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# USING CONFIDENCE INTERVALS TO OBTAIN A FAMILY OF ESTIMATORS OF THE INTRACLASS CORRELATION COEFFICIENT (OR HERITABILITY)

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## ABSTRACT

A family of point estimators is presented for the intraclass correlation coefficient (or heritability) in the balanced one-way random effects model. The family is obtained by equating a pivotal quantity to different values of the pivoting distribution, and includes the familiar ML and REML estimators. In terms of mean-squared error, most members of the family of estimators are admissible within the family. A sire model is used to illustrate the estimation of heritability. The authors provide guidance concerning the choice of an individual member of the family for estimation purposes and indicate how the method can be extended to unbalanced designs.

## 1 Introduction

Pivotal quantities have a long history of use in inferential statistics. For the most part they have been employed to construct confidence intervals or conduct hypothesis tests for population parameters (see Lehmann (1986)). As shown in this paper, pivotal quantities can also be used to build a family of point estimators for a parameter.

This paper illustrates the use of pivotal quantities to obtain estimates of the intraclass correlation coefficient, denoted by  $\rho$ , in a balanced one-way random effects model. This problem has received extensive study due to its numerous applications. For instance, the intraclass correlation coefficient is often used to measure the degree of familial resemblance. In plant and animal breeding studies, the intraclass correlation coefficient is related to heritability, the proportion of variation in a trait due to additive genetic effects.

Olkin and Pratt (1958) derive the uniformly minimum variance unbiased estimator of the intraclass correlation coefficient. This estimator has a positive probability of taking negative values. Donner (1986) presents a comprehensive review of the intraclass correlation coefficient in one-way random effects models, and concludes that the maximum likelihood estimator of  $\rho$  is preferable to the analysis of variance estimator when  $\rho$  is small. Donner and Wells (1986) provide a comparison of confidence interval methods for the intraclass correlation coefficient focusing on the unbalanced one-way random effects model.

On a closely related topic, Loh (1986), Das, Meneghini and Giri (1990), Das (1992), and Ye (1994) examine inferences of a ratio of variance components in a balanced one-way random effects model. In particular, Loh (1986) develops a point estimator that has uniformly smaller mean-squared error than the maximum likelihood, restricted maximum

likelihood, and Bayesian (using a non-informative prior) estimators.

This paper is organized as follows. Section 2 introduces the pivotal method of equating a pivot quantity to a value from its pivoting distribution. It is shown that this inference technique generates a family of estimators. Members of the family may be thought of as point estimators obtained by letting the coverage probability of a confidence interval shrink to zero. The method is then applied to estimating the intraclass correlation coefficient in the balanced one-way random effects model.

Section 3 investigates the performance of the estimators in the family by comparing their mean-squared errors. The results indicate there are a number of estimators of  $\rho$  that perform better than the maximum likelihood estimator for small values of the parameter. In addition, there are estimators that perform admirably for large values of  $\rho$ . The family members are indexed by a parameter  $F$ , and it is shown that in most cases there exists a value  $F_-$  such that all members with  $F \geq F_-$  are admissible within the family. The mean squared-error of selected estimators from the family are displayed as a function of  $\rho$ . In doing so, one can identify estimators which perform well over specific regions of the parameter space. Suggestions are made concerning which members of the family should be used in a given application.

In Section 4 an example concerning heritability is provided to help illustrate the use of the estimating procedures described in this paper. Section 5 provides some insight as to how the pivotal quantity may be applied to models that are more complex than the balanced one-way random effects model. Section 6 presents a short discussion and summary.

## 2 Relationship between Confidence Intervals and Point Estimators

Consider a situation in which a vector of statistics denoted by  $\mathbf{S}$  is used to estimate a vector of parameters denoted by  $\boldsymbol{\theta}$ . Suppose that one can transform  $\mathbf{S}$  to a vector of pivots  $\mathbf{T} = \mathbf{T}(\mathbf{S}, \boldsymbol{\theta})$ , that is,  $\mathbf{T}$  has density  $f_{\mathbf{T}}(\mathbf{t})$  that does not depend on  $\boldsymbol{\theta}$ . The *pivotal method* then is to equate  $\mathbf{T}$  with a value  $\mathbf{t}$  from the domain of the pivoting distribution, and to solve for each element of  $\boldsymbol{\theta}$ . For each parameter, this generates a family of estimators whose individual members depend on the value of  $\mathbf{t}$  selected.

Some motivation for using these estimators can be obtained by considering them as the limit points of confidence intervals (or more generally confidence regions) as the coverage probability is reduced to zero. Consider the case where a statistic  $S$  (an element of  $\mathbf{S}$ ) is a pivot for a parameter  $\theta$  (an element of  $\boldsymbol{\theta}$ ). Let  $F_T(\cdot)$  be the cumulative distribution function of  $T = T(S, \theta)$  and define  $p = F_T(t)$  where  $0 < p < 1$ . Construct a confidence interval with coverage probability  $1 - \alpha$  by inverting the statement

$$P(t_l < T(s, \theta) \leq t_u) = 1 - \alpha,$$

where  $F_T(t_l) = p\alpha$  and  $1 - F_T(t_u) = (1 - p)\alpha$ . Letting  $\alpha$  approach one leads to a single point for the interval, denoted by  $t$ , and the estimator of  $\theta$  is  $\hat{\theta}_t$  which satisfies  $T(s, \hat{\theta}_t) = t$ .

