ADDITIVE TWISTS AND A CONJECTURE BY MAZUR, RUBIN AND STEIN

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Abstract. In this paper, a conjecture of Mazur, Rubin and Stein concerning certain averages of modular symbols is proved. To cover levels that are important for elliptic curves, namely those that are not square-free, we establish results about L-functions with additive twists that are of independent interest.

Contents

1. Introduction

Motivated by a question regarding ranks of elliptic curves defined over cyclic extensions of $\mathbb{Q}$, B. Mazur and K. Rubin [10] studied the statistical behaviour of modular symbols associated to a weight 2 cusp form corresponding to an elliptic curve. Based on both theoretical and computational arguments (the latter jointly with W. Stein) they formulated a number of precise conjectures. We state one of them in its formulation given in [12].

For a positive q, let $\Gamma_0(q)$ denote the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant 1 with $a, b, c, d \in \mathbb{Z}$ and $q \mid c$. Let

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} = \sum_{n=1}^{\infty} A(n) n^{1/2} e^{2\pi i n z} \tag{1.1}$$

be a newform of weight 2 for $\Gamma_0(q)$.

For each $r \in \mathbb{Q}$, we set

$$\langle r \rangle^{+} = 2\pi \int_{i\infty}^{r} \Re(i f(z) dz) \qquad \text{and} \qquad \langle r \rangle^{-} = 2\pi i \int_{i\infty}^{r} \Re(f(z) dz).$$

For each $x \in [0,1]$ and $M \in \mathbb{N}$, set

$$G_M^{\pm}(x) = \frac{1}{M} \sum_{0 \le a \le Mx} \left\langle \frac{a}{M} \right\rangle^{\pm}.$$

Mazur, Rubin and Stein, in [10] stated the following conjecture:

Conjecture 1.1. *For each $x \in [0,1]$, we have*

$$\lim_{M \to \infty} G_M^{+}(x) = \frac{1}{2\pi} \sum_{n \ge 1} \frac{a(n) \sin(2\pi n x)}{n^2}; \tag{1.2}$$

$$\lim_{M \to \infty} G_M^{-}(x) = \frac{1}{2\pi i} \sum_{n \ge 1} \frac{a(n)(\cos(2\pi n x) - 1)}{n^2}.$$

The heuristic for this conjecture can be seen by the computation

$$G_M^{+}(x) = \frac{1}{M} \sum_{0 \le a \le Mx} \left\langle \frac{a}{M} \right\rangle^{+} = 2\pi \Re \left(i \int_{\infty}^{0} \frac{1}{M} \sum_{0 \le a \le Mx} f\left(\frac{a}{M} + iy \right) i \, dy \right).$$

The inner sum is a Riemann sum for the horizontal integral $\int_0^x$. As a heuristic let us replace the sum with the integral, even though the error is not controlled for small y. Upon doing this, computing the integral using the Fourier expansion of f gives us the right hand side of (1.2).

An average version of this conjecture in the case of square-free levels was proved in [12]. The same paper contains the proofs of other conjectures from the original set listed in [10] (see [11] for a recent presentation of the conjectures in the form of an article). More recently, one of the original conjectures of [10], namely the one dealing with the variance of the modular symbols, was proved in [4]. The authors also established a form of Conjecture 1.1 in the special case that $x = 1$ and M goes to infinity over a sequence of primes.

In this paper, we prove Conjecture 1.1 for an arbitrary level q, each $x \in [0,1]$ and as M goes to infinity over any sequence of integers. Our main theorem is as follows.

Theorem 1.2. *For each $x \in [0,1]$, as M goes to infinity, we have*

$$G_M^{+}(x) = \frac{1}{2\pi} \sum_{n \ge 1} \frac{a(n) \sin(2\pi n x)}{n^2} + \mathcal{O}_\epsilon\Bigg((Mq)^\epsilon M^{-\frac{1}{4}} q^{\frac{1}{4}} \prod_{\substack{p \mid (q,M) \\ p^2 \mid q}} p^{\frac{1}{2}} \Bigg);$$

$$G_M^{-}(x) = \frac{1}{2\pi i} \sum_{n \ge 1} \frac{a(n)(\cos(2\pi n x) - 1)}{n^2} + \mathcal{O}_\epsilon\Bigg((Mq)^\epsilon M^{-\frac{1}{4}} q^{\frac{1}{4}} \prod_{\substack{p \mid (q,M) \\ p^2 \mid q}} p^{\frac{1}{2}} \Bigg),$$

for any $\epsilon > 0$.

Very recently, H.-S. Sun [13] announced a similar statement in the special case of q square-free, with a slightly weaker exponent in q. The main reason for the difference in our results is that we develop an approximate functional equation for the additive twists of L-functions applicable to all levels and additive conductors and that we are then able to solve the difficult problem of bounding the Fourier coefficients of the "dual" function (Proposition 3.6)

The starting point of our method was the use of *Eisenstein series with modular symbols* in [12] combined with the computation of its Fourier coefficients in terms of shifted convolution series in [6]. In this paper we succeed in avoiding its use and this simplifies our argument. In an earlier version of the paper, the shifted convolution series itself remained a key tool, but we are now able to circumvent those too. (In this respect, our method parallels that of [13]). However, the part we no longer require for the proof of our main theorem contains several methods and results of independent interest and novelty, including *double* shifted convolution series. It is one of the themes of work in progress.

