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Meyer, J. C.; Needham, D. J.

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Author for correspondence:
D. J. Needham
e-mail: d.j.needham@bham.ac.uk

Extended weak maximum principles for parabolic partial differential inequalities on unbounded domains

J. C. Meyer and D. J. Needham

School of Mathematics, University of Birmingham, Watson Building, Edgbaston, Birmingham B15 2TT, UK

In this paper, we establish extended maximum principles for solutions to linear parabolic partial differential inequalities on unbounded domains, where the solutions satisfy a variety of growth/decay conditions on the unbounded domain. We establish a conditional maximum principle, which states that a solution u to a linear parabolic partial differential inequality satisfies a maximum principle whenever a suitable weight function can be exhibited. Our extended maximum principles are then established by exhibiting suitable weight functions and applying the conditional maximum principle. In addition, we include several specific examples, to highlight the importance of certain generic conditions, which are required in the statements of maximum principles of this type. Furthermore, we demonstrate how to obtain associated comparison theorems from our extended maximum principles.

1. Introduction

Maximum principles are primarily used in the study of initial-boundary value problems to obtain *a priori* bounds on solutions, comparison theorems and uniqueness results (for example, see the established texts [1,2]). A secondary application of maximum principles can be found in the qualitative study of solutions to initial-boundary problems; some recent trends and open problems can be found in the texts [3–5] as well as in numerous others.

In this paper, we consider maximum principles for linear parabolic operators on unbounded domains.

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Specifically, let $\Omega \subseteq \mathbb{R}^n$ be an unbounded open connected set with boundary $\partial\Omega$. Associated with Ω, we introduce

$$D_T = \Omega \times (0,T], \quad D_T^X = \{(x,t) \in D_T : |x| < X\}, \quad \partial D_T = (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T]),$$

for $T, X > 0$, with closures $\bar{D}_T$ and $\bar{D}_T^X$. Here, $\bar{D}_T = D_T \cup \partial D_T$. In addition, let L be an operator that acts on sufficiently smooth functions $u : D_T \to \mathbb{R}$, given by

$$L[u] := u_t - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} - \sum_{i=1}^{n} b_i u_{x_i} - cu \quad \text{on } D_T, \tag{1.1}$$

where $a_{ij}, b_i, c : D_T \to \mathbb{R}$ $(1 \le i, j, \le n)$ are prescribed functions on D_T. When the matrix $A(x,t) = (a_{ij}(x,t))$ is symmetric and positive semi-definite for each $(x,t) \in D_T$, then we refer to L as a linear parabolic operator. The primary purpose of this paper is to extend the relationship between allowable spatial growth/decay as $|x| \to \infty$, of solutions to linear parabolic partial differential inequalities ($L[u] \le 0$ on D_T) and the conditions on the coefficients of the linear parabolic operator L, for which a maximum principle holds on $\bar{D}_T$. In this respect, it is convenient to introduce E_α^λ, for $\alpha \in [0,\infty)$, $\lambda \in [0,\infty)$ as the set of continuous functions $u : \bar{D}_T \to \mathbb{R}$ (some $T > 0$) such that $u \in C^{2,1}(D_T)$ and

$$u(x,t) \le k_1 \, \mathrm{e}^{k_2 (1+|x|^2)^\alpha (1+\ln(1+|x|^2))^\lambda} \quad \forall (x,t) \in \bar{D}_T \tag{1.2}$$

for some $k_1, k_2 > 0$. Additionally, we also refer to E_α^λ, for $\alpha \in (-\infty, 0]$, $\lambda \in (-\infty, 0]$ as the set of continuous functions $u : \bar{D}_T \to \mathbb{R}$ (some $T > 0$) such that $u \in C^{2,1}(D_T)$ and

$$u(x,t) \le k_1 \, \mathrm{e}^{-k_2 (1+|x|^2)^{|\alpha|} (1+\ln(1+|x|^2))^{|\lambda|}} \quad \forall (x,t) \in \bar{D}_T \tag{1.3}$$

for some $k_1, k_2 > 0$. A secondary purpose of the paper is to highlight the importance of certain generic conditions on the linear parabolic operators L, for maximum principles to hold, via the provision of specific examples.

We first give a brief summary of the development of maximum principles (occasionally referred to as Phragmèn Lindelöf principles) for linear parabolic partial differential inequalities on unbounded domains [1,6] related to those established in this paper. In [7], a maximum principle for a linear parabolic partial differential inequality on an unbounded domain was obtained, which complemented the non-uniqueness result for the linear heat equation obtained in [8]. Specifically, this maximum principle was designed for linear parabolic partial differential inequalities to allow uniqueness to be established for classical solutions to the linear heat equation, under the weakest possible growth conditions as $|x| \to \infty$. Following these works, maximum principles for linear parabolic partial differential inequalities on unbounded domains, with specific growth conditions on the solutions as $|x| \to \infty$, which have the general form given in (1.2) (for various values of $\alpha, \lambda \ge 0$), were extensively developed (in particular, see [1,9–15] and references therein). In the development of this body of work, considerations regarding the optimum conditions on the associated maximum principles are rare; it is typical for a maximum principle to be established, without any discussion regarding limitations to the extension of the maximum principle, beyond the limitations of the method of proof.

More recently, in [16–20], via an alternative approach to that adopted in this paper, uniqueness results for initial-boundary value problems for parabolic partial differential equations have been established, with growth conditions specified on the solutions as $|x| \to \infty$ and $t \to 0^+$. To obtain these results, additional regularity on the coefficients a_{ij}, b_i and c in the linear parabolic operator L must be imposed, which we do not require for the results obtained in this paper. We also note that maximum principles for operators which have an additional coefficient $d : D_T \to \mathbb{R}$, multiplying the term u_t in (1.1), have been considered in [21], and although we do not consider these operators here, the approach we use can be readily adapted to accommodate these operators.

The main achievements of this paper comprise a generalization of the maximum principles established in [13,15] (which subsumed the results in [1,7,9,10,12]) for solutions to linear parabolic partial differential inequalities on unbounded domains with growth conditions on the solutions as $|x| \to \infty$ of the form given in (1.2), which we henceforth refer to as type (1.2) growth conditions.

We achieve this via a relaxation of the conditions in [13,15], on the coefficients b_i in the linear operator L. In addition, we extend the maximum principles established in [13,15] for solutions to linear parabolic partial differential inequalities on unbounded domains with decay conditions on the solutions as $|x| \to \infty$ of the form given in (1.3), which we henceforth refer to as type (1.3) decay conditions, which have not been not considered in any of the previously mentioned works. We highlight this because, in numerous applications of maximum principles, the rate of decay of the solution as $|x| \to \infty$ to the parabolic partial differential inequality is known, and hence, our results may be applicable, whereas the maximum principles designed for solutions with growth conditions as $|x| \to \infty$ of type (1.2) may be inapplicable. Additionally, we have constructed specific examples to highlight that extensions to these maximum principles, in certain generic directions, are not possible.

The structure of the paper is as follows. In §2, we establish a weak maximum principle for a linear parabolic operator on unbounded domains, which is an extension of the classical weak maximum principle [22] onto unbounded domains. From this weak maximum principle, we obtain a widely applicable conditional maximum principle, and in doing so, illustrate how to obtain maximum principles for linear parabolic operators on unbounded domains with varying growth/decay conditions as $|x| \to \infty$. This maximum principle is conditional because it depends on the existence of a suitable weight function ϕ. We also provide a subtle example to illustrate the importance of the conditions under which these maximum principles are obtained. In §3, we establish new maximum principles which generalize and extend the maximum principles contained in [13,15] by relaxing the conditions on the first-order coefficients b_i in the linear parabolic operator L, and considering additional classes of solutions of type (1.3), which are at most decaying as $|x| \to \infty$. We achieve this by establishing the existence of suitable weight functions ϕ that allow applications of the conditional maximum principle established in §2. We complete the section by providing a function which demonstrates that our relaxation on the first-order coefficient in the linear parabolic operator is in a sense optimal, in that, at most it can be relaxed by a logarithmic growth in the spatial variables. In §4, we demonstrate briefly how these maximum principles can be applied to obtain comparison theorems and uniqueness results for a class of semi-linear parabolic initial-boundary value problems.

2. The conditional maximum principle

Here, we establish a conditional maximum principle for linear parabolic operators on an unbounded domain. This is in the spirit of those available for elliptic operators on bounded domains [1, ch. 2, section 9] and for parabolic operators on unbounded domains [6, pp. 211–214]. This conditional result reduces the proof of a maximum principle for a specified linear parabolic operator L to establishing the existence of a suitable weight function ϕ. First, we have the following.

