection [44]), crossover operation (like simulate binary crossover [45]), mutation operation (like polynomial mutation [1]). After that, unite populations P and Q like $R = P \cup Q$. The next step is environmental selection, selecting elite individuals from united population R and updating the next generation population P. Then repeat the above steps until reach the termination. Finally, when the termination is satisfied, the output is the final obtained population.

It can be seen from the framework that the final population inherits the elite genes after a series of environmental selections. In other words, it updates the new generations by choosing the elite solutions into the new-born populations. However, in the many-objective optimization, the mutual relations between the solutions are hard to be differentiated during the environmental selection since most solutions are non-dominated in Pareto dominance sense. Moreover, the population sometimes can be optimized slowly as the dominance resistant solutions (DRSs) appear easily and will damage the environmental selection seriously. Thus, how to choose elite solutions properly during the selection is the key issue in dealing with the many-objective problems, and the main efforts of this paper are also made to improve the environmental selection.

Algorithm 2 : Environmental Selection

Input: Population: $R = \{x_1, x_2, \cdots, x_l\}$, Output Population Size: $n \leq l$

Output: Population: $P = \{x_1, x_2, \cdots, x_n\}$

1: $P = \emptyset$

2: $\mathscr{L} = NondominateSort\,(R) = \{L_1, L_2, \cdots\}$

3: **for each** $L_\imath \in \mathscr{L} \wedge |P| < n$ **do**

4:     **if** $|P| + |L_\imath| \leq n$ **then**

5:         $P = P \cup L_\imath$

6:     **else**

7:         $P = P \cup BS\,(L_\imath, n - |P|)$

8:     **end if**

9: **end for**

4.2. Environmental selection

As shown in Algorithm 2, firstly, the fast non-dominated sorting strategy [3] is applied to sort all the solutions in the mixed population R into different layers like $\{L_1, L_2, \cdots\}$. Then, add the solutions (from the sorted layers) into the elite population P as shown in Line 5. Notably, when $(|P| + |L_\imath| > n) \wedge (|P| < n)$, then select $n - |P|$ solutions from the critical layer $L_\imath$.

In NSGA-II [3], the fast non-dominated sorting strategy is used to sort the well-converged solutions out, then the crowding distance is applied to maintain the diversity of the solutions in the critical layer, so as to reach the balance of convergence and diversity. However, on the condition that the solutions are non-dominant, the diversity mechanism would play a more important role in the environmental selection. Thus, when it comes to deal with many-objective problems since it is often only one layer after the sorting process, the method coping with the critical layer should be better designed not only to maintain the diversity but also to enhance the convergence meanwhile. On the other hand, in the many-objective optimization, the diversity maintain strategy could lead to a dilemma that more DRSs will easily appear and do harm to convergence and diversity in return.

Therefore to the above points, this paper adopts the binary search (BS) especially designed in the environmental selection to deal with the critical layer as shown in Line 7 of Algorithm 2. Instead of the crowding distance strategy, the aim of BS is designed to maintain the size of population and prompt diversity and convergence meanwhile. The following subsection will detail the strategy of BS.

4.2.1. Binary search

As shown in Algorithm 3, the environmental selection adopts the binary search (BS) to adjust the ϵ value during the rehearsal stage. The stage is mainly to search for a proper ϵ value for the boundary elimination selection with two reasons. On the one hand, it is known that the diversity of the remained solutions depends on the value of ϵ [13]. Besides, the properly adjusted ϵ value can keep the size of the elite population stable after the selection. From another aspect, the BS serves for the boundary elimination selection by providing a proper ϵ value.

In Algorithm 3, firstly, the upper limit and the lower limit values of ϵ are obtained respectively in terms of the solutions in the critical layer, where the upper limit is the maximum spread of all objectives as shown in Line 2, and the lower limit is 0. Then, in each round, the binary search adjusts the upper limit and the lower limit values in terms of the size of output population $|P|$. Specifically, if the ϵ value is too large, then more solutions will be dominated and less solutions will be remained after the boundary elimination selection, and vise versa. From Line 4 to Line 17, if $|P| < n$ (which denotes that the ϵ value is too large that extra adjacent individuals are dominated), thus, the upper limit will be reduced by $\epsilon_{\uparrow} = \frac{\epsilon_{\uparrow}+\epsilon_{\downarrow}}{2}$; if $|P| > n$, the lower limit should be increased like $\epsilon_{\downarrow} = \frac{\epsilon_{\uparrow}+\epsilon_{\downarrow}}{2}$; otherwise, the ϵ is found and return the value. After r[1] rounds, the final ϵ is obtained. Finally, the obtained ϵ will be applied into the boundary elimination selection to select demanding size of elite solutions

[1] r is set to be 20.

Algorithm 3 : Binary Search (BS)

Input: Population: $F = \{x_1, x_2, \cdots, x_l\}$, Output Population Size: $n \leq l$, Rounds: r

Output: Population: $P = \{x_1, x_2, \cdots, x_n\}$

$\epsilon_{\downarrow} = 0$

$\epsilon_{\uparrow} = \max_{\imath \in \{1,2,\cdots,m\}} \left(\max_{x_{\uparrow} \in F} f_{\imath}(x_{\uparrow}) - \min_{x_{\downarrow} \in F} f_{\imath}(x_{\downarrow}) \right)$

$P = \emptyset$

while $r > 0$ **do**

  $\epsilon = \frac{\epsilon_{\uparrow} + \epsilon_{\downarrow}}{2}$

  $F = F \cup P$

  $P = \emptyset$

  $P = BES(F, \epsilon)$

  if $|P| < n$ **then**

    $\epsilon_{\uparrow} = \epsilon$

  else if $|P| > n$ **then**

    $\epsilon_{\downarrow} = \epsilon$

  else

    return

  end if

  $r = r - 1$

end while

$F = F \cup P$

$P' = \emptyset$

while $|P'| < n$ **do**

  $P' = P' \cup BES(F, \epsilon_{\uparrow})$

end while

$P = \{x'_1, x'_2, \cdots, x'_n\} \subset P'$

into the new population P as shown from Line 20 to Line 23.

In conclusion, the binary search consists of two main parts. The first part is to find the most suitable ϵ for the boundary elimination selection. The second part is to apply the boundary elimination selection to choose elite individuals from F (the critical layer) in Line 21 of Algorithm 3. The first part serves for the second part by providing a suitable ϵ. It is noted that one advantage of the binary search is to avoid the setting of ϵ when encountering different problems, and it is only related to the boundary elimination selection and population size. The contributions of the boundary elimination selection will be illustrated in the following section.

4.2.2. Boundary elimination selection

The procedure of the boundary elimination selection (BES) is presented in Algorithm 4. The method is mainly designed to choose the elite solutions with good distribution and good convergence from the critical layer, and try to avoid the impact of the dominance resistance solutions (DRSs [2]) during the selection.

Definition (Fitness of a solution): Let individual $x \in \Omega$, the fitness of x is defined as follows:

$$F_a(x) = d_1 + \lambda * d_2 \tag{5}$$

where

$$d_1 = \sqrt{\Sigma_{\imath \in \{1,2,\cdots,m\} \setminus \{a\}} f'_\imath(x)^2}$$
$$d_2 = f'_a(x)$$

and $a \in \{1, 2, \cdots, m\}$ (m is the number of objectives). $\lambda \in (0, 1]$ is a penalty parameter. $\imath \in \{1, 2, \cdots, m\} \setminus \{a\}$ means $\forall \imath \in \{1, \cdots, a-1, a+1, \cdots, m\}$.

[2]DRS is a dominance resistance solution far away to the optimal front and cannot be Pareto dominated by any other solutions in the population.

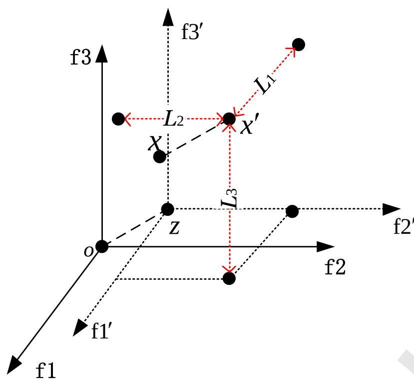


Figure 3: Illustration of the fitness in 3-objective model. z is the ideal point. $z - f1^{'}f2^{'}f3^{'}$ is the translational coordinate system, and $x^{'}$ is the translated solution of solution x. L_1, L_2, L_3 are the distances from $x^{'}$ to the planes respectively.

$f_\imath'(x)$ is defined as follows:

$$f_\imath'(x) = f_\imath(x) - \min_{x' \in \Omega} f_\imath(x') \quad \imath = \{1, 2, \cdots, m\} \tag{6}$$

From Eq. 5, the fitness of a solution x is composed of two parts (d_1 and d_2). Previously, it is to normalize each individual in the population, which is to make sure that the individuals are located in the first quadrant according to the Eq. 6. In other words, the original point is translated to the ideal point in the transferred coordinate system. $d_1 = \sqrt{\Sigma_{\imath \in \{1,2,\cdots,m\} \setminus \{a\}} f_\imath'(x)^2}$ denotes the distance from the point x to the ideal point [3] on the $m-1$ dimensional hyperplane of $O - f_1 \cdots f_m \setminus f_a$, and $d_2 = f_a'(x)$ is the distance from the point of x to the ideal point in the sense of the translated coordinate axis f_a. For example, in the translational coordinate system $z - f1^{'}f2^{'}f3^{'}$ in Fig. 3, if f_1 was selected, then $d_1 = \sqrt{{L_2}^2 + {L_3}^2}$ and $d_2 = L_1$; else if f_2 was selected, then $d_1 = \sqrt{{L_1}^2 + {L_3}^2}$ and $d_2 = L_2$.

[3]ideal point: $z = (\min_{i=1}^n f_1(x^i), \cdots, \min_{i=1}^n f_m(x^i))$, where m, n are the number of objectives and individuals respectively.

It is known that the DRSs are the non-dominated solutions with big values in some specific objectives, and easily appear in the corner or boundary of the objective space. So this is the difference between them and normal elite solutions. Therefore, during the selection process, the a in Eq. 5 will be selected from the shuffled sequence set of $\{1, 2, \cdots, m\}$ to reduce the impact from the DRSs since they can appear in different objectives. Besides, a penalty on the d_2 is set, which is to narrow the difference between DRSs and normal elite solutions. So there is a possibility that the DRSs can be dominated by others, especially by their adjacent solutions.

In the definitions (Eq. 5, 6), it can be seen that both d_1 and d_2 can reflect the convergence by estimating the distances to the ideal point from different aspects respectively. It means that smaller values of d_1 and d_2 denote that the solutions are more approximating to the ideal point but from different dimensions. On the other hand, when d_2 gets a penalty, it means that the superiority of the solutions on the a-th objective will be decreased, and more emphasis is put on d_1. Moreover, if the objectives are selected by turns, then the dominated position of the selected objective will be reduced to some extent. Thus, the impact of the DRSs will be eliminated. In return, then more trade-off solutions will be reserved in the population. Although the diversity of the population could be improved, in order to increase the convergence pressure, one way is to compare the adjacent solutions in a small group. Thereby, the boundary elimination selection (BES) is designed to choose elite solutions from a small region separated by a ϵ grid, so that the adjacent solutions with small difference will be eliminated.

From the above analysis, Algorithm 4 presents the framework of BES in detail. Firstly, it randomly shuffles the objective indexes sequence $A = \{1, 2, \cdots, m\}$ as shown in Line 3. For example, $(2, 1, 3)$ is the randomly shuffled sequence on 3-objective problem. Then, in Line 6, select index from the shuffled sequence set A in turns. According to the sequence set, calculate the fitness of each solution in the population via Eq. 5 and find the best solution x^* according to the fitness estimation in Line 7. Then, solution x^* is added into the next

Algorithm 4 : Boundary Elimination Selection (BES)

Input: Population: $F = \{x_1, x_2, \cdots, x_l\}$, Parameter ϵ

Output: Population: P

```
P = ∅
A = {1, 2, ···, m}
RandomShuffle(A)
F' = ∅
while F ≠ ∅ do
    a = CycleNext(A)
    x* = argmin_{x∈F} F_a(x)
    P = P ∪ {x*}
    F = F \ {x*}
    for each x ∈ F do
        if x* ≺_ε x then
            F' = F' ∪ {x}
            F = F \ {x}
        end if
    end for
end while
F = F ∪ F'
```

population P. From Lines 10-15, the solutions ϵ-dominated (definition 3) by x^* will be eliminated. After that, the next index will be selected and repeat the procedure until the population F is empty.