As noted above, previous progress towards the Mazur, Rubin and Stein conjecture concerned only the case of square-free level (or prime M). Extending to non-square-free levels proved much less routine than we expected and it led to results of independent interest. For example, in Proposition 3.8 we prove a general bound for antiderivatives of weight 2 newforms

$$\int_{\infty}^{\frac{a}{d}} f(z)\, dz$$

that holds for all rational values a/d and levels q. The proof of this bound is based on another result of independent importance, namely Proposition 3.6. As mentioned in [8, Section 14.9], the Ramanujan-Petersson bound for Fourier coefficients of a Dirichlet twist of f holds even when the twist is not a newform, but there is an implied constant which may depend on the level badly. In Proposition 3.6 we make that dependence entirely explicit.

The twisted cusp form that is the subject of Proposition 3.6 appears as the "dual" function in a functional equation for additive twists of L-functions for general levels and weights (Theorem 3.1). Theorem 3.1 is another result of independent interest and can be viewed as a Voronoi type formula. This is a very well-studied formula in analytic number theory but, whereas there are various instances of it proved for combinations of the twist and the level of the newform satisfying certain conditions (e.g. [9]), Theorem 3.1 seems, to our knowledge, to be the first general result that applies to all twists and levels in this explicit form. A referee of this work has brought to our attention a Voronoi type formula [1] that appeared after we announced our results and which is closely related to our Theorem 3.1. Specifically, the formula of [1] relates additively twisted Fourier coefficients of a Hecke eigenform to a dual sum of its Fourier coefficients at another cusp, related with the original additive twists. In our Theorem 3.1, we relate the additively twisted L-function of a Hecke eigenform to a linear combination of multiplicatively twisted L-functions. This approach, together with the Atkin–Lehner–Li theory [2], leads to an explicit functional equation, where the "dual" functions are expressed in terms of a linear combination of the original Hecke eigenvalues which are multiplicatively twisted and also additively twisted. This explicit form turns out to be precisely the formulation required, because we need bounds for the coefficients of the "dual" cusp forms (see the proof of Proposition 3.6).

1.1. **Outline of the proof of Theorem 1.2.** The technical aspects of the proof of Theorem 1.2 are quite complex, and for that reason we supply here a high level roadmap that we hope will make our proof a bit easier to understand.

The first step in the proof is to express $G_M^{\pm}(x)$ as a sum of modular symbols weighted by a family of smooth functions h_δ that approximates the characteristic function of $[0, x]$. This is done in Section 2, where an explicit family $\{h_\delta\}$ is constructed. In Lemma 2.1 it is shown that for any fixed $\delta = \delta_M < 1$ we have

$$G_M^{\pm}(x) = \frac{1}{M} \sum_{0 \le a \le M} \left\langle \frac{a}{M} \right\rangle^{\pm} h_\delta\left(\frac{a}{M}\right) + \text{error, uniform in } q \text{ and } M.$$

In view of this expression, in the following sections we focus on sums of the form

$$\frac{1}{M}\sum_{0\le a\le M}\left\langle\frac{a}{M}\right\rangle^{\pm} h\left(\frac{a}{M}\right)$$

for an arbitrary smooth period function h on $\mathbb{R}$. We have

$$\frac{1}{M}\sum_{0\le a\le M}\left\langle\frac{a}{M}\right\rangle^{\pm} h\left(\frac{a}{M}\right) = \text{an explicit series involving } L(1,f,a/M) \tag{1.3}$$

(see (2.15) and (2.14)). Here $L(1,f,a/M)$ is the central value of the additive twist of L-function.

To study the asymptotics of $L(1,f,a/M)$, which is required for the completion of the proof of Theorem 1.2, we need a functional equation for $L(s,f,a/M)$ applying to all levels q and integers M. The functional equation and the explicit Ramanujan-type bound for the Fourier coefficients of the twisted cusp forms in the case we need them is the content of Corollary 3.7.

Two important implications of the functional equation (also of independent interest) are the bound (3.27) for modular symbols and the approximate functional equation (3.35), both of which apply to arbitrary levels.

In Section 4 we substitute $L(1,f,a/M)$ in the right hand side of (1.3), using the approximate functional equation (3.35), and this leads us to an expression (4.3) consisting of two parts.

The first part is shown (in Lemma 4.2) to contribute the main term. The second part is complicated, but can be bounded using Weil's bound for Kloosterman sums and the explicit Ramanujan-type bound for the Fourier coefficients of twisted cusp forms proved in Corollary 3.7. Combining these two pieces together, we deduce

$$\frac{1}{M}\sum_{0\le a\le M}\left\langle\frac{a}{M}\right\rangle^{\pm} h_\delta\left(\frac{a}{M}\right) = \frac{1}{2}\Big\{\sum_{n\ge1}\Big(\hat{h}_\delta(-n)\pm\hat{h}_\delta(n)\Big)\frac{a(n)}{n}\Big\} + \text{explicit error term depending on } q \text{ and } M, \tag{1.4}$$

where $\hat{h}_\delta(n)$ stands for the n-th Fourier coefficient of the periodic function h_δ.

As the functions h_δ approach the characteristic function of $[0,x]$ as $\delta\to0$, $\hat{h}_\delta(n)$ approaches $(1-e^{-2\pi inx})/(2\pi in)$. Applying this to (1.4) and using the explicit form of the error term, we prove Theorem 1.2. The details of this final computation are shown in Section 5.

