1-1-2007

Intrinsic Linking and Knotting in Virtual Spatial Graphs

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We introduce a notion of intrinsic linking and knotting for virtual spatial graphs. Our theory gives two filtrations of the set of all graphs, allowing us to measure, in a sense, how intrinsically linked or knotted a graph is; we show that these filtrations are descending and nonterminating. We also provide several examples of intrinsically virtually linked and knotted graphs. As a byproduct, we introduce the *virtual unknotting number* of a knot, and show that any knot with nontrivial Jones polynomial has virtual unknotting number at least 2.

05C10; 57M27

1 Introduction

A *spatial graph* is an embedding of a graph G in R^3. Spatial graphs are a natural generalization of knots, which can be viewed as the particular case when G is an n–cycle. Over the past twenty years, there has been considerable work looking at linked and knotted cycles in spatial graphs, beginning with Conway and Gordon's proof that every embedding of K_6 has a pair of linked cycles, and every embedding of K_7 has a knotted cycle [1]. We say that these are examples of *intrinsically linked* and *intrinsically knotted* graphs, respectively. Since Conway and Gordon's paper, Robertson, Seymour and Thomas [12] have classified the intrinsically linked graphs, but the problem of intrinsically knotted graphs is still open, and is an active area of research, along with various variations on these problems (see, for example, Flapan, Foisy, Naimi and Pommersheim [3] and Foisy [5; 6]).

Another recent approach to generalizing knot theory is Kauffman's theory of *virtual knots* [8]. In previous work, the authors combined these ideas to introduce *virtual spatial graphs* [4]. The purpose of this paper is to extend some of the problems and theory of classical spatial graphs to this new realm of virtual spatial graphs. In particular, we will define a notion of *intrinsically virtually linked* graphs of various degrees, show that these induce a descending filtration on the set of graphs, and prove some existence results about the filtration. This gives us a way to talk about "degree" of intrinsic

linking – for example, in this theory K_6 is "more" intrinsically linked than the Petersen graph.

In Section 2 we review the definition of virtual spatial graphs. In Section 3 we introduce the notion of an *n–intrinsically virtually linked (n IVL)* graph and give examples of graphs which are *n* IVL but not $(n + 1)$ IVL for every $n \geq 1$. In Section 4 we take a short detour into virtual knots to define the *virtual unknotting number* and show that every classical knot with nontrivial Jones polynomial has virtual unknotting number at least 2. Finally, in Section 5 we define *n–virtually intrinsically knotted (n IVK)* graphs, show that all known intrinsically knotted graphs are also 1 IVK, and give examples of graphs which are $(2n - 1)$ IVK but not $(2n)$ IVK for every $n \geq 1$.

2 Virtual spatial graphs

In this section, we will briefly review the definition of a virtual spatial graph, from [4]. First, we recall the definition of a classical spatial graph. A *graph* is a pair $G = (V, E)$ of a set of *vertices V* and *edges* $E \subset V \times V$. Unless otherwise stated, our graphs are connected and *directed*, so that each edge is an *ordered* pair of vertices. An embedding of G in $\mathbb{R}^3$ maps the vertices of G to points in $\mathbb{R}^3$ and an edge (u, v) to an arc in $\mathbb{R}^3$ whose endpoints are the images of the vertices u and v, and that is oriented from u to v. We will consider these embeddings modulo equivalence by ambient isotopy. We can always represent such an embedding by projecting it to a plane so that each vertex neighborhood is a collection of rays with one end at the vertex and crossings of edges of the graph are transverse double points in the interior of the edges (as in the usual knot and link diagrams) [7]. An example of such a diagram is shown in Figure 1.
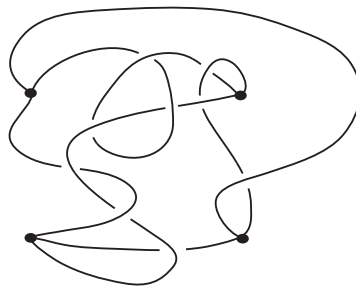

Figure 1: A graph diagram

Kauffman [7] and Yamada [13] have shown that ambient isotopy of spatial graphs is generated by a set of local moves on these diagrams which generalize the Reidemeister moves for knots and links. These Reidemeister moves for graphs are shown in Figure 2. The first five moves (moves (I)–(V)) generate *rigid vertex isotopy*, where the cyclic order of the edges around each vertex is fixed. Moves (I)–(VI) generate *pliable isotopy*, where the order of the vertices around each edge can be changed using move (VI).
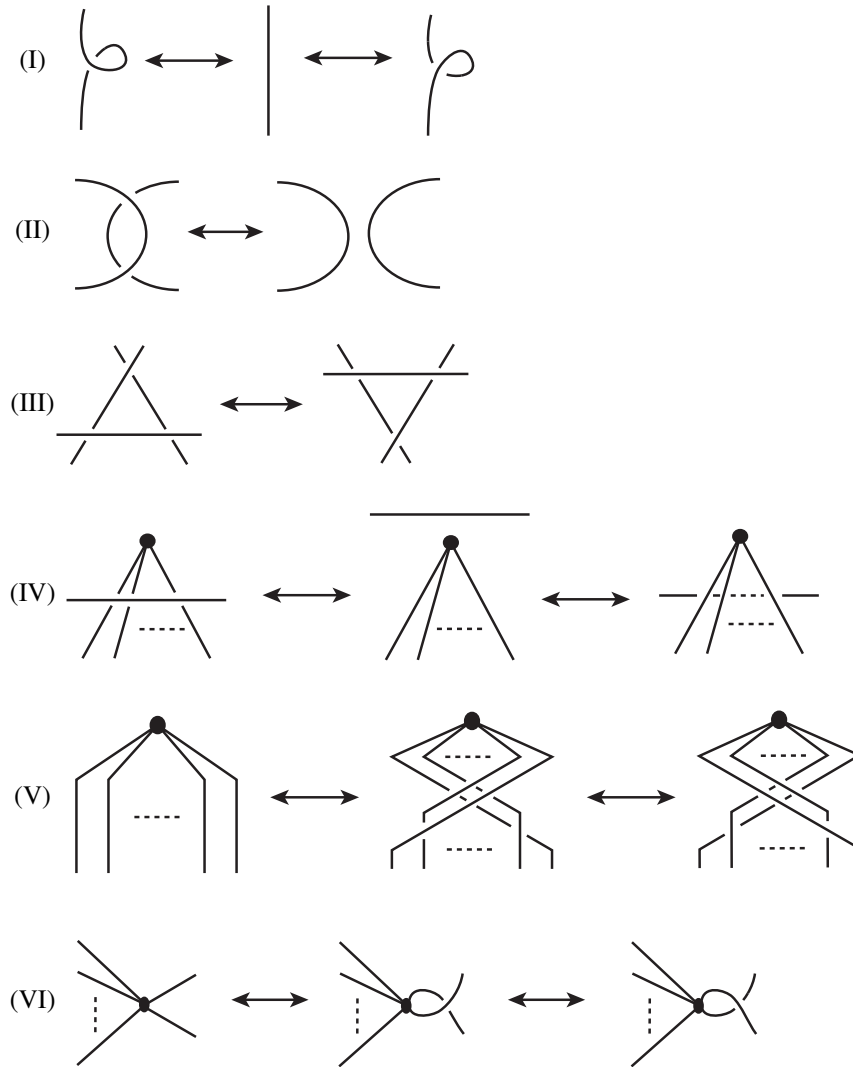

