1-1-2006

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International Journal of Pure and Applied Mathematics
Volume 32 No. 1 2006, 117-138

LINKED EXACT TRIPLES OF TRIANGULATED CATEGORIES AND A CALCULUS OF t-STRUCTURES

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Abstract: We introduce a new formalism of exact triples of triangulated categories arranged in certain types of diagrams. We prove that these arrangements are well-behaved relative to the process of gluing and ungluing t-structures defined on the indicated categories and we connect our constructs to a problem (from number theory) involving derived categories. We also briefly address a possible connection with a result of R. Thomason.

AMS Subject Classification: 18E30
Key Words: exact triples of triangulated categories, t-structures

1. Introduction

We examine arrangements of exact triples of triangulated categories in the setting of what might be called an initial assignment problem, adapting the phrase "initial value problem" from the theory of differential equations. The idea, worked out in Section 3, below, is to develop a yoga of gluing (and, so to speak, ungluing) t-structures on a prototypical arrangement of four

Received: September 28, 2006

such triples, starting with an assignment of three (out of seven) t-structures as the aforementioned initial assignments. It turns out that this arrangement of triples is not unnatural, and neither is the particular assignment of initial t-structures: recent and ongoing number-theoretic work [2, 3] by this author concerns a set-up of precisely this type for not just triangulated, but derived categories (of bounded sheaf complexes on special topological spaces). The larger context of that work is actually the open problem of the analytic proof of higher, i.e. general, reciprocity laws for algebraic number fields, seeing that the *raison d'être* of the cited papers is to go at Kubota's "blueprint" for a solution [10] by sheaf-theoretic methods. Specifically, we show in [3] that n-Hilbert reciprocity obtains as a consequence of the vanishing, or degeneration, of a particular "vertex" in $n-1$ (of n) arrangements of linked exact triples of the sort considered below. We say more about this number-theoretic connection in Section 5; it provides the justification for the t-structure calculus we introduce in what follows.

Because the particular notation attached to the corresponding arrangements of exact triples of derived categories of the sort just mentioned is dauntingly cumbersome we have opted, instead, to work in the present article with suitably restricted exact triples of triangulated categories. And, as we just indicated, the circumstances that these hypotheses are in fact met by certain players occurring in [3] yields *ab initio* that our discourse is not vacuous.

Additionally, we address in Section 4 a quasi-conjectural connection between our work and what we will call Thomason's correspondence. This correspondence, going back to [11], engenders a dictionary between strictly full triangulated subcategories of a given triangulated category with an essentially small object class and subgroups of the latter category's Grothendieck group. We convey, accordingly, what our results should look like in terms of corresponding Abelian groups, at least if certain largely set-theoretic criteria are met.

Finally, regarding the remaining structure of this paper, we devote Section 2 to the requisite background material covering triangulated (and occasionally derived) categories, exact triples and diagrams linking them, and t-structures and gluing and ungluing them in the setting of an exact triple. We spend a considerate amount of time on the different presentations of gluing (and of gluing data, of course) since we aim for the simplest possible formulation of the ensuing yoga of t-structures. Accordingly we impose certain important hypotheses on our categories and morphisms. However, as already observed, these restrictions do not degenerate to vacuity. The heart of the paper is Section 3.

2. Triangulated Categories, t-Structures and the Process of Gluing

Our primary reference is [5] and we take the liberty of outlining the relevant parts of the theory of triangulated categories, in the extended sense indicated, as presented there, without citing chapter and verse. When other sources are indicated we mention them explicitly, of course.

By definition, an additive category, $\mathfrak{D}$, is said to be a triangulated category if it comes equipped with a translation autofunctor

$$\begin{aligned} \_[1]:\ \mathfrak{D} &\longrightarrow \mathfrak{D}\,, \\ X &\mapsto X[1]\,, \end{aligned} \tag{2.1}$$

and a class of triangles,

$$X \to Y \to Z \to X[1], \tag{2.2}$$

also rendered as $X \to Y \to Z \xrightarrow{+1}$ or even simply (X, Y, Z), where the arrows are morphisms in $\mathfrak{D}$; morphisms of triangles are commutative diagrams of the form

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array} \tag{2.3}$$

and (2.3) gives an isomorphism of distinguished triangles if and only if the vertical arrows are isomorphisms in $\mathfrak{D}$. Beyond this, in the class of such triangles the following four axioms cut out a subclass of distinguished (or exact) triangles:

(TR1) *For every $X \in \mathfrak{D}$, the triangle $X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow X[1]$, i.e. $X = X \to 0 \xrightarrow{+1}$, is distinguished. If we have an isomorphism of triangles in (2.3) and one of these is distinguished, so is the other. For every $X \to Y$ in $\mathfrak{D}$ there is at least one $Z \in \mathfrak{D}$ such that (X, Y, Z) is distinguished.*

(TR2) (Rotation) $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *is distinguished if and only if* $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ *is distinguished, too.*

(TR3) *With the rows of (2.4), below, distinguished and with the solid vertical arrows, $f, g, f[1]$, given, we always get a completed diagram via the* *dotted arrow, h,*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1], \end{array} \tag{2.4}$$

making for a morphism of distinguished triangles.

(TR4) (The Octahedron - Draw a Picture!) *Given a pair of morphisms* $X \xrightarrow{u} Y, Y \xrightarrow{v} Z$ *in* $\mathfrak{D}$ *and three distinguished triangles*

$$X \xrightarrow{u} Y \xrightarrow{x} A \xrightarrow{+1}, Y \xrightarrow{v} Z \xrightarrow{y} B \xrightarrow{+1},$$

and $X \xrightarrow{v \circ u} Z \longrightarrow C \xrightarrow{+1}$, *there are morphisms* $A \xrightarrow{a} C, C \xrightarrow{b} B$ *in* $\mathfrak{D}$, *yielding triangle morphisms* $(id_X, v, a), (u, id_Z, b)$ *and the distinguished triangle* $A \xrightarrow{a} C \xrightarrow{b} B \xrightarrow{x[1] \circ y} A[1]$ *(this compact phrasing of the octahedron axiom is adapted from [4]).*

One establishes quickly that a distinguished triangle (X, Y, Z) gives rise, for any $A \in \mathfrak{D}$, to two long exact sequences,

$$\begin{aligned} \ldots \to \operatorname{Hom}_{\mathfrak{D}}(A, X[i]) \to \operatorname{Hom}_{\mathfrak{D}}(A, Y[i]) \to \operatorname{Hom}_{\mathfrak{D}}(A, Z[i]) \\ \to \operatorname{Hom}_{\mathfrak{D}}(A, X[i+1]) \to \ldots, \end{aligned} \tag{2.5^a}$$

$$\begin{aligned} \ldots \to \operatorname{Hom}_{\mathfrak{D}}(X[i+1], A) \to \operatorname{Hom}_{\mathfrak{D}}(Z[i], A) \to \operatorname{Hom}_{\mathfrak{D}}(Y[i] A) \\ \to \operatorname{Hom}_{\mathfrak{D}}(X[i], A) \to \ldots, \end{aligned} \tag{2.5^b}$$

$$H : \mathfrak{D} \longrightarrow \mathfrak{A}, \tag{2.6}$$

is a cohomological functor from $\mathfrak{D}$ to some Abelian category, $\mathfrak{A}$, meaning that H maps any distinguished triangle (X, Y, Z) to an exact sequence $H(X) \to H(Y) \to H(Z)$, then, via TR2,

$$\ldots \to H(X[i]) \to H(Y[i]) \to H(Z[i]) \to H(X[i+1]) \to \ldots \tag{2.7}$$

is a long exact sequence in $\mathfrak{A}$.

