THE ROW-COLUMN CONFOUNDED $2^m X4^n$ FACTORIAL DESIGN AND ITS DIFFERENT FRACTIONS

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ABSTRACT

The row-column confounded $2^m X4^n$ factorial designs and its different fractions can be constructed and analyzed using pseudo factors in the 4^n portion. (Examples are shown)

INTRODUCTION

Rao (1946), Kempthorne (1947), McLean and Anderson (1984) and John (1971) have discussed the latin square / quasi-latin square forms of factorial designs. Bailey (1977) and Patterson and Bailey (1978) have discussed the row-column form of the same designs with one treatment per cell. John and Lewis (1983) considered n-cyclic designs in row-column factorial experiments. Aggarwal and Shamsuddin (1987) and Shamsuddin (1994) constructed row-column confounded symmetrical factorial designs with single / multiple treatment per cell. John (1970) and Anderson and McLean (1974) used pseudo factors for constructing some factorials. Nath (1993) used pseudo factors in constructing factorial designs. Shamsuddin (1994) used pseudo factors for indirect construction and analysis of block and row-column factorial designs of prime power of levels. Haque (1996) and Shamsuddin and Haque (1997) has used pseudo factors for obtaining different fractions of $2^m x4^n$ designs. But none has used pseudo factors in constructing the row-column confounded $2^m x4^n$ factorial designs and its different fractions like square ring patterned and cross-diagonal fractions.

CONSTRUCTION.

The $2^m x4^n$ factorial can be constructed by use of pseudo factors partially i.e. by use of pseudo factors in the part of 4^n factorial of $2^m x4^n$ design. The 4^n factorial of n real factors can be obtained by the method of rotation conversion of 2n pseudo factors (Shamsuddin 1994).

The Row-column confounded $2^m x4^n$ Design.

The row-column confounded $2^m x4^n$ design is made by 2^{m+2n} design with 2^r rows and 2^k columns with $2^{m+2n-r-k}$ treatments per cell. The 2^{m+2n} design (say, $2^{m+2n} = 2^s$ design, where $s \geq r + k$) confounding 2^r-1 factorial effects belonging to 2^r-1 d.f. with its 2^r rows and 2^k-1 factorial effects belonging to 2^k-1 d.f. with its 2^k columns is obtained by the following sets of equations (Aggarwal and Shamsuddin 1987)

$PX = U_h$, h= 0,1,2,.. ,2^k-1, where $P=(p_{ij})_{kxs}, Q=[q_{ij}]_{rxs}, X' =[x_i]_{1xs}, U_0 =(u_o,..u_o)'$
$QX = V_0$ ……(i) $U_0 =(u_o,..u_o)', U_1=(u_1,..u_o)',…, U_{2^k-1}=(u_1,..u_1)'$,
representing the first row with 2^k cells,

$PX = U_h$, h= 0,1,2,.. ,2^k-1 where $V_1 = (v_1, v_o,..v_o)'$, all p_{ij}, q_{ij}, x_i, u_i,v_i Є GF(2),
$QX = V_1$ ……(ii)
representing the second row with 2^k cells, so on the others in which the equation sets
$PX = U_h$, h= 0,1,2,.. ,2^k-1
$QX = V_{2^r-1}$ ……(iii) where $V_{2^r-1} = (v_1,…,v_1)'$,
representing the last row with 2^k cells.

Writing all the rows, each of 2^k cells, we shall get 2^s design into $2^r x2^k$ setup of row-column design with 2^{s-r-k} treatment per cell (Table 1a or 2a). That is, we are to construct first the 2^{m+2n} factorial of two-leveled m real factors and 2n pseudo factors into the structure of 2^r rows and 2^k columns confounding 2^r-1 pseudo factorial effects of 2^{r-1} d.f. with rows and other 2^k-1 pseudo factorial effects of 2^k-1 d.f. with columns and then to correspond the levels of a real factors with the level combinations of 2 pseudo factors (Shamsuddin 1994) within the portion of 2^{2n} or 4^n factorial only for getting the row- column confounded $2^m x4^n$ factorial design (Table 1b or 2b) in full.