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2. An expression of $G_M^{\pm}(x)$

For a fixed $x\in[0,1]$, consider the characteristic function $1_{[0,x]}$ of $[0,x]$ extended to $\mathbb{R}$ periodically with period 1. We will construct a family of complex valued smooth functions on $\mathbb{R}/\mathbb{Z}$ approximating $1_{[0,x]}$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth, non-negative function, compactly supported in $(-1/4, 1/4)$ with $\int_{-1/2}^{1/2} \phi(t)dt = 1$ and $\phi(0) = 1$. For each $\delta < 1$ and $t \in (-1/2, 1/2)$, set

$$\phi_\delta(t) = \delta^{-1}\phi(t/\delta) \tag{2.1}$$

and extend this to $\mathbb{R}$ periodically, with period 1. The approximating functions are h_δ defined by

$$h_\delta(t) := 1_{[-\delta,x+\delta]} \star \phi_\delta(t) = \int_{-\delta}^{x+\delta} \phi_\delta(t-v)\, dv = \int_{t-x-\delta}^{t+\delta} \phi_\delta(v)\, dv,$$

where $\star$ denotes the convolution. This function is smooth, periodic and satisfies $0 \le h_\delta(t) \le 1$. Further,

$$h_\delta(t) = 0 \qquad \text{for } (5\delta/4 + x, 1 - 5\delta/4) \text{ and its translates.} \tag{2.2}$$

Indeed, for $5\delta/4 + x < t < -5\delta/4 + 1$, we have $\delta/4 < t - x - \delta < t + \delta < 1 - \delta/4$. Since the support of $\phi_\delta(v)$ is contained in $(-\delta/4, \delta/4)$ and its translations, (2.1) implies that $\phi_\delta(t)$ vanishes in that range.

We further have

$$h_\delta(t) = 1, \qquad \text{for } t \in [0, x] \tag{2.3}$$

and

$$\widehat{h_\delta}(n) = \int_{-1/2}^{1/2} h_\delta(x)e^{-2\pi i n x}\, dx = \widehat{1_{[-\delta,x+\delta]}}(n) \cdot \widehat{\phi_\delta}(n), \tag{2.4}$$

for the corresponding nth Fourier coefficients. This implies that, for $n \neq 0$,

(2.5)
$$\begin{aligned}\widehat{h_\delta}(n) = \frac{e^{2\pi i n\delta} - e^{-2\pi i n(x+\delta)}}{2\pi i n} \int_{-1/2}^{1/2} \phi_\delta(t)e^{-2\pi i n t}\, dt &= \frac{e^{2\pi i n\delta} - e^{-2\pi i n(x+\delta)}}{2\pi i n} \int_{\frac{-1}{2\delta}}^{\frac{1}{2\delta}} \phi(t)e^{-2\pi i n\delta t}\, dt \\ &= \frac{e^{2\pi i n\delta} - e^{-2\pi i n(x+\delta)}}{2\pi i n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(t)e^{-2\pi i n\delta t}\, dt.\end{aligned}$$

The last equality follows because ϕ is supported in $(-1/4, 1/4)$. With the smoothness of h_δ we deduce that, for each $K \ge 0$ and $n \neq 0$,

$$|\widehat{h_\delta}(n)| \ll_K (|n| + 1)^{-1}(\delta(1 + |n|))^{-K}. \tag{2.6}$$

This inequality combines a bound that is uniform in δ with a stronger one that, however, is not uniform in δ. Let

$$\frac{1}{2}A_h^\pm(M) = \frac{1}{M} \sum_{0 \le a \le M} \left\langle \frac{a}{M} \right\rangle^\pm h\left(\frac{a}{M}\right). \tag{2.7}$$

With these notations, we have the following

Lemma 2.1. *For $M > 1$, consider any fixed $\delta = \delta_M < 1$. Then,*

$$G_M^\pm(x) = \frac{1}{2}A_{h_\delta}^\pm(M) + \mathcal{O}\Bigg(\delta_M M^{\frac{1}{2}} q^{\frac{1}{4}} (qM)^\epsilon \prod_{\substack{p \mid M \\ \mathrm{ord}_p(M) < \mathrm{ord}_p(q)}} p^{\frac{1}{4}}\Bigg).$$

Note that the product over primes $p \mid M$ equals 1 if q is square-free or $(q, M) = 1$.

Proof. If $a \le Mx$, then $\frac{a}{M} \le x$ and thus $h_\delta(t) = 1$ by (2.3). These terms give us $G_M^{\pm}(x)$.

The error term is obtained by studying the case $xM < a \le xM + \frac{5}{4}M\delta_M$. Then $x < \frac{a}{M} \le x + \frac{5}{4}\delta_M$. By definition, $\left\langle \frac{a}{M} \right\rangle^{\pm}$ is a linear combination of $\int_\infty^{a/M} f(z)\, dz$ and its complex conjugate (see (2.11)). In (3.27), we prove a bound for this modular symbol that implies

$$\left\langle \frac{a}{M} \right\rangle^{\pm} h_\delta\left(\frac{a}{M}\right) \ll M^{\frac12} q^{\frac14} (Mq)^{\epsilon} \prod_{\substack{p \mid M \\ \mathrm{ord}_p(M) < \mathrm{ord}_p(q)}} p^{\frac14},$$

and thus

$$\begin{aligned} \frac{1}{M} \sum_{Mx < a \le Mx + \frac54 M\delta_M} \left\langle \frac{a}{M} \right\rangle^{\pm} h_\delta\left(\frac{a}{M}\right) &\ll \frac{1}{M} M^{\frac12} q^{\frac14} (qM)^{\epsilon} M\delta_M \prod_{\substack{p \mid M \\ \mathrm{ord}_p(M) < \mathrm{ord}_p(q)}} p^{\frac14} \\ &= M^{\frac12} q^{\frac14} (qM)^{\epsilon} \delta_M \prod_{\substack{p \mid M \\ \mathrm{ord}_p(M) < \mathrm{ord}_p(q)}} p^{\frac14}. \end{aligned} \tag{2.8}$$