Next, a class, S, of morphisms in a triangulated category, $\mathfrak{D}$, is called a localizing class if $S \in \mathcal{S} \Leftrightarrow S[1] \in \mathcal{S}$, and if we have that for $f, g \in \mathcal{S}$ in (2.4), above, we also obtain that $h \in \mathcal{S}$. The localization of $\mathfrak{D}$ at $\mathcal{S}$ is obtained by formally inverting the morphisms in $\mathcal{S}$ and is denoted by $\mathfrak{D}[\mathcal{S}^{-1}]$ (properly speaking this is achieved through the services of a functor $\mathfrak{D} \to \mathfrak{D}[\mathcal{S}^{-1}]$

replete with the obvious universality property; see [5], pp. 87-88). One says that a triple of triangulated categories,

$$\mathfrak{C} \xrightarrow{P} \mathfrak{D} \xrightarrow{Q} \mathfrak{E}, \tag{2.8}$$

is exact if, with P just inclusion, $\mathfrak{C}$ is a thick subcategory of $\mathfrak{D}$ and $\mathfrak{E} = \mathfrak{D}[\varphi(\mathfrak{C})^{-1}]$ is the localization of $\mathfrak{D}$ at the set

$$\varphi(\mathfrak{C}) := \{s \in \text{Mor } (\mathfrak{C}) \mid \exists \text{ a distinguished triangle, } X \xrightarrow{s} Y \longrightarrow Z \xrightarrow{+1}, \text{ with } Z \in \mathfrak{C}\}. \tag{2.9}$$

Recall that $\mathfrak{C}$ is a thick subcategory of $\mathfrak{D}$ if and only if we have that for any distinguished triangle $X \xrightarrow{f} Y \longrightarrow Z \xrightarrow{+1}$ for which f factors through some object in $\mathfrak{D}$, if $Z \in \mathfrak{C}$ then $X, Y \in \mathfrak{C}$.

As far as our purposes are concerned the *raison d'être* for (2.8) is of course the mechanics of gluing t-structures, so we begin this development be recalling that, again by definition, a t-structure on a triangulated category, $\mathfrak{D}$, is a pair of full subcategories, $(\mathfrak{D}^{\leq 0}, \mathfrak{D}^{\geq 0})$, of $\mathfrak{D}$, obeying the following rules: if $\mathfrak{D}^{\leq n} := \mathfrak{D}^{\leq 0}[-n]$ and $\mathfrak{D}^{\geq n} := \mathfrak{D}^{\geq 0}[-n]$, then:

(t1) $\mathfrak{D}^{\leq 1} \subseteq \mathfrak{D}^{\leq 0}$ (so, by iteration, $\mathfrak{D}^{\leq a} \subseteq \mathfrak{D}^{\leq b}$, if $a \leq b$) and $\mathfrak{D}^{\geq 1} \subseteq \mathfrak{D}^{\geq 0}$ (so $\mathfrak{D}^{\geq b} \subseteq \mathfrak{D}^{\geq a}$, if $a \leq b$).

(t2) If $X \in \mathfrak{D}^{\leq 0}, Y \in \mathfrak{D}^{\geq 1}$ then $\text{Hom}_{\mathfrak{D}}(X, Y) = 0$.

(t3) If $X \in \mathfrak{D}^{\leq 0}$, then there exist $X_0 \in \mathfrak{D}^{\leq 0}$ and $X_1 \in \mathfrak{D}^{\geq 1}$ such that $X_0 \longrightarrow X \longrightarrow X_1 \xrightarrow{+1}$ is a distinguished triangle in $\mathfrak{D}$.

Here we have used the definition given by Kashiwara and Schapira (see [10, p. 11]), who merely require $\mathfrak{D}^{\leq 0}$ and $\mathfrak{D}^{\geq 0}$ to be full. Usually one requires strict fullness and this is certainly the case in Beilinson-Bernstein-Deligne [1], p. 29, the standard reference for this material. Gel'fand and Manin also use the latter more stringent characterization ([5, p. 133], [6, p. 278]). We write $t(\mathfrak{D})$ (for $\mathfrak{D}^{\leq 0}, \mathfrak{D}^{\geq 0}$), for stylistic reasons which will become clear presently, and convey the fact that we have a t-structure on $\mathfrak{D}$ by the notation

$$\begin{array}{c} t(\mathfrak{D}) \\ \downarrow \\ \mathfrak{D} \end{array} \tag{2.10}$$

The core of $t(\mathfrak{D})$ is then just the intersection

$$\text{core } t(\mathfrak{D}) = \mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0}, \tag{2.11}$$

and is always an Abelian category (see [5], p. 134; [6], pp. 278-279, [1], p. 31). In connection with what we mentioned in Introduction, namely, that our broader objective is to employ t-structures to establish that a particular derived (whence triangulated) category is void, we note that one line of attack we are currently investigating involves using the core (2.11) as an operator on $\mathfrak{D}$, i.e. on triangulated categories, tagging the desired degeneracy in set-theoretic terms. However, this line is separate from our immediate concerns.

Therefore we now turn to the matter of gluing (*recollement*) of t-structures. Suppose we are given t-structures on $\mathfrak{C}$, $\mathfrak{E}$ in (2.8) and seek to construct from those a resultant t-structure on $\mathfrak{D}$:

$$\begin{array}{ccccc} t(\mathfrak{C}) & & t(\mathfrak{D}) & & t(\mathfrak{E}) \\ \downarrow & & \vdots\downarrow & & \downarrow \\ \mathfrak{C} & \xrightarrow{P} & \mathfrak{D} & \xrightarrow{Q} & \mathfrak{E} \end{array} \tag{2.12}$$

If this arrangement exists, i.e., if we have $t(\mathfrak{D})$ such that, in fact, $t(\mathfrak{C}) = t(\mathfrak{D}) \cap \mathfrak{C}$ (or, as P is inclusion, $P(t(\mathfrak{C})) = t(\mathfrak{D})$), meaning that $\mathfrak{C}^{\leq 0} = \mathfrak{D}^{\leq 0} \cap \mathfrak{C}$ and $\mathfrak{D}^{\geq 0} = \mathfrak{D}^{\geq 0} \cap \mathfrak{C}$, and $t(\mathfrak{E}) = Q(t(\mathfrak{D}))$, meaning that $\mathfrak{E}^{\geq 0} = Q(\mathfrak{C}^{\geq 0})$, we need only restate this compatibility of $t(\mathfrak{D})$ with $t(\mathfrak{C})$ and $t(\mathfrak{E})$ to get a condition on (2.8) providing us with so-called gluing data: the functors P, Q should be t-exact, where, generally, a functor $F : \mathfrak{D}_1 \to \mathfrak{D}_2$ of triangulated categories is t-exact if it is exact, i.e. F commutes with translation and distinguished triangles, and $F(\mathfrak{D}_1^{\leq 0}) \subseteq \mathfrak{D}_2^{\leq 0}$ and $F(\mathfrak{D}_1^{\geq 0}) \supseteq \mathfrak{D}_2^{\geq 0}$.

