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A Rush of Connections and Insights, a Glorious Moment of Clarity

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Introduction

For several years we have been interested in pre-service teachers' memory for mathematical episodes. Partly this is because memory is such a vital aspect of mathematical problem solving. Long-term declarative memory is the sort of memory involved when a person talks, writes, draws, or otherwise consciously represents their recollections. Warner, Coppolo & Davis (2002) identify long-term declarative memory as a key ingredient in flexible mathematical thinking - the ability to apply mathematical solution processes in different settings and across different representations - and Davis, Hill & Smith (2000) emphasize long-term declarative memory as a key feature in effective teaching of mathematics. Memory, broadly speaking, has three aspects: formation, storage and retrieval (Squire,1994). An aspect of memory that is of particular importance for this study is that of the "emotional color" of a memory. Le Doux (1998, 2002) has argued in recent years for the importance of emotional color in all aspects of memory, and Thurston (1997) alludes to colorizing his memory for written mathematics in order to "*try to understand what it's really getting at rather than just what it says*."

We examine in some detail a deep example of productive long-term memory for mathematics. This example occurred during our regular teaching sessions and was noticed by us because of our focus on how memory formation occurs in classroom settings. It was something we noticed explicitly because of our theoretical orientation to memory. The student, Sandra, who exhibited this productive memory was enrolled in a mathematics methods course for pre-service elementary teachers. Her experiences and written reflections during the semester indicate that she imagined herself to have a particularly bad memory for mathematics. This focus, on what pre-service elementary teachers imagine mathematics is about and what needs to be remembered is typical in our experience (Davis & McGowen, 2001; McGowen & Davis, 2001) It is also what Shapiro (1992) observed to be typical for relatively under-achieving mathematics students.

Prior to our focus on memory, our teaching was concerned with presenting mathematical material in a logically coherent manner. For example, in teaching college level mathematics we would, when teaching elementary descriptive statistics, first introduce the idea of a function as an organizing idea (McGowen, 1998) and then introduce the average as a particular type of function. This sort of teaching is powerful because it introduces concepts, such as that of function, as a general organizing principal for a lot of other mathematics. With our focus on memorable episodes, however, we wanted students to begin to look more deeply and to see connections where previously they saw none. So the sequence of tasks we present in the next section was designed to explicitly provoke, without our telling, students' understanding of the connectedness of many parts of mathematics. Their ability to express these connections, we argued, would be evidence for long-term powerful memory for mathematics in a way that students had not encountered before.

A sequence of tasks

In Fall 2000, we introduced a sequence of combinatorial problems for pre-service elementary teachers (Davis & McGowen, 2001; McGowen & Davis, 2001). These problems, described below, were designed to tackle in depth a feature that is not common in the thinking of these prospective teachers, and that is looking deeply and with an eye to connections into a mathematical problem and its solution. Typically such students believe they are bad at mathematics, in part because they cannot recall formulas in detail nor recall under which circumstances a formula might apply:

- "I was under the impression that finding a formula to solve a problem was, in reality, the answer to the problem."
- "I was used to having a formula and all I cared about was getting the right answer."
- "We all came in with our preconceived notions of mathematics as simply finding a formula and getting the right answer."
- "All throughout school, we have been taught that mathematics is simply just plugging numbers into a learned equation. The teacher would just show us the equation dealing with what we were studying and we would complete the equation given different numbers because we were shown how to do it."
- "As long as I could remember I have seen math as getting the right answer, and that being the only answer."

(Quotes: pre-service elementary teachers at Harper College, Fall, 2000)

Yet pre-service elementary teachers are notoriously bad at recalling the formulas they think they need to solve problems. It rarely occurs to them that in trying to recall involved formulas they are focusing on something that is generally quite unproductive. The sequence of problems, shown below, was designed expressly to tackle this mental model of mathematics.

1. How many towers of height 4 can be made from blocks of 2 colors? How many towers of height 5? (*see figure 1, below*).
2. On the grid shown below, indicate on each dot how many different ways there are to walk to that dot from "home" given that you can only walk UP or RIGHT.
3. How many different ways are there to run through a series of 4 tunnels if each tunnel can be either white or black? See figure 3, below:
4. (a+b)2 = *a*2+*2ab+b*2. What are the expansions for *(a+b)*3 and *(a+b)*4 ?
5. What are the connections between towers, grid walks, tunnels and binomial expansions?

These problems were unlikely to allow a solution by remembered formulas and, typically, students in Fall 2000 and in ensuing semesters have not tackled these problems by recalling formulas - largely because they cannot. What then do they get from a sequence of problems like this and what are they capable of recalling explicitly and using later in the semester?