Similarly,

$$\frac{1}{M} \sum_{M - \frac54 M\delta_M < a \le M} \left\langle \frac{a}{M} \right\rangle^{\pm} h_\delta\left(\frac{a}{M}\right) \ll M^{\frac12} q^{\frac14} (qM)^{\epsilon} \delta_M \prod_{\substack{p \mid M \\ \mathrm{ord}_p(M) < \mathrm{ord}_p(q)}} p^{\frac14}.$$

If $xM + \frac54 M\delta_M < a \le M - \frac54 M\delta_M$, then $x + \frac54\delta_M < \frac{a}{M} \le 1 - \frac54\delta_M$ and thus, by (2.2), $h_\delta(a/M)$ vanishes. Therefore

$$\begin{aligned} &\frac{1}{M} \sum_{0 \le a \le M} \left\langle \frac{a}{M} \right\rangle^{\pm} h_\delta\left(\frac{a}{M}\right) \\ &= \frac{1}{M} \left(\sum_{0 \le a \le Mx} + \sum_{xM < a \le Mx + \frac54 M\delta_M} + \sum_{Mx + \frac54 M\delta_M < a \le M - \frac54 M\delta_M} + \sum_{M - \frac54 M\delta_M < a \le M} \right) \left\langle \frac{a}{M} \right\rangle^{\pm} h_\delta\left(\frac{a}{M}\right) \\ &= \frac{1}{M} \sum_{0 \le a \le Mx} \left\langle \frac{a}{M} \right\rangle^{\pm} \cdot 1 + \mathcal{O}\Bigg(\delta_M M^{\frac12} q^{\frac14} (qM)^{\epsilon} \prod_{\substack{p \mid M \\ \mathrm{ord}_p(M) < \mathrm{ord}_p(q)}} p^{\frac14} \Bigg) \end{aligned}$$

as required. ∎

In view of this lemma, we will initially study this average for an arbitrary smooth periodic h. For each smooth $h : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ and each positive integer M, we have

$$\frac12 A_h^{\pm}(M) = \sum_{n \in \mathbb{Z}} \hat{h}(n) \frac{1}{M} \sum_{0 \le a \le M} \left\langle \frac{a}{M} \right\rangle^{\pm} e^{2\pi i n \frac{a}{d}} = \sum_{n \in \mathbb{Z}} \hat{h}(n) \frac{1}{M} \sum_{d \mid M} \sum_{\substack{a \bmod d \\ (a,d)=1}} \left\langle \frac{a}{d} \right\rangle^{\pm} e^{2\pi i n \frac{a}{d}}. \tag{2.9}$$

We will express the right-hand side of (2.9) in terms of additive twists of the L-function of f, whose definition we now recall. Let f be a cusp form of weight k for $\Gamma_0(q)$. For a positive integer d and $a \in \mathbb{Z}$, let

$$L(s, f, a/d) = \sum_{n=1}^{\infty} \frac{a(n) e^{2\pi i n \frac{a}{d}}}{n^s}$$

be the additive twist of the L-function for f, and

$$\Lambda(s,f,a/d) = \int_0^\infty f\left(\frac{a}{d}+iy\right) y^s \,\frac{dy}{y} = (2\pi)^{-s}\,\Gamma(s)\,L(s,f,a/d). \tag{2.10}$$

The series defining $L(s,f,a/d)$ is sometimes called a Voronoi series. It converges absolutely for $\Re(s) > 1+(k-1)/2$. For consistency with the formulation of the Mazur-Rubin-Tate conjecture, we normalise the series so that the central point is at $k/2$.

Both $L(s,f,a/d)$ and $\Lambda(s,f,a/c)$ have analytic continuation to $s\in\mathbb{C}$. Further properties are studied in Section 3.

With the notations above, we have

$$\begin{aligned}\left\langle\frac{a}{d}\right\rangle^{\pm} &= -\pi\int_\infty^0 \Big(f(\frac{a}{d}+iy) \pm f(-\frac{a}{d}+iy)\Big)\;dy \\ &= \pi\Big(\Lambda(1,f,\frac{a}{d}) \pm \Lambda(1,f,-\frac{a}{d})\Big) = \frac{1}{2}\Big(L(1,f,\frac{a}{d}) \pm L(1,f,-\frac{a}{d})\Big).\end{aligned} \tag{2.11}$$

Here we used $\overline{f(\frac{a}{d}+ix)} = f(-\frac{a}{d}+ix)$. This implies

$$\sum_{\substack{a \bmod d\\(a,d)=1}} \left\langle\frac{a}{d}\right\rangle^{\pm} e^{2\pi i n\frac{a}{d}} = \pi \sum_{\substack{a \bmod d\\(a,d)=1}} \Big(\Lambda(1,f,\frac{a}{d}) \pm \Lambda(1,f,-\frac{a}{d})\Big)\, e^{2\pi i n\frac{a}{d}}. \tag{2.12}$$

Applying (2.12) to (2.9),

$$A_h^{\pm}(M) = \sum_{n\in\mathbb{Z}} \hat{h}(n)\frac{1}{M}\sum_{d|M}\sum_{\substack{a \bmod d\\(a,d)=1}} \Big(L(1,f,\frac{a}{d}) \pm L(1,f,-\frac{a}{d})\Big)\, e^{2\pi i n\frac{a}{d}}. \tag{2.13}$$

Let

$$\alpha_{n,M}(t) = \frac{1}{M}\sum_{a \bmod M} e^{-2\pi i n\frac{a}{M}} L(t,f,\frac{a}{M}) = \frac{1}{M}\sum_{d|M}\sum_{\substack{a \bmod d\\(a,d)=1}} e^{-2\pi i n\frac{a}{d}} L(t,f,\frac{a}{d}). \tag{2.14}$$

Then we get

$$A_h^{\pm}(M) = \sum_{n\in\mathbb{Z}} \hat{h}(n)\big(\alpha_{-n,M}(1) \pm \alpha_{n,M}(1)\big). \tag{2.15}$$

We will study the properties of $L(t,f,a/d)$, the additive twist of an L-function twists in the next section. As mentioned in the introduction, we prove our results for general levels and weights. We summarize the results for the special case of interest of weight 2 in Section 3.4.

