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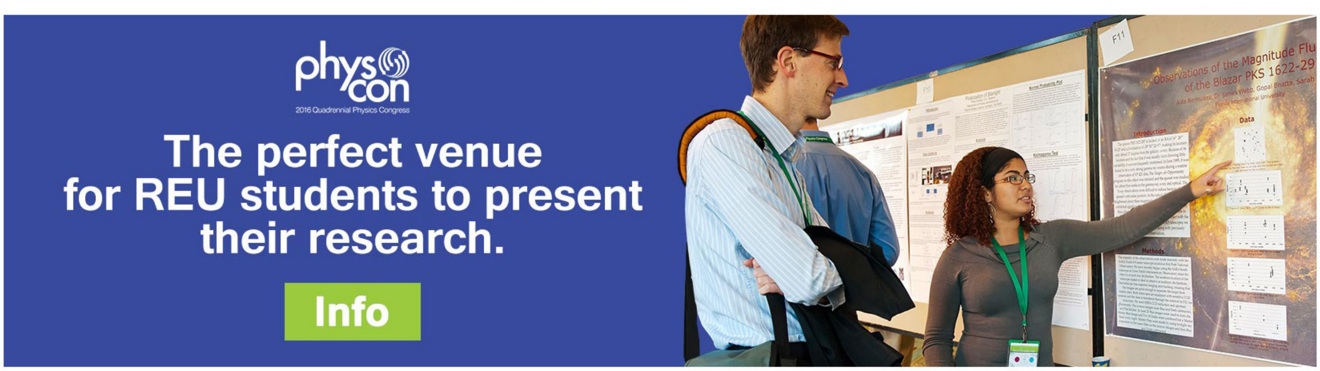

Intonation and compensation of fretted string instruments

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We discuss theoretical and physical models that are useful for analyzing the intonation of musical instruments such as guitars and mandolins and can be used to improve the tuning on these instruments. The placement of frets on the fingerboard is designed according to mathematical rules and the assumption of an ideal string. The analysis becomes more complicated when we include the effects of deformation of the string and inharmonicity due to other string characteristics. As a consequence, perfect intonation of all the notes on the instrument cannot be achieved, but complex compensation procedures can be introduced to minimize the problem. To test the validity of these procedures, we performed extensive measurements using standard monochord sonometers and other acoustical devices, confirming the correctness of our theoretical models. These experimental activities can be integrated into acoustics courses and laboratories and can become a more advanced version of basic experiments with monochords and sonometers.

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I. INTRODUCTION

The physics of musical instruments is an interesting subfield of acoustics and connects the theoretical models of vibrations and waves to the world of art and musical performance.[1–4] In the sixth century B.C., the mathematician and philosopher Pythagoras was fascinated by music and by the intervals between musical tones. He was probably the first to perform experimental studies of the pitches of musical instruments and relate them to ratios of integer numbers.

This connection between sound pitch and numbers is the origin of the diatonic scale, which dominated much of Western music, and also of the "just intonation system" based on perfect ratios of whole numbers, which was used for many centuries to tune musical instruments. Eventually, this system was abandoned in favor of a more refined method for intonation and tuning, the equal temperament system, which was introduced by scholars such as Vincenzo Galilei (Galileo's father), Marin Mersenne, and Simon Stevin in the 16th and 17th centuries, and strongly advocated by musicians such as J. S. Bach. In the equal-tempered scale, the interval of one octave is divided into 12 equal subintervals (semitones), achieving a more uniform intonation of musical instruments, especially when using all the 24 major and minor keys, as in Bach's the "Well Tempered Clavier." Historical discussion and reviews of the different intonation systems can be found in Refs. 5–7.

The 12-tone equal temperament system requires the use of irrational numbers because the ratio of the frequencies of two adjacent notes corresponds to $\sqrt[12]{2}$. On a fretted string instrument such as a guitar, lute, or mandolin, this intonation system is accomplished by placing the frets along the fingerboard according to these ratios. However, even with the most accurate fret placement, perfect instrument tuning is never achieved mainly because of the mechanical action of the player's fingers, which need to press the strings down on the fingerboard while playing, thus altering the string length and tension and changing the frequency of the sound being produced. Other causes of imperfect intonation include the inharmonicity of the strings due to their intrinsic stiffness and other more subtle effects. A discussion of these effects can be found in Refs. 8 and 9.

Experienced luthiers and guitar manufacturers usually correct for these effects by introducing *compensation*, that is, they slightly increase the string length to compensate for the increased sound frequency, resulting from the effects we have mentioned (see instrument building techniques in Refs. 10–14). Other solutions have been given[15–19] and in commercially patented devices.[20–22] These empirical solutions can be improved by studying the problem more systematically by modeling the string deformation, leading to a new type of fret placement that is more effective.

Some theoretical studies of the problem have appeared in specialized journals for luthiers and guitar builders,[23,24] but they are targeted to luthiers and manufacturers of a specific instrument (typically the classical guitar). In general physics journals we have found only basic studies on guitar intonation and strings[25–33] and no detailed analysis of the intonation.

Our objective is to review and improve the existing models of compensation for fretted string instruments and to perform experimental measures to test these models. The experimental activities described in this paper were performed using standard laboratory equipment (sonometers and other basic acoustic devices). These experimental activities can be introduced into standard laboratory courses on sound and waves as an interesting variation of experiments usually performed with classic sonometers.

II. GEOMETRICAL MODEL OF A FRETTED STRING

We introduce here a geometrical model of a guitar fingerboard, review the practical laws for fret placement, and study the deformations of a "fretted" string, that is, when the string is pressed onto the fingerboard by the mechanical action of the fingers.