Definition 2.1. A linear parabolic operator L (defined on D_T) is said to satisfy condition (H) on a set $E \subseteq D_T$ when $c : D_T \to \mathbb{R}$ is bounded above on E.

Definition 2.1 is associated with the classical maximum principle for a linear parabolic operator on a compact domain [1, pp. 174–175]. We now review a well-established maximum principle that plays a crucial role in obtaining our conditional maximum principle (for a similar result, see [11, p. 43]).

Lemma 2.2. *Suppose that the linear parabolic operator L satisfies condition (H) on $E = D_T$. Moreover, suppose that $u : \bar{D}_T \to \mathbb{R}$ is continuous with $u \in C^{2,1}(D_T)$ and*

$$L[u] \leq 0 \quad \text{on } D_T, \tag{2.1}$$

and

$$\liminf_{r\to\infty} \sup_{\substack{(x,t)\in\bar{D}_T \\ |x|=r}} u(x,t) \leq 0 \tag{2.2}$$

while $u \leq 0$ on ∂D_T. Then, $u \leq 0$ on $\bar{D}_T$.

Proof. It follows from condition (H) that there exists a constant $C > 0$ such that

$$c(x,t) < C \quad \forall\,(x,t) \in D_T. \tag{2.3}$$

Now, we define the function $w : \bar{D}_T \to \mathbb{R}$ given by

$$w(x,t) = u(x,t)\,\mathrm{e}^{-Ct} \quad \forall\,(x,t) \in \bar{D}_T. \tag{2.4}$$

It follows immediately that w is continuous on $\bar{D}_T$, $w \in C^{2,1}(D_T)$ and $w \leq 0$ on ∂D_T. Additionally, via (2.1) and (2.4), it follows that

$$w_t - \sum_{i,j=1}^{n} a_{ij} w_{x_i x_j} - \sum_{i=1}^{n} b_i w_{x_i} - (c - C)w \leq 0 \quad \text{on } D_T, \tag{2.5}$$

where $a_{ij}, b_i, c : D_T \to \mathbb{R}$ are the coefficients in L. Furthermore, it follows from (2.2) and (2.4) that

$$\liminf_{r\to\infty} \sup_{\substack{(x,t)\in\bar{D}_T\\ |x|=r}} w(x,t) \leq 0. \tag{2.6}$$

Therefore, there exists a sequence of real numbers $\{X_n\}_{n\in\mathbb{N}}$ such that $X_n \to \infty$ as $n \to \infty$ and

$$\sup_{\substack{(x,t)\in\bar{D}_T\\ |x|=X_n}} w(x,t) \leq \frac{1}{n}. \tag{2.7}$$

We now show that

$$w(x,t) \leq \frac{1}{n} \quad \forall\,(x,t) \in \bar{D}_T^{X_n} \tag{2.8}$$

for any $n \in \mathbb{N}$ and hence that $w \leq 0$ on $\bar{D}_T$. Suppose that (2.8) is false. Then, via (2.7), because w is bounded and continuous on $\bar{D}_T^{X_n}$, it follows that there exists $(x^*,t^*) \in D_T^{X_n}$ such that

$$\sup_{(x,t)\in\bar{D}_T^{X_n}} w(x,t) = w(x^*,t^*) > \frac{1}{n}. \tag{2.9}$$

Then, via (2.9), (2.5) and (2.3), it follows that

$$w_t(x^*,t^*) - \sum_{i,j=1}^{n} a_{ij}(x^*,t^*) w_{x_i x_j}(x^*,t^*) \leq (c(x^*,t^*) - C)w(x^*,t^*) < 0. \tag{2.10}$$

Now, because the matrix $A(x^*,t^*) = (a_{ij}(x^*,t^*))$ is symmetric and positive semi-definite, it follows that there exists an invertible linear coordinate change

$$x_i = \sum_{j=1}^{n} c_{ij} y_j$$

such that

$$\sum_{i,j=1}^{n} a_{ij}(x^*,t^*) w_{x_i x_j} = \sum_{r=1}^{n} \lambda_r^* w_{y_r y_r} \tag{2.11}$$

with $\lambda_r^* \geq 0$, $r = 1, \ldots, n$, being the eigenvalues of $A(x^*,t^*)$. Thus, it follows from (2.10) and (2.11) that

$$w_t(x^*,t^*) - \sum_{i=1}^{n} \lambda_i^* w_{y_i y_i}(x^*,t^*) < 0. \tag{2.12}$$

Now, because $(x^*,t^*) \in D_T^{X_n}$ is a local maxima of w, then it follows that

$$w_t(x^*,t^*) \geq 0 \quad \text{and} \quad w_{y_i y_i}(x^*,t^*) \leq 0, \tag{2.13}$$

and so,

$$w_t(x^*,t^*) - \sum_{i=1}^{n} \lambda_i^* w_{y_i y_i}(x^*,t^*) \geq 0$$

which contradicts (2.12). We conclude that

$$\sup_{(x,t)\in\bar{D}_T^{X_n}} w(x,t) \le \frac{1}{n},$$

for each $n \in \mathbb{N}$, and so

$$w(x,t) \le 0 \quad \forall\,(x,t) \in \bar{D}_T. \tag{2.14}$$

The result follows from (2.4) and (2.14). ■

From lemma 2.2, we can now establish a conditional maximum principle that can be used to obtain maximum principles for parabolic operators not necessarily satisfying condition (H). This maximum principle is conditional as its application relies on the construction of a suitable weight function. We note that a similar concept is introduced in [6, p. 213].

Lemma 2.3. *Let $u : \bar{D}_T \to \mathbb{R}$ be continuous with $u \in C^{2,1}(D_T)$ and $u \le 0$ on ∂D_T. In addition, let L be a linear parabolic operator with $L[u] \le 0$ on D_T. Suppose there exists a continuous function $\phi : \bar{D}_T \to \mathbb{R}$ such that $\phi > 0$ on $\bar{D}_T$ with $\phi \in C^{2,1}(D_T)$ and*

$$\liminf_{r\to\infty} \sup_{\substack{(x,t)\in\bar{D}_T\\ |x|=r}} \frac{u(x,t)}{\phi(x,t)} \le 0,$$

$$\frac{-L[\phi]}{\phi} \text{ is bounded above on } D_T.$$

Then, $u \le 0$ on $\bar{D}_T$.

Proof. First, we define the function $w : \bar{D}_T \to \mathbb{R}$ such that

$$w(x,t) = \frac{u(x,t)}{\phi(x,t)} \quad \forall\,(x,t) \in \bar{D}_T. \tag{2.15}$$

It follows immediately that w is continuous, $w \in C^{2,1}(D_T)$, $w \le 0$ on ∂D_T and

$$\liminf_{r\to\infty} \sup_{\substack{(x,t)\in\bar{D}_T\\ |x|=r}} w(x,t) \le 0. \tag{2.16}$$

Moreover, we observe that w satisfies

$$\tilde{L}[w] := w_t - \sum_{i,j=1}^{n} a_{ij} w_{x_i x_j} - \sum_{i=1}^{n}\left(b_i + \sum_{j=1}^{n} 2a_{ij}\frac{\phi_{x_i}}{\phi}\right) w_{x_i} - \left(-\frac{1}{\phi}L[\phi]\right) w \le 0 \quad \text{on } D_T. \tag{2.17}$$

Because the linear parabolic operator $\tilde{L}$ satisfies condition (H) on D_T, via (2.15)–(2.17), an application of lemma 2.2 gives

$$w \le 0 \quad \text{on } \bar{D}_T$$

and hence, via (2.15), that $u \le 0$ on $\bar{D}_T$, as required. ■

It follows that the establishment of a maximum principle for a specific function $u : \bar{D}_T \to \mathbb{R}$ and a specific linear parabolic operator L is reduced to finding a function $\phi : \bar{D}_T \to \mathbb{R}$ which satisfies the conditions in lemma 2.3. An advantage of this conditional maximum principle is that not only can it be used to develop generic maximum principles, as we demonstrate in §3, but it can also be used to obtain maximum principles for specific problems which do not adhere to the conditions of the available generic maximum principles, if a suitable weight function $\phi : \bar{D}_T \to \mathbb{R}$ can be found. We also note that, without further technical difficulties, lemma 2.3, with suitable minor modifications in statement, can be established when $u : \bar{D}_T \to \mathbb{R}$ is replaced by $u : \bar{\Omega} \times (0,T] \to \mathbb{R}$, with u being bounded above on D_T^x for each $x > 0$, and $u \le 0$ on $\Omega \times \{0\}$ is replaced by

$$\liminf_{t\to 0} \sup_{\substack{(x,s)\in D_T^X \\ s=t}} u(x,s) \le 0 \quad \forall X > 0.$$

