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# A Map on the Space of Rational Functions

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# A MAP ON THE SPACE OF RATIONAL FUNCTIONS

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ABSTRACT. We describe dynamical properties of a map $\mathfrak{F}$ defined on the space of rational functions. The fixed points of $\mathfrak{F}$ are classified and the long time behavior of a subclass is described in terms of Eulerian polynomials.

**1. Introduction.** Let $\mathfrak{R}$ denote the space of rational functions with complex coefficients. The Taylor expansion at $x = 0$ of $R \in \mathfrak{R}$ is written as

$$R(x) = \sum_{n \gg -\infty} a_n x^n \tag{1.1}$$

where $n \gg -\infty$ denotes the fact that the coefficients vanish for large negative $n$. We consider the map $\mathfrak{F} : \mathfrak{R} \to \mathfrak{R}$ defined by

$$\mathfrak{F}(R(x)) = \sum_{n \gg -\infty} a_{2n+1} x^n. \tag{1.2}$$

The map $\mathfrak{F}$ can be given explicitly by

$$\mathfrak{F}(R(x)) = \frac{R(\sqrt{x}\,) - R(-\sqrt{x}\,)}{2\sqrt{x}} \tag{1.3}$$

and it appeared in this form in the description of a general procedure for the exact integration of rational functions [**2**]. The splitting of an arbitrary function $R$ into its even and odd parts $R(x) = R_e(x) + R_o(x)$ yields

$$\int_0^\infty R(x)\,dx = \int_0^\infty R_e(x)\,dx + \int_0^\infty R_o(x)\,dx. \tag{1.4}$$

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We regret to inform you that G. Boros passed away before publication of this paper.

The integral of the even part can be analyzed with the methods described in [**2**], and the integral of the odd part can be transformed by $x \mapsto \sqrt{x}$ to produce

$$\int_0^\infty R(x)\,dx = \int_0^\infty R_e(x)\,dx + \frac{1}{2}\int_0^\infty \mathfrak{F}(R(x))\,dx. \tag{1.5}$$

Here we consider dynamical properties of the map $\mathfrak{F} : \mathfrak{R} \to \mathfrak{R}$. Section 2 characterizes rational functions $R$ for which the orbit

$$\text{Orb}\,(R) := \left\{\mathfrak{F}^{(k)}(R(x)) :\ k \in \mathbf{N}\right\} \tag{1.6}$$

ends at the fixed point $0 \in \mathfrak{R}$. Section 3 describes the dynamics of a special class of functions with all their poles restricted to the unit circle. We establish explicit formulas for the asymptotic behavior of their orbits, expressed in terms of Eulerian polynomials $A_m(x)$ defined by the generating function

$$\frac{1-x}{1-x\exp\left[\lambda(1-x)\right]} = \sum_{m=0}^{\infty} A_m(x)\,\frac{\lambda^m}{m!}. \tag{1.7}$$

The proof only employs the classical result [**5**, p. 243]:

$$A_m(x) = (1-x)^{m+1}\sum_{k=0}^{\infty} k^m x^k. \tag{1.8}$$

Section 4 contains a description of all the fixed points of $\mathfrak{F}$.

The map $\mathfrak{F}$ can be supplemented by

$$\mathfrak{E}(R(x)) := \frac{R(\sqrt{x}\,) + R(-\sqrt{x}\,)}{2} \tag{1.9}$$

for which similar results can be established. See [**3**] for details. These transformations can be written as

$$\mathfrak{F}(R(x)) = \frac{R(x^{1/2}) + \omega_2 R(\omega_2 x^{1/2})}{2x^{1/2}}$$

and

$$\mathfrak{E}(R(x)) = \frac{R(x^{1/2}) + R(\omega_2 x^{1/2})}{2}$$

where $\omega_2 = -1$. The extensions to higher degree will be considered in future work. For instance, in the case of degree 3 we consider the maps:

$$\begin{aligned}
\mathfrak{T}_1(R(x)) &:= \frac{1}{3}\left[R(x^{1/3}) + R(\omega_3 x^{1/3}) + R(\omega_3^2 x^{1/3})\right] \\
\mathfrak{T}_2(R(x)) &:= \frac{1}{3x^{1/3}}\left[R(x^{1/3}) + \omega_3^2 R(\omega_3 x^{1/3}) + \omega_3 R(\omega_3^2 x^{1/3})\right] \\
\mathfrak{T}_3(R(x)) &:= \frac{1}{3x^{2/3}}\left[R(x^{1/3}) + \omega_3 R(\omega_3 x^{1/3}) + \omega_3^2 R(\omega_3^2 x^{1/3})\right]
\end{aligned}$$

where $\omega_3 = e^{2\pi i/3}$. These maps correspond to subseries of the expansion of $R$ taken along a fixed class modulo 3.