3. Properties of the additive twist of an L-function

In this section we bound Fourier coefficients of certain twisted newforms that will appear in an application of the approximate functional equation (Proposition 3.6). Our bounds are uniform in terms of the level and they will be crucial for the proof of the main theorem. Those twisted newforms arise in the context of a general functional equation for the additive twist of an L-function. Our functional equation is of independent interest because all references we are aware of give the functional equation only for special combinations of the level and the denominator of the additive twist [9]. As mentioned in the introduction, a very recent paper by Assing and Corbett [1] also contains the proof of a Voronoi type summation formula which is based on a functional equation for L-series with additive twists. Their functional equation relates the additively twisted coefficients of the given L-function with Fourier coefficients at another cusp, related with the additive twists. Our proof makes use of the fact that additive twists can be represented as a linear combination of multiplicative twists by Dirichlet characters, by the orthogonality of characters. The explicit form in our theorem is precisely what we need to prove Proposition 3.6 and Corollary 3.7, which are crucial for our main result.

3.1. **Notations.** We closely follow [2]. Let k be an integer. For any function $h : \mathbb{H} \to \mathbb{C}$ and any matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$, define

$$(h \mid \gamma)(z) = \det(\gamma)^{\frac{k}{2}}(cz+d)^{-k} h\left(\frac{az+b}{cz+d}\right).$$

For a positive integer q and a Dirichlet character $\xi \bmod q$, let $M_k(q, \xi)$ (resp. $S_k(q, \xi)$) be the space of holomorphic modular forms (resp. cusp forms) of level q, weight k and central character ξ. Then $f \in S_k(q, \xi)$ has the following Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}.$$

The Hecke operators T_n for $(n, q) = 1$, U_d and B_d for $d \mid q$ are given by:

$$f \mid T_n = n^{\frac{k}{2}-1} \sum_{ac=n} \sum_{b=0}^{c-1} \xi(a) f \mid \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

$$f \mid U_d = d^{\frac{k}{2}-1} \sum_{b=0}^{d-1} f \mid \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$$

$$f \mid B_d = d^{-\frac{k}{2}} f \mid \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}.$$

For a primitive Dirichlet character χ mod r, we define

$$f \mid R_\chi = \sum_{u \bmod r} \overline{\chi(u)} f \mid \begin{pmatrix} r & u \\ 0 & r \end{pmatrix}.$$

Let $N_k(q, \xi)$ denote the set of Hecke-normalized (i.e. the first Fourier coefficient is 1) cuspidal newforms of weight k and level q and central character ξ. If $f \in N_k(q, \xi)$ then $f \in S_k(q, \xi)$ is an eigenform of all Hecke operators T_n for $(n, q) = 1$ and U_d for $d \mid q$ ([2, p. 222]).

For a primitive Dirichlet character $\chi \bmod r$, we define the multiplicative twist of $f \in N_k(q, \xi)$,

$$f^\chi(z) := \sum_{n=1}^{\infty} a(n)\chi(n) e^{2\pi i n z} = \frac{1}{\tau(\overline{\chi})}(f \mid R_\chi)(z). \tag{3.1}$$

where $\tau(\bar{\chi}) = \sum_{\alpha \bmod r} \overline{\chi}(\alpha) e^{2\pi i \frac{\alpha}{r}}$ is the Gauss sum for $\bar{\chi}$. From [2, Proposition 3.1], we can deduce that $f^\chi \in S_k([q, \mathrm{cond}(\xi) r, r^2], \xi\chi^2)$. (Here $[a, b]$ stands for the leact common multiple of a, b.) We will further be using [5, Lemma 1.4], where tight bounds for the level of a twist of a newform are shown.

It should be stressed that the twist f^χ need not be a newform even if f is a newform and χ is primitive. The main aim of this section is to address this problem in the case of interest, by decomposing the relevant twist (acted upon by an involution) in terms of newforms.

3.2. **The Atkin–Lehner–Li-operator and additive twists.** Assume that $R \mid q$ and $(R, q/R) = 1$. Then a Dirichlet character ξ modulo q can be written as a product of Dirichlet characters ξ_R modulo R and $\xi_{q/R}$ modulo q/R, i.e., $\xi = \xi_R \xi_{q/R}$.

Put

$$W_R = \begin{pmatrix} Rx_1 & x_2 \\ qx_3 & Rx_4 \end{pmatrix}, \tag{3.2}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{Z}$, $x_1 \equiv 1 \bmod q/R$, $x_2 \equiv 1 \bmod R$ and $\det(W_R) = R(Rx_1x_4 - \frac{q}{R}x_2x_3) = R$. We call the operator induced by W_R, the *Atkin–Lehner–Li-operator* associated to $R|q$. By [2, Proposition 1.1], for $f \in M_k(q, \xi)$ (resp. $S_k(q, \xi)$), we have $f \mid W_R \in M_k(q, \overline{\xi_R}\xi_{q/R})$ (resp. $S_k(q, \overline{\xi_R}\xi_{q/R})$) and

$$f \mid W_R \mid W_R = \xi_R(-1)\overline{\xi_{q/R}(R)} f.$$

For $f \in S_k(q, \xi)$, let

$$\tilde{f}_R = f \mid W_R \in S_k(q, \overline{\xi_R}\xi_{q/R}). \tag{3.3}$$

The aim of this section is to prove the following theorem.

