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How Spherical Is a Cube (Gravitationally)?

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An important concept that is presented in the discussion of Newton's law of universal gravitation is that the gravitational effect external to a spherically symmetric mass distribution is the same as if all of the mass of the distribution were concentrated at the center.[1,2] By integrating over ring elements of a spherical shell, we show that the gravitational force on a point mass outside the shell is the same as that of a particle with the same mass as the shell at its center. This derivation works for objects with spherical symmetry while depending on the fact that the gravitational force between two point masses varies inversely as the square of their separation.[3] If these conditions are not met, then the problem becomes more difficult. In this paper, we remove the condition of spherical symmetry and examine the gravitational force between two uniform cubes.

There have been a number of notable studies on the gravity of a cube. For example, I. R. Mufti developed approximation formulas for rapid evaluation of a cube's gravitational field[4] and, in a subsequent work, calculated the gravitational field of a body of arbitrary shape by using cubes as building blocks within the body.[5] In their works on the stability of orbits around irregular-shaped celestial bodies, Liu et al. first investigated the dynamics of a particle orbiting a fixed homogeneous cube,[6] and then orbiting a rotating homogeneous cube.[7] Finally, Chappell et al. calculated the Newtonian gravitational potential and field of a cubic homogeneous asteroid and applied it to the orbit of possible satellites.[8] The methods used in these studies have a somewhat higher degree of mathematical sophistication than our approach, which is more suitable for discussion in an intermediate-level course on mechanics. We introduce the student to the use of basic computational physics in the study of universal gravitation. In addition, our results can be used in a discussion comparing orbital dynamics of a satellite around a spherical body and a nonspherical body.

Basic calculations of orbital parameters can easily be made to an excellent approximation by modeling essentially spherical celestial objects as point masses at their centers. In situations involving objects with irregular shapes where a high degree of precision is required, more advanced techniques must be used, similar to those discussed in this paper. In 2001 the *NEAR Shoemaker* spacecraft made a historic landing on Eros, an asteroid about the size of Manhattan, slowing to less than 2 m/s before coming to rest on the surface.[9] In order to achieve a soft landing, it was necessary to develop a complex model of the gravitation of the asteroid based on its irregular shape and mass.

We will begin by determining numerically the gravitational force between unit cubes that are in contact on one face. This calculation is then repeated for cubes with increasing separation. Our results are then used to address the following questions:

(a) What is the difference in the gravitational force between two adjoining cubes and two adjoining spheres of the same mass and density as the cubes?

(b) What is the dependence of the gravitational force between the cubes on the separation between their centers?

(c) For a given separation, what is the offset between the center of a cube and the center of a sphere of the same mass and density that provides the same gravitational effect? When this offset becomes negligibly small, the cube "evolves" into a sphere since it may now be replaced by a point mass at its center for calculations of the gravitational force.

Analysis

For simplicity, we will use $G = 1$ in our calculations of the gravitational force $F = GM\,m/r^2$. The values of force in our calculations are therefore normalized by 6.67×10^{-11}. Mass, length, and force are all assumed to be in SI units. In order to calculate the force between the two unit cubes at various distances, place the first cube with x, y, and z all from 0 to 1. The second cube will have $d \le x \le d + 1$, with y and z from 0 to 1, as shown in Fig. 1. So d is the distance between the centers of the two cubes.

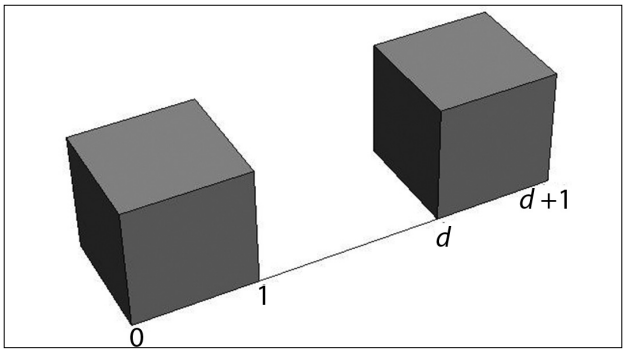

Fig. 1. Positions of the two unit cubes for which the gravitational force is calculated.

Each cube is assumed to have uniform density and a mass of 1. Let (x_1, y_1, z_1) be a point in the first cube and (x_2, y_2, z_2) be a point in the second. For these two points the contribution to the total force would be

$$\frac{1}{r^2} = \frac{1}{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

in the direction of the line connecting the two points.