Under these circumstances we obtain, with

$$^{\perp}(P(\mathfrak{C}^{>0})) := \{X \in \mathfrak{D} \,|\, \mathrm{Hom}\,(X, Y) = 0 \text{ for all } Y \in P(\mathfrak{C}^{>0})\}, \tag{2.13^a}$$

$$(P(\mathfrak{C}^{<0}))^{\perp} := \{X \in \mathfrak{D} \,|\, \mathrm{Hom}\,(Y, X) = 0 \text{ for all } Y \in P(\mathfrak{C}^{<0})\}, \tag{2.13^b}$$

that $t(\mathfrak{D}) = (\mathfrak{D}^{\leq 0}, \mathfrak{D}^{\geq 0})$ satisfies

$$\mathfrak{D}^{\leq 0} = \{X \in {}^{\perp}(P(\mathfrak{C}^{>0})) \,|\, Q(X) \in \mathfrak{E}^{\leq 0}\} = {}^{\perp}(P(\mathfrak{C}^{>0})) \cap Q^{-1}(\mathfrak{E}^{\leq 0}), \tag{2.14^a}$$

$$\mathfrak{D}^{\geq 0} = \{X \in (P(\mathfrak{C}^{<0}))^{\perp} \,|\, Q(X) \in \mathfrak{E}^{\geq 0}\} = (P(\mathfrak{C}^{<0}))^{\perp} \cap Q^{-1}(\mathfrak{E}^{\geq 0}). \tag{2.14^b}$$

But there is a sticky wicket in the game: the preceding construction presupposes $t(\mathfrak{D})$ whereas we actually seek an *à forteriori* construction of (a) $t(\mathfrak{D})$ from the initial data $t(\mathfrak{C}), t(\mathfrak{E})$, given only these latter t-structures. Happily this can be achieved if P, Q obey some additional hypotheses; moreover, imposing certain yet further restrictions, which turn out to be milder than they look, we obtain a very useful reformulation of $(2.14^{a,b})$. To wit:

Proposition 2.1. *Given an exact triple (2.8) with t-structures* $t(\mathfrak{C}), t(\mathfrak{E})$*:*

$$\begin{array}{ccccc} t(\mathfrak{C}) & & & & t(\mathfrak{E}) \\ \downarrow & & & & \downarrow \\ \mathfrak{C} & \xrightarrow{P} & \mathfrak{D} & \xrightarrow{Q} & \mathfrak{E} \end{array}$$

Suppose, too, that P *and* Q *possess left as well as right adjoint functors, as conveyed by the following notation:*

$$\begin{array}{ccccc} & \xleftarrow{{}^LP} & & \xleftarrow{{}^LQ} & \\ \mathfrak{C} & \xrightarrow{P} & \mathfrak{D} & \xrightarrow{Q} & \mathfrak{E} \\ & \xleftarrow{{}^RP} & & \xleftarrow{{}^RQ} & \end{array} \tag{2.16}$$

Then there exists a glued t-structure,

$$t(\mathfrak{D}) = t(\mathfrak{C}) \wedge t(\mathfrak{E}), \tag{2.17}$$

on $\mathfrak{D}$*, characterized by* $(2.13^{a,b}), (2.14^{a,b})$.

Proof. See reference [5], p. 137. □

It follows tautologically that

$$\Big[t(\mathfrak{C}) \wedge t(\mathfrak{E})\Big] \cap \mathfrak{C} = t(\mathfrak{C}), \tag{2.18^a}$$

$$Q\Big[t(\mathfrak{C}) \wedge t(\mathfrak{E})\Big] = t(\mathfrak{E}), \tag{2.18^b}$$

employing the same notational conveniences as before. In other words, to coin a phrase, ungluing undoes gluing. Furthermore, we mention for the sake of completeness that the existence of ${}^LP$ and ${}^RP$ is equivalent to the existence of ${}^LQ$ and ${}^RQ$: two for the price of one.

Our next goal is to bring about the aforementioned reformulation of $(2.14^{a,b})$. The most natural way to do this is to begin with the more restrictive case of derived categories. Accordingly we now recall a number of salient facts covering the latter.

If $\mathfrak{A}$ is any Abelian category, write $\mathrm{Kom}(\mathfrak{A})$ for the Abelian category of chain complexes of objects from $\mathfrak{A}$, and write $\mathbf{K}(\mathfrak{A})$ for the category of chain homotopy equivalence classes from $\mathrm{Kom}(\mathfrak{A})$; in other words, $\mathbf{K}(\mathfrak{A}) = \mathrm{Kom}(\mathfrak{A})/_{\simeq}$. By means of the mapping cone construction we get a notion of distinguished triangle for this category. Specifically, we require that if

$X^\bullet \xrightarrow{f} Y^\bullet \in \mathrm{Mor}\,(\mathbf{K}(\mathfrak{A}))$ then, $X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow$ cone $(f)^\bullet \longrightarrow X^\bullet[1]$ is a distinguished triangle (with $X^\bullet[1]$ defined by $(X^\bullet[1])^n = X^{n-1}$) and we require that any $Z^\bullet$ isomorphic to cone $(f)^\bullet$ gives a distinguished triangle in $(X^\bullet, Y^\bullet, Z^\bullet)$. Thus, $\mathbf{K}(\mathfrak{A})$ is a triangulated category (for the details, in addition to [5], see [9], p. 38 ff. Also, the reference [7] is a classic). Recall, next, that two chain complexes are quasi-isomorphic if and only if they possess isomorphic cohomology groups in all degrees. Let Qis denote the subclass of Mor $(\mathbf{K}(\mathfrak{A}))$ consisting of all morphisms which induce quasi-isomorphisms and note (e.g. [7], p. 35 ff) that Qis is a localizing class. Then

$$D(\mathfrak{A}) := \mathbf{K}(\mathfrak{A})\left[Qis^{-1}\right] \tag{2.19}$$

is the desired derived category and is triangulated *via* the preceding conventions.

One of the most natural occurrences of Abelian categories is of course provided by the theory of sheaves: if X is a topological space then the category $\mathfrak{Sh}/X$ is Abelian, taking our sheaves to have values in, say, the category of Abelian groups, $\mathbf{Z}$-modules, or, more generally, in R-$\mathfrak{Mod}$, the category of R-modules for a given commutative ring, R, with unity. In this setting it turns out that if Y is a closed subset of X then the stratification

$$Y \xrightarrow{i} X \xleftarrow{j} U := X \backslash Y \tag{2.20}$$

gives rise to the exact triple

$$\mathfrak{D}_Y \xrightarrow{i_*} \mathfrak{D}_X \xrightarrow{j^*} \mathfrak{D}_U \tag{2.21}$$

of derived (whence triangulated) categories, where we write, generally,

$$\mathfrak{D}_X := D(\mathfrak{Sh}/X) \tag{2.22}$$

and the exact functors i_* and j^* are of course nothing else than direct and inverse image, respectively. Therefore, not only is (2.21) an instance of (2.8), but the usual Grothendieck formalism gives the following counterpart to (2.16):

$$\mathfrak{D}_Y \underset{\xleftarrow{i^!}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i_*}}} \mathfrak{D}_X \underset{\xleftarrow{Rj*}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j^*}}} \mathfrak{D}_U \tag{2.23}$$

Here, each (exact) functor is left adjoint to the one directly below it, the appearance of "!" engenders restriction to sheaves with proper support in each degree, and R denotes Verdier's functor ([12], p. 300-03, ff.) In addition to the four adjointness isomorphisms afforded by this situation we have, courtesy of the specific definitions of these functors acting on sheaf complexes, that

$$i^*j_! = 0 = j^*i_* = i^!Rj_* \tag{2.24^{a,b,c}}$$

and

$$\begin{tikzcd} i^*i_* \arrow[r] \arrow[dr, "\cong"'] & id \arrow[d] \\ & i^!i_* \end{tikzcd} \qquad \begin{tikzcd} j^*Rj_* \arrow[r] \arrow[dr, "\cong"'] & id \arrow[d] \\ & j^*j_! \end{tikzcd} \tag{2.25^{a,b}}$$

as diagrams of natural transformations; also, there exist morphisms $w : i_*i^*\mathcal{F}^\bullet \longrightarrow j_!j^*\mathcal{F}^\bullet[1]$ and $w' : Rj_*j^*\mathcal{F}^\bullet \longrightarrow i_*i^!\mathcal{F}^\bullet[1]$, functorial in $\begin{matrix}\mathcal{F}^\bullet\\ \downarrow\\ X\end{matrix}$, such that, with u, v, u', v' the appropriate adjunction morphisms, the triangles

$$j_!j^*\mathcal{F}^\bullet \xrightarrow{u} \mathcal{F}^\bullet \xrightarrow{v} i_*i^*\mathcal{F}^\bullet \xrightarrow{w} i_!j^*\mathcal{F}^\bullet[1], \tag{2.26^a}$$

$$i_*i^!\mathcal{F}^\bullet \xrightarrow{u'} \mathcal{F}^\bullet \xrightarrow{v'} Rj_*j^!\mathcal{F}^\bullet \xrightarrow{w'} i_*i^!\mathcal{F}^\bullet[1], \tag{2.26^b}$$

are distinguished. And now we get:

Proposition 2.2. *If (2.20), (2.21), (2.23), whence $(2.24^{a,b,c}) - (2.26^{a,b})$, are in place, then*

$$\begin{tikzcd} t(\mathfrak{D}_Y) \arrow[d] & & t(\mathfrak{D}_U) \arrow[d] \\ \mathfrak{D}_y \arrow[r, "i_*"] & \mathfrak{D}_X \arrow[r, "j^*"] & \mathfrak{D}_U \end{tikzcd} \tag{2.27}$$

gives rise to

$$\begin{matrix} t(\mathfrak{D}_Y) \wedge t(\mathfrak{D}_U) =: t(\mathfrak{D}_X) \\ \downarrow \\ \mathfrak{D}_X \end{matrix} \tag{2.28}$$

by means of

$$\mathfrak{D}_X^{\leq 0} = \{\mathcal{F}^\bullet | j^*\mathcal{F}^\bullet \in \mathfrak{D}_U^{\leq 0}, i^*\mathcal{F}^\bullet \in \mathfrak{D}_Y^{\leq 0}\}, \tag{2.29^a}$$

$$\mathfrak{D}_X^{\geq 0} = \{\mathcal{F}^\bullet | j^*\mathcal{F}^\bullet \in \mathfrak{D}_U^{\geq 0}, i^!\mathcal{F}^\bullet \in \mathfrak{D}_Y^{\geq 0}\}. \tag{2.29^b}$$

Proof. This is Theorem 1.4 on p. 48 in [1]. □

With Proposition 2.2 in place, and returning to the arrangement (2.15), we generalize the indicated hypotheses as follows:

Definition 2.3. The gluing data (2.15), (2.15), i.e.

$$\begin{array}{ccccc} t(\mathfrak{C}) & & & & t(\mathfrak{E}) \\ \downarrow & & & & \downarrow \\ \mathfrak{C} & \xrightarrow{P} & \mathfrak{D} & \xrightarrow{Q} & \mathfrak{E} \end{array}$$

and

$$\begin{array}{ccccc} & \xleftarrow{{}^L P} & & \xleftarrow{{}^L Q} & \\ \mathfrak{C} & \xrightarrow{P} & \mathfrak{D} & \xrightarrow{Q} & \mathfrak{E}, \\ & \xleftarrow{{}^R P} & & \xleftarrow{{}^R Q} & \end{array}$$

is optimal if the counterparts of ($2.24^{a,b,c}$), ($2.25^{a,b}$), and ($2.26^{a,b}$), hold true, in addition to the condition that in (2.16) each functor is left adjoint to its downstairs neighbor (we leave it to the reader to write these conditions out, if so desired).

The idea is, of course, that for optimal gluing data the counterpart of Proposition 2.2 follows *mutatis mutandis*, which is to say that we have the following corollary.

Corollary 2.4. (2.15), (2.16) entails optimal gluing data then we obtain

$$\begin{array}{ccccc} t(\mathfrak{C}) & & t(\mathfrak{C}) \wedge t(\mathfrak{E}) & & t(\mathfrak{E}) \\ \downarrow & & \vdots \downarrow & & \downarrow \\ \mathfrak{C} & \xrightarrow{P} & \mathfrak{D} & \xrightarrow{Q} & \mathfrak{E} \end{array} \tag{2.29}$$

where, writing $t(\mathfrak{C}) \wedge t(\mathfrak{E}) = t(\mathfrak{D}) = (\mathfrak{D}^{\leq 0}, \mathfrak{D}^{\geq 0})$,

$$\mathfrak{D}^{\leq 0} = \{X \in \mathfrak{D} | Q(X) \in \mathfrak{E}^{\leq 0}, {}^L P(X) \in \mathfrak{C}^{\leq 0}\}, \tag{2.30^a}$$

$$\mathfrak{D}^{\geq 0} = \{X \in \mathfrak{D} | Q(X) \in \mathfrak{E}^{\geq 0}, {}^R P(X) \in \mathfrak{C}^{\geq 0}\}. \tag{2.30^b}$$

Proof. Clear. □

3. The Yoga of Gluing t Structures on Linked Exact Triples

Consider the arrangement

$$
\begin{tikzcd}
 & \mathfrak{A} \arrow[d] \arrow[dr] & & & \\
t(\mathfrak{B}) \arrow[r] & \mathfrak{B} \arrow[r] \arrow[d] & \mathfrak{D} \arrow[r] \arrow[dr] & \mathfrak{E} & t(\mathfrak{E}) \arrow[l] \\
 & \mathfrak{C} & & \mathfrak{F} \arrow[u] & \\
 & & & \mathfrak{G} \arrow[u] &
\end{tikzcd}
\tag{3.1}
$$

where all triples (of triangulated categories, of course) are exact and meet the hypotheses of Proposition 2.1: all inclusions and localization morphisms possess left and right adjoints. Thus we get, in particular, gluing data for $t(\mathfrak{B})$ and $t(\mathfrak{E})$ *vis à vis* the "vertex" $\mathfrak{D}$, so that we get, first, the glued t-structure $t(\mathfrak{B}) \wedge t(\mathfrak{E}) =: t(\mathfrak{D})$. Second, ungluing $t(\mathfrak{D})$ along the triple $\mathfrak{A} \to \mathfrak{D} \to \mathfrak{F}$, we get the t-structures $t(\mathfrak{D}) \cap \mathfrak{A}$ and $\mathrm{im}t(\mathfrak{D})$ (its meaning being obvious), respectively on $\mathfrak{A}$ and $\mathfrak{F}$. But then compatible t-structures on $\mathfrak{C}$ and $\mathfrak{G}$, being $t(\mathfrak{C})$ and $t(\mathfrak{G})$, are determined *à forteriori* such that $t(\mathfrak{A}) \wedge t(\mathfrak{C}) = t(\mathfrak{B})$ and $t(\mathfrak{G}) \wedge t(\mathfrak{E}) = t(\mathfrak{F})$. Using the indicated notational conventions we encode this behavior as follows:

$$
\begin{tikzcd}
{[t(\mathfrak{B}) \wedge t(\mathfrak{E})] \cap \mathfrak{A}} \arrow[r, dotted] & \mathfrak{A} \arrow[d] \arrow[dr, "t(\mathfrak{B}) \wedge t(\mathfrak{E})"] & & & \\
t(\mathfrak{B}) \arrow[r] & \mathfrak{B} \arrow[r] \arrow[d] & \mathfrak{D} \arrow[r] \arrow[dr] & \mathfrak{E} & t(\mathfrak{E}) \arrow[l] \\
im\ t(\mathfrak{B}) \arrow[r, dotted] & \mathfrak{C} & & \mathfrak{F} \arrow[u] & im\ [t(\mathfrak{B}) \wedge t(\mathfrak{E})] \arrow[l, dotted] \\
 & & & \mathfrak{G} \arrow[u] & im[t(\mathfrak{B}) \wedge t(\mathfrak{E})] \cap \mathfrak{G} \arrow[l, dotted]
\end{tikzcd}
\tag{3.2}
$$

