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The Intersection Graph Conjecture for Loop Diagrams

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THE INTERSECTION GRAPH CONJECTURE FOR LOOP DIAGRAMS

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ABSTRACT

The study of Vassiliev invariants for knots can be reduced to the study of the algebra of chord diagrams modulo certain relations (as done by Bar-Natan). Chmutov, Duzhin and Lando defined the idea of the intersection graph of a chord diagram, and conjectured that these graphs determine the equivalence class of the chord diagrams. They proved this conjecture in the case when the intersection graph is a tree. This paper extends their proof to the case when the graph contains a single loop, and determines the size of the subalgebra generated by the associated "loop diagrams." While the conjecture is known to be false in general, the extent to which it fails is still unclear, and this result helps to answer that question.

1   Introduction

In 1990, V.A. Vassiliev introduced the idea of *Vassiliev* or *finite type* knot invariants, by looking at certain groups associated with the cohomology of the space of knots. Shortly thereafter, Birman and Lin [2] gave a combinatorial description of finite type invariants. We will give a very brief overview of this combinatorial theory. For more details, see Bar-Natan [2] and Chmutov, Duzhin and Lando [4].

We first note that we can extend any knot invariant to an invariant of *singular* knots, where a singular knot is an immersion of S^1 in 3-space which is an embedding except for a finite number of isolated double points. Given a knot invariant v, we extend it via the relation:

An invariant v of singular knots is then said to be of *finite type*, if there is an integer n such that v is zero on any knot with more than n double points. v is then said to be of *type n*. The smallest such n is called the *order* of v. We denote by V_n

the space generated by finite type invariants of type n (i.e., whose order is $\leq n$). We can completely understand the space of finite type invariants by understanding all of the vector spaces V_n/V_{n-1}. An element of this vector space is completely determined by its behavior on knots with exactly n singular points. In addition, since such an element is zero on knots with more than n singular points, any other (non-singular) crossing of the knot can be changed without affecting the value of the invariant. This means that elements of V_n/V_{n-1} can be viewed as functionals on the space of *chord diagrams*:

Definition 1 *A* **chord diagram of degree n** *is an oriented circle, together with n chords of the circles, such that all of the $2n$ endpoints of the chords are distinct. The circle represents a knot, the endpoints of a chord represent 2 points identified by the immersion of this knot into 3-space. The diagram is determined by the order of the $2n$ endpoints.*

Note that since rotating a chord diagram does not change the order of the endpoints of the chords, it does not change the diagram. In particular, turning a diagram "upside down" by rotating it 180 degrees gives the same diagram. We will use this fact later in the paper.

Functionals on the space of chord diagrams which are derived from knot invariants will satisfy certain relations. This leads us to the definition of a *weight system*:

Definition 2 *A* **weight system of degree n** *is a linear functional W on the space of chord diagrams of degree n (with values in an associative commutative ring $\mathbf{K}$ with unity) which satisfies 2 relations:*

- *(1-term relation)*
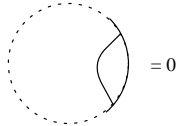
- *(4-term relation)*
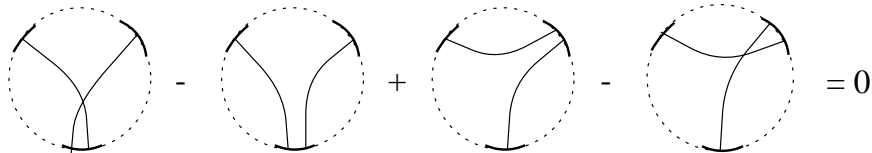
   Outside of the solid arcs on the circle, the diagrams can be anything, as long as it is the same for all four diagrams.

It can be shown (see [1, 2, 13]) that the space W_n of weight systems of degree n is isomorphic to V_n/V_{n-1}. For convenience, we will usually take the dual approach, and simply study the space of chord diagrams of degree n modulo the 1-term and 4-term relations. Bar-Natan [2] and Kneissler [9] have computed the dimension of these spaces for $n \leq 12$. It is useful to combine all of these spaces into a graded module via direct sum. We can give this module a Hopf algebra structure by defining an appropriate product and co-product:

- We define the product $D_1 \cdot D_2$ of two chord diagrams D_1 and D_2 as their connect sum. This is well-defined modulo the 4-term relation (see [2]).
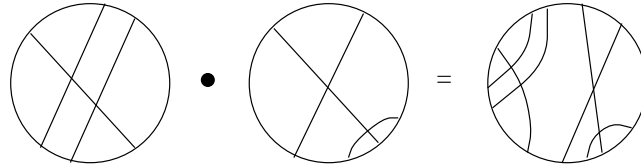
- We define the co-product $\Delta(D)$ of a chord diagram D as follows:

$$\Delta(D) = \sum_J D'_J \otimes D''_J$$

where J is a subset of the set of chords of D, D'_J is D with all the chords in J removed, and D''_J is D with all the chords not in J removed.


It is easy to check the compatibility condition $\Delta(D_1 \cdot D_2) = \Delta(D_1) \cdot \Delta(D_2)$. The rest of this paper is concerned with studying parts of this Hopf algebra.

It is often useful to consider the Hopf algebra of bounded unitrivalent diagrams, rather than chord diagrams. These diagrams, introduced by Bar-Natan [2] (there called *Chinese Character Diagrams*), can be thought of as a shorthand for writing certain linear combinations of chord diagrams. We define a *bounded unitrivalent diagram* to be a unitrivalent graph, with oriented vertices, together with a bounding circle to which all the univalent vertices are attached. We also require that each component of the graph have at least one univalent vertex (so every component is connected to the boundary circle). We define the space A of bounded unitrivalent diagrams as the quotient of the space of all bounded unitrivalent graphs by the *STU* relation:
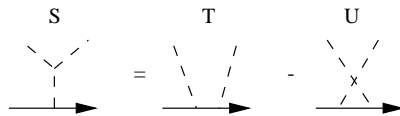
As consequences of the *STU* relation, the anti-symmetry (AS) and IHX relations also hold in A:
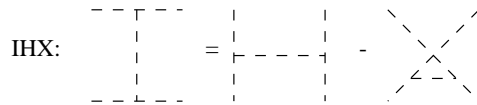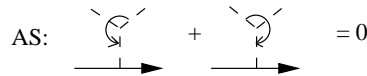
Bar-Natan shows that A is isomorphic to the algebra of chord diagrams.
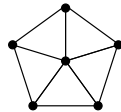
2 Intersection Graphs

Definition 3 *Given a chord diagram D, we define its* **intersection graph** $\Gamma(D)$ *as the graph such that:*