We start our analysis by recalling Mersenne's law, which describes the frequency ν of sound produced by a vibrating string,[8,9]

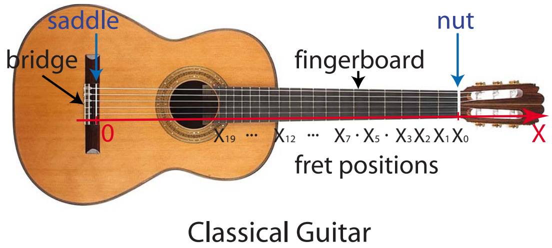

Fig. 1. Illustration of a classical guitar showing the coordinate system, from the saddle toward the nut, used to measure the fret positions on the fingerboard (guitar by Michael Peters; photo by Trilogy Guitars, reproduced with permission).

$$\nu_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}, \tag{1}$$

where $n=1$ refers to the fundamental frequency and $n=2,3,\ldots$ to the overtones. L is the string length, T is the tension, μ is the linear mass density of the string (mass per unit length), and $v=\sqrt{T/\mu}$ is the wave velocity.

In the equal-tempered musical scale an octave is divided into 12 semitones,

$$\nu_i = \nu_0 2^{i/12} \simeq \nu_0(1.05946)^i, \tag{2}$$

where ν_0 and ν_i are respectively the frequencies of the first note in the octave and of the ith note ($i=1,2,\ldots,12$). For $i=12$ we obtain a frequency, which is double that of the first note, as expected. Because Eq. (1) states that the fundamental frequency of the vibrating string is inversely proportional to the string length L, we combine Eqs. (1) and (2) to determine the string lengths for the different notes ($i=1,2,3,\ldots$) as a function of L_0 (the open string length, producing the first note of the octave considered), assuming that the tension T and the mass density μ are kept constant,

$$L_i = L_0 2^{-i/12} \simeq L_0(0.943874)^i. \tag{3}$$

Equation (3) can be used to determine the fret placement on a guitar or a similar instrument because the frets subdivide the string length into the required sublengths.

In Fig. 1 we show a picture of a classical guitar as a reference. The string length is the distance between the saddle[34] and the nut, and the frets are placed on the fingerboard at appropriate distances. We use the coordinate X, as illustrated in Fig. 1, to denote the position of the frets, measured from the saddle toward the nut position. X_0 denotes the position of the nut (the "zero" fret), and X_i, $i=1,2,\ldots$, are the positions of the frets of the instrument. On a classical guitar there are usually 19–20 frets on the fingerboard. They are realized by inserting thin pieces of a special metal wire in the fingerboard so that the frets will rise about 1.0–1.5 mm above the fingerboard.

The positioning of the frets follows Eq. (3), which we rewrite in terms of X,

$$X_i = X_0 2^{-i/12} \simeq X_0(0.943874)^i \simeq X_0\left(\frac{17}{18}\right)^i, \tag{4}$$

where the last approximation in Eq. (4) is the one employed by luthiers to locate the fret positions. Equation (4) is usually called the "rule of 18," which requires placing the first fret at a distance from the nut corresponding to 1/18 of the string length (or 17/18 from the saddle); second fret is placed at a distance from the first fret corresponding to 1/18 of the remaining length between the first fret and the saddle, and so on. Because $17/18=0.944444\approx 0.943874$, this empirical method is usually accurate enough for practical fret placement,[35] although modern luthiers use fret placement templates based on the decimal expression in Eq. (4).

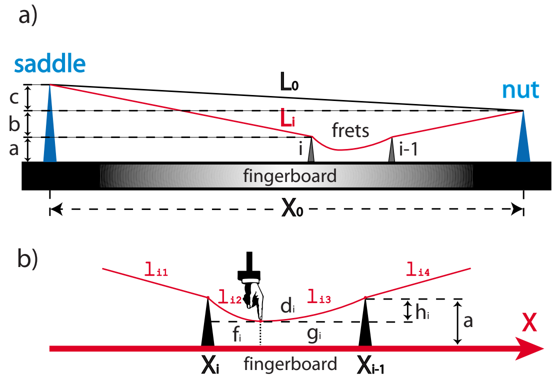

Fig. 2. Geometrical deformation model of a guitar string. (a) Original string (of length L_0) and the deformed string (of length L_i) when it is pressed between frets i and $i-1$. (b) The deformation model in terms of the four different sublengths $l_{i1}-l_{i4}$ of the deformed string.

Figure 2 illustrates the geometrical model of a fretted string, that is, when a player's finger or other device presses the string to the fingerboard until the string rests on the desired ith fret, thus producing the ith note when the string is plucked. In Fig. 2 we use a notation similar to the one developed in Refs. 23 and 24, but we will introduce a different deformation model.

Figure 2(a) shows the geometrical variables for a guitar string. The distance X_0 between the saddle and the nut is called the scale length of the guitar (typically between 640 and 660 mm for a modern classical guitar). The distance X_0 is not exactly the same as the real string length L_0 because the saddle and the nut usually have slightly different heights above the fingerboard surface. The connection between L_0 and X_0 is

$$L_0 = \sqrt{X_0^2 + c^2}. \tag{5}$$

The metal frets rise above the fingerboard by the distance a as shown in Fig. 2. The heights of the nut and saddle above the top of the frets are labeled in Fig. 2 as b and c, respectively. These heights are greatly exaggerated; they are usually small compared to the string length. The standard fret positions are again denoted by X_i, and in particular, we show the case where the string is pressed between frets i and $i-1$, thus reducing the vibrating portion of the string to the part between the saddle and the ith fret.

