THE ANALYSIS OF THE ADDITIVE MIXED MODEL FOR CLASSES OF NON ORTHOGONAL DESIGNS

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Recommended Citation

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Christine A. Stidley

New Mexico Tumor Registry
Cancer Treatment and Research Center
University of New Mexico Medical Center
Albuquerque, NM 87131

ABSTRACT

Tests for fixed and random effects can be difficult to derive for nonorthogonal designs with mixed models. However, extensions of the intra-block and inter-block analyses of Balanced Incomplete Block Designs can often be obtained. Here we derive the extensions for the broad class of Group Divisible Designs. Decompositions of the design space are used to develop exact tests for fixed and random effects in the additive mixed model with random block effects. Conditions on the design which permit the standard use of the intra-block and inter-block test statistics are given. Important subclasses of Group Divisible Designs include Equireplicate Variance Balanced Block Designs and Group Divisible Partially Balanced Incomplete Block Designs with Two Associate Classes. These two subclasses are also examined. An example from the literature of an experiment on fruit trees is used to illustrate the methods.

Key words: Analysis of variance; Group Divisible Designs; Intra-block analysis; Inter-block analysis; Variance Balanced Block Designs.

1. INTRODUCTION

We consider the class of Group Divisible Designs (GDDs), which includes such designs as Group Divisible Partially Balanced Incomplete Block Designs with Two Associate Classes (GDPBIBDs), Equireplicate Variance Balanced Block Designs (EVBDs) and Balanced Incomplete Block Designs (BIBDs). While much attention has been given to the construction of these designs and subclasses (see, for example, Freeman (1976), Gupta and Jones (1983), Hedayat and Federer (1974), Kageyama (1981), Kageyama and Mohan (1985), Sinha (1987), and Tyagi (1979)), distribution theory has not been obtained for the general class of designs or for many of the subclasses.

First, we describe this class of designs. Then by using extensions of the intra-block and inter-block analyses for BIBDs, we derive exact distribution theory for tests for fixed and random effects for the additive mixed model with random block effects. Details of the derivations are given in the Appendix. An example is also given.

2. DESCRIPTION OF GROUP DIVISIBLE DESIGNS

Let t be the number of treatment levels, b the number of blocks, n the number of observations, k_j the size of the jth block, $k = (k_1, \ldots, k_b)'$ and $r_0 = (r_1, \ldots, r_t)'$, the vector of treatment replicates. Let X_A and X_B be the zero-one design matrices for treatments and blocks, respectively.

Consider the t x t matrix Ω^{-1}, which is given by

$$\Omega^{-1} = R - NK^{-1}N' + r_0 r_0'/n \qquad (2.1)$$

where R and K are diagonal matrices with diagonals of r_0 and k, respectively, and $N = X_A'X_B$, the treatment-block incidence matrix. The matrix Ω^{-1} is important because of its role in the intra-block estimation of treatment effects (Tocher (1952)). Pearce (1963) uses this matrix to classify designs.

In Pearce's classification scheme a GDD is any design such that

$$\Omega^{-1} = c_1 I_t + c_2 I_m \otimes J_s + c_3 J_t, \qquad (2.2)$$

where I_t is the t x t identity matrix, J_t is a t x t matrix of ones, c_1, c_2 and c_3 are scalars and $\otimes$ denotes direct product. From (2.2) it can be seen that the t treatments are divided into m groups of s treatments. Treatments are thus nested within groups. Let X_M be the zero-one design matrix for groups. Differences in treatment effects are estimated with the same variance if the treatments belong to the same group, while the variance of the difference for treatments that are members of different groups is the same for all such differences. From (2.1) and (2.2) it is easy to see that every treatment is equally replicated r times, where $r = c_1 + sc_2 + tc_3$. Finally, we consider only connected designs, so the rank of Ω^{-1} is t (Rasch and Herrendörfer (1986, page 40)).