Figure 2: Reidemeister moves for graphs

A *virtual graph diagram* is just like a classical graph diagram, with the addition of *virtual crossings*. We will represent a virtual crossing as an intersection of two edges surrounded by a circle, with no under/over information. So we now have three kinds of crossings: positive and negative classical crossings and virtual crossings (see Figure 3).
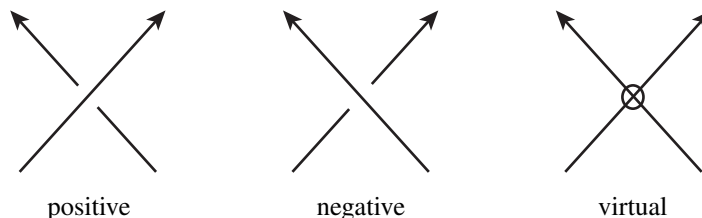

positive   negative   virtual

Figure 3: Types of crossings

The idea is that the virtual crossings are not really there (hence the name "virtual"). To make sense of this, we extend our set of Reidemeister moves for graphs to include moves with virtual crossings. We need to introduce 5 more moves, (I*)–(V*), shown in Figure 4. Notice that moves (I*)–(IV*) are just the purely virtual versions of moves (I)–(IV); move (V*) is the only move which combines classical and virtual crossings (in fact, there are two versions of the move, since the classical crossing may be either positive or negative). We do *not* allow the purely virtual version of move (V), since it changes the cyclic order the edges at a vertex (see our paper [4] for a more detailed discussion).

3   Intrinsically virtually linked graphs

A graph G is *intrinsically linked* if every classical diagram of the graph contains a nonsplit link whose components are disjoint cycles in the graph. Robertson, Seymour and Thomas [12] have shown that every intrinsically linked graph has a minor homeomorphic to a graph in the *Petersen family* of graphs (see Figure 5).

We will look at *virtual* diagrams of the graph G.

Definition 3.1 A graph G is *intrinsically virtually linked of degree n (n IVL)* if every virtual diagram of G with at most n virtual crossings contains a nontrivial virtual link whose components are disjoint cycles in G.

If we let IVL_n denote the set of intrinsically virtually linked graphs of degree n, it is clear that IVL_0 is simply the set of classically intrinsically linked graphs, and that
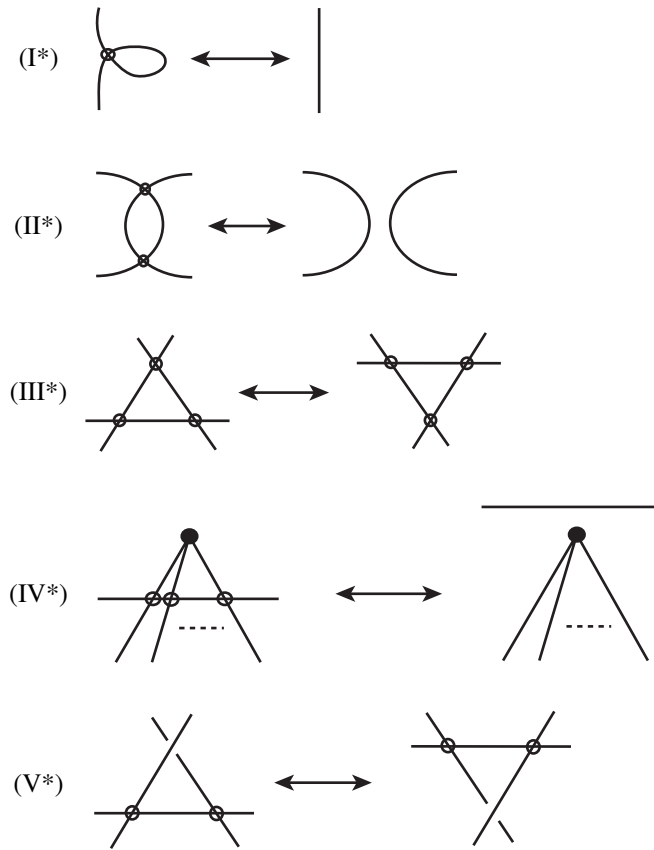
Figure 4: Reidemeister moves for virtual graphs

IVL_{n+1} is a subset of IVL_n. We would like to know whether it is a *proper* subset. We will find that $IVL_0 = IVL_1$, but that for $n \geq 1$, IVL_{n+1} is a proper subset of IVL_n.