Figure 2(b) shows the details of the deformation caused by the action of a finger between two frets. Previous work[23,24] modeled this shape as "knife-edge" deformation, which is not quite comparable to the action of a fingertip. We improved on this assumed shape by using a more rounded deformation and considered a curved shape as in Fig. 2(b). The action of the finger depresses the string behind the ith fret by an amount h_i below the fret level (not necessarily corresponding to the full height a) and at a distance f_i, compared to the distance d_i between consecutive frets.

It is necessary for our compensation model to calculate exactly the length of the deformed string for any fret value i.

As shown in Fig. 2, the deformed length L_i of the entire string is the sum of the lengths of the four different parts,

$$L_i = l_{i1} + l_{i2} + l_{i3} + l_{i4}, \tag{6}$$

where the four sublengths can be evaluated from the geometrical parameters as follows:

$$l_{i1} = \sqrt{(X_0 2^{-i/12})^2 + (b+c)^2}, \tag{7}$$

$$l_{i2} = h_i\sqrt{1 + \frac{f_i^2}{4h_i^2}} + \frac{f_i^2}{4h_i}\ln\left[\frac{2h_i}{f_i}\left(1 + \sqrt{1 + f_i^2/4h_i^2}\right)\right], \tag{8}$$

$$l_{i3} = h_i\sqrt{1 + \frac{g_i^2}{4h_i^2}} + \frac{g_i^2}{4h_i}\ln\left[\frac{2h_i}{g_i}\left(1 + \sqrt{1 + g_i^2/4h_i^2}\right)\right], \tag{9}$$

$$l_{i4} = \sqrt{X_0^2(1 - 2^{-(i-1)/12})^2 + b^2}. \tag{10}$$

In Eqs. (8) and (9) the sublengths l_{i2} and l_{i3} were obtained by using a simple parabolic shape for the rounded deformation shown in Fig. 2(b) due to the action of the player's fingertip. They were calculated by integrating the length of the two parabolic arcs shown in Fig. 2(b) in terms of the distances f_i, g_i, and h_i.

The distances between consecutive frets are calculated as

$$d_i = f_i + g_i = X_{i-1} - X_i = X_0 2^{-i/12}(2^{1/12} - 1), \tag{11}$$

so that given the values of X_0, a, b, c, h_i, and f_i, we can calculate for any fret number i, the values of all the other quantities, and the deformed length L_i. We will see in Sec. III that the fundamental geometrical quantities of the compensation model are defined as

$$Q_i = \frac{L_i - L_0}{L_0}, \tag{12}$$

and they can also be calculated for any fret i using Eqs. (5)–(11).

III. COMPENSATION MODEL

In this section we will describe the model used to compensate for the string deformation and for the inharmonicity of a vibrating string, basing our analysis on the work done by Byers.[16,24]

The strings used in musical instruments are not perfectly elastic but possess a certain amount of stiffness or inharmonicity, which affects the frequency of the sound produced. Equation (1) needs to be modified to include this property, yielding the result (see Ref. 36, Chap. 4, Sec. 16)

$$\nu_n \simeq \frac{n}{2L}\sqrt{\frac{T}{\rho S}}\left[1 + \frac{2}{L}\sqrt{\frac{ESk^2}{T}} + \left(4 + \frac{n^2\pi^2}{2}\right)\frac{ESk^2}{TL^2}\right], \tag{13}$$

where we have rewritten the linear mass density of the string as $\mu = \rho S$ (ρ is the string density and S the cross section area). The correction terms inside the square brackets are due to the string stiffness and related to the modulus of elasticity (or Young's modulus) E and to the radius of gyration k (equal to the string radius divided by two for a simple unwound steel or nylon string). Following Ref. 36, we will use cgs units in the rest of the paper and in all calculations, except when quoting some geometrical parameters for which it will be more convenient to use millimeters.

Equation (13) is valid for $ESk^2/TL^2 < 1/n^2\pi^2$, a condition that is usually satisfied in practical situations.[37] When the stiffness factor ESk^2/TL^2 is negligible, Eq. (13) reduces to Eq. (1). When this factor increases and becomes important, the allowed frequencies also increase, and the overtones ($n=2,3,\ldots$) increase in frequency more rapidly than the fundamental tone ($n=1$). The sound produced is no longer harmonic because the overtone frequencies are no longer simple multiples of the fundamental one, as seen from Eq. (13). In addition, the deformation of the fretted string will alter the string length L and, as a consequence of this effect, will also change the tension T and the area S in Eq. (13). These are the main causes of the intonation problem being studied. Additional causes that we do not address in this paper are the imperfections of the strings (nonuniform cross section or density), the motion of the end supports (especially the saddle and the bridge) transmitting the vibrations to the rest of the instrument, which also changes the string length, and the effects of friction.

Following Byers[24] we define $\alpha_n = (4 + n^2\pi^2/2)$ and $\beta = \sqrt{ESk^2/T}$ so that we can simplify Eq. (13),

$$\nu_n \simeq \frac{n}{2L}\sqrt{\frac{T}{\rho S}}\left[1 + 2\frac{\beta}{L} + \alpha_n\frac{\beta^2}{L^2}\right]. \tag{14}$$

We consider just the fundamental tone ($n=1$) as being the frequency of the sound perceived by the human ear,[38]

$$\nu_1 \simeq \frac{1}{2L}\sqrt{\frac{T}{\rho S}}\left[1 + 2\frac{\beta}{L} + \alpha\frac{\beta^2}{L^2}\right], \tag{15}$$

where $\alpha = \alpha_1 = (4 + \pi^2/2)$. In Eq. (15) L represents the vibrating length of the string, which in our case is the length l_{i1} when the string is pressed onto the ith fret. To further complicate the problem, the quantities T, S, and β in Eq. (15) depend on the actual total length of the string L_i, as calculated in Eq. (6). In other words, we tune the open string of original length L_0 at the appropriate tension T, but when the string is fretted, its length is changed from L_0 to L_i, thus slightly altering the tension, the cross section, and β, which is a function of the previous two quantities. This dependence is the origin of the lack of intonation, common to all fretted instruments, which calls for a compensation mechanism.