*Notation.* For $n \in \mathbf{N}$, $m_1, \dots, m_n$ are odd integers.

$$L = (m_1 m_2 \cdots m_n)^{-1} \quad \text{and} \quad m^* = m_1 + m_2 + \cdots + m_n - 2.$$

For $j \in \mathbf{Z}$ and $n \in \mathbf{N}$

$$Q_n(x) = \prod_{k=1}^{n} (x^{m_k} - 1) \tag{1.10}$$

and

$$R_{j,n}(x) = \frac{x^j}{Q_n(x)}. \tag{1.11}$$

**2. The kernel of the iterates.** Even rational functions can be characterized as the elements of the kernel of $\mathfrak{F}$. In this section we characterize those functions that vanish precisely after $n$ applications of $\mathfrak{F}$.

The sets

$$K_n := \operatorname{Ker} \mathfrak{F}^{(n)} = \left\{ S \in \mathfrak{R} : \ \mathfrak{F}^{(n)}(S) = 0 \right\}$$

form a nested sequence of vector spaces. We now describe the class of functions $J_n := K_n - K_{n-1}$, i.e., those functions that vanish after precisely $n$ applications of the map $\mathfrak{F}$. In particular we show that $J_n$ is not empty.

The decomposition of $S \in \mathfrak{R}$ into its even and odd parts can be expressed as

$$S(x) = S_{1,1}(x^2) + xS_{2,1}(x^2) \tag{2.1}$$

where $S_{1,1},\ S_{2,1} \in \mathfrak{R}$. This decomposition applied to $S_{1,1},\ S_{2,1}$ yields

$$S(x) = S_{1,2}(x^4) + xS_{2,2}(x^4) + x^2S_{3,2}(x^4) + x^3S_{4,2}(x^4).$$

Iterating this procedure produces a general decomposition.

**Lemma 2.1.** *Given $n \in \mathbf{N}$ and $S \in \mathfrak{R}$ there is a unique set of rational functions $\{S_{j,n} :\ 0 \le j \le 2^n - 1\}$ such that*

$$S(x) = \sum_{j=1}^{2^n} x^{j-1} S_{j,n}(x^{2^n}). \tag{2.2}$$

*Proof.* Split the sum in

$$S(x) = \sum_{k \gg -\infty} a_k x^k \tag{2.3}$$

according to the residue of $k$ modulo $2^n$. □

We now show that the functions $S \in J_n$ are precisely those for which $S_{2^n,n} = 0$ and $S_{2^{n-1},n} \neq 0$. This generalizes the case $n = 1$ that states that $\mathfrak{F}(S) = 0$ precisely when $S$ is even.

**Theorem 2.2.** *The rational functions that vanish after precisely $n$ applications of $\mathfrak{F}$ are those of the form*

$$S(x) = \sum_{j=1}^{2^n-1} x^{j-1} S_{j,n}(x^{2^n}), \tag{2.4}$$

*where $S_{1,n}, \ldots, S_{2^n-2,n}$ are arbitrary rational functions and $S_{2^n-1,n} \neq 0$.*

*Proof*. A direct calculation from (2.2) shows that, for $1 \leq k$,

$$\mathfrak{F}^{(k)}(S)(x) = \sum_{j=1}^{2^{n-k}} x^{j-1} S_{2^k j,n}(x^{2^{n-k}}). \tag{2.5}$$

The statement now follows from

$$\mathfrak{F}^{(n-1)}(S)(x) = S_{2^{n-1},n}(x^2) + x S_{2^n,n}(x^2) \tag{2.6}$$

and

$$\mathfrak{F}^{(n)}(S)(x) = S_{2^n,n}(x). \qquad \square \tag{2.7}$$

*Note.* We now state the result of Theorem 2.2 in the language of dynamical systems. Let $f : X \to X$ be a map on a set $X$, and let $x_0 \in X$ be a fixed point of $f$, i.e., $f(x_0) = x_0$. We say that a sequence $\{x_1, x_2, \ldots, x_n\}$ of elements of $X$ is a *prefixed sequence of length* $n$ *attached to* $x_0$ if $x_{i+1} = f(x_i)$ and $f(x_n) = x_0$. Theorem 2.2 states that $\mathfrak{F}$ admits prefixed sequences attached to 0 of arbitrary length.

**3. The dynamics of a special class.** The asymptotic behavior of $\mathfrak{F}$ can be described in complete detail for rational functions in the class

$$\mathfrak{C}_n := \mathfrak{C}_n(m_1, \ldots, m_n) = \left\{ \frac{P(x)}{Q_n(x)} : P \text{ is a Laurent polynomial} \right\}, \tag{3.1}$$

where $m_1, \ldots, m_n$ are *odd positive integers* and $Q_n(x)$ is defined in (1.10). A *Laurent polynomial* is a rational function of the form $a_{-k}x^{-k} + a_{-k+1}x^{-k+1} + \cdots + a_{j-1}x^{j-1} + a_j x^j$ with $k, j \in \mathbf{N}$.