There are many important subclasses of GDDs. GDPBIBDs are the only GDDs that are proper and binary (Stidley (1988)). Treatments within the same group appear in λ_1 blocks together, while treatments in different groups appear together in λ_2 blocks. Let k_1 denote the common block size. It is easy to show that for GDPBIBDs, $c_1 = r - (r - \lambda_1)/k_1$, $c_2 = (\lambda_2 - \lambda_1)/k_1$ and $c_3 = r/t - \lambda_2/k_1$.

If $c_2 = 0$ then the design is an EVBD or, equivalently, a Totally Balanced Design under Pearce's classification scheme. This class includes BIBDs and Extended Complete Block Designs, which were introduced by John (1963). Orthogonal designs are simply GDDs with $c_2 = c_3 = 0$. Note that for any GDD with $c_2 = 0$ either (i) $m = 1$ and $s = t$ or (ii) $s = 1$ and $m = t$. Both conditions imply that there is only a treatment or group factor, but not both. Without loss of generality when $c_2 = 0$, we refer to this factor as the treatment factor and assume $m = 1$ and $s = t$.

The model that we consider is the additive mixed model such that

$$Y = \mu 1_n + X_M \tau + X_A \alpha + X_B \beta + \epsilon,$$

where Y is the n-vector of observations, μ is the mean, 1_n is an n-vector of ones, τ is the m-vector of group effects, α is the t-vector of treatment effects, β is the b-vector of random block effects and ϵ is the vector of pure error terms. We assume that

(i) $\tau' 1_m = 0$

(ii) $\alpha' I_m \otimes 1_s = 0_m'$

(iii) $\beta \sim N_b(0_b, \sigma_b^2 I)$ where $\sigma_b^2 \geq 0$

(iv) $\epsilon \sim N_n(0_n, \sigma_e^2 I)$ where $\sigma_e^2 > 0$

and (v) β and ϵ are independent.

Thus, $E(Y) = \mu 1_n + X_M \tau + X_A \alpha$ and $cov(Y) = \sigma_b^2 X_B X_B' + \sigma_e^2 I_n$. The constraints placed on the fixed effects are made strictly for convenience and

do not affect the outcome of the tests of hypotheses that we discuss later.

3. DECOMPOSITION OF THE OBSERVATION SPACE

We use the projective method (Christensen (1987)). Let S be a (linear) subspace of the observation space, $\Re^n$. Then the subspace of $\Re^n$ that is orthogonal to S (with orthogonality defined with respect to the standard inner product) is denoted by $S^\perp$. If T is a subspace orthogonal to S, then this relationship is written as $T \perp S$. Let U be a subspace contained in S. Then $S|U$ denotes the subspace of S that is orthogonal to U. The projection operator onto a subspace S is denoted by P_S.

The observation space, $\Re^n$, is first decomposed into two orthogonal subspaces, the design space V and the error space $V^\perp$. The design space, $V = \text{colsp}[1_n|X_M|X_A|X_B]$, which is the column space of the design matrices used in the model, regardless of the types of effects with which they are associated.

Let $A=\text{colsp}(X_A)$, $M=\text{colsp}(X_M)$ and $B=\text{colsp}(X_B)$. M and B are the group and block spaces, respectively. Since the treatment effects are nested within the group effects, A is actually the group and treatment space, with $A = M \oplus A|M$, where $A|M$ is the treatment adjusted for group space and $\oplus$ indicates the sum of orthogonal subspaces.

For orthogonal designs the design space is easily decomposed, since the relevant subspaces are orthogonal to one another. However, the subspaces are not orthogonal for GDDs. As a result, we consider three decompositions of the design space:

$$V = 1 \oplus B|1 \oplus V|B \tag{3.1}$$

$$V = 1 \oplus M|1 \oplus V|M \tag{3.2}$$

$$\text{and} \quad V = 1 \oplus A|M \oplus V|(A|(M|1)), \tag{3.3}$$

where 1 denotes the mean space, $B|1$ and $M|1$ are the block and group spaces adjusted for the mean, and the spaces $V|B$, $V|M$ and $V|(A|(M|1))$ are the adjusted treatment and group space, adjusted block and treatment space and adjusted block and group space, respectively. The decomposition given in (3.1) is the intra-block decomposition, while the decompositions given in (3.2) and (3.3) are the intra-group and intra-treatment adjusted for groups decompositions. The sums of squares resulting from the intra-block, intra-group and intra-treatment adjusted for groups decompositions for nonorthogonal GDDs are given in the analysis of variance table in Table 1.

