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1-1-2013

Twisted Alexander polynomials of 2-bridge knots

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June 12, 2012

Abstract

We investigate the twisted Alexander polynomial of a 2-bridge knot associated to a Fox coloring. For several families of 2-bridge knots, including but not limited to, torus knots and genus-one knots, we derive formulae for these twisted Alexander polynomials. We use these formulae to confirm a conjecture of Hirasawa and Murasugi for these knots.

1   Introduction

Associated to every knot K is its classical Alexander polynomial $\Delta_K(t)$. This well-known polynomial invariant encodes metabelian information of the knot group. That is, one can think of the Alexander polynomial as describing the representations of the knot group into a metabelian subgroup of a linear group. Knot groups, however, allow for many other types of non-abelian linear representations. From this perspective, the Alexander polynomial is just a special case of a more general invariant called the twisted Alexander polynomial introduced by Xiao-Song Lin [6]. The twisted Alexander polynomial $\tilde{\Delta}_{K,\phi}(t)$ depends not only on the knot K but also on a choice of some linear representation $\phi : \pi_1(\mathbb{S}^3 - K) \to GL(n, \mathbb{C})$ and it has been studied extensively in recent years.

In this article, we study twisted Alexander polynomials of 2-bridge knots associated with a linear representation coming from a Fox coloring of the knot. If $K_{p/q}$ is a 2-bridge knot and ℓ is an odd prime dividing the determinant q of $K_{p/q}$, then it is well-known that there is a Fox ℓ-coloring of the knot. This coloring provides a linear representation of the knot group that factors through the dihedral group. We call the twisted Alexander polynomial associated with this linear representation the ℓ-*twisted* Alexander polynomial of $K_{p/q}$ and denote it by $\tilde{\Delta}^\ell_{p/q}(t)$. Our study of these polynomials was motivated by the following conjecture of Hirasawa and Murasugi [5].

Conjecture 1 [Hirasawa and Murasugi] *Let K be a knot and $\phi : \pi_1(S^3 - K) \to D_\ell \to GL(\ell, \mathbb{C})$ a non-abelian linear representation, where D_ℓ is the dihedral group of order 2ℓ with ℓ an odd prime. Then*

$$\tilde{\Delta}_{K,\phi}(t) = \frac{\Delta_K(t)}{t-1} f(t) f(-t)$$

for some integer Laurent polynomial $f(t)$ and furthermore

$$f(t) \equiv \left(\frac{\Delta_K(t)}{t+1} \right)^{\frac{\ell-1}{2}} \pmod{\ell}.$$

Hirasawa and Murasugi verify this conjecture for 2-bridge knots whose fundamental groups map onto certain free products but do so without finding a specific formula for the twisted Alexander polynomial. We will extend their work but employ a different method in this paper. In particular, we will derive a specific formula for the twisted Alexander polynomial of the knots we study and then use it to verify the conjecture directly. Before we proceed, however, it is important to note that there is some potential ambiguity in the statement of the conjecture. First of all, neither the Alexander nor the twisted Alexander polynomial is well-defined. The former is only well-defined up to multiples of $\pm t$ while the latter up to multiples of $\pm t^\ell$. Thus, the Laurent polynomial $f(t)$ in the conjecture also fails to be well-defined. Because of this there are two different interpretations of the conjecture. A weak form of the conjecture would allow for some choices of $\tilde{\Delta}_{K,\phi}(t)$, $\Delta_K(t)$, and $f(t)$ that give equality in the first part and possibly different choices of $\Delta_K(t)$ and $f(t)$ that give congruency in the second part. A stronger form of the conjecture would require the same choices in both parts. For the families of knots studied in this paper, we will prove the stronger version of the conjecture.

Motivated by a beautiful formula for $\Delta_{K_{p/q}}(t)$, we develop a computational method and accompanying formula for $\tilde{\Delta}^\ell_{p/q}(t)$. Combining this formula with a recursive technique, we derive specific formulae for several bi-infinite families of 2-bridge knots. In particular, given a specific choice of p, q, and ℓ, we derive a formula for the ℓ-twisted Alexander polynomial of the 2-bridge knot associated to the fraction

$$\frac{p'}{q'} = \frac{p + 2\ell j r}{q + 2\ell(kp + jar + 2\ell jkr)},$$

where k and j are any non-negative integers and $q = ap + r$ with $0 < r < a$. For any 2-bridge knot, the Alexander polynomial is always a factor of its twisted Alexander polynomial and, for these bi-infinite families, the formula for $\tilde{\Delta}^\ell_{p'/q'}(t)/\Delta_{K_{p'/q'}}(t)$ depends on p, q, ℓ, and k but not on j. With this formula in hand, we can verify Conjecture 1 for every knot in the family. From a practical point of view, one can compute the ℓ-twisted Alexander polynomial for as many bi-infinite families of this kind as one likes. Furthermore, if we fix p and ℓ, then only a finite number of computations are needed to derive a formula for $\tilde{\Delta}^\ell_{p/q}(t)$ for all possible q. We do this for a small sample of (p, ℓ), namely: $(5, 3)$, $(7, 3)$, $(3, 5)$, and $(3, 7)$.

The paper will proceed as follows. In Section 2, we define the ℓ-twisted Alexander polynomial and find a formula for it which is derived from a simple graph associated to p/q which we call the epsilon graph. In Section 3, we establish several properties of this graph which are used in Section 4 to determine how $\tilde{\Delta}^\ell_{p/q}(t)$ changes when p and q are changed in certain ways. Finally, in Section 5, we apply these results to establish Conjecture 1 for several infinite families of 2-bridge knots.

2   The ℓ-twisted Alexander polynomial

Consider the 2-bridge knot $K_{p/q}$ determined by the integers p and q where $0 < p < q$, both p and q are odd, and $\gcd(p, q) = 1$. It is well known that the fundamental group is presented as

$$G_{p/q} = < a, b \,|\, aw = wb >$$

where $w = b^{\epsilon_1} a^{\epsilon_2} \dots a^{\epsilon_{q-1}}$ and $\epsilon_i = (-1)^{\lfloor ip/q \rfloor}$. Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. As an aside, it is easy to show that the ϵ_i's are symmetric, that is, $\epsilon_i = \epsilon_{q-i}$.

The Alexander polynomial can be derived from this presentation using the Fox free differential calculus [2]. Letting $\mathcal{R}$ be the relator $\mathcal{R} = awb^{-1}w^{-1}$ in $G_{p/q}$, a calculation shows that the matrix $\mathcal{F} = \begin{pmatrix} \frac{\partial \mathcal{R}}{\partial a} & \frac{\partial \mathcal{R}}{\partial b} \end{pmatrix}$ of Fox derivatives is

$$\mathcal{F} = \left(1 + (a-1)\frac{\partial w}{\partial a} \quad (a-1)\frac{\partial w}{\partial b} - w \right)$$

where

$$\frac{\partial w}{\partial a} = \epsilon_2 b^{\epsilon_1} a^{\frac{\epsilon_2 - 1}{2}} + \epsilon_4 b^{\epsilon_1} a^{\epsilon_2} b^{\epsilon_3} a^{\frac{\epsilon_4 - 1}{2}} + \dots + \epsilon_{q-1} b^{\epsilon_1} a^{\epsilon_2} b^{\epsilon_3} \dots b^{\epsilon_{q-2}} a^{\frac{\epsilon_{q-1} - 1}{2}} \tag{1}$$

and

$$\frac{\partial w}{\partial b} = \epsilon_1 b^{\frac{\epsilon_1 - 1}{2}} + \epsilon_3 b^{\epsilon_1} a^{\epsilon_2} b^{\frac{\epsilon_3 - 1}{2}} + \dots + \epsilon_{q-2} b^{\epsilon_1} a^{\epsilon_2} b^{\epsilon_3} \dots b^{\frac{\epsilon_{q-2} - 1}{2}}. \tag{2}$$

To find the Alexander polynomial, we replace a and b in $\mathcal{F}$ with their images in the abelianization of $G_{p/q}$, whose generator we call t, to obtain the 1×2 matrix A. In this case both a and b have image t. Now striking either entry of A and taking the determinant of the remaining 1×1 matrix gives the following formula for the Alexander polynomial of an arbitrary 2-bridge knot. (See [7] and compare [3]).

$$\Delta_{K_{p/q}}(t) = 1 - t^{\epsilon_1} + t^{\epsilon_1 + \epsilon_2} - t^{\epsilon_1 + \epsilon_2 + \epsilon_3} + \dots + t^{\epsilon_1 + \epsilon_2 + \epsilon_3 + \dots + \epsilon_{q-1}}. \tag{3}$$

For any 2-bridge knot, we can use (3) to produce a canonical choice for the Alexander polynomial and we shall do so unless otherwise noted throughout the rest of the paper. Formula (3) can also be interpreted graphically in a beautiful way using what we call the *epsilon* graph, $E(p, q)$. (A similar graph is studied by Hirasawa and Murasugi in [4].)

Definition 2 Given $0 < p < q$, pq odd, and $\gcd(p, q) = 1$, the *epsilon graph*, $E(p, q)$, is the set of q vertices $\{v_i\}_{i=0}^{q-1}$ in the plane whose coordinates are $v_i = (i, \sum_{j=1}^{i} \epsilon_j)$, together with the set of $q - 1$ edges that connect consecutive vertices. The i-th edge has slope ϵ_i. A maximal set of consecutive edges in $E(p, q)$ having the same slope is called a *segment*.

We illustrate the case of $p/q = 11/19$ in Figure 1. Summing the number of vertices at each horizontal level of $E(11, 19)$ gives the alternating coefficients of the Alexander polynomial. That is, $\Delta_{K_{11/19}}(t) = -t^{-1} + 5 - 7t + 5t^2 - t^3$. This method of deriving $\Delta_{K_{p/q}}(t)$ from $E(p, q)$ follows directly from (3).
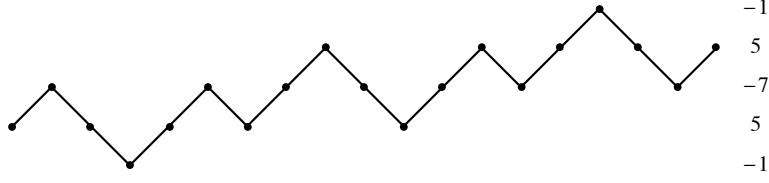
3

Figure 1: The epsilon graph of the 2-bridge knot $K_{11/19}$.

The twisted Alexander polynomial can also be obtained using the Fox differential calculus. See [6] and [8]. We first choose a linear representation $\phi : G_{p/q} \to GL(n, \mathbb{C})$. The *twisted Alexander matrix* $\hat{A}_\phi$ is obtained by replacing a, b, and 1 in the first column of $\mathcal{F}$ with $t\phi(a)$, $t\phi(b)$, and the identity matrix I, respectively. That is,

$$\hat{A}_\phi = I + (t\phi(a) - I) \left. \frac{\partial w}{\partial a} \right|_{a=t\phi(a), b=t\phi(b)}. \tag{4}$$

The *twisted Alexander polynomial with respect to ϕ* is then defined (see [8]) as

$$\tilde{\Delta}_{K,\phi}(t) = \frac{\det \hat{A}_\phi}{\det(t\phi(b) - I)}. \tag{5}$$

Furthermore, if ϕ is unimodular, this is an invariant of K up to multiples of $\pm t^n$.

We now focus on linear representations obtained from Fox colorings. If ℓ is an odd prime, then a *Fox coloring*, or an *ℓ-coloring*, is a homomorphsim ρ from $G_{p/q}$ onto the dihedral group D_ℓ of 2ℓ elements. Recall that the dihedral group has a presentation

$$D_\ell = <x, y \,|\, x^2 = 1, y^\ell = 1, yx = xy^{-1} > .$$

(Note that the presentation used in [5] uses generators $y^{-1}xy$ and xy in place of x and y, respectively.) It is well known that the determinant of the 2-bridge knot $K_{p/q}$ is q and that an ℓ-coloring exists if and only if ℓ divides q. Furthermore, if an ℓ-coloring exists, then, up to conjugation, we may assume that a and b are sent to x and xy^m, respectively, for some $0 < m < \ell$. Since the twisted Alexander polynomial is defined using determinants, this conjugation will not change $\tilde{\Delta}_{K,\phi}(t)$. We now choose, and fix, the following faithful, linear representation $\pi : D_\ell \to GL(\ell, \mathbb{C})$ defined by

$$\pi(x) = F = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad \pi(y) = R = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{6}$$

Observe that F and R are obtained from the identity matrix by reversing the rows and cyclically permuting the rows, respectively. Thus, an ℓ-coloring ρ determines a linear representation

$$\phi = \pi \circ \rho : G_{p/q} \to GL(\ell, \mathbb{C}),$$

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and hence an associated twisted Alexander polynomial.

To now determine the twisted Alexander matrix $\hat{A}_\phi$, we replace $\phi(a)$ and $\phi(b)$ in (4) with the matrices F and FR^m, respectively. Using the formula for $\partial w / \partial a$ in (1) and the fact that both F and FR^m are their own inverses, we see that

$$\hat{A}_\phi = I + \sum_{k=1}^{(q-1)/2} t^{\sum_{j=1}^{2k} \epsilon_j} (I - t^{-\epsilon_{2k}} F) R^{mk\epsilon_{2k}}. \tag{7}$$

The numerator of the twisted Alexander polynomial is the determinant of $\hat{A}_\phi$. By examining the form of FR^m, it follows that the denominator is

$$\det(tFR^m - I) = -(1+t)^{\frac{\ell-1}{2}}(1-t)^{\frac{\ell+1}{2}}. \tag{8}$$

Our strategy for finding the twisted Alexander polynomial is to compute the determinant of a similar matrix $P^T \hat{A}_\phi P$ which has a particularly nice form. If $mn \equiv 1 \pmod{\ell}$ and $\lambda = e^{2\pi i/\ell}$, then we define P to be the $\ell \times \ell$ symmetric matrix given by $P_{ij} = \lambda^{-(i-1)(j-1)n}$. Now from (7), we see that in order to compute the determinant of $P^T \hat{A}_\phi P = P \hat{A}_\phi P$ we need to determine the matrices $PR^{mk}P$ and $PFR^{mk}P$.