From symmetry, the overall force in the y and z directions will be zero, so we only need to determine the x-component of the force. That will be $(x_2 - x_1)/r \times 1/r^2$. Integrating over all the possible values for the six x-y-z variables gives the total force between the cubes:[8]

$$F = \int_0^1 \int_0^1 \int_d^{d+1} \int_0^1 \int_0^1 \int_0^1 \frac{x_2 - x_1}{\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{3/2}} \, dx_1 \, dy_1 \, dz_1 \, dx_2 \, dy_2 \, dz_2 \, .$$

This is messy enough that we will use an approximate integration method from numerical analysis. For a classroom exercise, a standard method like Mathematica's NIntegrate function can be used. It is a hard problem to do numerically when $d = 1$ and the two cubes are in contact on one face, since the function we are integrating has singularities.[10] The integral is still finite when $d = 1$, but the numerical methods have much slower convergence. To check that Mathematica's NIntegrate function would give at least the three or four significant digit accuracy needed here, we compared the results to those obtained from a Fortran program using composite Gauss quadrature.

More realistic examples of computing gravitational forces due to bodies with shapes that are not close to spherical come from space missions to asteroids and comets. For example, the asteroid Eros has an irregular shape and is roughly $33 \times 13 \times 13$ km.

The *NEAR Shoemaker* spacecraft was first put into a 321×366 km elliptical orbit, where the gravitational field could be approximated by replacing the asteroid by a point mass. Later, when the spacecraft was moved closer and finally landed on the asteroid, more accurate gravitational forces were needed. From the initial mapping of the shape of the asteroid, the force could be done as a three-dimensional integral similar to the one above, except the spacecraft could be considered as a point mass. Ultimately, the asteroid's gravity field was modeled as a spherical or ellipsoidal harmonic expansion,[11] which is beyond the scope of this project.

When $d = 1$, the cubes are in contact and the force between them is found to be 0.92568. For comparison, we consider the force between two touching spheres each of mass 1 and the same density as the cubes. The volume of each sphere is the same as that of the cube, so the diameter of the sphere is $(6/\pi)^{1/3}$, which is also the separation of their centers. Applying Newton's law of universal gravitation to the two point masses at the centers of the spheres, we find that the force between the spheres is 0.64963. As expected this is smaller than the force between the cubes, where the number of points in close proximity is greater.

We wish to compute the gravitational force between the cubes for various values of $d \geq 1$ to see how it differs from the force between spheres with mass 1. We expect that as the cubes get farther apart, the force between them will approach $1/d^2$, so the cubes, like spheres, can be replaced by point masses at their centers when calculating the gravitational force. Figure 2 shows a comparison of the calculated gravitational force between our two unit cubes with the inverse-square force between the equivalent point masses at their centers for $1.0 \leq d \leq 1.5$. The calculated force is smaller than the inverse-square force, with the difference ranging from −7.40% when the cubes are in contact to −2.32% at $d = 1.5$.

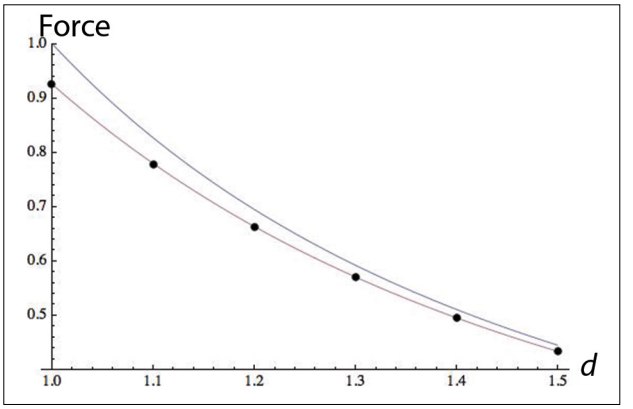

Fig. 2. Comparison of the calculated gravitational force between two unit cubes and the inverse-square force between their centers.

We now consider another way to look at how well these cubes can be approximated by spheres from a gravitational standpoint. This is done by examining the offset between the center of the cube and the center of a sphere with the same mass and uniform density. When $d = 1$, the force would be the same if we replaced the cubes with centers at $x = 0.5$ and $x = 1.5$ by two spheres of mass 1 whose centers were $1/\sqrt{0.92598} = 1.0392$ units apart. Centering the spheres would put their centers $(1.0392 - 1)/2 = 0.0196$ offset from the respective cube centers. These offsets will tend to zero as d increases.