From low expectations to high insight

Sandra began the semester with low expectations for a mathematics methods class:

When I enrolled in this class back in September, I thought I was signing up for a three-month crash course in the mechanics of mathematics--something which had always eluded me. At that time I could not imagine math as anything more than the relentless stream of equations I used to agonize over as a young student.

However, Sandra's weekly written reflections reveal that she soon came to class with a specific intention of gaining enlightenment. She was not a student who simply attended class in order to have her presence recorded. She came with specific intentions, one of which was the expectation of moments of insight:

Last week's class was wonderful. For the first time since we began, I experienced the succession of 'ah ha!' moments I had been waiting for so eagerly.

In the episode leading up to Sandra's big moment of insight, we were unhappy that students were not expressing clear and unambiguous connections between the combinatorial problems presented to them. A modification of the grid walk problem was presented to the students, in which they were to repeatedly toss 4 coins and walk UP if a coin turned up heads, and RIGHT if it turned up tails. The expected number of times one ends up at any given spot on the grid is given by the corresponding term in Pascal's triangle. Pascal's triangle was discovered by Blaise Pascal in the 17th century, although versions of it were known to the Chinese mathematician Yanghui in the 13th century, and to the Persian mathematician Omar Khayyám. It is, in essence, the grid in the grid walk, problem, above, with the dots filled-in with the number of ways of walking to each dot from "home":

Pascal's triangle is usually drawn with facing downwards, so that the "home" dot would be at the top, and with only the numbers remaining:

The characteristic feature of Pascals' triangle is that each entry is obtained from the two above it by addition. This feature is readily seen from the grid model, because each dot on the grid is reached from only the dot to the left (if there is one) or the dot below (if there is one), due to the fact that only walking UP and RIGHT is allowed.

A focus of attention is a critical necessary condition for the formation, storage, and retrieval of long-term declarative memories. Equally, *intention* is a necessary precursor to an appropriate focus of attention. Sandra demonstrates this clearly, and relates it powerfully in her written reflection, when she discusses how she saw a connection between the individual terms in a diagonal of the grid, and the sum of those terms being a specific power of 2. Sandra begins by reflecting on the difficulty she had seeing what some other students in the class seemed to be seeing, namely a connection between grid walks, towers, and powers of two:

We had been encouraged to think of the two colors in the towers as similar to the 'Up' vs. 'Right' options in the dot matrix, but this was difficult for me. In the 'Towers' activity, we used powers of two and increased them based on the number of blocks in the towers. But with the matrix *(= grid*), we plotted the number of options to each dot ...I did not see powers of two in any of the numbers I was writing in the dots.

Sandra resolved her dilemma in not seeing a connection to powers of two during the class activity in which student took turns in tossing 4 coins, and another student marked which spot on the grid that toss of 4 coins indicated. Sandra's attention was drawn to the perceptual data of the marks on the grid due to her intention to make sense of something that had puzzled her:

This changed when we did the Quarters Problem'. As we did the activity, tossing a quarter four times each, I saw that there were only a limited number of answers for each toss based on the constraints of the problem...only 2 sides to the quarter and only four tosses. And I quickly saw that those dots were beginning to arrange themselves along a diagonal. And what did all those dots along that diagonal have in common? They were the only solution you could have when tossing a quarter exactly four times. And if you added up the number of options along that diagonal, you got a number which was 2, 4, 8, 16, 32 & 64 respectively. Powers of two. Suddenly I saw why the '*Towers Problem*' could be solved using powers of two ... and I saw how increasing the blocks in the towers moved the answers to the next diagonal and thereby to the next power of two. And I saw how the number of options increased systematically as each square's options compounded with the addition of more variables.

Sandra understood - she saw - that the rows of Pascal's triangle add to powers of two. However, Sandra's deepening perception and understanding of connections did not stop there. With the sudden understanding for how the powers of two entered the picture she made another fundamental realization about the proportions of marks on the grid spots 4 steps from "home":

Now I could see fractions expressed in our grid, because each dot represented a specific fraction of chances that it could be expressed given the intervals.

This realization was both emotional and liberating. In one short episode of about 20 minutes Sandra woke from her mathematical slumbers:

And I knew then, that different approaches worked for different people, and that's why we need to acknowledge all the different ways we solve a problem. For some people building towers worked, for some it was the grid matrix and for others it was (a+b). it was tremendous to feel something abstract becoming concrete...to finally see algebra as a way of making sense of the world around you. I saw the connection between the 4 toss quarter game and the dot grid. I saw the way the number of tosses translated into a diagonal line across the grid. And I saw how certain dots were 'hit' more than others due to the number ways (*sic*) to reach that dot. And what was most momentous to me was seeing that the number of dots or 'hits' along that diagonal added up to a number that equaled the power of two associated with the number of 'intervals' or 'jumps'.