The case $n = 1$ has been described in [**1**]. The results are expressed in terms of the function

$$\begin{aligned} \gamma_m(j) &= m \left\lfloor \frac{j}{2} \right\rfloor - \frac{1}{2}(m-1)(j-1) \\ &= \begin{cases} \dfrac{m-1+j}{2} & \text{if } j \text{ is even} \\ \dfrac{j-1}{2} & \text{if } j \text{ is odd.} \end{cases} \end{aligned} \tag{3.2}$$

**Theorem 3.1.** *Let $m$ be an odd positive integer. Then*

1) *For $j \in \mathbf{N}$*

$$\mathfrak{F}\left(\frac{x^j}{x^m-1}\right) = \frac{x^{\gamma_m(j)}}{x^m-1}. \tag{3.3}$$

*Thus, the study of the dynamics of $\mathfrak{F}$ on $\mathfrak{C}_1$ is reduced to that of $\gamma_m$ on $\mathbf{Z}$.*

2) *The iterates $\left\{\gamma_m^{(p)}(j) : p = 0, 1, 2, \dots\right\}$ reach the set*

$$\mathfrak{A}_m := \{0, 1, 2, \dots, m-2\} \tag{3.4}$$

*or the fixed points $-1$ and $m-1$ in a finite number of steps. Moreover, $\mathfrak{A}_m$ is invariant under the action of $\gamma_m$. This action partitions $\mathfrak{A}_m$ into orbits.*

3) *The inverse of the restriction of $\gamma_m$ to $\mathfrak{A}_m$ is given by*

$$\delta_m(k) = \begin{cases} 2k+1 & \text{if } 0 \le k \le (m-2)/2 \\ 2k+1-m & \text{if } (m-1)/2 \le k \le m-2. \end{cases} \tag{3.5}$$

*Note* 3.2. The explicit form of $\delta_m$ permits the explicit computation of the orbits of $\gamma_m$ on the invariant set $\mathfrak{A}_m$. For example, if $m$ is prime then every orbit of $\gamma_m$ is of length $\mathrm{Ord}\,(2; m)$. In particular, there is a single orbit if and only if 2 is a primitive root modulo $m$.

The iterates $\gamma_m^{(p)}(j)$ can be characterized by the congruence (3.6) below. This will be used in the determination of the limiting behavior of the iterates of $\mathfrak{F}$ below. The proof of this congruence and the numerical and symbolic evidence of the asymptotic behavior of the iterates of $\mathfrak{F}$ on $R_{j,n}(x)$ were part of a SIMU 2002 project. Details will appear in [**3**].

**Lemma 3.3.** *Let $m$ be an odd positive number, $0 \le j < m$, and $p \in \mathbf{N}$. The unique solution of*

$$2^p(x+1) \equiv j+1 \mod m \tag{3.6}$$

*in $0 \le x < m$ is given by $x = \gamma_m^{(p)}(j)$.*

*Proof.* The proof is by induction on $p$. Note first that the solution of the congruence (3.6) is unique because $\gcd(m, 2^p) = 1$. The base case, $p = 0$, is

$$x + 1 \equiv j + 1 \mod m$$

with unique solution $j = \gamma_m^{(0)}(j)$. To complete the inductive step observe that

$$\begin{aligned}
2^{p+1}&\left(\gamma_m^{(p+1)}(j) + 1\right) \\
&= 2^{p+1}\left(m\left\lfloor\frac{\gamma_m^{(p)}(j)}{2}\right\rfloor - \frac{1}{2}(m-1)(\gamma_m^{(p)}(j) - 1) + 1\right) \\
&= 2^{p+1}m\left\lfloor\frac{\gamma_m(j)}{2}\right\rfloor - 2^p m\left(\gamma_m^{(p)}(j) - 1\right) + 2^p\left(\gamma_m^{(p)}(j) + 1\right) \\
&\equiv j + 1 \mod m. \qquad \square
\end{aligned}$$

The map $\mathfrak{F}$ has a very rich dynamical structure, even in the case $n = 1$.

**Theorem 3.4.** *Let $r \in \mathbf{N}$. Then $\mathfrak{F}$ has at least one periodic orbit of length $r$.*

*Proof.* For $r \in \mathbf{N}$ define $m = 2^r - 1$. Then the orbit of $1/(x^m - 1)$ under $\mathfrak{F}$ is

$$\frac{1}{x^m - 1} \longmapsto \frac{x^{2^{r-1}-1}}{x^m - 1} \longmapsto \frac{x^{2^{r-2}-1}}{x^m - 1} \longmapsto \cdots \longmapsto \frac{x}{x^m - 1} \longmapsto \frac{1}{x^m - 1}. \qquad \square$$

We now consider the properties of the class $\mathfrak{C}_n$ for $n > 1$.

**Lemma 3.5.** *The class $\mathfrak{C}_n$ is invariant under the action of $\mathfrak{F}$.*

*Proof.* We show that $\mathfrak{F}(R_{n,j}(x)) \in \mathfrak{C}_n$. The linearity of $\mathfrak{F}$ yields the result.

Introduce the notation

$$S_l := \sum x^{(m_{i_1}+m_{i_2}+\cdots+m_{i_l})/2} \tag{3.7}$$

where the sum is taken over all subsets of $\{m_1, \ldots, m_n\}$ containing $l$ elements, the empty sum giving $S_0 = 1$.