- $\Gamma(D)$ *has a vertex for each chord of D.*

- *Two vertices of* $\Gamma(D)$ *are connected by an edge if and only if the corresponding chords in D intersect, i.e. their endpoints on the bounding circle alternate.*
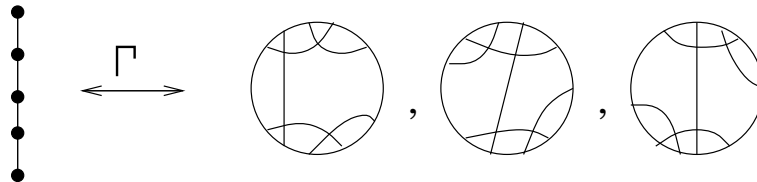
For example:
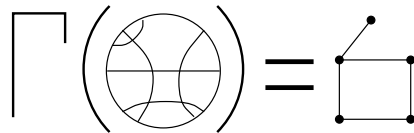
Not all graphs can be the intersection graph for a chord diagram. The simplest example of a graph which cannot be an intersection graph occurs with 6 vertices:

Also, a graph can be the intersection graph for more than one chord diagram. For example, there are three different chord diagrams with the following intersection graph:

However, these chord diagrams are all equivalent modulo the 4-term relation. Chmutov, Duzhin and Lando [4] conjectured that intersection graphs actually determine the chord diagram, up to the 4-term relation. In other words, they proposed:

Conjecture 1 *If D_1 and D_2 are two chord diagrams with the same intersection graph, i.e.* $\Gamma(D_1) = \Gamma(D_2)$*, then for any weight system W, $W(D_1) = W(D_2)$.*

This Intersection Graph Conjecture is now known to be false in general. Morton and Cromwell [12] found a finite type invariant of type 11 which can distinguish some mutant knots, and Le [10] and Chmutov and Duzhin [3] have shown that mutant knots cannot be distinguished by intersection graphs. However, the conjecture is true in many special cases, and the exact extent to which it fails is still unknown, and potentially very interesting.

The conjecture is known to hold in the following cases:

[share.pic]

Figure 1: Share marked by dashed line

- For chord diagrams with 8 or fewer chords (checked via computer calculations);

- For the weight systems coming from the defining representations of Lie algebras *gl(N)* or *so(N)* as constructed by Bar-Natan in [2];

- When $\Gamma(D_1) = \Gamma(D_2)$ is a tree (or, more generally, a linear combination of forests) (see [4]).

The main result of this paper is to add one more case to the above list; namely, when the intersection graph contains a single loop:

Theorem 1 *If D_1 and D_2 are two chord diagrams such that $\Gamma(D_1) = \Gamma(D_2) = \Gamma$, and Γ has at most one loop, i.e. $\pi_1(\Gamma) = \mathbf{Z}$ or 0, then for any weight system W, $W(D_1) = W(D_2)$.*

Our proof of this fact will closely follow the arguments of Chmutov et. al. [5], so we will begin by recalling some definitions and results from that article, and then generalize them to prove our result.

3 Shares, Elementary Transformations, and Tree Diagrams

We begin with the important idea of a *share* of a chord diagram:

Definition 4 *Let D be a chord diagram, and C its collection of chords. A **share** is a subset $S \subset C$ such that there exist four points x_1, x_2, x_3, x_4 in order along the circle so that:*

- *x_i is not an endpoint of a chord;*

- *For any chord $c \in S$, the endpoints of c are in the arcs $x_1 x_2$ and $x_3 x_4$;*

- *For any chord $c \in C - S$, the endpoints of c are in the arcs $x_2 x_3$ and $x_4 x_1$.*

In other words, we can divide the circle into 4 arcs so that no chord connects adjacent arcs.

For example, in Figure 1 the chords contained in the dotted region form a share. But the three thick chords on their own do **not** form a share, because there is a chord which separates their endpoints.

Note that any single chord is a share, and the complement $C - S$ of a share S is also a share. There is one more important case:

Lemma 1 *Let D be a chord diagram with $\Gamma(D)$ connected. Suppose D has a distinguished chord t (called the **trunk**), and $v(t)$ is the vertex of $\Gamma(D)$ corresponding to t. Then the chords of D corresponding to the vertices in a single component of $\Gamma(D) - v(t)$ form a share. Such a share is called a **bough** of t.*

PROOF: Let S be a set of chords corresponding to a single component of $\Gamma(D)-v(t)$. Then no chord of $S^c = \Gamma(D) - \{S \cup v(t)\}$ intersects any chord of S. In addition, since S is connected, chords of S cannot be separated by chords of S^c; otherwise, a chord of S^c would have to intersect some chord on a path in S connecting two chords of S. Now we can choose the minimal arcs x_1x_2 and x_3x_4 containing all the endpoints of S. There will be two arcs, because t intersects at least one chord of S, so the arcs must be separated by the endpoints of t. Therefore, there is a chord of S which connects the two arcs (namely, the one intersecting t). t will have one endpoint in x_2x_3 and one in x_4x_1, while all arcs of S^c will have both their endpoints in one of these two arcs. Therefore, S is a share, and we are done. $\square$

Now we will consider a particular type of chord diagram called a *tree diagram*:

Definition 5 *A* **tree diagram** *is a chord diagram* D *whose intersection graph* $\Gamma(D)$ *is a tree.*

The first question we ask is: When do two tree diagrams have the same intersection graph? It is clear that we can permute the order of boughs along some trunk of D without changing the intersection graph. For example:
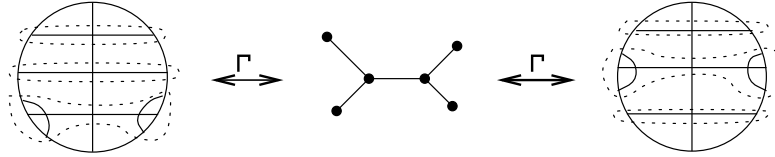
In these diagrams the boughs correspond to the dotted regions. We permute the second and third boughs along the trunk.

We call such a permutation of boughs an *elementary transformation*. Chmutov, Duzhin and Lando [5] prove the following propositions:

Proposition 1 *If* D_1 *and* D_2 *are tree diagrams such that* $\Gamma(D_1) = \Gamma(D_2)$, *then* D_1 *can be transformed into* D_2 *by a sequence of elementary transformations.*

Proposition 2 *If* D_1 *and* D_2 *are tree diagrams which differ by an elementary transformation, then* D_1 *and* D_2 *are equivalent modulo the 1-term and 4-term relations.*

Clearly, combining these two propositions gives us the Intersection Graph Conjecture for tree diagrams. The proof of the first proposition is straightforward (see [5]). The proof of the second proposition is much more complicated. It involves dividing the boughs along the trunk into "upper boughs" and "lower boughs", where the permutation affects solely the upper boughs, and then inducting on the number of chords in the upper boughs. Along the way, Chmutov et al. prove several lemmas (see [5]):

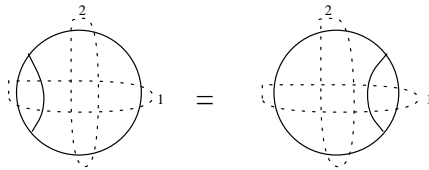**Lemma 2** *(The Generalized 4-term relation)*
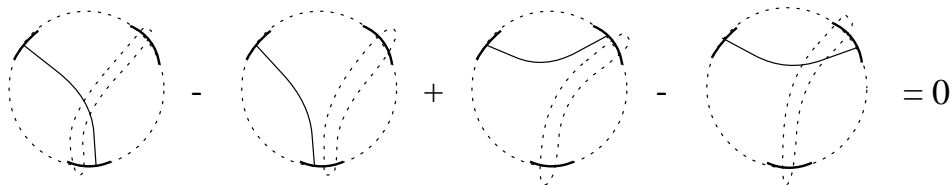
[counterexample.pic]

Figure 2: Not a 4-term relation

As with the usual 4-term relation, the diagrams may be anything outside of the solid arcs of the circle.