That moment of seeing a connection - of the mind and brain constructing a new vision of the perceptual material already in front of the student - is powerfully emotional:

SO what have I learned? Well, that sometimes, it takes a number of different approaches or methods to help students understand math. It didn't become clear to me until we did the coin toss. And there was a rush of connections and insights.

Sandra's "rush of connections and insights" were emotionally laden cognitive insights that etched themselves deep in her memory. They impressed themselves so powerfully in her memory that she was able, during the remainder of the semester to bring these memories back to mind and to use them in differing settings, some not directly related to the context in which the memories were formed. However Sandra's recollections of those memories was, in her view, "poor". She still equated good mathematical memory with rote recollection of complicated formulas or facts, in precise detail, rather than a recollection of a broad cognitive road-map, based on an emotional episode.

Long-term flexible memory

Sandra's insight into connections between towers, grid walks and binomial expansions, and coin tossing formed a durable memory which she used in a different context later in the semester. The problem presented to the students was to determine the expected value of the sum of the numbers showing face up on two independently rolled dice. Sandra was reminded of Pascal's triangle while attempting this problem, so utilizing her durable knowledge in a flexible manner in a new context (Warner, Coppolo & Davis, 2002):

.. while playing the dice game, I noticed that certain numbers were more likely to be rolled than others because there were multiple ways of adding up to those numbers. This reminded me of Pascal's triangle, which functioned in a very similar way. I remembered that the dots in the middle of the triangle occurred more frequently than those at the periphery, much the same way that the numbers in the middle of one to twelve range appeared more frequently on a pair of dice in the dice game. Making that connection enabled me to make a prediction about how frequently certain numbers would be rolled in the dice game. I simply recalled the number of dots in each place along Pascal's triangle and applied those numbers to the dice game. I think that connection reflected a level of comfort that I was beginning to acquire with the concept of exponents. It was a glorious moment of clarity, which inspired me to want to think even more about probability, a math topic which had always been intimidating to me as a learner.

Toward the end of a fifteen week semester - some 9 weeks after she had a sudden insight into connections between towers, grid walks and binomial expansions, and coin tossing - Sandra worked in a group on determining empirically the distribution of black and white sides showing when *6 Othello*TM pieces (black on one side, white on the other) were tossed randomly from a bag. Sandra, unlike others in her group, detached from the empirical activity and began drawing a grid, labeling the spots on the grid with the number of different ways of reaching that spot by walking UP or RIGHT (the entries in Pascal's triangle). She worked at this until she completed the *6* th diagonal. When the class results for the empirical activity were pooled the result was *3* whites and *3* blacks occurring most often - *29 %* of the time. The instructor, noting Sandra's approach, asked her what she had obtained. She replied with an answer of *20* divided by 26, which another student calculated as *0.31*. This was a stunning example of flexible thinking, using long-term declarative knowledge in a new context. Yet Sandra was critical of her memory in a reflection written after this episode. She apologized, following the class, for not being able to remember the entries in the *6* th row of Pascal's triangle, and described her memory as "poor".

Sandra showed other examples of long-term memory in the service of flexible thought. For example, she speculated on powers of two, raising a number of hypothetical situations, a characteristic marker of flexible mathematical thought (Warner, Coppolo & Davis, 2002):

I also learned that powers are not the random activity I thought they were. Finding the power of a number is a way of determining patterns, fractions and probability. And maybe there's more than that. What else can powers give us? But what if you do 3 to the 1st, 2nd, third or fourth power? A grid is no longer easy to use because there would need to be a new dimension introduced and most grids are best at displaying length and width. What would suit a third element? The towers I guess. An algebraic expression, but that's hard for me to grasp in concrete ways. Are there other ways that we have not explored? My guess as to an algebraic expression: (a+b+c) to the second power.

This is strong evidence that Sandra's memories were durable and flexible - she could both bring previous experiences back to mind, and apply them in novel situations. Her thinking at this point had come a long way from the view she expressed at the beginning of he semester:

I could not imagine math as anything more than the relentless stream of equations I used to agonize over as a young student.