Now

$$\prod_{k=1}^{n} \left( x^{m_k/2} + 1 \right) = 1 + \sum_{l=1}^{n} S_l \tag{3.8}$$

and

$$\prod_{k=1}^{n} \left( x^{m_k/2} - 1 \right) = (-1)^n + \sum_{l=1}^{n} (-1)^{n-l} S_l, \tag{3.9}$$

so that

$$\begin{aligned} \mathfrak{F}(R_{j,n}(x)) &= \frac{x^{(j-1)/2}}{2Q_n(x)} \left\{ \sum_{l=0}^{n} S_l + (-1)^{j+1} \sum_{l=0}^{n} (-1)^l S_l \right\} \\ &=: \frac{x^{(j-1)/2}}{2Q_n(x)} \times S. \end{aligned}$$

The sum $S$ contains all the terms $S_l$ with $l$ of the opposite parity of $j$ so that $x^{(j-1)/2} \times S$ is a polynomial. $\square$

We now consider the orbit of a general rational function $R$ in the class $\mathfrak{C}_n$. The case $n = 2$ illustrates the general situation. A direct calculation shows that

$$\mathfrak{F}\left( \frac{x^j}{(x^{m_1}-1)(x^{m_2}-1)} \right) = \frac{x^{(j-1+m_1)/2} + x^{(j-1+m_2)/2}}{(x^{m_1}-1)(x^{m_2}-1)} \quad \text{if } j \text{ is even}$$

and

$$\mathfrak{F}\left( \frac{x^j}{(x^{m_1}-1)(x^{m_2}-1)} \right) = \frac{x^{(j-1)/2} + x^{(j-1+m_1+m_2)/2}}{(x^{m_1}-1)(x^{m_2}-1)} \quad \text{if } j \text{ is odd.}$$

Thus $\mathfrak{F}$ preserves the denominator of $R$ (as was shown in Lemma 3.5) and each monomial of $R$ yields two monomials. We now show that the exponents of these monomials are always bounded.

**Proposition 3.6.** *Let $m^* = \sum_{k=1}^{n} m_k - 2$ and define*

$$\mathfrak{A} := \left\{ \frac{P}{Q_n} \in \mathfrak{C}_n : \ P \text{ is a polynomial with } deg(P) \leq m^* \right\}.$$

*Then $\mathfrak{A}$ is invariant under $\mathfrak{F}$, and every orbit starting at $R \in \mathfrak{C}_n$ reaches it in a finite number of steps.*

*Proof.* Let $R = P/Q_n \in \mathfrak{C}_n$ and write $\mathfrak{F}(R) = P_1/Q_n$. The largest exponent in $P_1$, call it $j$, appears from the sum $S_n$. The inequality $j > m^*$ implies $(j - 1 + m_1 + \cdots + m_n)/2 < j$. The case $j < 0$ is similar. $\square$

Now we consider the asymptotic behavior of the iterates of $\mathfrak{F}$ starting at $R \in \mathfrak{C}_n$. This is expressed in terms of the Eulerian polynomials $A_m(x)$ defined in (1.7). The discussion is divided into two cases depending upon whether the number

$$d := \gcd(m_1, m_2, \ldots, m_n) \tag{3.10}$$

is 1 or not. In Theorem 3.7 we prove that if $d = 1$ then

$$\mathfrak{F}^{(p)}(R(x)) \sim \frac{2^{(n-1)p}}{(n-1)!\, m_1 \cdots m_n} \frac{A_{n-1}(x)}{(1-x)^n}. \tag{3.11}$$

The case $d > 1$ is described in Theorem 3.8. We prove that the sequence of iterates of $\mathfrak{F}$ applied to the function $x^j/Q_n(x)$ taken along a fixed residue class, defined in (3.18), has an asymptotic behavior as in the case $d = 1$, with limit points expressed in terms of Eulerian polynomials.

This suggests the existence of an arbitrary number of limit points, but we have not ruled out the possibility that all these could coincide.

The proof employs the observation that if

$$R(x) = \sum_k f(k) x^k \tag{3.12}$$

is the expansion of $R$, then

$$\mathfrak{F}^{(p)}(R)(x) = \sum_k f(2^p(k+1) - 1) x^k. \tag{3.13}$$

**Theorem 3.7.** *Let $R \in \mathfrak{A}$ and suppose $d = 1$. Then*

$$\lim_{p\to\infty} \frac{\mathfrak{F}^{(p)}(R)(x)}{2^{(n-1)p}} = \frac{L\, A_{n-1}(x)}{(1-x)^n\,(n-1)!} \tag{3.14}$$

*where $L = 1/(m_1 \cdots m_n)$.*

*Proof.* Since $d = 1$, the rational function

$$R(x) = \frac{P(x)}{(x^{m_1}-1)\cdots(x^{m_n}-1)} \tag{3.15}$$

has a pole of order $n$ at $x = 1$ and poles of order less than $n$ at the other zeros of $Q_n(x)$, all of which are roots of unity. Thus the partial fraction expansion of $R$ has the form

$$R(x) = \frac{L}{(1-x)^n} + G(x) \tag{3.16}$$

where the term $G(x)$ contains all the terms of order lower than $n$. Hence

$$R(x) = L \sum_{k\geq 0} \binom{n+k-1}{n-1} x^k + G(x) \tag{3.17}$$

where the coefficient of $x^k$ in $G(x)$ is $O(k^{n-2})$. Thus

$$\begin{aligned} \mathfrak{F}^{(p)}(R)(x) &= L \sum_k \binom{n+2^p k+2^p-2}{n-1} x^k + \mathfrak{F}^{(p)}(G)(x) \\ &= \frac{L\,2^{(n-1)p}}{(n-1)!} \sum_k k^{n-1} x^k + O(2^{(n-2)p}) \end{aligned}$$

