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A Kinematical Approach to Conformal Cosmology

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Abstract

We present an alternative cosmology based on conformal gravity, as originally introduced by H. Weyl and recently revisited by P. Mannheim and D. Kazanas. Unlike past similar attempts our approach is a purely kinematical application of the conformal symmetry to the Universe, through a critical reanalysis of fundamental astrophysical observations, such as the cosmological redshift and others.

As a result of this novel approach we obtain a closed-form expression for the cosmic scale factor $R(t)$ and a revised interpretation of the space-time coordinates usually employed in cosmology. New fundamental cosmological parameters are introduced and evaluated. This emerging new cosmology does not seem to possess any of the controversial features of the current standard model, such as the presence of dark matter, dark energy or of a cosmological constant, the existence of the horizon problem or of an inflationary phase. Comparing our results with current conformal cosmologies in the literature, we note that our kinematic cosmology is equivalent to conformal gravity with a cosmological constant at late (or early) cosmological times.

The cosmic scale factor and the evolution of the Universe are described in terms of several dimensionless quantities, among which a new cosmological variable δ emerges as a natural cosmic time. The mathematical connections between all these quantities are described in details and a relationship is established with the original kinematic cosmology by L. Infeld and A. Schild.

The mathematical foundations of our kinematical conformal cosmology will need to be checked against current astrophysical experimental data, before this new model can become a viable alternative to the standard theory.

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Contents

I. INTRODUCTION

Modern cosmology has advanced very rapidly during these last decades, producing an impressive model of the Universe, but our current understanding is still troubled by many open questions and puzzles. Since the original observations of cosmological redshift in spectral lines, done by V. M. Slipher and E. P. Hubble almost one century ago and since the application of Einstein's General Relativity to cosmological theoretical models, we have progressed a long way towards our current picture, where the contents of the Universe are described in terms of two main components, *dark matter* and *dark energy*, accounting for most of the observed Universe, with ordinary matter just playing a minor role.

The history of recent experimental observations which led to postulate the existence of these two components is well known, as well as the many past and current theoretical explanations (see for example [1], [2], [3], [4], [5], [6], [7], [8], [9]), but since there is no evidence of the real nature of dark matter and dark energy, we have to conclude that our comprehension of the natural world is limited to only a very small percentage of it (the ordinary matter component), a statement potentially very embarrassing for cosmology and physics, if taken at face value.

Several alternatives to dark matter and dark energy have been proposed (for comprehensive reviews see for example [10], [11], [12], [13], [14] and references therein) which can be approximately divided into two categories: those retaining the Newton-Einstein gravitational paradigm, while introducing ad hoc corrections to explain dark matter and dark energy and those breaking away substantially from established gravitational theories. Following this second line of thought, we will concentrate our attention on the theory of Conformal Gravity (CG), a fourth order extension of Einstein's second order General Relativity (GR) as a possible framework for the solution of current cosmological problems.

II. CONFORMAL GRAVITY

A. Weyl's original proposal

The idea of a possible "conformal" generalization of Einstein's relativity was first developed by Hermann Weyl in 1918 ([15], [16], [17]). In his pioneering work, Weyl introduced the so-called *conformal* or *Weyl tensor*, a special combination of the Riemann tensor $R_{\lambda\mu\nu\kappa}$, the Ricci tensor $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ and the curvature (or Ricci) scalar $R = R^{\mu}{}_{\mu}$ (see [18] p. 145):

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) + \frac{1}{6}R\ (g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}), \quad (1)$$

where, in particular, $C^{\lambda}{}_{\mu\lambda\nu}(x)$ is invariant under the local transformation of the metric

$$g_{\mu\nu}(x) \rightarrow \widehat{g}_{\mu\nu}(x) = e^{2\alpha(x)} g_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x). \tag{2}$$

The factor $\Omega(x) = e^{\alpha(x)}$ represents the amount of local "stretching" of the geometry, hence the name "conformal" for a theory invariant under all possible local stretchings of the space-time.[1]

Weyl's ambitious original program was to introduce a new kind of geometry, in relation to a unified theory of gravitation and electromagnetism where, in addition to Eq. (2), the electromagnetic field would transform as $A_\mu(x) \rightarrow \widehat{A}_\mu(x) = A_\mu(x) - e\ \partial_\mu \alpha(x)$. This theory was later abandoned with the advent of modern gauge field interpretations of electrodynamics, retaining only terms such as "gauge transformation" or "gauge invariance," which were introduced in reference to Eq. (2) (for a brief history of conformal theories of gravitation from 1918 to 1988 see [11], [20]).

B. Fourth order metric theories

Following Weyl's idea, the conformally invariant generalizations of the gravitational theory were found to be fourth order theories, as opposed to the standard second order General Relativity. In other words, the field equations originating from a conformally invariant Lagrangian contain derivatives up to the fourth order of the metric with respect to the space-time coordinates.

Initially there was some ambiguity in the specific choice of the Lagrangian and the related action for these new theories, but following work done by Rudolf Bach [21], Cornel Lanczos [22] and others,[2] conformal gravity was ultimately based on the conformal (or Weyl) action:[3]

$$I_W = -\alpha_g \int d^4x\ (-g)^{1/2}\ C_{\lambda\mu\nu\kappa}\ C^{\lambda\mu\nu\kappa}, \tag{3}$$

[1] The name *conformal* derives more precisely, "from the property that the transformation does not affect the angle between two arbitrary curves crossing each other at some point, despite a local dilation: the conformal group preserves angles" (quoted from [19]).

[2] Even Albert Einstein used a conformally invariant formulation in one of his papers in 1921 [23].

[3] In this paper we use a metric signature (-,+,+,+) and we follow the sign conventions of Weinberg [18]. We will use c.g.s. units when needed and all fundamental constants, such as c and h, will always be explicitly introduced in every equation.

or on the following equivalent expression (which differs from the previous one by a topological invariant):

$$I_W = -2\alpha_g \int d^4x \, (-g)^{1/2} \, \left(R_{\mu\kappa} R^{\mu\kappa} - \frac{1}{3} R^2 \right), \tag{4}$$

where $g \equiv \det(g_{\mu\nu})$ and α_g is a gravitational coupling constant (see [10], [20], [24], [25]).[4] Under the conformal transformation of Eq. (2), the conformal tensor transforms as $C_{\lambda\mu\nu\kappa} \to \widehat{C}_{\lambda\mu\nu\kappa} = e^{2\alpha(x)} C_{\lambda\mu\nu\kappa} = \Omega^2(x) C_{\lambda\mu\nu\kappa}$, while the Conformal Gravity action I_W above is completely locally conformal invariant and is actually the unique general coordinate scalar action with such properties.

Variation of the Weyl action with respect to the metric led R. Bach [21] to rewrite the gravitational field equation in the presence of an energy-momentum tensor[5] $T_{\mu\nu}$:

$$W_{\mu\nu} = \frac{1}{4\alpha_g} \, T_{\mu\nu} \tag{5}$$

as opposed to the "standard" Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, R = -\frac{8\pi G}{c^3} \, T_{\mu\nu}, \tag{6}$$

where the "Bach tensor" $W_{\mu\nu}$ [21] plays the role of the combination of the Ricci tensor and curvature scalar on the left-hand side of Eq. (6). This tensor $W_{\mu\nu}$ has a much more complex structure than those appearing in Einstein's field equation. It is defined in a compact way as [26]:

$$W_{\mu\nu} = 2C^{\alpha}{}_{\mu\nu}{}^{\beta}{}_{;\beta;\alpha} + C^{\alpha}{}_{\mu\nu}{}^{\beta} \, R_{\beta\alpha}, \tag{7}$$

but if one requests a form where the Weyl tensor does not explicitly appear, the more complex structure for the Bach tensor will emerge ([24], [27]):

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[4] In these cited papers, α_g is referred to as a "dimensionless constant," by working with natural units. Alternatively, working with c.g.s. units, one can assign dimensions of an action to the constant α_g, so that the dimensionality of Eq. (5) will also be correct.

[5] We follow here the convention [10] of introducing the energy-momentum tensor $T_{\mu\nu}$ so that the quantity cT_{00} has the dimension of an energy density. For example, we write the perfect fluid energy-momentum tensor as: $T_{\mu\nu} = \frac{1}{c}[(\rho + p)U_\mu U_\nu + p g_{\mu\nu}]$.

$$W_{\mu\nu} = -\frac{1}{6} g_{\mu\nu}\ R^{;\lambda}{}_{;\lambda} + \frac{2}{3} R_{;\mu;\nu} + R_{\mu\nu}{}^{;\lambda}{}_{;\lambda} - R_{\mu}{}^{\lambda}{}_{;\nu;\lambda} - R_{\nu}{}^{\lambda}{}_{;\mu;\lambda} + \frac{2}{3} R\ R_{\mu\nu} - 2R_{\mu}{}^{\lambda}\ R_{\lambda\nu} + \\ + \frac{1}{2} g_{\mu\nu}\ R_{\lambda\rho}\ R^{\lambda\rho} - \frac{1}{6} g_{\mu\nu}\ R^2, \tag{8}$$

so that it involves derivatives up to the fourth order of the metric with respect to space-time coordinates.

The mathematical complexity of the Bach tensor and of Eq. (5) was one of the main reasons why the conformal theory of gravitation lost its attractiveness, between the thirties and the sixties, while quantum field theories were quickly progressing. A comprehensive review of the use of conformal invariance in physics up to the 1960s can be found in Ref. [28] and references therein. Only in the seventies, it was found that the fourth order theory is one-loop renormalizable [29], in contrast to standard general relativity, yielding a revival of conformal gravity.

C. Solutions to Conformal Gravity equations

It was already known to Bach in 1921, that every static spherically symmetric space-time, conformally related to the Schwarzschild-de Sitter solution, is a static spherically symmetric solution of the Bach equation. In 1962, the converse statement was shown by H. Buchdahl [30]: every static spherically symmetric solution of the Bach equation is conformally related to the Schwarzschild-de Sitter solution ([31], [32]).

In this line of research, a solution of Bach's equation was published by P. Mannheim and D. Kazanas (MK solution in the following) in 1989 ([24], [25]) and also studied by R. Riegert in his doctoral thesis [33]. This was the exact and complete exterior solution for a static, spherically symmetric source, in locally conformal invariant Weyl gravity, i.e., the fourth order analogue of the Schwarzschild exterior solution in General Relativity.

Solving Bach's Eq. (5), in the case $T_{\mu\nu} = 0$, Mannheim and Kazanas obtained a line element of the form

$$ds^2 = -B(r)\ c^2 dt^2 + \frac{dr^2}{B(r)} + r^2 d\psi^2 \tag{9}$$

where $d\psi^2 = d\theta^2 + \sin^2\theta\ d\phi^2$ in spherical coordinates and

$$B(r) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - \kappa r^2, \tag{10}$$

with the parameters $\beta = \frac{MG}{c^2}$ (cm), γ (cm^{-1}), κ (cm^{-2}) (again, we prefer to show explicitly constants such as the speed of light c in all formulas), where M is the mass of the (spherically symmetric) source. The familiar Schwarzschild solution is recovered in the limit for $\gamma, \kappa \rightarrow 0$, in the equations above. The other two parameters are interpreted by MK [24] in the following way: κ and the corresponding term κr^2 should indicate a background De Sitter space-time which would be important only at cosmological distances, since κ should have a very small value. On the other hand, γ measures the departure from the Schwarzschild metric, with the γr term becoming significant over galactic distance scales.

In other words, for values of $\gamma \approx 10^{-28} - 10^{-30}$ cm^{-1}, which is about the value of the inverse Hubble length, the standard Newtonian $\frac{1}{r}$ term still dominates at smaller distances, so that this theory would yield the same experimental success of General Relativity at the scale of the solar system. The three classic tests of GR, namely the gravitational redshift, the gravitational bending of light and the precession of planetary orbits, would still hold for conformal gravity at the solar system scale [10]. The only additional test of the gravitational theory, at this distance scale, which has not been analyzed yet in the MK theory, is the well-known decay of the orbit of a binary pulsar ([34], [35], [36], [37]).

Considering larger galactic distances, the contribution of the additional γr term might explain the flat galactic rotation curves, without the need of dark matter. This important connection to the dark matter problem and the galactic rotation curves was subsequently studied in great detail by Mannheim in a series of papers ([38], [39], [40], [41], [42], [43], [44], [10]), showing that it is possible to fit the experimental galactic rotation data with theoretical curves based on conformal gravity, with the same level of accuracy of current dark matter theories (see Fig. 1 of Ref. [44] or Ref. [10], for example), thus establishing conformal gravity as a viable alternative to the dark matter hypothesis.

When we apply conformal gravity to a galaxy, we need to specify in more details the role of the parameters β and γ. Again, Mannheim has shown that the Newtonian $\frac{1}{r}$ potential can be recovered for short distances, as a solution of a fourth order Poisson equation for the gravitational potential ϕ, as opposed to the standard second order equation (see [10], Sect. 4.2 for details). The resulting exterior potential for a single star source is of the form:

$$\phi^*(r > R) = -\frac{\beta^* c^2}{r} + \frac{\gamma^* c^2 r}{2} \tag{11}$$

where β^* and γ^* are the individual parameters for a system composed of a single star (i.e.,

$\beta^* = \frac{M_\odot G}{c^2}$, where we use the solar mass $M_\odot$ as a reference mass for a stellar object). In first approximation, for a system of N^* stars in a galaxy, we would expect to introduce overall β and γ parameters which are linear in the number of sources: $\beta = N^*\beta^*$; $\gamma = N^*\gamma^*$.

A more detailed analysis was done by Mannheim ([43],[44]) on a representative sample of eleven spiral galaxies, fitting their rotational velocity curves using the conformal gravity approach (Fig. 1 of Ref. [44] or Ref. [10] illustrates this detailed fitting). The galaxies were modeled with a thin/thick disk potential with the addition of a spherical central bulge region if necessary. The luminous Newtonian contribution was found to account well for the initial rise of the rotation curve from the center of the galaxy ($r = 0$) up to a peak at $r = 2.2\ r_0$, where r_0 is the scale length of the galaxy and r is the radial coordinate. The centripetal acceleration due to just the luminous matter distribution would yield the standard Keplerian term $\frac{v_{lum}^2}{r} \rightarrow \frac{N^*\beta^*c^2}{r^2}$, outside the optical disk. The number of stars N^* in each galaxy was computed by fitting the rotational curve, just due to the luminous Newtonian contribution, to the experimental value at the peak for $r = 2.2\ r_0$.

The discrepancy observed between the experimental data and the Keplerian prediction, for distances larger than the peak distance, was then modeled with parameters from conformal gravity. In particular, the last experimentally observed value for the rotational acceleration $\frac{v_{last}^2}{r}$ of the sample galaxies, was found to be well explained by a two parameter formula (in addition to the standard Keplerian term introduced above):

$$\frac{v_{last}^2}{r} = \frac{N^*\beta^*c^2}{r^2} + \frac{N^*\gamma^*c^2}{2} + \frac{\gamma_0 c^2}{2}. \tag{12}$$

In the previous equation, the first term on the right-hand side is the standard Keplerian one, while the two additional terms come from the conformal theory, without any need of dark matter contributions. The two additional universal parameters are evaluated from the detailed fitting of the experimental curves as follows [44]:

$$\gamma^* = 5.42 \times 10^{-41}\,\mathrm{cm}^{-1};\ \gamma_0 = 3.06 \times 10^{-30}\,\mathrm{cm}^{-1} \tag{13}$$

and their interpretation is analogous to the γ parameter of the MK solution of Eq. (10).

The presence of two gamma parameters is also explained by Mannheim: the γ^*N^* term is the gamma parameter of the specific galaxy being analyzed, being the product of the single star contribution γ^* times the number of stars N^* in the galaxy being considered. The more universal $\gamma_0 \simeq 3.06 \times 10^{-30}\,\mathrm{cm}^{-1}$ represents a cosmological gamma parameter, presumably due to the combined effect of all the galaxies (see discussion on page 416 of [10]). This term would affect the space-time geometry even in regions far away from matter sources, introducing an "universal acceleration" $\frac{\gamma_0 c^2}{2} = 1.38 \times 10^{-9}\ \frac{\mathrm{cm}}{\mathrm{s}^2}$ which is close to similar universal acceleration parameters, such as those introduced by the Modified Newtonian Dynamics (MOND) theory by M. Milgrom and others ([45], [46], [47], [48]).

In view of the very successful fitting of the experimental galactic rotation curves, shown in [44], we will consider here the Conformal Gravity model as a viable alternative to the dark matter hypothesis. In particular, we will retain the cosmological parameter γ_0, which will be used in our subsequent analysis, but we will need to reconsider its meaning and value later in this work.

III. CONFORMAL COSMOLOGY

As outlined in the previous section, we will assume that Conformal Gravity is a possible alternative gravitational theory, therefore the next logical step is to construct a cosmology based on these new ideas. In fact, many conformal cosmologies exist in the literature, including the one proposed by Mannheim in another series of papers ([49], [50], [51], [52], [53], [54], [55], [56], [57], [58]). Mannheim's cosmology is based on the construction of a traceless (as required by the conformal theory) energy-momentum tensor $T_{\mu\nu}$, in a theory in which the action is built out of fields rather than particles, using a spontaneous symmetry breaking mechanism in order to obtain particle masses. This modern approach elegantly overcomes the original objection to a conformal, scaleless theory, which would strictly require all particles to be massless, but is not free from theoretical controversy ([57], [59]).

Other "conformal" cosmologies exist in the literature (see for example [60], [61], [11]), based on similar approaches, but none of these has become a popular cosmological alternative to the standard model or even to cosmologies based on the MOND approach, including its latest relativistic version (Tensor-Vector-Scalar gravity, TeVeS, [48]). In our opinion, all these conformal cosmologies do not fully explain the connection between the assumed conformal symmetry and the physical reality of our Universe, as determined by cosmological observations. Therefore, we seek here an alternative approach, which doesn't require the field theory formalism, but is based on a critical analysis of the foundations of observational cosmology, starting with cosmological redshift.

A. From Static Standard Coordinates to the Robertson-Walker Metric

To introduce the discussion of cosmological redshift, it is necessary to analyze here in more details the transformation of the coordinates related to the MK solution, in particular the transformation from Static Standard Coordinates (SSC) to the Robertson-Walker (RW) metric. This is another fundamental aspect of Conformal Gravity: the CG solution is able to interpolate smoothly between the static Schwarzschild solution and the classical Robertson-Walker metric. We will follow again Mannheim and Kazanas ([10], [24], [25]), but we will use a slightly different notation and interpretation, for the different sets of coordinates used in the following. Another complete description of the necessary coordinate and conformal transformations from the Schwarzschild-de Sitter solution to the Mannheim-Kazanas solution can be found in Ref. [32].

We start again from the line element given by Eqs. (9) and (10), but we consider now regions far away from matter distributions, thus ignoring the matter dependent β term. In view of the discussion in the previous section, we could identify the γ parameter in Eq. (10) with $\gamma_0 \simeq 3.06 \times 10^{-30}\,\mathrm{cm}^{-1}$, as in Eq. (13). However, this value refers to a sample of eleven galaxies, where the rotational motion data being fitted by the conformal gravity theory cover a range of distances of a few kiloparsec, from the center of each galaxy.