Notice that the share is fixed while the chord moves around it. The lemma fails if we try to reverse these roles. For example, if we apply the weight system generated by the Kauffman polynomial (see Section 6) to the linear combination of chord diagrams in Figure 2 we obtain $y^3x + 4y^2x^2 - yx^3 \neq 0$.

Lemma 3

This lemma is a corollary of the generalized 4-term relation. Notice that neither of these lemmas make any assumptions about the diagram (i.e. it is *not* necessarily a tree diagram).

Lemma 4 *If D is the diagram on the left hand side below, and D-L is a tree, then:*
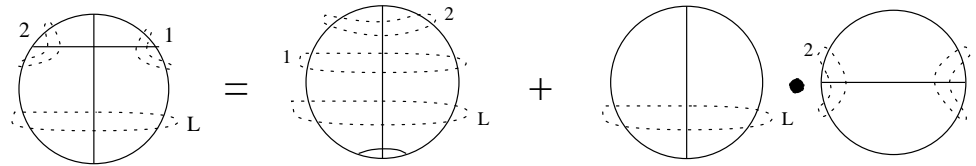
Chmutov et al. prove Lemma 4 only for tree diagrams, but a careful examination of their proofs shows that we can weaken this hypothesis as above. It is only necessary that the *upper boughs* form a tree diagram. The components of $\Gamma(D) - v(t)$ corresponding to the lower boughs (L in the statements above) may contain loops.

4 Loop Diagrams: Definitions and Results

We closely model our treatment of loop diagrams on the treatment of tree diagrams in [5].

Definition 6 *A chord diagram D is called a* **loop diagram** *if $\pi_1(\Gamma(D)) = \mathbf{Z}$, i.e. the intersection graph has a single loop.*

Our first task is to determine the elementary transformations for loop diagrams. As with tree diagrams, we can certainly permute boughs along a trunk. However, these moves are not sufficient. We can also reflect a bough across a trunk, as shown in
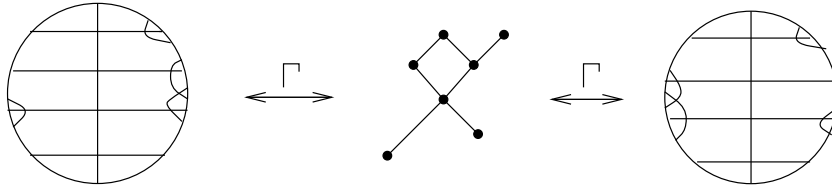
Figure 3: Reflecting a bough across a trunk

Figure 3. In the case of tree diagrams, this second move is a result of the first one, but in the case of loop diagrams, there may be a special bough which intersects the trunk twice, in which case the second move is independent of the first one. These two moves are now sufficient, so we define:

Definition 7 *The* **elementary transformations** *of a loop diagram are:*

- *permuting boughs with respect to some trunk;*

- *reflecting a bough with respect to some trunk.*

Now we prove three propositions:

Proposition 3 *If D_1 and D_2 are loop diagrams with $\Gamma(D_1) = \Gamma(D_2)$, then D_2 can be obtained from D_1 by a sequence of elementary transformations.*

Proposition 4 *If loop diagrams D_1 and D_2 differ by a permutation of boughs, then D_1 and D_2 are equivalent modulo 1-term and 4-term relations.*

Proposition 5 *If loop diagrams D_1 and D_2 differ by a reflection of a bough, then D_1 and D_2 are equivalent modulo 1-term and 4-term relations.*

Clearly, combining these three propositions and the Intersection Graph Conjecture for tree diagrams will give us Theorem 1.

5 Proof of Intersection Graph Conjecture for loop diagrams

PROOF OF PROPOSITION 3: Let Γ denote the intersection graph $\Gamma(D_1) = \Gamma(D_2)$. Γ has a unique minimal loop of length ≥ 3. Choose a vertex v_1 on the loop, and a direction around the loop, and use these choices to order the other vertices on the loop $v_1, v_2, \ldots, v_n$. Now let t_i and t_i' be the chords corresponding to v_i in D_1 and D_2 respectively. The ordering of the vertices gives an ordering of these chords in each diagram. If $n > 3$, this ordering induces an orientation on the bounding circles of the two diagrams, as shown in Figure 4. If $n = 3$, we also induce an orientation on the bounding circle, though it's not so easy to see. In this case we have three chords t_i, t_2, t_3, and the order of their endpoints moving clockwise around the bounding
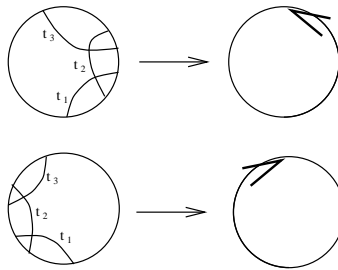
Figure 4: Order of chords induces orientation

[n3.pic]

Figure 5: The case when n=3

circle is either 123123 or 132132 (see Figure 5). In the first case, we will say the chords induce a clockwise orientation on the bounding circle, and in the second case the chords induce a counterclockwise orientation.

If these induced orientations do not agree, we can reflect the bough of t_1 in D_1 containing $\{t_2, \ldots, t_n\}$ across t_1, which will reverse the induced orientation. So, via this elementary transformations, we may assume that D_1 and D_2 are both oriented counterclockwise, as shown in Figure 6.