Before we establish new generic maximum principles in the following section, we give an example to illustrate the importance of condition (2.2) in lemma 2.2. Specifically, we produce a function $u : \bar{D}_T \to \mathbb{R}$ and a linear parabolic operator L for which all of the conditions in lemma 2.2 are satisfied *except* that condition (2.2) is marginally violated, and for which the conclusion of lemma 2.2 is false. To begin, let $\Omega = \mathbb{R}$ (and so $\partial\Omega = \emptyset$) and introduce $u : \bar{D}_1 \to \mathbb{R}$ defined as

$$u(x,t) = \begin{cases} -1 + \dfrac{2\sqrt{2}}{(1+t)^{1/2}}\, \mathrm{e}^{-((x-\ln(t))^2/4(1+t))}; & (x,t) \in D_1 \\ -1; & (x,t) \in \partial D_1. \end{cases} \tag{2.18}$$

It is readily established that u is continuous on $\bar{D}_1$. Moreover, $u \in C^{2,1}(D_1)$, with

$$u_x(x,t) = \frac{-\sqrt{2}(x-\ln(t))}{(1+t)^{3/2}}\, \mathrm{e}^{-((x-\ln(t))^2/4(1+t))}, \tag{2.19}$$

$$u_{xx}(x,t) = \frac{\sqrt{2}}{(1+t)^{3/2}} \left(-1 + \frac{(x-\ln(t))^2}{2(1+t)} \right) \mathrm{e}^{-((x-\ln(t))^2/4(1+t))} \tag{2.20}$$

and

$$u_t(x,t) = \frac{\sqrt{2}}{(1+t)^{3/2}} \left(-1 + \frac{(x-\ln(t))}{t} + \frac{(x-\ln(t))^2}{2(1+t)} \right) \mathrm{e}^{-((x-\ln(t))^2/4(1+t))} \tag{2.21}$$

for all $(x,t) \in D_1$. Furthermore,

$$|u(x,t)| \le 2\sqrt{2} - 1 \tag{2.22}$$

for all $(x,t) \in \bar{D}_1$ and so u is bounded on $\bar{D}_1$. Additionally,

$$\sup_{x\in\mathbb{R}} u(x,t) = -1 + \frac{2\sqrt{2}}{(1+t)^{1/2}} \quad \text{for } t \in (0,1], \tag{2.23}$$

and

$$\inf_{x\in\mathbb{R}} u(x,t) = -1 \quad \text{for } t \in (0,1]. \tag{2.24}$$

We observe that

$$\sup_{x\in\mathbb{R}} u(x,t) \ge 1 \quad \text{for all } t \in (0,1], \tag{2.25}$$

and

$$\sup_{x\in\mathbb{R}} u(x,0) = -1. \tag{2.26}$$

Moreover, via (2.19)–(2.21), we have

$$L[u] := u_t - u_{xx} + \left(\frac{1}{t}\right) u_x = 0, \tag{2.27}$$

for all $(x,t) \in D_1$, and so (2.27) corresponds to the inequality (1.1) with

$$a(x,t) = 1 \quad \forall\, (x,t) \in D_1, \tag{2.28}$$

$$b(x,t) = -\frac{1}{t} \quad \forall\, (x,t) \in D_1 \tag{2.29}$$

and

$$c(x,t) = 0 \quad \forall\, (x,t) \in D_1. \tag{2.30}$$

Thus, we have constructed a function $u : \bar{D}_1 \to \mathbb{R}$, and a linear parabolic operator L with $a, b, c : D_1 \to \mathbb{R}$ as given in (2.28)–(2.30) respectively, so that all the conditions of lemma 2.2 are satisfied, except condition (2.2), and for which the conclusion of lemma 2.2 is violated, via (2.25). We now consider how this example violates condition (2.2). We observe from (2.18) that

$$u(x,t) \to -1 \quad \text{as } |x| \to \infty \; \forall\, t \in [0,1].$$

However, this limit is *not* uniform for $t \in [0,1]$. Moreover,

$$\sup_{\substack{(x,t)\in\bar{D}_1\\ |x|=r}} u(x,t) \geq 1 \quad \forall\, r \geq 1,$$

and so,

$$\liminf_{r\to\infty} \sup_{\substack{(x,t)\in\bar{D}_1\\ |x|=r}} u(x,t) \geq 1$$

which violates condition (2.2). This feature is related to the unboundedness of $b(x,t)$ as $t \to 0^+$ in D_1 and leads to the resulting failure of lemma 2.2.

3. Maximum principles

Here, we apply lemma 2.3 to recover and extend the maximum principles developed in [1,7,9,10,12–15] for linear parabolic operators L, whose coefficients are constrained by the growth conditions of the unbounded solutions. For $\alpha, \lambda \geq 0$, we obtain maximum principles for successively smaller sets of functions E_α^λ (as in (1.2)) where the conditions on the coefficients of the linear parabolic operators L are dependent on the set of functions E_α^λ. In particular, we establish a generalization of the maximum principle in [15] (which itself, recovered and generalized the results in [1,7,9,10,12–14]), via a relaxation of the condition on the first-order coefficients in the linear parabolic operator L. Moreover, for E_α^λ, as in (1.3), with $\alpha < 0$ or $\lambda < 0$, we establish maximum principles of a form which have not been considered in any of the above works. We are able to make these extensions, following the careful consideration of the conditions on the first-order coefficients $b_i : D_T \to \mathbb{R}$ in the linear parabolic operator. At the end of this section, we give examples of functions which exhibit that the conditions under which the following maximum principles are established are, in some sense, optimal, namely that the conditions on b_i in theorems 3.5 and 3.4 are logarithmically sharp and algebraically sharp, respectively. To begin, we have the following.

Definition 3.1. Let $\psi \in C^2([1,\infty))$ be a positive strictly increasing function such that there exist constants $\mu, p_1, p_2 > 0$, for which,

$$\eta\psi''(\eta) \leq p_1\psi(\eta)\psi'(\eta) \tag{3.1}$$

and

$$0 < \eta\psi'(\eta) \leq p_2(\psi(\eta))^{2-\mu} \tag{3.2}$$

for all $\eta \in [1,\infty)$. A linear parabolic operator L is said to satisfy condition $(H)'$ with μ and ψ, when there exists constants $\bar{A}, \bar{B}, \bar{C} \geq 0$ such that for $1 \leq i \leq n$,

$$0 \leq a_{ii}(x,t) \leq \frac{\bar{A}}{\psi'(1+|x|^2)}, \tag{3.3}$$

$$b_i(x,t)x_i \leq \bar{B}\frac{\psi(1+|x|^2)}{\psi'(1+|x|^2)} \tag{3.4}$$

and
$$c(x,t) \leq \bar{C}(\psi(1+|x|^2))^\mu, \tag{3.5}$$

for all $(x,t) \in D_T$.

We next establish the existence of a suitable weight function $\phi : \bar{D}_T \to \mathbb{R}$ which may be used in applications of lemma 2.3. In the following result, we provide an extension, for our purpose, of that in [15, lemma 2].