In our next paper [62] we will argue that parameters such as γ are better determined by "local" measurements on a short distance scale and not on the kiloparsec scale. Mannheim's value of $\gamma_0 \simeq 3.06 \times 10^{-30}\,\mathrm{cm}^{-1}$ can therefore provide a useful order of magnitude for this quantity, but we will determine later its "current" value from more local measurements.

We will use the greek letter κ for the additional integration constant in the MK solution, instead of k used in the original references. In particular, we retain here the κr^2 "cosmological background" term that was dropped by Mannheim in his latest analysis [10], which on the contrary will play an essential role in our cosmology. We therefore write $B(r)$ as:

$$B(r) = 1 + \gamma r - \kappa r^2 \tag{14}$$

so that the line element becomes:

$$ds^2 = -\left(1 + \gamma r - \kappa r^2\right)\ c^2 dt^2 + \frac{dr^2}{(1 + \gamma r - \kappa r^2)} + r^2 d\psi^2, \tag{15}$$

in what we will call the Static Standard Coordinates - SSC (r, t, θ, ϕ) in the following. These are the coordinates we use to carry out all our standard laboratory measurements, with our current units of length, time, mass and others.

With a first coordinate transformation:[6]

$$\begin{aligned} \rho &= \frac{4r}{2\sqrt{1 + \gamma r - \kappa r^2} + 2 + \gamma r} \\ \tau &= \int R(t)\ dt \end{aligned} \tag{16}$$

the metric, as a line element, becomes ([24], [25], [10]):

$$ds^2 = \frac{1}{R^2(\tau)}\ \frac{[1 - \rho^2(\frac{\gamma^2}{16} + \frac{\kappa}{4})]^2}{[(1 - \frac{\gamma\rho}{4})^2 + \frac{\kappa\rho^2}{4}]^2} \left\{ -c^2 d\tau^2 + \frac{R^2(\tau)}{[1 - \rho^2(\frac{\gamma^2}{16} + \frac{\kappa}{4})]^2}(d\rho^2 + \rho^2 d\psi^2) \right\}. \tag{17}$$

At this point it is convenient to redefine the combination of parameters γ and κ, in Eq. (17), as follows:

$$\frac{\gamma^2}{16} + \frac{\kappa}{4} = -\frac{k}{4}, \tag{18}$$

where k will be ultimately linked to the "trichotomy constant" of a Robertson-Walker (RW) metric. Equation (17) can be rewritten as:

$$ds^2 = \frac{1}{R^2(\tau)}\ \frac{[1 + \frac{k}{4}\rho^2]^2}{[1 - \frac{\gamma}{2}\rho - \frac{k}{4}\rho^2]^2} \left\{ -c^2 d\tau^2 + \frac{R^2(\tau)}{[1 + \frac{k}{4}\rho^2]^2}(d\rho^2 + \rho^2 d\psi^2) \right\}. \tag{19}$$

As noted by Mannheim and Kazanas [24], the metric above is conformal to a RW metric in isotropic form. All we need is to apply a conformal transformation, such as the one in Eq. (2), to the metric tensor $g_{\mu\nu}(\rho, \tau)$ defined through Eq. (19), to obtain a new metric $\widehat{g}_{\mu\nu}(\rho, \tau)$ in the RW isotropic form. Precisely, we will "stretch" the space-time fabric, multiplying the last equation by the factor

[6] The angular coordinates θ and ϕ, as well as the quantity $d\psi^2 = d\theta^2 + \sin^2\theta\ d\phi^2$, are not changed by any of the transformations performed in this section. Therefore, we will not rename these angular coordinates. Note also the inverse transformation: $r = \frac{\rho}{(1-\gamma\rho/4)^2 + \kappa\rho^2/4}$; $t = \int \frac{d\tau}{R(\tau)}$.

$$\Omega^2(\rho,\tau) = R^2(\tau)\ \frac{[1-\frac{\gamma}{2}\rho-\frac{k}{4}\rho^2]^2}{[1+\frac{k}{4}\rho^2]^2}, \tag{20}$$

which depends on the space-time coordinates. We will then replace the metric as follows:

$$g_{\mu\nu}(\rho,\tau) \rightarrow \widehat{g}_{\mu\nu}(\rho,\tau) = \Omega^2(\rho,\tau)\ g_{\mu\nu}(\rho,\tau) = R^2(\tau)\ \frac{[1-\frac{\gamma}{2}\rho-\frac{k}{4}\rho^2]^2}{[1+\frac{k}{4}\rho^2]^2}\ g_{\mu\nu}(\rho,\tau). \tag{21}$$

Therefore we obtain:

$$d\widehat{s}^2 = -c^2 d\tau^2 + \frac{R^2(\tau)}{[1+\frac{k}{4}\rho^2]^2}(d\rho^2+\rho^2 d\psi^2), \tag{22}$$

and the metric is now in the form known as the "isotropic" Robertson-Walker. In the previous equations we did not use different symbols for the coordinates after the conformal transformation, but we kept the previous set of ρ, τ coordinate. The theory of local conformal transformations of the metric indicates that we can always choose the new coordinates of a point, after the local stretching, so that they correspond to the old coordinates ρ, τ of the original point before the stretching (see [28] for a detailed discussion of conformal transformations in physics).

The above transformation implies a change of the line element itself, which is stretched by the same amount

$$d\widehat{s}^2 = \Omega^2(\rho,\tau)\ ds^2 \tag{23}$$

and this "gauge transformation" will ultimately result in a redefinition of the local measuring rods and clocks, which will be a key feature of our cosmology. Another coordinate transformation will lead from the isotropic form of RW metric to the standard RW metric:[7]

$$\rho' = \frac{\rho}{1+\frac{k}{4}\rho^2} \tag{24}$$

$$\tau' = \tau$$

[7] The inverse transformation of Eq. (24) is: $\rho = 2(\frac{1-\sqrt{1-k\rho'^2}}{k\rho'})$; $\tau = \tau'$, where the minus sign in front of the square root selects the correct branch of the graph of the function considered. For $k = 0$ it reduces simply to $\rho = \rho'$.

and the metric becomes

$$d\hat{s}^2 = -c^2 d\tau'^2 + R^2(\tau') \left[\frac{d\rho'^2}{1 - k\rho'^2} + \rho'^2 d\psi^2 \right]. \tag{25}$$

In this expression the parameter k is still linked to γ and κ, through Eq. (18), or equivalently:

$$k = -\frac{\gamma^2}{4} - \kappa. \tag{26}$$

It is customary for the so-called trichotomy constant of a Robertson-Walker (RW) metric to have values $0, \pm 1$. This can be accomplished with a final rescaling of the coordinates, of the constant k and of the scale factor R, as follows:

$$\begin{aligned} \mathbf{k} &= \frac{k}{|k|} = 0, \pm 1 \\ \mathbf{r} &= \sqrt{|k|}\rho' \\ \mathbf{t} &= \tau' \\ \mathbf{R}(\mathbf{t}) &= \frac{R(\tau')}{\sqrt{|k|}}, \end{aligned} \tag{27}$$

where we use bold symbols $\mathbf{k}, \mathbf{r}, \mathbf{t}, \mathbf{R}$ to denote quantities after this last transformation.[8] We can finally obtain the standard Robertson-Walker form of the metric:[9]

$$d\hat{s}^2 = -c^2 d\mathbf{t}^2 + \mathbf{R}^2(\mathbf{t}) \left[\frac{d\mathbf{r}^2}{1 - \mathbf{k}\mathbf{r}^2} + \mathbf{r}^2 d\psi^2 \right]; \ \mathbf{k} = 0, \pm 1. \tag{28}$$

We recall that the RW metric in the previous equation can be expressed equivalently in the so-called curvature normalized form:

$$\begin{aligned} d\hat{s}^2 &= -c^2 d\mathbf{t}^2 + \mathbf{R}^2(\mathbf{t}) \left[d\chi^2 + S_{\mathbf{k}}^2(\chi) d\psi^2 \right] \\ S_{\mathbf{k}}(\chi) &\equiv \left\{ \begin{array}{cc} \sin\chi\ ; & \mathbf{k} = +1 \\ \chi\ ; & \mathbf{k} = 0 \\ \sinh\chi\ ; & \mathbf{k} = -1 \end{array} \right\}, \end{aligned} \tag{29}$$

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[8] In the special case $k = 0$ the transformation in Eq. (27) should actually read: $\mathbf{k} = 0$; $\mathbf{r} = \rho'$; $\mathbf{t} = \tau'$ and $\mathbf{R}(\mathbf{t}) = R(\tau')$.

[9] We note that, due to the transformations of Eq. (27), the quantities $\mathbf{r}$ and $\mathbf{k}$ are now dimensionless, while the factor $\mathbf{R}$ acquires the dimension of length. We will not follow the common alternative normalization, with a dimensionless scale factor, which is sometimes found in the literature.

for closed, flat or open universes respectively. The *comoving coordinate* χ is also dimensionless and the connection with the $\mathbf{r}$ coordinate in Eq. (28) is due to the simple relation:

$$\int_{\mathbf{0}}^{\mathbf{r}} \frac{d\mathbf{r}'}{\sqrt{1-\mathbf{k}\mathbf{r}'^2}} = \left\{ \begin{array}{cc} \operatorname{arcsin} \mathbf{r} \ ; & \mathbf{k}=+1 \\ \mathbf{r} \ ; & \mathbf{k}=0 \\ \operatorname{arcsinh} \mathbf{r} \ ; & \mathbf{k}=-1 \end{array} \right\} = S_{\mathbf{k}}^{-1}(\mathbf{r}) = \chi. \tag{30}$$

Another important quantity for our discussion is the *conformal time* interval $d\eta$, usually defined as an interval $c\, d\mathbf{t}$ divided by the scale factor $\mathbf{R}(\mathbf{t})$:

$$d\eta = \frac{cd\mathbf{t}}{\mathbf{R}(\mathbf{t})} = \sqrt{|k|}\frac{cd\mathbf{t}}{R(t)} = \sqrt{|k|}cdt \tag{31}$$

which is essentially equivalent to the SSC time interval dt, in view of Eqs. (16), (24), (27) and was in fact already introduced by the transformations of Eq. (16). Using the RW metric in the form of Eq. (29) we obtain a well-known and simple expression for the null geodesic $d\widehat{s}^2 = 0$, corresponding to the propagation of a light signal in the radial direction ($d\psi = 0$):

$$d\chi = \frac{d\mathbf{r}}{\sqrt{1-\mathbf{k}\mathbf{r}^2}} = -\frac{cd\mathbf{t}}{\mathbf{R}(\mathbf{t})} = -\sqrt{|k|}\ cdt = -d\eta, \tag{32}$$

thus establishing a direct connection between the comoving coordinate χ, the conformal time η and the SSC time coordinate t.[10] We note that the second equality in Eq. (32) is only valid for a null geodesic, i.e., χ and η are simply related to each other only when describing the propagation of a light signal. In this case the minus signs in the previous equation indicate that we are following a light ray propagating in the negative radial direction ($d\chi < 0$) for increasing conformal time ($d\eta > 0$).

Summarizing this section: the coordinate transformations described above allowed us to connect the original Static Standard Coordinates (r, t, θ, ϕ), used by Conformal Gravity to solve the problem of the rotational galactic curves without resorting to dark matter, to the cosmological comoving coordinates $(\mathbf{r}, \mathbf{t}, \theta, \phi)$, commonly used together with Eq. (28) as the basis of standard cosmology. We will continue to use normal and bold characters in the following to differentiate between these two sets of coordinates.

[10] For $k = 0$ we recall that $\mathbf{R}(\mathbf{t}) = R(t)$, therefore Eq. (32) should be written as $d\chi = -cdt = -d\eta$, omitting the $\sqrt{|k|}$ factor.

B. An alternative interpretation of the cosmological redshift

One of the foundations of observational cosmology is the well known cosmological redshift of galaxies, which is usually related to the expansion of the Universe. It is customary (see [18], [63], [64], [65], [66], or any other General Relativity - Cosmology textbook) to consider light emitted by a distant galaxy at (comoving) coordinates $(\mathbf{r}, \mathbf{t}, \theta, \phi)$ and reaching us at the origin of the coordinates $\mathbf{r} = 0$ and at time $\mathbf{t}_0$ (present time). The time of emission $\mathbf{t}$ is therefore in the past, i.e., $\mathbf{t} < \mathbf{t}_0$, or $\mathbf{t}_0 - \mathbf{t} > 0$ is the "look-back" time.[11] The redshift parameter z is related to the cosmic scale factor $\mathbf{R}(\mathbf{t})$, or to the change in the radiation wavelength/frequency, through the standard expression:

$$1 + z = \frac{\mathbf{R}(\mathbf{t}_0)}{\mathbf{R}(\mathbf{t})} = \frac{\lambda_0}{\lambda} = \frac{\nu}{\nu_0}, \tag{33}$$

where, quoting from Weinberg (see [18], pages 416-417): "... ν and λ are the frequency and wavelength of the light if observed near the place and time of emission, and hence presumably take the values measured when the same atomic transition occurs in terrestrial laboratories, while ν_0 and λ_0 are the frequency and wavelength of the light observed after its long journey to us."

Given this standard view of the redshift, it has always been considered a serious misconception to interpret the expansion of the Universe as if, "space itself is swelling up," thus causing galaxies to separate. Numerous textbooks are quick to point out this potentially erroneous interpretation (see for example [66], [67]), explaining that galaxies separate, "like coins glued on an inflating balloon," without altering their intrinsic dimensions, or that two massless objects set up at rest with respect to each other will show no tendency to separate, due to cosmological expansion.

However, an analysis of the literature of cosmological theories also reveals that other possible interpretations of the redshift, apart from the standard general relativistic expansion, were considered. Many alternative theories exist such as the *kinematic cosmology* by Infeld and Schild ([68], [69], [70], [71]), which is also based on the conformal gauge transformation of Eq. (23) as well as the cosmological principle and the constancy of the speed of light. In

[11] In this way, integrating Eq. (32) for light emitted at coordinate χ, at conformal time η, and reaching us at the origin ($\chi = 0$) at our present conformal time η_0, we obtain: $\chi = \sqrt{|k|}\; c(t_0 - t) = \eta_0 - \eta$.

this theory all possible cosmological models based on these assumptions are analyzed and classified, leading to different possible interpretations of the redshift, ranging from standard Doppler effect to the purely "gravitational redshift" effect that we will also employ in our model. We will later compare our results to the different models proposed by Infeld and Schild.

These conformally-flat-spacetime models were recently also studied by others ([72], [73], [74]) and were also considered in other theories such as the Hoyle-Narlikar cosmology [75]. This model, for example, assumes a non standard interpretation of the cosmological redshift, i.e., since the atomic radiation wavelength is inversely proportional in first approximation to the mass of the electron involved in the atomic transition, the ratio λ_0/λ is simply assumed to correspond to the value of the (variable) electron mass at different epochs: $\lambda_0/\lambda = m_e(\mathbf{t})/m_e(\mathbf{t}_0)$. Hoyle-Narlikar then implemented their model, assuming a conformally invariant theory where masses scale as $\widehat{m} = \Omega^{-1}m$, adding a variable gravitational constant G, whose variation is based on a large numbers hypothesis, similar to the original Dirac argument ([76], [77]) and finally proposed mechanisms of particle creation, in line with previous steady-state cosmologies.

While we do not agree with such theories, we share the idea that the redshift ratio λ_0/λ might be disclosing to us information about the emission/absorption process at different cosmological epochs. In this line of reasoning, we recall that modern metrology (see metrology web-sites [78], [79] and references therein) defines our basic units of length and time using non-gravitational physics, through a reference atomic wavelength or frequency, so that our *meter*[12] is just some multiple n_m of an atomic reference wavelength λ_m, or equivalently the *second* is a multiple n_s of the inverse of some atomic reference frequency ν_s:

$$1\ meter \equiv n_m\ \lambda_m \qquad (34)$$
$$1\ second \equiv n_s\ \frac{1}{\nu_s}.$$

Since our space-time units ultimately have an atomic definition based on emission/absorption of radiation, a possible "swelling" or dilation of the space-time fabric at

[12] The meter was recently redefined as the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second. This definition assumes an (exact) speed of light in vacuum: $c = 299\ 792\ 458\ m\ s^{-1}$. In this way the unit of length is basically defined through the unit of time, therefore not altering the validity of our discussion.

any level, from the atomic to the galactic scale, could never be detected using currently defined meter sticks and clocks, because these would be "dilated" by the same amount.

In other words, it is only possible to base our space-time units on the *current* and *local* values of wavelength or frequency of some standard reference atomic transition, but we cannot be absolutely certain that these reference wavelengths or frequencies are invariable and constant throughout the Universe and at all cosmological times. A possible variation of these reference wavelengths and frequencies would be also related to the well-known problem of the time variation of the *universal constants* (for modern reviews see [80], [81], [82]).

The logical connection between a possible conformal symmetry of the Universe, dealing with stretchings and dilations of the metric, and the previous discussion of changes and variations in our meter sticks and clock rates, should induce a revision of the redshift mechanism. In particular, the observed galactic redshift might be interpreted, in part or completely, as due to a change of these reference wavelengths and frequencies over cosmological distances and times. We will adopt this possible interpretation in the following, altering the classical meaning of ν, ν_0 and λ, λ_0 in Eq. (33).

In our alternative redshift interpretation we assume that the observed quantities λ_0 and ν_0, are telling us about the radiation emitted by the source galaxy at the place and time of emission, while the reference quantities λ and ν are, by the same argument, characteristics of the same atomic radiation as measured here on Earth at present time. We therefore write:

$$\begin{aligned} \lambda_0 = \lambda(\mathbf{r}, \mathbf{t}) \ ; \ \nu_0 = \nu(\mathbf{r}, \mathbf{t}) \\ \lambda = \lambda(\mathbf{0}, \mathbf{t}_0) \ ; \ \nu = \nu(\mathbf{0}, \mathbf{t}_0) \end{aligned} \tag{35}$$

where again $\mathbf{r} = \mathbf{0}$ represents the Earth's observer position, $\mathbf{t}_0$ is the present time, while $\mathbf{r}$ is the position of the source galaxy and $\mathbf{t}$ is the time of emission, in the past. Since the units of length and time, defined in Eq. (34), are respectively proportional to the radiation wavelengths and inversely proportional to the radiation frequencies, they also become functions of the space-time coordinates:

$$\begin{aligned} 1 \ meter \equiv \delta l(\mathbf{r}, \mathbf{t}) = n_m \ \lambda_m(\mathbf{r}, \mathbf{t}) \\ 1 \ second \equiv \delta t(\mathbf{r}, \mathbf{t}) = \frac{n_s}{\nu_s(\mathbf{r}, \mathbf{t})}, \end{aligned} \tag{36}$$

where δl and δt will indicate the unit-length and the unit-time-interval in the following. Due to this new interpretation, we correct Eq. (33), combining it also with the previous equations:

$$1 + z = \frac{\mathbf{R}(\mathbf{t}_0)}{\mathbf{R}(\mathbf{t})} = \frac{\lambda(\mathbf{r}, \mathbf{t})}{\lambda(\mathbf{0}, \mathbf{t}_0)} = \frac{\delta l(\mathbf{r}, \mathbf{t})}{\delta l(\mathbf{0}, \mathbf{t}_0)} = \frac{\nu(\mathbf{0}, \mathbf{t}_0)}{\nu(\mathbf{r}, \mathbf{t})} = \frac{\delta t(\mathbf{r}, \mathbf{t})}{\delta t(\mathbf{0}, \mathbf{t}_0)}. \tag{37}$$

In view of the modern definition of the unit of length, based on a fixed value of c, we will consider the value of the speed of light in vacuum to be just a connecting factor between the units of length and time and we see no reason, at least for now, to assume that this factor might also change at different space-time locations. In our opinion, a variation of the speed of light (proposed by some alternative cosmologies [83], [84], [85]) would imply a substantial difference in the universal evolution of the units of length and time which seems an unnecessary complication, not supported by experimental observations. Therefore, we will consider $c = 2.99792458 \times 10^{10}\ \frac{cm}{s}$ as a constant value in the following, but we will continue to explicitly include c in every equation.