From Algorithm 4, after the sequence is determined at each round, it can be found that the algorithm will pick up the best solution in terms of the fitness function to enhance the convergence, and eliminate the adjacent solutions of the selected solution as to increase the diversity. Specifically, in terms of Eq. 5, 6, $x^* = argmin_{x \in F} F_a(x)$ favors the adjacent solution of the DRS close to the specified a-th coordinate axis because d_2 gets a penalty and solution with smaller d_1 thereby has higher priority. Thus it can be seen that such solutions are always distributed in the corner of the a-th coordinate axis, which denotes that it also favors the solutions with good spread performance to some extent. Thereby, the DRSs will be ϵ-dominated by the selected solutions close to the boundaries (coordinate axes). In other words, during the boundary elimination selection, the shuffled sequences enhance the selection by choosing the scattered solutions from different corners of coordinate axis to the middle objective space. To be noted, before the BES, ϵ value has been determined in the BS. During the BES, the ϵ value can not only maintain the stability of the population but also enhance the diversity.

In order to illustrate BES clearly, Fig. 4 gives an example . There are 15 solutions in the population, and the shuffled index sequence is $(2, 1)$. So, the coordinate axis f_2 was selected, which means that the distances from all solutions to the coordinate axis f_1 get a penalty. In other words, the solution closer to f_2 has higher priority to be selected. Thus, the first pick is the red point a_1 closest to coordinate axis f_2, meanwhile a_2 will be eliminated as $a_1 \prec_\epsilon a_2$. Similarly, the next pick is a_3. At the same time, a_4 and a_5 are deleted although a_3 cannot Pareto dominate a_5 but could ϵ-dominate a_5. Notably, a_4 is a dominance resistance solution (DRS). However, in this selection, a_3 is closer to the ideal point than a_4 so that solution a_3 is picked according to Line 7 in Algorithm 4, and a_4 will be eliminated as a_3 ϵ-dominates a_4. After 7 times selections, there are 7 individuals selected into the next generation as shown in

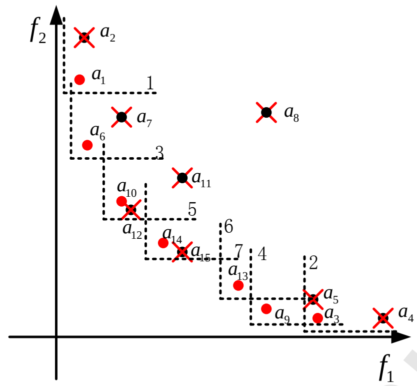


Figure 4: An example of BES. The red points are the reserved solutions into the next generation, and the points highlighted with a cross are eliminated. The number from 1-7 are the pick times.

Table 1: The result of the example with boundary elimination selection

Pick times	1	2	3	4	5	6	7
Sequences	f_2	f_1	f_2	f_1	f_2	f_1	f_2
Individuals	a_1	a_3	a_6	a_9	a_{10}	a_{13}	a_{14}

Table 1, such as a_1, a_3, a_6, a_9, a_{10}, a_{13}, and a_{14}.

From Algorithm 4 and Fig. 4, we can see that the boundary elimination selection tries to delete the adjacent individuals to keep diversity and to apply penalty based fitness to promote convergence. Besides, the shuffled sequence contributes to the BES to choose relatively scattered solutions in the corner and the DRSs will be eliminated to maintain convergence as well. Thus, during the selection, diversity and convergence could be maintained meanwhile.

4.3. Computational complexity analysis

The computational complexity of Algorithm 1 is mainly on mating selection and environmental selection.

Table 2: The settings of the test problems

Instances	No.of objectives (m)	No.of variables (n)	Parameter
DTLZ1	3,5,6,8,10	m-1+k	k=5
DTLZ2	3,5,6,8,10	m-1+k	k=10
DTLZ3	3,5,6,8,10	m-1+k	k=10
DTLZ4	3,5,6,8,10	m-1+k	k=10
DTLZ5	3,5,6,8,10	m-1+k	k=10
DTLZ6	3,5,6,8,10	m-1+k	k=10
DTLZ7	3,5,6,8,10	m-1+k	k=10

Here, n and m are the size of population and number of objectives respectively. In Algorithm 1, the MatingSelection (in Line 4 in Algorithm 1) applies the binary-tournament selection to produce new individuals, in which two previous individuals need to be compared with Pareto dominance relationship. The computational complexity of mating selection is $O(n \times m)$ [44]. As for the environmental selection in Algorithm 2, its computational complexity focuses on the fast non-dominated sorting and the boundary elimination selection. It is known that the complexity of the fast non-dominated sorting is $O(n^2 \times m)$. About the second part, the complexity is mainly on BES. To select one elite individual, it has to compare n individuals. As for the rest individuals, BES needs to identify the solutions with ϵ-dominance relationship, which costs $O(n \times m)$. Thus, to choose n elite individuals, the computational complexity is $O(n^2 \times m)$.

According to the above analysis, the sum of the computational complexity of BESBS is $O(n^2 \times m)$, where n and m are the size of population and number of objectives respectively.

5. Experiment design

In this section, systematic experiments are carried out to investigate the performance of BESBS and the settings about the experiments follow the default settings in the platform [46] (`https://github.com/O-T-L/OTL`). We firstly introduce the test problems and performance metrics. Then we will briefly

Table 3: The settings of the number of grids of DM

No.of objectives	3	5	6	8	10
No.of grids	10	4	3	3	3

introduce eight state-of-art MOEAs and the general settings. Finally, the experiments of the sensitivity of the parameter λ in BESBS will be presented.

5.1. Test problems and performance metrics

As the basis for experimental comparisons, the suit of DTLZ [47] is considered with two reasons: firstly, all of the problems can be scaled to any number of objectives and decision variables. Secondly, they are able to test different aspects of the algorithms by providing different challenges. The suit can be divided into two groups. The first group involves DTLZ2, DTLZ4, DTLZ5 and DTLZ7, which are designed to test algorithms' ability to address the problems with different shapes and locations of Pareto optimal front (PoF). The second group consists of DTLZ1, DTLZ3, and DTLZ6, which create more obstacles to impede the solutions converging into the PoF [48]. The relevant settings of the test problems are shown in Table 2.

In order to compare the performance of the algorithms, four widely-used performance metrics are applied in this paper, such as the generation distance (GD) [49], the diversity metrics (DM) [50], the inverted generation distance (IGD) [51], and the hypervolume indicator (HV) [52]. GD evaluates the convergence performance by computing the minimum distances from obtained solutions to the Pareto optimal front. Thus, the smaller GD value is, the better convergence the algorithm obtains. On the contrary, IGD evaluates the comprehensive performance by computing the minimum distances from the Pareto optimal solutions to the obtained solutions. The smaller IGD value means the better the comprehensive performance of the obtained solutions, since the pool convergence and pool diversity will reflect on a big value of IGD. All the settings about the experiments follow the default settings in the platform [46] (`https://github.com/O-T-L/OTL`). Two indicators (the HV [52] and the IGD

[51]) are widely applied to comprehensively evaluate the performance of the algorithms account for convergence and the distribution of the achieved non-dominated solutions.

Referring to the reference points used in the indicators, since the Pareto front of the DTLZ suite can be obtainable, a common way of estimating the reference set of the Pareto front is to uniformly select points in the decision variable space under the condition of $g(x) = 0$; for example for the 5-objective DTLZ2, if setting 10 uniformly-distributed values for each dimension regarding x_1 to x_4, then we can obtain 10000 points well-distributed in the Pareto front. However, this method may not be very accurate for problems with many objectives because uniformly-distributed points in the decision space are not always located uniformly when mapped into the objective space. So for the test problems with more than 5 objectives in our study, we randomly sample 10000 Pareto front points (by setting $g(x) = 0$) and then use the K-nearest neighbor method (introduced in SPEA2) to remove most crowded points one by one until the point size reduces to 5000. At last, we integrate them with the boundary points of the problem to construct the reference set of the Pareto front.

As for DM, the metric evaluates the diversity of obtained solutions by computing the ratio (between the number of obtained solutions in each grid and the number of Pareto optimal set in the corresponding grid) by meshing method [50]. Thus, the bigger DM values means the better diversity. The settings of the number of grids in DM are shown in Table 3.

5.2. Comparative algorithms and general settings

In many-objective optimization, this paper will introduce eight representative MOEAs, and they are ϵ-MOEA [14], NSGA-III [28], GrEA [6], MSOPS [27], HypE [24], SMS-EMOA [23], AR+DMO [9][20], and GDE-MOEA [38].

All the experimental results in this paper were obtained on average by executing 30 independent runs of each algorithm on each problem. The termination criterion is 30,000 evaluations for the first group of DTLZ (DTLZ2, DTLZ4, DTLZ5, and DTLZ7), and 100,000 evaluations for the second group of DTLZ

Table 4: The settings of ϵ on different problems

Instances	No.of objectives				
	3	5	6	8	10
DTLZ1	0.033	0.059	0.0554	0.0549	0.0565
DTLZ2,4	0.06	0.1927	0.234	0.29	0.308
DTLZ3	0.06	0.2	0.227	0.1567	0.85
DTLZ5	0.0052	0.0785	0.11	0.1272	0.1288
DTLZ6	0.0227	0.3552	0.75	1.15	1.45
DTLZ7	0.048	0.158	0.15	0.225	0.56

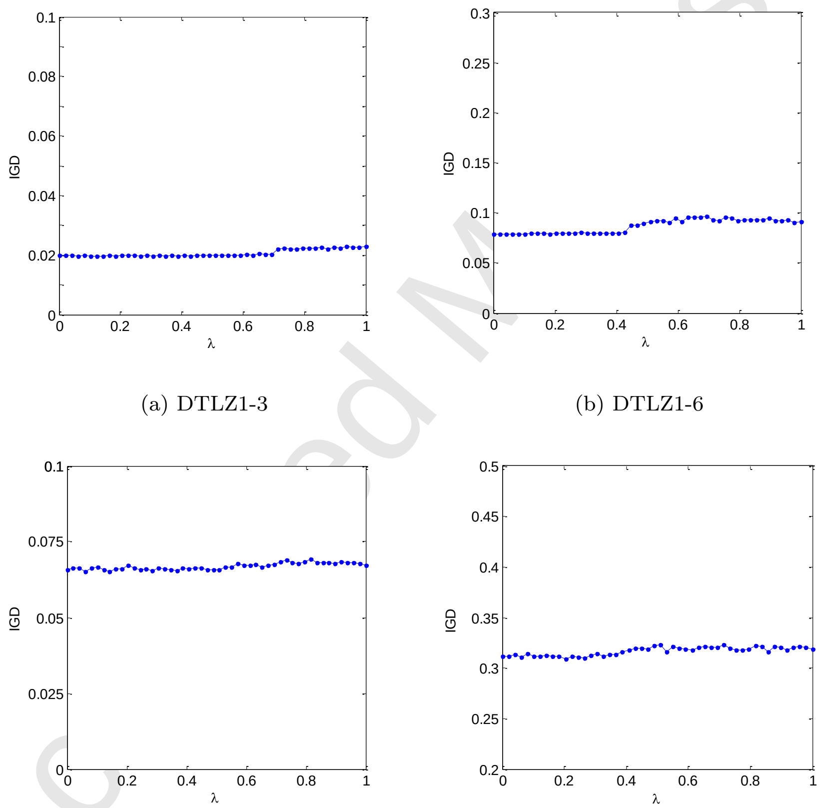
Table 5: The settings the reference point of NSGA-III and MOEA/D based on normal-boundary intersection (NBI)[53]

No.of objectives	3	5	6	8	10
Partitions	13	5	4	3	2,2
No.of reference points	105	126	126	120	55+55=110
Population size	108	128	128	120	112

(DTLZ1, DTLZ3, and DTLZ6). The corresponding settings are shown in Table 2.

Simulated binary crossover (SBX) [45] and polynomial mutation [1] are the crossover operator and mutation operator respectively with the same distribution indexes 20 (i.e. $\eta_c = 20$, $\eta_m = 20$). The crossover probability p_c is 1.0; the mutation probability p_m is $1/n$ where n is the number of decision variables of the problem.

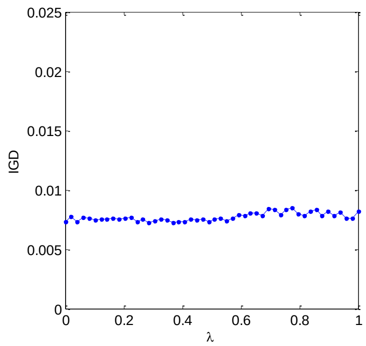
For general MOEAs, the population size is set to be 100. But for NSGA-III, since the population size is determined by the reference points, the relevant settings of its population size are shown in Table 5, which is referred to [6]. Here, the normal-boundary intersection [53] is selected to generate reference points in NSGA-III. About ϵ-MOEA, as its population size depends on the parameter ϵ, the settings of ϵ used in ϵ-MOEA are shown in Table 4, followed the practice in [6] in which we set the ϵ so that the archive of ϵ-MOEA is approximately of the same size as that of the other algorithms for a fair comparison. About MSOPS, it needs a set of reference vectors which are generated by the source code (link: `code.evanhughes.org`). The execution of SMS-EMOA with a large number of objectives can take unacceptable time. Therefore, for MOPs with five or more objectives, we approximately estimate the HV indicator in SMS-EMOA by the Monte Carlo sampling method where 10,000 sampling points are used. This approximation was often used for SMS-EMOA to deal with many-objective problems like [54]. For GrEA, the grid divisions are set to be 10 on all DTLZ problems.