Theorem 3.1. *For $q, M_1 \in \mathbb{N}$ let*

$$M = \prod_{\substack{p \mid M_1 \\ \mathrm{ord}_p(M_1) \geq \mathrm{ord}_p(q)}} p^{\mathrm{ord}_p(M_1)}$$

$$r = \prod_{\substack{p \mid M_1 \\ \mathrm{ord}_p(M_1) < \mathrm{ord}_p(q)}} p^{\mathrm{ord}_p(M_1)}$$

$$R = \prod_{p \mid (q,r)} p^{\mathrm{ord}_p(q)} \prod_{\substack{p \mid q \\ p \nmid M_1}} p^{\mathrm{ord}_p(q)}.$$

For each $n|r$, we set

$$r_n = \prod_{p|r,\ p \nmid n} p.$$

For any $\alpha \bmod M_1$, set $\alpha \equiv ar + uM \bmod M_1$ for $a \bmod M$ and $u \bmod r$. For a Hecke-normalized newform $f \in N_k(q, \xi)$, we have

$$\begin{aligned} \Lambda\big(f, s, \frac{\alpha}{Mr}\big) = \frac{i^k}{\varphi(r)} \sum_{n|r} \frac{r}{nr_n} \sum_{e|r_n} \sum_{\substack{\chi \bmod n \\ \textit{primitive}}} \chi(u\bar{e})\tau(\overline{\chi})\mu\left(\frac{r_n}{e}\right)\varphi\left(\frac{r_n}{e}\right) \\ \times (\xi_{R'}\chi^2)(-M)(M^2R')^{\frac{k}{2}-s} \frac{\overline{\xi_{q/R}}\left(\frac{r}{ne}a\right) a\left(\frac{r}{ne}\right)}{\left(\frac{r}{ne}\right)^s} \Lambda\big(\widetilde{f^\chi}_{R'}, k-s, -\frac{\overline{R'a\frac{r}{ne}}}{M}\big). \end{aligned} \tag{3.4}$$

Here $R' = [R, \mathrm{cond}(\xi_R)r, r^2]$, $\overline{R'a\frac{r}{ne}}$ is the inverse of $R'a\frac{r}{ne}$ modulo M and $\widetilde{f^\chi}_{R'} = f^\chi \mid W_{R'}$.

3.2.1. *Proof of Theorem 3.1.* We first note the following elementary facts we will be using in the sequel. We have $M_1 = rM$ and $r \mid q$, with $(r, M) = 1$. Also $R \mid q$, $r \mid R$ and $(R, q/R) = 1$. Moreover, $\frac{q}{R} \mid M$ and $r < R$, except for when $r = R = 1$ in which case $q \mid M$.

We next have the following lemma.

Lemma 3.2. *For $q \in \mathbb{N}$, assume that $R \mid q$ and $(R, q/R) = 1$. Take $M \in \mathbb{N}$ such that $\frac{q}{R} \mid M$ and $(R, M) = 1$. For $a \bmod M$ with $(a, M) = 1$, set*

$$V_{q,R}^{M,a} = \begin{pmatrix} R\overline{Ra} & \frac{1-Ra\overline{Ra}}{M} \\ -q\frac{M}{q/R} & Ra \end{pmatrix} \tag{3.5}$$

be an integral matrix with $\det(V_{q,R}^{M,a}) = R$. Here $Ra\overline{Ra} \equiv 1 \bmod M$.

When $f \in S_k(q, \xi)$, we have

$$f\big(\frac{a}{M} + iy\big) = \xi_R(-M)\overline{\xi_{q/R}(a)}i^k(MR^{\frac{1}{2}}y)^{-k}\tilde{f}_R\big(-\frac{\overline{Ra}}{M} + i\frac{1}{M^2Ry}\big). \tag{3.6}$$

Proof. Applying [2, Proposition 1.1],

$$\tilde{f}_R \mid V_{q,R}^{M,a} = \overline{\xi_R}(M)\xi_{q/R}(Ra)\xi_R(-1)\overline{\xi_{q/R}}(R)f = \overline{\xi_R}(-M)\xi_{q/R}(a)f.$$

Note that

$$V_{q,R}^{M,a}\left(\frac{a}{M} + iy\right) = -\frac{\overline{Ra}}{M} + i\frac{1}{M^2Ry}.$$

So we get

$$\begin{aligned} f\big(\frac{a}{M} + iy\big) &= \xi_R(-M)\overline{\xi_{q/R}}(a)\big(\tilde{f}_R \mid V_{q,R}^{M,a}\big)\left(\frac{a}{M} + iy\right) \\ &= \xi_R(-M)\overline{\xi_{q/R}}(a)R^{\frac{k}{2}}(-iMRy)^{-k}\tilde{f}_R\big(-\frac{\overline{Ra}}{M} + i\frac{1}{M^2Ry}\big). \end{aligned}$$

□

For $r \in \mathbb{N}$ and a Dirichlet character $\chi \bmod r$, define the generalized Gauss sum

$$c_\chi(n) = \sum_{u \bmod r} \chi(u)e^{2\pi in\frac{u}{r}}.$$

Then by orthogonality, for $a \in \mathbb{Z}$ with $(a, r) = 1$, we have

$$e^{2\pi in\frac{a}{r}} = \frac{1}{\varphi(r)}\sum_{\chi \bmod r} \overline{\chi}(a)c_\chi(n).$$