If $\vec{v}_j = (1, \lambda^{-jn}, \lambda^{-2jn}, \ldots, \lambda^{-(\ell-1)jn})^T$ for $0 \le j < \ell$, then a simple calculation shows that

$$R^m \vec{v}_j = \lambda^j \vec{v}_j.$$

Thus, λ^j is an eigenvalue of R^m with associated eigenvector $\vec{v}_j$. Furthermore, note that the j-th column of P is the eigenvector $\vec{v}_{j-1}$. Using the definition of λ, we see that

$$\vec{v}_i \cdot \vec{v}_j = \begin{cases} \ell & \text{for } i + j \equiv 0 \pmod{\ell} \\ 0 & \text{otherwise.} \end{cases}$$

Letting $\vec{e}_i$ denote the i-th standard basis vector, we then obtain

$$\begin{aligned} (PR^{mk}P)_{ij} &= \vec{e}_i \cdot (PR^{mk}P\vec{e}_j) \\ &= \vec{e}_i \cdot (PR^{mk}\vec{v}_{j-1}) \\ &= \lambda^{k(j-1)} (\vec{e}_i^T P) \vec{v}_{j-1} \\ &= \lambda^{k(j-1)} \vec{v}_{i-1} \cdot \vec{v}_{j-1} \\ &= \begin{cases} \ell \lambda^{k(j-1)} & \text{if } i + j \equiv 2 \pmod{\ell}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore,

$$PR^{mk}P = \ell \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda^{(\ell-1)k} \\ 0 & 0 & 0 & \cdots & \lambda^{(\ell-2)k} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \lambda^{2k} & \cdots & 0 & 0 \\ 0 & \lambda^k & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

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Furthermore, if we set $k = 0$, then we find that $\det(PP) = (-1)^{\frac{\ell-1}{2}} \ell^{\ell}$.

Consider now the product $PFR^{mk}P$. Then

$$
\begin{aligned}
(PFR^{mk}P)_{ij} &= \vec{e}_i \cdot (PFR^{mk}P\vec{e}_j) \\
&= (\vec{e}_i^T P)FR^{mk}\vec{v}_{j-1} \\
&= \lambda^{k(j-1)} \vec{v}_{i-1}^T F \vec{v}_{j-1} \\
&= \lambda^{k(j-1)}(\lambda^{-(\ell-1)(i-1)n}, \ldots, \lambda^{-2(i-1)n}, \lambda^{-(i-1)n}, 1) \cdot \vec{v}_{j-1} \\
&= \lambda^{(k+n)(j-1)} \sum_{l=1}^{\ell} \left(\lambda^{(i-j)n}\right)^l \\
&= \begin{cases} \ell\lambda^{(k+n)(j-1)} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\end{aligned}
$$

Therefore,

$$
PFR^{mk}P = \ell \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda^{k+n} & 0 & \cdots & 0 & 0 \\
0 & 0 & \lambda^{2(k+n)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda^{(\ell-2)(k+n)} & 0 \\
0 & 0 & 0 & \cdots & 0 & \lambda^{(\ell-1)(k+n)}
\end{pmatrix}.
$$

Combining all of the above, we see that $P\hat{A}_\phi P$ has the following form:

$$
P\hat{A}_\phi P = \ell \begin{pmatrix}
\Delta_{K_{p/q}} & 0 & 0 & \cdots & 0 & 0 \\
0 & d_1 & 0 & \cdots & 0 & e_{\ell-1} \\
0 & 0 & d_2 & \cdots & e_{\ell-2} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & e_2 & \cdots & d_{\ell-2} & 0 \\
0 & e_1 & 0 & \cdots & 0 & d_{\ell-1}
\end{pmatrix}
$$

where d_i and e_i are each Laurent polynomials in t with coefficients in $\mathbb{Z}[\lambda]$ given by

$$
d_i(\lambda, t) = -\lambda^{in} \sum_{k=1}^{\frac{q-1}{2}} t^{\sum_{j=1}^{2k-1} \epsilon_j} \lambda^{ik\epsilon_{2k}} \quad \text{and} \quad e_i(\lambda, t) = 1 + \sum_{k=1}^{\frac{q-1}{2}} t^{\sum_{j=1}^{2k} \epsilon_j} \lambda^{ik\epsilon_{2k}}. \tag{9}
$$

The determinant of $\hat{A}_\phi$ is now given by

$$
\begin{aligned}
\det \hat{A}_\phi &= \frac{\det\left(P\hat{A}_\phi P\right)}{\det\left(PP\right)} \\
&= (-1)^{\frac{\ell-1}{2}} \Delta_{K_{p/q}}(t) \prod_{i=1}^{\frac{\ell-1}{2}} \Big(d_i(\lambda, t) d_{\ell-i}(\lambda, t) - e_i(\lambda, t) e_{\ell-i}(\lambda, t)\Big).
\end{aligned}
$$

It is important to note that $d_i(\lambda, t) = d_1(\lambda^i, t)$ and $e_i(\lambda, t) = e_1(\lambda^i, t)$. Hence we will now write $d_{p/q}(\lambda, t)$ and $e_{p/q}(\lambda, t)$ in place of $d_1(\lambda, t)$ and $e_1(\lambda, t)$, respectively, to emphasize the dependance on p and q. Moreover, since the factors $-\lambda^{in}$ and $-\lambda^{-in}$ cancel in the product $d_{p/q}(\lambda^i, t)d_{p/q}(\lambda^{-i}, t)$ for each i, the determinant of $\hat{A}_\phi$ and therefore the twisted Alexander polynomial, $\tilde{\Delta}_{K_{p/q},\phi}(t)$, does not depend on n and m. In other words, there is no dependence on the choice of homomorphism of the knot group onto the dihedral group. Thus, we will henceforth assume that $m = n = 1$. The following definition fixes notation and terminology that we use throughout the rest of the paper.

Definition 3 Let $K_{p/q}$ be a 2-bridge knot, ℓ an odd prime dividing q, $\lambda = e^{2\pi i/\ell}$, $\rho : G_{p/q} \to D_\ell$ any surjective representation, and $\phi = \pi \circ \rho$ where π is the faithful linear representation defined in (6). The twisted Alexander polynomial of $K_{p/q}$ with respect to ϕ will be called the ℓ-*twisted Alexander polynomial* of $K_{p/q}$ and denoted by $\tilde{\Delta}^\ell_{p/q}(t)$.

The preceding discussion proves the following theorem. In this theorem, we use $\doteq$ to denote equality of Laurent polynomials up to factors of the form $\pm t^{r\ell}$ where $r \in \mathbb{Z}$.

Theorem 4 *If $K_{p/q}$ is a 2-bridge knot and ℓ an odd prime dividing q, then the ℓ-twisted Alexander polynomial of $K_{p/q}$ is*

$$\tilde{\Delta}^\ell_{p/q}(t) \doteq \frac{\Delta_{K_{p/q}}(t)}{t-1} \prod_{i=1}^{\frac{\ell-1}{2}} \frac{d_{p/q}(\lambda^i, t)d_{p/q}(\lambda^{-i}, t) - e_{p/q}(\lambda^i, t)e_{p/q}(\lambda^{-i}, t)}{t^2 - 1} \tag{10}$$

where $\Delta_{K_{p/q}}(t)$ is given by (3),

$$d_{p/q}(\lambda, t) = \sum_{k=1}^{\frac{q-1}{2}} t^{\sum_{j=1}^{2k-1} \epsilon_j} \lambda^{k\epsilon_{2k}}, \quad and \quad e_{p/q}(\lambda, t) = 1 + \sum_{k=1}^{\frac{q-1}{2}} t^{\sum_{j=1}^{2k} \epsilon_j} \lambda^{k\epsilon_{2k}}. \tag{11}$$

Since the determinant $d_{p/q}(\lambda, t)d_{p/q}(\lambda^{-1}, t) - e_{p/q}(\lambda, t)e_{p/q}(\lambda^{-1}, t)$ in (10) appears frequently in the remainder of the paper, we will denote it as $D_{p/q}(\lambda, t)$. Note that the right hand side of (10) gives a canonical form of $\tilde{\Delta}^\ell_{p/q}(t)$ that we will use throughout the rest of the paper.

For any 2-bridge knot, the formulae for d and e given in (11) can be read from the epsilon graph after it has been been labeled as follows. Label the vertex v_0 with 0. Next label both vertices v_{2i-1} and v_{2i} with $\pm i$, where the sign is the slope of the segment between these two vertices (which is ϵ_{2i}). Each vertex now contributes a term of the form $t^s \lambda^r$ to either d or e as follows. First, all vertices on the same horizontal level correspond to the same power t^s just as in the case of using the epsilon graph to compute the Alexander polynomial. Odd vertices contribute odd powers of t to d, while even vertices contribute even powers of t to e. Finally, the value of r at each vertex is the label of that vertex.
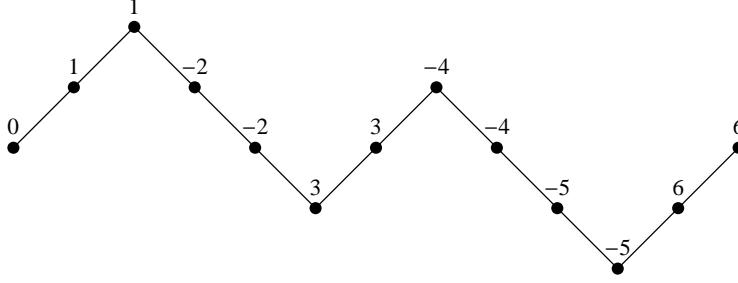
Figure 2: Labeling the epsilon graph of $K_{5/13}$ so as to read off d and e.

We give an example with $p/q = 5/13$ in Figure 2. Using the labeling, we obtain

$$e_{5/13}(\lambda, t) = t^{-2}\lambda^{-5} + 1 + \lambda^{-2} + \lambda^3 + \lambda^{-4} + \lambda^6 + t^2\lambda$$

and

$$d_{5/13}(\lambda, t) = t^{-1}(\lambda^3 + \lambda^{-5} + \lambda^6) + t(\lambda + \lambda^{-2} + \lambda^{-4}).$$

Hence

$$\prod_{i=1}^{6} \frac{D_{5/13}(\lambda^i, t)}{t^2 - 1} = \frac{(t-1)^6(t+1)^6 \left(t^6 - 2t^5 + t^3 - 2t + 1\right)^2 \left(t^6 + 2t^5 - t^3 + 2t + 1\right)^2}{t^{24}}$$

Now using Theorem 4

$$\tilde{\Delta}_{5/13}^{13}(t) \doteq \frac{\Delta_{K_{5/13}}(t)}{t - 1} f(t)f(-t),$$

where $\Delta_{K_{5/13}}(t) = t^{-2} - 3t^{-1} + 5 - 3t + t^2$ and $f(t) = t^{-12}(t+1)^6(t^6 - 2t^5 + t^3 - 2t + 1)^2$. Notice that the form of $\tilde{\Delta}_{5/13}^{13}(t)$ is as predicted by Conjecture 1 and it is easy to verify that the second part of the conjecture is true as well.

3 Properties of the Epsilon Graph

In this section we compile a number of properties of the epsilon graph that will be useful in computing d and e and hence ultimately in computing $\tilde{\Delta}_{p/q}^{\ell}(t)$ via Theorem 4. Many of the properties in Propositions 7 and 8 below have already appeared in print, see for example [4], but are included here for completeness.

It is nice to think of the ϵ_i's in the following way. On the interval from 0 to pq, mark all the points $p, 2p, 3p, \ldots, (q-1)p$. If ip lies between $2kq$ and $(2k+1)q$ for some k, then $\epsilon_i = 1$, while if ip lies between $(2k-1)q$ and $2kq$, then $\epsilon_i = -1$. With this perspective we see that $E(p, q)$ has p segments. This leads naturally to the following definition.

Definition 5 Given $0 < p < q$, pq odd, and $\gcd(p, q) = 1$, define the *sigma sequence* $\sigma(p, q) = \{\sigma_i\}_{i=1}^p$ by letting σ_i be the number of edges in the i-th segment of $E(p, q)$. Equivalently, σ_i is the number of multiples of p contained in the open interval $((i-1)q, iq)$.

The case of $p = 1$ is particularly simple. The epsilon graph consists of a single segment of length $q - 1$ because $\epsilon_i = 1$ for all i. Hence we will assume $p > 1$ for the rest of this section. We begin with a useful lemma.

Lemma 6 *Suppose $1 < p < q$ and $\gcd(p, q) = 1$. If $1 \leq k < p$, then*
$$\left\lfloor \frac{kq}{p} \right\rfloor = \left\lfloor \frac{kq - 1}{p} \right\rfloor.$$

Proof: We may write $kq = mp + r$ where $m > 0$ and $1 \leq r < p$. Now $\frac{kq}{p} = m + \frac{r}{p}$ so $\lfloor \frac{kq}{p} \rfloor = m$. But $kq - 1 = mp + r - 1$ so $\frac{kq-1}{p} = m + \frac{r-1}{p}$ and $0 \leq \frac{r-1}{p} < \frac{p-1}{p} < 1$. Thus $\left\lfloor \frac{kq-1}{p} \right\rfloor = m$ and $\left\lfloor \frac{kq}{p} \right\rfloor = \left\lfloor \frac{kq-1}{p} \right\rfloor$. $\qquad \square$

Proposition 7 *Suppose $1 < p < q$, pq odd, and $\gcd(p, q) = 1$. Further assume that $q = a_1 p + r_1$ with $0 < r_1 < p$. Then*

1. *The sigma sequence $\sigma(p, q)$ is symmetric.*

2. *The graph $E(p, q)$ has a central segment of even length. Moreover, the number of segments of even length is odd and the number of segments of odd length is even.*

3. *The total number of edges in $E(p, q)$ is $\sum_{i=1}^p \sigma_i = q - 1$.*

4. *For $1 \leq k \leq p$ we have $\sum_{i=1}^k \sigma_i = \left\lfloor \frac{kq-1}{p} \right\rfloor$. In particular, $\sigma_1 = a_1$.*

5. *For $1 \leq i \leq p$ we have $\sigma_i = \left\lfloor \frac{iq-1}{p} \right\rfloor - \left\lfloor \frac{(i-1)q-1}{p} \right\rfloor$.*

6. *The sum of any k consecutive σ_i's is equal to either $\sum_{i=1}^k \sigma_i$ or $1 + \sum_{i=1}^k \sigma_i$. In particular, for any i, $\sigma_i \in \{\sigma_1, 1 + \sigma_1\}$.*

Proof: Properties 1–3 are immediate.