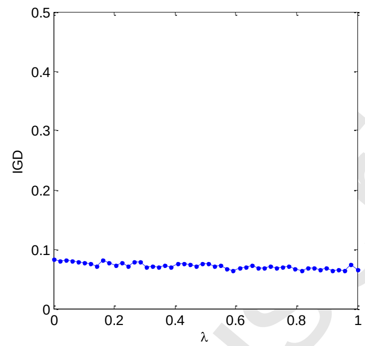
(a) DTLZ1-3 (b) DTLZ1-6

(c) DTLZ2-3 (d) DTLZ2-6

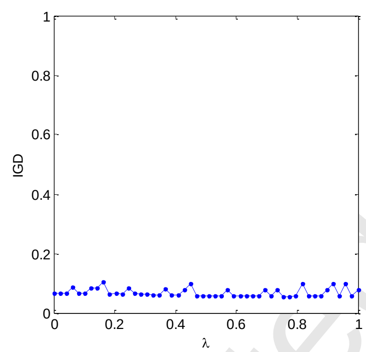
Figure 5: Study of different λ parameter configurations on DTLZ1, DTLZ2 problems with 3 and 6 objectives.



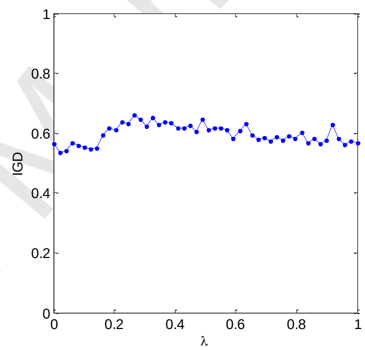
(e) DTLZ5-3



(f) DTLZ5-6



(g) DTLZ7-3



(h) DTLZ7-6

Figure 6: Study of different λ parameter configurations on DTLZ5, DTLZ7 problems with 3 and 6 objectives.

5.3. Sensitivity of Parameter λ in BESBS

In BESBS, λ is used as a penalty on d_2 in definition 5. In order to study the influence of different values of $\lambda \in (0, 1]$ on the performance of BESBS, we repeat the experiments independently 30 times on DTLZ1, DTLZ2, DTLZ5, DTLZ7 test problems with 3 and 6 objectives. Fig. 5 and Fig. 6 plot the mean IGD values with different λ values in $(0, 1]$.

Above all, with the increase of the objectives, the IGD values will be obviously increased since the searching space is enlarged and there are more difficulties for the optimization. But in dealing with the same problem with varying λ values, the effect from the λ varies from the problems.

From Fig. 5, the IGD values fluctuate in different ranges on varying problems. Specifically, on DTLZ1 problems with 3 and 6 objectives, the average IGD values are stable about 0.020 and 0.073 respectively with the increase of the λ value from 0 to 0.70 and 0.42 respectively. But when the λ values increase from 0.70 and 0.42 respectively, the IGD values will increase to another levels (0.023 and 0.085) respectively. In contrast, on DTLZ2 with 3 and 6 objectives, entirety, the IGD values tend to be more stable than that on DTLZ1, but the values fluctuate in small ranges in detail. Thus, from these two experiments on DTLZ1 and DTLZ2, the bigger values of λ would affect the performance more obvious, especially on dealing with the difficult multimodal DTLZ1 problems. But the small λ values could relatively less impact the performance.

From Fig. 6, interestingly on the degenerated problem DTLZ5 with 3 and 6 objectives, the increase of the value of λ does not show significant effect on the performance, which may be because the solutions will be distributed along the degenerated PoF after the sorting. The fitness embedded with the penalty λ will lose its advantage to favor the corner solutions. Thus, the effect from the λ values is not obvious during the optimization. On disconnected problem DTLZ7 with 3 and 6 objectives, the effect is more obvious. Especially on DTLZ7 with 6 objectives, the effect fluctuates from the beginning then increases sharply after 0.2, and ends with in a small degree gradual decrease. Thus, for disconnected problems, the effect from the λ can be serious, which may be resulted from the pool diversity of the solutions during the optimization. Thus, the increase of the penalty value will impact the performance especially for the disconnected problems.

In general, when $\lambda \in (0, 0.2]$, the effect of the λ seems to be in a low degree relatively. Thus $\lambda = 0.1$ is selected in the following experiments.

Table 6: The GD values (average values, variance values) of the obtained solutions of each algorithm on DTLZ1, DTLZ3, DTLZ6, where the best values are shown with a deep gray background and the second one with a gray background. † is a symbol of the algorithm which BESBS has better prominent performance than.

Problems	Obj.	BESBS	ϵ-MOEA	NSGA-III	MSOPS	HypE	SMS-EMOA	AR+DMO	GrEA	GDE-MOEA
DTLZ1	3	2.270E-4	2.375E-4	5.536E-4	3.064E-4	7.201E-2†	7.177E-3	2.307E-2	4.841E-2	2.302E-04
		9.949E-6	2.290E-5	1.703E-3	4.236E-4	2.123E-1†	3.587E-2	1.067E-1	2.072E-2	1.605E-05
	5	2.146E-3	2.633E-3	2.806E-2	4.471E-3	4.347E-1†	8.000E-3	5.087E-2	3.601E-2	3.246E-03
		6.877E-5	3.850E-4	8.846E-2	8.532E-3	6.237E-1†	2.289E-2	1.406E-1	3.710E-2	1.271E-04
	6	3.240E-3	3.587E-3	1.753E-1	2.582E-2	6.383E-0†	4.873E-2	1.070E-1	4.511E-2	9.853E-03
		1.029E-4	3.987E-4	3.306E-1	7.672E-2	1.501E-0†	1.574E-1	2.756E-1	2.962E-2	6.202E-03
	8	5.010E-3	8.623E-3	3.869E-1	1.008E-1	5.965E-0†	1.468E-1	6.037E-2	3.131E-2	6.206E-02†
		1.135E-4	9.132E-3	4.164E-1	1.620E-1	1.711E-0†	2.578E-1	1.801E-1	8.742E-3	7.951E-02†
	10	5.904E-3	7.969E-2	3.073E-1	1.202E-1	2.197E-0†	1.909E-1	1.323E-1	8.741E-2	1.837E-01†
		1.914E-4	5.423E-2	5.475E-1	1.608E-1	1.486E-0†	2.677E-1	2.870E-1	4.550E-2	2.588E-01†
DTLZ3	3	2.214E-4	1.485E-3	7.140E-2	5.227E-2	8.266E-2	2.779E-4	7.875E-2	1.607E-4	4.933E-05
		1.648E-4	4.542E-4	3.569E-1	2.776E-1	3.265E-1	1.535E-4	1.941E-1	6.648E-8	4.041E-05
	5	3.987E-4	1.209E-2	4.084E-1	1.750E-2	2.478E+1†	4.154E-2	1.063E-1	2.169E+0	2.996E+00†
		2.605E-4	7.867E-3	6.069E-1	6.749E-2	4.845E-0†	1.051E-1	3.754E-1	1.282E+1	1.827E+00†
	6	4.170E-4	1.749E-2	2.611E-0†	1.318E-1	2.639E+1†	8.531E-1	8.533E-2	1.980E+0	4.918E+00†
		2.093E-4	2.532E-2	2.011E-0†	3.536E-1	3.492E-0†	3.079E-0	2.651E-1	9.312E+0	7.795E-01†
	8	4.210E-4	1.218E-0	3.988E+1†	7.875E-1	2.351E+1†	7.800E-1	2.078E-1	2.578E+0	9.681E+00†
		2.149E-4	2.114E-0	1.369E+1†	9.384E-1	3.111E-0†	2.304E-0	4.392E-1	1.041E+1	1.465E+00†
	10	3.921E-4	3.201E-0	2.029E+1†	1.860E-0	1.875E+1†	2.314E-1	1.957E-1	2.418E-1	1.092E+01†
		2.522E-4	3.339E-0	1.489E+1†	1.361E-0	3.336E-0†	4.925E-1	5.824E-1	4.089E-1	2.207E+00†
DTLZ6	3	1.937E-3	5.481E-3†	1.278E-2†	2.054E-1†	7.191E-3†	3.732E-3	4.216E-3	4.598E-3	1.661E-03
		7.276E-4	5.841E-4†	9.991E-3†	3.772E-3†	5.355E-3†	8.752E-4	1.209E-3	1.675E-5	2.275E-03
	5	1.449E-1	1.478E-1	6.132E-1†	4.943E-1†	3.558E-1†	2.669E-1†	3.012E-1†	1.021E-1	1.813E-01
		1.396E-2	5.525E-3	1.611E-2†	1.107E-2†	2.663E-2†	1.722E-2†	1.841E-2†	5.913E-3	2.140E-01
	6	1.618E-1	2.498E-1†	8.629E-1†	5.547E-1†	3.759E-1†	2.458E-1†	3.444E-1†	3.066E-2†	2.628E-01
		1.263E-2	2.061E-2†	1.915E-2†	1.169E-2†	3.267E-2†	1.324E-2†	5.193E-2†	1.061E-2†	1.728E-01
	8	1.749E-1	3.835E-1†	9.142E-1†	7.717E-1†	4.654E-1†	2.229E-1	6.447E-1†	1.761E-1	2.243E-01
		2.276E-2	2.358E-1†	1.082E-1†	1.548E-2†	2.845E-2†	1.003E-2	3.860E-2	9.313E-2	2.249E-01
	10	1.769E-1	2.784E-1†	8.188E-1†	8.144E-1†	5.796E-1†	2.215E-1	7.539E-1†	3.264E-1	9.356E-02†
		1.558E-2	1.658E-1†	6.751E-2†	1.793E-2†	2.214E-2†	1.043E-2	5.337E-2†	1.506E-1	1.405E-01†

6. Experiment result and analysis

In this section, we investigate the performance of BESBS in comparison with 8 EMO algorithms. The results obtained by the algorithms which independently runs 30 times on each test problem, are listed in Tables 6-11. Both the mean and standard deviation of the metric values are presented in the boxes where the best and second best values are highlighted with the deep gray background and gray background respectively. Notably, the symbol "†" on the top right corner of the value indicates the remarkable p value (by Tamhane's T2 test [55]), and the significance level α is 0.05.

6.0.1. GD comparison for convergence

Table 6 and Table 7 give the GD values on the two groups of DTLZ problems respectively. It is obvious from Table 6 that BESBS achieved the smallest GD values on most DTLZ1, DTLZ3, and DTLZ6 problems, especially on the

Table 7: The GD values (average values, variance values) of the obtained solutions of each algorithm on DTLZ2, DTLZ4, DTLZ5, DTLZ7, where the best values are shown with a deep gray background and the second one with a gray background. † is a symbol of the algorithm which BESBS has better prominent performance than.