Computing a few offsets for d near 1 gives (1.0, 0.019600), (1.1, 0.016999), (1.2, 0.014478), (1.3, 0.012272), (1.4, 0.010401), (1.5, 0.008836). Even for cubes with zero separation ($d = 1$), the offset is less than 2% of the side length, decreasing to less than 1% at a separation of half a side length ($d = 1.5$). Next, we can try to estimate the rate at which the offsets go to zero as $d \to \infty$. Examining larger values of d can make the pattern easier to see. To do so, we compute the forces and offsets as the distance between centers of the cubes successively doubles. Along with the offset values, the ratio of one offset divided by the next offset gives information about the rate at which the offsets are approaching zero.

Table I shows that in the limit as d increases, the offset is

Table I. Rate of decrease of the offset.

d	offset	ratio
1	1.9600e-2	4.6888
2	4.1801e-3	7.4306
4	5.6256e-4	7.9197
8	7.1032e-5	7.9849
16	8.8959e-6	7.9965
32	1.1125e-6	7.9996
64	1.3907e-7	

divided by 8 when d is doubled. This suggests using $f(d) = a/d^3$ as the dominant term of a function to model the offsets, since then $f(2d) = a/(2d)^3 = f(d)/8$.

The ratio between 1 and 2 is not close to 8 because of the influence of the singularity when $d = 1$. We can add a second term to the model function to try to improve the agreement with the offsets near $d = 1$. It should go to zero faster than $1/d^3$, so that the asymptotic behavior as $d \to \infty$ is still $1/d^3$.

Doing some curve fitting and trying different powers of d for the second term gives b/d^6 as a likely second term. Fitting $f_1(d) = a/d^3 + b/d^6$ using the offset data from $d = 1.0, 1.1, \ldots, 1.5$ gives

$$f_1(d) = \frac{0.0332}{d^3} - \frac{0.0137}{d^6}.$$

Then a similar fit using the offset data from $d = 1, 2, \ldots, 64$ gives

$$f_2(d) = \frac{0.0354}{d^3} - \frac{0.0158}{d^6}.$$

The fact that the coefficients agree to within a few thousandths for the two very different ranges of d gives us some confidence that we can model the offsets fairly well. Our results are shown in Fig. 3.

By the time the centers of the cubes are 4 units apart, the offsets of the equivalent point masses (or sphere centers) from the centers are less than 0.0006 units away, so the cubes are essentially spherical from the standpoint of gravity.

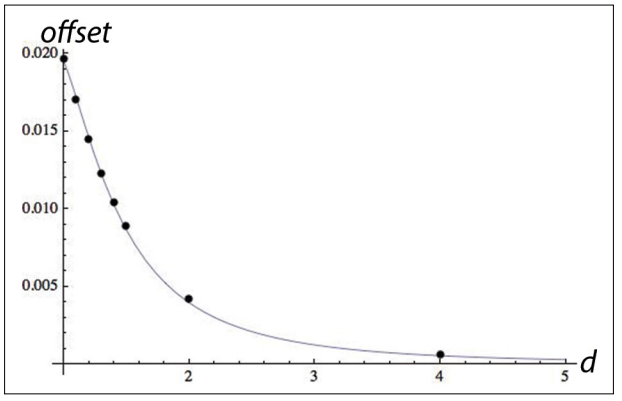

Fig. 3. Fitted curve to the decrease in the offset of the equivalent point mass from the cube center with increasing separation.

Conclusion

In this paper, we investigated the gravitational force between two identical uniform cubes of unit side length. To see how well cubes can be approximated by spheres from a gravitational standpoint, we examined the offset between the center of the cube and the center of a sphere with the same mass and uniform density that would provide the same gravitational effect. Even for cubes with zero separation, the offset was found to be less than 2% of the side length, decreasing to less than 1% at a separation of half a side length. In addition, we modeled the dependence of the offset as a function of the separation between the centers of the cubes. By the time the centers of the cubes are 4 units apart, the offsets of the equivalent point masses (or sphere centers) from the cube centers are less than 0.0006 units away, so the cubes are essentially spherical from the standpoint of gravity.

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