We first make the following observation.

Lemma 3.2 *If G is intrinsically linked, then every classical diagram of G contains a pair of linked cycles with odd linking number.*

Proof It is known that every diagram of K_6 and $K_{3,3,1}$ contains a pair of linked cycles with odd linking number [1; 11]. The other graphs in the Petersen family are derived from K_6 and $K_{3,3,1}$ by triangle-Y exchanges, in which a triangle (a cycle in the graph of length 3) is replaced by a Y (the three edges of the triangle are removed, and a new vertex is added adjacent to the three vertices of the triangle). Motwani et al [11] showed that these exchanges preserve the property that every diagram contains a pair of linked
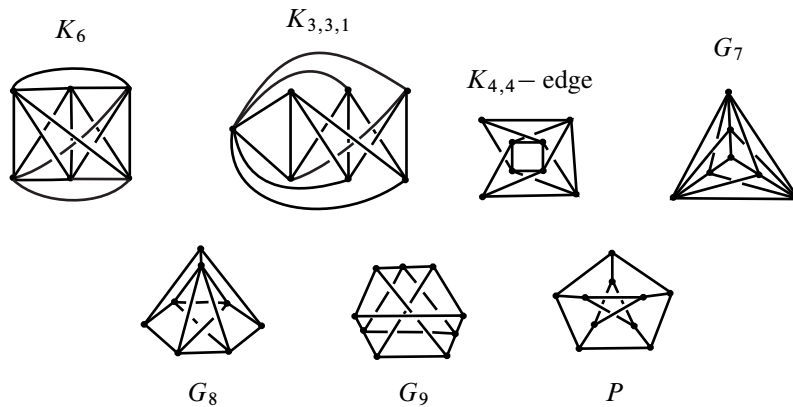
Figure 5: The Petersen family of graphs

cycles with odd linking number, so every graph in the Petersen family has this property. Since G is intrinsically linked, it has a subgraph H which can be transformed into one of the graphs in the Petersen family by edge contractions, by Robertson, Seymour and Thomas [12]. So every diagram of H contains a pair of linked cycles with odd linking number, and hence every diagram of G does as well. $\square$

Theorem 3.3 *If G is intrinsically linked, then G is also intrinsically virtually linked of degree 1, so* $\mathrm{IVL}_0 = \mathrm{IVL}_1$

Proof Since G is intrinsically linked, a diagram of G with no virtual crossings will contain a link. So consider a diagram V of G with exactly one virtual crossing c. Let V_+ be the result of replacing c by a positive classical crossing. By Lemma 3.2, V_+ contains a pair of linked cycles C_1 and C_2 with odd linking number.

If the crossing c does not involve an edge of C_1 and an edge of C_2, then the linking number of C_1 and C_2 is the same in both V_+ and V, so V is linked.

On the other hand, if c *does* involve an edge of C_1 and an edge of C_2, then the virtual linking number of C_1 with C_2 in V will be a half-integer, so V is again linked.

We conclude that every diagram of G with at most one virtual crossing contains a nontrivial link, and hence $G \in \mathrm{IVL}_1$. $\square$

Now that we know that $\mathrm{IVL}_0 = \mathrm{IVL}_1$, is IVL_2 any different? The answer is "Yes." Of the graphs in the Petersen family, G_8, G_9 and the Petersen graph P are in $\mathrm{IVL}_1 \setminus \mathrm{IVL}_2$, while the remaining four are in $\mathrm{IVL}_2 \setminus \mathrm{IVL}_3$. Figure 6 shows unlinked virtual diagrams
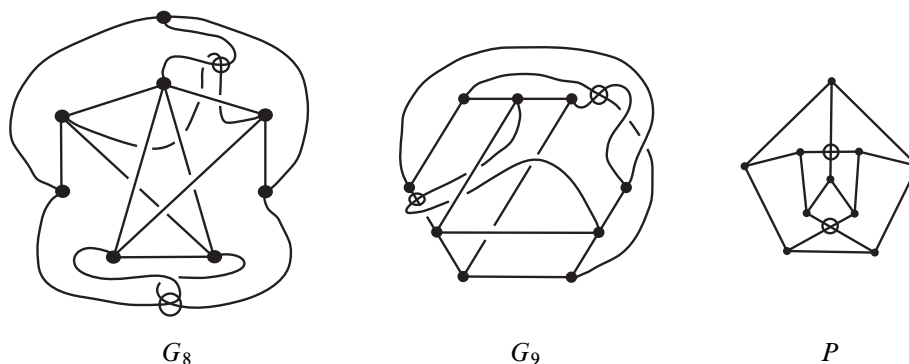
Figure 6: G_8, G_9 and the Petersen graph are not $2\,\mathrm{IVL}$.

of G_8, G_9 and P with two virtual crossings, showing that these graphs are not in IVL_2.