The proposed solution[24] to the intonation problem is to adjust the fret positions to correct for the frequency changes described in Eq. (15). The vibrating lengths l_{i1} are recalculated as $l'_{i1} = l_{i1} + \Delta l_{i1}$, where Δl_{i1} represents a small adjustment in the placement of the frets, so that the fundamental frequency from Eq. (15) matches the ideal frequency of Eq. (2) and the fretted note will be in tune.

The ideal frequency ν_i of the ith note can be expressed by combining Eqs. (2) and (15),

$$\nu_i = \nu_0 2^{i/12} \simeq \frac{1}{2L_0}\sqrt{\frac{T(L_0)}{\rho S(L_0)}} \times\left[1 + 2\frac{\beta(L_0)}{L_0} + \alpha\frac{[\beta(L_0)]^2}{L_0^2}\right]2^{i/12}, \tag{16}$$

where all the quantities on the right-hand side are related to the open string length L_0, because ν_0 is the frequency of the open string note. We can write the ν_i using Eq. (15) as

$$\nu_i \simeq \frac{1}{2l'_{i1}}\sqrt{\frac{T(L_i)}{\rho S(L_i)}}\left[1+2\frac{\beta(L_i)}{l'_{i1}}+\alpha\frac{[\beta(L_i)]^2}{l'^2_{i1}}\right], \tag{17}$$

where we have used the adjusted vibrating length l'_{i1} for the fretted note and all the other quantities on the right-hand side of Eq. (17) depend on the fretted string length L_i. By comparing Eqs. (16) and (17) we obtain the master equation for our compensation model,

$$\frac{1}{2L_0}\sqrt{\frac{T(L_0)}{\rho S(L_0)}}\left[1+2\frac{\beta(L_0)}{L_0}+\alpha\frac{[\beta(L_0)]^2}{L_0^2}\right]2^{i/12} = \frac{1}{2l'_{i1}}\sqrt{\frac{T(L_i)}{\rho S(L_i)}}\left[1+2\frac{\beta(L_i)}{l'_{i1}}+\alpha\frac{[\beta(L_i)]^2}{l'^2_{i1}}\right]. \tag{18}$$

We obtained an approximate solution[39] of Eq. (18) by expanding the right-hand side in terms of Δl_{i1} and by solving the resulting expression for the new vibrating lengths l'_{i1},

$$l'_{i1} \simeq l_{i1}\left\{1+\frac{\left[1+\frac{2\beta(L_0)}{l_{i1}}+\frac{\alpha[\beta(L_0)]^2}{l_{i1}^2}\right]-\frac{1}{[1+Q_i(1+R)]}\left[1+\frac{2\beta(L_0)}{L_0}+\frac{\alpha[\beta(L_0)]^2}{L_0^2}\right]}{\left[1+\frac{4\beta(L_0)}{l_{i1}}+\frac{3\alpha[\beta(L_0)]^2}{l_{i1}^2}\right]}\right\}. \tag{19}$$

In Eq. (19) the quantities Q_i are derived from Eq. (12) and from the new deformation model described in Sec. II. An additional experimental quantity R is introduced in Eq. (19) and defined as (see Ref. 24 for details)

$$R=\left[\frac{d\nu}{dL}\right]_{L_0}\frac{L_0}{\nu_0}, \tag{20}$$

and is the frequency change $d\nu$ relative to the original frequency ν_0 induced by an infinitesimal string length change dL relative to the original string length L_0.

The new vibrating lengths l'_{i1} from Eq. (19) correspond to new fret positions X'_i because $X'_i=\sqrt{l'^2_{i1}-(b+c)^2}\simeq l'_{i1}$ for $(b+c) \ll l'_{i1}$. A similar relation holds between X_i and l_{i1} (see Fig. 2) so that the same Eq. (19) can be used to determine the new fret positions from the old ones:

$$X'_i \simeq X_i\left\{1+\frac{\left[1+\frac{2\beta(L_0)}{l_{i1}}+\frac{\alpha[\beta(L_0)]^2}{l_{i1}^2}\right]-\frac{1}{[1+Q_i(1+R)]}\left[1+\frac{2\beta(L_0)}{L_0}+\frac{\alpha[\beta(L_0)]^2}{L_0^2}\right]}{\left[1+\frac{4\beta(L_0)}{l_{i1}}+\frac{3\alpha[\beta(L_0)]^2}{l_{i1}^2}\right]}\right\}. \tag{21}$$

At this point a luthier would position the frets on the fingerboard according to Eq. (21), which is not in the canonical form of Eq. (4). Moreover, each string would get slightly different fret positions because the physical properties such as tension and cross section are different for the various strings of a musical instrument. Therefore, this compensation solution would be very difficult to be implemented practically and would also affect the playability of the instrument.[40]

An appropriate compromise, also introduced by Byers,[24] is to fit the new fret positions $\{X'_i\}_{i=1,2,\ldots}$ to a canonical fret position equation [similar to Eq. (4)] of the form

$$X'_i = X'_0 2^{-i/12} + \Delta S, \tag{22}$$

where X'_0 is a new scale length for the string and ΔS is the "saddle setback," that is, the distance by which the saddle position should be shifted from its original position (usually $\Delta S > 0$ and the saddle is moved away from the nut). The nut position is also shifted, but we require keeping the string scale at the original value X_0. Therefore we need $X'_{\mathrm{nut}}+\Delta S = X_0$, where X'_{nut} is the new nut position in the primed coordinates. Introducing the shift in the nut position ΔN as $X'_{\mathrm{nut}} = X'_0+\Delta N$ and combining Eqs. (21) and (22), we obtain the definition of the "nut adjustment" ΔN as

$$\Delta N = X_0 - (X'_0 + \Delta S). \tag{23}$$

This quantity is typically negative, indicating that the nut has to be moved slightly forward toward the saddle.