To prove property 4, which immediately implies property 5, we look at the length of the interval $[0, kq]$ in two ways to obtain
$$p \sum_{i=1}^k \sigma_i + \delta = kq,$$
where $\delta = kq - p \sum_{i=1}^k \sigma_i$ and $0 < \delta \leq p$. Note that $\delta = p$ if and only if $k = p$. Now
$$\sum_{i=1}^k \sigma_i + \frac{\delta}{p} = \frac{kq}{p}.$$

9

If $1 \leq k < p$, then by Lemma 6

$$\sum_{i=1}^{k} \sigma_i = \left\lfloor \frac{kq}{p} \right\rfloor = \left\lfloor \frac{kq-1}{p} \right\rfloor.$$

If instead, $k = p$, then

$$\sum_{i=1}^{p} \sigma_i = q - 1 = \left\lfloor \frac{pq-1}{p} \right\rfloor.$$

Now consider property 6. The result is trivial if $k = p$, so assume that $1 \leq k < p$. Consider k consecutive σ_i's, say $\sigma_{j+1}, \sigma_{j+2}, \ldots, \sigma_{j+k}$. These correspond to multiples of p that lie in the open interval $(jq, (j+k)q)$. Let α be the distance between jq and the first multiple of p in $(jq, (j+k)q)$ and let β be the distance between $(k+j)q$ and the last multiple of p in $(jq, (j+k)q)$. We have that $0 < \alpha \leq p$ and $0 < \beta \leq p$. Now

$$kq = \alpha + p \left(\sum_{i=j+1}^{j+k} \sigma_i - 1 \right) + \beta.$$

From this we see that $0 < \alpha + \beta < 2p$. Now

$$\frac{kq}{p} = \frac{\alpha + \beta}{p} + \sum_{i=j+1}^{j+k} \sigma_i - 1,$$

which gives

$$\left\lfloor \frac{kq}{p} \right\rfloor = \left\lfloor \frac{\alpha + \beta}{p} \right\rfloor + \sum_{i=j+1}^{j+k} \sigma_i - 1.$$

Since $0 < \frac{\alpha+\beta}{p} < 2$ the floor $\left\lfloor \frac{\alpha+\beta}{p} \right\rfloor$ is 0 or 1. Hence $\sum_{i=j+1}^{j+k} \sigma_i$ is equal to either $\left\lfloor \frac{kq}{p} \right\rfloor$ or $1 + \left\lfloor \frac{kq}{p} \right\rfloor$. But $\left\lfloor \frac{kq}{p} \right\rfloor = \left\lfloor \frac{kq-1}{p} \right\rfloor = \sum_{i=1}^{k} \sigma_i$ by property 4 and Lemma 6. $\square$

Proposition 7 implies that segments come in two possible lengths, σ_1 or $1 + \sigma_1$. Call segments of length σ_1 *short* and segments of length $1 + \sigma_1$ *long*. Call a segment *isolated* if it is does not have an adjacent segment of the same length. Note that if no long segments exist, then vacuously, all long segments are isolated. By a *cluster* of segments we mean a maximal set of consecutive segments all of the same length. The next proposition lists more facts about $E(p, q)$.

Proposition 8 *Suppose $1 < p < q$, pq odd, and $\gcd(p, q) = 1$. Assume the Euclidean algorithm gives*

$$
\begin{array}{llll}
q & = & a_1 p + r_1, & 0 < r_1 < p \\
p & = & a_2 r_1 + r_2, & 0 \leq r_2 < r_1 \\
r_1 & = & a_3 r_2 + r_3, & 0 \leq r_3 < r_2 \\
& \vdots & & \\
r_n & = & a_{n+2} r_{n+1} + 1 &
\end{array}
$$

Then the epsilon graph has the following properties:

1. All short segments are isolated or all long segments are isolated (or both).

2. The number of long segments is $r_1 - 1$ and the number of short segments is $p - r_1 + 1$.

3. The initial cluster of short segments has length a_2.

4. If all long segments are isolated, then every cluster of short segments has length a_2 or $a_2 - 1$.

5. The number of clusters of short segments of length a_2 is equal to $r_2 + 1$. If $a_2 > 1$, then the number of clusters of short segments of length $a_2 - 1$ equals $r_1 - r_2 - 1$.

6. If there exists a nonisolated long segment and more than one cluster of long segments, then the first cluster of long segments has length a_3 and every cluster of long segments has length a_3 or $a_3 + 1$.

7. If there exists a nonisolated long segment and more than one cluster of long segments, then the number of clusters of long segments of length a_3 is $r_2 - r_3 + 1$ and the number of length $a_3 + 1$ is $r_3 - 1$.

Proof: If there are two short segments in a row and two long segments in a row, then this would give two pairs of consecutive σ_i's whose sums differ by 2, which is not possible by Proposition 7. This proves the first property.

To prove property 2, let L be the number of long segments and $p - L$ be the number of short segments. Then

$$(p - L)\sigma_1 + L(\sigma_1 + 1) = \sum_{i=1}^{p} \sigma_i = q - 1 = a_1 p + r_1 - 1.$$

Hence $L = r_1 - 1$ since $a_1 = \sigma_1$.

Property 3 is proven as follows. If there are no long segments, then $r_1 = 1$ and so $a_2 = p$ which is the length of the initial, and only, cluster of short segments. Alternatively, suppose the initial cluster of short segments has length k and is followed by a long segment. Now $r_1 > 1$. This gives the inequalities

$$
\begin{array}{ccccc}
k\sigma_1 p & < & kq & < & (k\sigma_1 + 1)p \\
((k+1)\sigma_1 + 1)p & < & (k+1)q & < & ((k+1)\sigma_1 + 2)p.
\end{array}
$$

If we now replace q with $\sigma_1 p + r_1$ we may derive that $k < \frac{p}{r_1} < k + 1$ and hence $k = \left\lfloor \frac{p}{r_1} \right\rfloor = a_2$.

One case of property 4 is easy. Suppose some cluster of short segments has length $k \leq a_2 - 2$. By property 4 it cannot be the initial or final cluster, and hence must have long segments on either end. Now the sum of these $k + 2$ consecutive σ_i's is $(k + 2)\sigma_1 + 2$. But, $\sum_{i=1}^{k+2} \sigma_i = (k + 2)\sigma_1$. This contradicts Proposition 7.

To prove the rest of property 4, we begin with the following useful observation. Let $0 < \alpha < p$ and consider the set of points $S_\alpha = \{\alpha, \alpha + p, \alpha + 2p, \alpha + 3p, \dots\}$. It is easy to show that if $0 < \alpha < r_1$, then exactly $\sigma_1 + 1$ points of S_α lie in the open interval $(0, q)$. On the other hand, if $r_1 < \alpha < p$, then exactly σ_1 points of S_α lie in the open interval $(0, q)$. Moreover, in this case,

$\alpha + (\sigma_1 + 1)p - q = \alpha - r_1$. Hence there are exactly σ_1 points of S_α contained in the open interval $(q, 2q)$ if and only if $\alpha - r_1 > r_1$.

Now suppose that there is a cluster of short segments of length more than a_2. Suppose the first segment of this cluster lies in the open interval $(jq, (j+1)q)$. Short segments continue at least as far as the interval $((j + a_2)q, (j + a_2 + 1)q)$. If α is the distance between jq and the first multiple of p in $(jq, (j+1)q)$, then $p > \alpha > r_1$. Furthermore, the distance between $(j + a_2)q$ and the first multiple of p in $((j + a_2)q, (j + a_2 + 1)q)$ is $\alpha - a_2 r_1$ and thus $\alpha - a_2 r_1 > r_1$. A similar analysis of the initial cluster of short segments implies that the distance between $a_2 q$ and the first multiple of p in $(a_2 q, (a_2 + 1)q)$ is $p - a_2 r_1$. Because the $a_2 + 1$'st segment is long, we have that $p - a_2 r_1 < r_1$. These inequalities imply that $\alpha > r_1 + a_2 r_1 > p$, a contradiction.

We now turn our attention to property 5. Suppose first that $a_2 = 1$. Let x be the number of clusters of short segments of length 1. Then x equals the number of short segments, so we have $x = p - (r_1 - 1)$ by property 2. But $p = r_1 + r_2$ so we obtain $x = r_2 + 1$. Suppose now that $a_2 > 1$. Let x and y be the number of clusters of short segments of lengths a_2 and $a_2 - 1$, respectively. Since long segments are isolated, the total number of clusters of short segments is $x + y = r_1$. Counting edges gives

$$q - 1 = \sigma_1 p + r_1 - 1 = x a_2 \sigma_1 + y(a_2 - 1)\sigma_1 + (r_1 - 1)(\sigma_1 + 1)$$

which simplifies to $x = r_2 + 1$. Furthermore, we have $y = r_1 - x = r_1 - r_2 - 1$.

For the first part of property 6, since long segments exist we have $r_1 > 1$. Furthermore, since long segments are nonisolated, short segments are isolated and so $a_2 = 1$. Suppose the first cluster of long segments has length k. This gives the following inequalities.

$$\begin{array}{ccccc}
((k+1)\sigma_1 + k)p & < & (k+1)q & < & ((k+1)\sigma_1 + k + 1)p \\
((k+2)\sigma_1 + k)p & < & (k+2)q & < & ((k+2)\sigma_1 + k + 1)p.
\end{array}$$

If we replace q with $\sigma_1 p + r_1$ and p with $r_1 + r_2$, we obtain $k < \frac{r_1}{r_2} < k + 1$ so that $k = \left\lfloor \frac{r_1}{r_2} \right\rfloor = a_3$.

For the second part of property 6, note that the epsilon graph always begins and ends with a cluster of short segments. Hence any cluster of long segments is preceded and followed by a cluster of short segments. Suppose two clusters of long segments have x and y segments respectively with $x + 2 \le y$. The cluster of length x has short segments on either side. This gives $x + 2$ consecutive σ_i's that sum to $(x + 2)\sigma_1 + x$. But the cluster of length y contains a set of $x + 2$ consecutive σ_i's that add to $(x + 2)(\sigma_1 + 1) = (x + 2)\sigma_1 + x + 2$. But now the sums of two sets of $x + 2$ consecutive σ_i's differ by 2, which is impossible by Proposition 7. Thus, all clusters of long segments have length $a_3 - 1, a_3$, or $a_3 + 1$. It remains to show that $a_3 - 1$ is impossible. Suppose instead that a cluster of long segments exists with length $a_3 - 1$. Assume the short segment that precedes this cluster lies in the open interval $(jq, (j+1)q)$ and that the short segment that follows it lies in the open interval $((j + a_3)q, (j + a_3 + 1)q)$. Let α be the distance between jq and the first multiple of p in $(jq, (j+1)q)$. Since a short segment is contained in this interval, we have $r_1 < \alpha < p$. The distance between $(j + a_3)q$ and the first multiple of p in the interval $((j + a_3)q, (j + a_3 + 1)q)$ is $\alpha + a_3(p - r_1) - p$ and so we must have $r_1 < \alpha + a_3(p - r_1) - p < p$. Now returning to the first cluster of long segments, we see that the distance between $a_3 q$ and the first multiple of p in the interval $(a_3 q, (a_3 + 1)q)$ is $a_3(p - r_1)$. Since the interval $(a_3 q, (a_3 + 1)q)$ contains a long segment, we have that $0 < a_3(p - r_1) < r_1$. But combining these inequalities leads to $p < \alpha$, a contradiction.

To prove property 7, let x equal the number of clusters of long segments of length a_3 and y equal the number of clusters of long segments of length $a_3 + 1$. As in property 6, $r_1 > 1$ and $a_2 = 1$. From property 5, the number of short segments is $r_2 + 1$ and hence $x + y = r_2$. Counting all the edges in the graph, we obtain

$$
\begin{aligned}
(r_2 + 1)a_1 + xa_3(a_1 + 1) + y(a_3 + 1)(a_1 + 1) &= q - 1 \\
(a_1 + 1) + (r_2 - y)a_3(a_1 + 1) + ya_3(a_1 + 1) + y(a_1 + 1) &= q - r_2 a_1 \\
(a_1 + 1) + r_2 a_3(a_1 + 1) + y(a_1 + 1) &= a_1(r_1 + r_2) + r_1 - r_2 a_1 \\
(a_1 + 1)(1 + r_2 a_3 + y) &= (a_1 + 1)r_1 \\
1 + r_1 - r_3 + y &= r_1 \\
y &= r_3 - 1.
\end{aligned}
$$

Therefore, the number of clusters of long segments of length a_3 is $r_2 - r_3 + 1$ and the number of clusters of length $a_3 + 1$ is $r_3 - 1$. $\qquad\square$

4 Relations among ℓ-twisted Alexander polynomials

Using Propositions 7 and 8, we can determine how $d_{p/q}(\lambda, t)$ and $e_{p/q}(\lambda, t)$ change when p and q are changed in certain ways. This will then allow us to compute the twisted Alexander polynomial for several different types of infinite families of 2-bridge knots. In order to do this, we need to investigate how $E(p, q)$ changes. It is useful to reformulate the contribution of each vertex in the graph $E(p, q)$ to d and e depending on the index of the vertex. Examining the formulae for d and e given in (11) again, we see that the k-th vertex contributes $t^s \lambda^{\pm \lfloor \frac{k+1}{2} \rfloor}$ where s depends on the t-level of the vertex as before and the plus or minus sign is determined as follows. If the vertex lies in the interior of a segment of slope 1, then the choice is plus. If it lies in the interior of a segment of slope -1, then the choice is minus. If instead, it is a local maxima or minima, then $k = \sigma_1 + \sigma_2 + \cdots + \sigma_i$, for some i, and the vertex contributes the term $t^{\sigma_1 - \sigma_2 + \cdots + (-1)^{i-1}\sigma_i} \lambda^{(-1)^{(k+i+1)} \lfloor \frac{k+1}{2} \rfloor}$.

Theorem 9 *Suppose $1 < p < q$, pq odd, and $\gcd(p, q) = 1$. Assume the Euclidean algorithm gives*

$$
\begin{aligned}
q &= a_1 p + r_1, &\quad 0 < r_1 < p \\
p &= a_2 r_1 + r_2, &\quad 0 \le r_2 < r_1
\end{aligned}
$$
$$\vdots$$

If ℓ is an odd prime dividing q, $p' = p + 2\ell j r_1$, and $q' = q + 2\ell j a_1 r_1$ for some positive integer j, then for the canonical forms from Theorem 4 we have

1. *$\Delta_{K_{p'/q'}}(t) \equiv \Delta_{K_{p/q}}(t) \pmod{\ell}$ and $\tilde{\Delta}^\ell_{p'/q'}(t) \equiv \tilde{\Delta}^\ell_{p/q}(t) \pmod{\ell}$.*

2. *If in addition, $\gcd(a_1, \ell) = 1$, then $\dfrac{\tilde{\Delta}^\ell_{p'/q'}(t)}{\Delta_{K_{p'/q'}}(t)} = \dfrac{\tilde{\Delta}^\ell_{p/q}(t)}{\Delta_{K_{p/q}}(t)}$.*

Proof: It suffices to prove the theorem for $j = 1$. Notice that

$$
\begin{aligned}
q' &= a_1 p' + r_1, &\quad 0 < r_1 < p \\
p' &= (a_2 + 2\ell)r_1 + r_2, &\quad 0 \le r_2 < r_1.
\end{aligned}
$$

This implies that $1 < p' < q'$, $p'q'$ odd, and $\gcd(p', q') = 1$.