Now we can permute the boughs of t_i so their order corresponds to the order of the boughs along t_i'. Since the other boughs of t_i only intersect t_i once (the intersection graph has only one loop), each has a distinguished trunk. We can then permute boughs along these trunks. As we continue inductively, all further boughs will have a distinguished trunk, so we can complete the transformation of D_1 to D_2 via permutations of boughs. $\square$

To prove Proposition 4, we will need the following lemma, which allows us to rotate boughs in our chord diagrams:
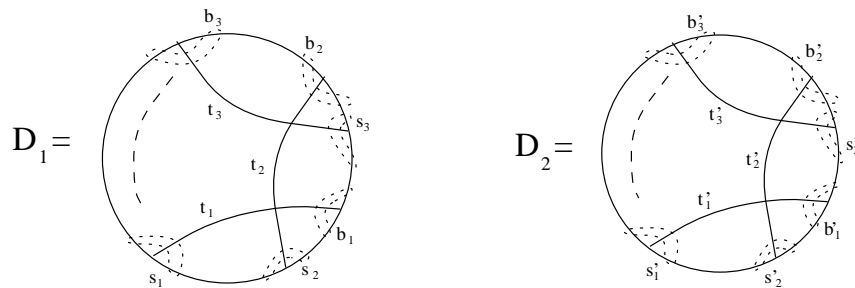

Figure 6: Counterclockwise orientation
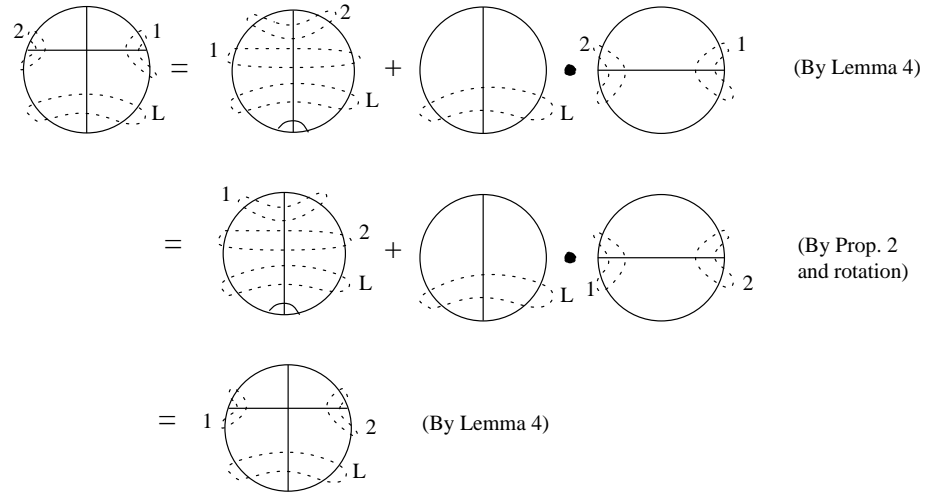
Figure 7: Proof of Lemma 5

Lemma 5
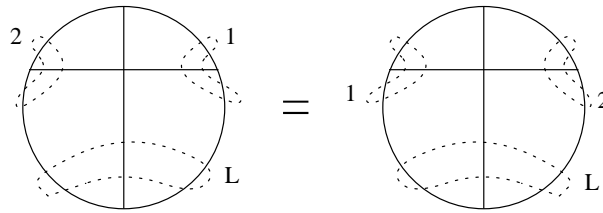
where L is any share, but D − L is a tree diagram. Here the shares 1 and 2 have each been rotated 180 degrees (not reflected).

PROOF: By Lemma 4, keeping in mind our observation that L can be any share, we get the equalities in Figure 7, which prove the lemma. □

PROOF OF PROPOSTION 4: First, we consider the case when one of the boughs being permuted contains the loop; i.e. we show:
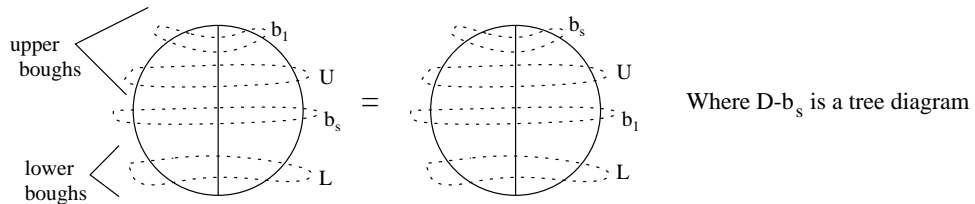
Following Chmutov et. al. [5] we show this by induction on the *complexity* of the diagrams. If we permute the boughs of D by a permutation π, the *lower boughs* are

the boughs below b_s in the diagram, and the *upper boughs* are the boughs above b_s. Then the *complexity* $c(D, \pi)$ is the total number of chords in the upper boughs.

BASE CASE: When $c(D, \pi) = 1$, then b_1 is just one chord. So we can move b_1 past b_s via Lemma 3. Here share 1 of Lemma 3 is $L \cup trunk$, and share 2 is b_s.

INDUCTIVE CASE: Assume the proposition is true for $c(D, \pi) < m$. We will show it holds for $c(D, \pi) = m$. The proof is by a chain of equalities of chord diagrams:

(permuting boughs, rotation)

Finally, we consider the case when neither of the boughs being permuted contains the loop. If L contains the loop, then we are done by Proposition 2 (using the weakened hypothesis). If U contains the loop, we simply rotate the diagram 180 degrees, permute the boughs using Proposition 2, and then rotate back.

This completes the proof of Propositon 4. □

Before we prove our final proposition, we will prove one more convenient lemma:

Lemma 6
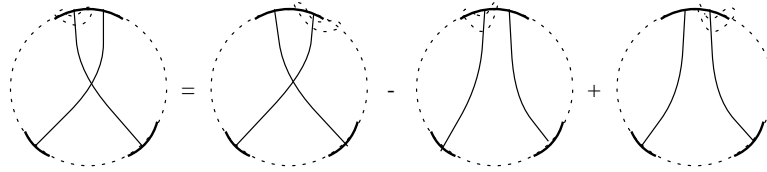
PROOF: The proof is simply an application of the Generalized 4-term relation:

(By the Generalized 4-term relation)

(Generalized 4-term relation)

□
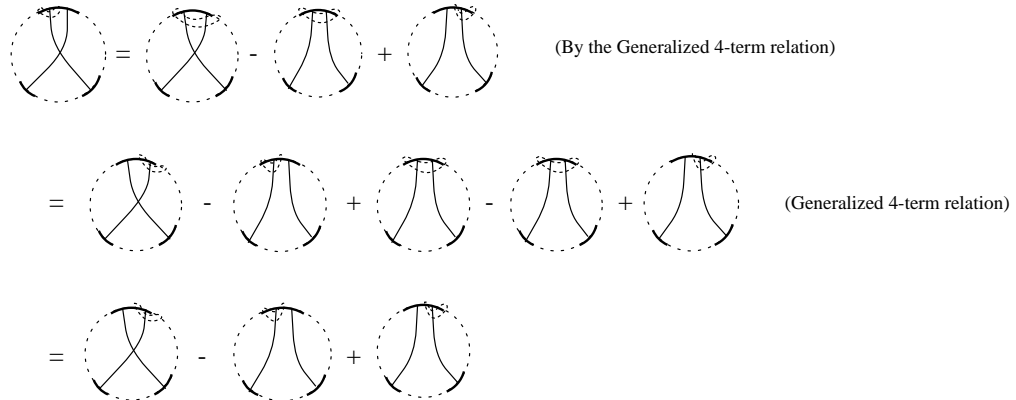
Now we can complete our argument by proving the final proposition:

PROOF OF PROPOSITION 5: We first consider the case when the bough does not contain the loop (and so intersects the trunk only once). In this case, reflecting the bough is just the result of permuting its sub-boughs to reverse their order (and doing the same on lower levels as necessary). So this case is a consequence of Proposition 4. The next case is when the bough does contain the loop, but only intersects the trunk once. In this case, we can either accomplish the reflection by permuting boughs, or we reach a stage when we wish to reflect the loop across a chord which it intersects twice. So we are reduced to the case of reflecting the loop across a trunk t which it intersects twice. By permuting boughs, we can put the
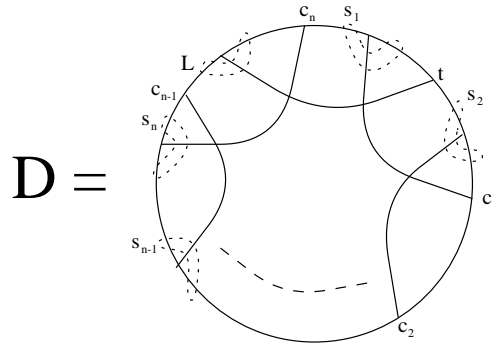
Figure 8: Normal Form

diagram into a "normal form", as shown in Figure 8. Again, the proof will work by induction. In this case, we will induct on $max\{n||s_n| > 0\}$, where the s_n's are numbered clockwise as shown, with $L = s_0$, and $|s_n|$ is the number of chords in the share s_n. We will call this the *heft* of the diagram.

BASE CASE: heft $= 0$. This case is easily proved using Proposition 4 and Lemma 6. See Figure 9.

INDUCTION: Assume Proposition 5 holds when the heft is less than m, and that the diagram D has heft m. Then Lemma 6 tells us that:
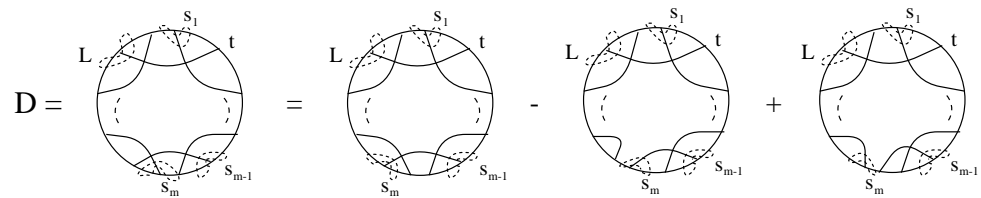
The last two diagrams on the left hand side are trees, so we can reflect the boughs through the trunk via permutations. The first diagram on the left hand side can be rewritten, permuting boughs, as:
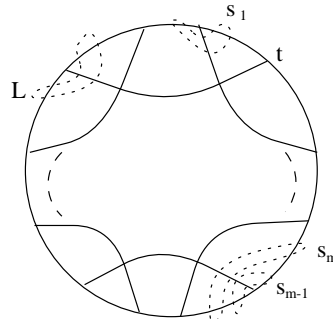
Therefore, this diagram has heft $m - 1$, and so the bough can be reflected by our inductive hypothesis. Doing the reflection on these three diagrams, and then
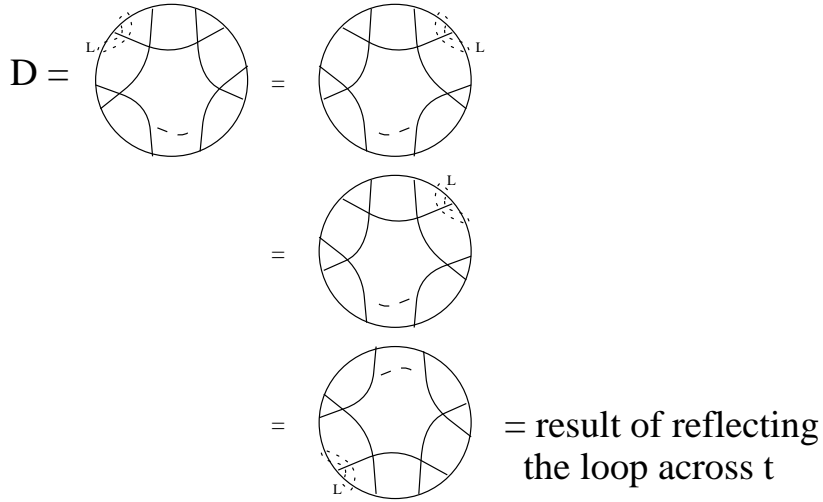
Figure 9: Proof of the Base Case

recombining them by Lemma 6, gives us the reflection in D:
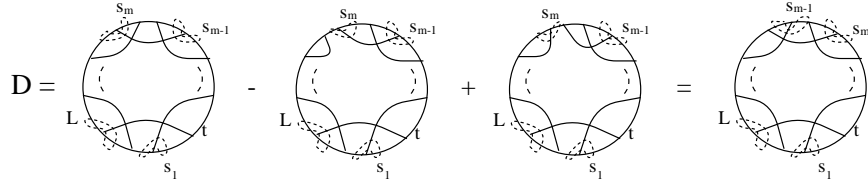
This completes the proof of Proposition 5, and hence of Theorem 1. □

6   Hopf Algebra of Tree and Loop Diagrams

Chmutov, Duzhin and Lando [6] use the fact that tree diagrams are determined by their intersection graphs to compute the subalgebra of the Hopf algebra of chord diagrams which is generated by the tree diagrams. They denote this subalgebra (the *forest subalgebra*) by A. To be precise, they prove:

Theorem 2 *The Hopf algebra A is isomorphic to the polynomial algebra $\mathbf{Q}[x_1, x_2, ...]$, where the grading of every x_n is n.*

In this section we will make the analogous computation for the algebra generated by both tree and loop diagrams. We will rather unimaginatively call this algebra the *loop algebra*, and denote it by B. Our goal in this section is to prove:

Theorem 3 *The Hopf algebra B is isomorphic to the polynomial algebra*

$$\mathbf{Q}[x_1, x_2, x_3, x_4^1, x_4^2, x_5^1, ..., x_n^1, ..., x_n^{n-2}, ...]$$
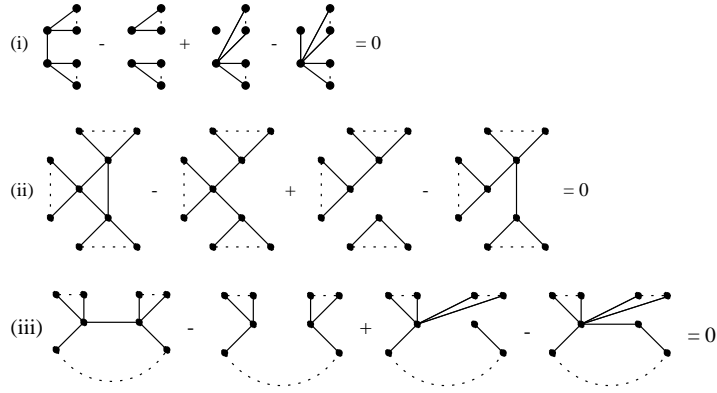
Figure 10: 4-term relation for intersection graphs

where the grading of every x_n^i is n, and the primitive space of grading n has dimension 1 if $n \leq 3$, and $n - 2$ if $n \geq 3$.