Finally, instead of adopting a new scale length X'_0, the luthier might want to keep the same original scale length X_0 and keep the fret positions according to Eq. (4). Because the corrections and the effects we have described are all linear with respect to the scale length chosen, it is sufficient to rescale the nut and saddle adjustment as follows:

$$\Delta S_{\mathrm{resc}} = \frac{X_0}{X'_0}\Delta S, \tag{24}$$

$$\Delta N_{\mathrm{resc}} = \frac{X_0}{X'_0}\Delta N. \tag{25}$$

This final rescaling is also needed on a guitar or other fretted instrument because the compensation procedure we have described has to be done independently on each string of the instrument. That is, all the quantities in the equations

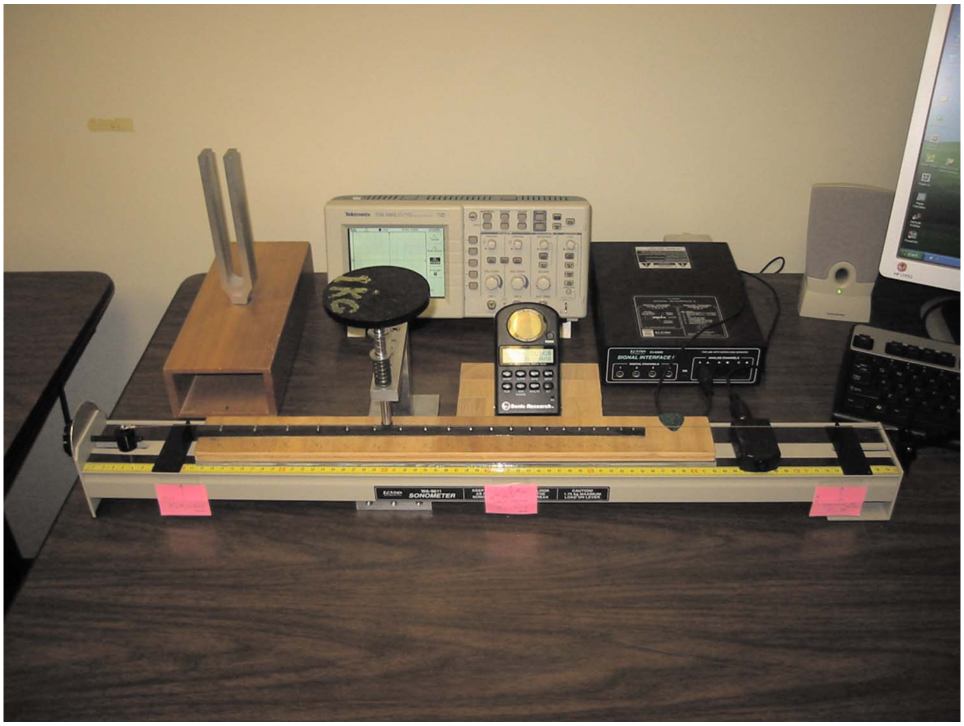

Fig. 3. Experimental apparatus composed of a standard sonometer to which we added a classical guitar fingerboard, visible as a thin black object with 20 metallic frets glued to a wooden board to raise it almost to the level of the string. Also shown is a mechanical device used to press the string on the fingerboard and several different instruments used to measure sound frequencies. This digital tuner is shown near the center, just behind the sonometer.

of this section should be rewritten adding a string index $j=1,2,\ldots,6$ for the six guitar strings. Each string would get a particular saddle and nut correction, but once these corrections are all rescaled according to Eqs. (24) and (25), the luthier can still set the frets according to Eq. (4). The saddle and nut will be shaped in a way to incorporate all the saddle-nut compensation adjustments for each string of the instrument (see Refs. 16 and 24 for practical illustrations of these techniques).

In practice, this compensation procedure does not change the original fret placement and the scale length of the guitar but requires very precise nut and saddle adjustments for each of the strings of the instrument using Eqs. (24) and (25). This procedure is a convenient approximation of the full compensation procedure, which would require repositioning all frets according to Eq. (21), but this solution would not be very practical.

IV. EXPERIMENTAL MEASUREMENTS

Because all our measurements were done using a monochord apparatus, we worked with a single string and not a set of six strings, as in a real guitar. Therefore, we will use all the equations without adding the additional string index j. However, it would be easy to modify our discussion to extend the deformation-compensation model to a multistring apparatus.

In Fig. 3 we show the experimental setup we used for our measurements. Because our goal was to test the physics involved in the intonation problem and not to build musical instruments or improve their construction techniques, we used standard laboratory equipment.

A standard PASCO sonometer WA-9613 (Ref. 41) was used as the main apparatus. This device includes a set of steel strings of known linear density and diameter and two adjustable bridges, which can be used to simulate the nut and saddle of a guitar. The string tension can be measured by using the sonometer tensioning lever or adjusted directly with the string tensioning screw (on the left of the sonometer, as seen in Fig. 3). In particular, this adjustment allowed the direct measurement for each string of the R parameter in Eq. (20) by slightly stretching the string and measuring the corresponding frequency change.