For example, selecting  $p = 0.5$  means that the confidence intervals have equal probability in each tail, and shrinking the interval to a point is equivalent to equating the pivot with its median value. The limit of the shortest length interval corresponds to selecting  $t = t_{mode}$ , where  $t_{mode}$  is the modal value of the pivotal quantity.

The following example illustrates the relationship between confidence intervals and point estimators. Let  $Y_1, \dots, Y_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$  where  $\theta = (\mu, \sigma^2)$ . The sufficient statistics are  $\bar{Y}$  and  $Q = \sum_i (Y_i - \bar{Y})^2$ . The corresponding pivots are  $T_1 = n^{1/2}(\bar{Y} - \mu)/\sigma$  and  $T_2 = Q/\sigma^2$  which are independent and have  $N(0, 1)$  and  $\chi_{n-1}^2$  distributions, respectively. Confidence intervals for  $\mu$  and  $\sigma^2$  may be obtained from  $\sqrt{n-1}T_1/\sqrt{T_2} \sim t_{n-1}$  and  $T_2 \sim \chi_{n-1}^2$ , respectively.

Because the  $t$ -distribution is symmetric, confidence intervals for  $\mu$  are symmetric and it is quite natural to think of these intervals as converging to  $\bar{Y}$ . Sprott (1990) refers to  $\bar{Y}$  as the convergence point of a nested set of confidence intervals. Confidence intervals for  $\sigma^2$ , however, are not symmetric and there is not a unique point of convergence as the coverage probability shrinks to zero.

Figure 1 illustrates the shrinking of a confidence interval to obtain a family of estimators of  $\sigma^2$ . Let  $\chi_L^2$  and  $\chi_U^2$  denote quantiles of the  $\chi_{n-1}^2$  distribution. A  $100(1 - \alpha)\%$  confidence interval for  $\sigma^2$  is  $(Q/\chi_U^2, Q/\chi_L^2)$  where

$$\begin{aligned} 1 - \alpha &= P \left[ \chi_L^2 < \frac{Q}{\sigma^2} < \chi_U^2 \right] \\ &= P \left[ \frac{Q}{\chi_U^2} < \sigma^2 < \frac{Q}{\chi_L^2} \right]. \end{aligned}$$

As the coverage probability shrinks to zero, the confidence interval converges to a point. In this manner a family of estimators of  $\sigma^2$  is obtained and is given by

$$\widehat{\sigma^2} = \frac{Q}{\chi_{value}^2}$$

where the value of  $\chi_{value}^2$  depends on the formulation of the confidence interval. For instance, an equal-tailed confidence interval converges to the median value,  $\chi_{value}^2 \approx n - 5/3$ . The limit of the shortest length confidence interval corresponds to the modal value,  $\chi_{value}^2 = n - 3$ . As depicted in Figure 1, the family of estimators of  $\sigma^2$  also contains the unbiased estimator ( $\chi_{value}^2 = n - 1$ ), the maximum likelihood estimator ( $\chi_{value}^2 = n$ ), and the minimum mean-squared error estimator ( $\chi_{value}^2 = n + 1$ ).

Harris and Burch (1998) show that an estimator which results from the maximization of a density function of a statistic  $S$  is a member of the family generated from a pivotal quantity based on that statistic  $S$ . If  $S$  is sufficient, then the family includes the MLE. In this case all members of the family converge to the MLE as sample size increases, and hence will be consistent, fully efficient, and asymptotically normally distributed under the usual regularity conditions. If the pivot is based on a non-sufficient statistic  $S$ , then typically the pivotal estimators will not be fully efficient (even asymptotically), although they could still be consistent and asymptotically normal. It is best to evaluate the asymptotic properties of estimators from such families on a case-by-case basis.

In this paper the authors will examine the estimates produced by the pivotal method applied to the balanced one-way random effects model given by

$$Y_{ij} = \mu + u_i + e_{ij}, \quad (1)$$

where  $i = 1, \dots, a$  and  $j = 1, \dots, b$ .  $Y_{ij}$  is the  $j^{th}$  observation associated with the  $i^{th}$  class (or group) of  $u_i$ . The classes of  $u_i$  in the model are assumed to be a random sample from some large population of classes.  $e_{ij}$  is often referred to as random error. It is assumed that  $u_i \stackrel{iid}{\sim} N(0, \sigma_1^2)$ ,  $e_{ij} \stackrel{iid}{\sim} N(0, \sigma_2^2)$ , and that  $u_i$  and  $e_{ij}$  are mutually independent. In addition,  $\sigma_1^2 \geq 0$  and  $\sigma_2^2 > 0$ .  $\mu$  is a fixed but unknown quantity that represents the overall mean of  $Y_{ij}$ .