By Proposition 8, the lengths of the short and long segments, respectively, of $E' = E(p', q')$ and $E = E(p, q)$ are the same. Secondly, the number of long segments is the same in each graph. Moreover, the length of the initial cluster of short segments in E' is 2ℓ segments longer than that in E. Additionally, while the long segments may not be isolated in E, they certainly are in E'. We will now show that E' is obtained from E by increasing the length of every cluster of short segments by 2ℓ. In the case where there is a nonisolated long segment of E, and hence short segments are isolated, this will mean lengthening each cluster of short segments to a length of $1 + 2\ell$ as well as inserting clusters of short segments of length 2ℓ between each adjacent pair of long segments.

Consider the i-th long segment in E. Suppose that it is preceded by S short segments and $L = i - 1$ long segments. This implies that

$$(S + L)q < (Sa_1 + L(a_1 + 1) + 1)p < (Sa_1 + L(a_1 + 1) + a_1 + 1)p < (S + L + 1)q.$$

These inequalities are equivalent to

$$(S' + L')q' < (S'a_1 + L'(a_1 + 1) + 1)p' < (S'a_1 + L'(a_1 + 1) + a_1 + 1)p' < (S' + L' + 1)q',$$

where $S' = S + (L + 1)2\ell$ and $L' = L$.

This proves that each long segment in E' is preceded by the correct number of long and short segments, namely, $L' = L$ and $S' = S + (L + 1)2\ell$, respectively. However, it remains to show that the arrangement of short and long segments in E' is obtained from that in E as claimed. We prove this by induction on i. When $i = 1$, we know by Proposition 8 that the first cluster of short segments is increased in length by 2ℓ. Assuming this is true up to the $(i - 1)$-st long segment, and given that the i-th long segment is preceded by the correct number of long and short segments, we see that either the i-th cluster of short segments is increased in length by 2ℓ or a new cluster of short segments of length 2ℓ is inserted between the $(i - 1)$-st and i-th long segments.

The first part of property 1 now follows since lengthening a single cluster of short segments by 2ℓ will increase two of the coefficients of the Alexander polynomial by ℓ and the $a_1 - 1$ remaining coefficients by 2ℓ. To prove the other properties, we must examine how d and e are effected.

We now show that d and e are the same for both E and E' if a_1 and ℓ are co-prime and are congruent modulo ℓ otherwise. Since the length of each cluster of short segments is increased by 2ℓ (and if the long segments are not isolated, new clusters of short segments of length 2ℓ are introduced between adjacent long segments) we see that many new summands are introduced into both d and e. However, each vertex in E continues to contribute the same summand to either d or e of E'. To see this, consider the k-th vertex of E which becomes the k'-th vertex of E'. Lengthening each cluster of short segments in E by 2ℓ means that $k' = k + 2j\ell$ for some j. Moreover, the vertex remains on the same t-level. If this vertex contributes $t^s \lambda^{\pm \lfloor \frac{k+1}{2} \rfloor}$ to d or e of E, then it will contribute $t^s \lambda^{\pm \lfloor \frac{k+1+2j\ell}{2} \rfloor} = t^s \lambda^{j\ell \pm \lfloor \frac{k+1}{2} \rfloor} = t^s \lambda^{\pm \lfloor \frac{k+1}{2} \rfloor}$ since $\lambda^\ell = 1$. Now we examine the contribution to d or e made by the insertion of a single set of 2ℓ consecutive short segments. Consider a vertex of E' in the first of these 2ℓ short segments. If this vertex is numbered k, then the reminding $\ell - 1$ vertices at the same t-level in this cluster are numbered $k + 2a_1 i$ for $i = 1, 2, \ldots, \ell - 1$. These vertices all

have λ exponents of the same sign and together they contribute the following to either d or e

$$t^s \left(\lambda^{\pm \lfloor \frac{k+1}{2} \rfloor} + \lambda^{\pm(a_1 + \lfloor \frac{k+1}{2} \rfloor)} + \cdots + \lambda^{\pm((\ell-1)a_1 + \lfloor \frac{k+1}{2} \rfloor)} \right)$$

$$= t^s \lambda^{\pm \lfloor \frac{k+1}{2} \rfloor} \left(1 + \lambda^{\pm a_1} + (\lambda^{\pm a_1})^2 + \cdots + (\lambda^{\pm a_1})^{\ell-1} \right)$$

$$= \begin{cases} 0 & \text{if } \gcd(a_1, \ell) = 1, \\ \ell\, t^s \lambda^{\pm \lfloor \frac{k+1}{2} \rfloor} & \text{otherwise.} \end{cases}$$

$\square$

As an immediate consequence of Theorem 9 we have the following corollary.

Corollary 10 *Assume the hypotheses of Theorem 9 and additionally that* $\gcd(a_1, \ell) = 1$. *If* $K_{p/q}$ *satisfies Conjecture 1, then so does* $K_{p'/q'}$.

In Theorem 9 we found that d and e did not change when p and q were increased in a certain way. In the following result we keep p fixed and increase q in a certain way. This will now change d and e but in a such a way that will allow us to generate an infinite family of 2-bridge knots for which we can compute d and e recursively. This in turn will provide a recursive computation of ℓ-twisted Alexander polynomials.

Proposition 11 *Suppose* $1 < p < q$, pq *odd,* $\gcd(p, q) = 1$, *and* ℓ *is an odd prime dividing* q. *Then for all* $k > 1$

1. $\Delta_{K_{p/(q+2k\ell p)}}(t) = \alpha_k\, \Delta_{K_{p/(q+2\ell p)}}(t) + (1 - \alpha_k)\Delta_{K_{p/q}}(t),$

2. $d_{p/(q+2k\ell p)}(\lambda, t) = \alpha_k\, d_{p/(q+2\ell p)}(\lambda, t) + (1 - \alpha_k)d_{p/q}(\lambda, t),$ *and*

3. $e_{p/(q+2k\ell p)}(\lambda, t) = \alpha_k\, e_{p/(q+2\ell p)}(\lambda, t) + (1 - \alpha_k)e_{p/q}(\lambda, t),$

where $\alpha_k = \frac{1 - t^{2k\ell}}{1 - t^{2\ell}}$.

Proof: We focus on the second and third properties. The first property is proven in a similar way. Let $d = d_{p/q}(\lambda, t), d' = d_{p/(q+2\ell p)}(\lambda, t)$, and $d'' = d_{p/(q+4\ell p)}(\lambda, t)$. Define e, e', and e'' analogously. We will show that

$$d'' - d' = t^{2\ell}(d' - d) \quad \text{and} \quad e'' - e' = t^{2\ell}(e' - e).$$

This will imply that

$$d'' = (1 + t^{2\ell})d' - t^{2\ell}d \quad \text{and} \quad e'' = (1 + t^{2\ell})e' - t^{2\ell}e.$$

Standard techniques of linear algebra can then be used to derive the recursion formula stated in the lemma.

Notice that d and e can be uniquely recovered from the sum $d + e$ by separating the terms of even and odd powers of t. Hence it suffices to prove that

$$(d'' + e'') - (d' + e') = t^{2\ell}[(d' + e') - (d + e)]. \tag{12}$$

Let $\sigma = \sigma(p, q) = \{\sigma_1, \sigma_2, \ldots, \sigma_p\}$, $\sigma' = \sigma(p, q + 2\ell p) = \{\sigma'_1, \sigma'_2, \ldots, \sigma'_p\}$, $\sigma'' = \sigma(p, q + 4\ell p) = \{\sigma''_1, \sigma''_2, \ldots, \sigma''_p\}$, and let E, E', and E'' be the associated epsilon graphs. It follows from property 5 of Proposition 7 that $\sigma'_i = \sigma_i + 2\ell$ and $\sigma''_i = \sigma'_i + 2\ell$. Hence E' is obtained from E by increasing each segment by 2ℓ edges. Similarly, E'' is obtained from E' by increasing each segment by 2ℓ edges. Superimposing the graphs, we see that the local minima of all three graphs coincide.

The k-th vertex of E contributes a term of the form $t^s \lambda^{\pm \lfloor \frac{k+1}{2} \rfloor}$ to $d + e$ as follows. For the moment, we focus only on the exponent of λ. To describe the sign of the exponent of λ better, we consider three cases: the vertex is in the interior of a segment with slope 1, the vertex is in the interior of a segment of slope -1, or the vertex is a local extrema. In the first case, the exponent of λ is $\lfloor \frac{k+1}{2} \rfloor$ and in the second case it is $-\lfloor \frac{k+1}{2} \rfloor$. If the vertex is a local extrema, then we must have that $k = \sigma_1 + \sigma_2 + \cdots + \sigma_i$ for some i. The exponent of λ associated to this term is now $(-1)^{k+i+1} \lfloor \frac{k+1}{2} \rfloor$. Notice that k is the total number of edges that precede the vertex while i is the total number of segments that precede the vertex.

Suppose v is a vertex that is a local minima of all three epsilon graphs. If v is the k-th vertex of E, then $k = \sigma_1 + \sigma_2 + \cdots + \sigma_{2i}$ for some i. If v is the k'-th and k''-th vertex of E' and E'', respectively, then $k \equiv k' \equiv k'' \pmod{2\ell}$ since $\sigma''_j \equiv \sigma'_j \equiv \sigma_j \pmod{2\ell}$. Hence v contributes the same term to $d + e$, $d' + e'$, and $d'' + e''$ because $\lambda^\ell = 1$. Thus these terms cancel in the differences on both sides of (12).

Consider now a local minima shared by the three epsilon graphs and the three segments of slope 1 that start at this minima and belong to the three graphs, respectively. Suppose v and v' are two vertices in the interior of the segments of E and E', respectively, that lie on the same t-level, say, t^s. Suppose v is the k-th vertex of E and v' is the k'-th vertex of E'. Again, since $\sigma'_j \equiv \sigma_j \pmod{2\ell}$, we see that $k' \equiv k \pmod{2\ell}$. The vertex v contributes the term $t^s \lambda^{\lfloor \frac{k+1}{2} \rfloor}$ to $d + e$ while the vertex v' contributes the term $t^s \lambda^{\lfloor \frac{k'+1}{2} \rfloor}$ to $d' + e'$. But, because $k' \equiv k \pmod{2\ell}$ and $\lambda^\ell = 1$, we see that the two contributions are the same. Thus they will cancel in the difference given in (12). The same is true of any two vertices at the same t-level but in the interior of segments of E'' and E'. Moreover, terms of this kind that lie at the same t-level on the interiors of segments of slope -1 that share the same local minima also cancel.

Next, we consider a local maxima of E. For such a vertex $k = \sigma_1 + \sigma_2 + \ldots \sigma_{2i+1}$. The corresponding maxima of E' is the vertex numbered $k' = \sigma'_1 + \sigma'_2 + \ldots \sigma'_{2i+1}$. If this is not the last vertex of E, then there exist two vertices of E' at the same t-level, namely the vertices numbered $k' - 2\ell$ and $k' + 2\ell$. The first contributes $t^s \lambda^{\lfloor \frac{k'-2\ell+1}{2} \rfloor}$ to $d' + e'$ and the second contributes $t^s \lambda^{-\lfloor \frac{k'+2\ell+1}{2} \rfloor}$ to $d' + e'$. These exponents of λ differ only in sign. Now, returning to the local maxima of E, we see that it contributes $t^s \lambda^{\pm \lfloor \frac{k+1}{2} \rfloor}$ and hence cancels with one of the two terms from E'. If in fact, the local maxima of E is the last vertex of E, then it contributes $t^s \lambda^{\lfloor \frac{k+1}{2} \rfloor}$ and again cancels with the term of E'. In conclusion we see that every term of $d + e$ cancels with some term of $d' + e'$.

Finally, it remains to see that each term of E'' that does not cancel with a term of E' is equal to $t^{2\ell}$ times a corresponding term of E' that does not cancel with a term of E. Consider for example $k'' = \sigma''_1 + \sigma''_2 + \ldots \sigma''_{2i+1}$ and the vertex v'' of E'' numbered $k'' - j$ for some $0 \le j < 2\ell$. Let v' be the vertex of E' numbered $k' - j$ where $k' = \sigma'_1 + \sigma'_2 + \ldots \sigma'_{2i+1}$. If the local maxima of E'' at position k'' has t-level s, then v'' contributes the term $t^{s-j} \lambda^{\lfloor \frac{k''-j+1}{2} \rfloor}$ to $(d'' + e'') - (d' + e')$ while

the vertex v' of E' contributes $t^{s-2\ell-j}\lambda^{\lfloor\frac{k'-j+1}{2}\rfloor}$ to $(d'+e')-(d+e)$. Similar pairings of vertices exist for the segments of slope -1. $\qquad\square$

5   Special families of ℓ-twisted Alexander polynomials

In this section we apply Theorems 4 and 9 as well as Proposition 11 to compute the ℓ-twisted Alexander polynomials for several infinite families of 2-bridge knots. Obtaining a specific formula for the ℓ-twisted Alexander polynomial allows us to verify Conjecture 1 for these families. We begin with 2-bridge torus knots.

5.1   2-bridge torus knots

If $p = 1$, the 2-bridge knot $K_{1/q}$ is the $(2, q)$-torus knot. Conjecture 1 was proven for these knots in the case where q is prime by Hirasawa and Murasugi [5]. However, they do this without producing an explicit formula for the twisted Alexander polynomial even though they provide a likely candidate. (See Remark 5.3 of [5].) The following theorem extends their result to any odd integer q and moreover confirms their conjectured formula in the case when q is prime.