where we have used

$$\binom{n+2^p k+2^p-2}{n-1} = \frac{2^{(n-1)p}}{(n-1)!}\, k^{n-1} + O(2^{(n-2)p})$$

as $p \to \infty$. The result now follows from (1.8). $\square$

The analysis of the case $d > 1$ involves the function

$$\rho(i) = \text{ Ord}\left(2; \frac{d}{\gcd(i+1, d)}\right) \tag{3.18}$$

with $d$ as in (3.10). The function $\rho$ appears in the dynamics of $\gamma_m$: for $0 \leq j \leq m-2$, the length of the orbit containing $j$ is $\rho(j)$. See [**1**] for details.

Introduce the notation

$$E_i := \{ j \in \mathbf{Z} : \text{ there exists } r \in \mathbf{N} : \gamma_d^{(r)}(j) = i \} \tag{3.19}$$

for the backward orbit of $\gamma_d$, and let

$$A := E_{-1} \cup E_{d-1} \tag{3.20}$$

be the integers that eventually are mapped to the fixed points of $\gamma_d$.

**Theorem 3.8.** *Let $n \in \mathbf{N}$.*

a) *If $j \in A$, then*

$$\lim_{p\to\infty} \frac{\mathfrak{F}^{(p)}(R_{j,n}(x))}{2^{(n-1)p}} = \frac{Ld^n A_{n-1}(x^d)}{(n-1)!\,(x^d-1)^n}.$$

b) *If $j \notin A$, choose $r^*$ such that*

$$j^* := \gamma_d^{(r^*)}(j) \in \{0, 1, \ldots, d-1\}.$$

*Then for any $q \in \{0, 1, \ldots, \rho(j^*) - 1\}$*

$$\lim_{p\to\infty} \frac{\mathfrak{F}^{(p\rho(j^*)+q)}(R_{j,n}(x))}{2^{(n-1)(p\rho(j^*)+q)}} = \frac{Lx^\gamma}{(n-1)!(x^d-1)^n} \times \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} (\gamma+1)^l\, d^{n-l} (x^d-1)^l A_{n-1-l}(x^d)$$

*where $\gamma = \gamma_d^{(q)}(j^*)$.*

*Proof.* Since $\gcd(m_1, m_2, \ldots, m_n) = d$, the function $R_{j,n}(x)$ has a pole of order $n$ at the $d$th roots of unity and a pole of strictly lower order at all other zeros of $Q_n(x)$. Thus its partial fraction decomposition has the form

$$R_{j,n}(x) = \frac{C_{0,j}}{(x-1)^n} + \frac{C_{1,j}}{(x-\omega_1)^n} + \cdots + \frac{C_{d-1,j}}{(x-\omega_{d-1})^n} \\ + \text{ lower order terms}$$

where $\omega_l = \exp(2\pi i l/d)$, $1 \le l \le d-1$. The coefficients $C_{l,j}$ are given by

$$C_{l,j} = \lim_{x\to\omega_l} \frac{(x-\omega_l)^n x^j}{Q_n(x)} = L\,\omega_l^{n+j}.$$

Thus

$$R_{j,n}(x) = \sum_{l=0}^{d-1} \frac{C_{l,j}}{(x-\omega_j)^n} + G(x), \tag{3.21}$$

where the error term $G(x)$ includes the polar parts of all poles of order less than $n$. Then

$$\begin{aligned} R_{j,n}(x) &= \sum_{l=0}^{d-1} C_{l,j}\,\frac{(-1)^n}{\omega_l^n} \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \frac{x^k}{\omega_l^k} + G(x) \\ &= (-1)^n L \sum_{k=0}^{\infty}\sum_{l=0}^{d-1} \binom{n+k-1}{n-1} \omega_l^{j-k} x^k + G(x). \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathfrak{F}^{(r)}(R_{j,n}(x)) \\ &= (-1)^n L \sum_{k=0}^{\infty} \left( \sum_{l=0}^{d-1} \omega_l^{j-2^r(k+1)+1} \right) \binom{n+2^r(k+1)-2}{n-1} x^k + \mathfrak{F}^{(r)}(G(x)). \end{aligned}$$

Now

$$\sum_{l=0}^{d-1} \omega_l^q = \begin{cases} 0 & \text{if } d \text{ does not divide } q \\ d & \text{if } d \text{ divides } q, \end{cases}$$

so the only values of $k$ that contribute to the sum are those for which

$$j+1 \equiv 2^r(k+1) \mod d.$$

The discussion is divided into two cases depending on whether some iterate of $j$ reaches one of the fixed points or not.