The intraclass correlation coefficient is defined as  $\rho = \sigma_1^2 / (\sigma_1^2 + \sigma_2^2)$ . This terminology is used since the correlation between two observations within the same class is  $\rho$ . That is, for  $j \neq j'$ ,  $Corr[Y_{ij}, Y_{ij'}] = \sigma_1^2 / (\sigma_1^2 + \sigma_2^2)$ . In addition, since  $Var[Y_{ij}] = \sigma_1^2 + \sigma_2^2$ ,  $\rho$  may be interpreted as the proportion of the total variation due to the random effect  $u_i$ . In most applications,  $0 \leq \rho < 1$ .

In animal breeding applications, (1) is referred to as a sire model where  $Y_{ij}$  is the trait (e.g., yearling weight) of the  $j^{th}$  offspring of the  $i^{th}$  sire,  $u_i$  is the effect of the  $i^{th}$  sire, and  $e_{ij}$  is composed of environmental and other effects. The heritability of a trait, denoted by  $h^2$ , is the proportion of total variation due to additive genetic effects. In (1), note that  $\sigma_1^2$  represents the sire's additive genetic variance as opposed to the additive genetic variance of the offspring of the sire. In theory, since an offspring receives half of its genetic material from the sire and half of its genetic material from the dam, it follows that one may equate the sire's additive genetic variance with one-fourth of the offspring's additive genetic variance. In other words,  $h^2 = 4\rho$  and thus a family of point estimators of the intraclass correlation coefficient corresponds to a family of estimators of heritability. By definition,  $0 \leq h^2 < 1$  so in this particular application  $0 \leq \rho < 0.25$ .

Using standard linear model results (see Graybill (1976)), the complete sufficient statistics associated with (1) are  $(\bar{Y}_{..}, Q_1, Q_2)$  where

$$\begin{aligned} \bar{Y}_{..} &= \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b Y_{ij} \sim N \left( \mu, \frac{1}{ab} \frac{\sigma_2^2}{1-\rho} (1 + \rho(b-1)) \right) \\ Q_1 &= \sum_{i=1}^a \sum_{j=1}^b (Y_{ij} - \bar{Y}_{i.})^2 \sim \sigma_2^2 \chi_{a(b-1)}^2 \end{aligned} \quad (2)$$

$$Q_2 = \sum_{i=1}^a \sum_{j=1}^b (\bar{Y}_{i.} - \bar{Y}_{..})^2 \sim \frac{\sigma_2^2}{1-\rho} (1 + \rho(b-1)) \chi_{a-1}^2. \quad (3)$$

Since the authors are interested in estimating a function of the variance components, the focus of attention is on the quadratic forms  $Q_1$  and  $Q_2$ . From (2) and (3),  $Q_1$  is the "within" sum of squares and  $Q_2$  is the "between" sum of squares in the standard analysis of variance table. The distributions of the quadratic forms involve the parameter of interest,

$\rho$ , and a nuisance parameter,  $\sigma_2^2$ . By considering a ratio involving  $Q_1$  and  $Q_2$ , the nuisance parameter is eliminated in the resulting distribution.

The pivotal quantity for  $\rho$  using the above results is

$$T(Q_1, Q_2, \rho) = \frac{\frac{Q_2}{1+\rho(b-1)}/(a-1)}{\frac{Q_1}{1-\rho}/a(b-1)} \sim F_{a-1, a(b-1)}. \quad (4)$$

Thus the pivotal method generates a family of estimators of  $\rho$  which are obtained by solving

$$T(Q_1, Q_2, \rho) = F,$$

where  $F$  is a value from the  $F$ -distribution mentioned above. Solving for  $\rho$  results in

$$\hat{\rho} = \frac{a(b-1)Q_2 - (a-1)FQ_1}{a(b-1)Q_2 + (a-1)(b-1)FQ_1}. \quad (5)$$

If  $F$  in (5) is selected to be the median of the  $F$ -distribution, the estimator of  $\rho$  is included in every equal-tailed confidence interval of  $\rho$  as the confidence coverage approaches zero. By taking  $F$  in (5) to be the mode of the  $F$ -distribution, an estimator of  $\rho$  that corresponds to the confidence interval having minimum length as the coverage probability converges to zero is obtained. It is interesting to note that the REML estimator, which is the same as the ANOVA estimator in the balanced design when constrained to the parameter space, corresponds to  $F = 1$ . The MLE corresponds to  $F = a/(a-1)$ .

Table 1 gives a selection of  $F$ -values corresponding to estimators that arise from the pivotal quantity approach. Note that the mode of the  $F$ -distribution is defined for  $a > 3$  and the  $F$ -values in Table 1 are increasing in order. That is,  $F_{Mode} < F_{Median} < F_{REML} < F_{MLE}$ .