Lemma 3.3. *Assume that q, M_1, M, r, r_n and R are as in the statement of Theorem 3.1. For any $\alpha \in \mathbb{Z}$ with $(\alpha, M_1) = 1$, let $a \bmod M$ and $u \bmod r$ be suh that $(a, M) = 1$, $(u, r) = 1$ such that $\alpha \equiv aq + uM \bmod Mr$. Then,*

$$f\big(\frac{\alpha}{Mr} + iy\big) = \frac{1}{\varphi(r)}\sum_{n|r}\frac{r}{nr_n}\sum_{e|r_n} a\left(\frac{r}{ne}\right)\mu\left(\frac{r_n}{e}\right)\varphi\left(\frac{r_n}{e}\right)\sum_{\substack{\chi \bmod n \\ \text{primitive}}}\tau(\bar{\chi})\chi(u\bar{e})f^\chi\big(\frac{a\frac{r}{ne}}{M} + i\frac{r}{ne}y\big). \tag{3.7}$$

Proof. Since $\frac{\alpha}{M_1} = \frac{\alpha}{Mr} = \frac{a}{M} + \frac{u}{r} \bmod 1$, we get

$$f\big(\frac{\alpha}{Mr} + iy\big) = \sum_{m=1}^\infty a(m)e^{2\pi im\frac{u}{r}}e^{2\pi im\frac{a}{M}+iy} = \frac{1}{\varphi(r)}\sum_{\chi \bmod r}\overline{\chi}(u)\sum_{m=1}^\infty a(m)c_\chi(m)e^{2\pi im\left(\frac{a}{M}+iy\right)}. \tag{3.8}$$

For a Dirichlet character $\chi \bmod r$, assume that χ is induced from a primitive character $\chi_* \bmod n$. Let $r_2 = \frac{r}{nr_n}$. By [3, Lemma 4.11], we have $c_\chi(m) = 0$ if $r_2 \nmid m$ and for any $m \in \mathbb{N}$,

$$c_\chi(mr_2) = r_2\chi_*(r_n)\tau(\chi_*)\overline{\chi_*}(m)\mu((r_n, m))\varphi((r_n, m)). \tag{3.9}$$

Applying this to (3.8), we have

$$\begin{aligned} f\Big(\frac{\alpha}{Mr} + iy\Big) &= \frac{1}{\varphi(r)} \sum_{\chi \bmod r} \overline{\chi}(u) \sum_{m=1}^{\infty} a(mr_2)c_\chi(mr_2)e^{2\pi i m r_2\left(\frac{a}{M}+iy\right)} \\ &= \frac{1}{\varphi(r)} \sum_{\chi \bmod r} \overline{\chi}(u) r_2\chi_*(r_n)\tau(\chi_*)a(r_2) \sum_{m=1}^{\infty} a(m)\overline{\chi_*}(m)\mu((r_n, m))\varphi((r_n, m))e^{2\pi i m\left(r_2\left(\frac{a}{M}+iy\right)\right)}. \end{aligned} \tag{3.10}$$

The last equality holds because $f \in N_k(q, \xi)$, so $f \mid U_p = a(p)f$ for any prime $p \mid q$, so $a(mr_2) = a(r_2)a(m)$. Note that $r_2 \mid r$ and $r \mid q$ so $r_2 \mid q$. By definition, r_n is square-free, so we have

$$\begin{aligned} \sum_{m=1}^{\infty} a(m)\overline{\chi_*}(m)\mu((r_n, m))\varphi((r_n, m))e^{2\pi i m r_2 z} &= \sum_{e \mid r_n} \mu(e)\varphi(e) \sum_{m=1}^{\infty} a(em)\overline{\chi_*}(em)e^{2\pi i e m r_2 z} \\ = \sum_{e \mid r_n} \mu(e)\varphi(e)a(e)\overline{\chi_*}(e) \sum_{m=1}^{\infty} a(m)\overline{\chi_*}(m)e^{2\pi i e m r_2 z} &= \sum_{e \mid r_0} \mu(e)\varphi(e)a(e)\overline{\chi_*}(e) f^{\overline{\chi_*}}(er_2 z). \end{aligned} \tag{3.11}$$

By applying (3.11) to (3.10) and taking $z = \frac{a}{M} + iy$, we get

$$\begin{aligned} f\Big(\frac{\alpha}{Mr} + iy\Big) &= \frac{1}{\varphi(r)} \sum_{\chi \bmod r} \overline{\chi(u)} r_2\tau(\chi_*)a(r_2) \sum_{e \mid r_n} \mu(e)\varphi(e)a(e)\chi_*\left(\frac{r_n}{e}\right) f^{\overline{\chi_*}}\left(er_2\left(\frac{a}{M} + iy\right)\right) \\ &= \frac{1}{\varphi(r)} \sum_{n \mid r} \frac{r}{nr_n} \sum_{e \mid r_0} a\left(\frac{r}{ne}\right)\mu\left(\frac{r_n}{e}\right)\varphi\left(\frac{r_n}{e}\right) \sum_{\substack{\chi \bmod n \\ \text{primitive}}} \tau(\overline{\chi})\chi(u\bar{e}) f^{\chi}\Big(\frac{a\frac{r}{ne}}{M} + i\frac{r}{ne}y\Big). \end{aligned}$$

□

Now we are ready to prove Theorem 3.1. Let $n \mid r$ and let χ be a primitive Dirichlet character $\bmod n$. By [2, Proposition 3.1], $f^\chi \in S_k(R'q/R, \xi\chi^2)$ and thus by (3.6),

$$\begin{aligned} &f^{\chi}\Big(\frac{\frac{r}{ne}a}{M} + i\frac{r}{ne}y\Big) \\ &\quad = i^k(\xi_{R'}\chi^2)(-M)\overline{\xi_{q/R}}\left(\frac{r}{ne}a\right)\left(MR'^{\frac{1}{2}}\frac{r}{ne}y\right)^{-k}\widetilde{f^{\chi}}_{R'}\Big(-\frac{\overline{R'a\frac{r}{ne}}}{M} + i\frac{1}{M^2R'\frac{r}{ne}y}\Big). \end{aligned} \tag{3.12}$$

Recall that $\widetilde{f^\chi}_{R'} = f^\chi \mid W_{R'} \in S_k(R'q/R, \overline{\xi_{R'}\chi^2}\xi_{q/R})$.