Table 1a: The row-column 2^6 factorial design confounding A_1B_2, $A_2B_1B_2$ and $A_1A_2B_1$ with 4 columns and YZ with 2 rows

000000	001000	000100	000001
000110	001110	000010	000111
001011	000011	001111	001010
001101	000101	001001	001100
110000	111000	110100	110001
110110	111110	110010	110111
111011	110011	111111	111010
111101	110101	111001	111100
100000	101000	100100	100001
100110	101110	100010	100111
101011	100011	101111	101010
101101	100101	101001	101100
010000	011000	010100	010001
010110	011110	010010	010111
011011	010011	011111	011010
011101	010101	011001	011100

Table 1b: The row-column 2^2x4^2 factorial design confounding AB^{u_2} with 4 columns and YZ with 2 rows

$00u_0u_0$	$00u_1u_0$	$00u_2u_0$	$00u_0u_2$
$00u_2u_1$	$00u_3u_1$	$00u_0u_1$	$00u_2u_1$
$00u_1u_3$	$00u_0u_3$	$00u_3u_3$	$00u_1u_1$
$00u_3u_2$	$00u_2u_2$	$00u_1u_2$	$00u_3u_0$
$11u_0u_0$	$11u_1u_0$	$11u_2u_0$	$11u_0u_2$
$11u_2u_1$	$11u_3u_1$	$11u_0u_1$	$11u_2u_1$
$11u_1u_3$	$11u_0u_3$	$11u_3u_3$	$11u_1u_1$
$11u_3u_2$	$11u_2u_2$	$11u_1u_2$	$11u_3u_0$
$10u_0u_0$	$10u_1u_0$	$10u_2u_0$	$10u_0u_2$
$10u_2u_1$	$10u_3u_1$	$10u_0u_1$	$10u_2u_1$
$10u_1u_3$	$10u_0u_3$	$10u_3u_3$	$10u_1u_1$
$10u_3u_2$	$10u_2u_2$	$10u_1u_2$	$10u_3u_0$
$01u_0u_0$	$01u_1u_0$	$01u_2u_0$	$01u_0u_2$
$01u_2u_1$	$01u_3u_1$	$01u_0u_1$	$01u_2u_1$
$01u_1u_3$	$01u_0u_3$	$01u_3u_3$	$01u_1u_1$
$01u_3u_2$	$01u_2u_2$	$01u_1u_2$	$01u_3u_0$

Table 2a: The row-column 2^2x4^2 factorial design confounding A_1B_2, $A_2B_1B_2$ and$A_1A_2B_1$ with 4 columns and $A_1B_1B_2$, A_2B_1 and$A_1A_2B_2$ with 4 rows

000000	001110	001001	000111
100000	101110	101001	100111
010000	011110	011001	010111
110000	111110	111001	110111
000110	001000	001111	000001
100110	101000	101111	100001
010110	011000	011111	010001
110110	111000	111111	110001
001101	000011	000100	001010
101101	100011	100100	101010
011101	010011	010100	011010
111101	110011	110100	111010
001011	000101	000010	001100
101011	100101	100010	101100
011011	010101	010010	011100
111011	110101	110010	111100

Table 2b: The row-column 2^2x4^2 factorial design confounding AB^{u_2} with 4 columns and AB^{u_3} with 4 rows

$00u_0u_0$	$00u_3u_1$	$00u_1u_2$	$00u_2u_3$
$10u_0u_0$	$10u_3u_1$	$10u_1u_2$	$10u_2u_3$
$01u_0u_0$	$01u_3u_1$	$01u_1u_2$	$01u_2u_3$
$11u_0u_0$	$11u_3u_1$	$11u_1u_2$	$11u_2u_3$
$00u_2u_1$	$00u_1u_0$	$00u_3u_3$	$00u_0u_1$
$10u_2u_1$	$10u_1u_0$	$10u_3u_3$	$10u_0u_1$
$01u_2u_1$	$01u_1u_0$	$01u_3u_3$	$01u_0u_1$
$11u_2u_1$	$11u_1u_0$	$11u_3u_3$	$11u_0u_1$
$00u_3u_2$	$00u_0u_3$	$00u_2u_0$	$00u_1u_1$
$10u_3u_2$	$10u_0u_3$	$10u_2u_0$	$10u_1u_1$
$01u_3u_2$	$01u_0u_3$	$01u_2u_0$	$01u_1u_1$
$11u_3u_2$	$11u_0u_3$	$11u_2u_0$	$11u_1u_1$
$00u_1u_3$	$00u_2u_2$	$00u_0u_1$	$00u_3u_0$
$10u_1u_3$	$10u_2u_2$	$10u_0u_1$	$10u_3u_0$
$01u_1u_3$	$01u_2u_2$	$01u_0u_1$	$01u_3u_0$
$11u_1u_3$	$11u_2u_2$	$11u_0u_1$	$11u_3u_0$

Fractional Factorial Designs:

There are different types fractional factorial designs, such as fraction containing one row or column, fraction having more than one row or column, square ring patterned fraction, cross-diagonal fraction and triangular fraction. We shall discuss here only two types of fractions.