*Case* 1. Suppose $j \in A$ and let $r^* \in \mathbf{N}$ be such that $\gamma_d^{(r^*)}(j) = -1$. Then Lemma 3.3 shows that $k = \gamma_d^{(r+r^*)}(j) + Nd = Nd - 1$. Thus

$$\begin{aligned}&\mathfrak{F}^{(r+r^*)}(R_{j,n}(x)) \\ &\quad = (-1)^n Ld \sum_{N=1}^{\infty} \binom{n-2+2^{r+r^*}Nd}{n-1} x^{Nd-1} + \mathfrak{F}^{(r+r^*)}(G(x)).\end{aligned}$$

In the limit as $r \to \infty$ the binomial coefficient is asymptotic to

$$\frac{2^{(r+r^*)(n-1)} N^{n-1} d^{n-1}}{(n-1)!}.$$

The number $r^*$ is fixed, so we have

$$\lim_{r\to\infty} \frac{\mathfrak{F}^{(r+r^*)}(R_{j,n}(x))}{2^{(r+r^*)(n-1)}} = \frac{(-1)^n Ld^n A_{n-1}(x^d)}{(n-1)!\,(1-x^d)^n} \tag{3.22}$$

as stated.

A similar argument shows that the same result is true if $j$ lies on the backward orbit of the second fixed point.

*Case* 2. Now assume $j \notin A$. Lemma 3.3 shows that

$$k = \gamma_d^{(r+r^*)}(j) + Nd = \gamma_d^{(r)}\left(\gamma_d^{(r^*)}(j)\right) + Nd.$$

Thus

$$\begin{aligned}&\mathfrak{F}^{(r+r^*)}(R_{j,n}(x)) \\ &\quad = (-1)^n Ld \sum_{N=0}^{\infty} \binom{n-2+2^{r+r^*}(Nd+\gamma_d^{(r+r^*)}(j)+1)}{n-1} x^{Nd+\gamma_d^{(r+r^*)}(j)}.\end{aligned}$$

In the limit as $r \to \infty$, the binomial coefficient is asymptotic to

$$\frac{2^{(r+r^*)(n-1)}(Nd+\gamma_1+1)^{n-1}}{(n-1)!}$$

where $\gamma_1 = \gamma^{(r+r^*)}(j)$. Thus

$$\frac{\mathfrak{F}^{(r+r^*)}(R_{j,n}(x))}{2^{(r+r^*)(n-1)}} = \frac{(-1)^n L d x^{\gamma_1}}{(n-1)!} \sum_{N=0}^{\infty} (Nd+\gamma_1+1)^{n-1}\, x^{Nd} + o(1).$$

as $r \to \infty$. Now

$$\begin{aligned} \sum_{N=0}^{\infty} (Nd+\gamma_1+1)^{n-1}\, x^{Nd} &= \sum_{N=0}^{\infty} x^{Nd} \sum_{l=0}^{n-1} \binom{n-1}{l} (1+\gamma_1)^{n-1-l} N^l d^l \\ &= \sum_{l=0}^{n-1} \binom{n-1}{l} (1+\gamma_1)^{n-1-l} d^l \frac{A_l(x^d)}{(1-x^d)^{l+1}}. \end{aligned}$$

As a function of $r$,

$$\gamma_1 = \gamma^{(r+r^*)}(j) = \gamma^{(r)}(j^*)$$

has period $\rho(j^*)$. Write $r = p\rho(j^*) + q$ with $0 \le q \le \rho(j^*) - 1$ and replace $\gamma_d^{(r)}(j^*)$ by $\gamma_d^{(q)}(j^*)$ to obtain

$$\begin{aligned} &\frac{\mathfrak{F}^{(p\rho(j^*)+q+r^*)}(R_{j,n}(x))}{2^{(p\rho(j^*)+q+r^*)(n-1)}} \\ &= \frac{(-1)^n L d x^{\gamma_d^{(q)}(j^*)}}{(n-1)!} \sum_{l=0}^{n-1} \binom{n-1}{l} \left(1+\gamma_d^{(q)}(j^*)\right)^l \cdot \frac{d^{n-1-l} A_{n-1-l}(x^d)}{(1-x^d)^{n-l}} + o(1). \end{aligned}$$

To conclude the proof, observe that as $q$ runs over the set of residues modulo $\rho(j^*)$, so does $q + r^*$. $\square$

**4. The fixed points of $\mathfrak{F}$.** A formal power series argument shows that any rational function $R$ fixed by $\mathfrak{F}$ must have an expansion of the form

$$xR(x) = c + \sum_{n=0}^{\infty} a_n \varphi(x^{2n+1}), \tag{4.1}$$

where

$$\varphi(x) = \sum_{n=0}^{\infty} x^{2^k} = x + x^2 + x^4 + x^8 + \cdots . \tag{4.2}$$

In particular, $R$ has at most a simple pole at the origin and if

$$R(x) = \sum_{n \geq -1} f(n) x^n$$

is such a fixed point, then $f(2n) = f(n)$ for $n \geq 0$. Thus the problem of finding fixed points of $\mathfrak{F}$ is reduced to finding sequences $\{a_n\}$ for which (4.1) is a rational function.