Now, working along $\mathfrak{A} \to \mathfrak{D} \to \mathfrak{F}$, the central vertex, $\mathfrak{D}$, obviously also supports the t-structure $\big([t(\mathfrak{B}) \wedge t(\mathfrak{E})] \cap \mathfrak{A}\big) \wedge \big(im[t(\mathfrak{B}) \wedge t(\mathfrak{E})]\big)$, and we should like there to be agreement: ${}_1t(\mathfrak{D}) := t(\mathfrak{B}) \wedge t(\mathfrak{E}) = 2t(\mathfrak{D}) := \big([t(\mathfrak{B}) \wedge t(\mathfrak{E})] \cap \mathfrak{A}\big) \wedge \big(im[t(\mathfrak{B}) \wedge t(\mathfrak{E})]\big)$. Since gluing and ungluing are mutually inverse operations, however, this condition, ${}_1t(\mathfrak{D}) =_2 t(\mathfrak{D})$, stating in essence that the initial data $(t(\mathfrak{B}), t(\mathfrak{E}))$ should determine only one glued

t-structure on $\mathfrak{D}$ regardless of whether the surrounding vertices are made to take part in the process, is tantamount to requiring that the initial data $\big(t(\mathfrak{B}), t(\mathfrak{C}), t(\mathfrak{G})\big)$ should yield but a single t-structure on $\mathfrak{D}$. This means that, if we label morphisms as follows,

$$\begin{tikzcd}
\mathfrak{A} \arrow[d, "\alpha"'] \arrow[dr, "\eta"] & & \\
\mathfrak{B} \arrow[r, "\gamma"] \arrow[d, "\beta"'] & \mathfrak{D} \arrow[r, "\delta"] \arrow[dr, "\theta"] & \mathfrak{E} \\
\mathfrak{C} & & \mathfrak{F} \arrow[u, "\epsilon"'] \\
 & & \mathfrak{G} \arrow[u, "\zeta"']
\end{tikzcd} \tag{3.3}$$

then $(2.30^{a,b})$ provides that ${}_1t(\mathfrak{D}) = {}_2t(\mathfrak{D})$ if and only if

$$ {}^L\gamma(X) \in \mathfrak{B}^{\leq 0} \Leftrightarrow {}^L\zeta\vartheta(X) \in \mathfrak{G}^{\leq 0} \text{ and } {}^L\eta(X) \in \mathfrak{B}^{\leq 0} \cap \mathfrak{A}\,, \tag{3.4^a}$$

$$ {}^L\gamma(X) \in \mathfrak{B}^{\geq 0} \Leftrightarrow {}^R\zeta\vartheta(X) \in \mathfrak{G}^{\geq 0} \text{ and } {}^R\eta(X) \in \mathfrak{B}^{\geq 0} \cap \mathfrak{A}\,, \tag{3.4^b}$$

and this sets the stage for the following proposition.

Proposition 3.1. *Consider the diagram*

$$\begin{tikzcd}
 & \mathfrak{A} \arrow[d, "\alpha"'] \arrow[dr, "\eta"] & & & \\
t(\mathfrak{B}) \arrow[r] & \mathfrak{B} \arrow[r, "\gamma"] \arrow[d, "\beta"'] & \mathfrak{D} \arrow[r, "\delta"] \arrow[dr, "\theta"] & \mathfrak{E} & t(\mathfrak{E}) \arrow[l] \\
 & \mathfrak{C} & & \mathfrak{F} \arrow[u, "\epsilon"'] & \\
 & & & \mathfrak{G} \arrow[u, "\zeta"'] & t(\mathfrak{G}) \arrow[l]
\end{tikzcd} \tag{3.5}$$

of linked exact triples supporting optimal gluing data and equipped with the initial assignment of t-structures $(t(\mathfrak{B}), t(\mathfrak{C}), t(\mathfrak{G}))$, as shown. Let

$$ {}_1t(\mathfrak{D}) = t(\mathfrak{B}) \wedge t(\mathfrak{C})\,, \tag{3.6^a}$$

$$ {}_2t(\mathfrak{D}) = \left(\Big[t(\mathfrak{B}) \wedge t(\mathfrak{E})\Big] \cap \mathfrak{A}\right) \wedge \Big[t(\mathfrak{G}) \wedge t(\mathfrak{E})\Big]\,, \tag{3.6^b}$$

and suppose, furthermore, that

$$ {}^L(\gamma\alpha) = {}^L\alpha {}^L\gamma\,, \tag{3.7^a} $$

$$ {}^R(\gamma\alpha) = {}^R\alpha {}^R\gamma\,. \tag{3.8^a} $$

Then

$$ {}_1t(\mathfrak{D}) = {}_2t(\mathfrak{D})\,. \tag{3.9} $$

Proof. For the sake of keeping tabs on what is happening inflate and complete (3.5) as follows:

$$
\begin{tikzcd}
t(\mathfrak{A}) = (\mathfrak{A}^{\le 0}, \mathfrak{A}^{\ge 0}) = t(\mathfrak{B}) \cap \mathfrak{A} \arrow[r] & \mathfrak{A} \arrow[d, "\alpha"'] \arrow[dr, "\eta"] & {}_1t(\mathfrak{D}) = ({}_1\mathfrak{D}^{\le 0},\ {}_1\mathfrak{D}^{\ge 0}) = t(\mathfrak{B} \wedge t(\mathfrak{E}) \arrow[d] & & \\
t(\mathfrak{B}) = (\mathfrak{B}^{\le 0}, \mathfrak{B}^{\ge 0}) \arrow[r] & \mathfrak{B} \arrow[r, "\gamma"] \arrow[d, "\beta"'] & \mathfrak{D} \arrow[r, "\delta"] \arrow[dr, "\theta"] & \mathfrak{E} & t(\mathfrak{E}) = (\mathfrak{E}^{\le 0}, \mathfrak{E}^{\ge 0}) \arrow[l] \\
t(\mathfrak{C}) = (\mathfrak{C}^{\le 0}, \mathfrak{C}^{\ge 0}) = \beta t(\mathfrak{B}) \arrow[r] & \mathfrak{C} & & \mathfrak{F} \arrow[u, "\epsilon"'] & t(\mathfrak{F}) = (\mathfrak{F}^{\le 0}, \mathfrak{F}^{\ge 0}) = = t(\mathfrak{G}) \wedge t(\mathfrak{E}) \arrow[l] \\
& {}_2t(\mathfrak{D}) = ({}_2\mathfrak{D}^{\le 0}, {}_2\mathfrak{D}^{\ge 0}) = = t(\mathfrak{A}) \wedge t(\mathfrak{F}) = = [t(\mathfrak{B}) \cap \mathfrak{A}] \wedge [t(\mathfrak{G}) \wedge t(\mathfrak{E})] \arrow[uur, bend left] & & \mathfrak{G} \arrow[u, "\zeta"] & t(\mathfrak{G}) = (\mathfrak{G}^{\le}, \mathfrak{G}^{\ge}) \arrow[l]
\end{tikzcd}
\tag{3.10}
$$

So we get, via (2.16) and Definition (2.3), that

$$
\mathfrak{G} \underset{\xleftarrow{{}^R\zeta}}{\overset{\xleftarrow{{}^L\zeta}}{\xrightarrow{\ \zeta\ }}} \mathfrak{F} \underset{\xleftarrow{{}^R\epsilon}}{\overset{\xleftarrow{{}^L\epsilon}}{\xrightarrow{\ \epsilon\ }}} \mathfrak{E}, \quad
\mathfrak{B} \underset{\xleftarrow{{}^R\gamma}}{\overset{\xleftarrow{{}^L\gamma}}{\xrightarrow{\ \gamma\ }}} \mathfrak{D} \underset{\xleftarrow{{}^R\delta}}{\overset{\xleftarrow{{}^L\delta}}{\xrightarrow{\ \delta\ }}} \mathfrak{E}, \quad
\mathfrak{A} \underset{\xleftarrow{{}^R\eta}}{\overset{\xleftarrow{{}^L\eta}}{\xrightarrow{\ \eta\ }}} \mathfrak{D} \underset{\xleftarrow{{}^R\theta}}{\overset{\xleftarrow{{}^L\theta}}{\xrightarrow{\ \theta\ }}} \mathfrak{E}
\tag{3.11^{a,b,c}}
$$

and then

$$ \mathfrak{F}^{\le 0} = \left\{F \in \mathfrak{F} \middle| \epsilon(F) \in \mathfrak{E}^{\le 0},\ \ {}^L\zeta(F) \in \mathfrak{G}^{\le 0}\right\}, \tag{3.12^a} $$