Problems	Obj.	BESBS	ϵ-MOEA	NSGA-III	MSOPS	HypE	SMS-EMOA	AR+DMO	GrEA	GDE-MOEA
DTLZ2	3	3.166E-4	7.292E-4†	1.346E-4	1.234E-4	3.412E-4	3.373E-4	5.236E-4†	3.894E-5†	1.971E-04
		7.263E-5	5.354E-5†	9.478E-5	1.163E-4	1.463E-4	7.496E-5	1.671E-4†	1.171E-9†	1.912E-04
	5	4.170E-4	4.311E-3†	9.743E-4†	3.649E-4	6.616E-4	1.247E-3†	4.042E-3†	4.523E-4	1.301E-04†
		1.063E-4	7.816E-4†	2.079E-4†	1.639E-4	2.288E-4	1.686E-4†	1.266E-3†	5.636E-8	8.951E-05†
	6	3.866E-4	5.232E-3†	1.889E-3†	5.450E-4	8.829E-4	1.743E-3†	8.398E-3†	6.273E-4	7.42E-05†
		1.109E-4	4.122E-4†	4.591E-4†	2.009E-4	2.014E-4	2.704E-4†	2.479E-3†	2.966E-8	7.453E-05†
	8	4.029E-4	6.789E-3†	6.571E-3†	1.028E-3	1.008E-3	2.428E-3†	1.902E-2†	2.111E-3†	7.613E-05†
		1.284E-4	7.161E-4†	2.038E-3†	2.639E-4	2.246E-4	3.417E-4†	3.081E-3†	6.712E-7†	6.882E-05†
	10	4.297E-4	5.409E-3†	3.781E-3†	1.747E-3	8.568E-4	3.158E-3†	2.918E-2†	1.808E-3†	5.771E-05†
		1.347E-4	5.329E-4†	3.604E-3†	3.248E-4	1.883E-4	7.073E-4†	3.367E-3†	1.890E-7†	1.032E-04†
DTLZ4	3	2.074E-4	8.453E-4	2.088E-4	6.591E-5	1.346E-3†	2.185E-4	3.412E-4	1.655E-4	2.033E-04
		1.116E-4	3.225E-4	1.673E-4	6.817E-5	2.973E-3†	1.102E-4	3.292E-4	1.650E-7	1.284E-04
	5	4.147E-4	5.414E-3†	1.923E-3†	3.422E-4	1.768E-3†	9.168E-4	3.062E-3†	5.474E-4	3.482E-04
		1.331E-4	1.795E-3†	1.085E-3†	1.367E-4	3.046E-3†	4.601E-4	2.071E-3†	1.699E-7	1.394E-04
	6	4.122E-4	9.714E-3†	3.065E-3†	6.339E-4	9.147E-4†	1.442E-3	4.492E-3†	7.785E-4	6.331E-04
		1.155E-4	7.773E-3†	1.364E-3†	2.804E-4	1.234E-3	5.226E-4	2.517E-3†	8.530E-8	9.201E-05
	8	4.044E-4	1.182E-2†	3.199E-3	1.529E-3	7.737E-4	2.301E-3	1.493E-2†	2.191E-3	8.131E-04
		1.236E-4	1.003E-2†	4.046E-3	6.809E-4	1.841E-4	3.740E-4	2.968E-3†	8.280E-7	1.554E-04
	10	3.754E-4	1.461E-2†	1.717E-2†	2.478E-3	5.430E-4	2.755E-3	2.860E-2†	1.718E-3	4.292E-03
		1.661E-4	1.218E-2†	7.845E-3†	9.414E-4	1.438E-4	5.072E-4	5.072E-3†	1.493E-7	1.971E-03
DTLZ5	3	3.555E-5	6.218E-5	2.170E-4	1.082E-1†	1.373E-3†	1.101E-4	4.377E-4	4.864E-5	8.761E-05
		1.895E-5	6.319E-6	8.139E-5	3.236E-3†	1.897E-3†	2.810E-5	6.331E-4	1.139E-9	5.652E-05
	5	4.908E-2	5.266E-2	6.833E-2†	1.897E-1†	1.371E-1†	1.013E-1†	2.593E-2	1.578E-2	1.031E-04†
		4.580E-3	2.731E-3	1.766E-2†	2.482E-3†	7.209E-3†	6.227E-3†	8.035E-3	2.641E-4	1.094E-04†
	6	5.199E-2	5.861E-2	8.931E-2†	2.042E-1†	1.502E-1†	1.115E-1†	4.344E-2	5.943E-2	1.122E-02
		5.413E-3	3.131E-3	2.221E-2†	2.836E-3†	5.309E-3†	4.397E-3†	1.576E-2	3.123E-5	1.533E-02
	8	5.691E-2	5.497E-2	1.160E-1†	2.293E-1†	1.630E-1†	1.185E-1†	1.459E-1†	1.027E-1†	2.381E-05†
		5.936E-3	4.951E-3	1.862E-2†	2.005E-3†	5.233E-3†	3.956E-3†	3.103E-2†	4.456E-5†	4.801E-05†
	10	6.693E-2	6.011E-2	1.504E-1†	2.345E-1†	1.808E-1†	1.181E-1†	1.804E-1†	1.095E-1†	1.812E-06†
		6.501E-3	6.555E-3	1.851E-2†	2.293E-3†	4.201E-3†	4.521E-3†	2.603E-2†	4.446E-5†	7.471E-07†
DTLZ7	3	6.388E-4	7.003E-4	1.668E-3†	4.219E-3†	1.604E-3†	7.172E-4	2.559E-3†	1.003E-3†	1.482E-03
		8.472E-5	4.524E-5	8.416E-4†	3.601E-4†	1.659E-3†	6.068E-6	1.111E-3†	3.095E-7†	1.012E-03
	5	1.186E-2	4.076E-3	1.664E-2	1.067E-2	1.089E-2	1.040E-2	7.147E-2†	1.095E-2	1.181E-02
		5.554E-4	1.303E-3	5.186E-3	1.787E-3	2.982E-3	7.750E-4	2.303E-2†	2.704E-7	3.432E-03
	6	1.587E-2	4.909E-3	7.182E-2†	2.249E-2	2.054E-2	1.591E-2	1.849E-1†	1.213E-2	3.301E-02
		9.711E-4	1.983E-3	4.186E-2†	1.202E-2	1.251E-2	3.096E-4	4.294E-2†	2.028E-7	9.283E-03
	8	2.201E-2	2.121E-2	1.003E-0†	2.035E-1†	4.026E-2	3.353E-2	3.412E-1†	2.655E-2	2.321E-01
		4.087E-3	1.496E-2	2.440E-1†	1.064E-1†	7.311E-3	1.011E-3	9.097E-2†	1.791E-6	1.733E-01
	10	4.869E-2	5.105E-2	2.924E-0†	6.755E-1†	6.625E-2	5.362E-2	6.411E-1†	5.016E-2	6.883E-01
		4.349E-3	4.123E-2	5.748E-1†	2.949E-1†	1.651E-2	3.329E-3	1.309E-1†	6.546E-6	3.751E-01

problems with 8- and 10-objective. From Table 6, BESBS ranks the first with 10 best and 4 second best out of 15 problems. ϵ-MOEA ranks the second with 6 second best problems. GrEA outperforms other algorithms on DTLZ3 with 3 objectives, and DTLZ6 with 5- and 6-objective. AR+DMO and SMS-EMOA have similar performance. GDE-MOEA has better performance on 3-objective DTLZ1, DTLZ3 and DTLZ6, as well as 10-objective DTLZ6, because the GDE-MOEA puts more emphasis on the convergence. The convergence of BESBS tends to be stable and not subject to the increase of the number of objectives, especially on DTLZ1 and DTLZ3 problems which are multi-modal problems containing a large number of local Pareto optimal fronts.

From Table 7, on most problems, the GD values of both GDE-MOEA and BESBS are smaller than that of other algorithms. Both of them outperform 10 out of 20 problems, which means that the solutions of them are relatively well-approximating to the Pareto optimal front on such problems. Among the 20 problems, BESBS outperforms on most DTLZ4 and DTLZ7 problems, but GDE-MOEA does on DTLZ2 and DTLZ5 problems. ϵ-MOEA did well especially on DTLZ7 with 5, 6, 8 objectives; MSOPS did well on DTLZ2 and DTLZ4 with 5 objectives; AR+DMO has better convergence on DTLZ5 with 6 objectives; GrEA outperforms on DTLZ2 and DTLZ5 with 3 and 5 objectives respectively. Specially, the performance of BESBS on the problems with different shapes and locations of PoF seems to be not better than ϵ-MOEA, which shows that experience-based ϵ-MOEA is well suited for dealing with MaOPs whose PoF is a degenerated curve or multi-model. This could be partly alleviated in BESBS by using a smaller λ as shown in Fig. 5 and Fig. 6, because more attention will be paid on the well-approximating solutions on d_1 rather than d_2 in Eq. 5.

In terms of Table 6 and Table 7, the records of best values in the two charts obtained by BESBS are 10 and 6 respectively on the two groups of DTLZ, and second better records are 4 on the two sets. Overall, BESBS did well on 24 out of 35 problems. It can be concluded that the BESBS better others on the convergence.

Table 8: The DM values (average values, variance values) of the obtained solutions of each algorithm on DTLZ1, DTLZ3, DTLZ6, where the best values are shown with a deep gray background and the second one with a gray background. † is a symbol of the algorithm which BESBS has better prominent performance than.

Problems	Obj.	BESBS	ϵ-MOEA	NSGA-III	MSOPS	HypE	SMS-EMOA	AR+DMO	GrEA	GDE-MOEA
DTLZ1	3	9.226E-1	1.001E-0	9.856E-1	7.257E-1†	6.451E-1†	8.070E-1†	4.577E-1†	5.561E-1	2.699E-01
		3.491E-1	1.196E-2	3.389E-2	7.566E-3†	1.176E-1†	3.348E-2†	1.309E-1†	4.491E-2	5.355E-02
	5	8.781E-1	7.906E-1†	9.463E-1	8.463E-1	3.150E-1†	7.976E-1†	2.656E-1†	7.551E-1	3.901E-01
		3.800E-2	1.065E-1†	9.616E-2	1.716E-2	1.631E-1†	4.410E-2†	7.816E-2†	4.560E-2	4.951E-02
	6	9.196E-1	7.803E-1†	7.725E-1†	9.848E-1	0†	9.265E-1	2.911E-1†	9.410E-1	2.286E-01
		7.406E-2	4.194E-2†	3.835E-1†	2.289E-2	0†	8.412E-2	9.139E-2†	3.711E-2	5.578E-02
	8	5.429E-1	3.179E-0	4.019E-2	6.807E-1	0	4.589E-1	2.251E-1	7.851E-1	1.079E-01
		4.484E-2	9.738E-0	6.773E-2	3.221E-2	0	6.922E-2	6.665E-2	2.701E-2	7.092E-02
	10	4.419E-1	1.840E-0	2.284E-2	4.488E-1	4.815E-2	3.121E-1	1.667E-1	9.182E-1	9.199E-02
		5.680E-2	7.519E-0	3.079E-2	2.633E-2	9.004E-2	5.618E-2	4.422E-2	1.449E-1	7.215E-02
DTLZ3	3	8.242E-1	8.727E-1	9.044E-1	5.965E-1†	4.053E-1†	7.679E-1	2.413E-1†	6.261E-1	2.599E-01
		3.347E-2	2.607E-2	1.217E-1	1.318E-2†	9.807E-2†	3.071E-2	1.367E-1†	1.281E-2	2.750E-02
	5	8.831E-1	7.316E-1†	1.858E-1†	6.514E-1†	0†	6.182E-1†	2.018E-1†	4.089E-1†	0†
		3.253E-2	2.741E-1†	2.664E-1†	1.985E-2†	0†	3.769E-2†	9.536E-2†	1.021E-1†	0†
	6	8.552E-1	8.403E-1	0†	6.736E-1†	0†	4.836E-1†	1.645E-1†	3.589E-1	0†
		3.989E-2	3.794E-1	0†	4.593E-2†	0†	2.036E-1†	6.779E-2†	1.071E-1	0†
	8	8.846E-1	4.643E-1†	0†	4.773E-1†	0†	3.515E-1†	1.308E-1†	2.621E-1†	0†
		2.713E-2	1.240E-0†	0†	1.954E-1†	0†	1.679E-1†	5.492E-2†	9.250E-2†	0†
	10	8.749E-1	2.737E-3†	0†	2.072E-1†	0†	3.986E-1†	1.161E-1†	6.289E-1†	0†
		2.661E-2	4.639E-3†	0†	2.185E-2†	0†	1.315E-1†	6.923E-2†	9.350E-2†	0†
DTLZ6	3	1.289E-0	1.401E-0	1.439E-1	1.754E-0	1.314E-0	1.512E-0	1.638E-0†	1.411E+0	6.642E-02†
		1.243E-1	9.472E-2	1.041E-1	8.322E-2	9.014E-2	8.517E-2	2.512E-1†	6.610E-3	5.084E-03†
	5	1.587E-0	0†	0†	1.272E-0†	2.555E-1†	1.454E-0†	0†	1.410E+0†	1.986E-01†
		1.903E-1	0†	0†	1.485E-1†	2.215E-1†	1.401E-1†	0†	6.610E-3†	1.671E-01†
	6	2.622E-0	0†	0†	8.803E-1†	2.447E-1†	1.962E-0†	0†	2.461E-1†	7.688E-01†
		3.117E-1	0†	0†	4.618E-1†	3.348E-1†	2.389E-1†	0†	5.371E-3†	4.553E-01†
	8	1.761E-0	9.177E-2†	0†	0†	0†	1.717E-0†	0†	2.112E-1†	3.079E-01†
		3.903E-1	1.005E-1†	0†	0†	0†	2.751E-1†	0†	1.430E-2†	3.367E-01†
	10	1.427E-0	3.815E-2†	0†	0†	0†	1.794E-0	0†	1.401E-1†	4.366E-01†
		4.094E-1	7.763E-2†	0†	0†	0†	2.520E-1	0†	1.661E-2†	2.205E-01†

Table 9: The DM values (average values, variance values) of the obtained solutions of each algorithm on DTLZ2, DTLZ4, DTLZ5, DTLZ7, where the best values are shown with a deep gray background and the second one with a gray background. † is a symbol of the algorithm which BESBS has better prominent performance than.