Lemma 3.2. *Let L be a linear parabolic operator which satisfies condition* $(H)'$ *with* μ *and* ψ*. Additionally, for any* $k>0$*, let*

$$\delta = \min\left\{T, \frac{1}{\mu(\tilde{A}+\tilde{B}+\tilde{C}+1)}\right\}, \tag{3.6}$$

where

$$\tilde{A} = 4n^2\bar{A}\left(\frac{|\mu-1|p_2}{(\psi(1))^{\mu}} + p_1 + k\mu e p_2\right), \quad \tilde{B} = 2n\left(\bar{B} + \frac{\bar{A}}{\psi(1)}\right), \quad \tilde{C} = \frac{\bar{C}}{k\mu}.$$

Then, the continuous function $\phi : \bar{D}_\delta \to \mathbb{R}$*, given by,*

$$\phi(x,t) = \mathrm{e}^{k(\psi(1+|x|^2))^{\mu}\,\mathrm{e}^{t/\delta}} \quad \forall\,(x,t) \in \bar{D}_\delta, \tag{3.7}$$

satisfies $\phi > 0$ *on* $\bar{D}_\delta$*, with* $\phi \in C^{2,1}(D_\delta)$*, and*

$$-\frac{L[\phi]}{\phi} \le 0 \quad \text{on } D_\delta.$$

Proof. Because $A(x,t) = (a_{ij}(x,t))$ is symmetric and positive semi-definite for all $(x,t) \in D_T$, then

$$|a_{ij}(x,t)x_ix_j| \le \sqrt{a_{ii}(x,t)a_{jj}(x,t)}(1+|x|^2) \le \bar{A}\frac{(1+|x|^2)}{\psi'(1+|x|^2)} \quad \forall\,(x,t) \in D_T. \tag{3.8}$$

Now, let $\phi : \bar{D}_\delta \to \mathbb{R}$ be as given in (3.7) and, for $(x,t) \in \bar{D}_T$, set $s = (1+|x|^2)$. Observe that $\phi \in C^{2,1}(D_\delta)$ and

$$\begin{aligned}
\phi_t(x,t) &= \frac{k}{\delta}(\psi(s))^{\mu}\,\mathrm{e}^{t/\delta}\phi(x,t)\\
\phi_{x_i}(x,t) &= 2k\mu(\psi(s))^{\mu-1}\psi'(s)x_i\,\mathrm{e}^{t/\delta}\phi(x,t)\\
\phi_{x_ix_j}(x,t) &= k\mu\,\mathrm{e}^{t/\delta}\phi(x,t)(4(\mu-1)(\psi(s))^{\mu-2}(\psi'(s))^2x_ix_j + 4(\psi(s))^{\mu-1}\psi''(s)x_ix_j\\
&\qquad + 2(\psi(s))^{\mu-1}\psi'(s)\delta_{ij} + 4k\mu\,\mathrm{e}^{t/\delta}(\psi(s))^{2\mu-2}(\psi'(s))^2x_ix_j)
\end{aligned}$$

for all $(x,t) \in D_\delta$. Thus, we have

$$\begin{aligned}
\frac{-L[\phi](x,t)}{\phi(x,t)} &= k\mu\,\mathrm{e}^{t/\delta}(\psi(s))^{\mu}\left(-\frac{1}{\delta\mu} + 2\sum_{i=1}^{n}(b_i(x,t)x_i + a_{ii}(x,t))\frac{\psi'(s)}{\psi(s)} + \frac{c(x,t)}{k\mu\,\mathrm{e}^{t/\delta}(\psi(s))^{\mu}}\right.\\
&\qquad \left. + 4\sum_{i,j=1}^{n}a_{ij}(x,t)x_ix_j\left(\frac{(\mu-1)(\psi'(s))^2}{(\psi(s))^2} + \frac{\psi''(s)}{\psi(s)} + k\mu\,\mathrm{e}^{t/\delta}\frac{(\psi'(s))^2}{(\psi(s))^{2-\mu}}\right)\right)
\end{aligned} \tag{3.9}$$

for all $(x,t) \in D_\delta$. It now follows from (3.8) and definition 3.1 that

$$\begin{aligned}
&4\sum_{i,j=1}^{n}a_{ij}(x,t)x_ix_j\left(\frac{(\mu-1)(\psi'(s))^2}{(\psi(s))^2} + \frac{\psi''(s)}{\psi(s)} + k\mu\,\mathrm{e}^{t/\delta}\frac{(\psi'(s))^2}{(\psi(s))^{2-\mu}}\right)\\
&\quad \le \frac{4n^2\bar{A}s}{\psi'(s)}\left(\frac{|\mu-1|(\psi'(s))^2}{(\psi(s))^2} + \frac{\max\{0,\psi''(s)\}}{\psi(s)} + k\mu\,\mathrm{e}^{t/\delta}\frac{(\psi'(s))^2}{(\psi(s))^{2-\mu}}\right)\\
&\quad \le 4n^2\bar{A}\left(\frac{|\mu-1|p_2}{(\psi(1))^{\mu}} + p_1 + k\mu e p_2\right) = \tilde{A} \quad \forall\,(x,t) \in D_\delta.
\end{aligned} \tag{3.10}$$

In addition, via definition 3.1, we have

$$2\sum_{i=1}^{n}(b_i(x,t)x_i + a_{ii}(x,t))\frac{\psi'(s)}{\psi(s)} \le 2n\left(\bar{B} + \frac{\bar{A}}{\psi(1)}\right) = \tilde{B} \quad \forall\,(x,t) \in D_\delta. \tag{3.11}$$

Furthermore, via definition 3.1, we have

$$\frac{c(x,t)}{k\mu\,\mathrm{e}^{t/\delta}(\psi(s))^{\mu}} \le \frac{\bar{C}}{k\mu} = \tilde{C} \quad \forall\,(x,t) \in D_\delta. \tag{3.12}$$

Therefore, it follows from (3.9)–(3.12), with (3.6) that

$$\begin{aligned}\frac{-L[\phi](x,t)}{\phi(x,t)} &\le k\mu\, \mathrm{e}^{t/\delta}(\psi(s))^{\mu}\left(-\frac{1}{\delta\mu}+\tilde{A}+\tilde{B}+\tilde{C}\right)\\ &\le k\mu\, \mathrm{e}^{t/\delta}(\psi(s))^{\mu}(-(\tilde{A}+\tilde{B}+\tilde{C}+1)+\tilde{A}+\tilde{B}+\tilde{C})\le 0 \quad \forall\,(x,t)\in D_\delta,\end{aligned}$$

as required. ∎

We can now establish a generalization of the maximum principle presented in [15]. We have the following.

Theorem 3.3. *Let $u:\bar{D}_T\to\mathbb{R}$ be continuous, with $u\in C^{2,1}(D_T)$ and $u\le 0$ on ∂D_T. In addition, let L be a linear parabolic operator which satisfies condition (H)′ with μ and ψ, and such that $L[u]\le 0$ on D_T. When there exists $k>0$ such that*

$$\liminf_{r\to\infty}\sup_{\substack{(x,t)\in\bar{D}_T\\ |x|=r}}\frac{u(x,t)}{\mathrm{e}^{k(\psi(1+|x|^2))^{\mu}}}\le 0, \tag{3.13}$$

then $u\le 0$ on $\bar{D}_T$.

Proof. Suppose there exists $k>0$ such that condition (3.13) is satisfied. With this value of $k>0$, set $\delta>0$ and $\phi:\bar{D}_\delta\to\mathbb{R}$ as in (3.6) and (3.7). It then follows from lemmas 3.2 and 2.3, together with condition (3.13), that $u\le 0$ on $\bar{D}_\delta$. If $\delta=T$, the proof is complete. If $\delta\ne T$, then

$$\delta=\delta'=\frac{1}{\mu(\tilde{A}+\tilde{B}+\tilde{C}+1)}<T.$$

We can then repeat the above step $N(\in\mathbb{N})$ times, with $\delta=\delta'$, and $0<T-(N+1)\delta'\le\delta'$. We may then take a final step with $\delta=T-(N+1)\delta'$, and so we have $u\le 0$ on $\bar{D}_T$ $(T=\delta'+N\delta'+(T-(N+1)\delta'))$. ∎

Next, we establish generalizations of the maximum principles given in [13,14] for solutions to partial differential inequalities in E^{λ}_{α} with $\alpha,\lambda\ge 0$. We present these maximum principles in descending order, in that the sets E^{λ}_{α} in the following theorems get subsequently smaller while the conditions on the coefficients in the linear parabolic operator relax, tighten and switch sign (see theorems 3.4, 3.5, 3.9 and 3.10).

Theorem 3.4. *Let $u:\bar{D}_T\to\mathbb{R}$ be continuous with $u\in E^{\lambda}_{\alpha}$ for $\alpha\in(0,\infty)$, $\lambda\in[0,\infty)$. In addition, let L be a linear parabolic operator which, for $A,B,C\ge 0$ satisfies*

$$\left.\begin{aligned}&0\le a_{ii}(x,t)\le A(1+|x|^2)^{1-\alpha}(1+\log(1+|x|^2))^{-\lambda}\\ &b_i(x,t)x_i\le B(1+|x|^2)\\ \text{and}\quad &c(x,t)\le C(1+|x|^2)^{\alpha}(1+\log(1+|x|^2))^{\lambda}\end{aligned}\right\} \tag{3.14}$$

for all $(x,t)\in D_T$ and $1\le i\le n$. When $u\le 0$ on ∂D_T and $L[u]\le 0$ on D_T, then $u\le 0$ on $\bar{D}_T$.