IV. EVALUATION OF THE COSMIC SCALE FACTOR

The alternative interpretation of the cosmological redshift, presented in the previous section, is actually an adaptation of the well-known *gravitational redshift* (or *gravitational time-dilation*) to the cosmological scale and was even considered in the 1920's as a possible origin of the observed redshift (see the historical discussion in Weinberg [18], page 417), but would have required very strong local gravitational fields, so this explanation was quickly abandoned in favor of a "cosmological" Doppler effect. Nevertheless, it is interesting to notice that this possibility was taken into account at the beginning of modern cosmology as well as many other explanations.

The gravitational redshift is a fundamental consequence of the equivalence principle which states that the rate of a clock at rest is affected by the presence of a gravitational field as follows:

$$\frac{\delta t}{\Delta t} = \frac{1}{\sqrt{-g_{00}(x)}}, \tag{38}$$

where δt and Δt are respectively the clock periods in the presence or in the absence of gravitation, and $g_{00}(x)$ is the value of the time component of the metric at the point where the clock is located (see [18], section 3.5 for a general discussion, or [65] for a review of experimental results). Since the "true" period Δt of a clock is unknown, we can only observe this effect by comparing the rate of the clock at two different locations x_1, x_2 in the gravitational field:

$$\frac{\delta t_1}{\delta t_2} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} = \frac{\nu_2}{\nu_1} = \frac{\lambda_1}{\lambda_2} \equiv 1 + z \tag{39}$$

and this quantity is related to the ratio of the frequencies or wavelengths of the same atomic transition observed at the two locations, which can also be described by a "redshift" parameter z. The connection between Eqs. (37) and (39) is immediate, identifying the two locations x_1, x_2 with $(\mathbf{r}, \mathbf{t})$ and $(\mathbf{0}, \mathbf{t}_0)$ respectively.

The *gravitational redshift* or *time dilation* has been tested repeatedly for the classic Schwarzschild solution of the metric, i.e., $g_{00}(x) = -B(r) = -(1 - \frac{2MG}{c^2 r})$. Using this expression inside Eq. (39) we obtain a redshift if the point of emission x_1 is closer to the massive source of the field, compared to the point of observation x_2, such as in the case of light emitted by the Sun or by white dwarfs and observed here on Earth [65]. A blueshift can be observed instead by using the Earth's gravity and by placing point x_1 at a higher level than point x_2, as in the classic experiment by Pound and Rebka (see description in [18]). These gravitational redshifts are very small (the one due to the Sun corresponds to $z \sim 10^{-6}$ and those related to white dwarfs are about two orders of magnitude bigger) and cannot produce any cosmological redshift, since they are just a local effect, predicted on the basis of the classic Schwarzschild solution for a static and spherically symmetric massive source, such as a planet or a star.

However, in view of the preceding discussion of the cosmological redshift and of the new MK solutions shown in Eq. (10) or Eq. (14), involving a cosmological generalization of the classic Schwarzschild solution through the cosmological parameters γ and κ, we can now propose a direct determination of the scale factor $\mathbf{R}(\mathbf{t})$ based on this "extended" interpretation of the *gravitational-cosmological redshift*. In other words, we will show that, assuming the validity of Conformal Gravity and of the interpolation between the Static Standard Coordinates and the Robertson-Walker metric explained in Sect. III A, our alternative redshift interpretation restricts the possible conformal transformations of the metric to just one possible case, i.e., just one possible function $\Omega(x)$ in Eq. (2), therefore also practically breaking this conformal symmetry without resorting to field-theory symmetry breaking procedures.

The function $\mathbf{R}(\mathbf{t})$ will be uniquely determined from these purely "kinematical" considerations and we will not need to obtain it from the solution of the "dynamic" field equation (5) of conformal gravity, as it is done in standard General Relativity using the Friedmann equations. We will then compare our solutions for $\mathbf{R}(\mathbf{t})$ with the corresponding solutions obtained by current conformal cosmologies in the literature, since the metric in Eqs. (9)-(10) is based on the same equation of motion of conformal gravity, i.e., Eq. (5) with $T_{\mu\nu} = 0$.

A. The cosmic scale factor as a function of the radial coordinates

It is immediate to obtain the cosmic scale factor as a function of the radial coordinates. We start by combining Eqs. (14) and (26), in order to rewrite $B(r)$ as:

$$B(r) = 1 + \gamma r + \left(\frac{\gamma^2}{4} + k\right) r^2 = -g_{00}(r) \tag{40}$$

in Static Standard Coordinates. Now we use Eq. (39) to compute the *gravitational-cosmological* time dilation for two points corresponding to the source galaxy space-time position and the Earth's observer placed at the origin at present time:

$$1 + z = \frac{R(0)}{R(r)} = \frac{\lambda(r,t)}{\lambda(0,t_0)} = \frac{\nu(0,t_0)}{\nu(r,t)} = \sqrt{\frac{-g_{00}(0)}{-g_{00}(r)}} = \frac{1}{\sqrt{1 + \gamma r + \left(\frac{\gamma^2}{4} + k\right) r^2}}, \tag{41}$$

which gives the redshift factor $(1+z)$ as a very simple function of the coordinate r in SSC. We also express the factor $(1+z)$ as a ratio of cosmic scale factors, computed at the two points of interest, although usually the scale factor is only introduced in the RW metric. We will show in the following that this function R can be expressed in any of the space-time coordinates of interest, therefore we can also introduce it in the SSC.

Our objective is now to transform this expression into RW coordinates, by using the transformations outlined in Sect. III A. Before doing this, we note that Eq. (41) should give the observed cosmological redshift (i.e., $1 + z > 1$) at least for some distance interval $r > r_{rs}$, where r_{rs} is the coordinate beyond which we start observing a cosmological redshift. In addition, we assume that the parameter γ is small and positive at the present time, probably close to Mannheim's value of $\gamma_0 \simeq 3.06 \times 10^{-30}\,\mathrm{cm}^{-1}$, while the parameter k is not yet restricted ($k \gtreqless 0$).

For $\gamma > 0$, a quick inspection of Eq. (41) shows that a solution allowing redshift is possible only in one case: for a negative k and more precisely for $k < -\frac{\gamma^2}{4}$. In this case the function in Eq. (41) is well defined for positive values of r in the interval $0 \leq r < 1/(\sqrt{|k|} - \frac{\gamma}{2})$. Moreover, we obtain:

$$r_{rs} = \gamma/(|k| - \frac{\gamma^2}{4}), \tag{42}$$

giving a blueshift ($z < 0$) for distances in the interval $0 < r < r_{rs}$, and a proper redshift ($z > 0$) for larger distances $r > r_{rs}$, which might correspond to the observed cosmological redshift.

Since the cosmic scale factor and all the other cosmological quantities of interest are usually expressed in Robertson-Walker coordinates, we have to convert the expression in Eq. (41) into these coordinate. This can be accomplished by using the transformations of Sect. III A. From Eq. (16) and its inverse transformation, it follows that

$$\left[1 + \gamma r + \left(\frac{\gamma^2}{4} + k\right) r^2\right] = \frac{[1 + \frac{k}{4}\rho^2]^2}{[1 - \frac{\gamma}{2}\rho - \frac{k}{4}\rho^2]^2}, \tag{43}$$

so that we can write

$$1 + z = \frac{1}{\sqrt{1 + \gamma r + \left(\frac{\gamma^2}{4} + k\right) r^2}} = \frac{[1 - \frac{\gamma}{2}\rho - \frac{k}{4}\rho^2]}{[1 + \frac{k}{4}\rho^2]}, \tag{44}$$

which is well defined for $0 \leq \rho < 2/\sqrt{|k|}$.

The conformal transformation of Eq. (21) will not alter the ρ coordinate, so we just need to apply the final two transformations of Eqs. (24) and (27) to obtain, after some algebraic work:

$$1 + z = \frac{\mathbf{R(0)}}{\mathbf{R(r)}} = \sqrt{1 - k\rho'^2} - \frac{\gamma}{2}\rho' = \sqrt{1 - \mathbf{k}\ \mathbf{r}^2} - \frac{\gamma}{2\sqrt{|k|}}\mathbf{r}, \tag{45}$$

where we use again $\mathbf{r}$ (in bold) to denote the radial coordinate in RW metric and $\mathbf{k} = \frac{k}{|k|} = 0, \pm 1$, following Eq. (27). We observe that the last term in the previous equation diverges for $k = 0$, but according to the note following Eq. (27) in this particular case the previous equation should simply become $1 + z = \frac{\mathbf{R(0)}}{\mathbf{R(r)}} = 1 - \frac{\gamma}{2}\rho' = 1 - \frac{\gamma}{2}\mathbf{r}$. Another way to obtain Eq. (45) from Eq. (41) is to use the direct connection between coordinates r and $\mathbf{r}$, which can be easily derived from the transformations of Sect. III A and is the following:

$$\sqrt{|k|}r = \left(\sqrt{\frac{1}{\mathbf{r}^2} - \mathbf{k}} - \frac{\gamma}{2\sqrt{|k|}}\right)^{-1}. \tag{46}$$

A more elegant way to write the previous fundamental equations is to introduce a dimensionless parameter:

$$\delta \equiv \frac{\gamma}{2\sqrt{|k|}}, \tag{47}$$

or $\delta \equiv \frac{\gamma}{2}$ for the particular case $k = 0$, and rewrite Eq. (45) as

$$1 + z = \frac{\mathbf{R(0)}}{\mathbf{R(r)}} = \sqrt{1 - \mathbf{k}\ \mathbf{r}^2} - \delta \mathbf{r}\ ; \quad \mathbf{k} = 0, \pm 1. \tag{48}$$

We have written the ratio of cosmic scale factors as a function of the radial coordinates of the points of emission and absorption of radiation, since the function on the right-hand side of the previous equation depends only on $\mathbf{r}$, although we are implicitly referring also to the times at which the radiation was emitted and absorbed. A more precise notation would be to write always these cosmological scale factors as $\mathbf{R}(\mathbf{r}, \mathbf{t})$ and $\mathbf{R}(\mathbf{0}, \mathbf{t}_0)$ in all our formulas, but we will continue to use our simplified notation also in the following.

In the last equations the parameter γ is positive and determined, at least for now, by Mannheim's fits of galactic rotational curves, while the other parameter $k = -\frac{\gamma^2}{4} - \kappa$ is still undetermined. Since a cosmological redshift is generally observed,[13] i.e., $\mathbf{R(0)}/\mathbf{R(r)} > 0$ in general, we have already remarked that this suggests a negative value of $k < -\frac{\gamma^2}{4} < 0$, which implies $0 < \delta^2 \equiv \frac{\gamma^2}{4|k|} < 1$, thus restricting in general δ to the interval $-1 < \delta < 1$ (but with a currently positive value).

For these values of the parameters the quantity $r_{rs} = \gamma/(|k| - \frac{\gamma^2}{4})$ introduced in Eq. (42) can be rewritten in RW coordinates as

$$\mathbf{r}_{rs} = 1/\left(\sqrt{|k|}/\gamma - \gamma/4\sqrt{|k|}\right) = \frac{2\delta}{1 - \delta^2}. \tag{49}$$

Again, we have a standard redshift for distances $\mathbf{r} > \mathbf{r}_{rs}$, but an unexpected blueshift at closer distances $0 < \mathbf{r} < \mathbf{r}_{rs}$. This possibility is particularly interesting in view of a recently

[13] Except for some nearby galaxies typically located in our Local Group, whose blueshift is presumably due to their peculiar motion, or for the "Pioneer anomalous blueshift" which will be considered in another paper [62].

discovered phenomenon, the so-called *Pioneer anomaly* ([86], [87], [88], [89], [90]) consisting in an anomalous blueshift observed in the navigation of the Pioneer spacecraft, just outside the Solar System.

B. Time-dependent form of the cosmic scale factor

The cosmic scale factor $\mathbf{R}$ is usually considered a function of some cosmic time coordinate $\mathbf{t}$, rather than a function of the radial coordinates, as introduced in the previous section. This is a consequence of the Cosmological Principle (i.e., the Universe being assumed spatially homogeneous and isotropic) and of the application of this principle to the hypersurfaces with constant cosmic standard time, which are maximally symmetric subspaces of the whole of space-time (see Chapters 13-14 in Ref. [18] for details). Following this standard hypothesis, the resulting metric takes the RW form of Eq. (28) and the redshift is described by the ratio of scale factors at two different cosmic times, as in Eq. (33).

On the contrary, our new interpretation assumes that the redshift is due to the stretching of the space-time fabric as described by Eqs. (41) and (45), which are essentially static solutions, derived from the Conformal Gravity theory. In order to retain the validity of the Cosmological Principle and, in particular, still assume the homogeneity of the Universe at a given cosmic time, we have to transform the space dependence of our new cosmic scale factors into a more traditional time dependence.

This can be accomplished by noting that the redshifted radiation described by Eqs. (41) or (45) is reaching us from past times and that light coming from a radial distance $\mathbf{r}$ is all emitted at the same time $\mathbf{t}$ in the past. Therefore, the scale factor $\mathbf{R}(\mathbf{r})$ can be associated with a corresponding factor $\mathbf{R}(\mathbf{t})$, at a given past cosmic time $\mathbf{t}$. This association is performed by computing the time it takes for a light signal emitted at radial distance $\mathbf{r}$ to reach the observer at the origin. It is then straightforward to turn Eqs. (41) or (45) into their time dependent equivalent, since we are following a light signal traveling in vacuum from a distant galaxy toward us, for which $ds^2 = 0$ or $d\widehat{s}^2 = 0$.

It is convenient to study the propagation of this light signal by using first our original Static Standard Coordinates (r, t, θ, ϕ). Combining Eqs. (15) and (40), for a light ray traveling along the $(-r)$ direction with θ and ϕ fixed, we have:

$$ds^2 = -[1 + \gamma r + (\frac{\gamma^2}{4} + k)\ r^2]\ c^2 dt^2 + \frac{dr^2}{[1 + \gamma r + (\frac{\gamma^2}{4} + k)\ r^2]} = 0, \tag{50}$$

or equivalently

$$c\ dt = -\frac{dr}{[1 + \gamma r + (\frac{\gamma^2}{4} + k)\ r^2]}, \tag{51}$$

where the negative sign comes from the $(-r)$ direction of light propagation.

Integrating between times t and t_0, corresponding to radial positions r and $r = 0$, we obtain different results, depending on the sign of the parameter k:

$$\begin{aligned}
x \equiv c\ (t_0 - t) = \int_0^r \frac{dr'}{[1 + \gamma r' + (\frac{\gamma^2}{4} + k)\ r'^2]} = \qquad\qquad \\
= \left[\frac{1}{\sqrt{k}} \tan^{-1}\left(\frac{4kr' + 2\gamma + \gamma^2 r'}{4\sqrt{k}}\right)\right]_0^r = \frac{i}{2\sqrt{k}} \ln\left[\frac{1 + (\frac{\gamma}{2} - i\sqrt{k})\ r}{1 + (\frac{\gamma}{2} + i\sqrt{k})\ r}\right]\ ;\ k > 0 \\
= \left[\frac{-4}{\gamma(2 + \gamma r')}\right]_0^r = \frac{r}{(1 + \frac{\gamma}{2}\ r)}\ ;\ k = 0 \\
= \left[\frac{1}{\sqrt{|k|}} \tanh^{-1}\left(\frac{4\,|k|\, r' - 2\gamma - \gamma^2 r'}{4\sqrt{|k|}}\right)\right]_0^r = \frac{1}{2\sqrt{|k|}} \ln\left[\frac{1 + (\frac{\gamma}{2} + \sqrt{|k|})\ r}{1 + (\frac{\gamma}{2} - \sqrt{|k|})\ r}\right]\ ;\ k < 0,
\end{aligned} \tag{52}$$

where we have introduced the useful quantity $x \equiv c\ (t_0 - t)$. It is possible to invert Eq. (52) in each case and obtain the distance r as a function of $x \equiv c\ (t_0 - t)$:

$$\begin{aligned}
r &= \frac{1}{\sqrt{k}}\ \frac{1}{\left[\cot\left(\sqrt{k}x\right) - \frac{\gamma}{2\sqrt{k}}\right]}\ ;\ k > 0 \\
r &= \frac{x}{\left[1 - \frac{\gamma}{2}x\right]}\ ;\ k = 0 \\
r &= \frac{1}{\sqrt{|k|}}\ \frac{1}{\left[\coth\left(\sqrt{|k|}x\right) - \frac{\gamma}{2\sqrt{|k|}}\right]}\ ;\ k < 0.
\end{aligned} \tag{53}$$

Finally, it takes a little more work to combine Eq. (53) together with Eq. (41), to obtain the explicit form of the cosmic scale factor:

$$
\begin{aligned}
1+z &= \frac{R(t_0)}{R(t)} = \left[\cos\left(\sqrt{k}x\right) - \frac{\gamma}{2\sqrt{k}}\sin\left(\sqrt{k}x\right)\right] \ ; \ k>0 \\
1+z &= \frac{R(t_0)}{R(t)} = \left[1 - \frac{\gamma}{2}x\right] \ ; \ k=0 \\
1+z &= \frac{R(t_0)}{R(t)} = \left[\cosh\left(\sqrt{|k|}x\right) - \frac{\gamma}{2\sqrt{|k|}}\sinh\left(\sqrt{|k|}x\right)\right] \ ; \ k<0,
\end{aligned} \tag{54}
$$

a remarkably compact expression in each case. To obtain the scale factor as a function of the cosmic time coordinate $\mathbf{t}$ we could repeat the same procedure, studying the propagation of light in RW metric, but this involves rather cumbersome integrals. It is easier to find a direct relation between the time coordinates t and $\mathbf{t}$.