The projection operators onto the spaces, 1, A, M, $A|M$, B, $A|1$, $M|1$ and $B|1$ are easily obtained. For example, $P_A = X_A(X_A'X_A)^{-1}X_A'$. The projection operator onto the adjusted treatment and group space is given by

$$P_{V|B} = (r/c_1)(P_{A|M} - P_{A|M}P_B - P_B P_{A|M} + P_B P_{A|M} P_B) + [r/(c_1 + sc_2)](P_{M|1} - P_{M|1}P_B - P_B P_{M|1} + P_B P_{M|1} P_B)$$

(Stidley (1988)). The expressions for the projection operators onto $V|A$, $V|M$, $V|(A|(M|1))$ and $V^\perp$ are easily obtained from the expression for $V|B$.

The inter-block decompositions are also given in Table 1. For these decompositions the block subspaces are further decomposed. If $c_3 \neq 0$ then let

$$P_{M°} = [r/(tc_3)]P_{B|1}P_M P_{B|1}$$

$$P_{EM} = P_{B|1} - P_{M°}$$

$$\text{and} \quad P_{BM} = P_{V|M} - P_{EM}.$$

If $sc_2 + tc_3 \neq 0$ then let

$$P_{A°} = [r/(sc_2 + tc_3)]P_{B|1}P_{A|M}P_{B|1}$$

$$P_{EA} = P_{B|1} - P_{A°}$$

$$\text{and} \quad P_{BA} = P_{V|(A|(M|1))} - P_{EA}.$$

It is easy to show that these matrices are projection operators onto subspaces of $B|1$, $V|M$ and $V|(A|(M|1))$. Let $M°$, EM, BM, $A°$, EA and BA denote the subspaces onto which these operators project. The ranks of the matrices are equal to the corresponding degrees of freedom given in Table 1.

Note that for many GDDs, the ANOVA table can be simplified. For instance, the following relationships exist among the factor spaces (this result follows easily from Proposition 1 in the Appendix):

(i) $M|1 \perp B$ if and only if $c_3 = 0$

(ii) $A|M \perp B$ if and only if $sc_2 + tc_3 = 0$

and (iii) $A|1 \perp B$ if and only if $c_2 = c_3 = 0$.

Since the orthogonal case given in (iii) is trivial, we assume that at least one of the two scalars, c_2 and c_3, is nonzero.

The three major classes of GDPBIBDs are singular, semi-regular and regular GDPBIBDs. From the relationships among the factor spaces, it is easy to see that a GDPBIBD is singular if and only if $A|M \perp B$, while it is semi-regular if and only if $M|1 \perp B$. For a regular GDPBIBD neither $M|1$ nor $A|M$ is orthogonal to B.

4. DISTRIBUTION THEORY

We are interested in testing for treatment, group and block effects. The specific hypotheses that we examine are:

(tg)   Treatment and group hypotheses:

H_0 : $\tau_i = \alpha_j = 0$ for all i, j vs

H_1 : $\tau_i \neq 0$ or $\alpha_j \neq 0$ for at least one i or j.

(g)   Group hypotheses:

H_0 : $\tau_i = 0$ for all i vs H_1 : $\tau_i \neq 0$ for at least one i.

(t)   Treatment hypotheses:

H_0 : $\alpha_j = 0$ for all j vs H_1 : $\alpha_j \neq 0$ for at least one j.