The diagram for the Petersen graph P illustrates an important observation: if a graph has minimal crossing number n, then it is not in IVL_n. This is because we can simply virtualize every crossing in a minimal diagram, so every link will be purely virtual, and hence trivial. The remaining graphs of the Petersen family (K_6, $K_{3,3,1}$, $K_{4,4} - \mathrm{edge}$ and G_7) all have minimal crossing number 3, and so have unlinked diagrams with three virtual crossings, as shown in Figure 7.
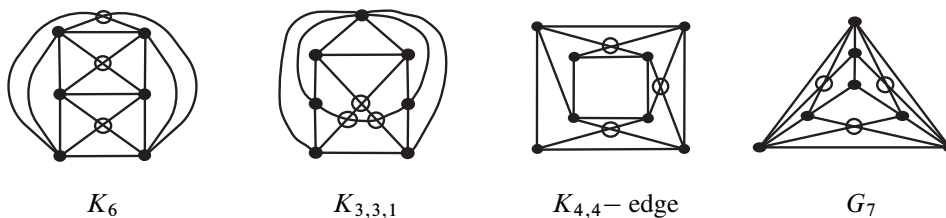
Figure 7: K_6, $K_{3,3,1}$, $K_{4,4} - \mathrm{edge}$ and G_7 are not $3\,\mathrm{IVL}$.

It only remains to show that these four graphs *are* $2\,\mathrm{IVL}$. We begin with K_6.

Proposition 3.4 K_6 *is in* $\mathrm{IVL}_2 \setminus \mathrm{IVL}_3$.

Proof Assume we have a diagram G of K_6 with exactly two virtual crossings c and d. Let G^* be the result of replacing both c and d with classical crossings. Then G^* contains a pair of disjoint cycles C_1 and C_2 with odd linking number. By the argument

in Theorem 3.3, G is linked unless both c and d involve an edge of C_1 and an edge of C_2. So we can assume that both crossings are between nonadjacent pairs of edges (since C_1 and C_2 are disjoint cycles).

We will number the vertices of K_6 from 1 to 6. Without loss of generality, we may assume that crossing c is between edges 1-2 and 4-5. If crossing d does not occur between the same two edges, then either the link composed of triangles (1-2-3, 4-5-6) or the link composed of triangles (1-2-6, 3-4-5) contains only one virtual crossing between the components (since the two links share only the edges 1-2 and 4-5), and hence has a half-integer virtual linking number in G, as in Theorem 3.3.

So we need only consider the case when both c and d are between edges 1-2 and 4-5. In this case, we can replace c and d by classical crossings of opposite sign, so the linking number of any pair of disjoint cycles in G^* is the same as in G. So C_1 and C_2 must also have odd linking number in G, and so G is a virtually linked diagram of K_6. This proves that K_6 is in IVL_2. Combined with Figure 7, we have that K_6 is in $\mathrm{IVL}_2 \setminus \mathrm{IVL}_3$.   □

Proposition 3.5 $K_{3,3,1}$, $K_{4,4}$–edge and G_7 are in $\mathrm{IVL}_2 \setminus \mathrm{IVL}_3$.

Proof   The proofs are similar to Proposition 3.4, and the details are left to the reader. In each case, the proof involves considering a virtual crossing between pairs of nonadjacent edges and showing that, given a diagram with two virtual crossings, either we can find a link with just one virtual crossing, or we can replace the two virtual crossings by classical crossings so as to preserve the sum of the linking numbers in the diagram. This shows that these graphs are in IVL_2, and Figure 7 shows that they are not in IVL_3.   □

We can use the examples of the Petersen graph and K_6 to construct graphs which are $n\,\mathrm{IVL}$ but not $(n+1)\,\mathrm{IVL}$ for all n. Given graphs G_1 and G_2, the *join* of the two graphs, denoted $G_1 \wedge G_2$, is the result of choosing a vertex from each graph and identifying them (for example, see Figure 8).
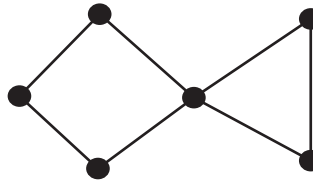

Figure 8: The join of a square and a triangle

We will join copies of the Petersen graph P and K_6. First consider $G = \bigwedge_{i=1}^{n} P$. Since P is in $\text{IVL}_1 \setminus \text{IVL}_2$, an unlinked diagram for G will require 2 virtual crossings in each copy of P, for a total of $2n$ virtual crossings. So G is $(2n-1)\,\text{IVL}$ but not $2n\,\text{IVL}$. Now consider $H = \bigwedge_{i=1}^{n-1} P \wedge K_6$. An unlinked diagram for H requires two virtual crossings in each copy of P and *three* virtual crossings in the copy of K_6, for a total of $2n+1$ virtual crossings. So H is $2n\,\text{IVL}$ but not $(2n+1)\,\text{IVL}$. Together, these constructions prove the following theorem:

Theorem 3.6 *For all $n \geq 1$, the set $\text{IVL}_n \setminus \text{IVL}_{n+1}$ is not empty. So the filtration of the sets IVL_n is strictly decreasing and never terminates.*

4   Virtual unknotting number

In this section we consider the operation of *virtualizing* classical crossings in classical or virtual knots and links, by which we mean replacing the classical crossing by a virtual crossing and leaving the remainder of the diagram unchanged. Since a purely virtual knot or link is trivial [8], virtualizing crossings is an unknotting operation. This naturally leads us to define the *virtual unknotting number*.

Definition 4.1 Given a virtual knot K and a diagram D, define $\text{vu}_D(K)$ to be the minimum number of classical crossings in D which need to be virtualized in order to unknot K. The *virtual unknotting number* $\text{vu}(K)$ is the minimum of $\{\text{vu}_D(K)\}$, taken over all diagrams D for K.

Notice that, in our definition, all virtualizations occur in a single diagram of the knot. Alternatively, we could allow Reidemeister moves to be performed between virtualizations of crossings, changing the diagram, and define the virtual unknotting number as the minimum number of virtualizations needed to unknot the knot by such a process. The following proposition shows that this gives the same virtual unknotting number, so the two definitions are equivalent.