We start combining together Eqs. (16), (24) and (27), obtaining

$$
d\mathbf{t} = R(t)\ dt \tag{55}
$$

or, integrating between times t, $\mathbf{t}$ in the past and present times t_0, $\mathbf{t}_0$,

$$
\mathbf{t}_0 - \mathbf{t} = \int_t^{t_0} R(t)\ dt \tag{56}
$$

where the cosmic scale factor $R(t)$ is expressed through Eq. (54):

$$
\begin{aligned}
R(t) &= R(t_0)/\left[\cos\left(\sqrt{k}x\right) - \frac{\gamma}{2\sqrt{k}}\sin\left(\sqrt{k}x\right)\right] \ ; \ k>0 \\
R(t) &= R(t_0)/\left[1 - \frac{\gamma}{2}x\right] \ ; \ k=0 \\
R(t) &= R(t_0)/\left[\cosh\left(\sqrt{|k|}x\right) - \frac{\gamma}{2\sqrt{|k|}}\sinh\left(\sqrt{|k|}x\right)\right] \ ; \ k<0.
\end{aligned} \tag{57}
$$

Integrating Eq. (56) with the expressions from Eq. (57) we obtain (using the more compact parameter $\delta \equiv \gamma/2\sqrt{|k|}$):

$$\mathbf{t}_0 - \mathbf{t} = \frac{2\mathbf{R}(\mathbf{t}_0)}{c\sqrt{1+\delta^2}} \operatorname{arccoth}\left[\frac{\cot\left(\frac{\sqrt{k}}{2}x\right) - \delta}{\sqrt{1+\delta^2}}\right] \ ; \ k > 0 \tag{58}$$

$$\mathbf{t}_0 - \mathbf{t} = -\frac{2\mathbf{R}(\mathbf{t}_0)}{c\gamma} \ln\left[1 - \frac{\gamma}{2}x\right] \ ; \ k = 0$$

$$\mathbf{t}_0 - \mathbf{t} = \frac{2\mathbf{R}(\mathbf{t}_0)}{c\sqrt{1-\delta^2}} \operatorname{arccot}\left[\frac{\coth\left(\frac{\sqrt{|k|}}{2}x\right) - \delta}{\sqrt{1-\delta^2}}\right] \ ; \ k < 0.$$

Inverting these expressions, we obtain the connections between the time coordinates:

$$x \equiv c\ (t_0 - t) = \frac{2}{\sqrt{k}} \operatorname{arccot}\left\{\sqrt{1+\delta^2}\coth\left[\frac{\sqrt{1+\delta^2}}{2}\frac{c\,(\mathbf{t}_0 - \mathbf{t})}{\mathbf{R}(\mathbf{t}_0)}\right] + \delta\right\} \ ; \ k > 0 \tag{59}$$

$$x \equiv c\ (t_0 - t) = \frac{2}{\gamma}\left[1 - e^{-\frac{\gamma}{2}\frac{c(\mathbf{t}_0 - \mathbf{t})}{\mathbf{R}(\mathbf{t}_0)}}\right] \ ; \ k = 0$$

$$x \equiv c\ (t_0 - t) = \frac{2}{\sqrt{|k|}} \operatorname{arccoth}\left\{\sqrt{1-\delta^2}\cot\left[\frac{\sqrt{1-\delta^2}}{2}\frac{c\,(\mathbf{t}_0 - \mathbf{t})}{\mathbf{R}(\mathbf{t}_0)}\right] + \delta\right\} \ ; k < 0.$$

Inserting this last equation into Eq. (57), we finally compute the cosmic scale factor in RW coordinates:

$$\mathbf{R}(\mathbf{t}) = \mathbf{R}(\mathbf{t}_0)\ \left\{\cosh\left[\sqrt{1+\delta^2}\frac{c(\mathbf{t}_0 - \mathbf{t})}{\mathbf{R}(\mathbf{t}_0)}\right] + \frac{\delta}{\sqrt{1+\delta^2}}\sinh\left[\sqrt{1+\delta^2}\frac{c(\mathbf{t}_0 - \mathbf{t})}{\mathbf{R}(\mathbf{t}_0)}\right]\right\};\ \mathbf{k} = 1 \tag{60}$$

$$\mathbf{R}(\mathbf{t}) = \mathbf{R}(\mathbf{t}_0)\ e^{\frac{\gamma}{2}\frac{c(\mathbf{t}_0 - \mathbf{t})}{\mathbf{R}(\mathbf{t}_0)}} \ ; \ \mathbf{k} = 0$$

$$\mathbf{R}(\mathbf{t}) = \mathbf{R}(\mathbf{t}_0)\ \left\{\cos\left[\sqrt{1-\delta^2}\frac{c\,(\mathbf{t}_0 - \mathbf{t})}{\mathbf{R}(\mathbf{t}_0)}\right] + \frac{\delta}{\sqrt{1-\delta^2}}\sin\left[\sqrt{1-\delta^2}\frac{c\,(\mathbf{t}_0 - \mathbf{t})}{\mathbf{R}(\mathbf{t}_0)}\right]\right\};\ \mathbf{k} = -1.$$

We will study all these functions in detail in the following sections. We remark here that we now have the cosmological scale factor expressed in four different types of coordinates. The space coordinates r and $\mathbf{r}$, entering Eqs. (41) and (48) respectively, are appropriate to measure distances in their respective coordinate systems: r (in centimeters or meters) refers to the Static Standard Coordinates and to the local "meter stick" being used, it is therefore suitable for local measurements. To measure distances on astronomical scales we will need to introduce revised expressions for the classic luminosity distance and for the other distances used in cosmology, which are all based on the Robertson-Walker $\mathbf{r}$ (dimensionless) coordinate. Therefore, when measuring distances to galaxies, supernovae, etc. (in Mpc, light years, or other suitable units) we will use the $\mathbf{r}$ coordinate and our fundamental cosmological scale factor will be given by Eq. (48).

A different situation exists for the two time coordinates t and $\mathbf{t}$, or for the equivalent look-back times $t_0 - t$ and $\mathbf{t}_0 - \mathbf{t}$. Although we measure both of them with the same units (seconds, years, etc.), the time interval between two events is expressed differently by the two time coordinates, as shown in Eqs. (58) and (59) or by the original connection in Eq. (56). In standard cosmology, when making time determinations such as estimating the age of the Universe or using radioactive decay for the age determination of astronomical objects, it is assumed that our local time t can be synchronized with the cosmological time $\mathbf{t}$, therefore no distinction is made between the two.

However, in our model these two quantities are different and their connection is given above. When using the comoving RW coordinates, one should refer to the cosmic time $\mathbf{t}$ (in bold) and use Eq. (60). On the contrary, when making contact with observations, such as age estimates or when taking time derivatives of the cosmic scale factor to obtain the Hubble and deceleration parameters, the local time coordinate t should be used, together with Eq. (54), since all these observations refer to our terrestrial clocks.

We now compare our solutions in Eq. (60) with the corresponding solutions obtained by current conformal cosmologies in the literature, as we already remarked that the metric in Eqs. (9)-(10) is based on the same equation of motion of conformal gravity, i.e., Eq. (5) with $T_{\mu\nu} = 0$.

Our solutions in Eq. (60) are similar to those proposed by Mannheim in Eq. (9) of Ref. [54] and also discussed in Ref. [53]. In particular, Mannheim analyzes the "de Sitter geometry in a purely kinematic way that requires no commitment to any particular dynamical equation of motion, neither that of conformal gravity nor that of the standard model" (quoted from [54]) and it is therefore related to a cosmology with just a cosmological constant source. In Robertson-Walker coordinates a simple kinematic relation is then obtained (see Eq. (8) of Ref. [54]):

$$\dot{\mathbf{R}}^{\mathbf{2}}(\mathbf{t}) + \mathbf{k}c^2 = \alpha c^2 \mathbf{R}^2(\mathbf{t}) \tag{61}$$

and five possible cosmological solutions (see Eq. (9) of Ref. [54]) are found by Mannheim, for the different cases related to the signs of the parameters α and $\mathbf{k}$:

$$\begin{aligned}
\mathbf{R}(\mathbf{t},\alpha<0,\mathbf{k}<0) &= \left(\frac{\mathbf{k}}{\alpha}\right)^{1/2} \sin\left[(-\alpha)^{1/2}\, c\mathbf{t}\right] \\
\mathbf{R}(\mathbf{t},\alpha=0,\mathbf{k}<0) &= (-\mathbf{k})^{1/2}\, c\mathbf{t} \\
\mathbf{R}(\mathbf{t},\alpha>0,\mathbf{k}<0) &= \left(-\frac{\mathbf{k}}{\alpha}\right)^{1/2} \sinh\left(\alpha^{1/2} c\mathbf{t}\right) \\
\mathbf{R}(\mathbf{t},\alpha>0,\mathbf{k}=0) &= \mathbf{R}(\mathbf{t}=0)\exp\left(\alpha^{1/2} c\mathbf{t}\right) \\
\mathbf{R}(\mathbf{t},\alpha>0,\mathbf{k}>0) &= \left(\frac{\mathbf{k}}{\alpha}\right)^{1/2} \cosh\left(\alpha^{1/2} c\mathbf{t}\right).
\end{aligned} \tag{62}$$

We can rewrite our three solutions in Eq. (60) so that they are equivalent to three of Mannheim's expressions in the previous equation, as long as we identify the time variables as follows: $(\mathbf{t}_0 - \mathbf{t}) \rightarrow \mathbf{t}$ and $\mathbf{t}_0 \rightarrow (\mathbf{t}=0)$. Our Eq. (60) then becomes:

$$\begin{aligned}
\mathbf{R}(\mathbf{t},\mathbf{k}=1) &= \frac{\mathbf{R}(\mathbf{t}=0)}{\sqrt{1+\delta^2}} \cosh\left[\epsilon + \frac{\sqrt{1+\delta^2}}{\mathbf{R}(\mathbf{t}=0)} c\mathbf{t}\right];\ \text{with } \epsilon = \sinh^{-1}\delta \\
\mathbf{R}(\mathbf{t},\mathbf{k}=0) &= \mathbf{R}(\mathbf{t}=0)\ \exp\left[\frac{\gamma}{2\mathbf{R}(\mathbf{t}=0)} c\mathbf{t}\right] \\
\mathbf{R}(\mathbf{t},\mathbf{k}=-1) &= \frac{\mathbf{R}(\mathbf{t}=0)}{\sqrt{1-\delta^2}} \sin\left[\epsilon + \frac{\sqrt{1-\delta^2}}{\mathbf{R}(\mathbf{t}=0)} c\mathbf{t}\right];\ \text{with } \epsilon = \cos^{-1}\delta.
\end{aligned} \tag{63}$$

These three solutions correspond respectively to the fifth, fourth and first expression in Eq. (62), with an additional quantity denoted by ϵ, in the first and third solution, which depends on our parameter δ. It can be easily checked that all three expressions in Eq. (63) verify the original kinematic relation in Eq. (61), with $\alpha = \frac{1+\delta^2}{\mathbf{R}^2(\mathbf{t}=0)}; \frac{\gamma^2}{4\mathbf{R}^2(\mathbf{t}=0)}; -\frac{1-\delta^2}{\mathbf{R}^2(\mathbf{t}=0)}$ respectively in the three cases. Therefore, our fundamental expressions in Eq. (60) are valid solutions of the problem analyzed by Mannheim and described above.

Following this discussion, it can be noted [31] that the solutions we obtained with our kinematical approach to conformal cosmology are equivalent to those of the standard conformal cosmology ([53], [54]) based solely on a cosmological constant. In this view, our approach could be valid in two separate epochs: a very early Universe which undergoes a cosmological constant dominated inflationary phase and a very late Universe in which the cosmological constant dominates the energy density.

The identification of the cosmological redshift with the gravitational redshift could then represent not the current state of the Universe, but rather the one into which the Universe might ultimately evolve, or the one that it may initially have evolved from [31]. Therefore, it is noteworthy that our kinematic cosmology is actually recovered from conformal gravity with a cosmological constant at late (or early) times and this result constitutes the central point of this paper.

Finally, in addition to the connecting relations between r, $\mathbf{r}$, t and $\mathbf{t}$ given by Eqs. (46), (52), (53), (58) and (59), a very simple connection exists between the RW $\mathbf{r}$ coordinate and the time t of the SSC, which are the most important observational quantities, as discussed in the previous paragraphs. This follows immediately from the relation between the comoving coordinate χ and the conformal time η found in Eqs. (30)-(32) for the propagation of a light signal. This relation can be re-written as

$$\chi = \sqrt{|k|}c(t_0 - t) = \eta_0 - \eta, \tag{64}$$

assuming that the emission of the light signal happens at coordinates (χ, η) and the signal is received at $(\chi = 0, \eta_0)$. The relation between $\mathbf{r}$ and the look-back time $t_0 - t$ is therefore:

$$\mathbf{r} = S_{\mathbf{k}}(\chi) = \left\{ \begin{array}{cc} \sin\left[\sqrt{|k|}c(t_0 - t)\right] \ ; & \mathbf{k} = +1 \\ c(t_0 - t) \ ; & \mathbf{k} = 0 \\ \sinh\left[\sqrt{|k|}c(t_0 - t)\right] \ ; & \mathbf{k} = -1 \end{array} \right\}. \tag{65}$$

These relations represent a simple and precise connection between the two coordinates for the motion of a light signal over cosmological distances, but with time measured by our current standard units.

C. Analysis of the solutions for the cosmic scale factor

Equations (41), (48), (54), (57), and (60) are our main result. They represent the closed-form expressions for the cosmic scale factor R, or rather its ratio to the present value, as a function of the "look-back time" or "look-back distance" in both SSC or RW coordinates. In the way they were derived such equations are also valid for space-time coordinates in the "future," so they represent the overall evolution of the Universe.

We recall once again that, of the two parameters in our fundamental equations, γ is approximately determined by Mannheim's fits as a small positive quantity. The other constant $k = -\frac{\gamma^2}{4} - \kappa$ is still undetermined, in both magnitude and sign, since it includes the still unknown κ quantity. We will not assume $\kappa = 0$, as in Mannheim's cosmology [10], since we note that, in the particular case of $\kappa = 0$, $k = -\frac{\gamma^2}{4} < 0$, our solution in Eq. (54) would yield a very simple (and unlikely) exponential solution

$$1 + z = \frac{R(t_0)}{R(t)} = \cosh[\frac{\gamma}{2}x] - \sinh[\frac{\gamma}{2}x] = e^{-\frac{\gamma}{2}c(t_0 - t)}, \tag{66}$$

reminiscent of the classic steady state cosmology, a theory that we don't believe can represent the physical reality of our Universe. We need therefore to study in detail our fundamental solutions and obtain more accurate values of the parameters from experimental observations [62].

To analyze the general form of our solutions it is better to plot the ratio R/R_0, which shows directly the past (and future) evolution of the cosmic scale factor. We also notice that all formulas can be rewritten using only dimensionless quantities which are particularly convenient. Eq. (41) can be written as:

$$\frac{R(\alpha)}{R(0)} = \sqrt{1 + \delta\alpha + \frac{1}{4}(\delta^2 + \mathbf{k})\alpha^2} \tag{67}$$

$$\delta \equiv \frac{\gamma}{2\sqrt{|k|}};\ \alpha \equiv 2\sqrt{|k|}r\ ;\ \mathbf{k} = \frac{k}{|k|} = 0, \pm 1$$

where α becomes a dimensionless coordinate and the parameter δ is thought of as a possible time varying quantity. In Fig. 1 we plot these solutions, for the three values of $\mathbf{k}$, assuming a positive current value $0 < \delta = \delta(t_0) < 1$.

We recall that the positive horizontal semi-axis is actually representing a "look-back" quantity, here expressed using the dimensionless $\alpha \equiv 2\sqrt{|k|}r$; on the contrary the negative horizontal semi-axis represents "future" values of this variable, so that the universal evolution of the cosmic scale factor should be observed following our curves from right to left. The dot on the vertical axis represents our "current" position in the universal evolution. We can clearly see that the only solution which indicates a redshift in the past (values below the black thin dashed line for positive α) is the one for $\mathbf{k} = -1$ (red-solid, in Fig. 1). We will therefore consider this particular solution as our fundamental candidate to represent

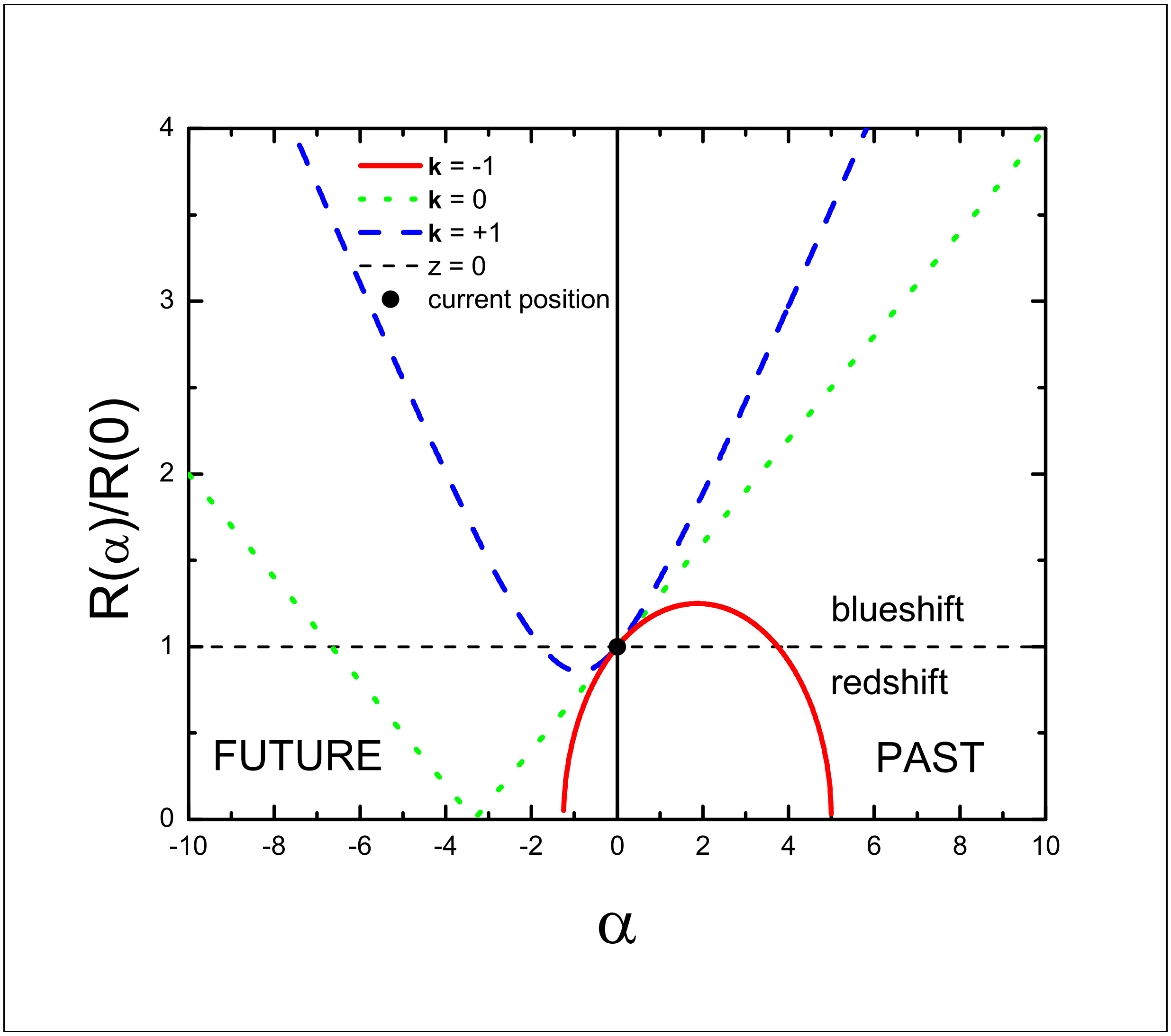

FIG. 1: R functions in Eq. (67) are shown here for different values of $\mathbf{k}$: $\mathbf{k} = -1$ in red (solid), $\mathbf{k} = 0$ in green (dotted), and $\mathbf{k} = +1$ in blue (dashed), and for a positive value of the parameter $\delta \simeq 0.6$ (an unrealistically large value; our current value $\delta = \delta(t_0)$ will be shown to be positive and close to zero [62]).

the evolution of the Universe.