(b)   Block hypotheses:

H_0 : $\sigma_b^2 = 0$ vs H_1 : $\sigma_b^2 > 0$.

We consider the various test statistics that can be constructed from the quadratic forms given in Table 1. Let P denote the projection operator onto a subspace and $X^2(d,\delta)$ denote the chi-squared distribution

with degrees of freedom d and noncentrality parameter δ. For the sub-spaces in Table 2, $Y'PY$ is distributed as $w_1 X^2(d, w_2/(2w_1))$ under the constraints listed in the table (see Proposition 2 in the Appendix). The parameters w_1 and w_2 are given in Table 2, while the degrees of freedom d are given in Table 1. Note that some of the quadratic forms in Table 1 are not listed in Table 2. Standard distribution theory can be obtained for these sums of squares only under constraints that are so extreme the designs are trivial.

For the distribution theory for test statistics we consider three subclasses of GDDs, along with the general GDD. The subclasses are GDDs such that (i) $M|1 \perp B$ (that is, $c_3 = 0$), (ii) $A|M \perp B$ (that is, $sc_2 + tc_3 = 0$) and (iii) $c_2 = 0$ (that is, $M|1 = \phi$, the null space). By the results in Table 2 and independence of the appropriate quadratic forms, we can use the following statistics to test for treatment, group and block effects for these four classes of GDDs.

(1) $\quad [Y'P_{V|B}Y/(t-1)]/[Y'P_{V\perp}Y/(n-b-t+1)]$

(2) $\quad [Y'P_{M|1}Y/(m-1)]/[Y'P_{V\perp}Y/(n-b-t+1)]$

(3) $\quad [Y'P_{A|M}Y/(t-m)]/[Y'P_{V\perp}Y/(n-b-t+1)]$

(4) $\quad [Y'P_{EA}Y/(b-t+m-1)]/[Y'P_{V\perp}Y/(n-b-t+1)]$

(5) $\quad [Y'P_{EM}Y/(b-m)]/[Y'P_{V\perp}Y/(n-b-t+1)]$

(6) $\quad [Y'P_{BA}Y/(t-1)]/[Y'P_{V\perp}Y/(n-b-t+1)]$

(7) $\quad [Y'P_{M\circ}Y/(m-1)]/[Y'P_{EM}Y/(b-m)]$

(8) $\quad [Y'P_{A\circ}Y/(t-m)]/[Y'P_{EA}Y/(b-t+m-1)]$

Let $\mathfrak{F}(d_1, d_2, \delta)$ denote the F-distribution with degrees of freedom d_1 and d_2 and noncentrality parameter δ. Under the constraints given in Table 2, test statistics (1) through (6) are distributed as $w\mathfrak{F}(d_1, d_2, \delta)$, where w is the ratio of the numerator and denominator values for w_1, d_1 and d_2 are the numerator and denominator degrees of freedom, respectively, and δ is the numerator noncentrality parameter, $w_2/(2w_1)$. Test statistics (7) and (8) are distributed as $w\mathfrak{F}(d_1, d_2, \delta)$ under the constraints in Table 2 and additional constraints that result in a zero value for the denominator noncentrality parameter. A derivation of these results is given in the Appendix. Recall that we are assuming that the design is nonorthogonal, so at least one of the scalars, c_2 and c_3, is nonzero.

Table 3 summarizes when these test statistics can be used to test the four hypotheses. We can test all of the hypotheses of interest for all three subclasses. Note the importance of the test statistics constructed from the inter-block decompositions. These decompositions are often ignored, but, obviously, can yield valuable information.

Note that when $c_2 = 0$, we have no group effect to test. Then statistics (1) and (8) can be used to test the same hypotheses. Since they are independent test statistics, they can be combined to obtain a possibly more powerful test (for example, by the method presented by Zelen (1957)) or the more powerful individual test may be used. With BIBDs, test statistic (1) always gives the more powerful test, but this result does not hold for all EVBDs (Stidley (1988)).