Proposition 4.2 *Say that K and J are diagrams of virtual knots, and that K can be transformed into J by a sequence of Reidemeister moves and crossing virtualizations, involving a total of n crossing virtualizations. Then there are diagrams K' and J' such that K is virtually equivalent to K', J is virtually equivalent to J', and K' can be transformed into J' by virtualizing n crossings (without any intervening Reidemeister moves).*

Proof   Consider the original transformation from K to J. Whenever a crossing is virtualized, instead perform a move of type (II*) to create a half-virtual clasp. We will keep track of this clasp by adding an edge to the diagram as shown in Figure 9 (creating a virtual spatial graph).
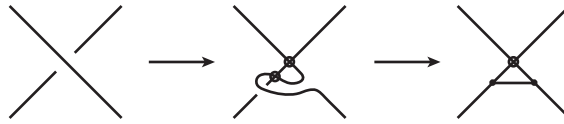

Figure 9: Adding an edge when virtualizing a crossing

Continue the original sequence of Reidemeister moves, only now perform them on the virtual spatial graph, retaining the added edge. We will end with the diagram J, together with n additional edges. Now replace each edge with a half-virtual clasp as shown in Figure 10.
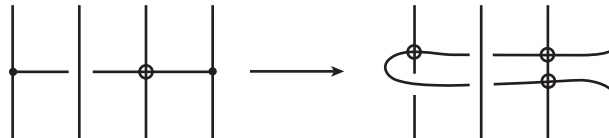

Figure 10: Turning an edge into a half-virtual clasp

The resulting knot diagram K' will be virtually equivalent to K, since all the moves that involved the added edge can be viewed as having involved the two parallel strands of the clasp instead. Virtualizing the classical crossing in each of the n clasps allows the clasp to be removed by a type (II*) Reidemeister move, giving a diagram J' equivalent to J.   □

Remarks

(1) Clearly, the virtual unknotting number is no greater than the crossing number. In fact, it is strictly less than the crossing number, since a virtual knot with exactly one classical crossing is always an unknot.

(2) It is also easy to see that, for a classical knot, the virtual unknotting number is no more than twice the classical unknotting number. This is because, as shown in Figure 11, we can change a single crossing by virtualizing a pair of crossings.

Like the classical unknotting number, the virtual unknotting number is generally very difficult to compute. The main result of this section will be to show that any classical knot with nontrivial Jones polynomial has virtual unknotting number at least 2. This will involve studying how virtualizing a classical crossing changes the Jones polynomial.
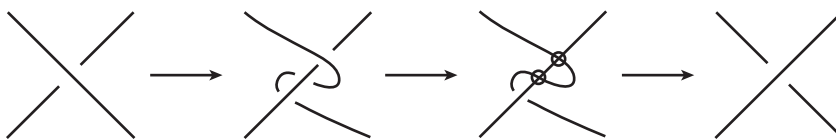
Figure 11: Changing a crossing by virtualizing a pair of crossings

4.1 Virtualizing crossings and the Jones polynomial

Kauffman [8] showed how to extend the bracket polynomial, and hence the Jones polynomial, to virtual knots.

Definition 4.3 The *bracket polynomial* $\langle K \rangle$ for an unoriented virtual knot K is determined by the following relations:

- $\langle K \rangle = A \langle K_a \rangle + A^{-1} \langle K_b \rangle$
- $\langle U_n \rangle = (-A^2 - A^{-2})^{n-1}$

where K, K_a and K_b are as shown in Figure 12, and U_n is the unlink of n components.
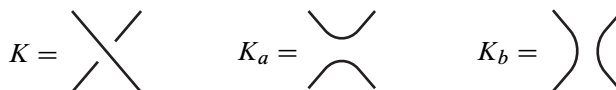

Figure 12: Bracket smoothings

The *writhe* $w(K)$ of an oriented virtual knot is just the sum of the signs of the classical crossings. The Jones polynomial $V_K(t)$ is then defined by:

$$V_K(t) = \left((-A^3)^{-w(K)} \langle K \rangle \right)_{t^{1/2} = A^{-2}}$$

In general, $V_K(t) \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$. However, in classical links with odd numbers of components, including knots, the Jones polynomial has only integer powers of t [10].

Now we will consider how this polynomial changes when we change the sign of a crossing, or virtualize it. It will be useful to also consider the Jones polynomial in the form $f_K(A) = (-A^3)^{-w(K)} \langle K \rangle$ (ie using the variable A instead of t). We will also want to consider the bracket polynomials of the unknots U_+, U_- and U_* and the virtual links H_+ and H_- shown in Figure 13. So U_+ is the unknot with a positive twist, U_- is the unknot with a negative twist, and U_* is the unknot with a virtual twist, while H_+ and H_- are the two virtual Hopf links.
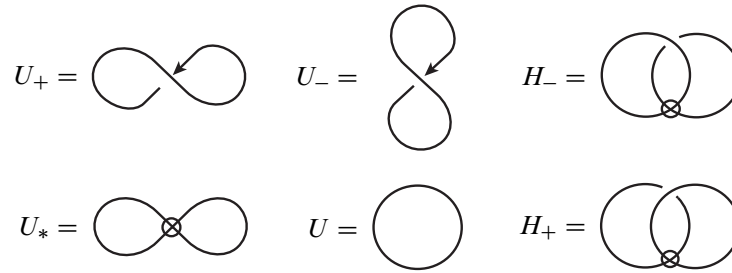
Figure 13: Unknots U_+, U_-, U_* and U, and virtual Hopf links H_+ and H_-