The analytic characteristics of this solution (for $\mathbf{k} = -1$) are easily determined for a given value of δ $(-1 < \delta < 1)$: the zeroes of this function are obtained for $\alpha = \frac{2}{1-\delta}; -\frac{2}{1+\delta}$ or for $r = 1/(\sqrt{|k|} - \frac{\gamma}{2}); -1/(\sqrt{|k|} + \frac{\gamma}{2})$, respectively in the "past" and in the "future." As discussed at the end of the previous section, the variable r (or α) does not represent a cosmological distance, but rather a simple coordinate in the Static Standard frame of reference, therefore the values given above might represent initial or final "singularities" of the Universe, but only if we were to measure the Universe with our fixed meter stick. It seems more appropriate to consider them just as limiting values for our r coordinate.

As already noted before, the region in the past closer to our current location, for $0 < \alpha < \alpha_{rs} = \frac{4\delta}{1-\delta^2}$, or $0 < r < r_{rs} = \gamma/(|k| - \frac{\gamma^2}{4})$, would yield a blueshift. The redshift region occurs for $\alpha_{rs} = \frac{4\delta}{1-\delta^2} < \alpha < \frac{2}{1-\delta}$ or $r_{rs} = \gamma/(|k| - \frac{\gamma^2}{4}) < r < 1/(\sqrt{|k|} - \frac{\gamma}{2})$. The blueshift region (greatly exaggerated in Fig. 1), could be just a small region around us, of the size of our Solar System or part of our galaxy, depending on the current values of the parameters γ and k. The red curve in Fig. 1 is also obviously symmetric around its point of maximum, located at $\alpha_{\max} = \frac{2\delta}{1-\delta^2} = \frac{\alpha_{rs}}{2}$, which corresponds to a maximum blueshift $(R(\alpha)/R(0))_{\max} = 1/\sqrt{1-\delta^2}$ or $z_{\min} = \sqrt{1-\delta^2} - 1$, the minimum value of z in the blueshift region (negative value).

In the previous paragraphs we started using dimensionless quantities, which greatly simplify the analysis of our solutions. In particular, the dimensionless $\delta \equiv \frac{\gamma}{2\sqrt{|k|}}$, which enters all our fundamental equations, seems to be more important than the single dimensionful parameters γ and k. Plotting the solid red curve of Fig. 1, for values of δ varying from -1 to $+1$, would show a family of curves of similar shape and interpretation, just with the point representing our "current" position ($R(r)/R(0) = 1$) shifting along the red curve from the limiting position on the right of the graph (for $\delta \rightarrow -1$) to the limiting position on the left (for $\delta \rightarrow +1$). Similar considerations would also apply for the other two curves in the figure.

We recall that in modern cosmology a cosmic standard time coordinate should be related to some property of the evolving Universe itself. Quoting from Weinberg ([18], page 409): "... several cosmic scalar fields... are everywhere decreasing monotonically; choose any one of these, say a scalar S, and let the time of any event be any definite decreasing function $t(S)$ of the chosen scalar, when and where the event occurs." In view of the preceding discussion, it seems possible that the role of the universal quantity S, only increasing rather than decreasing, might be taken by $\delta \equiv \gamma/2\sqrt{|k|}$, varying continuously from -1 to $+1$. Plotting our fundamental solution ($\mathbf{k} = -1$) for δ increasing monotonically from -1 to 1, we would observe at first (for values of δ close to -1) a plot similar to the one of Fig. 1, but with the limiting position on the right very close to the origin and a very steep slope of the initial part of the plot: this is equivalent to a very fast initial expansion of the Universe, a sort of "inflationary" situation.

As δ increases in the negative interval $-1 < \delta < 0$, the expansion of the Universe would seem to slow down: the limiting position in the past would shift to the right in the plot, the red curve would "slide" to the right, subject to the condition of always intersecting the vertical axis at $R(r)/R(0) = 1$, and the slope of the curve at the origin, corresponding to the expansion rate, would continuously decrease. Similar behavior would be followed also by the $\mathbf{k} = 0, +1$ solutions in Fig. 1, but we consider these solutions not to be physically relevant, therefore we concentrate our attention on the $\mathbf{k} = -1$ solution only (red-solid in all our figures). For $\delta = 0$ the fundamental ($\mathbf{k} = -1$) solution would reduce to:

$$\frac{R(t)}{R(t_0)} = \sqrt{1 - \frac{1}{4}\alpha^2}, \tag{68}$$

and the red curve in Fig. 1 would simply be shifted in a symmetric position with the maximum at $\alpha = 0$, signaling that the maximum possible expansion of the Universe has been reached (for an observer at the origin) and that the expansion rate at the origin would now be zero. The zeroes of the function would appear to be at $\alpha = \pm 2$, with the Universe half way through its cosmic evolution. The subsequent evolution would be seen by letting the parameter δ run over positive values from 0 to 1. This situation is again what is depicted in Fig. 1, corresponding to a current positive value of δ (the plot in Fig. 1 is actually for $\delta \simeq 0.6$, therefore greatly exaggerating the blueshift portion).

Our current value of $\delta = \delta(t_0)$, that we will estimate in our second paper [62], is probably positive and small, therefore determining already a local contraction of the Universe (blueshift) in a limited region around us, but still showing the past expansion (redshift) in most of our visible Universe. For increasing positive values of δ, approaching the final $+1$ value, the rate of universal contraction would increase, leading to the final situation in a totally symmetric way, compared to the initial expansion.

The role of "universal time" given to the dimensionless parameter δ, naturally prompts us to plot all our fundamental solutions in terms of this universal quantity. This can be done using Eq. (67) and by noting that the maximum of the red plot in Fig. 1 corresponds to a "universal time" $\delta = 0$, in the sense that an observer placed at that position, at the time of emission of the light which reaches us with the maximum possible blueshift, would measure $\delta = 0$ as his/her current cosmological time.

This maximum value occurs at $\alpha_{\max} = \frac{2\delta}{1-\delta^2}$ (or for $r_{\max} = \gamma/2(|k| - \frac{\gamma^2}{4})$) and the corresponding value is $R(\alpha_{\max}) = R(\alpha = 0)\frac{1}{\sqrt{1-\delta^2}} = R(\delta = 0)$. But $R(\alpha = 0)$ corresponds to the R factor evaluated at the current value of the parameter δ, i.e., $R(\delta) = R(\alpha = 0)$, thus obtaining:[14]

$$R(\delta) = R(\delta = 0)\sqrt{1-\delta^2};\ \mathbf{k} = -1, \tag{69}$$

whose plot as a function of the "universal time" δ is simply a semicircle of radius corresponding to the maximum "size" of the Universe $R(\delta = 0)$. Similar analysis would hold for the $\mathbf{k} = 0, +1$ solutions, giving respectively:

$$\begin{aligned} R(\delta) &= R(\delta = 0)\sqrt{1+\delta^2};\ \mathbf{k} = +1 \qquad (70) \\ R(\delta) &= R(\delta = 0);\ \mathbf{k} = 0. \end{aligned}$$

The three possible solutions for the evolution of the Universe, as a function of δ, are therefore remarkably simple, when plotted in these new coordinate and are summarized in Fig. 2.

In this figure, the "universal time" δ now runs from "past" to "future" for increasing values. The red-solid plot shows our currently favorite cosmology of simple "semi-circular" evolution. The current value of $\delta = \delta(t_0)$ should be positive, such as the value represented by the dot in the figure. Regions in the past below the horizontal black dashed line represent the observed redshift, while the region above the same horizontal thin black line indicates the local blueshift region. Again, the $\delta(t_0)$ value of the figure ($\delta(t_0) \simeq 0.6$) is greatly exaggerated. The actual value should be very small and positive [62].

The green-dotted $\mathbf{k} = 0$ and the blue-dashed $\mathbf{k} = +1$ solutions would yield respectively a static, constant size Universe or a parabolic, first-contracting then-expanding Universe. These are not favored by observation, in particular, for $\mathbf{k} = +1$, we would observe a redshift at "close" distances and then a blueshift at "large" cosmological distances. Therefore our choice of a $\mathbf{k} = -1$ cosmology seems to be confirmed.

Using Eq. (69) with our current value of $\delta = \delta(t_0)$, i.e., $R\left[\delta(t_0)\right] = R(\delta = 0)\sqrt{1-\delta^2(t_0)}$ and dividing through the same equation for an arbitrary value of δ, we obtain

[14] This argument is similar to the assumption we made earlier, when converting the scale factor from being a function of radial coordinates to a function of time. If δ plays the role of a universal time, the Cosmological Principle would suggest that, for a given value of δ, all the locations in space should be equivalent, therefore the scale factor R should be properly a function just of the cosmological time δ.

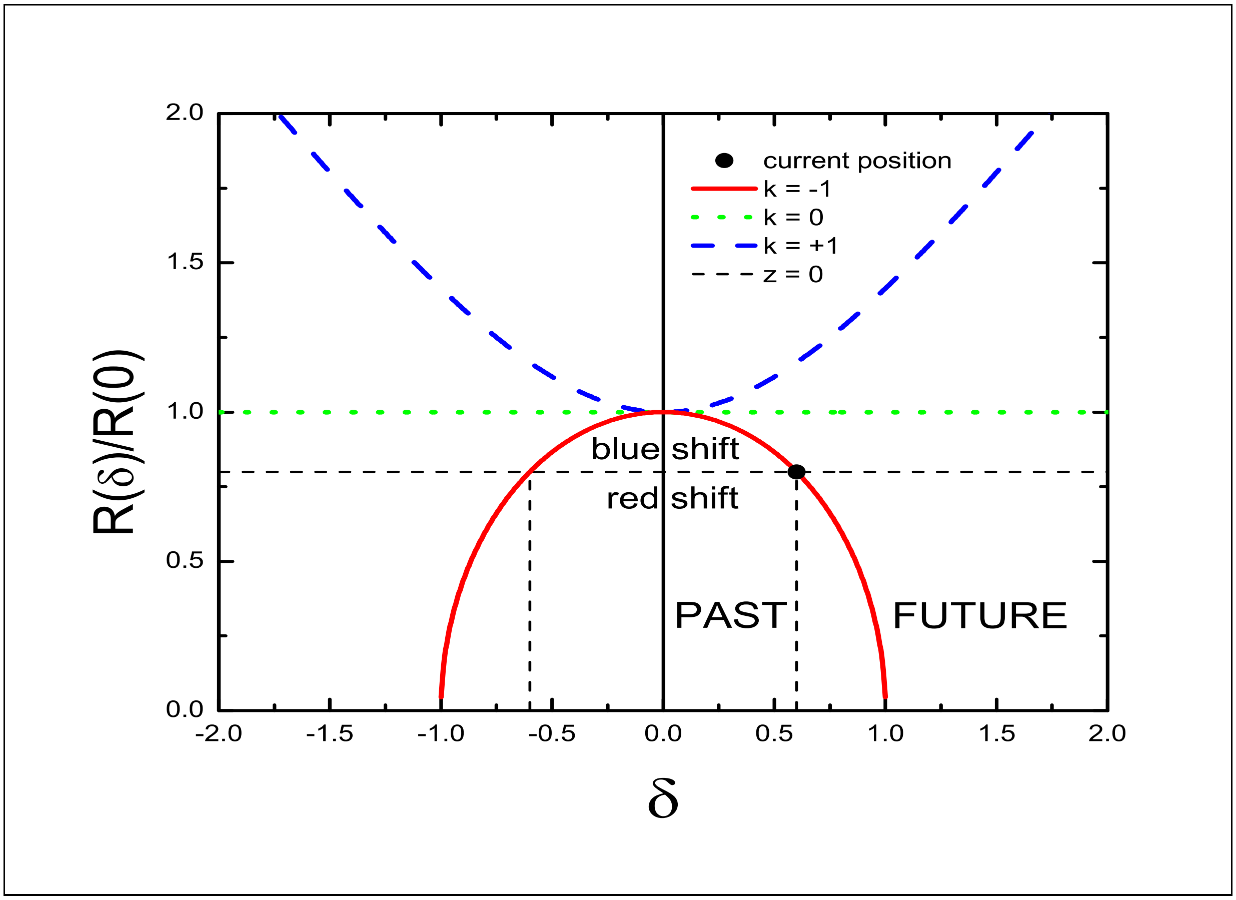

FIG. 2: R functions in Eqs. (69), (70) are shown here for different values of $\mathbf{k}$: $\mathbf{k} = -1$ in red (solid), $\mathbf{k} = 0$ in green (dotted), and $\mathbf{k} = +1$ in blue (dashed). The "cosmological time" δ is increasing from -1 to $+1$, so that the evolution of the Universe proceeds from left to right along the plotted curves. The present cosmological time is indicated as $\delta \simeq 0.6$, an unrealistically large value. Our current value of $\delta = \delta(t_0)$ will be shown to be positive and close to zero [62].

$$1 + z = \frac{R\left[\delta(t_0)\right]}{R(\delta)} = \sqrt{\frac{1 - \delta^2(t_0)}{1 - \delta^2}}, \tag{71}$$

from which we can derive the most general connection between $\delta(t_0)$, δ and z (for the case $\mathbf{k} = -1$):

$$\begin{aligned} z &= \sqrt{\frac{1 - \delta^2(t_0)}{1 - \delta^2}} - 1 \\ \delta &= \pm \frac{\sqrt{\delta^2(t_0) + z(z+2)}}{(1+z)}. \end{aligned} \tag{72}$$

Alternatively, we can combine Eq. (71) with Eq. (67), for the case $\mathbf{k} = -1$, namely $\sqrt{1-\delta^2}/\sqrt{1-\delta^2(t_0)} = \sqrt{1+\delta(t_0)\alpha + \frac{1}{4}\left[\delta^2(t_0)-1\right]\alpha^2}$, and solve it for α and δ:

$$\alpha = 2\frac{\delta(t_0)-\delta}{1-\delta^2(t_0)} \tag{73}$$
$$\delta = \delta(t_0) - \frac{1}{2}\left[1-\delta^2(t_0)\right]\alpha.$$

D. The other fundamental solutions

Most of the analysis in the previous section can be repeated also for the other fundamental solutions. From Eq. (48) we obtain:

$$\frac{\mathbf{R}(\mathbf{r})}{\mathbf{R}(\mathbf{0})} = \frac{1}{\sqrt{1-\mathbf{k}\,\mathbf{r}^2} - \delta\mathbf{r}} \;; \quad \mathbf{k} = 0, \pm 1, \tag{74}$$

as a function of dimensionless quantities (we recall that $\mathbf{r}$ can be considered dimensionless, following the note before Eq. (28)). In Fig. 3 we plot these solutions for a positive value of δ. The same comments of Fig. 1 are applicable here, only the shape of the curves is different due to the transformation between coordinates r and $\mathbf{r}$, described in Eq. (46). The $\mathbf{k} = -1$ solution, red-solid in the figure, is still the only cosmologically viable.

Again, the cosmological evolution from past to future is seen following the red curve from right to left. The past redshift region is followed by a local blueshift region. The black dot indicates the present situation, but we still used an unrealistic value $\delta \simeq 0.6$ to plot the curves. In the actual situation [62], for a small positive δ, we would be placed near the expansion maximum, at the beginning of the "contraction" phase. The main difference between Fig. 1 and Fig. 3 is that the red continuous curve, as a function of $\mathbf{r}$, does not show any initial or final "singularity," i.e., the evolution function only approaches zero for $\mathbf{r} \to \pm\infty$. Since astronomical distances, such as the luminosity distance or others, are defined using $\mathbf{r}$, the Universe will not show any initial or final "singularity" when measured using these coordinates. On the contrary, such initial and final points were shown to be present in Fig. 1 or Fig. 2, where the "universal time" δ was used.

In other words, the description of the Universe through the parameter δ would suggest the existence of an "initial" and a "final" singularity, thus not contradicting our standard "Big Bang" intuition, but tracing the origin of the Universe with the $\mathbf{r}$ coordinate would not

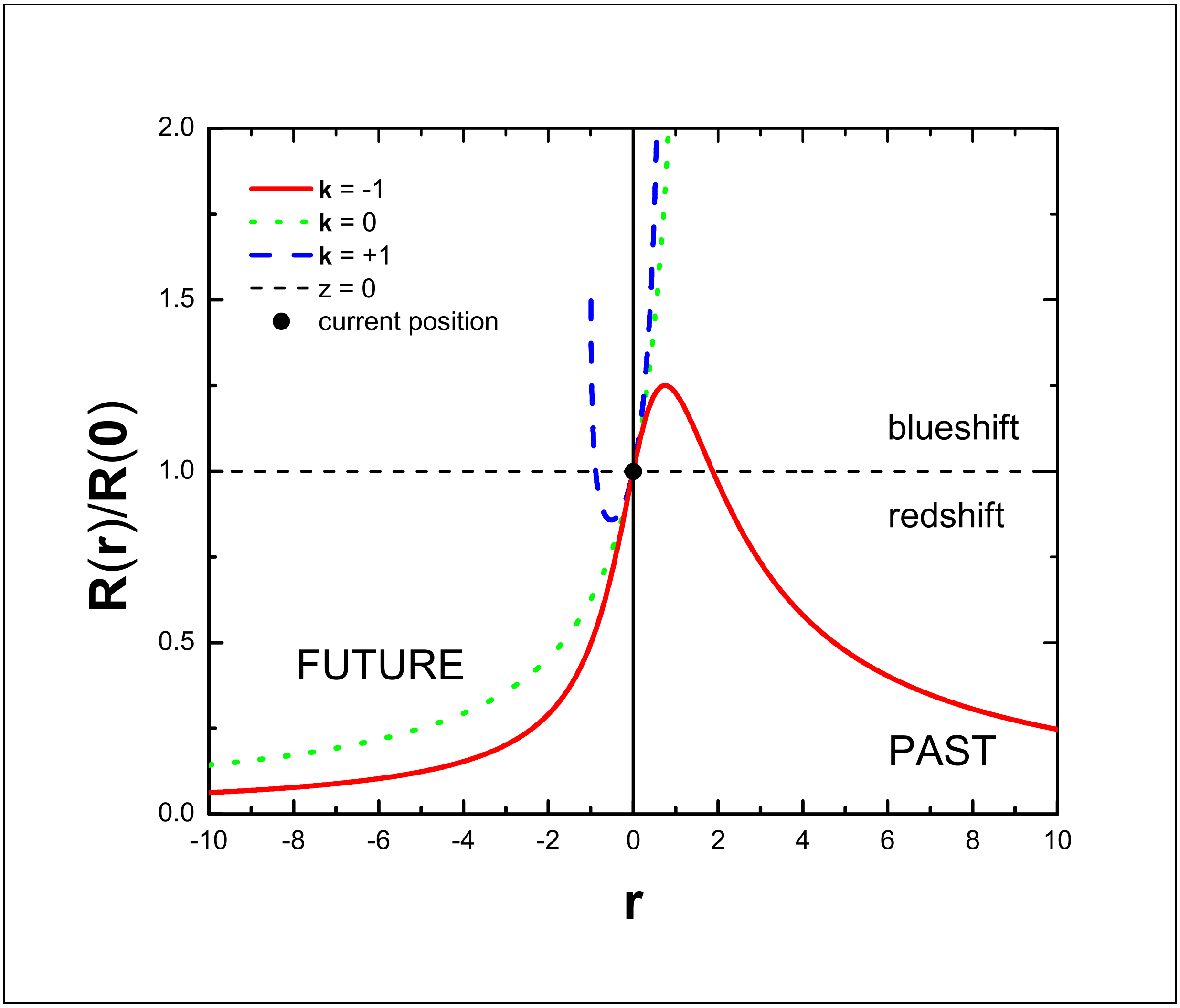

FIG. 3: R functions in Eq. (74) are shown for different values of $\mathbf{k}$: $\mathbf{k} = -1$ in red (solid), $\mathbf{k} = 0$ in green (dotted), and $\mathbf{k} = +1$ in blue (dashed). Again, we used an unrealistic $\delta \simeq 0.6$ to make the illustration more legible.

describe the initial singularity in terms of a finite "distance" corresponding to a zero of the $\mathbf{R}$ function. A similar "dual" interpretation will also be found for the time coordinates in the following paragraphs, although with an inverted role played by the coordinates t and $\mathbf{t}$.