In this paper the authors are concerned with estimators that are restricted to the parameter space of  $\rho$ . In doing so, the estimators of  $\rho$  are defined as

$$\hat{\rho}_F = \max \left\{ 0, \frac{a(b-1)Y - (a-1)F}{a(b-1)Y + (a-1)(b-1)F} \right\} \quad (6)$$

where  $Y = Q_2/Q_1$ . Using the relationship between the  $F$ -values in Table 1, note that  $\hat{\rho}_{F_{Mode}} \geq \hat{\rho}_{F_{Median}} \geq \hat{\rho}_{F_{REML}} \geq \hat{\rho}_{F_{MLE}}$ . Section 3 investigates the properties of these particular estimators as well as others in the family.

The asymptotic properties of the family of estimators depend on whether  $a$  (the number groups) goes to infinity, or whether  $a$  remains fixed and  $b$  (the number of observations per group) goes to infinity. As  $a$  approaches infinity,  $F$  in (6) approaches 1, and the family of estimators collapses to a single estimator, namely,  $\hat{\rho}_{F_{REML}}$ . Since  $\hat{\rho}_{F_{REML}}$  is a maximum likelihood estimator, it exhibits the properties common to the maximum likelihood method. That is, consistency and asymptotic normality. However, if  $a$  is fixed as  $b$  approaches infinity,  $F$  in (6) is distributed as a scaled chi-squared variate. The result is a family of pivotal estimators indexed by  $F \sim (a-1)^{-1} \chi_{a-1}^2$ . In this case, no member of the family, including the MLE, is consistent.

### 3 Comparisons of Estimators

This section investigates the estimators of  $\rho$  derived in Section 2 by using mean-squared error as the criterion for judging the estimators. Following a brief discussion about  $\hat{\rho}_F$ , some general results concerning mean-squared error are presented. The authors then proceed to compare some estimators selected from the family.

The estimator in (6) may be rewritten as

$$\hat{\rho}_F = \frac{a(b-1)Y - (a-1)F}{a(b-1)Y + (a-1)(b-1)F} I_{\left[\frac{(a-1)F}{a(b-1)}, \infty\right)}(Y)$$

where  $I$  denotes an indicator function. The expected value of  $\hat{\rho}$  is

$$E[\hat{\rho}_F] = \int_{\frac{(a-1)F}{a(b-1)}}^{\infty} \frac{a(b-1)y - (a-1)F}{a(b-1)y + (a-1)(b-1)F} f_Y(y) dy \quad (7)$$

where  $f_Y(y)$  is the probability density function of  $Y$ . From the equation for the pivotal quantity in (4) and using the fact that  $Y = Q_2/Q_1$ , one can show that  $Y = k(\rho)X$ , where

$$k(\rho) = \frac{(a-1)(1 + \rho(b-1))}{a(b-1)(1 - \rho)}$$

and  $X$  is an  $F$ -distributed variable with  $a-1$  and  $a(b-1)$  degrees of freedom.

A simple transformation of variables using the  $F$ -density shows that the density of  $Y$  evaluated at  $y$  is

$$f_Y(y) = \frac{1}{B\left(\frac{a-1}{2}, \frac{a(b-1)}{2}\right)} \frac{(1-\rho)^{(a-1)/2} (1+\rho(b-1))^{a(b-1)/2} y^{(a-3)/2}}{(1+y+(b-1-y)\rho)^{(ab-1)/2}}$$

where  $B$  is the beta function. The mean-squared error of  $\hat{\rho}_F$  can be computed in a manner similar to that of the expectation of  $\hat{\rho}_F$ .

Of major interest is whether there is one choice of  $F$ , i.e., one particular member of the family of estimators, which is superior to all others in terms of mean-squared error over the entire parameter space. In fact, this is not the case, and it appears that all members of the family with  $F$  sufficiently large are admissible. The following theorem (stated without proof) establishes admissibility for all members with sufficiently large  $F$  for  $a > 5$ .

**Theorem:**

(i) If  $a > 5$ ,

$$MSE(\hat{\rho}_F) = (1 - \rho)^2 \left( F^2 \left( Var[U] + E[U]^2 \right) - 2FE[U] + 1 \right) + o\left( (1 - \rho)^2 \right),$$

where  $U = ((1 + \rho(b-1))(a-1))/((1 - \rho)a(b-1)Y)$  is an  $F$ -distributed random variable with numerator and denominator degrees of freedom  $a(b-1)$  and  $a-1$ , respectively.

(ii) For  $a > 5$ , define  $F_- = ((a - 5)a(b - 1))/((a - 1)(a(b - 1) + 2))$ . Then  $\hat{\rho}_F$  is admissible for all  $F \geq F_-$ .

Further details concerning mean-squared error as well as bias may be found in Harris and Burch (1998).