Problems	Obj.	BESBS	ϵ-MOEA	NSGA-III	MSOPS	HypE	SMS-EMOA	AR+DMO	GrEA	GDE-MOEA
DTLZ2	3	8.340E-1 2.666E-2	8.808E-1 2.163E-2	9.565E-1 1.072E-2	5.938E-1† 1.197E-2†	4.614E-1† 3.137E-2†	7.646E-1† 2.995E-2†	3.069E-1† 1.145E-1†	6.882E-1 3.780E-4	2.312E-01 5.010E-02
	5	9.022E-1 2.131E-2	9.242E-1 7.157E-2	8.694E-1 8.663E-3	6.493E-1† 1.635E-2†	3.179E-1† 3.114E-2†	6.439E-1† 3.163E-2†	3.378E-1† 9.373E-2†	9.581E-1 6.901E-4	1.202E-01 3.973E-02
	6	8.485E-1 3.006E-2	8.432E-1 3.329E-2	7.978E-1† 1.328E-1†	6.815E-1† 3.019E-2†	3.536E-1† 3.280E-2†	6.545E-1† 4.770E-2†	3.126E-1† 5.551E-2†	9.446E-1 1.044E-3	1.105E-01 4.805E-02
	8	8.951E-1 2.323E-2	1.019E-0 1.237E-1	7.371E-1† 2.263E-1†	6.397E-1† 2.274E-2†	1.941E-1† 2.672E-2†	4.897E-1† 3.957E-2†	2.911E-1† 6.141E-2†	9.170E-1 4.909E-4	5.141E-02† 1.648E-02†
	10	8.934E-1 2.025E-2	1.082E-0 2.495E-1	1.638E-1† 2.385E-1†	6.622E-1† 2.407E-2†	8.719E-2† 8.549E-2†	4.191E-1† 4.703E-2†	2.822E-1† 6.136E-2†	9.783E-1 2.326E-4	2.750E-02† 1.568E-02†
DTLZ4	3	6.705E-1 3.046E-1	5.634E-1 3.928E-1	6.187E-1 4.251E-1	5.785E-1 6.319E-3	4.210E-1† 1.432E-1†	5.501E-1 3.092E-1	2.469E-1† 1.836E-1†	5.728E-1 5.176E-2	2.901E-01 1.497E-01
	5	8.348E-1 1.518E-1	5.444E-1† 3.261E-1†	6.513E-1† 2.666E-1†	6.232E-1† 2.163E-2†	3.164E-1† 3.042E-2†	5.420E-1† 2.019E-1†	3.402E-1† 1.947E-1†	8.701E-1 2.220E-2	2.304E-01 9.512E-02
	6	8.191E-1 7.509E-2	5.249E-1† 2.763E-1†	4.099E-1† 3.257E-1†	6.995E-1 2.762E-2	3.703E-1† 3.177E-2†	5.333E-1† 1.519E-1†	3.070E-1† 2.041E-1†	9.281E-1 1.375E-3	1.757E-01 5.303E-02
	8	8.729E-1 2.675E-2	7.899E-1 2.906E-1	5.570E-2† 1.550E-1†	6.248E-1† 3.401E-2†	2.418E-1† 3.440E-2†	6.093E-1† 3.829E-2†	3.545E-1† 9.916E-2†	9.221E-1 3.754E-4	8.384E-02† 2.809E-02†
	10	8.884E-1 1.667E-2	8.999E-1 2.919E-1	4.915E-2† 1.564E-1†	6.819E-1† 2.861E-2†	1.419E-1† 4.783E-2†	6.074E-1† 4.014E-2†	2.901E-1† 7.359E-2†	9.717E-1 8.113E-5	9.414E-02† 5.067E-02†
DTLZ5	3	9.668E-1 3.558E-2	9.387E-1 5.507E-3	8.663E-1† 1.078E-1†	1.366E-0 3.498E-2	8.708E-1† 6.963E-2†	9.368E-1 4.205E-2	9.917E-1 8.799E-2	9.219E-1 9.533E-4	5.323E-01† 4.952E-02†
	5	1.747E-0 1.896E-1	1.639E-0 1.009E-1	8.365E-1† 5.531E-1†	1.309E-0† 1.521E-1†	9.785E-1† 1.633E-1†	1.504E-0† 1.306E-1†	1.322E-0† 3.819E-1†	1.150E+0 6.431E-2	2.118E-01† 6.294E-02†
	6	3.066E-0 2.557E-1	2.807E-0 3.956E-1	1.399E-0† 9.602E-1†	1.579E-0† 2.540E-1†	1.117E-0† 1.416E-1†	2.031E-0† 1.971E-1†	1.303E-0† 5.974E-1†	2.657E+0 6.683E-2	4.057E-01† 1.236E-01†
	8	2.791E-0 4.200E-1	2.404E-0† 2.910E-1†	3.863E-1† 4.733E-1†	7.316E-1† 1.698E-1†	9.791E-1† 1.784E-1†	1.819E-0† 1.463E-1†	4.163E-2† 7.028E-2†	2.088E+0 5.923E-1	3.009E-01† 9.307E-02†
	10	2.224E-0 2.508E-1	2.305E-0 2.869E-1	1.463E-1† 2.558E-1†	4.259E-1† 8.533E-2†	1.611E-1† 4.365E-2†	1.903E-0† 1.728E-1†	0† 0†	2.753E+0 5.435E-1	2.177E-01† 5.781E-02†
DTLZ7	3	9.511E-1 9.108E-2	1.061E-0 1.152E-1	3.065E-1† 1.444E-1†	7.300E-1† 2.563E-2†	7.479E-1† 3.615E-2†	8.618E-1 1.914E-1	3.705E-1† 2.235E-1†	7.078E-1 2.109E-3	4.278E-01 1.964E-01
	5	9.599E-1 5.342E-2	1.125E-0 5.807E-1	8.799E-2† 2.937E-2†	4.530E-1† 1.963E-2†	6.833E-1† 4.441E-2†	8.373E-1 1.782E-1	4.214E-1† 1.209E-1†	8.381E-1 1.171E-3	3.252E-01 1.345E-01
	6	3.817E-1 1.046E-1	6.927E-1 3.941E-1	4.623E-2† 2.874E-2†	2.995E-1 3.245E-2	4.522E-1 1.170E-2	3.267E-1 1.063E-1	3.469E-1 1.413E-1	5.409E-1 2.200E-3	3.630E-01 1.276E-01
	8	7.185E-1 9.740E-2	1.355E-0 9.513E-1	0† 0†	1.470E-1† 4.142E-2†	8.043E-1 4.950E-2	6.239E-1 1.952E-1	6.669E-2† 8.116E-2†	8.668E-1 2.168E-3	1.253E-01 1.149E-01
	10	7.718E-1 7.162E-2	1.917E-0 1.622E-0	0† 0†	1.640E-2† 8.730E-3†	9.478E-1 8.215E-2	7.902E-1 2.997E-1	9.725E-3† 2.554E-2†	9.755E-1 2.281E-3	2.420E-01 1.914E-01

6.0.2. DM comparison for diversity

Table 8 and Table 9 present the mean and standard deviation of DM values of the eight algorithms on the two groups of DTLZ problems after 30 independent runs respectively. Similarly, the best values are shown with a deep gray background and the second best with a gray background.

In Table 8, both BESBS and ϵ-MOEA did well on 9 and 6 out of 15 problems respectively, following by NSGA-III, GrEA, MSOPS and SMS-EMOA. Specifically, NSGA-III and ϵ-MOEA achieved good performance on DTLZ1 and DTLZ3 with 3 and 5 objectives. BESBS outperformed others on DTLZ3 and DTLZ6 problems with more than 3 objectives. ϵ-MOEA and GrEA obtained relatively good performance on DTLZ1 problems with more than 5 objectives. The results reveal that BESBS is promising to deal with MOPs with more obstacles to converge, and the performance of ϵ-MOEA is also encouraging.

From Table 9, GrEA, ϵ-MOEA, and BESBS did well comparatively on the second group of DTLZ problems with varying shapes and locations of PoF. GrEA achieved the best DM values on DTLZ2 and DTLZ4 problems, which indicates that the improved grid-based approach is promising for the hypersphere MOPs. ϵ-MOEA gained the best DM values than others on DTLZ7 problems and a slight better than BESBS and GrEA. BESBS outperformed on most DTLZ4 and DTLZ5 problems with more than 3 objectives.

Among the 35 instances in Table 8 and Table 9, ϵ-MOEA and BESBS win the best on 11 and 10 instances in terms of DM values respectively, following by GrEA with 7, NSGA-III with 3, and MSOPS with 3. Considering both the best and second best, ϵ-MOEA ranks the first with 20 wins, following by BESBS with 19 and GrEA with 16. Thus, the results can illustrate that ϵ-MOEA, BESBS and GrEA can gain good diversity relatively in comparison with other algorithms.

6.0.3. IGD comparison for comprehensive performance

Table 10 and Table 11 list the mean and standard deviation of IGD values of eight algorithms after 30 independent runs on the DTLZ problems, where the

Table 10: The IGD values (average values, variance values) of the obtained solutions of each algorithm on DTLZ1, DTLZ3, DTLZ6, where the best values are shown with a deep gray background and the second one with a gray background. † is a symbol of the algorithm which BESBS has better prominent performance than.

Problems	Obj.	BESBS	ϵ-MOEA	NSGA-III	MSOPS	HypE	SMS-EMOA	AR+DMO	GrEA	GDE-MOEA
DTLZ1	3	1.969E-2	1.926E-2	2.041E-2	2.824E-2†	3.176E-2†	2.418E-2	5.678E-2†	5.698E-2	1.090E-01†
		4.567E-4	1.597E-4	2.904E-4	3.256E-4†	8.712E-3†	1.703E-3	1.902E-2†	3.437E-3	1.563E-02†
	5	6.239E-2	6.582E-2	6.781E-2	8.023E-2	2.613E-1†	7.697E-2	2.119E-1†	8.824E-2	1.570E-01
		1.071E-3	6.898E-3	4.785E-3	7.678E-4	3.046E-1†	2.845E-3	2.997E-2†	4.641E-3	1.186E-02
	6	7.882E-2	8.404E-2	2.372E-1	1.001E-1	1.018E+1†	1.019E-1	2.588E-1	9.475E-2	1.611E-01
		1.267E-3	6.234E-3	2.109E-1	1.242E-3	2.729E-0†	5.198E-3	3.122E-2	3.342E-3	1.306E-02
	8	1.038E-1	1.959E-1	1.319E-0	1.272E-1	9.126E-0†	1.479E-1	2.819E-1	1.126E-1	5.719E-01†
		1.543E-3	2.396E-1	1.117E-0	1.298E-3	4.283E-0†	7.733E-3	2.392E-2	5.448E-4	7.619E-01†
	10	1.198E-1	5.565E-1	7.563E-1	1.459E-1	2.597E-0†	1.806E-1	3.067E-1	2.416E-1	7.486E-01†
		2.679E-3	6.372E-1	3.212E-1	1.265E-3	2.085E-0†	1.379E-2	1.867E-2	1.695E-2	1.093E+00†
DTLZ3	3	6.679E-2	6.779E-2	5.631E-2	7.233E-2	1.267E-1†	7.074E-2	2.450E-1†	6.391E-1	3.599E-01†
		3.329E-3	3.261E-3	7.981E-3	1.006E-3	5.267E-2†	3.146E-3	7.684E-2†	3.212E-2	2.373E-02†
	5	1.766E-1	2.478E-1	1.131E-0	1.832E-1	4.707E+1†	2.044E-1	2.964E-1	4.368E-1	9.422E+00†
		4.528E-3	5.396E-2	8.538E-1	3.400E-3	8.189E-0†	6.374E-3	4.579E-2	5.920E-2	3.046E+00†
	6	3.184E-1	3.936E-1	4.538E-0	3.106E-1	6.614E+1†	1.019E-0	7.419E-1	6.216E-1	1.874E+01†
		7.006E-3	1.325E-1	2.820E-0	1.335E-2	1.533E+1†	2.233E+0	1.958E-1	8.453E-2	6.016E+00†
	8	4.410E-1	8.210E-0	1.096E+2†	5.690E-1	6.685E+1†	1.094E-0	8.642E-1	7.378E-1	2.697E+01†
		9.356E-3	1.357E+1	5.978E+1†	3.799E-1	1.916E+1†	1.636E-0	6.566E-2	4.665E-2	6.802E+00†
	10	5.242E-1	2.213E+1†	5.327E+1†	1.226E-0	4.524E+1†	7.678E-1	1.011E+0	6.391E-1	3.321E+01†
		1.386E-2	2.378E+1†	4.416E+1†	8.658E-1	1.293E+1†	2.952E-1	2.223E-1	3.212E-2	5.682E+00†
DTLZ6	3	2.513E-2	5.118E-2†	6.839E-2†	5.607E-2†	5.990E-2†	4.340E-2†	7.759E-2†	4.179E-2†	6.856E-01†
		7.007E-3	6.221E-3†	1.233E-2†	1.261E-2†	1.101E-2†	9.508E-3†	5.183E-2†	9.951E-5†	9.710E-02†
	5	1.296E-1	1.688E-0†	5.081E-0†	4.179E-1†	7.077E-1†	1.537E-1	2.503E-0†	2.490E-1†	1.008E+00†
		1.110E-2	1.729E-1†	2.842E-1†	4.919E-2†	9.928E-2†	1.465E-2	1.685E-1†	4.709E-2†	5.754E-01†
	6	1.746E-1	2.807E-0†	7.425E-0†	9.642E-1†	6.861E-1†	1.759E-1	2.851E-0†	4.720E-1†	4.894E-01
		1.619E-2	2.583E-1†	4.289E-1†	1.274E-1†	1.028E-1†	1.813E-2	4.458E-1†	3.040E-2†	1.051E-01
	8	2.252E-1	2.191E-0†	9.143E-0†	3.192E-0†	1.425E-0†	2.226E-1	5.522E-0†	9.382E-1†	6.791E-01
		1.945E-2	1.335E-0†	1.175E-0†	4.423E-1†	1.171E-1†	2.089E-2	3.640E-1†	1.270E+0†	2.903E-01
	10	2.812E-1	3.932E-0†	7.902E-0†	3.003E-0†	2.105E-0†	2.387E-1	6.296E-0†	1.389E+0†	5.741E-01
		3.289E-2	1.888E-0†	8.012E-1†	4.287E-1†	1.576E-1†	2.337E-2	4.517E-1†	1.944E+0†	1.281E-01

best values and second best values are highlighted with a deep gray background a gray background. The IGD values are to reflect the comprehensive performance of the comparable algorithms involving the convergence and diversity.