As for the other analytical properties of the $\mathbf{k} = -1$ solution in Fig. 3, the positions of zero redshift are located at the origin $\mathbf{r} = 0$ and at $\mathbf{r}_{rs} = \frac{2\delta}{1-\delta^2}$. The position of maximum expansion is $\mathbf{r}_{\max} = \frac{\delta}{\sqrt{1-\delta^2}}$, corresponding to a maximum blueshift $(\mathbf{R}(\mathbf{r})/\mathbf{R}(\mathbf{0}))_{\max} = 1/\sqrt{1-\delta^2}$ or $z_{\min} = \sqrt{1-\delta^2} - 1$, as already found with the graph in Fig. 1. The $\mathbf{R}(\mathbf{r})$ curve is not symmetric around its point of maximum, as it was for the curve in r, due to the transformation between these two coordinates. The points of inflection of the red curve in Fig. 3, representing the position of change between a positive and negative acceleration of the expansion, can also be easily found analytically from Eq. (74), but their expression is rather cumbersome and will be omitted here.

It is possible to repeat with the $\mathbf{r}$ coordinate the same reasoning done with r, to obtain the solution in terms of the universal parameter δ, by using the fact that the maximum expansion position $\mathbf{r}_{\max} = \frac{\delta}{\sqrt{1-\delta^2}}$ (for $\delta = \delta(\mathbf{t}_0)$ our local current value) is assumed to correspond to $\delta(\mathbf{r}_{\max}) = 0$, when this parameter is measured at the position of maximum expansion. In this way we find the same solutions already expressed in Eqs. (69), (70) and plotted in Fig. 2.

In the same way, as it was done for the variable α, we can obtain the connections between $\mathbf{r}$, δ and our current value $\delta(\mathbf{t}_0)$ (for the $\mathbf{k} = -1$ case):

$$
\begin{aligned}
\mathbf{r} &= \frac{\delta(\mathbf{t}_0) - \delta}{\sqrt{[1-\delta^2(\mathbf{t}_0)]\,[1-\delta^2]}} \\
\delta &= \frac{\mathbf{r} - \delta(\mathbf{t}_0)\sqrt{1+\mathbf{r}^2}}{\delta(\mathbf{t}_0)\mathbf{r} - \sqrt{1+\mathbf{r}^2}}.
\end{aligned}
\tag{75}
$$

Considering, for example, the current value as $\delta = \delta(\mathbf{t}_0) \cong 0.1$, the previous equation is plotted Fig. 4. We can see from this figure that the initial and final values for the cosmological time, $\delta = \mp 1$, correspond to infinite values of the $\mathbf{r}$ coordinate, as already remarked.

It is also immediate to determine the relation between $\mathbf{r}$ and the redshift parameter z. Directly from Eq. (48) for the case $\mathbf{k} = -1$, we obtain such relation and its inverse:

$$
\begin{aligned}
z &= \sqrt{1+\mathbf{r}^2} - \delta\mathbf{r} - 1 \\
\mathbf{r} &= \frac{\delta(1+z) \pm \sqrt{\delta^2 + z(z+2)}}{(1-\delta^2)} = \frac{\delta(1+z) \pm \sqrt{(1+z)^2 - (1-\delta^2)}}{(1-\delta^2)},
\end{aligned}
\tag{76}
$$

where the inverse expression holds for $z \geq z_{\min} = \sqrt{1-\delta^2} - 1$.

Similar considerations apply to the time dependent solutions. Eq. (57) can also be written in dimensionless form:

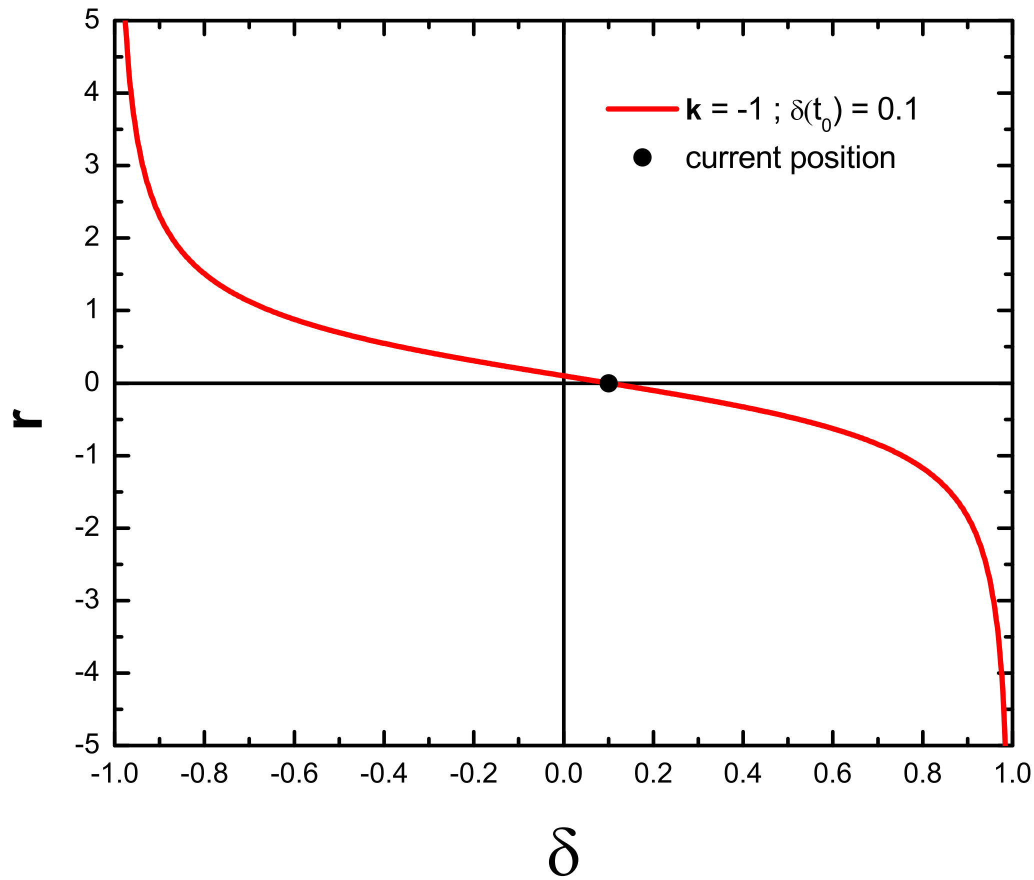

FIG. 4: The connection between **r** and δ is illustrated here for a value $\delta(\mathbf{t}_0) \simeq 0.1$. The correct value of the cosmological time will be determined later [62].

$$\frac{R(t)}{R(t_0)} = [\cos\chi - \delta\sin\chi]^{-1} \;\; ; \;\; k > 0 \tag{77}$$
$$\frac{R(t)}{R(t_0)} = [1 - \delta\chi]^{-1} \;\; ; \;\; k = 0$$
$$\frac{R(t)}{R(t_0)} = [\cosh\chi - \delta\sinh\chi]^{-1} \;\; ; \;\; k < 0$$
$$\chi \equiv \sqrt{|k|}x = \sqrt{|k|}c(t_0 - t)$$

and is illustrated in Fig. 5 (as usual for an unrealistically large value of $\delta \simeq 0.6$).

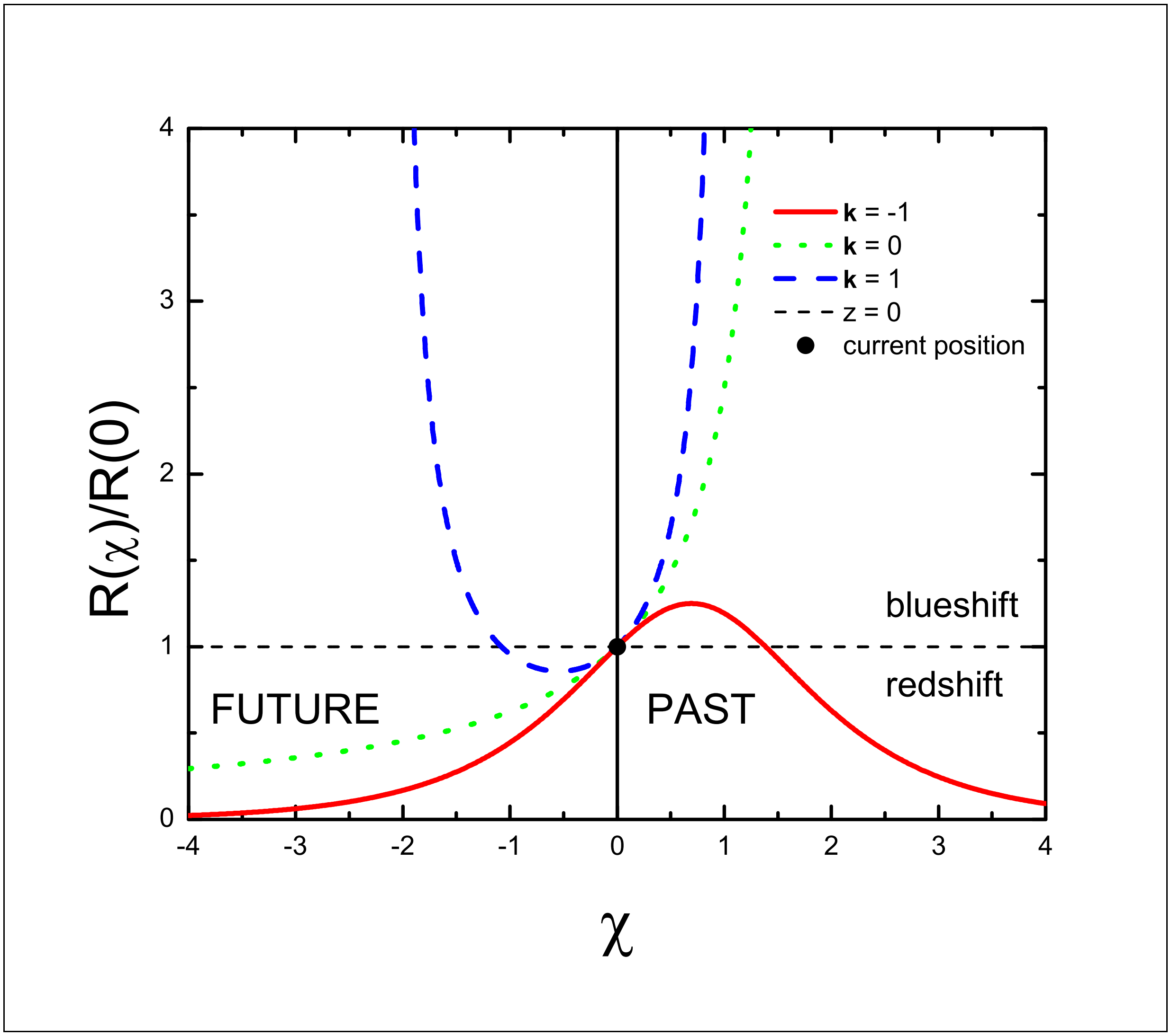

FIG. 5: R functions in Eq. (77) are shown here for different values of $\mathbf{k}$: $\mathbf{k} = -1$ in red (solid), $\mathbf{k} = 0$ in green (dotted), and $\mathbf{k} = +1$ in blue (dashed). We use again $\delta \simeq 0.6$, as in previous illustrations.

We can find the main characteristics of the $\mathbf{k} = -1$ solution (in red-solid) as we have done in the previous cases. We first remark that the variable $\chi \equiv \sqrt{|k|}x = \sqrt{|k|}c(t_0 - t)$ is proportional to the time interval $(t_0 - t)$ in Static Standard Coordinates which, as discussed at the end of Sect. IV B, is the time variable we will use in Cosmology for time measurements, age determinations, etc., so the expressions in Eq. (77) are actually more important than those in the RW time variable that we will discuss later. The χ coordinate used here also corresponds to the comoving coordinate of Eq. (29) as already discussed at the end of Sect.

III A.

As for the solution in $\mathbf{r}$, we note immediately that the red curve in Fig. 5 doesn't show initial or final singularities: the Universe appears to be infinitely old and will never end if measured with our time standard. We find that $\chi_{rs} = \operatorname{arccosh}\left[(1+\delta^2)/(1-\delta^2)\right] = 2\operatorname{arctanh}\delta$ or $(t_0 - t)_{rs} = \frac{2}{\sqrt{|k|}c}\operatorname{arctanh}\delta$, for the look-back time at which redshift starts being observed. The red curve has a maximum at $\chi_{\max} = \operatorname{arctanh}\delta$ or $(t_0 - t)_{\max} = \frac{1}{\sqrt{|k|}c}\operatorname{arctanh}\delta$ (again we find $(R(t)/R(0))_{\max} = 1/\sqrt{1-\delta^2}$ or $z_{\min} = \sqrt{1-\delta^2} - 1$) and it is evidently symmetric around this point of maximum expansion of the Universe.

In fact, it is easy to check that with a translation of the χ coordinate, $\chi = \chi_{\max} + \widetilde{\chi}$, which brings the origin of the new $\widetilde{\chi}$ coordinate to the point of maximum, we have $[\cosh(\chi) - \delta\sinh(\chi)] = \sqrt{1-\delta^2}\cosh(\widetilde{\chi})$ so that:

$$\frac{R(t)}{R(t_0)} = \left[\sqrt{1-\delta^2}\cosh(\widetilde{\chi})\right]^{-1} = \frac{\operatorname{sech}(\widetilde{\chi})}{\sqrt{1-\delta^2}}. \tag{78}$$

The time dependent evolution function becomes therefore a very simple function when described in terms of the $\widetilde{\chi}$ coordinate and we can obtain the δ dependent form of the evolution factor, namely $R(\delta) = R(\delta = 0)\sqrt{1-\delta^2}$ as in Eq. (69), by considering that the maximum for $\chi_{\max} = \operatorname{arctanh}\delta$ corresponds to $\delta = 0$ and using this information inside Eq. (78).

This yields also to the general connection between δ and χ, i.e., how the fundamental cosmological parameter δ changes with our time. Following the same steps which led us to Eq. (73), we obtain for $\mathbf{k} = -1$:

$$\begin{aligned} \chi &= \operatorname{arctanh}\delta(t_0) - \operatorname{arctanh}\delta \\ \delta &= -\tanh[\chi - \operatorname{arctanh}\delta(t_0)]. \end{aligned} \tag{79}$$

To provide a practical example, if we assume $\delta = \delta(t_0) \cong 0.1$, the previous equation is plotted in Fig. 6.

The connection between χ (or the look-back time $t_0 - t$) and z immediately follows from Eqs. (54) or (77), for $\mathbf{k} = -1$:

$$\begin{aligned} z &= [\cosh\chi - \delta\sinh\chi] - 1 \\ \chi &= \operatorname{arcsinh}\frac{\delta(1+z) \pm \sqrt{\delta^2 + z(z+2)}}{(1-\delta^2)} \end{aligned} \tag{80}$$

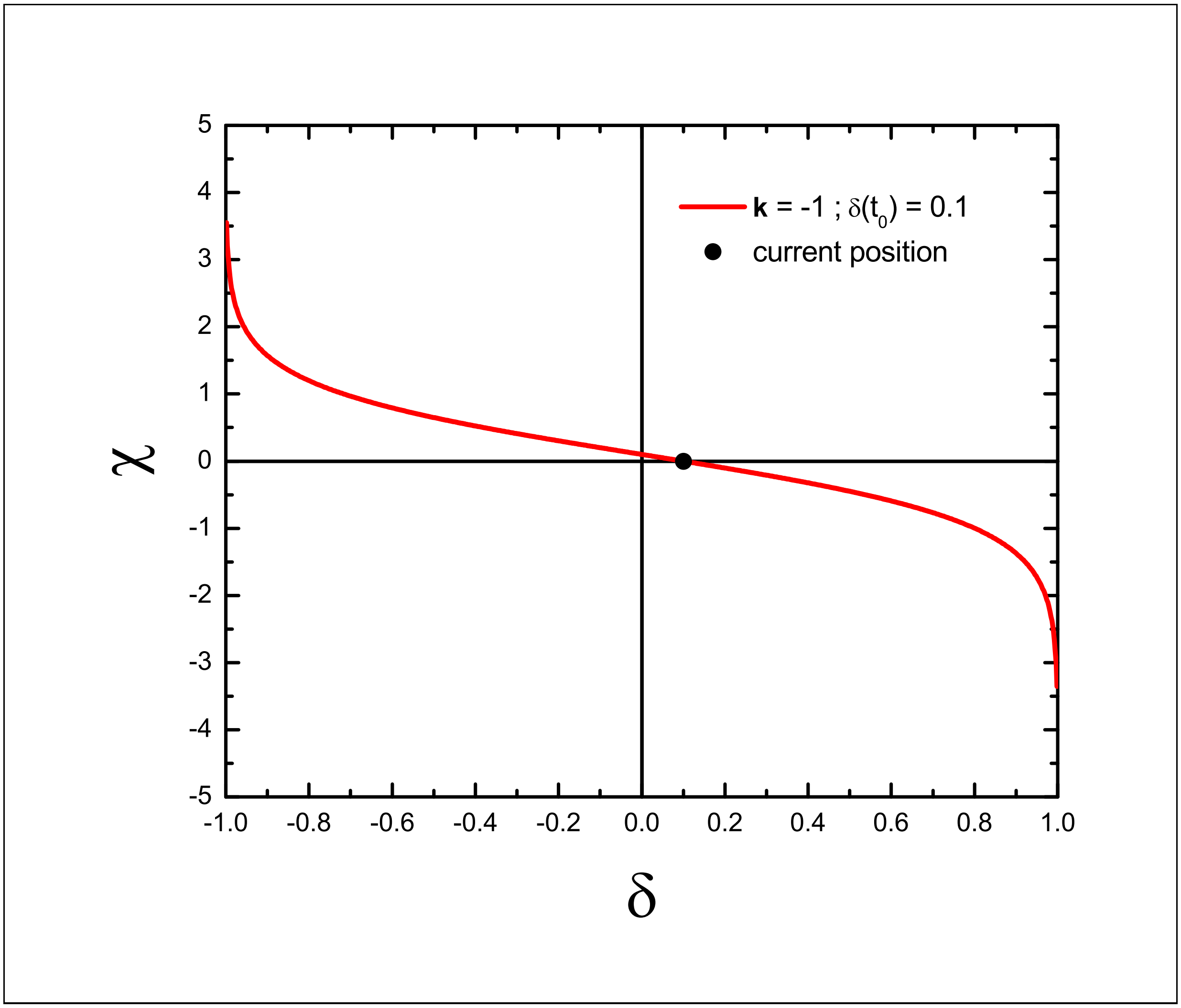

FIG. 6: The connection between χ and δ is illustrated here for a value $\delta(t_0) \cong 0.1$. The correct value of the cosmological time will be determined later [62].