The authors now compare the mean-squared errors of estimators selected from the family of estimators. The FORTRAN routine DQDAGI from the IMSL (1991) collection of codes is used to determine integrals such as the one in (7). The  $F$ -values in Table 1 are used along with  $F_-$  from the Theorem and  $F_+ = (a + 1)/(a - 1)$ .  $F_+$  is employed to point out the fact that for small values of  $\rho$ , there are members of the family that have smaller mean-squared error than the maximum likelihood estimator. In particular, since  $F_+ > F_{MLE}$ ,  $\hat{\rho}_{F_+} \leq \hat{\rho}_{F_{MLE}}$  across the entire parameter space and  $MSE(\hat{\rho}_{F_+}) < MSE(\hat{\rho}_{F_{MLE}})$  when  $\rho$  is small.

Figure 2 displays the mean-squared errors of the selected estimators across the parameter space for  $a = 10$  and  $b = 5$ . Recall that  $F_{REML} = 1$  which does not depend on  $a$  and  $b$ . Also,  $F_- = 0.53$ ,  $F_{Mode} = 0.74$ ,  $F_{Median} = 0.94$ ,  $F_{MLE} = 1.11$ , and  $F_+ = 1.22$ . As Figure 2 confirms, the members of the family considered here are admissible. It is interesting to note that the estimators associated with large  $F$ -values perform well for small values of  $\rho$  and the estimators associated with small  $F$ -values perform well for large values of  $\rho$ . This certainly agrees with the results in the Theorem.

Figure 3 displays the mean-squared errors of the estimators relative to the mean-squared error of  $\hat{\rho}_{F_{REML}}$ . This figure indicates, for example, that the mean-squared error of  $\hat{\rho}_{F_+}$  is approximately 50% smaller than the mean-squared error of  $\hat{\rho}_{F_{REML}}$  when  $\rho = 0$  and about 50% larger than the mean-squared error of  $\hat{\rho}_{F_{REML}}$  as  $\rho$  approaches one. The above results suggest that a sensible strategy that uses informal prior belief about  $\rho$  is to use  $F_-$  if one believes  $\rho$  is "large",  $F = 1$  (which corresponds to the REML estimator) if  $\rho$  is "intermediate", and  $F_+$  if  $\rho$  is "small". Note that in heritability studies  $0 \leq \rho < 0.25$  so one may be inclined to select  $F_+$ .

## 4 Example

To illustrate how one may apply the above results to compute estimators of  $\rho$ , consider a subset of data from Harville and Fenech (1985). The model employed is the balanced one-way random effects model for the birth weights (lbs) of lambs from six sires. The data is displayed in Table 2. The model used by Harville and Fenech (1985) also takes into account population lines and age of dam and is beyond the scope of this paper.

In this example,  $a = 6$ ,  $b = 4$ , and  $Y = Q_2/Q_1 = 0.527$ . Table 3 lists a selection of  $F$ -values and the corresponding values of  $\hat{\rho}_F$ . The estimators given by (6) are in decreasing order since their corresponding  $F$ -values are in increasing order. In this heritability example  $\hat{\rho}_{F_-}$  and  $\hat{\rho}_{F_{mode}}$  are outside the parameter space. Although not directly comparable to the results in Table 3, Harville and Fenech (1985) provide an ANOVA estimate of  $\rho$  equal to 0.217 based on the entire set of 62 birth weights and the more realistic model.



## 5 More Complex Models

In this paper the pivotal method has been employed to obtain a family of estimators of  $\rho$  for the balanced one-way random effects model. The general methodology of using pivotal quantities to obtain estimators of parameters is comprehensive and can be used in a variety of applications. In the context of estimating functions of variance components, additional research is needed to determine the feasibility of employing the pivotal method in more elaborate models.

For instance, consider the mixed linear model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e},$$

where  $\mathbf{Y}$  is a  $n \times 1$  vector of observable random variables,  $\boldsymbol{\beta}$  is a  $p \times 1$  vector of unknown parameters, and  $\mathbf{u}$  and  $\mathbf{e}$  are vectors of unobservable random variables of size  $m \times 1$  and  $n \times 1$ , respectively. The matrices  $\mathbf{X}$  and  $\mathbf{Z}$  are known and without loss of generality,  $\text{rank}(\mathbf{X}) = p$ . The random vectors  $\mathbf{u}$  and  $\mathbf{e}$  are taken to be independent with  $\mathbf{u} \sim MVN(\mathbf{0}, \sigma_1^2 \mathbf{A})$  and  $\mathbf{e} \sim MVN(\mathbf{0}, \sigma_2^2 \mathbf{I}_n)$ . In animal breeding contexts, the known matrix  $\mathbf{A}$  is referred to as the relationship matrix since it describes the degree to which the elements of  $\mathbf{u}$  are related.