Theorem 12 *If ℓ is an odd prime dividing q, then the ℓ-twisted Alexander polynomial of the 2-bridge, torus knot $K_{1/q}$ is,*

$$\tilde{\Delta}^\ell_{1/q}(t) \doteq \frac{\Delta_{K_{1/q}}(t)}{t-1}\left(\frac{(1+t)(1+t^q)^{\frac{\ell-1}{2}}}{1+t^\ell}\right)\left(\frac{(1-t)(1-t^q)^{\frac{\ell-1}{2}}}{1-t^\ell}\right).$$

Proof: Let ℓ be an odd prime dividing q. Since $\epsilon_i = 1$ for all i, we have from (11) that

$$d_{1/q}(\lambda, t) = \sum_{k=1}^{\frac{q-1}{2}} t^{2k-1}\lambda^k = \frac{t\lambda(t^{q-1}\lambda^{\frac{q-1}{2}}-1)}{t^2\lambda-1} \quad \text{and} \quad e_{1/q}(\lambda, t) = 1 + \sum_{k=1}^{\frac{q-1}{2}} t^{2k}\lambda^k = \frac{t^{q+1}\lambda^{\frac{q+1}{2}}-1}{t^2\lambda-1}.$$

From this and the identity $\lambda^{\frac{q+1}{2}} = \lambda^{\frac{-q+1}{2}}$ we see that

$$D_{1/q}(\lambda, t) = -\frac{(t^{2q}-1)(t^2-1)}{(t^2-\lambda)(t^2-\lambda^{-1})}.$$

This leads to

$$\begin{aligned}
\prod_{i=1}^{\frac{\ell-1}{2}} D_{1/q}(\lambda^i, t) &= \prod_{i=1}^{\frac{\ell-1}{2}} \frac{-(t^{2q}-1)(t^2-1)}{(t^2-\lambda^i)(t^2-\lambda^{-i})} \\
&= \prod_{i=1}^{\frac{\ell-1}{2}} \frac{-(t^{2q}-1)(t^2-1)}{(t^2-\lambda^i)(t^2-\lambda^{\ell-i})} \\
&= \frac{(-1)^{\frac{\ell-1}{2}}(t^{2q}-1)^{\frac{\ell-1}{2}}(t^2-1)^{\frac{\ell-1}{2}}}{\prod_{i=1}^{\ell-1}(t^2-\lambda^i)}.
\end{aligned}$$

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If ω is a primitive ℓ-th root of unity, then

$$\frac{x^\ell - 1}{x - 1} = x^{\ell-1} + \cdots + x^2 + x + 1 = (x - \omega)(x - \omega^2)\ldots(x - \omega^{\ell-1}). \tag{13}$$

Setting $x = t^2$ and $\omega = \lambda$ leads to

$$\prod_{i=1}^{\ell-1}(t^2 - \lambda^i) = \frac{t^{2\ell} - 1}{t^2 - 1}.$$

Combining these results with Theorem 4 completes the proof. $\qquad\square$

Corollary 13 *Conjecture 1 is true for all 2-bridge, torus knots.*

Proof: Letting

$$f(t) = \frac{(1+t)(1+t^q)^{\frac{\ell-1}{2}}}{1 + t^\ell},$$

the first part of the conjecture is clear from Theorem 12. Furthermore, since ℓ is prime, it follows that $(1+t)^\ell \equiv 1 + t^\ell \pmod{\ell}$. Hence,

$$\begin{aligned}
f(t) &\equiv \frac{(1+t)(1+t^q)^{\frac{\ell-1}{2}}}{(1+t)^\ell} \\
&\equiv \frac{(1+t^q)^{\frac{\ell-1}{2}}}{(1+t)^{\frac{\ell-1}{2}}(1+t)^{\frac{\ell-1}{2}}} \\
&\equiv \left(\frac{(1+t^q)}{(1+t)(1+t)}\right)^{\frac{\ell-1}{2}} \pmod{\ell}.
\end{aligned}$$

Using (3), we obtain

$$\Delta_{K_{1/q}}(t) = 1 - t + t^2 - \cdots + t^{q-1} = \frac{1+t^q}{1+t}.$$

Therefore, these choices of $\Delta_{K_{1/q}}(t)$ and $f(t)$ also satisfy the second part of the conjecture. $\qquad\square$

5.2 2-bridge knots with genus one

In this section we compute the ℓ-twisted Alexander polynomial for all 2-bridge knots with genus one. The 2-bridge knot $K_{p/q}$ has genus one if and only if

$$\frac{p}{q} = \frac{4rs - 2s \pm 1}{4rs \pm 1},$$

for some natural numbers r and s. (See Proposition 12.25 of [1].) We begin by determining the epsilon graph for these fractions. The following lemma follows easily from the facts in Propositions 7 and 8.

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Lemma 14 *Suppose r and s are natural numbers. Then*

1. $\sigma(4rs - 2s - 1, 4rs - 1) = \{[1]_{2r-2}, 2, [1]_{2r-2}, 2, [1]_{2r-2}, 2, \ldots, [1]_{2r-2}, 2, [1]_{2r-2}\}$ and

2. $\sigma(4rs - 2s + 1, 4rs + 1) = \{[1]_{2r-1}, 2, [1]_{2r-2}, 2, [1]_{2r-2}, 2, \ldots, [1]_{2r-2}, 2, [1]_{2r-1}\}.$

where $[n]_k = \underbrace{n, n, \ldots, n}_{k}$ and the number of 2's in each case is $2s - 1$.

Using Theorem 4 and Lemma 14 we are now prepared to find the ℓ-twisted Alexander polynomial of 2-bridge knots with genus one.

Theorem 15 *If $K_{p/q}$ is a 2-bridge knot with genus one and ℓ is an odd prime dividing q, then*

1. $\tilde{\Delta}^{\ell}_{p,q}(t) \doteq \dfrac{\Delta_{K_{p/q}}(t)}{t-1}(1+t)^{\frac{\ell-1}{2}}(1-t)^{\frac{\ell-1}{2}}, \qquad if \quad \dfrac{p}{q} = \dfrac{4rs - 2s - 1}{4rs - 1} \quad and$

2. $\tilde{\Delta}^{\ell}_{p,q}(t) \doteq \dfrac{\Delta_{K_{p/q}}(t)}{t-1}\left(\dfrac{1+t}{t}\right)^{\frac{\ell-1}{2}}\left(\dfrac{1-t}{-t}\right)^{\frac{\ell-1}{2}}, \quad if \quad \dfrac{p}{q} = \dfrac{4rs - 2s + 1}{4rs + 1}.$

Proof: Suppose first that $\frac{p}{q} = \frac{4rs-2s-1}{4rs-1}$. Then from Lemma 14 we have

$$\sigma(p, q) = \{[1]_{2r-2}, 2, [1]_{2r-2}, 2, [1]_{2r-2}, 2, \ldots, [1]_{2r-2}, 2, [1]_{2r-2}\}$$

where the number of isolated 2's is $2s - 1$. Now label the epsilon graph as described at the end of Section 2. Inspection of the vertices on the middle horizontal level of the epsilon graph gives

$$
\begin{aligned}
d_{p/q}(\lambda, t) \ = \ & t\big[(\lambda^{-1} + \cdots + \lambda^{-r+1}) + (\lambda^r + \cdots + \lambda^{2r-1}) + \\
& (\lambda^{-2r} + \cdots + \lambda^{-3r+1}) + (\lambda^{3r} + \cdots + \lambda^{4r-1}) + \ldots \\
& (\lambda^{-2rs+2r} + \cdots + \lambda^{-2rs+r+1}) + (\lambda^{2rs-r} + \cdots + \lambda^{2rs-1})\big].
\end{aligned}
$$

Rewriting this expression, multiplying by $(1 - \lambda)/(1 - \lambda)$, and then simplifying gives

$$
\begin{aligned}
d_{p/q}(\lambda, t) \ = \ & t\big[-1 + (1 + \cdots + \lambda^{-r+1}) + (\lambda^{-2r} + \cdots + \lambda^{-3r+1}) + \cdots + \\
& (\lambda^{-2rs+2r} + \cdots + \lambda^{-2rs+r+1}) + (\lambda^r + \cdots + \lambda^{2r-1}) + \\
& (\lambda^{3r} + \cdots + \lambda^{4r-1}) + \cdots + (\lambda^{2rs-r} + \cdots + \lambda^{2rs-1})\big] \\
= \ & \frac{t}{1 - \lambda}\Big[-(1 - \lambda) + (\lambda^{-r+1} - \lambda) + (\lambda^{-3r+1} - \lambda^{-2r+1}) + \cdots + \\
& (\lambda^{-2rs+r+1} - \lambda^{-2rs+2r+1}) + (\lambda^r - \lambda^{2r}) + (\lambda^{3r} - \lambda^{4r}) + \cdots + (\lambda^{2rs-r} - \lambda^{2rs})\Big].
\end{aligned}
$$

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Next we multiply by $(1 + \lambda^r)/(1 + \lambda^r)$ and use the fact that $\lambda^{4rs-1} = 1$ to obtain

$$
\begin{aligned}
d_{p/q}(\lambda, t) &= \frac{t}{(1-\lambda)(1+\lambda^r)}\left[-(1+\lambda^r)(1-\lambda) + (\lambda^{-2rs+r+1} - \lambda^{r+1}) + (\lambda^r - \lambda^{2rs+r})\right]\\
&= \frac{t}{(1-\lambda)(1+\lambda^r)}\left[-(1+\lambda^r)(1-\lambda) + \lambda^{2rs+r} - \lambda^{r+1} + \lambda^r - \lambda^{2rs+r}\right]\\
&= \frac{t}{(1-\lambda)(1+\lambda^r)}\left[-(1+\lambda^r)(1-\lambda) + \lambda^r(1-\lambda)\right]\\
&= \frac{-t}{1+\lambda^r}.
\end{aligned}
$$

Returning to the labeled epsilon graph, we find that $e_{p/q}(\lambda, t) = \alpha + \beta t^2$, where α and β are Laurent polynomials in λ. Furthermore, using the rotational symmetry of the epsilon graph, we see that $\alpha \lambda^{2rs-1} = \beta$, and so, $\alpha = \beta \lambda^{2rs}$. Reading β off from the graph, we have

$$
\begin{aligned}
\beta &= (\lambda^r + \lambda^{r+1} + \cdots + \lambda^{2r-1}) + (\lambda^{3r} + \lambda^{3r+1} + \cdots + \lambda^{4r-1}) + \cdots +\\
&\quad (\lambda^{2rs-r} + \lambda^{2rs-r+1} + \cdots + \lambda^{2rs-1})\\
&= \frac{1}{1-\lambda}[(\lambda^r - \lambda^{2r}) + (\lambda^{3r} - \lambda^{4r}) + \cdots + (\lambda^{2rs-r} - \lambda^{2rs})]\\
&= \frac{1 - \lambda^{2rs}}{(1-\lambda)(1+\lambda^{-r})}.
\end{aligned}
$$

Hence

$$
e_{p/q}(\lambda, t) = \frac{(\lambda^{2rs} + t^2)(1 - \lambda^{2rs})}{(1-\lambda)(1+\lambda^{-r})}.
$$

Using these formulas, we may now compute $D_{p/q}(\lambda, t)$.

$$
\begin{aligned}
D_{p/q}(\lambda, t) &= \frac{t^2}{(1+\lambda^r)(1+\lambda^{-r})} - \frac{(\lambda^{2rs} + t^2)(\lambda^{-2rs} + t^2)(1 - \lambda^{2rs})(1 - \lambda^{-2rs})}{(1-\lambda)(1-\lambda^{-1})(1+\lambda^r)(1+\lambda^{-r})}\\
&= \frac{(1 - \lambda^{4rs})(1 - \lambda^{-4rs})t^2 - (1 + (\lambda^{2rs} + \lambda^{-2sr})t^2 + t^4)(1 - \lambda^{2sr})(1 - \lambda^{-2sr})}{(1-\lambda)(1-\lambda^{-1})(1+\lambda^r)(1+\lambda^{-r})}\\
&= -\frac{(1 - \lambda^{2sr})(1 - \lambda^{-2sr})(t+1)^2(t-1)^2}{(1-\lambda)(1-\lambda^{-1})(1+\lambda^r)(1+\lambda^{-r})}.
\end{aligned}
$$

Here we have used the fact that $\lambda^{4rs} = \lambda$. Now

$$
\begin{aligned}
\prod_{i=1}^{\frac{\ell-1}{2}} D_{p/q}(\lambda^i, t) &= (-1)^{\frac{\ell-1}{2}}(1+t)^{\ell-1}(1-t)^{\ell-1}\prod_{i=1}^{\frac{\ell-1}{2}}\frac{(1 - \lambda^{2sri})(1 - \lambda^{-2sri})}{(1 - \lambda^i)(1 - \lambda^{-i})(1 + \lambda^{ri})(1 - \lambda^{-ri})}\\
&= (-1)^{\frac{\ell-1}{2}}(1+t)^{\ell-1}(1-t)^{\ell-1}\prod_{i=1}^{\ell-1}\frac{1 - \lambda^{2sri}}{(1 - \lambda^i)(1 + \lambda^{ri})}.
\end{aligned}
$$

Setting $x = 1$ and $x = -1$ in (13), we obtain the following two identities:

$$
\prod_{i=1}^{\ell-1}(1 - \omega^i) = \ell \qquad \text{and} \qquad \prod_{i=1}^{\ell-1}(1 + \omega^i) = 1.
$$

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Since both r and $2rs$ are relatively prime to ℓ, it follows that both λ^r and λ^{2rs} are also primitive. Hence we obtain

$$\prod_{i=1}^{\ell-1} \frac{1 - \lambda^{2sri}}{(1 - \lambda^i)(1 + \lambda^{ri})} = 1.$$

Substituting these results into (10) now gives

$$\begin{aligned}
\tilde{\Delta}_{p/q}^\ell(t) &\doteq \frac{\Delta_{K_{p/q}}(t)}{(t+1)^{\frac{\ell-1}{2}}(t-1)^{\frac{\ell+1}{2}}}(-1)^{\frac{\ell-1}{2}}(1+t)^{\ell-1}(1-t)^{\ell-1} \\
&\doteq \frac{\Delta_{K_{p/q}}(t)}{t-1}(t+1)^{\frac{\ell-1}{2}}(t-1)^{\frac{\ell-1}{2}}.
\end{aligned}$$

Using a similar analysis for the case $\frac{p}{q} = \frac{4rs-2s+1}{4rs+1}$ we find that

$$e_{p/q}(\lambda, t) = \frac{1}{1 + \lambda^r} \quad \text{and} \quad d_{p/q}(\lambda, t) = \frac{(t^{-1} + \lambda^{2rs}t)(1 - \lambda^{2rs})\lambda}{(1 - \lambda)(1 + \lambda^{-r})}.$$

From this we have

$$D_{p/q}(\lambda, t) = \frac{(1 - \lambda^{2sr})(1 - \lambda^{-2sr})(t+1)^2(t-1)^2}{t^2(1 - \lambda)(1 - \lambda^{-1})(1 + \lambda^r)(1 + \lambda^{-r})}$$

which is the determinant in the previous case divided by $-t^2$. This gives the second formula. $\square$

We conclude by demonstrating that the conjecture of Hirasawa and Murasugi is indeed true for 2-bridge knots with genus one.