From this expression we can also determine the following relation:

$$\sinh\chi - \delta\cosh\chi = \pm\sqrt{\delta^2 + z(z+2)} = \pm\sqrt{(1+z)^2 - (1-\delta^2)}, \tag{81}$$

which is useful to compute the time derivatives of the cosmic scale factor as a function also of the redshift z:

$$\dot{R}(t) = R(t_0)\sqrt{|k|}c\frac{[\sinh\chi - \delta\cosh\chi]}{[\cosh\chi - \delta\sinh\chi]^2} = \pm R(t_0)\sqrt{|k|}c\frac{\sqrt{(1+z)^2 - (1-\delta^2)}}{(1+z)^2} \tag{82}$$

$$\ddot{R}(t) = -R(t_0)\,|k|\,c^2\frac{[\cosh\chi - \delta\sinh\chi]^2 - 2\,[\sinh\chi - \delta\cosh\chi]^2}{[\cosh\chi - \delta\sinh\chi]^3} =$$

$$= R(t_0)\,|k|\,c^2\frac{(1+z)^2 - 2(1-\delta^2)}{(1+z)^3}.$$

Finally, Eq. (60) is easily expressed as

$$\frac{\mathbf{R(t)}}{\mathbf{R(t_0)}} = \left[\cosh\left(\sqrt{1+\delta^2}\zeta\right) + \frac{\delta}{\sqrt{1+\delta^2}}\sinh\left(\sqrt{1+\delta^2}\zeta\right)\right] \;;\; \mathbf{k} = 1 \tag{83}$$

$$\frac{\mathbf{R(t)}}{\mathbf{R(t_0)}} = e^{\delta\zeta} \;;\; \mathbf{k} = 0$$

$$\frac{\mathbf{R(t)}}{\mathbf{R(t_0)}} = \left[\cos\left(\sqrt{1-\delta^2}\zeta\right) + \frac{\delta}{\sqrt{1-\delta^2}}\sin\left(\sqrt{1-\delta^2}\zeta\right)\right] \;;\; \mathbf{k} = -1$$

$$\zeta \equiv \frac{\mathbf{x}}{\mathbf{R(t_0)}} = \frac{c(\mathbf{t_0} - \mathbf{t})}{\mathbf{R(t_0)}}$$

and is illustrated in Fig. 7 for $\delta \simeq 0.6$. In this case, for the $\mathbf{k} = -1$ solution, the ζ variable is limited within the interval $\frac{1}{\sqrt{1-\delta^2}}\arctan\left(-\frac{\sqrt{1-\delta^2}}{\delta}\right) < \zeta < \frac{1}{\sqrt{1-\delta^2}}\left[\arctan\left(-\frac{\sqrt{1-\delta^2}}{\delta}\right) + \pi\right]$, $\zeta_{\max} = \frac{1}{\sqrt{1-\delta^2}}\arctan\left(\frac{\delta}{\sqrt{1-\delta^2}}\right)$ and $\zeta_{rs} = \frac{2}{\sqrt{1-\delta^2}}\arctan\left(\frac{\delta}{\sqrt{1-\delta^2}}\right)$.

For all the solutions discussed in this section and described by Eqs. (74), (77), (83), it is possible to express them as a function of the "universal time" δ, as we have done for Eq. (67). In all cases the result is obviously always the same, as expressed by Eqs. (69), (70) and shown in Fig. 2. All our expressions of the cosmic scale factor R/R_0 just differ in the coordinates used, but they represent the same cosmological function.

In summary, our description of the past and future evolution of the Universe, through the cosmic scale factor R, can be done by using any of the six kinematical dimensionless variables α, $\mathbf{r}$, χ, ζ, z and δ. They are all related to one another by the transformations outlined above. In Table 1 we have included all the possible connecting formulas, for the $\mathbf{k} = -1$ case, adding those not explicitly analyzed in the current section.

In this Table, as well as in all the equations above, special care is to be given to the meaning of the universal time δ. Whenever we connect together any two of the dimensionless variables α, $\mathbf{r}$, χ, ζ, z, such as in Eqs. (48), (67), (76), (80), these relations are assumed to hold for a given, fixed value of the cosmological time. This value is simply indicated as δ in

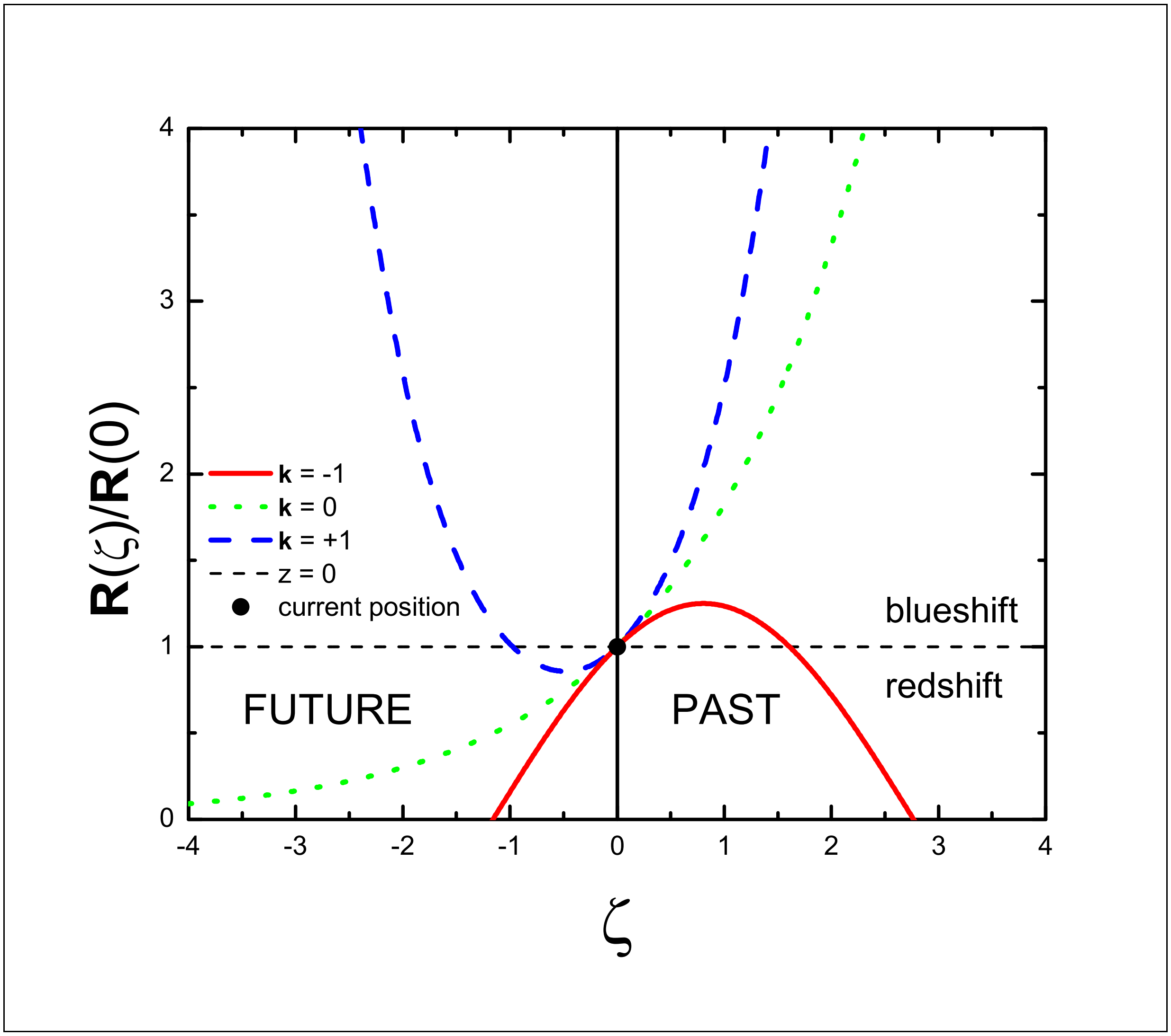

FIG. 7: R functions in Eq. (83) are shown here for different values of $\mathbf{k}$: $\mathbf{k} = -1$ in red (solid), $\mathbf{k} = 0$ in green (dotted), and $\mathbf{k} = +\mathbf{1}$ in blue (dashed). We use again $\delta \simeq 0.6$, as an example.

all these formulas, but typically refers to the current value $\delta = \delta(t_0)$, which will be studied in our next paper [62] (or can be assumed to be in the range $-1 < \delta < 1$).

On the contrary, when we specify a direct connection between one of the dimensionless variables α, $\mathbf{r}$, χ, ζ, z and the cosmological time δ, such as in Eqs. (75) or (79), we describe how this variable is changing together with δ, as seen from an observer at current time $\delta(t_0)$, so that we have to specify both quantities, δ and $\delta(t_0)$, in these expressions.

$\star$	$\boxed{\alpha}$	$\boxed{\mathbf{r}}$	$\boxed{\chi}$
$\boxed{\alpha}$	$\star$	$\alpha = \frac{2\mathbf{r}}{\sqrt{1+\mathbf{r}^2}-\delta\mathbf{r}}$	$\alpha = \frac{2}{\coth\chi-\delta}$
$\boxed{\mathbf{r}}$	$\mathbf{r}= \frac{\mathrm{signum}(\alpha)}{\sqrt{\left(\frac{2}{\alpha}+\delta\right)^2-1}}$	$\star$	$\mathbf{r}= \sinh\chi$
$\boxed{\chi}$	$\chi= \mathrm{arccoth}\left(\frac{2}{\alpha}+\delta\right)$	$\chi= \mathrm{arcsinh}\,\mathbf{r}$	$\star$
$\boxed{\zeta}$	$\zeta = \frac{2}{\sqrt{1-\delta^2}}\,\mathrm{arccot}\left[\frac{\frac{2}{\alpha}+\mathrm{signum}(\alpha)\sqrt{\left(\frac{2}{\alpha}+\delta\right)^2-1}}{\sqrt{1-\delta^2}}\right]$	$\zeta = \frac{2}{\sqrt{1-\delta^2}}\,\mathrm{arccot}\left(\frac{1+\sqrt{1+\mathbf{r}^2}-\delta\mathbf{r}}{\mathbf{r}\sqrt{1-\delta^2}}\right)$	$\zeta = \frac{2}{\sqrt{1-\delta^2}}\,\mathrm{arccot}\left(\frac{\coth\frac{\chi}{2}-\delta}{\sqrt{1-\delta^2}}\right)$
$\boxed{z}$	$z = \frac{1}{\sqrt{1+\delta\alpha+\frac{1}{4}(\delta^2-1)\alpha^2}}-1$	$z = \sqrt{1+\mathbf{r}^2}-\delta\mathbf{r}-1$	$z = \cosh\chi-\delta\sinh\chi-1$
$\boxed{\delta}$	$\delta = \delta(t_0)-\frac{1}{2}\left[1-\delta^2(t_0)\right]\alpha$	$\delta = \frac{\mathbf{r}-\delta(\mathbf{t}_0)\sqrt{1+\mathbf{r}^2}}{\delta(\mathbf{t}_0)\mathbf{r}-\sqrt{1+\mathbf{r}^2}}$	$\delta = -\tanh\left[\chi-\mathrm{arctanh}\,\delta(t_0)\right]$

$\star$	$\boxed{\zeta}$	$\boxed{z}$	$\boxed{\delta}$
$\boxed{\alpha}$	$\alpha = \frac{4\left[\sqrt{1-\delta^2}\cot\left(\sqrt{1-\delta^2}\frac{\zeta}{2}\right)+\delta\right]\sin^2\left(\sqrt{1-\delta^2}\frac{\zeta}{2}\right)}{(1-\delta^2)}$	$\alpha = 2\frac{\delta(1+z)\pm\sqrt{\delta^2+z(z+2)}}{(1-\delta^2)(1+z)}$	$\alpha = 2\frac{\delta(t_0)-\delta}{1-\delta^2(t_0)}$
$\boxed{\mathbf{r}}$	$\mathbf{r}= 2\frac{\left[\sqrt{1-\delta^2}\cot\left(\sqrt{1-\delta^2}\frac{\zeta}{2}\right)+\delta\right]}{\left[\sqrt{1-\delta^2}\cot\left(\sqrt{1-\delta^2}\frac{\zeta}{2}\right)+\delta\right]^2-1}$	$\mathbf{r}= \frac{\delta(1+z)\pm\sqrt{\delta^2+z(z+2)}}{(1-\delta^2)}$	$\mathbf{r}= \frac{\delta(\mathbf{t}_0)-\delta}{\sqrt{1-\delta^2(\mathbf{t}_0)}\sqrt{1-\delta^2}}$
$\boxed{\chi}$	$\chi= 2\,\mathrm{arccoth}\left[\sqrt{1-\delta^2}\cot\left(\sqrt{1-\delta^2}\frac{\zeta}{2}\right)+\delta\right]$	$\chi= \mathrm{arcsinh}\left[\frac{\delta(1+z)\pm\sqrt{\delta^2+z(z+2)}}{(1-\delta^2)}\right]$	$\chi= \mathrm{arctanh}\,\delta(t_0)-\mathrm{arctanh}\,\delta$
$\boxed{\zeta}$	$\star$	$\zeta= \frac{1}{\sqrt{1-\delta^2}}\left\{\pi-\arcsin\left[\sqrt{1-\delta^2}\frac{\delta+\sqrt{\delta^2+z(z+2)}}{(1+z)}\right]\right\}; \quad \frac{\pi}{2} < \zeta\sqrt{1-\delta^2}$ $\zeta= \frac{1}{\sqrt{1-\delta^2}}\arcsin\left[\sqrt{1-\delta^2}\frac{\delta\pm\sqrt{\delta^2+z(z+2)}}{(1+z)}\right]; \quad -\frac{\pi}{2} < \zeta\sqrt{1-\delta^2} < \frac{\pi}{2}$ $\zeta= \frac{1}{\sqrt{1-\delta^2}}\left\{-\pi-\arcsin\left[\sqrt{1-\delta^2}\frac{\delta-\sqrt{\delta^2+z(z+2)}}{(1+z)}\right]\right\}; \quad \zeta\sqrt{1-\delta^2} < -\frac{\pi}{2}$	$\zeta = \frac{\arccos\delta-\arccos\delta(\mathbf{t}_0)}{\sqrt{1-\delta^2(\mathbf{t}_0)}}$
$\boxed{z}$	$z = \frac{1}{\cos\left(\sqrt{1-\delta^2}\zeta\right)+\frac{\delta}{\sqrt{1-\delta^2}}\sin\left(\sqrt{1-\delta^2}\zeta\right)}-1$	$\star$	$z = \sqrt{\frac{1-\delta^2(t_0)}{1-\delta^2}}-1$
$\boxed{\delta}$	$\delta = \cos\left[\arccos\delta(\mathbf{t}_0)+\sqrt{1-\delta^2(\mathbf{t}_0)}\zeta\right]$	$\delta = \pm\frac{\sqrt{\delta^2(t_0)+z(z+2)}}{(1+z)}$	$\star$

TABLE I: Connecting formulas between the six kinematical dimensionless variables, for the k = -1 case, including those not explicitly analyzed in the current section.

E. The age of the Universe and the horizon problem

Two important issues which need to be addressed by any cosmological theory are the age of the Universe and the existence of particle or event "horizons," which might limit our "view" of light signals and events from the past. We can show that these two topics do not cause any problem in our conformal cosmology.

These issues were originally addressed by Mannheim ([44], [50], [52], [91]) by showing that the Conformal Gravity theory does not possess any horizon or flatness problem, and does not contradict current estimates of the age of the Universe. For completeness, we present in this section a similar analysis, based on our particular solutions for the cosmic scale factor.

The age of the Universe is determined experimentally from various observations, such as the age of chemical elements, which leads to age determinations of terrestrial rocks and meteorites thus determining the age of the solar system. Other sources of information include the age of the oldest star clusters, white dwarfs, etc. Without entering into the details of these determinations, they all give estimates or set lower limits for the age or the Universe of about $10\ Gyr \lesssim t_0 \lesssim 15\ Gyr$, assuming that our current time t_0 is measured from an initial singularity. These estimates essentially agree with the scientific consensus based on standard cosmology, which evaluates this age to be about 13.7 billion years [8].

In Sect. IV B we have already remarked that the local time variable t, which is used for experimental age determinations, cannot be identified with the cosmic standard time $\mathbf{t}$, since the former is essentially the *conformal time* of the latter, in the sense of Eqs. (31) and (32). As described graphically in Fig. 5 and Fig. 7, the cosmic evolution of the scale factor shows initial and final singularities when using the cosmic time $\mathbf{t}$, similar to the case of the evolution described in terms of the universal parameter δ, but does not show any initial or final singularity when using the other time variable t. This is due to the stretching of our local time coordinate t, done by the time conformal transformation just mentioned, so that the age of the Universe appears to be infinite in this temporal coordinate. Therefore current experimental determinations, also done using time t, will never contradict this infinitely lasting Universe.

On the contrary, the Universe appears to be limited temporally if a "cosmic" time $\mathbf{t}$ or δ is used. We have already described the limits of these two variables; the delta parameter varies in the range $-1 < \delta < 1$, while the limits for the $\mathbf{t}$ variable are more easily expressed by the corresponding limits for $\zeta = c(\mathbf{t}_0 - \mathbf{t})/\mathbf{R}(\mathbf{t}_0)$, resulting in $\frac{1}{\sqrt{1-\delta^2}} \arctan\left(-\frac{\sqrt{1-\delta^2}}{\delta}\right) < \zeta < \frac{1}{\sqrt{1-\delta^2}} \left[\arctan\left(-\frac{\sqrt{1-\delta^2}}{\delta}\right) + \pi\right]$ (in this last expression $\delta = \delta(t_0)$ is our current value of this parameter). We will estimate later [62] the values of all our cosmological parameters, but we can anticipate that the experimental range of the age of the Universe, $t_0 \approx 10\ Gyr - 15\ Gyr$ mentioned above, will translate in a value of $\delta \cong -1$, obtained using Eq. (79) after transforming the "age" t_0 into a corresponding value χ_0. Therefore, the experimental observations seem to point to cosmological times very close to the initial singularity, although they cannot be used directly to measure the age of the Universe.