From Burch and Iyer (1997), a collection of pivotal quantities used to construct confidence intervals for  $\rho$  are of the form

$$\frac{\sum_{i=k+1}^d \frac{Q_i}{1+\rho(\Delta_i-1)} / \sum_{i=k+1}^d r_i}{\sum_{i=1}^k \frac{Q_i}{1+\rho(\Delta_i-1)} / \sum_{i=1}^k r_i} \sim F \left( \sum_{i=k+1}^d r_i, \sum_{i=1}^k r_i \right) \quad (8)$$

where  $Q_1, \dots, Q_d$  are quadratic forms that make up a set of minimal sufficient statistics for the linear model void of the fixed effects. By construction, the quadratic forms are independent. Furthermore,  $0 \leq \Delta_1 < \dots < \Delta_d$  are the distinct eigenvalues of a covariance matrix related to a function of the observations. Each  $\Delta_i$  is repeated  $r_i$  times, where  $i = 1, \dots, d$ . See Burch and Iyer (1997) for specific details.

From (8) there are a total of  $d - 1$  individual pivotal quantities since the index  $k = 1, \dots, d - 1$  determines the number of quadratic forms in the denominator and hence the degrees of freedom of the associated  $F$ -distribution. For fixed  $k$ , the pivotal quantity may be set equal to a value from the corresponding  $F$ -distribution to obtain a point estimator of  $\rho$ . Except in simple cases, the estimator cannot be expressed in closed-form so numerical methods are required to find the actual estimate. Evaluating the performance of the estimators derived from various combinations of pivotal quantities and  $F$ -values is a subject of future work.

## 6 Discussion

In this paper a family of estimators was derived by setting the pivotal quantity equal to different values of its pivoting distribution. Each member of the family can be thought of as the limit point of a confidence interval as its coverage shrinks to zero. The family of estimators also include the ML and REML estimators. In general, estimators associated with large  $F$ -values have small mean-squared errors for small values of  $\rho$  and estimators associated with small  $F$ -values have small mean-squared errors for large values of  $\rho$ . For investigators focusing on estimators when  $\rho$  is small (such as heritability studies), an  $F$ -value greater than  $F_{MLE}$  results in an estimator that performs better than the MLE. As a guide the authors suggest the use of  $F = F_+$  if one believes  $\rho$  to be small,  $F = F_-$  if one believes  $\rho$  to be large, and  $F = F_{REML}$  for other cases.

This paper suggests using large  $F$  when  $\rho$  is small. However, increasing  $F$  leads to an increase in the probability that the estimator is zero. Since this may be of concern, an alternative criterion to mean-squared error is to examine the ratio of mean-squared error to the probability that the estimator is positive, and seek to minimize this quantity. Informal investigations suggest that the conclusions drawn from using this criterion are qualitatively the same as outlined in this paper, that is, large  $F$  is better for small  $\rho$  and vice versa.

Although the authors have addressed admissibility concerns for the family of estimators, this does not preclude the possibility that an estimator derived from a different approach may have smaller mean-squared error for all  $\rho$  than any member of the family discussed in this paper. This suggests that even more work is warranted in this well-studied area of inferential statistics.

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Table 1:  $F$ -values

Mode	$F_{Mode} = \frac{a(b-1)[(a-1)-2]}{(a-1)[a(b-1)+2]}$
Median	$F_{Median} = F_{0.5, a-1, a(b-1)}$
REML	$F_{REML} = 1$
MLE	$F_{MLE} = \frac{a}{a-1}$

Table 2: Birth weight (lbs) of lambs

	Sire					
	1	2	3	4	5	6
	10.1	10.1	10.1	10.6	10.0	10.7
	11.8	11.0	11.7	7.7	12.7	12.5
	12.9	14.0	8.8	10.0	13.2	9.0
	13.1	15.5	11.0	11.2	13.3	10.2

Table 3: Estimates of  $\rho$  in lamb data

$F$ -values	$\hat{\rho}_F$
$F_- = 0.18$	0.70
$F_{Mode} = 0.54$	0.39
$F_{Median} = 0.90$	0.21
$F_{REML} = 1.00$	0.18
$F_{MLE} = 1.20$	0.12
$F_+ = 1.40$	0.08

$$\hat{\sigma}^2 = Q / \chi^2_{\text{value}}$$

$$\begin{aligned} \chi^2_{\text{value}} = n - 3 & \quad (\text{mode}) \\ & n - 5/3 \quad (\text{median}) \\ & n - 1 \quad (\text{mean}) \\ & n \quad (\text{MLE}) \\ & n + 1 \quad (\text{min MSE}) \end{aligned}$$