Corollary 16 *Conjecture 1 is true for all 2-bridge knots with genus one.*

Proof: Letting $f(t) = (1 + t)^{\frac{\ell-1}{2}}$, the first part of the conjecture is clear from Theorem 15. Thus, it remains to show that

$$f(t) \equiv \left(\frac{\Delta_{K_{p/q}}(t)}{1 + t} \right)^{\frac{\ell-1}{2}} \pmod{\ell}.$$

If $\frac{p}{q} = \frac{4rs-2s-1}{4rs-1}$, then it is easy to verify from the epsilon graph that

$$\Delta_{K_{p/q}}(t) = rs - (2rs - 1)t + rst^2.$$

Since ℓ divides $4rs - 1$, we have $4rs \equiv 1 \pmod{\ell}$ and hence

$$\begin{aligned}
\Delta_{K_{p/q}}(t) &\equiv rs + 2rst + rst^2 \\
&\equiv rs(1 + t)^2 \pmod{\ell}.
\end{aligned}$$

Also, by Fermat's Little Theorem, we have $2^{\ell-1} \equiv 1 \pmod{\ell}$; hence

$$\begin{aligned}
1 &\equiv (4rs)^{\frac{\ell-1}{2}} \\
&\equiv 2^{\ell-1}(rs)^{\frac{\ell-1}{2}} \\
&\equiv (rs)^{\frac{\ell-1}{2}} \pmod{\ell}.
\end{aligned}$$

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Therefore,

$$\left(\frac{\Delta_K(t)}{1+t}\right)^{\frac{\ell-1}{2}} \equiv \left(\frac{rs(1+t)^2}{1+t}\right)^{\frac{\ell-1}{2}}$$
$$\equiv (rs)^{\frac{\ell-1}{2}}(1+t)^{\frac{\ell-1}{2}}$$
$$\equiv f(t) \pmod{\ell}.$$

In the case of $\frac{p}{q} = \frac{4rs-2s+1}{4rs+1}$, we have $\Delta_{K_{p/q}}(t) = -rst^{-1} + (2rs+1) - rst$, $f(t) = t^{\frac{1-l}{2}}(t+1)^{\frac{l-1}{2}}$, and the proof is similar. $\qquad\square$

5.3 Recursive Families

If we apply the change in p and q given in Theorem 9 to a genus-one 2-bridge knot, we produce another genus-one 2-bridge knot. Hence we cannot use Theorem 9 to extend the results of Theorem 15. In this sub-section, we show how to use Theorem 9 and Proposition 11 to derive formulae for ℓ-twisted Alexander polynomials of various bi-infinite families of 2-bridge knots. Using these formulae we can confirm Conjecture 1. From a practical point of view, we can do this for any 2-bridge knot with fixed values of p and ℓ. We begin with the case of $p = 5$ and $\ell = 3$.

Given any 2-bridge knot $K_{p/q}$ with $p = 5$ and q divisible by $\ell = 3$, if we reduce q by multiples of $2\ell p = 30$ we arrive at one of the following *root* fractions: 5/9, 5/21, 5/27, or 5/33. Consider first the fraction 5/9. Computing d and e for 5/9 and 5/39, and using the fact that $\lambda^3 = 1$, we obtain

$$d_{5/9}(\lambda,t) = -t^{-1} - t\lambda, \qquad\qquad e_{5/9}(\lambda,t) = -\lambda$$
$$d_{5/39}(\lambda,t) = -t^{-1} - t\lambda + t^3(1+\lambda) - t^5 - t^7\lambda, \qquad e_{5/39}(\lambda,t) = -\lambda - t^2(1+\lambda) - t^4 + t^6\lambda.$$

Propostion 11 now tells us that

$$d_{5/(9+30k)}(\lambda,t) = \alpha_k\, d_{5/39}(\lambda,t) + (1-\alpha_k)\, d_{5/9}(\lambda,t)$$
$$e_{5/(9+30k)}(\lambda,t) = \alpha_k\, e_{5/39}(\lambda,t) + (1-\alpha_k)\, e_{5/9}(\lambda,t)$$

where $\alpha_k = \frac{1-t^{6k}}{1-t^6}$. If we now simplify the determinant $D_{5/(9+30k)}(\lambda,t)$ we obtain

$$D_{5/(9+30k)}(\lambda,t) = \frac{(t-1)^2(t+1)^2\left(t^6\alpha_k - t^3\alpha_k + 1\right)\left(t^6\alpha_k + t^3\alpha_k + 1\right)}{t^2}$$

Thus the twisted Alexander polynomial is

$$\tilde{\Delta}^3_{5/(9+30k)}(t) \doteq \frac{\Delta_{5/(9+30k)}(t)}{t-1} t^4(t-1)(t+1)\left(t^6\alpha_k - t^3\alpha_k + 1\right)\left(t^6\alpha_k + t^3\alpha_k + 1\right).$$

If we let $f(t) = t^2(t+1)(t^6\alpha_k - t^3\alpha_k + 1)$, then the polynomial has the form given in Conjecture 1. To verify the rest of the conjecture we need to show that

$$f(t) \equiv \frac{\Delta_{5/(9+30k)}(t)}{1+t} \pmod{3}.$$

Computing the Alexander polynomials of $K_{5/9}$ and $K_{5/39}$ we obtain

$$
\begin{aligned}
\Delta_{5/9}(t) &= -2t^{-1} + 5 - 2t, \\
\Delta_{5/39}(t) &= -2t^{-1} + 5 - 5t + 5t^2 - 5t^3 + 5t^4 - 5t^5 + 5t^6 - 2t^7.
\end{aligned}
$$

Hence by Proposition 11,

$$
\Delta_{5/(9+30k)}(t) \doteq t^3 \left(\alpha_k(-2t^{-1} + 5 - 5t + 5t^2 - 5t^3 + 5t^4 - 5t^5 + 5t^6 - 2t^7) + (1 - \alpha_k)(-2t^{-1} + 5 - 2t) \right).
$$

It is now straightforward to check that this polynomial, divided by $1 + t$, is congruent modulo 3 to $f(t)$. Since $9 = 1 \cdot 5 + 4$ and $\gcd(1, 3) = 1$, we can combine this result with Theorem 9 to prove the following Theorem.

Theorem 17 *Let k and j be non-negative integers. Then*

$$
\tilde{\Delta}^3_{(5+24j)/(9+30k+144jk+24j)}(t) \doteq \frac{\Delta_K(t)}{t - 1} t^4 (t - 1)(t + 1) \left(t^6 \alpha_k - t^3 \alpha_k + 1 \right) \left(t^6 \alpha_k + t^3 \alpha_k + 1 \right),
$$

where $\alpha_k = \frac{1 - t^{6k}}{1 - t^6}$. Moreover, these knots satisfy Conjecture 1 with $f(t) = t^2(1 + t)(1 - t^3 \alpha_k + t^6 \alpha_k)$.

Continuing with $p = 5$ and $\ell = 3$, there are three more families to consider, namely those with root fractions $5/21$, $5/27$, and $5/33$. However, in the case of $5/33$ we have $33 = 6 \cdot 5 + 3$ and $\gcd(6, 3) \neq 1$. Hence we may apply Propostion 11 but not Theorem 9 to the root fraction $5/33$. For the root fractions $5/21$ and $5/27$, the 3-twisted Alexander polynomial of all the knots in the bi-infinite families indexed by j and k can be computed in a way completely similar to the case of $5/9$ and used to verify Conjecture 1. For the root fraction $5/33$ we obtain a similar result but must assume that $j = 0$. We omit the details and summarize our results in Theorem 18 and the Appendix. We include in the Appendix not only the remaining three families with $p = 5$ and $\ell = 3$, but a sampling of several more families.

Theorem 18 *For each root fraction p/q listed in the Appendix, the ℓ-twisted Alexander polynomial of $K_{p'/q'}$, where*

$$
\frac{p'}{q'} = \frac{p + 2\ell jr}{q + 2\ell(kp + jar + 2\ell jkr)},
$$

for any non-negative integers k and j, is of the form

$$
\tilde{\Delta}^\ell_{p'/q'}(t) \doteq \frac{\Delta_{K_{p'/q'}}(t)}{t - 1} f(t) f(-t)
$$

where $f(t)$ is given in the Appendix and $\alpha_k = \frac{t^{2\ell k} - 1}{t^{2\ell} - 1}$. Moreover

$$
f(t) \equiv \left(\frac{\Delta_{K_{p'/q'}}(t)}{t + 1} \right)^{\frac{\ell - 1}{2}} \pmod{\ell}.
$$

If the root fraction appears in bold-face, then we must assume that $j = 0$.

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Appendix

The following table gives the Laurent polynomial $f(t)$ described in Theorem 18 for given root faction p/q and odd prime ℓ.

p/q	ℓ	$f(t)$
5/9	3	$t^{-1}(t+1)\left(t^6\alpha_k - t^3\alpha_k + 1\right)$
5/21	3	$t(t+1)\left(t^6\alpha_k - t^3\alpha_k + 1\right)$
5/27	3	$-(t+1)\left(t^{10}\alpha_k - t^9\alpha_k - t^8\alpha_k - t^7\alpha_k + t^6\alpha_k + t^5\alpha_k + t^4 - t^3 - t^2 - t + 1\right)$
5/33	3	$t^{-2}(t+1)\left(t^{12}\alpha_k + t^{11}\alpha_k - t^{10}\alpha_k - t^9\alpha_k - t^8\alpha_k + t^7\alpha_k + t^6 + t^5 - t^4 - t^3 - t^2 + t + 1\right)$
7/9	3	$t^{-1}(t+1)\left(t^6\alpha_k + t^5\alpha_k - t^4\alpha_k - t^3\alpha_k - t^2\alpha_k + t\alpha_k + 1\right)$
7/15	3	$(t+1)\left(t^6\alpha_k - t^3\alpha_k + 1\right)$
7/27	3	$(t+1)\left(t^6\alpha_k - t^3\alpha_k + 1\right)$
7/33	3	$-t^{-1}(t+1)\left(t^{10}\alpha_k - t^9\alpha_k - t^8\alpha_k - t^7\alpha_k + t^6\alpha_k + t^5\alpha_k + t^4 - t^3 - t^2 - t + 1\right)$
7/39	3	$-t^{-3}(t+1)\left(t^{12}\alpha_k + t^{11}\alpha_k - t^{10}\alpha_k - 3t^9\alpha_k - t^8\alpha_k + t^7\alpha_k + 2t^6\alpha_k + t^6 + t^5 - t^4 \right.$ $\left. -3t^3 - t^2 + t + 1\right)$
7/45	3	$(t+1)\left(2t^{12}\alpha_k + t^{11}\alpha_k - t^{10}\alpha_k - 3t^9\alpha_k - t^8\alpha_k + t^7\alpha_k + t^6\alpha_k + 2t^6 + t^5 - t^4 - 3t^3 - t^2 \right.$ $\left. +t + 2\right)$
3/5	5	$t^{-2}(t+1)\left(t^{20}\alpha_k^2 - t^{18}\alpha_k^2 + t^{17}\alpha_k^2 - t^{16}\alpha_k^2 - t^{14}\alpha_k^2 + 2t^{13}\alpha_k^2 - 2t^{12}\alpha_k^2 + 2t^{11}\alpha_k^2 - 3t^{10}\alpha_k^2 \right.$ $+2t^9\alpha_k^2 - t^8\alpha_k^2 + t^7\alpha_k^2 - t^6\alpha_k^2 + 2t^5\alpha_k^2 - t^4\alpha_k^2 + 2t^{10}\alpha_k - t^8\alpha_k + t^7\alpha_k$ $\left. -2t^5\alpha_k + t^3\alpha_k - t^2\alpha_k + 1\right)$
3/25	5	$-(t+1)\left(t^{32}\alpha_k^2 - 2t^{31}\alpha_k^2 + t^{30}\alpha_k^2 - t^{29}\alpha_k^2 + t^{28}\alpha_k^2 - 2t^{27}\alpha_k^2 + 3t^{26}\alpha_k^2 - 2t^{25}\alpha_k^2 + 2t^{24}\alpha_k^2 \right.$ $-2t^{23}\alpha_k^2 + t^{22}\alpha_k^2 + t^{20}\alpha_k^2 - t^{19}\alpha_k^2 + t^{18}\alpha_k^2 - t^{16}\alpha_k^2 + 2t^{22}\alpha_k - 4t^{21}\alpha_k + 2t^{20}\alpha_k$ $-2t^{19}\alpha_k + 2t^{18}\alpha_k - 4t^{17}\alpha_k + 6t^{16}\alpha_k - 4t^{15}\alpha_k + 3t^{14}\alpha_k - 3t^{13}\alpha_k + 2t^{12}\alpha_k - 2t^{11}\alpha_k$ $+2t^{10}\alpha_k - t^9\alpha_k + t^8\alpha_k + t^{12} - 2t^{11} + t^{10} - t^9 + t^8 - 2t^7 + 3t^6 - 2t^5 + t^4 - t^3 + t^2 - 2t$ $\left. +1\right)$

p/q	ℓ	$f(t)$

7/15 5 $(t+1)\left(t^{20}\alpha_k^2 - t^{18}\alpha_k^2 + t^{17}\alpha_k^2 - t^{16}\alpha_k^2 - t^{14}\alpha_k^2 + 2t^{13}\alpha_k^2 - 2t^{12}\alpha_k^2 + 2t^{11}\alpha_k^2 - 3t^{10}\alpha_k^2\right.$
$+2t^9\alpha_k^2 - t^8\alpha_k^2 + t^7\alpha_k^2 - t^6\alpha_k^2 + 2t^5\alpha_k^2 - t^4\alpha_k^2 + 2t^{10}\alpha_k - t^8\alpha_k + t^7\alpha_k$
$\left.-2t^5\alpha_k + t^3\alpha_k - t^2\alpha_k + 1\right)$

7/25 5 $t^{-6}(t+1)\left(t^{28}\alpha_k^2 - 2t^{26}\alpha_k^2 + 2t^{24}\alpha_k^2 - t^{23}\alpha_k^2 - t^{22}\alpha_k^2 + 3t^{21}\alpha_k^2 - t^{20}\alpha_k^2 - t^{19}\alpha_k^2\right.$
$-2t^{18}\alpha_k^2 + 2t^{17}\alpha_k^2 + 2t^{15}\alpha_k^2 - 4t^{14}\alpha_k^2 + 3t^{13}\alpha_k^2 - t^{12}\alpha_k^2 - t^{11}\alpha_k^2 - t^{10}\alpha_k^2 + 3t^9\alpha_k^2$
$-t^8\alpha_k^2 + 2t^{18}\alpha_k - 4t^{16}\alpha_k + 5t^{14}\alpha_k - t^{13}\alpha_k - 3t^{12}\alpha_k + 3t^{11}\alpha_k + t^{10}\alpha_k - 4t^9\alpha_k$
$\left.-t^8\alpha_k + 3t^7\alpha_k + t^6\alpha_k - t^5\alpha_k - t^4\alpha_k + t^8 - 2t^6 + 3t^4 - 2t^2 + 1\right)$