The class of functions discussed in Section 3 yields examples of fixed points. Let $m$ be an odd positive integer. Then $m-1$ is fixed by $\gamma_m$, so $R_{1,m-1}(x) = x^{m-1}/(x^m - 1)$ is fixed by $\mathfrak{F}$. This example can be obtained by a different approach. First observe that if $\mathfrak{F}(R) = R$ and $r$ is any odd positive integer, then the function

$$\mathcal{B}_r(R(x)) = x^{r-1} R(x^r) \tag{4.3}$$

is also fixed by $\mathfrak{F}$. The function $g(x) = 1/(x-1)$ is fixed by $\mathfrak{F}$, so that $R_{1,m-1}(x) = \mathcal{B}_m(g(x))$ is also fixed.

The description of all the fixed points of $\mathfrak{F}$ requires the notion of *cyclotomic cosets*: given $n$, $r \in \mathbf{N}$ with $r$ odd and $0 \leq n \leq r-1$, the set

$$C_{r,n} = \{2^s n \mod r : \ s \in \mathbf{Z}\} \tag{4.4}$$

is the 2-cyclotomic coset of $n$ mod $r$. Observe that $C_{r,n}$ is a finite set. With $\lambda$ a fixed primitive $r$th root of unity, define

$$f_{r,n}(x) = \sum_{m \in C_{r,n}} \frac{\lambda^m}{1 - \lambda^m x}. \tag{4.5}$$

The partial fraction decomposition of the fixed point $\mathcal{B}_m(g(x))$ can be decomposed into a sum of rational functions each fixed by $\mathfrak{F}$. For example, consider $\mathcal{B}_7(g(x)) = x^6/(x^7-1)$, and let $\lambda = \exp(2\pi i/7)$ be a primitive 7th root of unity. Then

$$\begin{aligned} 7\mathcal{B}_7(g(x)) &= \sum_{k=0}^{6} \frac{\lambda^k}{1-\lambda^k x} \\ &= \frac{1}{1-x} + \left( \frac{\lambda}{1-\lambda x} + \frac{\lambda^2}{1-\lambda^2 x} + \frac{\lambda^4}{1-\lambda^4 x} \right) \\ &\quad + \left( \frac{\lambda^3}{1-\lambda^3 x} + \frac{\lambda^5}{1-\lambda^5 x} + \frac{\lambda^6}{1-\lambda^6 x} \right), \end{aligned} \tag{4.6}$$

where each of the sets of terms grouped together is a rational function fixed by $\mathfrak{F}$. In the notation introduced above, this decomposition is

$$7\mathcal{B}_7(g(x)) = f_{7,0}(x) + f_{7,1}(x) + f_{7,3}(x). \tag{4.7}$$

We now classify all the fixed points of $\mathfrak{F}$.

**Theorem 4.1.** *A rational function is fixed by $\mathfrak{F}$ if and only if it is a linear combination of $1/x$ and the functions $f_{r,n}(x)$ for $r$ odd and $0 \leq n \leq r-1$.*

*Proof.* The identity

$$\mathfrak{F}\left( \frac{\lambda}{1-\lambda x} \right) = \frac{\lambda^2}{1-\lambda^2 x}$$

shows that $\mathfrak{F}$ fixes the $f_{r,n}$ because the squaring map $S : z \mapsto z^2$ permutes the values $\lambda^m$ for $m \in C_{r,n}$.

We first establish the converse under the assumption that the poles of $R$ are simple. The final step of the proof consists of checking that this condition holds for any fixed point of $\mathfrak{F}$.

Let $R$ be a rational function, with simple poles, that is fixed by $\mathfrak{F}$. The partial fraction decomposition of $R$ is

$$R(x) = \frac{c}{x} + \sum_{j=1}^{J} \frac{\alpha_j \lambda_j}{1-\lambda_j x}, \tag{4.8}$$

which is unique up to order. Apply $\mathfrak{F}$ to produce

$$R(x) = \frac{c}{x} + \sum_{j=1}^{J} \frac{\alpha_j \lambda_j^2}{1 - \lambda_j^2 x}.$$

The uniqueness of (4.8) shows that

$$\{\lambda_1, \dots, \lambda_J\} = \{\lambda_1^2, \dots, \lambda_J^2\}, \tag{4.9}$$

that is, the set $\Lambda = \{\lambda_1, \dots, \lambda_J\}$ is permuted by the squaring map. We conclude that the set $\{\lambda_k^{2^n} : n \in \mathbf{N}\}$ is a finite set (a subset of $\Lambda$) and so every $\lambda_k$ is an $r_k$th root of unity, for some odd positive integer $r_k$.

Now group terms in the sum (4.8) according to the orbits of the squaring map $S$ on the set $\Lambda$. The coefficient $\alpha_j$ in (4.8) must be constant along each orbit, and moreover, the orbit of $\lambda_k$ under $S$ is precisely the set $\{\lambda_k^m : m \in C_{n,r_k}\}$ for some $n$. Therefore $R(x)$ can be decomposed as a linear combination of the required form.

The next result concludes the proof of the theorem.

**Proposition 4.2.** *Let $R(x)$ be a rational function that can be expressed in the form*

$$xR(x) = c + \sum_{n=0}^{\infty} a_n \varphi(x^{2n+1}) \tag{4.10}$$

*where $\varphi$ is given in* (4.2). *Then the poles of $R$ must be simple.*

The following result, used in the proof of Proposition 4.2, is demonstrated in [**8**, p. 202].