From Table 10, BESBS did the best on DTLZ1, DTLZ3, and DTLZ6 problems with 10 best and 5 second best out of 15 problems. SMS-EMOA performs best on 8- and 10-objective DTLZ6. ϵ-MOEA and MSOPS respectively performed well on DTLZ1, DTLZ3 problems with the number of objectives less than 8. This group of DTLZ problems are hard to converge, thereby it is mainly to challenge the convergence ability of algorithms. When comparing the Table 6, Table 8 and Table 10, a similar conclusion can be made that BESBS has better comprehensive performance on most problems since it has good convergence and diversity on most problems. Thus, BESBS is competitive on the first group of DTLZ problems.

In Table 11, varying strategies have different advantages on the second group

Table 11: The IGD values (average values, variance values) of the obtained solutions of each algorithm on DTLZ2, DTLZ4, DTLZ5, DTLZ7, where the best values are shown with a deep gray background and the second one with a gray background. † is a symbol of the algorithm which BESBS has better prominent performance than.

Problems	Obj.	BESBS	ϵ-MOEA	NSGA-III	MSOPS	HypE	SMS-EMOA	AR+DMO	GrEA	GDE-MOEA
DTLZ2	3	6.549E-2 1.916E-3	6.306E-2 9.763E-4	5.320E-2 1.241E-4	7.263E-2 9.108E-4	1.066E-1† 5.389E-3†	7.099E-2 2.581E-3	1.957E-1† 3.775E-2†	7.753E-2 2.176E-6	2.65E-01† 6.83E-02†
	5	1.781E-1 5.719E-3	1.905E-1† 9.616E-3†	1.745E-1 1.759E-3	1.938E-1† 4.559E-3†	2.553E-1† 8.487E-3†	2.034E-1† 5.720E-3†	2.494E-1† 1.821E-2†	1.750E-1 1.034E-5	6.41E-01 2.09E-01
	6	3.132E-1 6.929E-3	3.005E-1 9.980E-3	3.209E-1 6.663E-2	3.037E-1 2.441E-3	4.833E-1† 1.219E-2†	3.660E-1† 1.119E-2†	5.599E-1† 5.376E-2†	2.955E-1 4.728E-4	6.42E-01 1.19E-01
	8	4.333E-1 6.721E-3	4.064E-1 8.945E-3	4.989E-1† 9.633E-2†	4.189E-1 2.193E-3	6.493E-1† 1.525E-2†	5.046E-1† 9.935E-3†	7.694E-1† 5.217E-2†	3.963E-1 2.666E-5	8.12E-01† 1.16E-01†
	10	5.128E-1 9.953E-3	4.589E-1 1.121E-2	9.317E-1† 1.319E-1†	4.976E-1 3.602E-3	7.806E-1† 9.119E-3†	6.043E-1† 1.090E-2†	8.975E-1† 3.275E-2†	4.827E-1 2.205E-5	9.87E-01† 1.53E-01†
DTLZ4	3	2.042E-1 2.749E-1	2.984E-1 3.157E-1	3.195E-1 3.627E-1	7.349E-2 2.118E-4	1.845E-1 1.913E-1	2.675E-1 2.791E-1	4.823E-1† 3.735E-1†	1.737E-1 6.333E-2	3.11E-01 4.10E-01
	5	2.250E-1 1.312E-1	4.198E-1† 2.556E-1†	3.137E-1 2.420E-1	1.896E-1 4.225E-3	2.830E-1 5.346E-2	4.578E-1† 3.092E-1†	5.555E-1† 3.321E-1†	2.396E-1 2.520E-2	4.14E-01 1.15E-01
	6	3.443E-1 5.209E-2	4.721E-1† 1.582E-1†	5.874E-1† 2.474E-1†	3.075E-1 1.475E-3	4.777E-1† 2.440E-2†	4.717E-1† 1.153E-1†	6.345E-1† 2.287E-1†	3.020E-1 1.372E-5	7.01E-01† 9.33E-02†
	8	4.474E-1 1.879E-2	5.274E-1† 1.146E-1†	1.089E-0† 1.449E-1†	4.314E-1 2.014E-3	6.167E-1† 1.387E-2†	5.086E-1† 3.598E-2†	7.219E-1† 4.997E-2†	4.011E-1 1.567E-5	8.94E-01† 4.51E-02†
	10	5.288E-1 1.563E-2	6.119E-1† 8.546E-2†	1.209E-0† 1.446E-1†	5.159E-1 2.767E-3	7.493E-1† 2.842E-2†	6.003E-1† 2.715E-2†	9.005E-1† 4.501E-2†	4.907E-1 1.102E-5	9.84E-01† 1.07E-01†
DTLZ5	3	7.406E-3 6.488E-4	6.974E-3 3.226E-4	1.805E-2† 1.415E-2†	2.002E-2† 1.689E-4†	1.172E-2† 1.351E-3†	6.935E-3 4.711E-4	6.123E-3 3.395E-3	1.286E-2 1.911E-7	1.33E-01† 3.53E-02†
	5	6.112E-2 1.134E-2	9.398E-2 1.193E-2	5.111E-1† 3.121E-1†	5.664E-2 5.619E-3	7.865E-2 1.750E-2	2.237E-2 1.983E-3	6.403E-2 6.736E-2	3.788E-2 2.099E-4	4.10E-01† 1.55E-01†
	6	8.005E-2 1.059E-2	1.219E-1 1.429E-2	6.066E-1† 2.101E-1†	8.579E-2 4.849E-3	7.299E-2 1.202E-2	2.619E-2 1.803E-3	1.387E-1 1.785E-1	9.472E-2 2.045E-4	5.31E-01† 1.01E-01†
	8	1.061E-1 1.801E-2	1.609E-1 1.663E-2	1.308E-0† 2.286E-1†	1.961E-1 7.138E-3	1.345E-1 3.019E-2	3.213E-2 3.103E-3	1.062E-0† 3.523E-1†	2.354E-1 2.593E-3	4.84E-01† 1.43E-01†
	10	1.123E-1 1.621E-2	1.585E-1 2.175E-2	1.239E-0† 2.057E-1†	2.946E-1† 2.500E-2†	5.499E-1† 3.675E-2†	3.323E-2 3.242E-3	1.449E-0† 3.229E-1†	3.236E-1 3.031E-3	4.34E-01† 1.72E-01
DTLZ7	3	7.555E-2 5.283E-2	6.431E-2 5.767E-2	4.016E-1† 1.565E-1†	1.580E-1 1.164E-2	1.154E-1 7.857E-3	1.393E-1 1.196E-1	6.511E-1† 2.470E-1†	1.012E-1† 3.551E-5†	4.58E-01† 3.33E-01†
	5	3.664E-1 5.379E-2	6.331E-1† 2.003E-1†	1.395E-0† 2.861E-1†	5.154E-1† 2.233E-2†	3.648E-1 5.784E-3	4.669E-1 1.591E-1	1.042E-0† 2.515E-1†	3.142E-1 9.010E-5	7.60E-01† 4.93E-01†
	6	5.773E-1 1.537E-1	5.796E-1 2.937E-1	3.028E-0† 7.163E-1†	7.995E-1 3.546E-2	5.156E-1 7.610E-3	5.731E-1 8.633E-2	1.973E-0† 6.338E-1†	4.571E-1 1.554E-4	9.59E-01† 1.64E-01†
	8	6.806E-1 3.060E-2	9.004E-1 1.938E-1	1.348E+1† 2.383E-0†	1.299E-0 1.873E-1	7.472E-1 4.018E-2	8.029E-1 1.249E-1	6.213E-0† 1.008E-0†	7.054E-1 3.102E-4	3.54E+00† 2.08E+00†
	10	1.030E-0 3.977E-2	1.239E-0 2.361E-1	3.274E+1† 5.358E-0†	3.187E+0 2.927E-1	1.185E+0 1.255E-1	1.218E+0 2.755E-1	1.066E+1† 1.243E-0†	9.635E-1† 8.580E-4†	5.50E+00† 4.64E+00†

of DTLZ. Specifically, BESBS did well on DTLZ5 and DTLZ7 problems with 8 and 10 objectives. The improved dominance relationship based ϵ-MOEA did well on DTLZ2 problems; the aggregation based approaches (NSGA-III, MSOPS) respectively outperformed on DTLZ2 and DTLZ4 problems with 3 and 5 objectives; the indicator based SMS-EMOA did well on DTLZ5 problems. However, GrEA achieved the smallest IGD values on most problems. It can be seen that GrEA gained good diversity performance from Table 9 but pool on convergence according to Table 7. Thus, for such problems with different shape and locations of PoF mainly challenging the diversity ability of the algorithms, GrEA is more promising. Thus, BESBS needs to prompt its diversity in the boundary elimination selection, although it has promising convergence ability for such problems according to the Table 7.

From Table 10 and Table 11, it can be seen that BESBS relatively outperforms on the problems which are challenging to converge. And GrEA also has better performance on such problems with different shape and locations of PoF.

6.0.4. HV comparison for comprehensive performance

Table 12 and Table 13 list the mean and standard deviation of HV values of eight algorithms after 30 independent runs on the DTLZ problems, where the best values and second best values are highlighted with a deep gray background a gray background. The HV values are to reflect the comprehensive performance of the comparable algorithms involving the convergence and distribution of the obtained solutions.

In Table 12, BESBS outperforms other algorithms on most problems with 9 best and 3 second best records. The following is MSOPS, which has better performance on DTLZ1 with more than 6 objectives. Especially, NSGA-III did good on low-dimensional problems especially on DTLZ1 with 3 and 5 objectives and DTLZ3 with 3 objectives. GrEA also has good performance on DTLZ6 with more than 5 objectives.

In Table 13, BESBS and SMS-EMOA relatively have better performance than others. Specifically, BESBS has 4 best and 6 second best records, and

SMS-EMOA has 5 best and 4 second best. The following one is HypE with 5 best and 1 second best record. To be mentioned, GrEA has 2 best and 5 second best respectively. From the table, the HV based HypE and SMS-EMOA outperform others on most DTLZ5 and DTLZ7 problems. About the ϵ based algorithms, BESBS did better than ϵ-MOEA and GDE-MOEA.

From Fig. 7-8, the performance of the algorithms can be seen visually. Fig. 7 shows the performance of eight algorithms on 10-objective DTLZ1. In the chart, each line stands for a solution[4]. Only the solutions obtained by BESBS and AR+DMO have converged into the Pareto front. Apart from Table 10, Fig. 7 also demonstrated that BESBS achieved a good balance between convergence and diversity.

Fig. 8 demonstrates the performances of the obtained solutions by eight algorithms on DTLZ6 with 10 objectives. Notably, 10-objective DTLZ6 is one of the MOPs relatively difficult to converge. It still can be seen from the figure that BESBS and SMS-EMOA relatively achieved the best with better convergence and the obtained solutions have similar trend of distribution as the PF of DTLZ6 with better diversity. The convergence of GDE-MOEA ranks first, but the distribution of the solutions obtained by GDE-MOEA is pool. The others achieved bad convergence since some objective values are more than 5.