It is an easy exercise to compute $\langle U_+ \rangle = -A^3$, $\langle U_- \rangle = -A^{-3}$, $\langle U_* \rangle = \langle U \rangle = 1$, and $\langle H_+ \rangle = \langle H_- \rangle = A + A^{-1}$. Let K_+ be a virtual knot with a positive crossing c, K_- be the result of changing c to a negative crossing, and K_* be the result of virtualizing c. Now compute $\langle K_+ \rangle$, applying the bracket skein relation to every crossing *except* c. Each state in the resulting state sum consists of some number of unlinked circles together with H_+, U_+ or U_- (the crossing c may have changed its sign, depending on how the arcs were redrawn). If we collect the terms involving H_+, those involving U_+ and those involving U_-, we see that there will be some polynomials $r(A)$, $s(A)$ and $z(A)$ such that:

$$\langle K_+ \rangle = r(A)\langle H_+ \rangle + s(A)\langle U_+ \rangle + z(A)\langle U_- \rangle = (A+A^{-1})r(A) - A^3 s(A) - A^{-3}z(A)$$

Applying the same procedure to K_- and K_*, we get the same results, except that in K_- the roles of U_+ and U_- are switched, and H_+ is replaced by H_-, and in K_*, U_+ and U_- are both replaced by U_*, and H_+ is replaced by the unlink U_2. So,

$$\langle K_- \rangle = r(A)\langle H_- \rangle + s(A)\langle U_- \rangle + z(A)\langle U_+ \rangle = (A+A^{-1})r(A) - A^{-3}s(A) - A^3 z(A)$$

$$\langle K_* \rangle = r(A)\langle U_2 \rangle + s(A)\langle U_* \rangle + z(A)\langle U_* \rangle = (-A^2 - A^{-2})r(A) + s(A) + z(A)$$

Now, let w be the writhe of K_+, so the writhes of K_* and K_- are $w-1$ and $w-2$, respectively. Then we have the following:

$$
\begin{aligned}
f_{K_+}(A) &= (-A)^{-3w}((A+A^{-1})r(A) - A^3 s(A) - A^{-3}z(A)) \\
&= -((-A)^{-3w+1} + (-A)^{-3w-1})r(A) + (-A)^{-3w+3}s(A) + (-A)^{-3w-3}z(A) \\
&= (1+A^{-2})(-(-A)^{-3w+1}r(A)) + ((-A)^{-3w+3}s(A)) + ((-A)^{-3w-3}z(A)) \\
f_{K_*}(A) &= (-A)^{-3w+3}((-A^2-A^{-2})r(A) + s(A) + z(A)) \\
&= -((-A)^{-3w+5} + (-A)^{-3w+1})r(A) + (-A)^{-3w+3}s(A) + (-A)^{-3w+3}z(A) \\
&= (A^4+1)(-(-A)^{-3w+1}r(A)) + ((-A)^{-3w+3}s(A)) + A^6((-A)^{-3w-3}z(A))
\end{aligned}
$$

$$f_{K_-}(A) = (-A)^{-3w+6}((A+A^{-1})r(A) - A^{-3}s(A) - A^3 z(A))$$
$$= -((-A)^{-3w+7} + (-A)^{-3w+5})r(A) + (-A)^{-3w+3}s(A) + (-A)^{-3w+9}z(A)$$
$$= (A^6 + A^4)(-(-A)^{-3w+1}r(A)) + ((-A)^{-3w+3}s(A))$$
$$+ A^{12}((-A)^{-3w-3}z(A))$$

So we have proven the following result (where $t^{1/2} = A^{-2}$):

Theorem 4.4 *Given oriented virtual knots K_+, K_- and K_* which differ only in a single crossing (which is positive, negative and virtual in the three knots), there are polynomials $n(t) = -(-t^{-1/4})^{-3w+1}r(t^{-1/4})$, $p(t) = (-t^{-1/4})^{-3w+3}s(t^{-1/4})$ and $q(t) = (-t^{-1/4})^{-3w-3}z(t^{-1/4})$ in $\mathbb{Z}[t^{-1/2}, t^{1/2}]$ such that:*

$$V_{K_+}(t) = (1 + t^{1/2})n(t) + p(t) + q(t)$$
$$V_{K_*}(t) = (t^{-1} + 1)n(t) + p(t) + t^{-3/2}q(t)$$
$$V_{K_-}(t) = (t^{-3/2} + t^{-1})n(t) + p(t) + t^{-3}q(t)$$

Of particular interest for us is the case when K_+ and K_- are *classical* knots, so K_* has only one virtual crossing. In this case, the state sum for K_+ does *not* involve H_+, so $r(A) = 0$, and hence $n(t) = 0$ in Theorem 4.4. So we get the following:

$$V_{K_+}(t) = p(t) + q(t)$$
$$V_{K_*}(t) = p(t) + t^{-3/2}q(t)$$
$$V_{K_-}(t) = p(t) + t^{-3}q(t)$$

This allows us to write down various skein relations for the Jones polynomial for knots with a single virtual crossing.

Corollary 4.5 *Consider the oriented knots (or links) K_+, K_-, K_* and K_0 which differ only at one crossing, which is positive in K_+, negative in K_-, virtual in K_*, and replaced by the oriented smoothing in K_0, and with all other crossings classical. Then we have the following relations among the Jones polynomials of these links:*

- $V_{K_+} = (t^{3/2} + 1)V_{K_*} - t^{3/2}V_{K_-}$
- $V_{K_-} = (t^{-3/2} + 1)V_{K_*} - t^{-3/2}V_{K_+}$
- $V_{K_*} = (V_{K_+} + t^{3/2}V_{K_-})/(t^{3/2} + 1)$
- $(t^{-1} + t^{-1/2})V_{K_+} - (t + t^{-1/2})V_{K_*} = (t^{1/2} - t^{-1/2})V_{K_0}$
- $(t^{-1} + t^{1/2})V_{K_*} - (t + t^{1/2})V_{K_-} = (t^{1/2} - t^{-1/2})V_{K_0}$