Proof. Let $\psi:[1,\infty)\to\mathbb{R}$ be given by

$$\psi(\eta)=\eta^{\alpha}(1+\log(\eta))^{\lambda}\quad \forall\,\eta\in[1,\infty). \tag{3.15}$$

It follows that $\psi\in C^2([1,\infty))$, $\psi(\eta)\ge 1$ and

$$\psi'(\eta)=\eta^{\alpha-1}(1+\log(\eta))^{\lambda}\left(\alpha+\frac{\lambda}{(1+\log(\eta))}\right)>0 \tag{3.16}$$

and

$$\begin{aligned}\psi''(\eta)&=\eta^{\alpha-2}(1+\log(\eta))^{\lambda}\left(\alpha+\frac{\lambda}{(1+\log(\eta))}\right)\\ &\quad\times\left(\alpha-1+\frac{\lambda}{1+\log(\eta)}-\frac{\lambda}{(1+\log(\eta))(\alpha(1+\log(\eta))+\lambda)}\right)\end{aligned} \tag{3.17}$$

for all $\eta \in [1,\infty)$. We next verify that $\psi : [1,\infty) \to \mathbb{R}$, with $\mu = 1$, satisfies conditions (3.1) and (3.2) in definition 3.1. From (3.16) and (3.17), we have

$$\begin{aligned}\eta\psi''(\eta) &= \psi'(\eta)\left(\alpha - 1 + \frac{\lambda}{1+\log(\eta)} - \frac{\lambda}{(1+\log(\eta))(\alpha(1+\log(\eta))+\lambda)}\right)\\ &\le \psi'(\eta)(\alpha+\lambda)\\ &\le (\alpha+\lambda)\psi(\eta)\psi'(\eta)\end{aligned}$$

for all $\eta \in [1,\infty)$, which verifies (3.1). Additionally, via (3.16), we have

$$\eta\psi'(\eta) = \left(\alpha + \frac{\lambda}{(1+\log(\eta))}\right)\psi(\eta) \le (\alpha+\lambda)\psi(\eta)$$

for all $\eta \in [1,\infty)$, which verifies (3.2). It then follows directly from the conditions (3.14) and definition 3.1, that L satisfies condition $(H)'$ with $\mu = 1$ and ψ given by (3.15). Now, with $u \in E^{\lambda}_{\alpha}$, there exists $k > 0$ such that

$$\liminf_{r\to\infty} \sup_{\substack{(x,t)\in\bar{D}_T\\ |x|=r}} \frac{u(x,t)}{e^{k\psi(1+|x|^2)}} \le 0. \tag{3.18}$$

Therefore, because L satisfies condition $(H)'$ with $\mu = 1$ and ψ given by (3.15), an application of theorem 3.3, with (3.18), establishes that $u \le 0$ on $\bar{D}_T$, as required. ∎

Theorem 3.5. *Let $u : \bar{D}_T \to \mathbb{R}$ be continuous with $u \in E^{\lambda}_{\alpha}$ for $\alpha = 0$, $\lambda \in (1,\infty)$. In addition, let L be a linear parabolic operator which, for $A, B, C \ge 0$ satisfies*

$$0 \le a_{ii}(x,t) \le A(1+|x|^2)(1+\log(1+|x|^2))^{2-\lambda}$$
$$b_i(x,t)x_i \le B(1+|x|^2)(1+\log(1+|x|^2))$$
$$c(x,t) \le C(1+\log(1+|x|^2))^{\lambda}$$

for all $(x,t) \in D_T$ and $1 \le i \le n$. When $u \le 0$ on ∂D_T and $L[u] \le 0$ on D_T, then $u \le 0$ on $\bar{D}_T$.

Proof. Let $\psi : [1,\infty) \to \mathbb{R}$ be given by

$$\psi(\eta) = (1+\log(\eta))^{\lambda-1} \quad \forall\, \eta \in [1,\infty). \tag{3.19}$$

It is readily verified that L satisfies condition $(H)'$ with $\mu = \lambda/(\lambda-1)$ and ψ given by (3.19). The remainder of the proof follows that of theorem 3.4. ∎

Theorems 3.4 and 3.5 recover and extend the maximum principles, which have been developed chronologically in [7,9–12], and extend the maximum principles in [13,14]. We note that maximum principles are considered in [13], which have growth conditions which we have not considered here for the sake of brevity (these are obtained directly from lemma 2.3 with an appropriate weight function ϕ). We now focus our attention on the classes of solutions that decay as $|x| \to \infty$, which have received much less attention in the literature. Generally, when considering solutions in this class, results with similar coefficient conditions to those in lemma 2.2, theorems 3.4 or 3.5 are applied to obtain maximum principles; however, these can be considerably improved by *a priori* defining the decay of the solution as $|x| \to \infty$. To begin, we require the following.

Definition 3.6. Let $\psi \in C^2([1,\infty))$, $\mu > 0$ and the linear parabolic operator L be as in definition 3.1, with (3.1) replaced by

$$\eta\psi''(\eta) \ge -p_1\psi'(\eta)\psi(\eta) \tag{3.20}$$

for all $\eta \in [1,\infty)$, and (3.4) replaced by

$$b_i(x,t)x_i \ge -\bar{B}\frac{\psi(1+|x|^2)}{\psi'(1+|x|^2)} \tag{3.21}$$

for all $(x,t) \in D_T$ and $1 \le i \le n$. When conditions (3.20), (3.2), (3.3), (3.21) and (3.5) are satisfied, then the linear parabolic operator L is said to satisfy condition $(H)''$ with μ and ψ.

We now have the following.

Lemma 3.7. *Let L be a linear parabolic operator which satisfies condition $(H)''$ with μ and ψ. Additionally, for any $k<0$, let*

$$\delta=\min\left\{T,\frac{1}{\mu(|\tilde{A}|+|\tilde{B}|+|\tilde{C}|+1)}\right\}, \tag{3.22}$$

where

$$\tilde{A}=4n^2\bar{A}\left(\frac{-|\mu-1|p_2}{(\psi(1))^{\mu}}-p_1+k\mu p_2\right),\quad \tilde{B}=-2n\bar{B},\quad \tilde{C}=\frac{e\bar{C}}{k\mu}.$$

Then, the continuous function $\phi:\bar{D}_\delta\to\mathbb{R}$, given by,

$$\phi(x,t)=\mathrm{e}^{k(\psi(1+|x|^2))^{\mu}\,\mathrm{e}^{-t/\delta}}\quad\forall\,(x,t)\in\bar{D}_\delta, \tag{3.23}$$

satisfies $\phi>0$ on $\bar{D}_\delta$, with $\phi\in C^{2,1}(D_\delta)$, and

$$-\frac{L[\phi]}{\phi}\le 0\quad\text{on } D_\delta.$$

Proof. We proceed as in the proof of lemma 3.2 with $\phi:\bar{D}_\delta\to\mathbb{R}$ given by (3.23), and $k<0$. It then follows that

$$\begin{aligned}\frac{-L[\phi](x,t)}{\phi(x,t)}&=k\mu\,\mathrm{e}^{-t/\delta}(\psi(s))^{\mu}\left(\frac{1}{\delta\mu}+2\sum_{i=1}^{n}(b_i(x,t)x_i+a_{ii}(x,t))\frac{\psi'(s)}{\psi(s)}+\frac{\mathrm{e}^{t/\delta}c(x,t)}{k\mu(\psi(s))^{\mu}}\right.\\&\left.\quad+4\sum_{i,j=1}^{n}a_{ij}(x,t)x_ix_j\left(\frac{(\mu-1)(\psi'(s))^2}{(\psi(s))^2}+\frac{\psi''(s)}{\psi(s)}+k\mu\,\mathrm{e}^{-t/\delta}\frac{(\psi'(s))^2}{(\psi(s))^{2-\mu}}\right)\right)\end{aligned} \tag{3.24}$$

for all $(x,t)\in D_\delta$. Now, it follows from (3.8) and definition 3.6 that

$$\begin{aligned}&4\sum_{i,j=1}^{n}a_{ij}(x,t)x_ix_j\left(\frac{(\mu-1)(\psi'(s))^2}{(\psi(s))^2}+\frac{\psi''(s)}{\psi(s)}+k\mu\,\mathrm{e}^{-t/\delta}\frac{(\psi'(s))^2}{(\psi(s))^{2-\mu}}\right)\\&\quad\ge\frac{4n^2\bar{A}s}{\psi'(s)}\left(\frac{-|\mu-1|(\psi'(s))^2}{(\psi(s))^2}+\frac{\min\{0,\psi''(s)\}}{\psi(s)}+k\mu\frac{(\psi'(s))^2}{(\psi(s))^{2-\mu}}\right)\\&\quad\ge 4n^2\bar{A}\left(\frac{-|\mu-1|p_2}{(\psi(1))^{\mu}}-p_1+k\mu p_2\right)=\tilde{A}\quad\forall\,(x,t)\in D_\delta.\end{aligned} \tag{3.25}$$