From the above experiments in terms of the convergence, diversity, and comprehensive performance, it can be concluded that BESBS presented the most competitive performance on the set of DTLZ problems in comparison with the other eight compared algorithms. Also, its convergence on most problems is encouraging and not subject to the varying objectives on most problems.

7. Conclusion

In this paper, we proposed a novel selection strategy denoted by BESBS. In other words, the environmental selection mechanism is based on the boundary

[4]About each solution, the horizontal coordinate consists of items of the objectives, and the vertical coordinate records the solutions' objective values.

Table 12: The HV values (average values, variance values) of the obtained solutions of each algorithm on DTLZ1, DTLZ3, DTLZ6, where the best values are shown with a deep gray background and the second one with a gray background. † is a symbol of the algorithm which BESBS has better prominent performance than.

Problems	Obj.	BESBS	ϵ-MOEA	NSGA-III	MSOPS	HypE	SMS-EMOA	AR+DMO	GrEA	GDE-MOEA
DTLZ1	3	0.968948402 0.001050008	0.9532891197† 0.0041285452†	0.97479 0.000775242	0.97161 0.002789046	0.970152071 0.002688616	0.971829817 0.001455138	0.958876545 0.012556287	0.9470000718† 0.0221038481†	0.9123794557† 0.0419884591†
	5	0.997436644 0.000320784	0.951848333 0.017185335	0.998199859 0.000691353	0.99772 0.000220101	0.7531727176† 0.4260677743†	0.825432603 0.383426218	0.955023713 0.029116729	0.986180766 0.01396665	0.820863267 0.07704043
	6	0.998854347 0.000235752	0.920353709 0.03444032	0.93436652 0.111940859	0.99897 0.000182878	0† 0†	0.467822796 0.499420256	0.966239231 0.005388691	0.964391309 0.064698241	0.899793302 0.037555518
	8	0.999666943 0.00016893	0.919439813 0.061069853	0.6193963276† 0.4383510919†	0.99988 0.000168655	0† 0†	0† 0†	0.956303172 0.024342735	0.997131905 0.001164038	0.6466659145† 0.3731045245†
	10	0.999847514 0.000207956	0.749912737 0.254330876	0.3125915048† 0.3566563018†	0.99994 0.000126491	0.3273318454† 0.4396049654†	0.997950979 0.000843972	0.953562118 0.01801512	0.982115147 0.024537008	0† 0†
DTLZ3	3	7.398988307 0.005619046	7.377408708 0.012727219	7.41408 0.004773259	7.37776 0.006529795	7.388532448 0.028448434	7.400482595 0.011643099	7.122118514 0.123243431	6.908780147 0.853789475	5.17960065660† 0.54011064070†
	5	31.63582371 0.014818797	31.19177362 0.225270543	8.570713594 11.35887458	31.5392 0.007390083	0† 0†	7.21552933310† 13.10714863110†	27.62003206 1.63185327	29.3876405320† 24.89572570190†	0† 0†
	6	63.68717354 0.030687776	62.98805832 0.273011118	6.087674 9.250915494	63.6032 0.031207976	0† 0†	6.61761471140† 19.79081658340†	56.60241151 3.832670542	62.03790882 2.52855044	0† 0†
	8	255.6824557 0.074215643	100.95562570231† 29.052077988†	0† 0†	255.5648 0.154073345	0† 0†	0† 0†	218.8997714 13.65681062	239.171845 25.62487049	0† 0†
	10	1023.504159 0.178061246	240.1698807463† 7.5693639908†	0† 0†	748.34858103930† 82.02105386110†	0† 0†	0† 0†	885.0110042 50.08510327	1011.694176 16.71372973	0† 0†
DTLZ6	3	6.055447428 0.033465412	5.988325871 0.034517161	5.934921736 0.027126393	5.95096 0.054525411	5.95359158610† 0.05907307480†	5.984369877 0.033009152	5.75900048820† 0.25367427040†	5.962065940† 0.02141700490†	3.16945997660† 0.11347754410†
	5	22.44253141 0.077511703	2.68680979470† 1.11999712640†	0† 0†	18.054080† 0.70457584530†	14.31713780720† 1.76806491180†	14.07531319760† 1.18897884980†	0† 0†	20.31384398 4.105545193	6.48250040220† 5.26874056790†
	6	43.61424721 0.342050754	0.00444450010† 0.01122423110†	0† 0†	16.46805453450† 3.7378008070†	28.59032611660† 2.82075322360†	34.73846106690† 1.18811818380†	0† 0†	28.701560495110† 0.40331853410†	18.00544726730† 1.15849775580†
	8	164.1680343 2.303015436	34.195916530520† 6.27757431680†	0† 0†	0.01299675750† 0.02855480660†	31.600983371510† 9.29377531850†	80.88806518470† 11.79390158770†	0† 0†	89.122530705760† 1.58852782220†	48.96457616830† 19.77518574020†
	10	605.3229748 19.83300386	57.601176777690† 3.83006944740†	0† 0†	0† 0†	0† 0†	0.04780591860† 0.13616797970†	0† 0†	297.178322348720† 49.37646649680†	252.843013399820† 20.43118897680†

Table 13: The HV values (average values, variance values) of the obtained solutions of each algorithm on DTLZ2, DTLZ4, DTLZ5, DTLZ7, where the best values are shown with a deep gray background and the second one with a gray background. † is a symbol of the algorithm which BESBS has better prominent performance than.

Problems	Obj.	BESBS	ϵ-MOEA	NSGA-III	MSOPS	HypE	SMS-EMOA	AR+DMO	GrEA	GDE-MOEA
DTLZ2	3	7.438792559 0.015315049	7.380171563 0.015197873	7.412799221 0.025547548	7.38688 0.002901647	7.401472661 0.015110765	7.400092281 0.018078096	7.2904491056† 0.0372458217†	7.387952885 0.011774386	4.9079835898† 0.3830770165†
	5	31.60371126 0.02030101	31.42794846 0.040625758	31.63930606 0.010774033	31.56384 0.012625477	31.56739775 0.025796303	31.57528017 0.03792735	30.68557076 0.206893336	31.60551543 0.065018288	21.072714076† 3.8562576904†
	6	63.65772925 0.055317775	63.30259647 0.164162808	63.6945535 0.018192388	63.61856 0.089863739	63.63053892 0.051093192	63.62589253 0.02780218	61.38894183 0.668372438	63.69227149 0.030121736	40.02605323† 9.751591834†
	8	255.5929104 0.138083404	253.9873326 0.435437096	255.5695509 0.075264016	255.6953599 0.098374395	255.3982949 0.097483762	255.7080235 0.09870561	241.1849178364† 2.6922827274†	255.2791413 0.138464046	147.75624121143† 9.0003604238†
	10	1023.424385 0.212851112	981.8241148 11.4919839	883.95209233051† 12.4147754971†	1023.54944 0.120195845	1023.235932 0.15098585	1023.223771 0.2952268	964.0195588702† 10.6330510212†	1022.635216 0.585764029	546.01101132348† 2.8503030725†
DTLZ4	3	6.766884807 1.082174332	7.097885364 0.479858692	7.017709712 0.53080755	7.41152 0.024207106	6.991803027 0.888614876	6.548196755 1.01122667	5.62840514 1.446660546	7.261191868 0.321840817	6.510868945 1.412341168
	5	30.96177898 0.736974439	30.05425704 2.012078287	31.35638556 0.387205912	31.5584 0.032212627	31.4424226 0.335206728	30.68113757 1.56803601	29.68095445 2.143036244	31.68743325 0.024678464	29.15306327 0.980483932
	6	63.53124649 0.362459934	62.64276704 2.236574617	59.6957566255† 5.2106303214†	63.6448 0.051752574	63.65311619 0.055854911	63.07304266 0.63866408	58.98456686 8.347332687	63.69710479 0.037351458	58.1802410112† 2.1232665637†
	8	255.778289 0.061208358	254.5964259 0.626505046	194.77244515755† 0.1525878488†	255.63648 0.035799342	255.6006335 0.151306304	255.8025061 0.061566542	246.369355 3.461006554	255.7304491 0.058500901	219.7721333991† 2.5383061975†
	10	1023.88698 0.101723354	1019.862146 3.743924589	570.130203770918† 2.5771334384†	1023.68256 0.075557354	1023.214913 0.358144702	1022.917897 1.024497533	965.4888877 22.21011859	1023.777626 0.111899599	816.408339925815† 7.8966265894†
DTLZ5	3	6.088799247 0.004509215	6.066741979 0.016924233	6.050743044 0.071541227	6.076639999 0.01316048	6.062244841 0.02707819	6.107240694 0.023536244	6.084705607 0.037966797	6.091594506 0.020010828	4.4362474386† 0.3570627504†
	5	23.57012725 0.116018286	22.3471429611† 0.1378484114†	21.003799519† 1.6834987397†	22.91039988 0.143594624	23.01512473 0.344354962	23.47995639 0.112577842	21.0137520217† 1.7754349714†	23.19811096 0.157490387	10.8727601999† 0.578873186†
	6	46.20574343 0.386867029	43.72028042 1.252493994	37.6846393829† 4.1853856637†	44.07167999 0.354233565	45.7451265 0.314443997	46.98460342 0.188540207	43.1194728182† 4.2334116187†	45.4353755 0.420505521	21.0609571118† 1.8536718636†
	8	180.3811692 13.71304413	166.7166296 4.752182972	106.3012338033† 16.4448050264†	155.62161 3.208383874	173.8037243 3.100643606	186.0530386 0.76130198	74.36960538135† 5.1197870778†	159.787835 5.645024405	73.1790565794† 4.0655018265†
	10	706.5569865 10.60942408	663.182954 7.536353622	130.4526726635† 79.8765690631†	559.6897199625† 46.2910789297†	614.3562582771† 23.5531898767†	674.2628318 19.99222467	64.72711152041† 26.2189488063†	615.4813544 36.54667809	336.32136585351† 49.0193809815†
DTLZ7	3	13.09985813 0.963011931	12.87489874 0.994669515	13.14414538 0.058707963	13.27892523 0.052927373	13.45109857 0.081006559	12.80734869 1.310195406	7.7588373212† 1.3007448733†	13.0077546721† 0.1553615055†	9.7854891868† 2.6231083413†
	5	82.91708391 0.758361856	59.1696735033† 8.8574801753†	54.2063736534† 4.1990031087†	77.40284772 1.411923484	86.5755695 0.70377041	74.64066917 13.46846404	48.8831083427† 5.5048557654†	79.58096868 1.176672948	58.0588675011† 6.8432151894†
	6	184.4018277 19.34296803	139.30479109442† 0.6695204353†	55.7836202118† 19.8674736409†	156.6369750276† 7.6800019796†	198.2482407 0.834906169	184.4320405 27.21005047	83.34711870792† 1.5428955643†	172.7598402 1.951197925	114.6559063861† 9.3008231578†
	8	690.0375538 30.05166319	546.5398096663† 95.8013991085†	0† 0†	454.7983691161† 68.5997539748†	955.318468573† 47.9402835519†	940.6206427 46.00388041	103.0609555217† 60.2415619016†	727.1814429 12.04822884	281.54366708921† 28.2353958272†
	10	2982.992004 441.4748645	1879.2386557792† 388.6265479033†	0† 0†	782.5486418823† 268.5594002259†	3831.1116525369† 115.3952355744†	2522.180739462† 364.8413550541†	72.0677481308† 134.6043181585†	2722.264003 119.7407248	395.6039062618† 380.1349827149†