7/45 5 $-(t+1)\left(t^{32}\alpha_k^2 - 3t^{31}\alpha_k^2 + t^{30}\alpha_k^2 + t^{29}\alpha_k^2 + t^{28}\alpha_k^2 - 3t^{27}\alpha_k^2 + 4t^{26}\alpha_k^2 - 2t^{25}\alpha_k^2\right.$
$-2t^{23}\alpha_k^2 + 2t^{22}\alpha_k^2 + t^{21}\alpha_k^2 + t^{20}\alpha_k^2 - 3t^{19}\alpha_k^2 + t^{18}\alpha_k^2 + t^{17}\alpha_k^2 - 2t^{16}\alpha_k^2 + 2t^{14}\alpha_k^2$
$-t^{12}\alpha_k^2 + 2t^{22}\alpha_k - 6t^{21}\alpha_k + 2t^{20}\alpha_k + 2t^{19}\alpha_k + 2t^{18}\alpha_k - 6t^{17}\alpha_k + 7t^{16}\alpha_k - 5t^{15}\alpha_k$
$+t^{14}\alpha_k - t^{13}\alpha_k + 3t^{12}\alpha_k - 2t^{11}\alpha_k + 3t^{10}\alpha_k - 3t^9\alpha_k - t^8\alpha_k + t^7\alpha_k + t^6\alpha_k$
$\left.+t^{12} - 3t^{11} + t^{10} + t^9 + t^8 - 3t^7 + 3t^6 - 3t^5 + t^4 + t^3 + t^2 - 3t + 1\right)$

7/55 5 $-t^{-2}(t+1)\left(t^{32}\alpha_k^2 - 2t^{31}\alpha_k^2 + t^{30}\alpha_k^2 - t^{29}\alpha_k^2 + t^{28}\alpha_k^2 - 2t^{27}\alpha_k^2 + 3t^{26}\alpha_k^2 - 2t^{25}\alpha_k^2\right.$
$+2t^{24}\alpha_k^2 - 2t^{23}\alpha_k^2 + t^{22}\alpha_k^2 + t^{20}\alpha_k^2 - t^{19}\alpha_k^2 + t^{18}\alpha_k^2 - t^{16}\alpha_k^2 + 2t^{22}\alpha_k - 4t^{21}\alpha_k$
$+2t^{20}\alpha_k - 2t^{19}\alpha_k + 2t^{18}\alpha_k - 4t^{17}\alpha_k + 6t^{16}\alpha_k - 4t^{15}\alpha_k + 3t^{14}\alpha_k - 3t^{13}\alpha_k + 2t^{12}\alpha_k$
$-2t^{11}\alpha_k + 2t^{10}\alpha_k - t^9\alpha_k + t^8\alpha_k + t^{12} - 2t^{11} + t^{10} - t^9 + t^8 - 2t^7 + 3t^6 - 2t^5 + t^4$
$\left.-t^3 + t^2 - 2t + 1\right)$

7/65 5 $-t^{-2}(t+1)\left(t^{36}\alpha_k^2 - 2t^{33}\alpha_k^2 + t^{32}\alpha_k^2 - 2t^{31}\alpha_k^2 + t^{30}\alpha_k^2 - t^{29}\alpha_k^2 + 3t^{28}\alpha_k^2 - 2t^{27}\alpha_k^2\right.$
$+t^{26}\alpha_k^2 - 2t^{25}\alpha_k^2 + 2t^{24}\alpha_k^2 + t^{22}\alpha_k^2 + t^{20}\alpha_k^2 - t^{19}\alpha_k^2 - t^{18}\alpha_k^2 + 2t^{26}\alpha_k - 4t^{23}\alpha_k$
$+2t^{22}\alpha_k - 4t^{21}\alpha_k + 2t^{20}\alpha_k - 2t^{19}\alpha_k + 6t^{18}\alpha_k - 3t^{17}\alpha_k + 2t^{16}\alpha_k - 4t^{15}\alpha_k + 3t^{14}\alpha_k$
$-2t^{13}\alpha_k + t^{12}\alpha_k + 2t^{10}\alpha_k - t^9\alpha_k + t^{16} - 2t^{13} + t^{12} - 2t^{11} + t^{10} - t^9 + 3t^8 - t^7 + t^6$
$\left.-2t^5 + t^4 - 2t^3 + 1\right)$

7/75 5 $t^{-2}(t+1)\left(t^{40}\alpha_k^2 + t^{39}\alpha_k^2 - t^{38}\alpha_k^2 - t^{36}\alpha_k^2 - 2t^{34}\alpha_k^2 + 2t^{33}\alpha_k^2 - t^{32}\alpha_k^2 + 2t^{31}\alpha_k^2\right.$
$-3t^{30}\alpha_k^2 + t^{29}\alpha_k^2 - t^{28}\alpha_k^2 + 2t^{27}\alpha_k^2 - t^{26}\alpha_k^2 + 2t^{25}\alpha_k^2 - t^{22}\alpha_k^2 + 2t^{30}\alpha_k + 2t^{29}\alpha_k$
$-2t^{28}\alpha_k - 2t^{26}\alpha_k - 4t^{24}\alpha_k + 4t^{23}\alpha_k - 2t^{22}\alpha_k + 4t^{21}\alpha_k - 4t^{20}\alpha_k + 3t^{19}\alpha_k - 2t^{18}\alpha_k$
$+4t^{17}\alpha_k - 3t^{16}\alpha_k + 2t^{15}\alpha_k - t^{14}\alpha_k - 2t^{12}\alpha_k + t^{11}\alpha_k + t^{20} + t^{19} - t^{18} - t^{16} - 2t^{14}$
$\left.+2t^{13} - t^{12} + 2t^{11} - t^{10} + 2t^9 - t^8 + 2t^7 - 2t^6 - t^4 - t^2 + t + 1\right)$

p/q	ℓ	$f(t)$

$3/7$ 7 $(t+1)\left(t^{42}\alpha_k^3 - 2t^{40}\alpha_k^3 + 3t^{39}\alpha_k^3 - 4t^{38}\alpha_k^3 + 5t^{37}\alpha_k^3 - 6t^{36}\alpha_k^3 + 6t^{35}\alpha_k^3 - 8t^{34}\alpha_k^3\right.$

$\qquad +11t^{33}\alpha_k^3 - 13t^{32}\alpha_k^3 + 15t^{31}\alpha_k^3 - 17t^{30}\alpha_k^3 + 19t^{29}\alpha_k^3 - 23t^{28}\alpha_k^3 + 23t^{27}\alpha_k^3 - 21t^{26}\alpha_k^3$

$\qquad +21t^{25}\alpha_k^3 - 21t^{24}\alpha_k^3 + 21t^{23}\alpha_k^3 - 21t^{22}\alpha_k^3 + 23t^{21}\alpha_k^3 - 21t^{20}\alpha_k^3 + 17t^{19}\alpha_k^3 - 15t^{18}\alpha_k^3$

$\qquad +13t^{17}\alpha_k^3 - 11t^{16}\alpha_k^3 + 9t^{15}\alpha_k^3 - 6t^{14}\alpha_k^3 + 5t^{13}\alpha_k^3 - 5t^{12}\alpha_k^3 + 4t^{11}\alpha_k^3 - 3t^{10}\alpha_k^3$

$\qquad +2t^9\alpha_k^3 - t^8\alpha_k^3 - t^7\alpha_k^3 + t^6\alpha_k^3 + 3t^{28}\alpha_k^2 - 4t^{26}\alpha_k^2 + 6t^{25}\alpha_k^2 - 7t^{24}\alpha_k^2$

$\qquad +7t^{23}\alpha_k^2 - 7t^{22}\alpha_k^2 + 4t^{21}\alpha_k^2 - 7t^{20}\alpha_k^2 + 11t^{19}\alpha_k^2 - 13t^{18}\alpha_k^2 + 14t^{17}\alpha_k^2 - 14t^{16}\alpha_k^2$

$\qquad +14t^{15}\alpha_k^2 - 17t^{14}\alpha_k^2 + 14t^{13}\alpha_k^2 - 10t^{12}\alpha_k^2 + 8t^{11}\alpha_k^2 - 7t^{10}\alpha_k^2 + 7t^9\alpha_k^2 - 7t^8\alpha_k^2$

$\qquad +10t^7\alpha_k^2 - 7t^6\alpha_k^2 + 3t^5\alpha_k^2 - t^4\alpha_k^2 + 3t^{14}\alpha_k - 2t^{12}\alpha_k + 3t^{11}\alpha_k - 3t^{10}\alpha_k$

$\qquad \left.+2t^9\alpha_k - 3t^7\alpha_k + 2t^5\alpha_k - 3t^4\alpha_k + 3t^3\alpha_k - 2t^2\alpha_k + 1\right)$

$3/35$ 7 $-t^{-3}(t+1)\left(t^{72}\alpha_k^3 - t^{71}\alpha_k^3 - t^{70}\alpha_k^3 + 2t^{69}\alpha_k^3 - 3t^{68}\alpha_k^3 + 4t^{67}\alpha_k^3 - 5t^{66}\alpha_k^3\right.$

$\qquad +5t^{65}\alpha_k^3 - 6t^{64}\alpha_k^3 + 9t^{63}\alpha_k^3 - 11t^{62}\alpha_k^3 + 13t^{61}\alpha_k^3 - 15t^{60}\alpha_k^3 + 17t^{59}\alpha_k^3 - 21t^{58}\alpha_k^3$

$\qquad +23t^{57}\alpha_k^3 - 21t^{56}\alpha_k^3 + 21t^{55}\alpha_k^3 - 21t^{54}\alpha_k^3 + 21t^{53}\alpha_k^3 - 21t^{52}\alpha_k^3 + 23t^{51}\alpha_k^3 - 23t^{50}\alpha_k^3$

$\qquad +19t^{49}\alpha_k^3 - 17t^{48}\alpha_k^3 + 15t^{47}\alpha_k^3 - 13t^{46}\alpha_k^3 + 11t^{45}\alpha_k^3 - 8t^{44}\alpha_k^3 + 6t^{43}\alpha_k^3 - 6t^{42}\alpha_k^3$

$\qquad +5t^{41}\alpha_k^3 - 4t^{40}\alpha_k^3 + 3t^{39}\alpha_k^3 - 2t^{38}\alpha_k^3 + t^{36}\alpha_k^3 + 3t^{58}\alpha_k^2 - 3t^{57}\alpha_k^2 - 3t^{56}\alpha_k^2$

$\qquad +6t^{55}\alpha_k^2 - 9t^{54}\alpha_k^2 + 12t^{53}\alpha_k^2 - 15t^{52}\alpha_k^2 + 15t^{51}\alpha_k^2 - 18t^{50}\alpha_k^2 + 27t^{49}\alpha_k^2 - 33t^{48}\alpha_k^2$

$\qquad +39t^{47}\alpha_k^2 - 44t^{46}\alpha_k^2 + 48t^{45}\alpha_k^2 - 56t^{44}\alpha_k^2 + 59t^{43}\alpha_k^2 - 56t^{42}\alpha_k^2 + 56t^{41}\alpha_k^2 - 56t^{40}\alpha_k^2$

$\qquad +55t^{39}\alpha_k^2 - 53t^{38}\alpha_k^2 + 55t^{37}\alpha_k^2 - 52t^{36}\alpha_k^2 + 43t^{35}\alpha_k^2 - 37t^{34}\alpha_k^2 + 31t^{33}\alpha_k^2 - 26t^{32}\alpha_k^2$

$\qquad +22t^{31}\alpha_k^2 - 17t^{30}\alpha_k^2 + 14t^{29}\alpha_k^2 - 11t^{28}\alpha_k^2 + 8t^{27}\alpha_k^2 - 5t^{26}\alpha_k^2 + 3t^{25}\alpha_k^2 - 2t^{24}\alpha_k^2$

$\qquad +3t^{44}\alpha_k - 3t^{43}\alpha_k - 3t^{42}\alpha_k + 6t^{41}\alpha_k - 9t^{40}\alpha_k + 12t^{39}\alpha_k - 15t^{38}\alpha_k + 15t^{37}\alpha_k$

$\qquad -18t^{36}\alpha_k + 27t^{35}\alpha_k - 33t^{34}\alpha_k + 39t^{33}\alpha_k - 43t^{32}\alpha_k + 45t^{31}\alpha_k - 49t^{30}\alpha_k + 49t^{29}\alpha_k$

$\qquad -49t^{28}\alpha_k + 49t^{27}\alpha_k - 49t^{26}\alpha_k + 47t^{25}\alpha_k - 43t^{24}\alpha_k + 41t^{23}\alpha_k - 35t^{22}\alpha_k + 29t^{21}\alpha_k$

$\qquad -25t^{20}\alpha_k + 20t^{19}\alpha_k - 16t^{18}\alpha_k + 13t^{17}\alpha_k - 10t^{16}\alpha_k + 7t^{15}\alpha_k - 4t^{14}\alpha_k + 3t^{13}\alpha_k$

$\qquad -t^{12}\alpha_k + t^{30} - t^{29} - t^{28} + 2t^{27} - 3t^{26} + 4t^{25} - 5t^{24} + 5t^{23} - 6t^{22} + 9t^{21} - 11t^{20} + 13t^{19}$

$\qquad -14t^{18} + 14t^{17} - 14t^{16} + 13t^{15} - 14t^{14} + 14t^{13} - 14t^{12} + 13t^{11} - 11t^{10} + 9t^9 - 6t^8 + 5t^7$