Proof From Theorem 4.4, and remembering that $n(t) = 0$, we have that:

$$V_{K_+} - V_{K_-} = (1 - t^{-3})q(t) = (1 + t^{-3/2})(V_{K_+} - V_{K_*})$$

The first three relationships come from solving this equation for V_{K_+}, V_{K_-} and V_{K_*}. The last two relationships come from combining the first three with the well-known skein relation for the Jones polynomial: $t^{-1}V_{K_+} - tV_{K_-} = (t^{1/2} - t^{-1/2})V_{K_0}$. $\square$

Our next corollary uses the following fact:

Lemma 4.6 *If K_+ and K_- are classical knots which differ in a single crossing, and $p(t)$ and $q(t)$ are polynomials such that $V_{K_+}(t) = p(t) + q(t)$ and $V_{K_-}(t) = p(t) + t^{-3}q(t)$ (as in Theorem 4.4), then $p(t)$, $q(t) \in \mathbb{Z}[t^{-1}, t]$ (so all powers of t in $p(t)$ and $q(t)$ are integers).*

Proof Since K_+ and K_- are classical knots, we know that their Jones polynomials have only integral powers of t [10]. Say that $p(t) = p_1(t) + t^{1/2}p_2(t)$ and $q(t) = q_1(t) + t^{1/2}q_2(t)$, where $p_i(t)$, $q_i(t) \in \mathbb{Z}[t^{-1}, t]$. Then $t^{1/2}p_2(t) + t^{1/2}q_2(t) = 0$ and $t^{1/2}p_2(t) + t^{1/2}t^{-3}q_2(t) = 0$ (so that V_{K_+} and V_{K_-} are left with only integer powers of t). This means that $p_2(t) = -q_2(t) = -t^{-3}q_2(t)$. But this is only possible if $q_2(t) = 0$, and hence $p_2(t) = 0$. We conclude that $p(t)$, $q(t) \in \mathbb{Z}[t^{-1}, t]$. $\square$

Corollary 4.7 *If K_+ and K_- are classical knots which differ in a single crossing, K_* is the result of virtualizing that crossing, and $V_{K_+}(t) \neq V_{K_-}(t)$, then K_* is a nonclassical (and hence nontrivial) virtual knot.*

Proof From Theorem 4.4, setting $n(t) = 0$, we have that $V_{K_+}(t) = p(t) + q(t)$, $V_{K_-}(t) = p(t) + t^{-3}q(t)$ and $V_{K_*}(t) = p(t) + t^{-3/2}q(t)$. By Lemma 4.6, we have $p(t)$, $q(t) \in \mathbb{Z}[t^{-1}, t]$. Moreover, since $V_{K_+}(t) \neq V_{K_-}(t)$, $q(t) \neq 0$. Therefore, $V_{K_*}(t) = p(t) + t^{-3/2}q(t) \notin \mathbb{Z}[t^{-1}, t]$, which implies that K_* is not a classical knot. $\square$

Remark Dye and Kauffman [2] have used similar techniques to look at the effect on the Jones polynomial of a different way of "virtualizing" a crossing.

4.2 Results on the virtual unknotting number

Our main result in this section is the following theorem:

Theorem 4.8 *If K is a classical knot with $vu(K) = 1$, then $V_K(t) = 1$.*

Proof We proceed by contradiction. Assume $\mathrm{vu}(K) = 1$, but $V_K(t) \neq 1$. Since K is a knot, the powers of t in $V_K(t)$ are all integers. Since $\mathrm{vu}(K) = 1$, we can virtualize some crossing c to get a trivial knot K_*, so $V_{K_*}(t) = 1$. We will consider the case when c is a positive crossing in K (the case when c is negative is similar).

By Theorem 4.4, with $n(t) = 0$, $V_K - V_{K_*} = (1 - t^{-3/2})q(t) = V_K - 1$, so $t^{-3/2}q(t) = q(t) + 1 - V_K$. Since $V_K \neq 1$, $q(t) \neq 0$. By Lemma 4.6, $q(t)$ contains only integer powers of t. But then $q(t) + 1 - V_K$ contains only integer powers of t, while $t^{-3/2}q(t)$ contains *no* integer powers of t, so these cannot be equal. This is the desired contradiction. □

Conjecture 4.9 *If K is a nontrivial classical knot, then* $\mathrm{vu}(K) > 1$.

A counterexample to this conjecture would also be an example of a nontrivial knot with trivial Jones polynomial.

As an application of Theorem 4.8, consider the twist knots K_n, shown in Figure 14. This class of knots includes the trefoil knot as K_0.
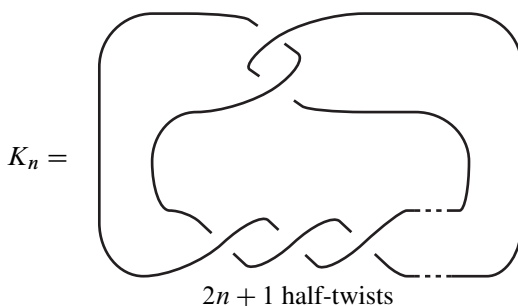
$$K_n =$$

2n + 1 half-twists

Figure 14: The twist knot K_n

Using the skein relation $t^{-1}V_{K_+} - tV_{K_-} = (t^{1/2} - t^{-1/2})V_{K_0}$, a straightforward induction shows that $V_{K_n} \neq 1$, so the virtual unknotting number of a twist knot is at least 2. On the other hand, it is clear that virtualizing the two crossings in the clasp will trivialize K_n, so the virtual unknotting number must be exactly 2.

Theorem 4.10 *The virtual unknotting number of the twist knot K_n is 2.*

5   Intrinsically virtually knotted graphs

The idea of an *intrinsically virtually knotted* graph is completely analogous to the idea of an intrinsically virtually linked graph in Section 3.

Definition 5.1   A graph G is *intrinsically virtually knotted of degree n (n IVK)* if every virtual diagram of G with *at most n* virtual crossings contains a virtually knotted cycle.