On top of the sonometer we placed a piece of a classical guitar fingerboard with scale length $X_0=645$ mm. The geometrical parameters in Fig. 2 were $a=1.3$ mm (fret thickness), $b=1.5$ mm, and $c=0.0$ mm (because we used two identical sonometer bridges as nut and saddle). This arrangement ensured that the metal strings produced a good quality sound, without "buzzing" or undesired noise when the sonometer was played like a guitar by gently plucking the string. Also, because we set $c=0$, the open string length is equal to the scale length: $L_0=X_0=645$ mm.

The mechanical action of the player's finger pressing on the string was produced by using a spring loaded device (also shown in Fig. 3, pressing between the sixth and seventh fret) with a rounded end to obtain the deformation model illustrated in Fig. 2(b). Although we tried different possible ways of pressing on the strings, for the measurements described in this section, we always pressed halfway between the frets ($f_i=g_i=d_i/2$) and all the way down on the fingerboard ($h_i=a=1.3$ mm). In this way, all the geometrical parameters of Fig. 2 were defined and the fundamental quantities Q_i of Eq. (12) could be determined.

The sound produced by the plucked string (which was easily audible due to the resonant body of the sonometer) was analyzed with different devices to accurately measure its frequency. At first we used the sonometer detector coil or a microphone connected to a digital oscilloscope or to a computer through a digital signal interface, as shown also in Fig. 3. All these devices could measure frequencies accurately,

Table I. Summary of the physical characteristics and the compensation parameters for the three steel strings used in our experimental tests.

	String 1	String 2	String 3
Open string note	C_3	F_3	C_4
Open string frequency (Hz)	130.813	174.614	261.626
Radius (cm)	0.0254	0.0216	0.0127
Linear density μ (g/cm)	0.0150	0.0112	0.0039
Tension (dyne)	5.16×10^6	5.88×10^6	4.41×10^6
Young's modulus E (dyne/cm^2)	2.00×10^{12}	2.00×10^{12}	2.00×10^{12}
R	130	199	78.7
Rescaled saddle setback ΔS_{resc} (cm)	0.733	0.998	0.518
Rescaled nut adjustment ΔN_{resc} (cm)	−2.31	−2.41	−1.35

but we used a professional digital tuner,[42] which could discriminate frequencies at the level of ±0.1 cents[43] for most of our measurements.

V. STRING PROPERTIES AND EXPERIMENTAL RESULTS

For our experimental tests we chose three of the six steel guitar strings included with the PASCO sonometer. Their physical characteristics and the compensation parameters are described in Table I.

The open string notes and related frequencies were chosen so that the sound produced using all the 20 frets of our fingerboard would span from two to three octaves, and the tensions were set accordingly. We used a value for Young's modulus typical of steel strings, and we measured the R parameter in Eq. (20). The rescaled saddle setback ΔS_{resc} and the rescaled nut adjustment ΔN_{resc} from Eqs. (24) and (25) were calculated for each string using the procedure outlined in Sec. III.

We then carefully measured the frequency of the sounds produced by pressing each string onto the twenty frets of the fingerboard in two modes: Without any compensation, that is, setting the frets according to Eq. (4), and with compensation, that is, after shifting the position of saddle and nut by the amounts specified in Table I and retuning the open string to the original note.

Table II illustrates the frequency values for string 1, obtained in the two modes and compared to the theoretical values of the same notes for a "perfect intonation" of the instrument. The measurements were repeated several times and the quantities in Table II represent average values. Fret number zero represents the open string being plucked, so there is no difference in frequency for the three cases. For all the other frets, the frequencies without compensation are

Table II. Frequencies of the different notes obtained with string 1. Theoretical perfect intonation values (in hertz) are compared to the experimental values with and without compensation. Also shown are the frequency deviations (in cents) from the theoretical values for both cases.

Fret No.	Note	Frequency, perfect intonation	Frequency, no compensation	Frequency deviation, no compensation	Frequency with compensation	Frequency deviation with compensation
0	C_3	130.813	130.813	0	130.813	0
1	$C_3^\#$	138.591	143.832	64.3	137.958	−7.9
2	D_3	146.832	150.551	43.3	147.323	5.8
3	$D_3^\#$	155.563	159.126	39.2	155.363	−2.2
4	E_3	164.814	168.407	37.3	164.070	−7.8
5	F_3	174.614	178.348	36.6	173.933	−6.8
6	$F_3^\#$	184.997	188.754	34.8	184.763	−2.2
7	G_3	195.998	200.386	38.3	195.878	−1.1
8	$G_3^\#$	207.652	212.105	36.7	207.632	−0.2
9	A_3	220.000	224.644	36.2	220.081	0.6
10	$A_3^\#$	233.082	237.495	32.5	233.136	0.4
11	B_3	246.942	252.345	37.5	247.123	1.3
12	C_4	261.626	266.338	30.9	261.505	−0.8
13	$C_4^\#$	277.183	281.958	29.6	277.076	−0.7
14	D_4	293.665	298.545	28.5	293.688	0.1
15	$D_4^\#$	311.127	315.276	22.9	311.463	1.9
16	E_4	329.628	334.822	27.1	329.787	0.8
17	F_4	349.228	353.408	20.6	348.785	−2.2
18	$F_4^\#$	369.994	373.545	16.5	370.330	1.6
19	G_4	391.996	396.597	20.2	393.335	5.9
20	$G_4^\#$	415.305	418.742	14.3	417.068	7.3