The square ring patterned fraction:
This square ring pattern contains the first row, last row, first column, and last column of 2^2x4^2 factorial design. These rows and columns are represented (Shamsuddin and Aggarwal 1987, 1988) by the following sets of equations respectively

$QX = V_g$, $g= 0,1,2,.. ,2^r-1$, where $P=(p_{ij})_{kxs}$,$Q=[q_{ij}]_{rxs,}$$X'=[x_i]_{1xs}$,$U_0 =(u_o,..u_o)'$
$PX = U_0$ ……(iv) $V_0 =(v_o,..v_o)'$,$V_1=(v_1,..v_o)'$,…, $V_{2^r-1}=(v_1,..v_1)'$,
representing the first column with 2^r cells,

$QX = V_g$, $g= 0,1,2,.. ,2^r-1$
$PX = U_{2^k-1}$ ……(v)
representing the last column with 2^r cells.
The first row and last row are given by the equation sets (i) and (iii)
Solving these equation sets respectively, we get the first row, first column, last row and last column connected each other to have the square ring patterned fractional factorial (Table 3a)

Table 3a: The 4x4 square ring patterned irregular fraction defined by the column defining contrast 1 $=A_1B_2=A_2B_1B_2=A_1A_2B_1$ and row defining contrast 1 $=A_1B_1B_2$, $=A_2B_1=A_1A_2B_2$ derived from 2^6 design of 2 real factors Y,Z and 4 pseudo factors A_1, A_2, B_1,B_2

000000	001110	001001	000111
100000	101110	101001	100111
010000	011110	011001	010111
110000	111110	111001	110111
000110			000001
100110			100001
010110			010001
110110			110001
001101			001010
101101			101010
011101			011010
111101			111010
001011	000101	000010	001100
101011	100101	100010	101100
011011	010101	010010	011100
111011	110101	110010	111100

Table 3b: The 4x4 square ring patterned irregular fraction defined by the column defining contrast 1= AB^{u_2} and the row defining contrast 1= AB^{u_3} indirectly derived from 2^2x4^2 design.

$00u_0u_0$	$00u_3u_1$	$00u_1u_2$	$00u_2u_3$
$10u_0u_0$	$10u_3u_1$	$10u_1u_2$	$10u_2u_3$
$01u_0u_0$	$01u_3u_1$	$01u_1u_2$	$01u_2u_3$
$11u_0u_0$	$11u_3u_1$	$11u_1u_2$	$11u_2u_3$
$00u_2u_1$			$00u_0u_1$
$10u_2u_1$			$10u_0u_1$
$01u_2u_1$			$01u_0u_1$
$11u_2u_1$			$11u_0u_1$
$00u_3u_2$			$00u_1u_1$
$10u_3u_2$			$10u_1u_1$
$01u_3u_2$			$01u_1u_1$
$11u_3u_2$			$11u_1u_1$
$00u_1u_3$	$00u_2u_2$	$00u_0u_1$	$00u_3u_0$
$10u_1u_3$	$10u_2u_2$	$10u_0u_1$	$10u_3u_0$
$01u_1u_3$	$01u_2u_2$	$01u_0u_1$	$01u_3u_0$
$11u_1u_3$	$11u_2u_2$	$11u_0u_1$	$11u_3u_0$

Cross-Diagonal Fractions:

The cross-diagonal fraction is the combination of two diagonal intersecting each other at their center cell. The first diagonal is given by

$PX = U_h$, $h= 0,1,2,.. ,2^k1$, where$P=(p_{ij})_{kxs}$,$Q=[q_{ij}]_{rxs,}$$X'=[x_i]_{1xs}$,$U_0=(u_o,..u_o)'$,$U_1=(u_1,..u_o)'$,..,
$QX= V_g$,$g= 0,1,2,.. ,2^r-1$, ….(vi) $U_{2^k-1}=(u_1,..u_1)'$,$V_0 =(v_o,..v_o)'$,$V_1=(v_1,..v_o)'$.. $V_{2^r-1}=(v_1,..v_1)'$

and the second diagonal is represented by
$PX = U_h$, $h= 0, 1, .. , 2^k1$,
$QX= V_g$, $g= 2^r-1, 2^r-2,…, 0$, ….(vii)