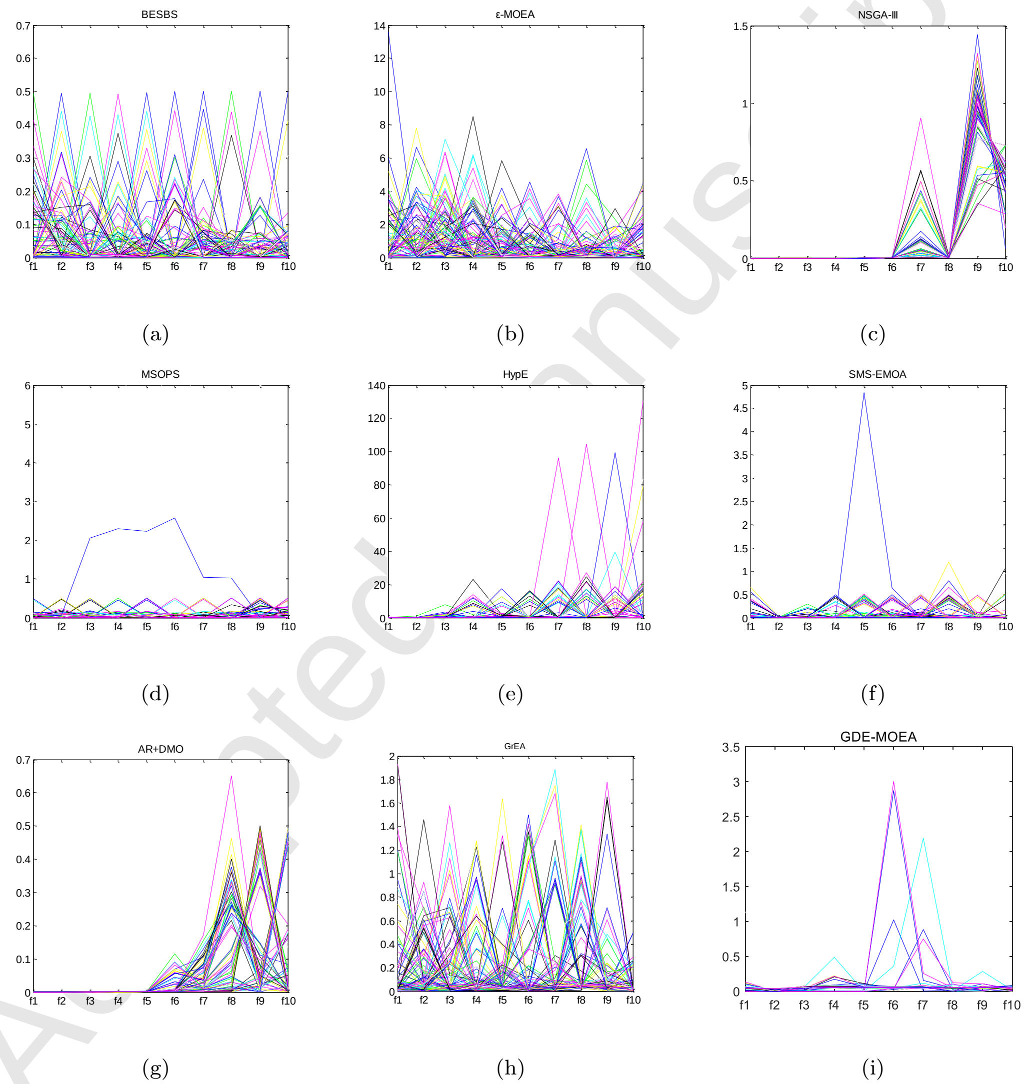
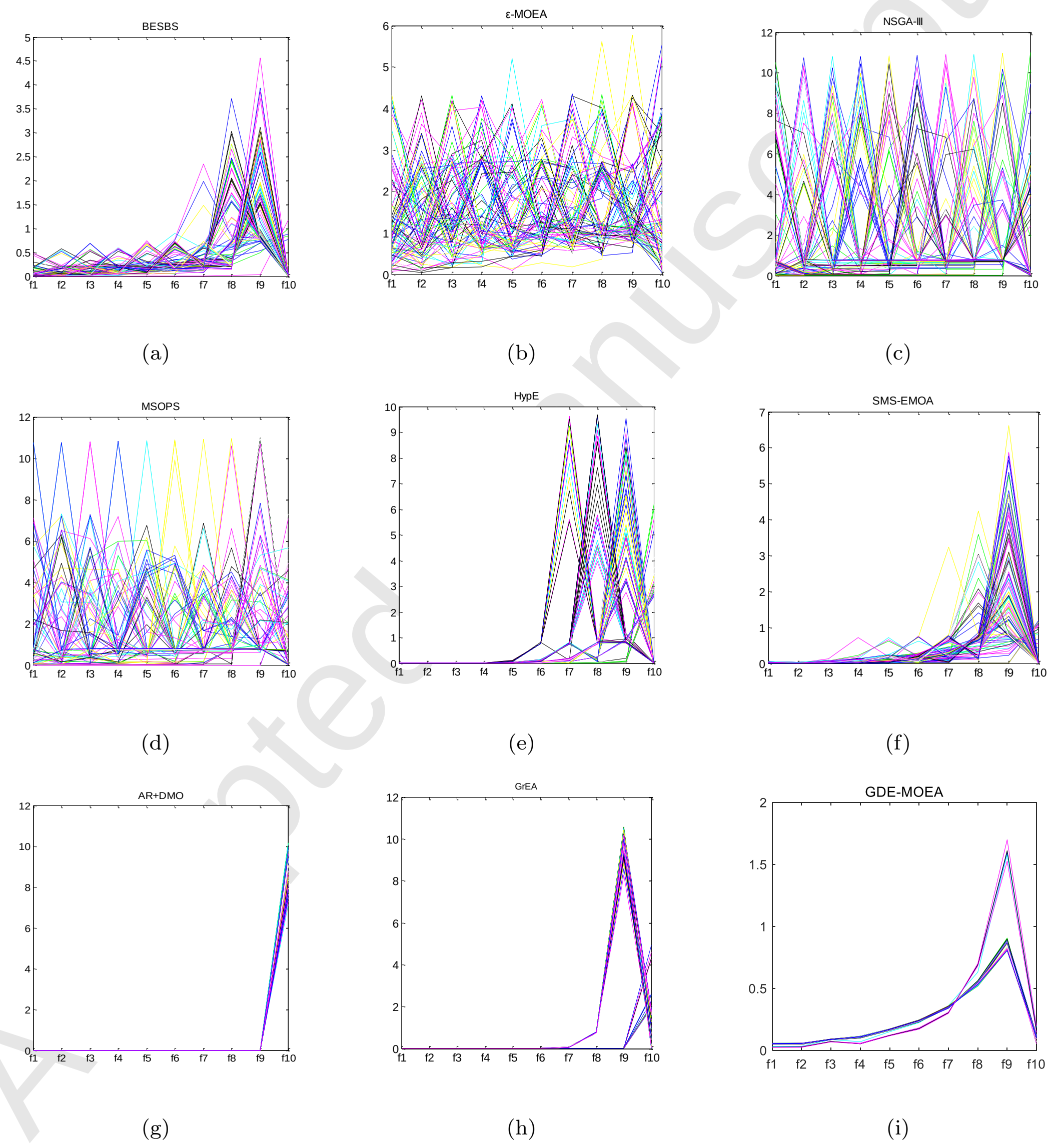


Figure 7: The performances of the obtained solutions by eight algorithms on DTLZ1 with 10 objectives.



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Figure 8: The performances of the obtained solutions by eight algorithms on DTLZ6 with 10 objectives.

elimination selection and the binary search. The binary search previously adjusts the ϵ value to keep the stability of the population and then transfers the ϵ value to the boundary elimination selection. Boundary elimination selection picks the elite solutions near the transferred coordinate according to their fitness by turns. Because of the assist from the penalty of fitness and shuffled coordinate indexes, solutions with good convergence will be selected, meanwhile, the DRSs will be eliminated by means of the adjusted ϵ-dominance relationship. Thereby the impact from the DRSs will be avoided and the solutions with good balance between the convergence and diversity will be chosen as much as possible. Overall, the main contribution of this paper is to propose a new idea to deal with multi- and many-objective problems that consider to retain the convergence information when promote the diversity during the second selection.

Systematic experiments were carried out to compare BESBS with other eight state-of-the-art EMO algorithms. Widely used test problems are chosen for challenging different abilities of the algorithms. The experimental results demonstrated that BESBS is very competitive on most testing instances in terms of finding well-approximating and well-distributed solutions in many-objective optimization. Although most algorithms are designed to reach the balance of convergence and diversity, the experiments also reveal that none of algorithms can beat all algorithms on any of the instances. In other words, when addressing a many-objective problem, more emphasis on the perspective of the decision maker put on the advantages of the algorithms is still needed.

However, BESBS has some disadvantages to address. From the experimental analysis, the penalty parameter λ has an impact on the selection to some extent. We still need to investigate the performance of BESBS with a flexible penalty. Besides, the diversity on the problems with varying shapes and locations is not competitive with GrEA. Thus, we need to conduct further experiments on the search behavior of BESBS so as to improve its performance.

8. Compliance with Ethical Standards

The authors declare that there is no conflict of interests regarding the publication of this paper. This article does not contain any studies with human participants or animals performed by any of the authors.

9. Acknowledgement

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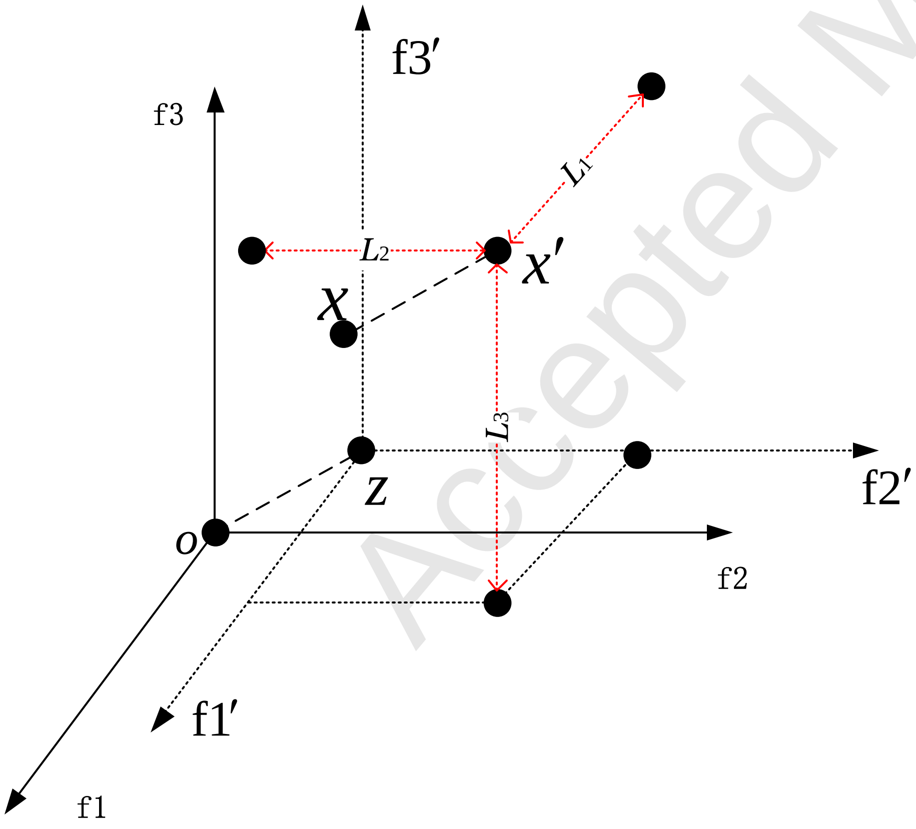
Highlights:

- This paper proposed a novel selection strategy -- the boundary elimination selection based on the binary search.
- This paper uses the binary search during the environmental selection to adjust the ε value to keep the stability of the population size.
- The binary search also transfers the ε value to the boundary elimination selection to increase the diversity of the population.
- The boundary elimination selection could enhance the convergence of the population and avoid the impact from the dominance resistant solutions.
- This paper could avoid the impact of DRSs during the optimization and achieve good balance between the convergence and diversity simultaneously.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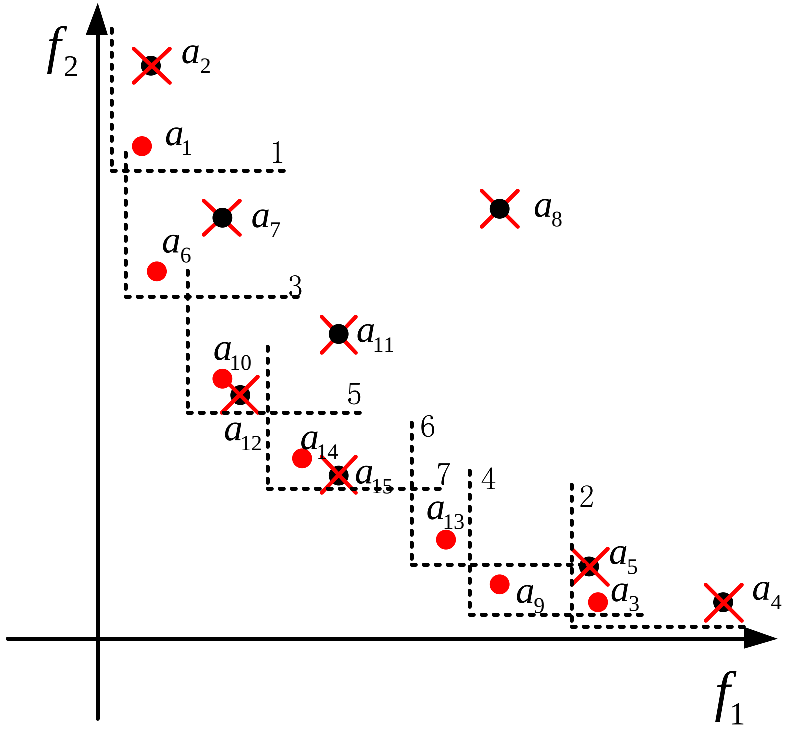
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