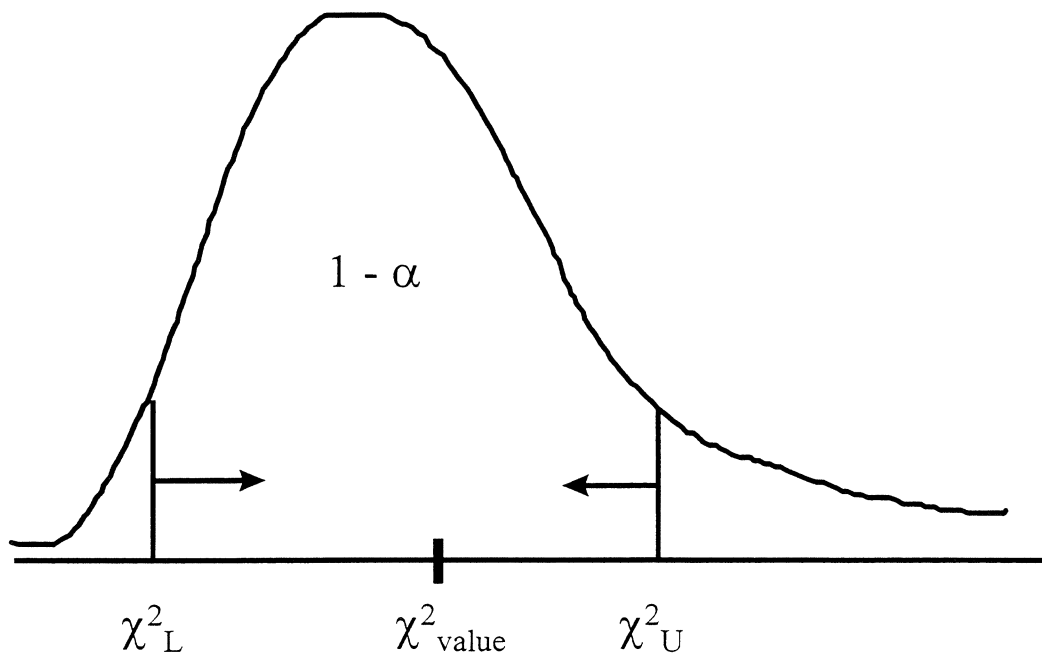


Figure 1: Family of estimators of  $\sigma^2$

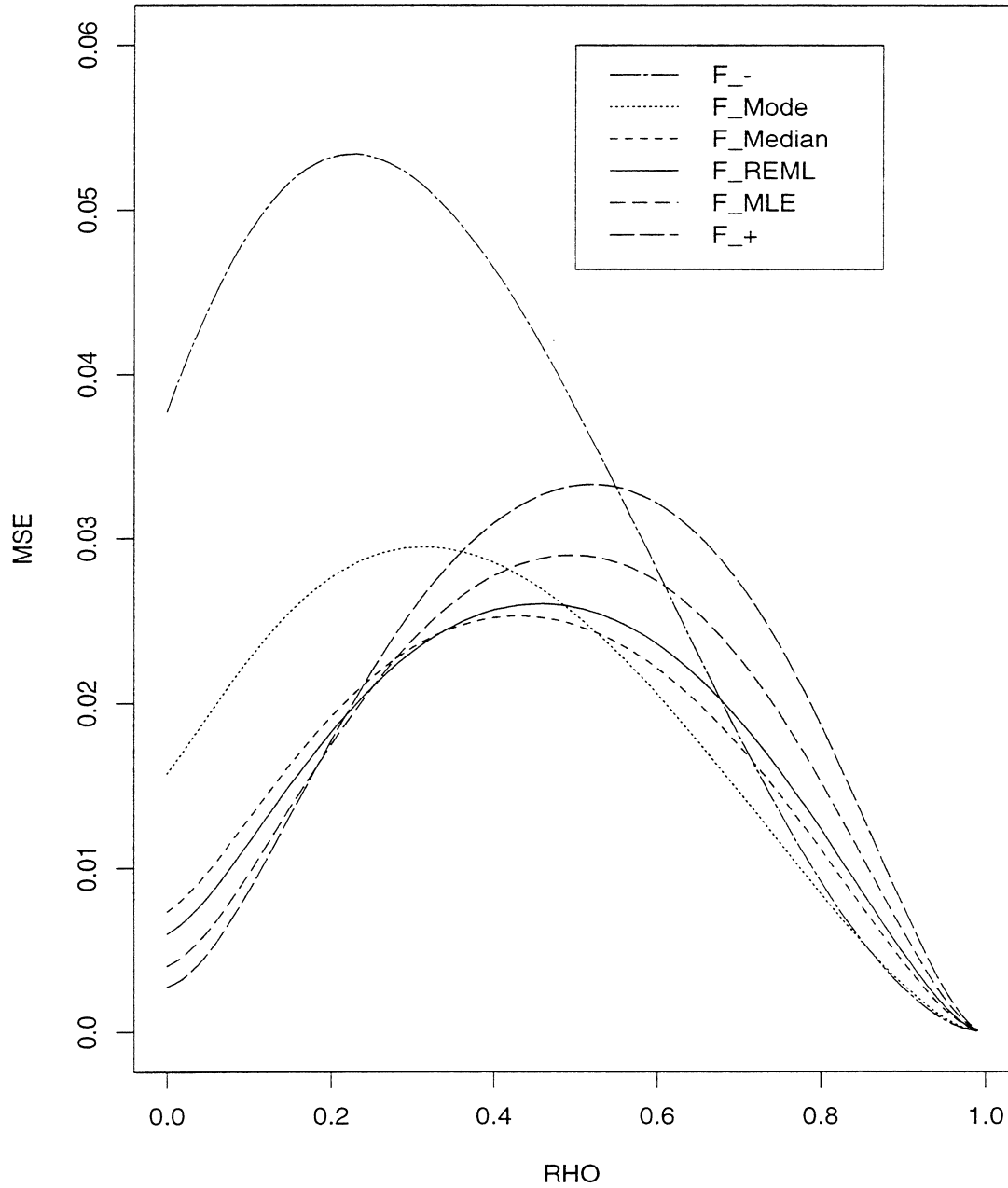


Figure 2: MSE of estimators