Journal of Physics Special Topics
An undergraduate physics journal

A1_3 Hydrogen Atom in One Dimension

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December 2, 2017

Abstract

We investigated the ground state energy of the hydrogen atom if it existed in a world with only one spatial dimension. Using variational principle, we found that the ground state energy of a 1D hydrogen atom is less than or equal to 0.17 eV.

Introduction

It is always interesting to think about how physics would work under different circumstances. In this article we will explore the consequences of having only one spatial dimension; in particular we will investigate the hydrogen atom and see how its ground state energy differs.

Coulomb Force in 1D

Inverse square laws such as the Coulomb force have a very different form in a 1D world. Here, we will derive the Coulomb force in 1D from Gauss' law. Using the integral form of Gauss' law, we can find the Electric field at a distance x from a point charge q_1:

$$\oint_S \vec{E} \cdot \hat{n} \ dA = \frac{q_1}{\epsilon_0} \tag{1}$$

Where $\vec{E}$ is the electric field, $\hat{n}$ is the normal vector of the surface and ϵ_0 is the electric constant. For this system, we will use a spherical Gaussian surface centered on the point charge. The electric field radiates evenly in both directions (since there are only two directions in a 1D world) and is always parallel to the normal vector of a 1D sphere (the Gaussian surface). The surface area of a 1D sphere is 2 units.[1] Therefore:

$$\oint_S \vec{E} \cdot \hat{n} \ dA = E \oint_S \ dA = 2E \tag{2}$$

Substitute Eq.(2) into Eq.(1) we get:

$$2E = \frac{q_1}{\epsilon_0} \tag{3}$$

and the magnitude of the Coulomb force in 1D acting upon a second point charge q_2 is:

$$F = q_2 E = \frac{q_1 q_2}{2\epsilon_0} \tag{4}$$

Test particles feel the same amount of force, provided that they are at the same distance from the source. In 1D, only two points satisfy this. In this case, the force is evenly distributed among the test particles. It is important that the units of the electric constant change according to the world's dimensions. This is in order to conserve the units on both sides of Eq.(4). However whether or not the magnitude will change with dimension is unclear; in this article we will assume the constant has the same magnitude but different units.

Coulomb Potential Energy in 1D

The potential energy (V) is given by:

$$V = \int_C \vec{F} \cdot d\vec{l} \tag{5}$$

This is a path integral connecting the point with zero potential to a point located at a distance x from the source. Using Eq.(4)and Eq.(5), we find the Coulomb potential energy in 1D is:

$$V = \frac{q_1 q_2}{2\epsilon_0}|x| \tag{6}$$

In 3D, the point of zero potential is at infinity, since $\vec{F}$ is proportional to $1/x^2$. In the 1D case, $\vec{F}$ is independent of x, therefore V is directly proportional to x. Thus the point of zero potential is now located at $x = 0$.

Schrödinger Equation for Hydrogen in 1D

There is no rigorous derivation for the Schrödinger equation, so it is unclear if it changes depending on the number of dimensions. All that we know is that it works in our 3D world. That being said, the Schrödinger equation is ultimately a statement about the conservation of energy, which we can assume to hold in any number of dimensions. Thus, in 1D, the Schrödinger equation of a hydrogen atom takes the form:

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + \frac{q_1 q_2}{2\epsilon_0}|x|\Psi = E\Psi \tag{7}$$

where we use Eq.(6) for the potential energy and the other symbols have their usual meanings. Eq.(7) has no trivial solution, but we can impose a limit on the ground state energy using the variational principle. For this system with a uniform restoring force, we use a trial function:

$$v(x) = \begin{cases} a^2 - x^2 & \text{if } |x| \leq a \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

From the variational principle we can show that:

$$E_0 \leq \langle \widehat{H} \rangle = \frac{15}{16}\left(\frac{\hbar^2 b^2}{m}\right)^{1/3}, b = \frac{q_1 q_2}{2\epsilon_0} \tag{9}$$

Where E_0 is the ground state energy of the hydrogen atom, m is the electron mass, and q_1 q_2 are the charges of the proton and electron. [2]

Using numerical values, we found that the ground state energy must be less than or equal to 0.17 eV.

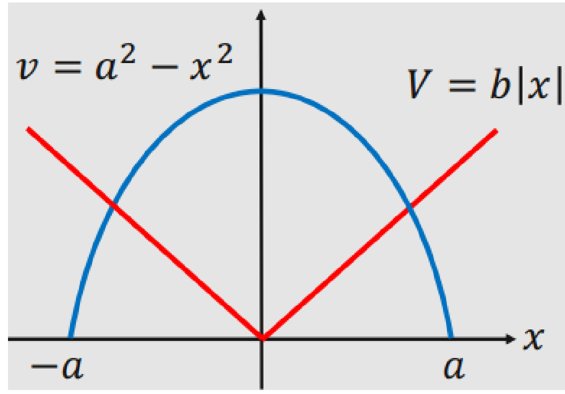

Figure 1: Graph of $|x|$ potential with trial function.[2]

Conclusion and Discussion

In conclusion we have shown that the inverse square laws are unique to the 3D world. In a 1D world, forces such as the Coulomb force no longer decrease with distance between the particles. As a result, the energy levels of the hydrogen atoms must change to reflect the new potential imposed by the 1D Coulomb force, as shown by the decrease of ground state energy from 13.6 eV (3D) [3] to 0.17 eV (1D).

It is easy to assume that physics is simpler in lower dimensions; this is generally true when we are talking about exploiting the symmetry of a system and modeling it with a 1D system embedded in a 3D world. However, when we are actually dealing with an entire world with different dimensions to ours, things can become very strange.

References

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[3] Tipler, P. and Mosca, G. (2008). Physics for scientists and engineers. New York: W. H. Freeman. Sixth Edition, pg 1237