Calculations to obtain these test statistics can be obtained with any matrix programming language, such as Gauss (Aptech Systems (1988)). The intra-block, intra-group and intra-treatment statistics can be

automatically obtained from standard analysis of variance packages, such as GLM in SAS (SAS (1985)), but the inter-block test statistics require further programming.

5. EXAMPLE

To illustrate these results we consider the example given in Table 4, which is a modification of an example given by Pearce (1965). The experiment consists of treating cherry trees with six fungicides, blocking for the treatment that was used on the trees during the previous year. Another blocking factor was also used, but as Pearce initially did, we omit this factor. We further modify the design by changing the block in which four observations occurred. The original two factor design is a GDD such that it is a generalization of an extended complete block design. The modification also has these characteristics, along with the property that $A|M$ is orthogonal to B. Thus, from Table 3 we see that we can test all four hypotheses. The calculations were done using Gauss (Aptech Systems (1988)). Table 5 gives the calculated values of each of the test statistics along with the p-values. For the treatment and group tests, the p-values are quite large, giving no evidence of a treatment or group effect. However, due to the small p-value for the block test, we conclude that $\sigma_b^2 > 0$.

6. CONCLUDING REMARKS

The set of GDDs contains many important subsets of designs, such as BIBDs, GDPBIBDs and EVBDs. Although the class of designs is broad, §4 gives a general distribution theory for testing treatment, group and block effects for the mixed model with random block effects. These test statistics are extensions of the intra-block and inter-block test statistics used in the analysis of BIBDs. The results presented in §4 and §5 show the importance of the inter-block test statistics in an analysis of nonorthogonal designs. However, while standard computer packages readily supply information on the intra-block type of test statistics, the inter-block test statistics are not given. With the results presented here the inter-block test statistics for GDDs can be obtained by further programming.

ACKNOWLEDGEMENTS

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APPENDIX

The distribution theory for the quadratic forms and the test statistics is derived. First, the following results are needed.

Proposition 1.

$$P_A P_B P_A = [(sc_2 + tc_3)/r]P_{A|M} + (tc_3/r)P_{M|1} + P_1$$

$$P_{M|1} P_B P_{M|1} = (tc_3/r)P_{M|1}$$

and $P_{A|M} P_B P_{A|M} = [(sc_2 + tc_3)/r]P_{A|M}.$

Proof. $\qquad P_A P_B P_A = (1/r^2)X_A N K^{-1} N' X_A'.$ $\qquad\qquad$ (A.1)

Note that for equireplicate designs

$$\Omega^{-1} = rI_t - NK^{-1}N' + (r/t)J_t,$$

while for GDDs

$$\Omega^{-1} = c_1 I_t + c_2 I_m \otimes J_s + c_3 J_t.$$

Since $r = c_1 + sc_2 + tc_3$, from the above equations we obtain

$$NK^{-1}N' = (sc_2 + tc_3)I_t - c_2 I_m \otimes J_s + [(c_1 + sc_2)/t]J_t. \qquad (A.2)$$

The result for $P_A P_B P_A$ is obtained by substituting this expression into A.1 and simplifying.

Since M is contained in A, $P_{M|1} P_B P_{M|1} = P_{M|1} P_A P_B P_A P_{M|1} = (tc_3/r)P_{M|1}$. The result for $P_{A|M} P_B P_{A|M}$ follows similarly. $\qquad\square$

The following corollary is obtained from repeated applications of Proposition 1.

Corollary 1. If $k = k_1 l_b$ then let $M_1 = [(k_1 tc_3/r)\sigma_b^2 + \sigma_e^2]^{-1} P_{M|1} cov(Y)$ and $M_2 = \{[k_1(sc_2 + tc_3)/r]\sigma_b^2 + \sigma_e^2\}^{-1} P_{A|M} cov(Y)$, while if $k = k_1 l_b$ and $c_2 = 0$, then let $M_3 = [(k_1 c_1/r)\sigma_b^2 + \sigma_e^2]^{-1} P_{BA} cov(Y)$. Then the matrices M_1, M_2 and M_3 are idempotent.