In addition, via definition 3.6, we have

$$2\sum_{i=1}^{n}(b_i(x,t)x_i+a_{ii}(x,t))\frac{\psi'(s)}{\psi(s)}\ge-2n\bar{B}=\tilde{B}\quad\forall\,(x,t)\in D_\delta, \tag{3.26}$$

and

$$\frac{\mathrm{e}^{t/\delta}c(x,t)}{k\mu(\psi(s))^{\mu}}\ge\frac{e\bar{C}}{k\mu}=\tilde{C}\quad\forall\,(x,t)\in D_\delta. \tag{3.27}$$

Therefore, it follows via (3.24)–(3.27) that

$$\frac{-L[\phi](x,t)}{\phi(x,t)}\le k\mu\,\mathrm{e}^{-t/\delta}(\psi(s))^{\mu}\left(\frac{1}{\delta\mu}+\tilde{A}+\tilde{B}+\tilde{C}\right)\le 0\quad\forall\,(x,t)\in D_\delta,$$

as required. ∎

We now make a further extension of the maximum principle contained in [15] for solutions that satisfy a specified decay condition as $|x|\to\infty$.

Theorem 3.8. *Let $u : \bar{D}_T \to \mathbb{R}$ be continuous, $u \in C^{2,1}(D_T)$ and $u \leq 0$ on ∂D_T. In addition, let L be a linear parabolic operator which satisfies condition $(H)''$ with μ and ψ, and such that $L[u] \leq 0$ on D_T. When there exists $k < 0$ such that*

$$\liminf_{r\to\infty} \sup_{\substack{(x,t)\in\bar{D}_T\\ |x|=r}} \frac{u(x,t)}{e^{k(\psi(1+|x|^2))^{\mu}}} \leq 0,$$

then $u \leq 0$ on $\bar{D}_T$.

Proof. The proof follows the same steps as the proof of theorem 3.3. ∎

We are now in a position to establish new maximum principles, of the type considered in [13,14] for solutions which satisfy specified decay conditions as $|x| \to \infty$ of type (1.3), and which complement theorems 3.4 and 3.5. Such maximum principles have not been considered in any of the previously mentioned works, with the exception of results relating to lemmas 2.2 and 2.3. The novelty of these new maximum principles can be observed in the sign change in the condition on the first-order coefficients b_i. We now have the following.

Theorem 3.9. *Let $u : \bar{D}_T \to \mathbb{R}$ be continuous with $u \in E^{\lambda}_{\alpha}$ for $\alpha = 0$, $\lambda \in (-\infty, -1)$. In addition, let L be a linear parabolic operator which, for $A, B, C \geq 0$ satisfies*

$$0 \leq a_{ii}(x,t) \leq A(1+|x|^2)(1+\log(1+|x|^2))^{2-|\lambda|}$$

$$b_i(x,t)x_i \geq -B(1+|x|^2)(1+\log(1+|x|^2))$$

$$c(x,t) \leq C(1+\log(1+|x|^2))^{|\lambda|}$$

for all $(x,t) \in D_T$ and $1 \leq i \leq n$. When $u \leq 0$ on ∂D_T and $L[u] \leq 0$ on D_T, then $u \leq 0$ on $\bar{D}_T$.

Proof. For $\lambda < -1$, let $\psi : [1,\infty) \to \mathbb{R}$ be given by

$$\psi(\eta) = (1+\log(\eta))^{|\lambda|-1} \quad \forall\, \eta \in [1,\infty). \tag{3.28}$$

and $\mu = |\lambda|/(|\lambda|-1)$. It follows that $\psi \in C^2([1,\infty))$, $\psi(\eta) \geq 1$ and

$$\psi'(\eta) = \frac{(|\lambda|-1)(1+\log(\eta))^{|\lambda|-2}}{\eta} > 0 \tag{3.29}$$

and

$$\psi''(\eta) = \frac{(|\lambda|-1)(1+\log(\eta))^{|\lambda|-2}}{\eta^2}\left(\frac{(|\lambda|-2)}{(1+\log(\eta))} - 1\right) \tag{3.30}$$

for all $\eta \in [1,\infty)$. We now verify conditions (3.20) and (3.2) in definition 3.6 for $\psi : [1,\infty) \to \mathbb{R}$ given by (3.28) and $\mu = |\lambda|/(|\lambda|-1)$. It follows from (3.29), (3.30) and (3.28) that

$$\eta\psi''(\eta) = \psi'(\eta)\left(\frac{|\lambda|-2}{(1+\log(\eta))} - 1\right) \geq -3\psi'(\eta) \geq -3\psi'(\eta)\psi(\eta)$$

for all $\eta \in [1,\infty)$, which verifies (3.20). Additionally, via (3.29) and (3.28),

$$0 < \eta\psi'(\eta) = (|\lambda|-1)(\psi(\eta))^{2-\mu}$$

for all $\eta \in [1,\infty)$, which verifies (3.2). Therefore, via the additional conditions in the statement, L satisfies condition $(H)''$ with $\mu = |\lambda|/(|\lambda|-1)$ and ψ given by (3.28). Furthermore, because $u \in E^{\lambda}_{0}$, there exists $k < 0$ such that

$$\liminf_{r\to\infty} \sup_{\substack{(x,t)\in\bar{D}_T\\ |x|=r}} \frac{u(x,t)}{e^{k\psi(1+|x|^2)^{\mu}}} \leq 0.$$

The result then follows from theorem 3.8. ∎

Complementary to this, we also have the following.

Theorem 3.10. *Let $u:\bar{D}_T\to\mathbb{R}$ be continuous with $u\in E_\alpha^\lambda$ for $\alpha\in(-\infty,0)$, $\lambda\in(-\infty,0]$. In addition, let L be a linear parabolic operator which, for $A,B,C\geq 0$ satisfies*

$$\begin{aligned}0\leq a_{ii}(x,t)&\leq A(1+|x|^2)^{1-|\alpha|}(1+\log(1+|x|^2))^{-|\lambda|}\\ b_i(x,t)x_i&\geq -B(1+|x|^2)\\ c(x,t)&\leq C(1+|x|^2)^{|\alpha|}(1+\log(1+|x|^2))^{|\lambda|}\end{aligned}$$

for all $(x,t)\in D_T$ and $1\leq i\leq n$. When $u\leq 0$ on ∂D_T and $L[u]\leq 0$ on D_T, then $u\leq 0$ on $\bar{D}_T$.

Proof. Let $\psi:[1,\infty)\to\mathbb{R}$ be given by

$$\psi(\eta)=\eta^{|\alpha|}(1+\log(\eta))^{|\lambda|}\quad\forall\,\eta\in[1,\infty). \tag{3.31}$$

It is readily verified that L satisfies condition $(H)''$ with $\mu=1$ and ψ given by (3.31). The remainder of the proof follows that of theorem 3.9. ∎

It is worth remarking that in [7,9–14,21,22], maximum principles are obtained where the condition on the first-order coefficient $b_i:D_T\to\mathbb{R}$ in the linear operator L, is bounded in modulus, namely for $B\geq 0$,

$$|b_i(x,t)|\leq B(1+|x|)\quad\forall\,(x,t)\in D_T.$$

This contrasts to the one-sided bounds in the statements of the maximum principles obtained here.