$$ \mathfrak{F}^{\ge 0} = \left\{F \in \mathfrak{F} \middle| \epsilon(F) \in \mathfrak{E}^{\ge 0},\ \ {}^R\zeta(F) \in \mathfrak{G}^{\ge 0}\right\}, \tag{3.12^b} $$

$$ \mathfrak{A}^{\le 0} = \mathfrak{B}^{\le 0} \cap \mathfrak{A},\ \mathfrak{A}^{\ge 0} = \mathfrak{B}^{\ge 0} \cap \mathfrak{A}\,, \tag{3.13^{a,b}} $$

$$ \mathfrak{C}^{\le 0} = \beta\mathfrak{B}^{\le 0},\ \ \mathfrak{C}^{\ge 0} = \beta\mathfrak{C}^{\ge 0}\,, \tag{3.14^{a,b}} $$

$$ {}_1\mathfrak{D}^{\leq 0} = \left\{ D \in \mathfrak{D} \middle| \delta(D) \in \mathfrak{E}^{\leq 0},\ \ {}^L\gamma(D) \in \mathfrak{B}^{\leq 0} \right\}, \tag{3.15^a} $$

$$ {}_1\mathfrak{D}^{\geq 0} = \left\{ D \in \mathfrak{D} \middle| \delta(D) \in \mathfrak{E}^{\geq 0},\ \ {}^R\gamma(D) \in \mathfrak{B}^{\geq 0} \right\}, \tag{3.15^b} $$

$$ {}_2\mathfrak{D}^{\leq 0} = \left\{ D \in \mathfrak{D} \middle| \theta(D) \in \mathfrak{F}^{\leq 0},\ {}^L\eta(D) \in \mathfrak{A}^{\leq 0} \right\}, \tag{3.16^a} $$

$$ {}_2\mathfrak{D}^{\geq 0} = \left\{ D \in \mathfrak{D} \middle| \theta(D) \in \mathfrak{F}^{\geq 0}, {}^R\eta(D) \in \mathfrak{A}^{\geq 0} \right\}. \tag{3.16^b} $$

But now $(3.12^{a,b}) - (3.14^{a,b})$ imply that

$$ {}_2\mathfrak{D}^{\leq 0} = \left\{ D \middle| \epsilon\theta(D) \in \mathfrak{E}^{\leq 0},\ {}^L\zeta\theta \in \mathfrak{G}^{\leq 0},\ {}^L\eta(D) \in \mathfrak{B}^{\leq 0} \cap \mathfrak{A} \right\}, \tag{3.17^a} $$

$$ {}_2\mathfrak{D}^{\geq 0} = \left\{ D \middle| \epsilon\theta(D) \in \mathfrak{E}^{\geq 0},\ {}^R\zeta\theta(D) \in \mathfrak{G}^{\geq 0},\ {}^R\eta(D) \in \mathfrak{B}^{\geq 0} \cap \mathfrak{A} \right\}, \tag{3.17^b} $$

which provides that ${}_1t(\mathfrak{D}) = {}_2t(\mathfrak{D})$ if and only if

$$ \begin{aligned} \delta(D) \in \mathfrak{E}^{\leq 0},\ {}^L\gamma(D) \in \mathfrak{B}^{\geq 0} \Leftrightarrow \epsilon\theta(D) \in \mathfrak{E}^{\leq 0},\ {}^L\zeta\theta(D) \in \mathfrak{G}^{\leq 0}, \\ {}^L\eta(D) \in \mathfrak{G}^{\leq 0} \cap \mathfrak{A}, \end{aligned} \tag{3.18^a} $$

$$ \begin{aligned} \delta(D) \in \mathfrak{E}^{\geq 0},\ {}^R\gamma(D) \in \mathfrak{B}^{\geq 0} \Leftrightarrow \epsilon\theta(D) \in \mathfrak{E}^{\geq 0},\ {}^R\zeta\theta(D) \in \mathfrak{G}^{\geq 0}, \\ {}^R\eta(D) \in \mathfrak{B}^{\geq 0} \cap \mathfrak{A}. \end{aligned} \tag{3.18^b} $$

First, ${}^L\zeta\theta(D) \in \mathfrak{G}^{\leq 0}$, ${}^R\zeta\theta(D) \in \mathfrak{G}^{\geq 0}$ obtain tautologically from (3.10), of course. Additionally, (3.10) yields that $\delta = \epsilon\theta$ leaving us the task of verifying that ${}^L\gamma(D) \in \mathfrak{B}^{\leq 0}$ is equivalent to ${}^L\eta(D) \in \mathfrak{B}^{\leq 0} \cap \mathfrak{A}$, and similarly with ${}^L\gamma$ (resp. $\mathfrak{B}^{\leq 0}$) replaced by ${}^R\gamma$ (resp. $\mathfrak{B}^{\geq 0}$). But (3.10) also gives that $\eta = \gamma\alpha$ whence using (3.7^a) and the fact that α is just inclusion (cf. (2.8)), ${}^L\gamma(D) \in \mathfrak{B}^{\geq 0} \Leftrightarrow {}^L\alpha\ {}^L\gamma(D) = {}^L(\gamma\alpha)(D) = {}^L\eta(D) \in \mathfrak{B}^{\geq 0} \cap \mathfrak{A}$. The other result proceeds in exactly the same way. This completes the proof. □

It turns out that our chosen arrangement (3.1) is actually slightly more general than what we encounter in applications, as will become evident presently. Specifically, we should specialize to the case where $\mathfrak{C} = \mathfrak{G}$ in (3.1), and tailor our initial t-structure data accordingly. Diagram (3.2) then becomes

$$ \begin{tikzcd} & & & t(\mathfrak{B}) \wedge t(\mathfrak{E}) \arrow[dd, dashed] & & & \\ {[t(\mathfrak{B}) \wedge t(\mathfrak{E})] \cap \mathfrak{A}} \arrow[r, dashed] & \mathfrak{A} \arrow[drr] & & & & \mathfrak{E} & t(\mathfrak{E}) \arrow[l] \\ & & & \mathfrak{D} \arrow[urr] \arrow[dr] & & & \\ t(\mathfrak{B}) \arrow[rr] & & \mathfrak{B} \arrow[ur] \arrow[uul] & & \mathfrak{F} \arrow[uur] & & im[t(\mathfrak{B}) \wedge t(\mathfrak{E})] \arrow[ll, dashed] \\ & im\ t(\mathfrak{B}) \arrow[rr, dashed] & & \mathfrak{C} \arrow[ul] \arrow[ur] & & & im[t(\mathfrak{B}) \wedge t(\mathfrak{E})] \cap \mathfrak{C} \arrow[lll, dashed] \end{tikzcd} \tag{3.19} $$

which apparently precipitates the requirement that $im\ [t(\mathfrak{B}) \wedge t(\mathfrak{C})] \cap \mathfrak{C}$ and $im\ t(\mathfrak{B})$ should agree, at least if we demand that (3.19) should qualify as a commutative diagram. In view of (3.3) this would mean that we should require that $\theta\gamma = \zeta\beta$. In point of fact, however, it is not clear that this is always the case, and, to boot, there is no binding requirement in place either, as far as forcing $im\ t(\mathfrak{B}) = im[t(\mathfrak{B}) \wedge t(\mathfrak{C})] \cap \mathfrak{C}$ is concerned. This having been said, we contend that our initial arrangement, as given by (3.1), (3.2), (2.3), and (3.5), is the proper one to go with, in that it certainly subsumes the arrangement (3.19).