Discussion

Sandra's view of mathematical memory is, as we have stated above, common among elementary pre-service teachers. In our experience students are concerned at their poor memory for algebraic formulas. This, they often write, is why they are so relatively bad at mathematics. Yet Shapiro's work in Russia in the 1960's, carried out as part of his PhD thesis under the direction of Krutetskii, shows convincingly that talented mathematics students do not recall complicated expressions that are used only rarely (Shapiro, 1992). Rather, they focus on broad schemes, in which cognitive road maps are built, recalled, and used as needed. This vision of mathematics, in which the focus of attention is on perceived structure, is generally not available to pre-service elementary teachers as they begin mathematics content and mathematics methods courses.

We have attempted, through a structured sequence of combinatorial activities, to give pre-service elementary teachers a different vision of mathematical thinking, one based on insight and connections, as much as on calculation. Not all students obtain as intensely emotional understandings as did Sandra. Yet in our experience over *3* years this approach to a new understanding of mathematics affects many students deeply, and sets up long lasting productive memories of mathematics and ways of thinking mathematically. Is our focus on the formation of productive mathematical memories useful for a pre-service teacher? Sandra wrote:

I look forward to those rare, but beautiful opportunities to make connections across the discipline, knowing that it is those eureka moments which make it all worth while. And I can't wait until the day when I have the privilege of seeing my students experiencing those same moments of clarity and comprehension, exhilaration and accomplishment.

We have found that experiences such as Sandra's are not common for all pre-service teachers, not even for a majority. We have found that we have to work hard, with continual monitoring of verbal and written responses, to get students to a point over several weeks where they begin to have insights such as Sandra expressed. Usually these insights come in sudden, but small pieces. Rarely are they expressed as vividly as those of Sandra. The experiences described here, their emotional color and the role that seemed to play in the flexible recall of knowledge in other problem solving settings, raises for us a critical issue of mathematical education. Sandra, and pre-service teachers like her, through a process of intention to look for deeper meanings behind both the use manipulative, concrete materials, and the symbols of mathematics, seem to develop a different focus of attention in class, That focus of attention allows them to suddenly see connections they had not seen before. And this seeing of new connections, as Sandra's case shows, can be emotionally very powerful. How can we, as teachers of pre-service teachers affect this intentional stance? What steps can we take at the beginning of an instructional sequence to assist students to even imagine that there might be connections worth knowing about? Such experience are, in our experience, critical for pre-service teachers because they act as a powerful catalyst in turning the teachers away from a view of mathematics as a subject of tedious, dry calculations, to one of mathematics as an exciting subject with numerous deep connections.

Without our having thought long and hard about the nature of memory, and how it is formed and retrieved to be used in novel problems settings, we most likely would not have constructed the sequence of tasks described above, nor would we have noticed the importance and deep nature of Sandra's episodic memory. Prior to conceptualizing a role for long-term episodic memory in mathematics learning our teaching was focussed to a large extent on the presentation of logical and coherent sequences of tasks to illustrate or exemplify one or more points. For example, we might have stressed the importance of being systematic when counting different possibilities. A good example of this would be the number of different handshakes required if everyone in a room is to shake hands with everyone else. This problem yields simply to a systematic approach in which the people are numbered, sequentially 1, 2, 3, and so on.

What occurred with the sequence of problems we listed at the beginning of this article, and with our simultaneous focus on long-term memory, was that we were now much more focussed on students seeing connections. Thus, we did not any longer simply present mathematical problems, such as the handshake problem, above, because it is a good counting problem that yields to a systematic approach. Rather we looked for connections between problems and focussed in our teaching on a sequence of instruction that highlighted those connections. For example, in the case where there are 4 people, the handshake problem is the tunnel problem (problem 3 above) in disguise.

Our experience in teaching this way, focussing on connections between different mathematical problems, is that we cannot "teach" students to see connections. We can set up conditions under which their sudden enlightenment is more likely, but ultimately it is not a matter of learned behavior, rather of the students' direct noticing and seeing. This is a point made clearly by Terrence Deacon (1998) in his account of developing symbolic competence. These moments of vision on the part of students are usually accompanied by expressions of intense pleasure and wonder. It is those moments we seek to capture and to highlight, to refer our students back to "when you saw ...". This gives our students the possibility of a new way of understanding mathematics, and a new way of remembering. Those, relatively few, who have clear visions of connections, and who forge strong episodic memories no longer see mathematics as a collection of unrelated facts and formulas. Things make sense for them, and it is exactly the memory of the sense-making that constitutes what mathematics is for them now. The advantages of this are enormous - instead of mathematics being laborious and hard to remember, it is now easy and pleasurable. So we now teach to foster such memories: strong, vivid, episodic memories of seeing connections between different parts of mathematics.

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