These two equation sets (vi) and (vii) intersect each other at the center.
Solving the equation sets (vi) and (vii) we get the cross-diagonal fractional factorial (Table 4a)

Table 4a: The cross-diagonal 2^{6-1} fractional factorial of 2 real factor Y,Z and 4 pseudo factors A_1, A_2, B_1, B_2.

000000			000111
100000			100111
010000			010111
110000			110111
	001000	001111	
	101000	101111	
	011000	011111	
	111000	111111	
	000011	000100	
	100011	100100	
	010011	010100	
	110011	110100	
001011			001100
101011			101100
011011			011100
111011			111100

Table 4b: The cross diagonal fraction showing the half-replicate of 2^2x4^2 design

$00u_0u_0$			$00u_2u_3$
$10u_0u_0$			$10u_2u_3$
$01u_0u_0$			$01u_2u_3$
$11u_0u_0$			$11u_2u_3$
	$00u_1u_0$	$00u_3u_3$	
	$10u_1u_0$	$10u_3u_3$	
	$01u_1u_0$	$01u_3u_3$	
	$11u_1u_0$	$11u_3u_3$	
	$00u_0u_3$	$00u_2u_0$	
	$10u_0u_3$	$10u_2u_0$	
	$01u_0u_3$	$01u_2u_0$	
	$11u_0u_3$	$11u_2u_0$	
$00u_1u_3$			$00u_3u_0$
$10u_1u_3$			$10u_3u_0$
$01u_1u_3$			$01u_3u_0$
$11u_1u_3$			$11u_3u_0$

ESTIMATION and ANALYSIS

We can estimate all factorial effects in all two-leveled full factorials of pseudo factors by direct method or Yates method and in the fractions of pseudo factorials by direct method.
Then these estimated values are converted to real factorial effects by conversion procedure (Shamsuddin 1994).
In the fractional factorial the same procedure of conversion is followed as that for a 4^n design. After estimating the pseudo factorial effects (main effect or interaction effect) group-wise, followed by the conversion procedure, we get the estimates of real main effects and interaction effects by conversion for the 4^n factorial design. The same conversion procedure of pseudo factors into the real factors is followed in the case of 2^mx4^n design and its different fractions by applying the conversion in the portion of 4n only without changing the level shapes of 2^m portion.

Estimation in full factorial of 2^mx4^n design:

Using the conversion method all effects of this full factorial design are easily estimated by 2^{m+2n} design of m real factor and 2n pseudo factors except the confounded effects with rows and those with columns. Here in the 2^2x4^2 design confounding effect with column we get the estimate of Y, Z, A, B, AB, AB^{u_2}, AB^{u_3} applying the rotation-conversion procedure (Shamsuddin 1994) of pseudo factors A_1,A_2,B_1,B_2 in the portion 4^2 factorial.

Estimation technique in square ring patterned fractional factorials:

In the Table 3a showing the square ring patterned in the 2^6 factorial design of 2 real factor Y,Z and 4 pseudo actors A_1,A_2,B_1,B_2 the first row is represented by the first row defining contrast defined by $1 = -A_1B_1B_2 = +A_2B_1 = -A_1A_2B_2$ ….(viii) and the last row defining contrast defined by $1 = +A_1B_1B_2 = -A_2B_1 = -A_1A_2B_2$ …(ix). Adding these two row contrasts we get the half replicate defined by $1 = A_1A_2B_2$ from where we can estimate A_1, A_2, A_1A_2, Y, Z, YZ, YA_1,YA_2,YA_1A_2 , ZA_1, ZA_2, ZA_1A_2 , YZA_1,YZA_2, YZA_1A_2 . Similarly the first column is represented by the first column defining contrast defined by $1 = +A_1B_2 = -A_2B_1B_2 = -A_1A_2B_1$ ……. (x) and the last column contrast is defined by $1 = -A_1B_2 = +A_2B_1B_2 = -A_1A_2B_1$ ……. (xi). Adding also these two column contrasts we get another half replicate defined by $1 = A_1A_2B_1$ from where we can obtain another set of estimates B_1, B_2, B_1B_2, Y, Z, YZ, YB_1,YB_2,YB_1B_2 , ZB_1, ZB_2, ZB_1B_2 , YZB_1,YZB_2, YZB_1B_2. On average we get the estimate in all from both half replicates Y, Z, YZ, A_1, A_2, A_1A_2, B_1, B_2, B_1B_2, YA_1, YA_2, YA_1A_2 , ZA_1, ZA_2, ZA_1A_2 , YB_1,YB_2,YB_1B_2 , ZB_1, ZB_2, ZB_1B_2 , YZA_1,YZA_2, YZA_1A_2, YZB_1,YZB_2, YZB_1B_2.from the square-ring patterned fraction of 2^6 design of 2 real factors Y,Z and 4 pseudo factors A_1,A_2,B_1,B_2. Now using the conversion method we get indirectly the estimates of all real factorial effects A, B, YA, ZA, YB, ZB, YZA, YZB of 2^2x4^2 factorial design.