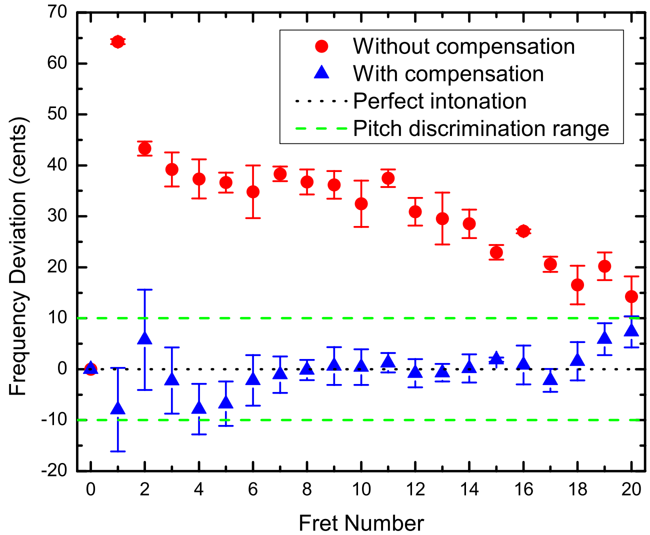

Fig. 4. Frequency deviation from perfect intonation level (black dotted line) for notes obtained with string 1. Red circles denote results without compensation, while blue triangles denote results with compensation. Also shown (region between green dashed lines) is the approximate pitch discrimination range for frequencies related to this string.

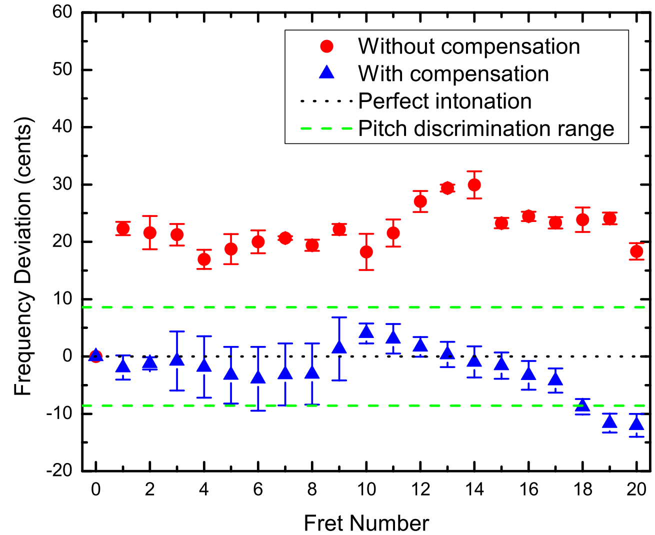

Fig. 5. Frequency deviation from perfect intonation level (black dotted line) for notes obtained with string 2. Red circles denote results without compensation, while blue triangles denote results with compensation. Also shown (region between green dashed lines) is the approximate pitch discrimination range for frequencies related to this string.

considerably higher than the theoretical values for a perfectly intonated instrument, which results in the pitch of these notes perceived as being higher (or sharper) than the correct pitch.[44] When we played our monochord sonometer in the first mode, it sounded out of tune. The frequency values obtained by using our compensation correction sounded much closer to the theoretical values, thus effectively improving the overall intonation of our monochord instrument.

In Table II we show the frequency deviation of each note from the theoretical value of perfect intonation with and without compensation. The frequency shifts are expressed in cents[43] rather than in hertz because the former unit is a more suitable measure of how the human ear perceives different sounds to be in or out of tune. The frequency deviation values illustrate more clearly the effectiveness of the compensation procedure: Without compensation the deviation from perfect intonation ranges between 14.3 and 64.3 cents; with compensation this range is reduced to between −7.9 and +7.3 cents.

We plot our results for string 1 in terms of the frequency deviation of each note from the theoretical value of perfect intonation. Figure 4 shows these frequency deviations for each fret number (corresponding to the different musical notes in Table II) without compensation (circles) and with compensation (triangles). Error bars come from the standard deviations of the measured frequency values.

We also show in Fig. 4 the pitch discrimination range (the region between the dashed lines), that is, the difference in pitch that an individual can effectively detect when hearing two different notes in rapid succession. Notes within this range will not be perceived as different in pitch by the ear. It can be easily seen in Fig. 4 that all the values without compensation are well outside the pitch discrimination range and thus will be perceived as out of tune (in particular as sharper sounds). In contrast, the values with compensation are within the dashed discrimination range of about ±10 cents.[45] The compensation procedure has almost made them equivalent to the perfect intonation values (corresponding to the zero cent deviation, perfect intonation level, dotted line in Fig. 4). Note that fret number zero corresponds to playing the open string note, which is always perfectly tuned; therefore the experimental points for this fret do not show any frequency deviation.

We repeated the same type of measurements for strings 2 and 3, which were tuned at higher frequencies as open strings (respectively, as F_3 and C_4; see Table I). In this way we obtained measured frequencies with and without compensation for these two other strings, similar to those presented in Table II. For brevity, we omit these numerical values, but we present in Figs. 5 and 6 the frequency deviation plots, as we did for string 1 in Fig. 4.