$\qquad \left.-5t^6 + 4t^5 - 3t^4 + 2t^3 - t^2 - t + 1\right)$

p/q	ℓ	$f(t)$

$5/7$ 7

$$(t+1)\left(t^{42}\alpha_k^3 + 2t^{41}\alpha_k^3 - 4t^{40}\alpha_k^3 + 2t^{39}\alpha_k^3 - 3t^{38}\alpha_k^3 + 5t^{37}\alpha_k^3 - 6t^{36}\alpha_k^3 + 6t^{35}\alpha_k^3\right.$$
$$-10t^{34}\alpha_k^3 + 13t^{33}\alpha_k^3 - 12t^{32}\alpha_k^3 + 14t^{31}\alpha_k^3 - 17t^{30}\alpha_k^3 + 19t^{29}\alpha_k^3 - 23t^{28}\alpha_k^3 + 19t^{27}\alpha_k^3$$
$$-17t^{26}\alpha_k^3 + 23t^{25}\alpha_k^3 - 23t^{24}\alpha_k^3 + 21t^{23}\alpha_k^3 - 21t^{22}\alpha_k^3 + 23t^{21}\alpha_k^3 - 17t^{20}\alpha_k^3 + 13t^{19}\alpha_k^3$$
$$-17t^{18}\alpha_k^3 + 15t^{17}\alpha_k^3 - 11t^{16}\alpha_k^3 + 9t^{15}\alpha_k^3 - 6t^{14}\alpha_k^3 + 7t^{13}\alpha_k^3 - 7t^{12}\alpha_k^3 + 3t^{11}\alpha_k^3$$
$$-2t^{10}\alpha_k^3 + 2t^9\alpha_k^3 - t^8\alpha_k^3 - t^7\alpha_k^3 - t^6\alpha_k^3 + 2t^5\alpha_k^3 + t^4\alpha_k^3 - t^3\alpha_k^3$$
$$+3t^{28}\alpha_k^2 + 4t^{27}\alpha_k^2 - 7t^{26}\alpha_k^2 + 3t^{25}\alpha_k^2 - 4t^{24}\alpha_k^2 + 6t^{23}\alpha_k^2 - 7t^{22}\alpha_k^2 + 4t^{21}\alpha_k^2$$
$$-11t^{20}\alpha_k^2 + 14t^{19}\alpha_k^2 - 10t^{18}\alpha_k^2 + 11t^{17}\alpha_k^2 - 13t^{16}\alpha_k^2 + 14t^{15}\alpha_k^2 - 17t^{14}\alpha_k^2 + 10t^{13}\alpha_k^2$$
$$-7t^{12}\alpha_k^2 + 11t^{11}\alpha_k^2 - 10t^{10}\alpha_k^2 + 8t^9\alpha_k^2 - 7t^8\alpha_k^2 + 10t^7\alpha_k^2 - 3t^6\alpha_k^2 - 4t^4\alpha_k^2$$
$$+3t^3\alpha_k^2 - t^2\alpha_k^2 + 3t^{14}\alpha_k + 2t^{13}\alpha_k - 3t^{12}\alpha_k + 3t^9\alpha_k - 2t^8\alpha_k - 3t^7\alpha_k$$
$$\left.-2t^6\alpha_k + 3t^5\alpha_k - 3t^2\alpha_k + 2t\alpha_k + 1\right)$$

$5/21$ 7

$$-(t+1)\left(t^{48}\alpha_k^3 - 4t^{47}\alpha_k^3 + 6t^{46}\alpha_k^3 - 7t^{45}\alpha_k^3 + 8t^{44}\alpha_k^3 - 9t^{43}\alpha_k^3 + 10t^{42}\alpha_k^3\right.$$
$$-12t^{41}\alpha_k^3 + 16t^{40}\alpha_k^3 - 19t^{39}\alpha_k^3 + 21t^{38}\alpha_k^3 - 23t^{37}\alpha_k^3 + 25t^{36}\alpha_k^3 - 27t^{35}\alpha_k^3 + 27t^{34}\alpha_k^3$$
$$-23t^{33}\alpha_k^3 + 21t^{32}\alpha_k^3 - 21t^{31}\alpha_k^3 + 21t^{30}\alpha_k^3 - 21t^{29}\alpha_k^3 + 21t^{28}\alpha_k^3 - 19t^{27}\alpha_k^3 + 13t^{26}\alpha_k^3$$
$$-9t^{25}\alpha_k^3 + 7t^{24}\alpha_k^3 - 5t^{23}\alpha_k^3 + 3t^{22}\alpha_k^3 - t^{21}\alpha_k^3 - t^{19}\alpha_k^3 + t^{18}\alpha_k^3 - t^{16}\alpha_k^3$$
$$+2t^{15}\alpha_k^3 - 3t^{14}\alpha_k^3 + 3t^{13}\alpha_k^3 - t^{12}\alpha_k^3 + 3t^{34}\alpha_k^2 - 12t^{33}\alpha_k^2 + 18t^{32}\alpha_k^2 - 21t^{31}\alpha_k^2$$
$$+22t^{30}\alpha_k^2 - 22t^{29}\alpha_k^2 + 21t^{28}\alpha_k^2 - 24t^{27}\alpha_k^2 + 33t^{26}\alpha_k^2 - 39t^{25}\alpha_k^2 + 42t^{24}\alpha_k^2 - 43t^{23}\alpha_k^2$$
$$+43t^{22}\alpha_k^2 - 42t^{21}\alpha_k^2 + 39t^{20}\alpha_k^2 - 30t^{19}\alpha_k^2 + 24t^{18}\alpha_k^2 - 21t^{17}\alpha_k^2 + 20t^{16}\alpha_k^2 - 20t^{15}\alpha_k^2$$
$$+21t^{14}\alpha_k^2 - 18t^{13}\alpha_k^2 + 9t^{12}\alpha_k^2 - 3t^{11}\alpha_k^2 + t^9\alpha_k^2 - t^8\alpha_k^2 + 3t^{20}\alpha_k - 12t^{19}\alpha_k$$
$$+18t^{18}\alpha_k - 21t^{17}\alpha_k + 20t^{16}\alpha_k - 17t^{15}\alpha_k + 12t^{14}\alpha_k - 12t^{13}\alpha_k + 17t^{12}\alpha_k - 20t^{11}\alpha_k$$
$$+21t^{10}\alpha_k - 20t^9\alpha_k + 17t^8\alpha_k - 12t^7\alpha_k + 9t^6\alpha_k - 5t^5\alpha_k + 2t^4\alpha_k + t^6$$
$$\left.-4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 1\right)$$

$5/49$ 7

$$-t^{-3}(t+1)\left(t^{66}\alpha_k^3 - 3t^{65}\alpha_k^3 + 3t^{64}\alpha_k^3 - 2t^{63}\alpha_k^3 + t^{62}\alpha_k^3 - t^{60}\alpha_k^3 + t^{59}\alpha_k^3\right.$$
$$+t^{57}\alpha_k^3 - 3t^{56}\alpha_k^3 + 5t^{55}\alpha_k^3 - 7t^{54}\alpha_k^3 + 9t^{53}\alpha_k^3 - 13t^{52}\alpha_k^3 + 19t^{51}\alpha_k^3 - 21t^{50}\alpha_k^3$$
$$+21t^{49}\alpha_k^3 - 21t^{48}\alpha_k^3 + 21t^{47}\alpha_k^3 - 21t^{46}\alpha_k^3 + 23t^{45}\alpha_k^3 - 27t^{44}\alpha_k^3 + 27t^{43}\alpha_k^3 - 25t^{42}\alpha_k^3$$
$$+23t^{41}\alpha_k^3 - 21t^{40}\alpha_k^3 + 19t^{39}\alpha_k^3 - 16t^{38}\alpha_k^3 + 12t^{37}\alpha_k^3 - 10t^{36}\alpha_k^3 + 9t^{35}\alpha_k^3 - 8t^{34}\alpha_k^3$$
$$+7t^{33}\alpha_k^3 - 6t^{32}\alpha_k^3 + 4t^{31}\alpha_k^3 - t^{30}\alpha_k^3 + 3t^{52}\alpha_k^2 - 9t^{51}\alpha_k^2 + 9t^{50}\alpha_k^2 - 6t^{49}\alpha_k^2$$
$$+3t^{48}\alpha_k^2 - 3t^{46}\alpha_k^2 + 3t^{45}\alpha_k^2 + 3t^{43}\alpha_k^2 - 10t^{42}\alpha_k^2 + 16t^{41}\alpha_k^2 - 21t^{40}\alpha_k^2 + 24t^{39}\alpha_k^2$$
$$-30t^{38}\alpha_k^2 + 39t^{37}\alpha_k^2 - 42t^{36}\alpha_k^2 + 43t^{35}\alpha_k^2 - 43t^{34}\alpha_k^2 + 42t^{33}\alpha_k^2 - 39t^{32}\alpha_k^2 + 39t^{31}\alpha_k^2$$
$$-42t^{30}\alpha_k^2 + 39t^{29}\alpha_k^2 - 32t^{28}\alpha_k^2 + 26t^{27}\alpha_k^2 - 21t^{26}\alpha_k^2 + 18t^{25}\alpha_k^2 - 15t^{24}\alpha_k^2 + 12t^{23}\alpha_k^2$$
$$-9t^{22}\alpha_k^2 + 5t^{21}\alpha_k^2 - 2t^{20}\alpha_k^2 + 3t^{38}\alpha_k - 9t^{37}\alpha_k + 9t^{36}\alpha_k - 6t^{35}\alpha_k + 3t^{34}\alpha_k$$
$$-3t^{32}\alpha_k + 3t^{31}\alpha_k + 3t^{29}\alpha_k - 11t^{28}\alpha_k + 17t^{27}\alpha_k - 21t^{26}\alpha_k + 21t^{25}\alpha_k - 21t^{24}\alpha_k$$
$$+21t^{23}\alpha_k - 21t^{22}\alpha_k + 23t^{21}\alpha_k - 23t^{20}\alpha_k + 21t^{19}\alpha_k - 17t^{18}\alpha_k + 14t^{17}\alpha_k - 12t^{16}\alpha_k$$
$$+9t^{15}\alpha_k - 6t^{14}\alpha_k + 3t^{13}\alpha_k - t^{11}\alpha_k + t^{10}\alpha_k + t^{24} - 3t^{23} + 3t^{22} - 2t^{21} + t^{20}$$
$$-t^{18} + t^{17} + t^{15} - 4t^{14} + 6t^{13} - 7t^{12} + 6t^{11} - 4t^{10} + t^9 + t^7 - t^6 + t^4 - 2t^3 + 3t^2 - 3t$$
$$\left.+1\right)$$

p/q	ℓ	$f(t)$

$5/63$ $\quad 7 \quad$ $t^{-6}\left(t+1\right)\left(t^{78}\alpha_k^3 - t^{77}\alpha_k^3 - 2t^{76}\alpha_k^3 + t^{75}\alpha_k^3 + t^{74}\alpha_k^3 + t^{73}\alpha_k^3 - 2t^{72}\alpha_k^3\right.$

$\qquad + 2t^{71}\alpha_k^3 - 3t^{70}\alpha_k^3 + 7t^{69}\alpha_k^3 - 7t^{68}\alpha_k^3 + 6t^{67}\alpha_k^3 - 9t^{66}\alpha_k^3 + 11t^{65}\alpha_k^3 - 15t^{64}\alpha_k^3$

$\qquad + 17t^{63}\alpha_k^3 - 13t^{62}\alpha_k^3 + 17t^{61}\alpha_k^3 - 23t^{60}\alpha_k^3 + 21t^{59}\alpha_k^3 - 21t^{58}\alpha_k^3 + 23t^{57}\alpha_k^3 - 23t^{56}\alpha_k^3$

$\qquad + 17t^{55}\alpha_k^3 - 19t^{54}\alpha_k^3 + 23t^{53}\alpha_k^3 - 19t^{52}\alpha_k^3 + 17t^{51}\alpha_k^3 - 14t^{50}\alpha_k^3 + 12t^{49}\alpha_k^3 - 13t^{48}\alpha_k^3$

$\qquad + 10t^{47}\alpha_k^3 - 6t^{46}\alpha_k^3 + 6t^{45}\alpha_k^3 - 5t^{44}\alpha_k^3 + 3t^{43}\alpha_k^3 - 2t^{42}\alpha_k^3 + 4t^{41}\alpha_k^3 - 2t^{40}\alpha_k^3$

$\qquad - t^{39}\alpha_k^3 + 3t^{64}\alpha_k^2 - 3t^{63}\alpha_k^2 - 6t^{62}\alpha_k^2 + 3t^{61}\alpha_k^2 + 3t^{60}\alpha_k^2 + 3t^{59}\alpha_k^2 - 6t^{58}\alpha_k^2$

$\qquad + 6t^{57}\alpha_k^2 - 9t^{56}\alpha_k^2 + 21t^{55}\alpha_k^2 - 21t^{54}\alpha_k^2 + 18t^{53}\alpha_k^2 - 27t^{52}\alpha_k^2 + 32t^{51}\alpha_k^2 - 42t^{50}\alpha_k^2$

$\qquad + 47t^{49}\alpha_k^2 - 39t^{48}\alpha_k^2 + 48t^{47}\alpha_k^2 - 59t^{46}\alpha_k^2 + 56t^{45}\alpha_k^2 - 55t^{44}\alpha_k^2 + 59t^{43}\alpha_k^2 - 58t^{42}\alpha_k^2$

$\qquad + 44t^{41}\alpha_k^2 - 47t^{40}\alpha_k^2 + 52t^{39}\alpha_k^2 - 43t^{38}\alpha_k^2 + 38t^{37}\alpha_k^2 - 31t^{36}\alpha_k^2 + 26t^{35}\alpha_k^2 - 25t^{34}\alpha_k^2$

$\qquad + 19t^{33}\alpha_k^2 - 14t^{32}\alpha_k^2 + 11t^{31}\alpha_k^2 - 9t^{30}\alpha_k^2 + 5t^{29}\alpha_k^2 - 3t^{28}\alpha_k^2 + 5t^{27}\alpha_k^2 - 2t^{26}\alpha_k^2$

$\qquad + 3t^{50}\alpha_k - 3t^{49}\alpha_k - 6t^{48}\alpha_k + 3t^{47}\alpha_k + 3t^{46}\alpha_k + 3t^{45}\alpha_k - 6t^{44}\alpha_k + 6t^{43}\alpha_k$

$\qquad - 9t^{42}\alpha_k + 21t^{41}\alpha_k - 21t^{40}\alpha_k + 18t^{39}\alpha_k - 27t^{38}\alpha_k + 31t^{37}\alpha_k - 39t^{36}\alpha_k + 43t^{35}\alpha_k$

$\qquad - 39t^{34}\alpha_k + 45t^{33}\alpha_k - 49t^{32}\alpha_k + 49t^{31}\alpha_k - 47t^{30}\alpha_k + 49t^{29}\alpha_k - 47t^{28}\alpha_k + 37t^{27}\alpha_k$

$\qquad - 37t^{26}\alpha_k + 35t^{25}\alpha_k - 31t^{24}\alpha_k + 28t^{23}\alpha_k - 20t^{22}\alpha_k + 16t^{21}\alpha_k - 14t^{20}\alpha_k + 10t^{19}\alpha_k$

$\qquad - 7t^{18}\alpha_k + 6t^{17}\alpha_k - 6t^{16}\alpha_k + t^{15}\alpha_k + t^{13}\alpha_k + t^{36} - t^{35} - 2t^{34} + t^{33} + t^{32}$

$\qquad + t^{31} - 2t^{30} + 2t^{29} - 3t^{28} + 7t^{27} - 7t^{26} + 6t^{25} - 9t^{24} + 10t^{23} - 12t^{22} + 13t^{21} - 13t^{20}$

$\qquad + 14t^{19} - 13t^{18} + 14t^{17} - 13t^{16} + 13t^{15} - 12t^{14} + 10t^{13} - 9t^{12} + 6t^{11} - 7t^{10} + 7t^9$

$\qquad \left. - 3t^8 + 2t^7 - 2t^6 + t^5 + t^4 + t^3 - 2t^2 - t + 1\right)$