On the more general subject of linked exact triples arranged as in (3.3), leaving aside for the moment any consideration of t-structures and gluing data, it is also the case that these occur in more or less natural contexts. For example, if we have an ambient topological space, X, with closed subspaces F and Y such that Y is relatively closed in F, then $W := F \backslash Y$ is relatively open in F, $U := X \backslash Y$ is open, $\check{U} := X \backslash F$ is open in X and relatively open in U, and $Z := U \backslash \check{U}$ is relatively closed in U. This makes for a diagram

$$\begin{tikzcd}
Y \arrow[d] \arrow[dr] & & \\
F \arrow[r] & X & \check{U} \arrow[l] \arrow[d] \\
W \arrow[u] & & U \arrow[ul] \\
& & Z \arrow[u]
\end{tikzcd} \tag{3.20}$$

where each triple $\bullet \rightarrow \bullet \leftarrow \bullet$ is an instance of (2.20). Accordingly we need only invoke (2.21) to get

$$
\begin{tikzcd}
\mathfrak{D}_Y \arrow[d] \arrow[dr] & & \\
\mathfrak{D}_F \arrow[r] \arrow[d] & \mathfrak{D}_X \arrow[r] \arrow[dr] & \mathfrak{D}_{\breve{U}} \\
\mathfrak{D}_W & & \mathfrak{D}_U \arrow[u] \\
& & \mathfrak{D}_Z \arrow[u]
\end{tikzcd}
\tag{3.21}
$$

as an instance of (3.3), given that derived categories are triangulated. One explicit instance of this formalism (where, it turns out, $W = Z$, and so $\mathfrak{D}_W = \mathfrak{D}_Z$) is realized by the assignments $X = \mathbf{R}^2, Y = \{(0,0)\}$, F is the x-axis; another instance, the motivation for the present investigation, is forthcoming in [3].

It is of course true that the arrangement (3.3) we have chosen to focus on here is not the only way in which to interlink exact triples of triangulated categories: however, it is clear that it is a minimal arrangement for four such triples. Also, our assignment of initial t-structures as given in (3.5) is certainly not the only available option, but it is evidently typical for a trio, and this configuration is featured in [3] for reasons belonging to the number-theoretic problem considered there (and discussed at somewhat greater length in Section 5). In any case, with the foregoing analysis of the chosen diagrams we proper to lay a foundation for a general yoga, or calculus, of initial assignment problems for t-structures situated on diagrams of linked exact triples of triangulated categories.

4. Regarding Thomason's Correspondence

In his famous paper [11] Thomason showed (very expeditiously) that for an essentially small triangulated category, $\mathfrak{D}$, the covariant functor K_0 sets up a bijective correspondence between $\mathfrak{D}$'s strictly full and dense triangulated subcategories and the subgroups of the (Abelian) Grothendieck group. It should obviously be very useful indeed if we could somehow bring this correspondence, which we will refer to as Thomason's correspondence, to bear on the yoga of gluing and ungluing t-structures on diagrams like (3.5) as developed in Section 3. This kind of application of Thomason's correspondence would of course involve a number of verifications or additional restrictions, given that we would have to make sense of not only the image of gluing in the target category $\mathfrak{Ab}$ (of Abelian groups), but, before that, of the image of localization in the sense of the morphism Q as per (2.8) and (2.9). But if a subclass of triangulated categories could be identified, characterized by being altogether amenable to Thomason's correspondence, then calculations on diagrams like (3.5) should be directly transferrable to suitable lattices of subgroups of Abelian groups. This applies specifically to the problem we address in the next section, arising in number theory. This said, we start the ball rolling in the present section by investigating a number of preliminary questions along the indicated lines.

First, regarding the question of essential smallness of an ambient triangulated category, $\mathfrak{D}$, i.e. the stipulation that its object class, taken modulo isomorphism, can be taken to be a set, for present convenience we posit that $\mathfrak{D}$'s object class is in fact a set already, i.e. $\mathfrak{D}$ is small. Since this is really quite a binding hypothesis, steps will have to be taken in order to apply this line to future applications but we do not address this contingency here. However, once we have sets to deal with, we are on safe ground as far as taking intersections is concerned and we observe that this is an extremely useful condition given that so many of the operations involved in the yoga of Sections 2 and 3 involve intersection. Indeed, both localization and gluing qualify under this heading.

Second, Thomason's correspondence proper presupposes that we are dealing with triangulated subcategories of $\mathfrak{D}$ which are also strictly full and dense, the latter adjective meaning that every object in $\mathfrak{D}$ can be realized as a direct summand of an object isomorphic to an object in the given dense subcategory. Manifestly, then, if Thomason's correspondence is to be brought to bear on t-structures, $t(\mathfrak{D})$, then we would have to have that both $\mathfrak{D}^{\leq 0}$ and $\mathfrak{D}^{\geq 0}$ are strictly full, dense and triangulated. For the moment, we will just agree to stipulate that our t-structures obey these requirements, although this will have to be checked carefully as far as applications are concerned.

Now we come to

Definition 4.1. If $\mathfrak{D}$ is a small triangulated category, its Grothendieck group, $K_0(\mathfrak{D})$ is the free Abelian group of isomorphism classes in $\mathfrak{D}$, modulo the Euler relations, entailing that $\ulcorner Y \urcorner = \ulcorner X \urcorner \oplus \ulcorner Z \urcorner$ in $K_0(\mathfrak{D})$ if and only if (X, Y, Z) is a distinguished triangle in $\mathfrak{D}$.

The requisite universality property attached to K_0, which acts covariantly on exact functors, is this: for every $\mathfrak{D}$, as above, we get a mapping

$$\mathfrak{D}/_{\cong} \xrightarrow{\epsilon_{\mathfrak{D}}} K_0(\mathfrak{D}) \tag{4.1}$$

satisfying the condition that if A is any Abelian group in which the Euler relations hold, and if $\mathfrak{D}/_{\cong} \xrightarrow{F} A$ is arbitrary, then we get

$$\begin{array}{ccc} \mathfrak{D}/_{\cong} & \xrightarrow{F} & A \\ \epsilon_{\mathfrak{D}} \downarrow & & \downarrow \exists! \tilde{F} \\ K_0(\mathfrak{D}) & = & K_0(\mathfrak{D}) \end{array} \tag{4.2}$$

In other words, every mapping from $\mathfrak{D}$'s isomorphism classes to an Abelian group with Euler relations factors through $\epsilon_{\mathfrak{D}sub}$. It turns out, however, that K_0 is more than a covariant functor equipped with a universality condition, and this is part and parcel of the Thomason correspondence. Specifically, Thomason showed in Section 3 of [11] that there is an induced bijective correspondence between the strictly full, dense, triangulated subcategories, $\mathfrak{A} \subset \mathfrak{D}$, and the subgroups of $K_0(\mathfrak{A}) \lhd K_0(\mathfrak{D})$. And the correspondence is completely natural: inverse to the association of $\mathfrak{A}$ to $K_0(\mathfrak{A})$, we have the association of any $H \lhd K_0(\mathfrak{D})$ to the subcategory

$$\mathfrak{T}_H := \{X \in \mathfrak{D} | \ulcorner X \urcorner \in H\}. \tag{4.3}$$

Thus we have the mutually inverse relations

$$\mathfrak{T}_{K_0(\mathfrak{A})} = \mathfrak{A}, \quad K_0(\mathfrak{T}_H) = H \tag{$4.4^{a,b}$}$$

for $\mathfrak{A}$ a subcategory of $\mathfrak{D}$ of the given type and $H \lhd K_0(\mathfrak{A})$. This puts us in a position to prove, by way of an illustration of what might be had down the line,

Proposition 4.2. *If $\mathfrak{A}, \mathfrak{B}$ are strictly full, dense, triangulated subcategories of the ambient small triangulated category, $\mathfrak{D}$, then $K_0(\mathfrak{A} \cap \mathfrak{B}) = K_0(\mathfrak{A}) \cap K_0(\mathfrak{B})$.*

Proof.

$$\begin{aligned} \mathfrak{T}_{K_0(\mathfrak{A}) \cap K_0(\mathfrak{B})} &= \{X \in \mathfrak{D} | \ulcorner X \urcorner \in K_0(\mathfrak{A}) \cap K_0(\mathfrak{B})\} \\ &= \{X | \ulcorner X \urcorner \in K_0(\mathfrak{A})\} \cap \{X | \ulcorner X \urcorner \in K_0(\mathfrak{B})\} = \mathfrak{T}_{K_0(\mathfrak{A})} \cap \mathfrak{T}_{K_0(\mathfrak{B})} = \mathfrak{A} \cap \mathfrak{B} \\ &= \mathfrak{T}_{K_0(\mathfrak{A} \cap \mathfrak{B})}, \end{aligned}$$

by (4.3) and (4.4^a). But now we note that the condition that the relations ($4.4^{a,b}$) entail a pair of inverses to conclude that $\mathfrak{T}$ must be injective. So $K_0(\mathfrak{A}) \cap K_0(\mathfrak{B}) = K_0(\mathfrak{A} \cap \mathfrak{B})$, as required. □

As we mentioned above, we offer this facile result as a first step toward realizing images of localization, gluing, and eventually the full yoga of t-structures introduced above, in the evocative setting of the category of Abelian groups. However, we postpone these undoubtedly rather ambitious pursuits till a future writing.