The Hopf algebra will certainly be isomorphic to some such polynomial algebra. The content of the theorem is in computing the exact dimension of the primitive space in each grading. We do this by first finding an upper bound for this dimension, and then finding sufficiently many independent primitive elements to show that this upper bound is in fact the dimension of the primitive space. The first half of this program is accomplished via the following proposition:

Proposition 6 *The 4-term relation on B induces three relations on intersection graphs with at most one loop, shown in Figure 10. In these pictures, the graphs are identical outside of the region shown, and the dotted lines indicate a group of edges incident to a single vertex. In the third relation, the dotted line along the bottom indicates that there is another path in the graph connecting the two vertices.*

PROOF: The key to this proposition is Theorem 1. This allows us to say that if the equalities hold for *any* chord diagrams with the given intersection graphs, then they will hold for *all* such diagrams, since the diagrams will be equivalent by Theorem 1. Then we will have the desired relations induced on intersection graphs. So we prove the equalities by choosing nice chord diagrams for which the proofs are easy.

The first relation is induced by Lemma 4 (with our weakened hypotheses), with an extra term which arises if we repeat the proof without using the 1-term relation (see [5]).

The second relation results from the following equalities of chord diagrams:

$$\bigcirc - \bigcirc + \bigcirc - \bigcirc$$

$$= \bigcirc - \bigcirc + \bigcirc - \bigcirc = 0$$

The last relation is the most complicated to prove. We first consider the following two 4-term relations:

$$\bigcirc - \bigcirc + \bigcirc - \bigcirc = 0$$

$$\bigcirc - \bigcirc + \bigcirc - \bigcirc = 0$$

By Theorem 1, the last two terms on the left-hand side of the second equation can be rewritten as:

$$\bigcirc - \bigcirc$$

So we consider a third 4-term relation:

$$\bigcirc - \bigcirc + \bigcirc - \bigcirc$$

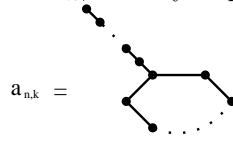Subtracting the second 4-term relation from the sum of the other two, we get:

$$\bigcirc - \bigcirc + \bigcirc - \bigcirc$$

which induces the desired relation on intersection graphs. □

This proposition tells us that any tree with n vertices is equivalent, modulo decomposable elements, to $a_n =$. Similarly, any graph with n vertices whose only loop is a triangle is equivalent to $2a_n$, and any loop graph

$a_{n,k} =$

with n vertices and a loop of length k is equivalent to . This last equivalence is because the middle two terms of part (iii) above cancel modulo decomposable elements (they are both trees with n vertices), and then by repeated use of part (i). Inductively, we see that $\{a_n, a_{n,k}\}$ are generators for the algebra of chord diagrams (though we don't yet know they are independent). We would like to find a set of independent *primitive* generators in each degree. Recall that an element p of a Hopf algebra is *primitive* if $\Delta(p) = 1 \otimes p + p \otimes 1$, so no decomposable element can be primitive. The set of independent primitive elements can be no larger than the set of graphs $\{a_n, a_{n,k}\}$ described above, so we have placed an upper limit on the dimension of the primitive space of grading n of the intersection graphs of loop diagrams, and hence (by Theorem 1) on the dimension of the primitive space of grading n of B. This upper limit is 1 when $n \leq 3$, and $n - 2$ when $n \geq 3$. So our goal is to show that these upper limits are in fact the dimensions of the spaces by exhibiting a sufficient number of primitive elements.

Definition 8 $p_n = \sum_J (-1)^{|J|} a_{n,J}$, *where J is some subset of the edges of a_n, and* $a_{n,J} = a_n - J$. $p_{n,k} = \sum_J (-1)^{|J|} a_{n,k,J}$.

We can show that all the elements $\{p_n, p_{n,k}\}$ are primitive directly, but it is more elegant to use bounded unitrivalent diagrams. Bar-Natan [2] has shown that an element of A is primitive if and only if it is a linear combination of bounded unitrivalent diagrams with *connected* interiors (i.e. the diagram minus its boundary circle is connected). So it suffices to show that $\{p_n, p_{n,k}\}$ have this form. It is useful to recall the "wheel with k spokes," introduced by Chmutov and Varchenko [7]:

Proposition 7

$p_n = (-1)^{n-1}$
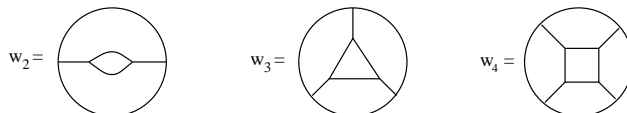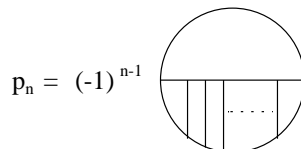
Figure 11: Links of the skein relation

$$p_{n,k} = (-1)^n \left(\ldots + \ldots \right)$$

$$p_{n,n} = (-1)^n w_n$$

Therefore, all the elements p_n and $p_{n,k}$ are primitive.

PROOF: The first part of the proposition was noticed by Chmutov and Varchenko [7]. To prove it, we pick an edge e of a_n and rewrite:

$$p_n = \sum_{J \ni e} (-1)^{|J|} (a_{n,J} - a_{n,J-e})$$

Each term of this sum is an STU relation, so we are left with a sum of bounded unitrivalent diagrams which have inserted a "leg" in place of the edge e of a_n; there are half as many terms in this sum as in the original one. We continue this process with each edge in turn, eventually obtaining $p_n = (-1)^{|all\ edges|} D = (-1)^{n-1} D$, where D is the bounded unitrivalent diagram in the proposition. Via exactly the same argument, we find that $p_{n,n} = (-1)^n w_n$. The proof for $p_{n,k}$ is the same, except that we do not consider the edge of $a_{n,k}$ where the loop reattaches, so we end up with 2 terms instead of one. In this case, the final factor of -1 (to give a coefficient of $(-1)^n$ rather than $(-1)^{n-1}$) comes from the anti-symmetry relation. □

It only remains to show that $\{p_n, p_{n,k}\}_{k=4}^n$ is independent. Chmutov, Duzhin and Lando [6] show that the p_n's are non-zero using weighted graphs, but this approach seems difficult to generalize. Instead, we take a more direct approach, and find a weight system W such that $\{W(p_n), W(p_{n,k})\}$ is independent. We will use the weight system which arises from the Kauffman polynomial, as described by Meng [11]. We say that a knot is *semi-oriented* if it is continuously oriented except at a finite number of points. Next, we recall the defining skein relation for the Kauffman polynomial [8]. We modify the relation slightly to give an invariant, up to *ambient* isotopy, of semi-oriented knots. We define links $L+, L-, L*, L\#andL!$ as in Figure 11 (where the links are the same outside of the region shown).