**Lemma 4.3.** *Let $q_1, q_2, \dots, q_d$ be a fixed sequence of complex numbers, $d \geq 1$, and $q_d \neq 0$. The following conditions on a function $f : \mathbf{N} \to \mathcal{C}$ are equivalent*:

(1)
$$\sum_{n\geq 0} f(n)x^n = \frac{P(x)}{Q(x)},$$
*where* $Q(x) = 1 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_dx^d$.

(2) *For* $n \gg 0$,
$$f(n) = \sum_{i=1}^{k} P_i(n)\lambda_i^n,$$
*where* $1 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_dx^d = \prod_{i=1}^{k}(1-\lambda_i x)^{d_i}$, *the* $\lambda_i$*'s are distinct, and* $P_i(n)$ *is a polynomial in* $n$ *of degree less than* $d_i$.

*Proof of Proposition* 4.2. Assume $P(x), Q(x)$ are relatively prime, and that $f(n)$ is the generating function for $P(x)/Q(x)$ written in the form promised by the lemma above, for $n \gg 0$. Since $f(n) = f(2n)$,
$$Q(x) = \prod_{i=1}^{k}(1-\lambda_i x)^{d_i} = \prod_{i=1}^{k}(1-\lambda_i^2 x)^{e_i},$$
so
$$\{\lambda_1, \dots, \lambda_k\} = \{\lambda_1^2, \dots, \lambda_k^2\}.$$
As in the proof of Theorem 4.1, we conclude that each $\lambda_i$ is a primitive $r_i$th root of unity for some positive integers $r_1, \dots, r_k$.

Let $M = \operatorname{lcm}(r_1, \dots, r_k)$ and for $a \in \mathbf{N}$, define
$$R_a = \{m \in \mathbf{N} : m \equiv a \mod M\}$$
and $f_a = f|_{R_a}$ to be the restriction of the function $f : \mathbf{N} \to \mathcal{C}$ to the set $R_a$. Then
$$f_a(a+jM) = \sum_{i=1}^{k} P_i(a+jM)\lambda_i^{a+jM} = \sum_{i=1}^{k} P_i(a+jM)\lambda_i^{a},$$
so each $f_a$ has a representation as a polynomial in the variable $j$ since $\lambda_i^a$ is constant on the set $R_a$. We denote the natural extension of this map to an element of $\mathcal{C}[j]$ by $F_a$. Note that the restriction of $F_a$ to $\mathbf{N}$ will not be $f$ in general. Our goal is to prove that each $F_a$ is a

constant function, with corresponding constant denoted by $c_a$. Once this is shown, we have

$$\frac{P(x)}{Q(x)} = \sum_{a=1}^{M} c_a \sum_{j=0}^{\infty} x^{a+jM} = \sum_{a=1}^{M} \frac{c_a x^a}{1-x^M},$$

so $P(x)/Q(x)$ is a rational function with only simple poles, as desired.

It remains to show that each polynomial map $F_a : \mathcal{C} \to \mathcal{C}$ is a constant function. For each positive integer $n$, define $S_n = \{2^t n : t \in \mathbf{N}\}$. We say that $a$ has an *infinite cross-section* if $R_a \cap S_n$ is an infinite set for some $n \in \mathbf{N}$. We proceed by considering two cases, depending on whether $a$ has an infinite cross-section or not.

*Case* 1. Suppose $a$ has an infinite cross-section, i.e., $R_a \cap S_n$ is an infinite set. Since $f(s) = f(2s)$ for all $s \in \mathbf{N}$, $F_a$ is constant on $R_a \cap S_n$. Since $R_a \cap S_n$ is an infinite set, $F_a$ is a constant polynomial.

*Case* 2. Suppose $a$ does not have an infinite cross-section, i.e., $R_a \cap S_n$ is finite for all positive integers $n$. Then $R_a \cap S_n$ must be nonempty for infinitely many values of $n$. Since there are only finitely many distinct sets of the form $R_b$, it follows that for each $S_n$, there exists $b \in \mathbf{N}$ such that $R_b \cap S_n$ is infinite. Moreover, since there are only finitely many choices for $R_b$, there is at least one $b \in \mathbf{N}$ such that there exist infinitely many values of $n$ where $R_a \cap S_n$ is nonempty and $R_b \cap S_n$ is infinite. Since $b$ has an infinite cross-section, an application of Case 1 demonstrates that the restriction of $f$ to $R_b$ is the constant function $c_b$. Since $f$ is constant on each $S_n$, the restriction of $f$ to $S_n$ is the constant $c_b$. Thus $F_a$ achieves the value $c_b$ infinitely many times, and so $F_a$ must be a constant polynomial. $\square$

The proof of Theorem 4.1 is complete.

*Note* 4.4. For each fixed point $R_f$ we construct the rational function

$$R(x) = \sum_{j=1}^{2^n-1} x^{j-1} R_{j,n}(x^{2^n}) - R_f(x), \tag{4.11}$$

where $R_{1,n}, \ldots, R_{2^n-2,n}$ are arbitrary rational functions and $R_{2^n-1,n} \neq 0$. Theorem 2.2 shows that $R$ is the general form of a prefixed sequence of length $n$ attached to $R_f$.

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