As before, we denote the set of n IVK graphs by IVK_n. So IVK_0 is the set of (classically) intrinsically knotted graphs, and $IVK_{n+1} \subset IVK_n$, so we have a filtration of sets. In general, intrinsic knotting is much more difficult to work with than intrinsic linking, and even IVK_0 is not yet fully understood. Our results in this section are not as strong as in Section 3, and even those we have require the machinery developed in Section 4. In particular, Theorem 4.8 allows us to prove:

Theorem 5.2   *If G is an intrinsically knotted graph such that every diagram of G contains a knot with nontrivial Jones polynomial, then G is intrinsically virtually knotted of degree 1.*

Proof   Consider a diagram D of G with exactly one virtual crossing. If we replace the virtual crossing with a positive crossing, we get a classical diagram D_+ of G. So D_+ contains a knotted cycle C with nontrivial Jones polynomial. By Theorem 4.8, the virtual unknotting number of C is greater than 1.

If the virtual crossing in D is not in the cycle C, then D contains C as a knotted cycle. On the other hand, if the virtual crossing *is* in C, then the resulting virtually knotted cycle is still nontrivial, since $vu(C) > 1$. In either case, D contains a virtually knotted cycle, and we conclude that G is intrinsically virtually knotted of degree 1.   □

It is conjectured that the Jones polynomial distinguishes the unknot, which leads us to conjecture:

Conjecture 5.3   *If a graph G is intrinsically knotted, then G is intrinsically virtually knotted of degree 1.*

While we cannot prove this, we can verify that it holds for all known examples of intrinsically knotted graphs.

Theorem 5.4   *The complete graph on 7 vertices, K_7, is intrinsically virtually knotted of degree 1.*

Proof Conway and Gordon [1] showed that every classical diagram of K_7 contains a knotted Hamiltonian cycle with Arf invariant 1, so $K_7 \in \mathrm{IVK}_0$. However, there is a well-known connection between the Arf invariant of a knot K and its Jones polynomial V_K: $V_K(\sqrt{-1}) = (-1)^{\mathrm{Arf}(K)}$ [10]. Since $\mathrm{Arf}(C) = 1$, V_C must be nontrivial. So, by Theorem 5.2, K_7 is intrinsically virtually knotted of degree 1. □

The same argument shows that $K_{3,3,1,1}$ and Foisy's graph H are also intrinsically virtually knotted of degree 1, since the intrinsic knottedness of both graphs was proved by Foisy using the Arf invariant [5; 6]. The other known intrinsically knotted graphs are derived from K_7 and $K_{3,3,1,1}$ using triangle-Y exchanges, in which a triangle (a cycle in the graph of length 3) is replaced by a Y (the three edges of the triangle are removed, and a new vertex is added adjacent to the three vertices of the triangle). Motwani et al [11] showed that these exchanges preserve intrinsic linking, intrinsic knotting, and many other graph properties. Their proof easily generalizes to show that if every diagram of a graph G contains a knotted cycle with nontrivial Jones polynomial, and G' is the result of a triangle-Y exchange, then every diagram of G' will also contain a knotted cycle with nontrivial Jones polynomial. So every graph which can be derived from $K_{3,3,1,1}$ and K_7 by triangle-Y exchanges is also intrinsically virtually linked of degree 1. Since these are all the known minor-minimal intrinsically knotted graphs, we have verified Conjecture 5.3 for all known intrinsically knotted graphs.

In particular, this means the graph C_{14} shown in Figure 15 is intrinsically virtually knotted of degree 1, since it is one of the 13 graphs which can be obtained from K_7 by triangle-Y exchanges (it is one of the two such graphs which are triangle-free) [9].
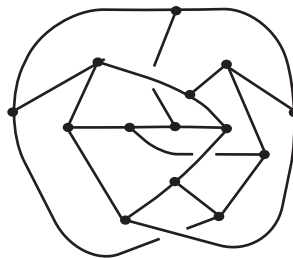

Figure 15: The graph C_{14}

Since the diagram of C_{14} shown in Figure 15 has only three crossings, virtualizing two of them will mean that the diagram of any cycle contains at most one classical crossing, and is therefore unknotted (the cycle is virtually isotopic to either U_+ or U_- of Figure 13). So C_{14} is *not* intrinsically virtually knotted of degree 2. This allows us to prove that the filtration of intrinsically virtually knotted graphs never terminates, similarly to the proof of Theorem 3.6.

Theorem 5.5 *For all $n \geq 1$, the set $\mathrm{IVK}_{2n-1} \setminus \mathrm{IVK}_{2n}$ is not empty.*

Proof Let $G = \bigwedge_{i=1}^{n} C_{14}$. Then an unknotted diagram of G requires two virtual crossings in each copy of C_{14}. We conclude that $G \in \mathrm{IVK}_{2n-1} \setminus \mathrm{IVK}_{2n}$. $\qquad\square$

It is still open whether $\mathrm{IVK}_{2n} \setminus \mathrm{IVK}_{2n+1}$ is likewise nonempty, though it seems likely. We would need to find a graph which is intrinsically virtually knotted of degree 2, but not of degree 3 (analogous to K_6 in the case of intrinsic virtual linking). One candidate may be the graph C_{13} shown in Figure 16.
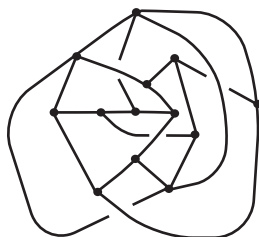

Figure 16: The graph C_{13}

Since C_{13} is derived from K_7 by triangle-Y exchanges, it is $1\,\mathrm{IVK}$; but since the diagram shown in Figure 16 has only four crossings, it is *not* $3\,\mathrm{IVK}$. It is not known, however, whether it is $2\,\mathrm{IVK}$.

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