for  $a = 10$ ,  $b = 5$

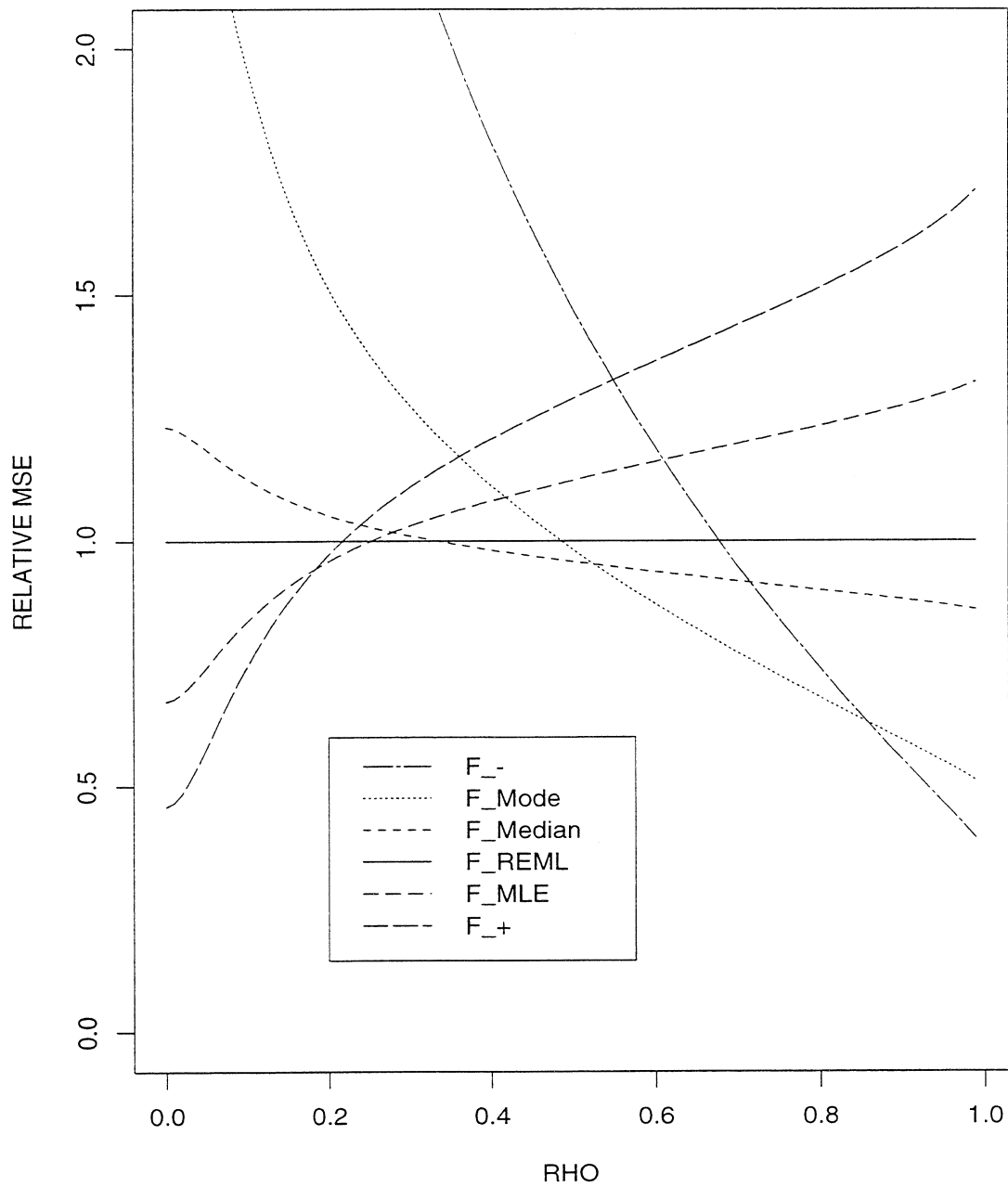


Figure 3: MSE of estimators relative to MSE of  $\hat{\rho}_{F_{REML}}$  for  $a = 10, b = 5$