The following results give the distribution theory presented in §4.

Proposition 2. Let S be one of the subspaces listed in Table 2. Then, subject to the conditions given in Table 2, $Y'P_S Y$ is distributed as $w_1 X^2(d, w_2/(2w_1))$, where w_1 and w_2 are given in Table 2 and d is the degrees of freedom given in Table 1.

Proof. Let P be the projection operator onto one of the subspaces. Then

$$P cov(Y) = \sigma_b^2 P X_B X_B' + \sigma_e^2 P.$$

Under the conditions given in Table 2, it can be shown that either

(i) $P cov(Y) = w_1 P$

or (ii) Corollary 1 holds.

Thus, $P cov(Y)/w_1$ is idempotent, so $Y'PY$ is distributed as $w_1 X^2(d, \delta)$, where $\delta = E(Y)' P E(Y)/(2w_1)$. The derivation of the expressions for the noncentrality parameter uses the expression for $NK^{-1}N'$ given in (A.2) and the relationship between X_A and X_M, which is, $X_M = X_A(I_m \otimes l_s)$. □

Corollary 2. Let F be one of the eight listed test statistics. Consider the conditions for the numerator and denominator quadratic forms given in Table 2. For test statistic (7) add the constraint that $sc_2 + tc_3 = 0$, while for (8) consider two situations. First, add the constraint that $c_3 = 0$, while for the second case let the additional constraint be that $c_2 = 0$. Let w be the ratio of the numerator w_1 to the denominator w_1. Then, under these conditions, F is distributed as $w \mathcal{F}(d_1, d_2, \delta)$.

Proof. Let P denote the projection operator onto the subspace corresponding to one of the numerator sums of squares. Then the column space of P is contained in V and $P cov(Y) P_{V^\perp} = 0$. If $k = k_1 l$ and $c_3 \neq 0$, then M° is orthogonal to EM. Then

$$P_{M^\circ} cov(Y) P_{EM} = (k_1 \sigma_b^2 + \sigma_e^2) P_{M^\circ} P_{EM} = 0.$$

A similar result holds for P_{A° and P_{EA}. Thus, the numerator and denominator quadratic forms are independent. The result then follows from Proposition 2. □

Table 1

Analysis of Variance Table for Nonorthogonal Group Divisible Designs

Table 1.1   Intra-block decomposition

Source	SS	df
Blocks, ignoring treatments and groups	$Y'P_{B\mid 1}Y$	$b-1$
Inter-block decompositions:		
Decomposition 1: Group component	$Y'P_{M\circ}Y$	$m-1$
Inter-block/group error	$Y'P_{EM}Y$	$b-m$
Decomposition 2: Treatments nested within groups component	$Y'P_{A\circ}Y$	$t-m$
Inter-block/treatment error	$Y'P_{EA}Y$	$b-t+m-1$
Treatments & groups, eliminating blocks	$Y'P_{V\mid B}Y$	$t-1$
Intra-block error	$Y'P_{V\perp}Y$	$n-b-t+1$
Total, adjusted for the mean	$Y'P_{1\perp}Y$	$n-1$

Table 1.2   Intra-group decomposition

Source	SS	df
Groups, ignoring treatments and blocks	$Y'P_{M\mid 1}Y$	$m-1$
Treatments & blocks, eliminating groups	$Y'P_{V\mid M}Y$	$b+t-m-1$
Inter-block decomposition: Block component	$Y'P_{BM}Y$	$t-1$
Inter-block/group error	$Y'P_{EM}Y$	$b-m$
Intra-block error	$Y'P_{V\perp}Y$	$n-b-t+1$
Total, adjusted for the mean	$Y'P_{1\perp}Y$	$n-1$