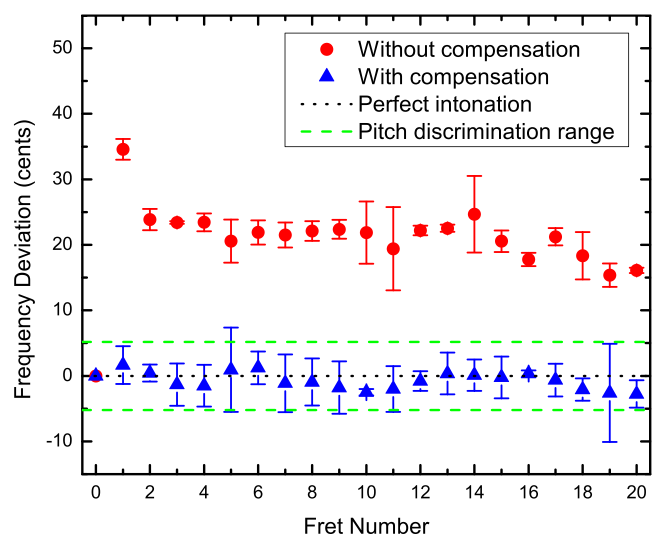

Fig. 6. Frequency deviation from perfect intonation level (black dotted line) for notes obtained with string 3. Red circles denote results without compensation, while blue triangles denote results with compensation. Also shown (region between green dashed lines) is the approximate pitch discrimination range for frequencies related to this string.

The results in Figs. 5 and 6 are similar to those in Fig. 4: The frequencies without compensation are much higher than the perfect intonation level, and the compensation procedure is able to reduce almost all the frequency values to the region within the dashed curves (the pitch discrimination range). The discrimination ranges in Figs. 5 and 6 were calculated respectively as ± 8.6 and ± 5.2 cents due to the different frequencies produced by these two other strings.

For the three cases we analyzed we conclude that the compensation procedure is very effective in improving the intonation of each of the strings. Although more work on the subject is needed (in particular we need to test nylon strings, which are more commonly used in classical guitars), we have shown that the intonation problem of fretted string instruments can be analyzed and solved using physical and theoretical models, which are more reliable than the empirical methods developed by luthiers during the historical development of these instruments.

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[34]The saddle is the white piece of plastic or other material located near the bridge on which the strings are resting. The strings are usually attached to the bridge, which is located on the left of the saddle. On other type of guitars or other fretted instruments, the strings are attached directly to the bridge (without using any saddle). In this case the string length would be the distance between the bridge and the nut. Our analysis would not be different in this case: The bridge position would replace the saddle position.
[35]Following Eq. (4), frets number 5, 7, 12, and 19 are particularly important because they (approximately) correspond to vibrating string lengths, which are respectively 3/4, 2/3, 1/2, and 1/3 of the full length, consistent with the Pythagorean theory of monochords.
[36]P. M. Morse, *Vibration and Sound* (American Institute of Physics, New York, 1983).
[37]The condition is equivalent to $n^2 < TL^2/ESk^2\pi^2 \approx 369$, 803, and 5052, where the numerical values are related to the three steel strings we present in Table I and calculated for the shortest possible vibrating length $L \simeq L_0/3 \simeq 21.5$ cm. The approximation in Eq. (13) is valid for our strings for at least $n \lesssim 19$.
[38]This statement is also an approximation because the pitch (or perceived frequency) is affected by the presence of the overtones. See, for example, the discussion of the psychological characteristics of music in Ref. 4.
[39]Our solution in Eq. (19) differs from the similar solution obtained in Ref. 24, Eq. (17). We believe that this difference is due to a minor error in their calculation, which causes only minimal changes in the numerical results. Therefore, the compensation procedure used by Byers (Ref. 24) in his guitars is practically very effective in improving the intonation of his instruments.
[40]Nevertheless some luthiers actually construct guitars where the individual frets under each string are adjustable in position by moving them slightly along the fingerboard. Each note of the guitar is then individually fine-tuned to achieve the desired intonation, requiring a very time consuming tuning procedure.
[41]PASCO, WA-9613 Sonometer Instruction Manual (1988).
[42]TurboTuner, Model ST-122 True Strobe Tuner, ⟨www.turbo-tuner.com⟩.
[43]The cent is a logarithmic unit of measure used for musical intervals. The octave is divided into 12 semitones, each of which is subdivided in 100 cents; thus the octave is divided into 1200 cents. Because an octave corresponds to a frequency ratio of 2:1, 1 cent is precisely equal to an interval of $2^{1/1200}$. Given two frequencies a and b of two different notes, the number n of cents between the notes is $n = 1200 \log_2(a/b) \simeq 3986 \log_{10}(a/b)$. Alternatively, given a note b and the number n of cents in the interval, the second note a of the interval is $a = b \times 2^{n/1200}$.
[44]We note that the frequency of the sound produced is the physical quantity we measured in our experiments. The pitch is defined as a sensory

characteristic arising out of frequency but also affected by other subjective factors, which depend on the individual. It is beyond the scope of this paper to consider these subjective factors.

[45] This discrimination range was estimated for the frequencies of string 1, according to the discussion in Ref. 4, pp. 248–252. This range usually varies from about (±5) to (±10) cents for frequencies between 1000 and 2000 Hz to even larger values of (±40)–(±50) cents at lower frequencies between 60 and 120 Hz.

Laboratory Clock. This clock was purchased by the Kenyon College physics department in 1926 as part of the fittings for the new Samuel Mather Science Hall. Attached to the bottom of its meter-long pendulum is a sharp needle that passes through a mercury bubble once per second, thus completing an electrical circuit. Along with a power supply, a series of runs of bell wire and numerous telegraph sounders, this was used to provide an audible tick all over the physics department. At a time when stopwatches were expensive, this provided a standard time base for timing pendulums using beats between the ticks and the motion of the pendulum. The clock has been restored and is now in the Greenslade Collection, still keeping excellent time. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)