We now provide an example that illustrates the optimality of our condition on the first-order term $b_i:D_T\to\mathbb{R}$ in both theorems 3.4 and 3.5. With $\Omega=\mathbb{R}$, we consider $w:\bar{D}_1\to\mathbb{R}$ given by,

$$w(x,t)=\begin{cases}-1+2\,\mathrm{e}^{-(1/\gamma(\log(1+x^2))^\gamma+1-t)^2}; & (x,t)\in\bar{D}_1\backslash(\{0\}\times[0,1])\\ -1; & (x,t)\in\{0\}\times[0,1],\end{cases} \tag{3.32}$$

where $\gamma>0$ is constant. Observe that w is continuous on $\bar{D}_1$ and $w\in C^{2,1}(D_1)$, where $w_t:D_1\to\mathbb{R}$ and $w_x:D_1\to\mathbb{R}$ are given by,

$$w_t(x,t)=\begin{cases}4\,\mathrm{e}^{-(1/\gamma(\log(1+x^2))^\gamma+1-t)^2}\left(\dfrac{1}{\gamma(\log(1+x^2))^\gamma}+1-t\right); & (x,t)\in D_1\backslash(\{0\}\times(0,1])\\ 0; & (x,t)\in\{0\}\times(0,1],\end{cases} \tag{3.33}$$

and

$$w_x(x,t)=\begin{cases}4\,\mathrm{e}^{-(1/\gamma(\log(1+x^2))^\gamma+1-t)^2}\left(\dfrac{1}{\gamma(\log(1+x^2))^\gamma}+1-t\right)\\ \quad\times\left(\dfrac{2x}{(\log(1+x^2))^{1+\gamma}(1+x^2)}\right); & (x,t)\in D_1\backslash(\{0\}\times(0,1])\\ 0; & (x,t)\in\{0\}\times(0,1].\end{cases} \tag{3.34}$$

In addition,

$$|w(x,t)|\leq 1\quad\forall\,(x,t)\in\bar{D}_1, \tag{3.35}$$

and so $w\in E_0^\lambda$ for all $\lambda\geq 0$ (and hence $w\in E_\alpha^\lambda$ for all $\alpha,\lambda\geq 0$). In addition,

$$w(x,t)\to -1+2\,\mathrm{e}^{-(1-t)^2}\quad\text{as } |x|\to\infty \text{ uniformly for } t\in[0,1], \tag{3.36}$$

and

$$w(x,0)=\begin{cases}-1+2\,\mathrm{e}^{-(1/\gamma(\log(1+x^2))^\gamma+1)^2}; & x\in\mathbb{R}\backslash\{0\}\\ -1; & x=0.\end{cases} \tag{3.37}$$

Thus, via (3.33) and (3.34), we observe that

$$L[w]:=w_t-bw_x=0\quad\text{on } D_1, \tag{3.38}$$

where $b : D_1 \to \mathbb{R}$ is given by,

$$b(x,t) = \begin{cases} \dfrac{(\log(1+x^2))^{1+\gamma}(1+x^2)}{2x}; & (x,t) \in D_1 \backslash (\{0\} \times (0,1]) \\ 0; & (x,t) \in \{0\} \times (0,1], \end{cases} \tag{3.39}$$

and $L[\cdot]$ is a linear parabolic operator of the form (1.1), with $a, c : D_1 \to \mathbb{R}$ given by,

$$a(x,t) = 0 \quad \forall (x,t) \in D_1, \tag{3.40}$$

and

$$c(x,t) = 0 \quad \forall (x,t) \in D_1. \tag{3.41}$$

and $b : D_1 \to \mathbb{R}$ given by (3.39). Observe from (3.39) that

$$b(x,t)x = \frac{(\log(1+x^2))^{1+\gamma}(1+x^2)}{2} \quad \forall (x,t) \in D_1, \tag{3.42}$$

Thus, we have constructed a function $w : \bar{D}_1 \to \mathbb{R}$, with $a, b, c : D_1 \to \mathbb{R}$ as given in (3.40), (3.39) and (3.41), respectively, so that all the conditions of theorem 3.5 are satisfied except for the condition on $b : D_1 \to \mathbb{R}$, and for which theorem 3.5 (and theorem 3.4) fails.

Remark 3.11. Observe that it is the growth rate of $b : D_1 \to \mathbb{R}$ given by (3.39) as $|x| \to \infty$, and not the behaviour as $x \to 0$ that leads to the resulting failure of theorem 3.5 (and theorem 3.4). Moreover, it follows that the condition on $b_i : D_T \to \mathbb{R}$ in theorem 3.5 is logarithmically sharp, namely the condition on $b_i : D_T \to \mathbb{R}$ cannot be relaxed to allow larger logarithmic growth as $|x| \to \infty$, without altering other conditions. Additionally, it follows that the condition on $b_i : D_T \to \mathbb{R}$ in theorem 3.4 is algebraically sharp, namely the condition on $b_i : D_T \to \mathbb{R}$ cannot be relaxed to allow larger algebraic growth as $|x| \to \infty$, without altering other conditions. However, additional logarithmic growth, as in the conditions of theorem 3.5, is perhaps possible.

It should also be noted that if a function $u : \bar{D}_1 \to \mathbb{R}$ satisfies the conditions of theorem 3.5, with coefficients $a, b, c : D_1 \to \mathbb{R}$ given by (3.40)

$$b(x,t) = -k(\log(1+x^2))^{\gamma} x^3 \quad \forall (x,t) \in D_1, \tag{3.43}$$

and (3.41) respectively (with constants $k, \gamma > 0$), then theorem 3.5 implies that $u \le 0$ on $\bar{D}_1$, despite the superlinear growth of $b : D_1 \to \mathbb{R}$ as $|x| \to \infty$, given by (3.43), because the inequality on $xb(x,t)$ in theorem 3.5 only requires the growth rate as $|x| \to \infty$ to be limited from above. Such cases would be precluded in the maximum principles in [15], which require growth rate limitations on $|xb(x,t)|$ as $|x| \to \infty$. We also note that in [10, p. 17], an example is given that violates the conclusion of theorem 3.5; however, in this example, the conditions on both $a : D_1 \to \mathbb{R}$ and $b : D_1 \to \mathbb{R}$ are violated, and hence, it is more difficult to draw conclusions from it.

To contextualize the nature of theorem 3.10 as an extension of the maximum principles in [15], it is illustrative to consider the following example. Let $\Omega = \mathbb{R}$ and introduce the linear parabolic operator

$$L[u] := u_t - u_{xx} - bu_x - cu \quad \text{on } D_1, \tag{3.44}$$

where $b, c : D_1 \to \mathbb{R}$ are such that

$$b(x,t) = \tilde{b}(x) \quad \forall (x,t) \in D_1$$
$$c(x,t) = (1+x^2)^{\beta} \quad \forall (x,t) \in D_1$$

with $\tilde{b} : \mathbb{R} \to \mathbb{R}$ an increasing function without growth limitations as $|x| \to \infty$, and $\beta \in (0,1]$. For L given by (3.44), theorems 3.4 and 3.5 cannot be applied, owing to the unspecified growth of $\tilde{b}$ as $|x| \to \infty$. Moreover, lemma 2.2 cannot be applied since c is not bounded above on D_1. However, it follows that L given by (3.44) satisfies the conditions of theorem 3.10 with $\alpha = -\beta$ and $\lambda = 0$, and hence, if $u \in E^0_{-\beta}$ satisfies $L[u] \le 0$ with $u \le 0$ on ∂D_1, then $u \le 0$ on $\bar{D}_1$. In addition, note that if we consider L given by (3.44) but with $\beta > 1$, then L would not satisfy theorem 3.10, owing to the constant coefficient of the second-order term together with the growth of the coefficient of the zeroth-order term. Conversely, if we consider L given by (3.44) with $\tilde{b}:\mathbb{R}\to\mathbb{R}$ being a decreasing function, then L would satisfy the conditions of theorem 3.4 with $\alpha=\beta$ and $\lambda=0$, and hence, if $u\in E^0_\beta$ satisfies $L[u]\le 0$ with $u\le 0$ on ∂D_1, then $u\le 0$ on $\bar{D}_1$.

4. Applications

Here, we demonstrate how the maximum principles, we have developed in §3, can be used to establish comparison theorems. These comparison theorems can then be used to establish uniqueness results for the following semi-linear parabolic initial-boundary value problem, which commonly arises in both applied and theoretical studies of partial differential equations (see, for example, the recent texts [3–5], and the classical texts [6,11]). We restrict attention to bounded solutions (that is, in E^0_0) of initial-boundary value problems for semi-linear parabolic equations, for brevity, with results for unbounded/decaying solutions following similarly. Additionally, we note that comparison theorems and uniqueness results can be established for bounded solutions to initial-boundary value problems for quasi-linear/nonlinear parabolic equations via a similar approach to that which follows, provided appropriate restrictions on the quasi-linear/nonlinear terms hold (see, for example, [1] or [6]). Now, let $u:\bar{D}_T\to\mathbb{R}$ be continuous and bounded, and $u\in C^{2,1}(D_T)$, such that

$$L[u]=f(x,t,u) \quad \text{on } D_T, \tag{4.1}$$

where L is a linear parabolic operator as in (1.1), and

$$f:D_T\times\mathbb{R}\to\mathbb{R} \tag{4.2}$$

is a prescribed function, while

$$u=g \quad \text{on } \partial D_T, \tag{4.3}$$

where $g:\partial D_T\to\mathbb{R}$ is a given function, which is bounded and continuous. A continuous and bounded function $u:\bar{D}_T\to\mathbb{R}$ with $u\in C^{2,1}(D_T)$, and which satisfies (4.1) and (4.3) is referred to as a solution of the initial-boundary value problem (IBVP) with linear parabolic operator L, nonlinearity f and initial-boundary data g. Before we establish our results relating to (IBVP), we require two definitions.