Table 1.3   Intra-treatment adjusted for group decomposition

Source	SS	df
Treatments, ignoring groups and blocks	$Y'P_{A\mid M}Y$	$t-m$
Groups & blocks, eliminating treatments nested within groups	$Y'P_{V\mid(A\mid(M\mid 1))}Y$	$b+m-2$
Inter-block decomposition: Block component	$Y'P_{BA}Y$	$t-1$
Inter-block/treatment error	$Y'P_{EA}Y$	$b-t+m-1$
Intra-block error	$Y'P_{V\perp}Y$	$n-b-t+1$
Total, adjusted for the mean	$Y'P_{1\perp}Y$	$n-1$

Table 2

Distribution Parameters and Constraints

Sub-space	w_1	w_2	Constraints
B|1	$k_1\sigma_b^2 + \sigma_e^2$	$stc_3\tau'\tau + (sc_2 + tc_3)\alpha'\alpha$	$k = k_1 1_b$
M|1	$(k_1 tc_3/r)\sigma_b^2 + \sigma_e^2$	$rs\tau'\tau$	$k = k_1 1_b$
A|M	$[k_1(sc_2 + tc_3)/r]\sigma_b^2 + \sigma_e^2$	$r\alpha'\alpha$	$k = k_1 1_b$
V|B	σ_e^2	$s(c_1 + sc_2)\tau'\tau + c_1\alpha'\alpha$	none
M°	$k_1\sigma_b^2 + \sigma_e^2$	$stc_3\tau'\tau$	$k = k_1 1_b; \; c_3 \neq 0$
EM	$k_1\sigma_b^2 + \sigma_e^2$	$(sc_2 + tc_3)\alpha'\alpha$	$k = k_1 1_b; \; c_3 \neq 0$
A°	$k_1\sigma_b^2 + \sigma_e^2$	$(sc_2 + tc_3)\alpha'\alpha$	$k = k_1 1_b; \; sc_2 \neq -tc_3$
EA	$k_1\sigma_b^2 + \sigma_e^2$	$stc_3\tau'\tau$	$k = k_1 1_b; \; sc_2 \neq -tc_3$
BA	$(k_1 c_1/r)\sigma_b^2 + \sigma_e^2$	0	$k = k_1 1_b; \; c_2 = 0$
V⊥	σ_e^2	0	none

Table 3

Summary of Tests of Fixed and Random Effects
for Subclasses of Group Divisible Designs

Subclass of Group Divisible Designs

Hypothesis	general	M|1 ⊥ B $(c_3 = 0)$	A|M ⊥ B $(sc_2 = -tc_3)$	$c_2 = 0$ $(M\|1 = \phi)$
t&g	1	1	1	
g		2[†]	7[†]	
t		8[†]	3[†]	1, 8[‡*]
b		4[†]	5[†]	4[‡], 6[†]

[†] Additional constraint required: design is proper.
[‡] Additional constraints required: design is proper and $b \neq t$.
[*] Test statistics 1 and 8 are independent for this subclass.

Note: Any number within the table refers to one of the eight test statistics. For the appropriate design, this statistic can be used to test the corresponding hypothesis.

Table 4

Example

Cherry crop yield (pounds)

Fungicide

Blocks	A	B	C	D	E	F
1	16.7	18.6	13.1	8.9	12.8,13.6	12.5,10.9
2	7.2	8.1	7.6	13.8	9.9,9.4	1.2,15.9
3	12.3,10.9	10.1,5.4	16.3	17.9	15.3	7.8
4	11.7	15.3	13.5,19.3	14.6,15.5	12.2	9.0
5	15.0	7.3	7.5,8.9	9.8,12.7	10.0	10.9
6	9.2,10.4	13.3,16.7	15.7	9.6	17.4	10.3

Table 5

Summary of Hypothesis Testing for Example

Hypothesis	numerator SS	df	denominator SS	df	F	p-value
tg	49.33	5	478.7	37	0.76	0.58
g	11.18	2	127.5	3	0.13	0.88
t	30.70	3	478.7	37	0.79	0.51
b	127.50	3	478.7	37	3.29	0.03