5. A Connection with Number Theory

Other recent and ongoing work of ours [2, 3], already cited repeatedly in the foregoing, is aimed at attacking a central open problem in analytic number theory through sheaf theoretic means. The configurations of linked exact triples of triangulated categories considered in the preceding sections, specifically (3.3) and (for derived categories) (3.20), arise in [3], where the proximate objective is to use the initial t-structures assignment problem posed by the set-up (3.5) to establish that if certain as yet unspecified conditions hold for some of these initial t-structures, the upper left vertex of (3.20) must in fact be empty. In the context of our so-called quasi-dual to Kubota's formalism for n-Hilbert reciprocity, developed in [2], this vanishing (for $n-1$ of n assignments of this upper left vertex) is enough to yield n-Hilbert reciprocity; therefore, if this is carried out through the services of a suitable Grothendieck-Deligne-type generalization of the Fourier transform we should have a solution of Hecke's open problem, going back to [8], of the analytic proof of higher reciprocity for a number field (the central problem alluded to above).

In [3] we consider instances of (3.21), completed in the sense of (3.5), of the following sort (with "p" indicating perversity):

$$\begin{tikzcd}
 & \mathfrak{D}_{\tilde{Y}_{\xi_0}} \arrow[d, "\tilde{i}_{\xi_0,*}"'] \arrow[dr, "i_{\xi_0,*}"] & & & \\
{}^{p}t(\mathfrak{D}_{\check{X}_{\xi_0}}) \arrow[r] & \mathfrak{D}_{\check{X}_{\xi_0}} \arrow[r, "\dot{i}_{\xi_0,*}"] \arrow[d, "j^*"'] & \mathfrak{D}_{\check{X}^2_{\Lambda}} \arrow[r, "\check{i}^*_{\xi_0}"] \arrow[dr, "j^*_{\xi_0}"'] & \mathfrak{D}_{\check{U}_{\xi_0}} & {}^{p}t(\mathfrak{D}_{\check{U}_{\xi_0}}) \arrow[l] \\
 & \mathfrak{D}_{W_{\xi_0}} & & \mathfrak{D}_{\check{U}_{\xi_0}} \arrow[u, "\tilde{j}^*_{\xi_0}"'] & \\
 & & & \mathfrak{D}_{Z_{\xi_0}} \arrow[u, "i_*"'] & {}^{p}t(\mathfrak{D}_{Z_{\xi_0}}) \arrow[l]
\end{tikzcd} \tag{5.1}$$

with the following diagram of stratified topological spaces (as per (3.10)) as foundation:

$$\begin{tikzcd}
\bar{Y}_{\xi_0} \arrow[d, "\hat{\imath}_{\xi_0}"'] \arrow[dr, "\hat{\imath}_{\xi_0}"] & & \\
\check{X}_{\xi_0} \arrow[r, "\check{\imath}_{\xi_0}"] & \tilde{X}^2_{\mathbf{A}} & \check{U}_{\xi_0} \arrow[l, "\check{\jmath}_{\xi_0}"'] \arrow[d, "\hat{\jmath}_{\xi_0}"] \\
W_{\xi_0} \arrow[u, "j"] & & U_{\xi_0} \arrow[ul, "j_{\xi_0}"] \\
 & & Z_{\xi_0} \arrow[u, "i"']
\end{tikzcd} \tag{5.2}$$

Here $\tilde{X}_{\mathbf{A}} = SL_2(k)_{\mathbf{A}} \underset{c_{\mathbf{A}}^{(n)}}{\times} U_n$, $c_{\mathbf{A}}^{(n)} \in H^2(SL_2(k)_{\mathbf{A}}, \mu_n)$ being Kubota's adèlic 2- cocycle [10], and we suppress group structure; of course μ_n is the group of n-th roots of 1 situated (by hypothesis) in the ambient algebraic number field k then, for $\xi_0 \in \mu_n$, the space $\bar{Y}_{\xi_0}$ is the closure of the intersection (Y_{ξ_0}) of $(c_{\mathbf{A}}^{(n)})^{-1}(\xi_0)$ with $(SL_2(k) \underset{c_{\mathbf{A}}^{(n)}}{\times} \mu_n)^2$, and the space $\check{X}_{\zeta_0}$ is defined by the characterization $(c_{\mathbf{A}}^{(n)})^{-1}(\xi_0) \times \mu^2$. The remaining four spaces obtain by the complementation rules engendered by the stipulation that each of the four triples $\bullet \longrightarrow \bullet \longleftarrow \bullet$ in (5.2) should be instances of (2.20). It follows that the corresponding four triples $\bullet \longrightarrow \bullet \longrightarrow \bullet$ in (5.1) are instances of (2.21), and exact, and, as we indicated in Section 2, the gluing data (cf. (2.23)) supported by these four exact triples is optimal. One infers immediately that Proposition 3.1 applies:

$$\begin{aligned}
&{}^pt(\mathfrak{D}_{\check{X}_{\xi_0}}) \wedge {}^pt(\mathfrak{D}_{\check{U}_{\xi_0}}) \\
&\qquad = ([{}^pt(\mathfrak{D}_{\check{X}_{\xi_0}}) \wedge {}^pt(\mathfrak{D}_{\check{U}_{\xi_0}})] \cap \mathfrak{D}_{\bar{Y}_{\xi_0}}) \wedge [{}^pt(\mathfrak{D}_{Z_{\xi_0}}) \wedge {}^pt(\mathfrak{D}_{Z_{\xi_0}})] .
\end{aligned} \tag{5.3}$$

The idea is to show that if $\xi_0 \neq 1$ then $\bar{Y}_{\xi_0} = \phi$, which would follow quickly if $\mathfrak{D}_{\bar{Y}_{\xi_0}} = \phi$. In turn the latter condition is implied by the circumstance that, for $\xi_0 \neq 1$, the derived category $\mathfrak{D}_{\bar{Y}_{\xi_0}}$ fails to support a t-structure. Taking the data (5.3), or (5.1), of course, *in toto*, with ξ_0 varying over μ_n, we seek (in [3] and future work) a set of initial assignments $({}^pt(\mathfrak{D}_{\check{X}_{\xi_0}}), {}^pt(\mathfrak{D}_{\check{U}_{\xi_0}}), {}^pt(\mathfrak{D}_{Z_{\xi_0}}))$, tailored to the arithmetic aspects of the specially engineered spaces in (5.2), and a particular "t-structure calculation", yielding the aforementioned "collapse" of all but one of the vertices $\bar{Y}_{\xi_0}$, namely $\bar{Y}_1$. It would then follow immediately from the definition of the Y_{ξ_0} that $c_{\mathbf{A}}^{(n)} \equiv 1$ on $SL_2(k)^2$, which, following Kubota [10] is precisely n-Hilbert reciprocity.

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