Definition 4.1. Let $\bar{u},\underline{u}:\bar{D}_T\to\mathbb{R}$ be continuous and bounded, and $\bar{u},\underline{u}\in C^{2,1}(D_T)$. Suppose further that

$$\begin{aligned} L[\bar{u}]-f(x,t,\bar{u})&\ge 0 \quad \text{on } D_T,\\ L[\underline{u}]-f(x,t,\underline{u})&\le 0 \quad \text{on } D_T,\\ \underline{u}\le g&\le \bar{u} \quad \text{on } \partial D_T, \end{aligned}$$

where L is a linear parabolic operator and, $f:D_T\times\mathbb{R}\to\mathbb{R}$ and $g:\partial D_T\to\mathbb{R}$ are prescribed functions. Then, on $\bar{D}_T$, $\underline{u}$ is called a *regular subsolution* and $\bar{u}$ is called a *regular supersolution* to (IBVP) with linear parabolic operator L, nonlinearity f and initial-boundary data g.

Definition 4.2. The function $f:D_T\times\mathbb{R}\to\mathbb{R}$ is said to satisfy the condition $(H)_\alpha$ with $\alpha\ge 0$ when for any closed bounded interval $M\subset\mathbb{R}$, there exists a constant $k_M>0$ such that for all $u,v\in M$ with $u\ge v$, f satisfies the inequality

$$f(x,t,u)-f(x,t,v)\le k_M(1+x^2)^\alpha(u-v) \quad \forall\,(x,t)\in D_T.$$

The following observation is useful.

Remark 4.3. Let f satisfy condition $(H)_\alpha$ with $\alpha \geq 0$, then on every closed bounded interval $M \subset \mathbb{R}$, there exists a constant $k_M > 0$ such that for all $u, v \in M$ with $u \neq v$, then,

$$\frac{(f(x,t,u) - f(x,t,v))}{(u-v)} \leq k_M(1+x^2)^\alpha \quad \forall\,(x,t) \in D_T.$$

Furthermore, it follows that if f is locally Lipschitz continuous in u, uniformly on D_T, namely for all $u, v \in M$, there exists a constant $k_M > 0$ such that

$$|f(x,t,u) - f(x,t,v)| \leq k_M|u-v| \quad \forall\,(x,t) \in D_T,$$

then f satisfies condition $(H)_\alpha$ for all $\alpha \geq 0$.

We now establish the following comparison theorem for (IBVP).

Theorem 4.4. *Let $\bar{u} : \bar{D}_T \to \mathbb{R}$ and $\underline{u} : \bar{D}_T \to \mathbb{R}$ be a regular supersolution and a regular subsolution to (IBVP) with linear parabolic operator L, nonlinearity f and initial-boundary data g, respectively. Moreover, suppose that for some $\alpha \geq 0$, f satisfies condition $(H)_\alpha$, and there exists constants $A, B, C \geq 0$ such that the coefficients of the linear parabolic operator L satisfy*

$$0 \leq a_{ii}(x,t) \leq A(1+|x|^2)^{1-\alpha}$$

$$b_i(x,t)x_i \leq B(1+|x|^2)$$

$$c(x,t) \leq C(1+|x|^2)^\alpha$$

for all $(x,t) \in D_T$ and $1 \leq i \leq n$. Then, $\underline{u} \leq \bar{u}$ on $\bar{D}_T$.

Proof. Define $w : \bar{D}_T \to \mathbb{R}$, to be

$$w(x,t) = \underline{u}(x,t) - \bar{u}(x,t) \quad \forall\,(x,t) \in \bar{D}_T, \tag{4.4}$$

and it follows immediately that $w : \bar{D}_T \to \mathbb{R}$ is continuous and bounded, and hence, that $w \in E_0^0 \subset E_\alpha^0$. Moreover, it follows that there exists a closed bounded interval $M \subset \mathbb{R}$, such that $w(x,t) \in M$ for all $(x,t) \in \bar{D}_T$. Now, on D_T, we have via definition 4.1,

$$L[w] - (f(x,t,\underline{u}) - f(x,t,\bar{u})) = w_t - \sum_{i,j=1}^{n} a_{ij} w_{x_i x_j} - \sum_{i=1^n} b_i w_{x_i} - (c + \tilde{c})w \leq 0, \tag{4.5}$$

where $a_{ij}, b_i, c : D_T \to \mathbb{R}$ are the coefficients in the linear parabolic operator L, and

$$\tilde{c}(x,t) = \begin{cases} 0; & \text{when } \bar{u}(x,t) = \underline{u}(x,t) \text{ on } D_T \\ \left(\dfrac{f(x,t,\underline{u}) - f(x,t,\bar{u})}{\underline{u}(x,t) - \bar{u}(x,t)}\right); & \text{when } \underline{u}(x,t) \neq \bar{u}(x,t) \text{ on } D_T. \end{cases}$$

It follows, via remark 4.3, that there exists $k_M > 0$ such that

$$\tilde{c}(x,t) \leq k_M(1+x^2)^\alpha \quad \forall\,(x,t) \in D_T.$$

Therefore, it follows that the linear parabolic operator $L - \tilde{c}$ in (4.5) satisfies the conditions of theorem 3.4 when $\alpha > 0$ or theorem 3.5 when $\alpha = 0$. Moreover, via definition 4.1,

$$w \leq 0 \quad \text{on } \partial D_T. \tag{4.6}$$

A direct application of theorem 3.4 ($\alpha > 0$) or theorem 3.5 ($\alpha = 0$), with (4.5) and (4.6), establishes that

$$w \leq 0 \quad \text{on } \bar{D}_T,$$

and via (4.4), we have

$$\underline{u} \leq \bar{u} \quad \text{on } \bar{D}_T,$$

as required. ■

We are now able to establish uniqueness of solutions to IBVP.

Theorem 4.5. *Suppose that $f : D_T \times \mathbb{R} \to \mathbb{R}$ satisfies condition $(H)_\alpha$ for some $\alpha \geq 0$, and there exists constants $A, B, C \geq 0$ such that the coefficients of the linear parabolic operator L satisfy*

$$0 \leq a_{ii}(x,t) \leq A(1+|x|^2)^{1-\alpha}$$

$$b_i(x,t)x_i \leq B(1+|x|^2)$$

$$c(x,t) \leq C(1+|x|^2)^{\alpha}$$

for all $(x,t) \in D_T$ and $1 \leq i \leq n$. Then, (IBVP) with linear parabolic operator L, nonlinearity f and initial-boundary data g has at most one solution on $\bar{D}_T$.

Proof. Let $u^{(1)} : \bar{D}_T \to \mathbb{R}$ and $u^{(2)} : \bar{D}_T \to \mathbb{R}$ both be solutions to (IBVP) with linear parabolic operator L, nonlinearity f and initial-boundary data g on $\bar{D}_T$. It is trivial to show that if u is a solution to IBVP with linear parabolic operator L, nonlinearity f and initial-boundary data g on $\bar{D}_T$ then, via Definition 4.1, u is both a regular supersolution and a regular subsolution to IBVP with linear parabolic operator L, nonlinearity f and initial-boundary data g on $\bar{D}_T$. On taking $u^{(1)}$ and $u^{(2)}$ to be a regular subsolution and a regular supersolution to IBVP with linear parabolic operator L, nonlinearity f and initial-boundary data g, respectively, then via theorem 4.4 we have,

$$u^{(1)} \leq u^{(2)} \quad \text{on } \bar{D}_T. \tag{4.7}$$

A symmetrical argument establishes that

$$u^{(2)} \leq u^{(1)} \quad \text{on } \bar{D}_T, \tag{4.8}$$

and therefore, via (4.7) and (4.8), it follows that $u^{(1)} = u^{(2)}$ on $\bar{D}_T$, as required. ■

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