Definition 9 *The Kauffman polynomial F is the invariant of links, up to ambient isotopy, defined by the skein relation:*

$$bF(L+) - b^{-1}F(L-) = v(F(L\#) - F(L*))$$

In particular, this means that:

$$F(0|) = \left(1 + \frac{b^{-1} - b}{v}\right) F(|)$$

To obtain the weight system used by Meng, we make the following substitutions:

$$b = exp(-\frac{1}{2}yh); \ v = exp(-\frac{1}{2}xh) - exp(\frac{1}{2}xh)$$

This immediately gives us the formula:

$$F(L!) = (exp(\frac{1}{2}(y-x)h) - exp(\frac{1}{2}(y+x)h))(F(L\#) - F(L*)) + (exp(yh)-1)F(L-)$$

It is clear that the coefficient of h^n is a finite type invariant of type n. To compute a weight system W associated to the invariant, we isolate the first non-zero coefficient, obtaining the relations:

$$W(L!) = yW(L-) + xW(L*) - xW(L\#)$$

$$W(O|) = (1 - \frac{y}{x})W(|)$$

Now we want to evaluate this weight system on our primitive elements. As an example, we will compute $W(p_2)$ explicitly. Note that dots on the boundary circles indicate where the orientation reverses (so they always arise in pairs). If two such dots are connected by an arc containing no endpoints of chords, they can be moved together by an isotopy of the diagram and cancelled.

$$W(p_2) = W\left(\bigoplus - \bigcircledcirc\right) = W\left(\bigoplus\right)$$

$$W\left(\bigoplus\right) = yW\left(\bigcircledcirc\right) + xW\left(\text{O—O}\right) - xW\left(\bigcircledcirc\right)$$

$$= x(yW(OO) + xW(O) - xW(O)) - x(yW(O) + xW(O) - xW(OO))$$

$$= x(y(1 - \frac{y}{x})) - x(y + x - x(1 - \frac{y}{x}))$$

$$= y(x-y) - x(2y) = -y(x+y)$$

Lemma 7 $W(p_n) = -y(x+y)^{n-1}$ *for all* n. $W(p_{n,k}) = (x+y)^{n-k}W(p_{k,k})$ *for all* $k < n$.

PROOF: We prove this lemma by induction. We begin with the first statement. First, we recall that (by Theorem 1):

The second and fourth terms disappear by the 1-term relation. From the remaining terms, we see (for $n > 2$):

$$W(p_n) = yW(p_{n-1}) + xW\left(\text{}\right) - xW(p_{n-1}^*) = yW(p_{n-1}) - xW(p_{n-1}^*)$$

Where p_n^* is defined by:
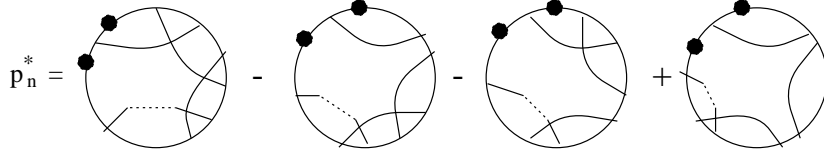
The remainder of the diagrams are the same as p_n.

Similarly, noting that $W\left(\text{}\right) = 2y$, and that the changes of orientation in p_n^* effectively reverse the signs of the last two terms of the weight system relation, we find that (for $n > 2$):

$$W(p_n^*) = -yW(p_{n-1}) + xW(p_{n-1}^*) = -W(p_n)$$

A direct computation shows that $W(p_2^*) = -W(p_2)$. Therefore,

$$W(p_n) = (x + y)W(p_{n-1})$$

A simple induction, together with our computation of $W(p_2)$, then gives the result.

The second statement is proved similarly, except that for the base case $p_{k+1,k}$, some additional analysis of $W(p_{k,k})$ is required. $\square$

Lemma 8 $W(p_{n,n}) = -2W(p_n) + yW(p_{n-1}) + 2x^2W(p_{n-2}) + 3x^2W(p_{n-2,n-2}) - 2x^3W(p_{n-3,n-3})$ *for* $n > 6$.

PROOF: The proof of this lemma, while long, is elementary. It is similar in concept to the proof of Lemma 7, and is left to the industrious reader. $\square$

Meng [11] noted that for any chord diagram D, $W(D)$ has a factor $y(x+y)$. Together with our results above, this implies that any $W(p_n)$, $n > 2$ or $W(p_{n,k})$, $n > k$ has a factor $(x + y)^2$. However, this is not the case for $W(p_{n,n})$:

Lemma 9 $W(p_{n,n})$ *is not divisible by* $(x + y)^2$.

PROOF: For n = 4,5,6 we show this by direct computation.

$$W(p_{4,4}) = y(x + y)(6x^2 + 3xy + y^2)$$

$$W(p_{5,5}) = y(x + y)(-x^3 + 6x^2y + 4xy^2 + y^3)$$

$$W(p_{6,6}) = y(x + y)(16x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4)$$

In general, $W(p_{n,n}) = y(x+y)Q_n(x,y)$, where Q_n is a polynomial in x and y of degree $n-2$. $W(p_{n,n})$ has a factor of $(x+y)^2$ if and only if Q_n has a factor of $(x+y)$; i.e. if $Q_n(-y,y) = 0$. From the Lemma 8, we know:

$$Q_n = y(3x+y)(x+y)^{n-4} + 3x^2 Q_{n-2} - 2x^3 Q_{n-3}$$

In particular, for $n > 6$ we have $Q_n(-y,y) = 3y^2 Q_{n-2}(-y,y) + 2y^3 Q_{n-3}(-y,y)$. And from above, we see that $Q_4(-y,y) = 4y^2$, $Q_5(-y,y) = 4y^3$, $Q_6(-y,y) = 12y^4$; in particular, the coefficients are all *positive*. By induction, the coefficient c_n of $Q_n(-y,y) = c_n y^{n-2}$ is monotonically increasing and always positive, and therefore can never be 0. We conclude that Q_n does *not* have a factor of $(x+y)$, which completes the lemma. $\square$

So now we can prove our final proposition:

Proposition 8 $\{p_n, p_{n,k}\}_{k=4}^n$ *is an independent set for all* n.