Another important topic is the analysis of possible horizons which might limit our perception of past events and light signals. The so-called horizon problem was one of the main reasons why an inflationary phase of the Universe was proposed in the standard model. A particle horizon is usually referred to the comoving radius χ_H:

$$\chi_H = S_{\mathbf{k}}^{-1}(\mathbf{r}_H) = \int_{\mathbf{0}}^{\mathbf{r}_H} \frac{d\mathbf{r}}{\sqrt{1 - \mathbf{k}\mathbf{r}^2}} = c \int_{\mathbf{t}_H}^{\mathbf{t}_0} \frac{d\mathbf{t}}{\mathbf{R}(\mathbf{t})}, \tag{84}$$

assuming the integral in cosmic time $\mathbf{t}$ converges for $\mathbf{t}_H \rightarrow \mathbf{0}$ in models with an initial singularity, or for $\mathbf{t}_H \rightarrow -\infty$ in models without initial singularity, thus yielding a finite value for χ_H. On the contrary, if the integral on the right-hand side of Eq. (84) is diverging for the same limits for the variable $\mathbf{t}_H$, the horizon problem disappears altogether.

In our model, this integral diverges to $+\infty$ when $\mathbf{t}_H$ approaches its lower bound, given by the corresponding (upper) limit for the ζ variable mentioned above. This can be checked directly by computing the integral, or simply by recalling that $\chi_H = c\sqrt{|k|}(t_0 - t_H)$ in standard time t, following Eq. (64), and that this variable is not bounded in the past or the future, thus for $t_H \rightarrow -\infty$ we immediately get $\chi_H \rightarrow \infty$ and no particle horizon is present. In this way all regions in the Universe can be causally connected by light signals, including the epoch when the Cosmic Microwave Background (CMB) was generated. We do not need to invoke inflationary phases to justify the highly homogeneous nature of the CMB.

For the same reason, no event horizons appear in our formulation. These are associated with a similar integral:

$$\chi_{EH} = S_{\mathbf{k}}^{-1}(\mathbf{r}_{EH}) = \int_{\mathbf{0}}^{\mathbf{r}_{EH}} \frac{d\mathbf{r}}{\sqrt{1 - \mathbf{k}\mathbf{r}^2}} = c \int_{\mathbf{t}}^{\mathbf{t}_{MAX}} \frac{d\mathbf{t}'}{\mathbf{R}(\mathbf{t}')} \tag{85}$$

for an event which occurred at $(\mathbf{r}_{EH}, \mathbf{t})$ to be detected at a later time through light signals, but before time $\mathbf{t}_{MAX}$ which can be be either infinity or the time of a full contraction of the Universe. Again, the last integral simply gives $\chi_{EH} = c\sqrt{|k|}(t_{MAX} - t)$ in the SSC coordinate, so that $\chi_{EH} \to \infty$ for $t_{MAX} \to +\infty$, and we can therefore receive information from any event in the past if we wait a long enough time interval.

It is beyond the scope of this paper to investigate also the other reasons which led to postulate the inflationary scenario, such as the flatness problem, the fine tuning of parameters and others, but some of these issues do not seem in any case to be significant in our kinematical cosmology, where most of the physical parameters are varying with the cosmic time δ and therefore do not require any particular explanation for the values they currently hold. The cosmic time δ, or any similar parameter, seems to be driving the evolution of the Universe from its initial to its final value and most of the other physical quantities just follow this evolution.

V. CONNECTION WITH KINEMATIC COSMOLOGY

We have already remarked in Sect. III B that a *kinematic cosmology* was introduced by L. Infeld and A. Schild (I-S in the following) in 1945 ([68], [69], [70], [71]). These physicists were focusing their attention on "that part of relativistic cosmology which deals with the metric form of our Universe, characterized by a four-dimensional space-time manifold, and with the motion of free particles and light rays," [70] thus dealing with the kinematical description of cosmology while ignoring its dynamical aspect.

By assuming three fundamental postulates, the first one on light-geometry (namely the existence of a cosmological coordinate system - CCS, conformal to flat Minkowski space), plus the usual postulates of isotropy and homogeneity of the Universe, Infeld and Schild introduced the following metric:[15]

$$ds^2 = \gamma(\mathfrak{t}, \mathfrak{r}) \left(-c^2 d\mathfrak{t}^2 + d\mathfrak{r}^2 + \mathfrak{r}^2 d\psi^2\right), \tag{86}$$

where $\gamma(\mathfrak{t}, \mathfrak{r})$ is a dimensionless conformal factor. Conformally Flat Space-time (CFS) is the modern term used today to denote such metrics, which were also studied in more recent

[15] We will use a different type of characters ($\mathfrak{t}$,$\mathfrak{r}$) to distinguish this new set of coordinates (CCS or CFS type) from all the other coordinates used in this paper. The angular part of the metric will remain unaffected by all the transformations in this section.

works ([72], [73], [74], [92], [93]). In addition, Infeld and Schild were able to restrict the possible $\gamma(\mathfrak{t},\mathfrak{r})$ functions satisfying the three postulates, to just three fundamental classes corresponding to standard closed, open and flat universes (in the notation of Ref. [70], case I - $K > 0$, case II - $K < 0$ and case III - $K = 0$, respectively).

Therefore, it is important to check our cosmological solutions against the general classes proposed by Infeld and Schild, to make sure that they have the general form required by the three fundamental postulates mentioned above. A similar check should be performed for any other cosmological solution obtained using other conformally-invariant theories in the literature.

We will consider here only case II - $K < 0$, since this will correspond to our preferred cosmological solution, for $\mathbf{k} = -1$, presented in Sect. IV. For this particular case the conformal factor γ must be of the form [70]:

$$\gamma(\mathfrak{t},\mathfrak{r}) = f\left[\frac{c\mathfrak{t}/2\alpha}{1+\left(c^2\mathfrak{t}^2-\mathfrak{r}^2\right)/4\alpha^2}\right]\left[1+\frac{\left(c^2\mathfrak{t}^2-\mathfrak{r}^2\right)}{4\alpha^2}\right]^{-2}, \tag{87}$$

where 2α is a convenient *natural cosmological unit* (with dimension of length) introduced by Infeld and Schild, so that the resulting γ factor will be dimensionless, for any choice of the arbitrary function f ($K = \pm 1/\alpha^2$ in cases I and II of Ref. [70]).

The importance of the work by Infeld and Schild is also due to their original derivation of the finite coordinate transformations from Conformally Flat Space-time (CFS) to standard Robertson-Walker (RW) metric as described in our Eq. (28) or (29), thus proving that RW space-time is conformally flat, i.e., equivalent to flat Minkowski space-time up to a conformal factor represented by $\gamma(\mathfrak{t},\mathfrak{r})$ in Eq. (86). We will review this important coordinate transformation in the following section.

A. From Conformally Flat Space-time to the Robertson-Walker Metric

The full transformation from CFS to RW can be found in the original 1945 paper by Infeld-Schild [70], as well as in other references ([72], [74], [94]). Again, we will consider in the following this transformation just for the particular $\mathbf{k} = -1$ case, which will be compared to our cosmological solutions presented in Sect. IV. We start from the CFS metric in Eq. (86) and, following Ref. [70], we apply a first transformation simply to introduce dimensionless coordinates $\widetilde{t}, \widetilde{r}$:

$$\begin{aligned}\widetilde{t} &= \frac{c\mathfrak{t}}{2\alpha} \\ \widetilde{r} &= \frac{\mathfrak{r}}{2\alpha}\end{aligned} \tag{88}$$

using the *natural cosmological unit* 2α. The metric then becomes

$$ds^2 = \widetilde{\gamma}(\widetilde{t}, \widetilde{r})\left(-d\widetilde{t}^2 + d\widetilde{r}^2 + \widetilde{r}^2 d\psi^2\right), \tag{89}$$

where $\widetilde{\gamma}(\widetilde{t}, \widetilde{r}) = 4\alpha^2\gamma(\mathfrak{t}, \mathfrak{r})$ has now the dimension of a length squared, while the coordinates are all dimensionless. In this new units Eq. (87) is rewritten as

$$\widetilde{\gamma}(\widetilde{t}, \widetilde{r}) = \widetilde{f}\left[\frac{\widetilde{t}}{1 + \left(\widetilde{t}^2 - \widetilde{r}^2\right)}\right]\left[1 + \left(\widetilde{t}^2 - \widetilde{r}^2\right)\right]^{-2} \tag{90}$$

where $\widetilde{f} = 4\alpha^2 f$ also acquires dimension of a squared length. This is followed by a second transformation to another set of dimensionless quantities:

$$\begin{aligned}X &= \widetilde{t} + \widetilde{r} \\ Y &= \widetilde{t} - \widetilde{r}\end{aligned} \tag{91}$$

which transforms the metric as follows,

$$\begin{aligned}ds^2 &= \widetilde{\gamma}(X, Y)\left[-dXdY + \frac{1}{4}(X - Y)^2 d\psi^2\right] \\ \widetilde{\gamma}(X, Y) &= \widetilde{f}\left[\frac{X + Y}{2\,(1 + XY)}\right](1 + XY)^{-2}.\end{aligned} \tag{92}$$

Two additional transformations are required to obtain the RW metric. The next step is:

$$\begin{aligned}u &= \tanh^{-1} X \\ v &= \tanh^{-1} Y\end{aligned} \tag{93}$$

which yields a new form of the metric,

$$ds^2 = \frac{1}{4}\widetilde{f}\left[\frac{1}{2}\tanh(u + v)\right]\operatorname{sech}^2(u + v)\left[-4dudv + \sinh^2(u - v)d\psi^2\right], \tag{94}$$

where $\operatorname{sech}(x) = 1/\cosh(x)$, is the hyperbolic secant. The last transformation is:

$$\begin{aligned}\eta &= u + v \\ \chi &= u - v\end{aligned} \tag{95}$$

which finally takes us to a RW metric, expressed in terms of the conformal time η and the comoving coordinate χ, introduced in Eqs. (30) - (32):

$$\begin{aligned}ds^2 &= \mathbf{R}^2(\eta)\left(-d\eta^2 + d\chi^2 + \sinh^2\chi d\psi^2\right) \\ \mathbf{R}^2(\eta) &= \frac{1}{4}\widetilde{f}\left(\frac{1}{2}\tanh\eta\right)\operatorname{sech}^2\eta.\end{aligned} \tag{96}$$

The previous equation also restricts possible functions for the cosmic scale factor $\mathbf{R}^2(\eta)$, to be of the form specified above, just leaving the function $\widetilde{f}$ totally arbitrary. One further step can bring the metric of Eq. (96) into the standard Robertson-Walker metric of Eq. (28) expressed in terms of our variables $\mathbf{t}$ and $\mathbf{r}$. We just need to apply the inverse of Eqs. (30) and (31), namely:

$$\begin{aligned}\mathbf{t} &= \frac{1}{c}\int \mathbf{R}(\eta)d\eta \\ \mathbf{r} &= \sinh\chi\end{aligned} \tag{97}$$

and the previous metric will become the standard RW expression for the $\mathbf{k} = -1$ case,

$$ds^2 = -c^2 d\mathbf{t}^2 + \mathbf{R}^2(\mathbf{t})\left[\frac{d\mathbf{r}^2}{1+\mathbf{r}^2} + \mathbf{r}^2 d\psi^2\right]. \tag{98}$$

B. Comparison with our cosmological solution

In order to compare the Infeld-Schild version of kinematical cosmology outlined in the previous section with our model, it is easier to use cosmological equations expressed in terms of the variables η and χ. We can compare the I-S expression of the cosmic scale factor $\mathbf{R}(\eta)$ given in Eq. (96) with our expression, for the $k < 1$ case, from Eq. (77). This equation can be written also as

$$\mathbf{R}(\eta) = \frac{\mathbf{R}(\chi = 0)}{\cosh\chi - \delta\sinh\chi} = \frac{\mathbf{R}(\eta_0)}{a\cosh\eta + b\sinh\eta} = \frac{\mathbf{R}(\eta_0)}{a + 2b\frac{\tanh\eta}{2}}\operatorname{sech}\eta \tag{99}$$
$$a = \cosh\eta_0 - \delta\sinh\eta_0 \ ; \ b = \delta\cosh\eta_0 - \sinh\eta_0$$

where we used $\mathbf{R}$ instead of R, dividing both sides of Eq. (77) by $\sqrt{|k|}$ and substituted the comoving coordinate χ with the conformal time η, namely $\chi = \eta_0 - \eta$, in view of Eq. (64). In this way we notice that our expression in the last equation is precisely of the type expected by the I-S models, as in Eq. (96), and we can uniquely determine the function $\widetilde{f}$ as follows,

$$\widetilde{f}\,(x) = \frac{4\mathbf{R}^2(\eta_0)}{(a + 2bx)^2}. \tag{100}$$

The quantities a and b are defined in Eq. (99) as a function of the current conformal time η_0, depending on the arbitrary choice of the zero for this variable. A better choice would be to measure both χ and η variables from the point of maximum expansion of the Universe. This was done explicitly in Sect. IV D and led to Eq. (78), which can be easily rewritten in terms of a new variable $\widetilde{\eta}$, also measured from the position of maximum expansion and defined as $\widetilde{\eta} = -\widetilde{\chi} = \chi_{\max} - \chi = \operatorname{arctanh}\delta - \chi$. With this new choice of variable, Eq. (78) would simply become $\mathbf{R}(\widetilde{\eta}) = \frac{\mathbf{R}(\eta_0)}{\sqrt{1-\delta^2}}\operatorname{sech}\widetilde{\eta}$. This expression is consistent with our general form in Eq. (99) for the particular case of $\eta_0 = \operatorname{arctanh}\delta$, which leads to $a = \sqrt{1-\delta^2}$ and $b = 0$. In this particular case, the function $\widetilde{f}$ in Eq. (100) would become simply a constant,

$$\widetilde{f}\,(x) = \frac{4\mathbf{R}^2(\eta_0)}{1 - \delta^2}. \tag{101}$$

These results will also lead to a unique expression for the conformal factors $\widetilde{\gamma}$ or γ, from Eqs. (90) and (87) respectively:

$$\begin{aligned} \widetilde{\gamma}(\widetilde{t},\widetilde{r}) &= 4\mathbf{R}^2(\widetilde{t}_0)\left\{a\left[1 + \left(\widetilde{t}^2 - \widetilde{r}^2\right)\right] + 2b\widetilde{t}\right\}^{-2} \\ \gamma(\mathfrak{t},\mathfrak{r}) &= \frac{\widetilde{\gamma}(\widetilde{t},\widetilde{r})}{4\alpha^2} = \frac{\mathbf{R}^2(\mathfrak{t}_0)}{\alpha^2}\left\{a\left[1 + \frac{(c^2\mathfrak{t}^2 - \mathfrak{r}^2)}{4\alpha^2}\right] + b\frac{c\mathfrak{t}}{\alpha}\right\}^{-2}, \end{aligned} \tag{102}$$

so that the original I-S metric of Eq. (86) is now completely defined, except for the dimensionless ratio $\mathbf{R}^2(t_0)/\alpha^2$. The previous expressions can be further simplified, by measuring our space-time variables from the point of maximum expansion of the Universe, so that $a = \sqrt{1-\delta^2}$ and $b = 0$, as discussed above.

It is beyond the scope of this work to analyze in more details the implications of the Infeld-Schild kinematical cosmology, once their conformal factor is fixed in the form of our Eq. (102). The objective of this section was simply to show that our model is fully consistent with the original I-S kinematical cosmology, as shown in the previous discussion.

We remark here that the CFS metric is just another way to describe our Universe and that the I-S coordinates $(\mathfrak{t}, \mathfrak{r})$ are different from the RW coordinates $(\mathbf{t}, \mathbf{r})$, or our original SSC coordinates (t, r), although they are all connected by the set of transformations outlined in Sect. III A and Sect. V A.

The original goal of kinematical cosmology, as introduced by Infeld and Schild, was to describe the Universe through the motion of its *fundamental particles* (the galaxies, or nebulae as they were called in 1945), rather than have fixed, comoving galaxies in RW metric and describe the evolution of the Universe through the cosmic scale factor $\mathbf{R}$. In other words, Infeld and Schild traded the advantage of the RW description, i.e., to have the fundamental particles at rest (comoving), for a metric conformal to Minkowski flat space-time, thus equivalent to the space of special relativity, where the speed of light is simply constant, light propagates as in flat space, and standard physical laws, such as Maxwell's equations, can be extended without modifications from Minkowski space to CFS.

This different way of describing the Universe implies a motion of the fundamental particles, as observed in CFS. This motion was also studied in detail by I-S. In particular, our cosmological solution presented in Eq. (102) belongs to the I-S Case II ([70]), which is called a "converging-diverging" model, since the fundamental particles move radially in a way that brings them together towards the space-time origin and then away from it. Again, we will leave to future studies a more complete analysis of our cosmological solution, in view of the Infeld-Schild theory.

VI. CONCLUSIONS

We presented in this work the mathematical foundations of a new kinematical approach to conformal cosmology. This was based on the assumption that the observed redshift is mainly due to a gravitational origin and that the gravitational potential of conformal gravity might be the cause of the observed "expansion" of the Universe.

We have seen how the original Mannheim-Kazanas potential can support this explanation and how the chain of transformations, from Static Standard Coordinates to the Robertson-Walker metric, can lead to a unique expression of the cosmic scale factor. In particular, the $\mathbf{k} = -1$ solution is the only capable of producing the observed galactic redshift and describes the evolution of the Universe in terms of an initial expansion phase (which can still be traced back to an initial singularity) up to a point of maximum expansion, then followed by a symmetrical contraction phase, towards a final singularity.

We have compared our solutions with those of current conformal cosmologies and noted that our kinematic cosmology is recovered from conformal gravity with a cosmological constant at late (or early) cosmological times. This implies that during these cosmological epochs the cosmological and gravitational redshifts might be in fact equivalent, as assumed in our kinematical approach to conformal cosmology.

Our detailed analysis of the solutions has also shown the importance of using dimensionless quantities and, in particular, of the cosmological variable $\delta \equiv \gamma/2\sqrt{|k|}$ which effectively combines together the two parameters γ and k (or κ) originally introduced by Mannheim and Kazanas. We also introduced the hypothesis that δ might actually represent a true cosmic time, in terms of which the evolution of the Universe would be described by the very simple Eq. (69). If this interpretation is correct, all the space-time dimensionless variables α, r, χ, ζ, z, δ are linked together and also related to the current value $\delta(t_0)$ of the cosmic time, through the transformations outlined in Table I.

The current value $\delta(t_0)$ should be small and positive, as indicated by Mannheim's estimate of the parameter γ_0, but its actual value has to be determined by more precise fitting of astrophysical data, such as the luminosities of type Ia Supernovae or others. This will be the objective of a second part of this project [62], where we will also try to explain the anomalous local blueshift region, which is implied by our cosmological solution.

If our model will be successful in explaining current experimental data, *kinematical conformal cosmology* might become a viable alternative model for the description of the Universe, with the advantage of avoiding altogether the introduction of dark matter and dark energy, or other controversial features of the current standard model.

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