Estimation in cross-diagonal fraction:

In the 2^6 design (Table 1a) the first row is defined by the first row defining contrast $1= -A_1B_1B_2 =+ A_2B_1 =-A_1A_2B_2$ and the first column by the first column defining contrast $1 = +A_1B_2 = -A_2B_1B_2 = -A_1A_2B_1$ and multiplying directly (in the same direction) we get the first diagonal defining contrast defined by $= -B_1= -B_2 = +B_1B_2$ …..(xii), again multiplying first row defining contrast $1= -A_1B_1B_2 =+ A_2B_1 =-A_1A_2B_2$ with the last column defining contrast defined by $1 = -A_1B_2 = +A_2B_1B_2 = -A_1A_2B_1$ we the second diagonal defining contrast $1 = +B_1 = +B_2 = +B_1B_2$ ……(xiii) . Now adding these two diagonal contrasts we get the half replicate defined by $1= B_1B_2$ from we can estimate A_1, A_2, A_1A_2, Y, Z, YZ, YA_1,YA_2,YA_1A_2 , ZA_1, ZA_2, ZA_1A_2 , YZA_1,YZA_2, YZA_1A_2 . With help of conversion method we get the estimate of Y, Z,YZ, A, YA, ZA, YZA, etc

Analysis:

The sum of square of a pseudo factorial effect in these two leveled full or fractional factorials is estimated as usual procedure by [square of total factorial effect / total d.f.]

In analysis the real factor ss or the real factorial effect ss is obtained as the total of the ss´s of all pseudo factors or pseudo factorial effects used for conversion of the real factor or real factorial effect. Thus, as for example in calculating the sum of squares of 4^2 factorials, we see ss(A) is ss(A_1)+ss(A_2) +ss(A_1A_2) and ss(AB^{u_2}) is the total of ss(A_1B_2), ss($A_2B_1B_2$,) and ss($A_1A_2B_1$) as shown in the conversion procedure.

As an example we show here the ANOVA table of 4^2 factorial design of 2 real factors A and B confounding AB^{u_2} with 4 columns and AB^{u_3} with 4 rows using the 2^4 design of 4 pseudo factors A_1, A_2, B_1, B_2 through the conversion rule.

Table 5. The Analysis of Variance Table of 4^2 Design

2^4 factorial design S.V.	D.F.	SS	4^2 factorial design S.V.	D.F.	SS
A_1	1	$S(A_1)$			
A_2	1	$S(A_2)$	A	3	S(A)
A_1A_2	1	$S(A_1A_2)$			
B_1	1	$S(B_1)$			
B_2	1	$S(B_2)$	B	3	S(B)
B_1B_2	1	$S(B_1B_2)$			
A_1B_1	1	$S(A_1B_1)$			
A_2B_2	1	$S(A_2B_2)$	AB	3	S(AB)
$A_1A_2B_1B_2$	1	$S(A_1A_2B_1B_2)$			
Row effect	3	S_r	Row effect	3	S_r
Col. effect	3	S_c	Col. Effect	3	S_c
Error	$r2^4$-1-other d.f	S_e	Error	$r4^2$-1-other d.f..	S_e

Summary

The construction, estimates and analysis of row-column 2^m x 4^n factorial designs and its two types of fractions were shown is detail. This row-column confounded mixed factorial design helps to estimate different main effects and interactions for both 2 level and 4 level-series are shown here separately as well as jointly.

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