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P4_7 Greatest Gravity

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Abstract

Earth is not of uniform density, but consists of many layers, causing acceleration due to free fall to be a non-linear function of radius. A model is constructed to evaluate at which radius the greatest gravitational acceleration is felt. This point was found to be at the meeting point of the lower mantle and the outer core, where the acceleration peaks at a = 10.8 ms^{-2}.

Introduction

The concept of free-falling through Earth is not new to scientists; Newton and Hooke argued on the subject many centuries ago [1]. However, rather than using the typical model of a uniform density Earth, this paper attempts to model Earth's construction as accurately as possible, to find the point at which gravity is at its strongest.

Earth's Layers

If an object fell radially through a uniform density Earth with no other forces acting, the acceleration it experiences would decrease linearly with its distance from Earth's centre (Earth's rotation is negated). All the matter outside the radius at which the object has reached, pulls the object in every direction with an equal magnitude, thus the gravitational attraction of all the mass in the outer shell sums to zero, leaving only the sphere of mass below the object's position to cause acceleration.

In reality, various different compounds exist within Earth, each with their own characteristics, which have formed distinct layers. Due to the pressure gradient caused by gravity, the density through a single layer is not believed to be constant, but increases linearly towards the centre [2]. However, including this gradient in the mathematics adds great complexity to the problem, and so it will be assumed that each layer is of constant density.

Earth is comprised of five distinct layers: the crust, the upper mantle, the lower mantle, the outer core and the inner core. Each layer's density, radial range and mass are listed in Table 1. It is assumed Earth is spherically symmetric, since without this assumption, the objective would require a complex three dimensional model of gravity, not appropriate for this report.

Layer	Av. Density (kg/m^3)	Radial Range (km)	Mass ($\times 10^{24}$ kg)
Crust	2600	6371 – 6341	0.0396
Upper Mantle	3900	6341 – 5651	1.217
Lower Mantle	5000	5651 – 3451	2.919
Outer Core	11100	3451 – 1191	1.833
Inner Core	13000	1191 – 0	0.092

Table 1 – The average density and radial range for each layer [3].

Using these data, Earth's mass was found to be 6.09×10^{24} kg; within 2% of the accepted value of 5.972×10^{24} kg [4]. Thus, it can be concluded that this model is a good approximation.

Acceleration Equations

The acceleration at a point within Earth can be expressed as a function of distance to the Earth's centre, r. This expression can be derived from Newton's law of gravitation:

$$a = -\frac{GM}{r^2}, \qquad (1)$$

where a is the acceleration of a massive object, M is the mass inducing the acceleration and G is the gravitational constant. Above Earth's surface, the whole of Earth's mass acts to induce an acceleration, but beneath the surface this equation must be augmented so that the shell of matter above r is negated. The acceleration equation within a uniform density Earth is

$$a = -\frac{G\rho V}{r^2} = -\frac{4}{3}\frac{G\rho\pi r^3}{r^2} = -\frac{4}{3}G\rho\pi r, \qquad (2)$$

where V is the volume of mass enclosed by r and ρ is the density of the matter.

Now Earth's layers can be considered. If it is assumed an object is within the crust, all the mass of the lower layers are acting on the body in their entirety, since their whole volume is within the radius of the object's position. However, only the portion of the crust that is within the object's radial distance contributes. This portion can be considered a shell of thickness equal to the distance between the object and the 'floor' (i.e. lowest point) of the crust. Added to the contribution from the lower layers, this gives a curve for the acceleration through the crust, given in equation 3:

$$a_c = -\left(\frac{4\pi G\rho_c(r^3 - r_{um}{}^3)}{3r^2} + \frac{G(M_{um}+M_{lm}+M_{oc}+M_{ic})}{r^2}\right), \quad (3)$$

where r_{um} is the upper mantle's 'ceiling' radius; this convention holds for any r with a subscript. Similarly, the subscripts give the initials of the layer to which the parameter corresponds (e.g. M_{um} is the mass of the upper mantle layer). The first term describes the acceleration caused by a shell of crust matter, of thickness $r - r_{um}$. If the object is now taken to be inside the upper mantle, the crust vanishes from the equation and the upper mantle portion within r must be considered a shell;

$$a_{um} = -\left(\frac{4\pi G\rho_{um}(r^3 - r_{lm}{}^3)}{3r^2} + \frac{G(M_{lm}+M_{oc}+M_{ic})}{r^2}\right). \quad (4)$$

A similar equation can be constructed for the acceleration curve through each layer.

Peak Acceleration

To find the point of maximum acceleration, peaks in the curve must be found. This is done by differentiating to find the extremum points, and differentiating again to deduce which of these are maxima. If no maximum exists within the boundaries for that layer (the range of r where the layer lies), then the peak acceleration must exist at either the floor or ceiling of the layer, since the gradient of acceleration remains in the same direction throughout that range. This procedure is carried out for the crust as follows:

$$\frac{da_c}{dr} = \frac{6G(M_{um}+M_{lm}+M_{oc}+M_{ic}) - 8\pi G\rho_c r_{um}{}^3}{3r^3} - \frac{4\pi G\rho_c}{3},$$

$$\text{where } \frac{da_c}{dr} = 0 \text{ at extremum.} \quad (5a, 5b)$$

$$r'_c = \sqrt[3]{\frac{6(M_{um}+M_{lm}+M_{oc}+M_{ic}) - 8\pi\rho_c r_{um}{}^3}{4\pi\rho_c}}, \quad (6)$$

where r'_c is the radius at an extremum point of acceleration for the crust. This gives r'_c = 8.45 × 10^6 m, which lies outside range of r applicable to the crust. After analysing all layers, only the lower mantle exhibited an extremum within its range. This extremum was found to be a minimum, thus not of interest to this model.

The surface acceleration is well known to be 9.81 ms^{-2}. According to equation (1), the value is 9.82 ms^{-2}. This discrepancy is caused by the two assumptions made when constructing the model, and can be used as a rough estimate for the error margins on the data. Table 2 gives values for the acceleration magnitude at each layer's floor.

Position	Acceleration (ms^{-2})	Radius (km)
Crust Floor	9.84 ± 0.01	6341
Upper Mantle Floor	10.1 ± 0.01	5651
Lower Mantle Floor	10.8 ± 0.01	3451
Outer Core Floor	4.33 ± 0.01	1191

Table 2 – The gravitational acceleration calculated at specified radii.

Conclusions

The results show that gravitational acceleration reaches its peak at the meeting point of the mantle and the core. This is the point closest to the higher density regions, whilst the entire body of high-density matter is still acting to accelerate, thus the conclusion that this is the point of greatest acceleration seems reasonable. The steep drop in acceleration from the outer core to the inner core is expected, due to the fact that as the high density regions are traversed, a lot of matter is passed in a short distance, which then no longer induces acceleration in the object. Hence, this model seems to predict the behaviour of gravity within Earth to a good approximation.

References

[1] PA3830 Physics Challenge Question Book, Dept. Physics and Astronomy, University of Leicester, 2012.

[2] Tenzer, R., et al, Contributions to Geophysics and Geodesy, **42** (2012).

[3] http://pubs.usgs.gov/gip/interior/, accessed on 13/11/2013.

[4] Tipler and Mosca (2008), "*Physics with Modern Physics for Scientists and Engineers*", 6th edition, W.H. Freeman and Company.