PROOF: We will prove this by showing that $\{W(p_n), W(p_{n,k})\}_{k=4}^n$ is an independent set. Again, we use induction. The base case is clear from our computations. Assume that $\{W(p_{n-1}), W(p_{n-1,k})\}_{k=4}^{n-1}$ is an independent set. Then Lemma 7 implies that $\{W(p_n), W(p_{n,k})\}_{k=4}^{n-1}$ is an independent set. It remains only to show that $W(p_{n,n})$ is independent from the other elements. But we have just seen that the other elements are all divisible by $(x+y)^2$, whereas $W(p_{n,n})$ is not (by Lemma 9), so it cannot be a linear combination of the others. Hence $\{W(p_n), W(p_{n,k})\}_{k=4}^n$ is independent for all n, so $\{p_n, p_{n,k}\}_{k=4}^n$ must also be independent for all n. $\square$

This completes the proof of Theorem 3.

7   Questions and Conjectures

Although we know from studying mutant knots that the Intersection Graph Conjecture fails in general (see [3], [10] and [12]), there are still many questions left to be asked. It is still unknown how badly the conjecture fails. In order to determine exactly how useful intersection graphs are in the study of chord diagrams and finite type invariants, we would need to answer the following question:

Question 1 *What is the kernel of the map* Γ*, in each degree, from the space of chord diagrams modulo the 4-term relation to the space of intersection graphs, modulo the images of all 4-term relations?*

The results of Chmutov et. al. [5] and Theorem 1 show that this kernel is trivial if the map is restricted to the space of tree and loop diagrams. Chmutov et. al. [4] have shown via computer calculations that the Intersection Graph Conjecture holds for chord diagrams of degree ≤ 8.

Proposition 9 *The kernel of* Γ *is trivial when restricted to chord diagrams of degree* ≤ 8.

PROOF: Assume that a linear combination $\sum k_i D_i$ of chord diagrams of degree $n \leq 8$ is in the kernel of Γ. Then the image $\sum k_i \Gamma(D_i)$ is trivial modulo the images of all 4-term relations. So by adding the images of some 4-term relations to the

linear combination of intersection graphs, we can cancel all of the graphs. Since the Intersection Graph Conjecture holds for diagrams of degree ≤ 8, each graph that we add corresponds to a unique chord diagram; so adding the images of the 4-term relations to the combination of intersection graphs corresponds to adding a unique set of 4-term relations to the linear combination of chord diagrams. Since all the graphs in the resulting combination of intersection graphs cancel, so must the corresponding chord diagrams in the combination of chord diagrams (again, because each graph corresponds to a unique chord diagram). Therefore, $\sum k_i D_i$ is trivial modulo the 4-term relation, and we are done. □

However, by Morton and Cromwell [12], the kernel is known to be non- trivial in degree 11. Nothing else is known; in particular, we would like to know if the kernel is trivial in degrees 9 and 10.

In addition, while the kernel is known to be non-trivial in degree 11, no-one has exhibited two explicit inequivalent chord diagrams of degree 11 which have the same intersection graph. However, we can glean some obvious possibilities from Morton and Cromwell [12]. Since they show that the Conway and Kinoshita-Terasaka knots can be distinguished by a finite-type invariant of type 11, we can begin by looking at the planar projections of these knots, which are then singular knots with 11 double points. The chord diagrams for these singular knots are:

These chord diagrams differ just by rotating one share about another (the fixed share is the one enclosed by the dashed line), so they have the same intersection graph. The labeling of the chords in the chord diagrams corresponds to the labeling of the chords in the intersection graph:

We can then conjecture that:

Conjecture 2 *The chord diagrams C and KT are not equivalent.*

We can also ask how far the approach of this paper, looking at the rank of the fundamental group of the intersection graph, rather than at its degree, can be

extended. The fact that the conjecture fails at degree 11 means that there is a counterexample with at most $\binom{11}{3} = 165$ loops. The counterexample proposed above has 15 loops. However, it does not seem that the arguments used in this paper can be easily extended to the case of diagrams with two loops, so perhaps there is counterexample there.

**Question 2** *Is there a counterexample to the IGC involving 2-loop diagrams?*

A first step towards answering Question 1 would be to determine in general the group of elementary transformations which can be performed on a chord diagram without changing its intersection graph (i.e., find some set of generators for this group). It seems likely that we would have to describe such a set of generators in terms of shares:

Conjecture 3 *The group of elementary transformations is generated by transformations of the following two types:*

- *Reflecting a share across another share:*
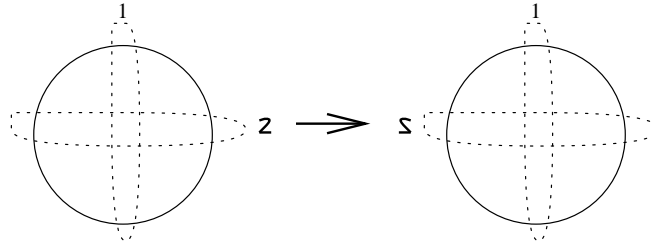
- *Turning a share upside down:*
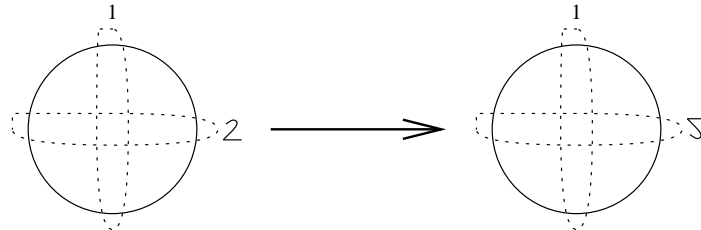
These transformations will certainly generate all transpositions, and hence all permutations, of shares along a chord (or another share); so they generate all the elementary transformations used in [5] and in this paper.

A final question concerns the size of the equivalence classes of intersection graphs under the relations induced by Γ. The results of Chmutov et. al. [5] and this paper have demonstrated that there are different chord diagrams (individual chord diagrams, not linear combinations) which are equivalent modulo the 4-term relation - namely, tree and loop diagrams sharing the same intersection graph. But these results do not help us answer the analogous question in the space of intersection graphs:

Question 3 *Are there two different intersection graphs (individual graphs, not linear combinations) which are equivalent modulo the 4-term relations (i.e. the relations induced by the 4-term relations via* Γ*)?*

It seems likely that the answer is "Yes," as it is for chord diagrams, but it would be interesting to have an explicit example. It would be even better to discover the "elementary transformations" between equivalent intersection graphs.

Intersection graphs may still have much to offer us as a tool for studying the space of chord diagrams, but there is still a lot of work to be done before we can exploit them fully.

8   Acknowledgements

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