Applying Lemma 3.3, we get

$$\begin{aligned} \Lambda\Big(s, f, \frac{\alpha}{Mr}\Big) &= \int_0^{\infty} f\left(\frac{\alpha}{Mr} + iy\right) y^s \,\frac{dy}{y} \\ &= \frac{1}{\varphi(r)} \sum_{n \mid r} \frac{r}{nr_n} \sum_{e \mid r_n} \sum_{\substack{\chi \bmod n \\ \text{primitive}}} \chi(u\bar{e})\tau(\overline{\chi})\mu\left(\frac{r_n}{e}\right)\varphi\left(\frac{r_n}{e}\right)a\left(\frac{r}{ne}\right)\int_0^{\infty} f^{\chi}\Big(\frac{\frac{r}{ne}a}{M} + i\frac{r}{ne}y\Big)y^s\,\frac{dy}{y}. \end{aligned}$$

By (3.12),

$$\int_0^\infty f^\chi\Big(\frac{\frac{r}{ne}a}{M}+i\frac{r}{ne}y\Big)y^s\,\frac{dy}{y}$$
$$=i^k(\xi_{R'}\chi^2)(-M)\overline{\xi_{q/R}}\left(\frac{r}{ne}a\right)\int_0^\infty\big(MR'^{\frac12}\frac{r}{ne}y\big)^{-k}\widetilde{f^\chi}_{R'}\Big(-\frac{\overline{R'a\frac{r}{ne}}}{M}+i\frac{1}{M^2R'\frac{r}{ne}y}\Big)y^s\,\frac{dy}{y}$$
$$=i^k(\xi_{R'}\chi^2)(-M)(M^2R')^{\frac{k}{2}-s}\frac{\overline{\xi_{q/R}}\left(\frac{r}{ne}a\right)}{\left(\frac{r}{ne}\right)}^s\Lambda(k-s,\widetilde{f^\chi}_{R'},-\frac{\overline{R'a\frac{r}{ne}}}{M}).$$

This implies (3.4).

3.3. **Decomposition of $\widetilde{f^\chi}_{R'}$ and its Fourier coefficients.** In this section, we restrict to the case of trivial central character ξ, which is the case we need for the proof of our main theorem. We do so to avoid further complicating the presentation. The results, appropriately adjusted, hold for general central characters too.

The aims of this section are to decompose $\widetilde{f^\chi}_{R'}$ in terms of newforms and to bound its Fourier coefficients. The former aim will be achieved by Lemma 3.4 and Proposition 3.5, whereas the latter is the subject of Proposition 3.6.

We first fix some notation we will be using throughout the section:

- $q\in\mathbb{N}$;
- $r\mid q$ and for any prime $p\mid r$, $\mathrm{ord}_p(r)<\mathrm{ord}_p(q)$. (Thus, if q is square-free then $r=1$);
- $R\mid q$ such that $r\mid R$ and $(R,q/R)=1$;
- χ is a primitive Dirichlet character modulo $r_*\mid r$. (When $r_*=1$ then $\chi=1$);
- $R'=[R,r^2]$;
- R_* is the r_*-primary factor of q, i.e., $R_*=\prod_{p\mid r_*}p^{\mathrm{ord}_p(q)}$.

With these notations we have the following lemma.

Lemma 3.4. *Let f be a Hecke-normalized newform $f\in N_k(q)$. Then there exist $q'\mid[q,r_*^2]$ with $\frac{q}{R_*}\mid q'$ and $F_\chi\in N_k(q',\chi^2)$, such that*

$$f^\chi(z)=\sum_{\ell\mid r_*}\mu(\ell)(F_\chi\mid U_\ell\mid B_\ell)(z).$$

We set

$$F_\chi(z)=\sum_{n=1}^\infty a_\chi(n)e^{2\pi inz},\tag{3.13}$$

and define a multiplicative function β_{F_χ}: for each prime $p\mid r_*$ satisfying $F_\chi\mid U_p\neq0$

$$\beta_{F_\chi}(p^j)=\begin{cases}1 & \text{if } j=0\\ -a_\chi(p) & \text{if } j=1\\ -p^{k-1}\chi^2(p) & \text{if } j=2 \text{ and } p\nmid q',\\ 0 & \text{otherwise.}\end{cases}\tag{3.14}$$

Let

$$r_{*0}=\prod_{\substack{p\mid r_*,\ p\nmid q'\\ F_\chi\mid U_p\neq0}}p^2\prod_{\substack{p\mid(r_*,q')\\ F_\chi\mid U_p\neq0}}p.$$

Then

$$f^\chi = \sum_{\ell \mid r_{*0}} \beta_{F_\chi}(\ell) F_\chi \mid B_\ell \in S_k([q, r^2], \chi^2). \tag{3.15}$$

Proof. The proof of the first assertion is based on a repeated use of [2, Theorem 3.2]. For each $p \mid r_*$, let χ_p be the primitive Dirichlet character of conductor $p^{\mathrm{ord}_p(r_*)}$ so that $\chi = \prod_{p \mid r_*} \chi_p$. By [2, Theorem 3.2], there exists a newform $F_{\chi_p} \in N_k(q_p', \chi_p^2)$, for some level q_p' such that $(q/p^{\mathrm{ord}_p(q)}) \mid q_p'$

$$f^{\chi_p} = F_{\chi_p} - F_{\chi_p} \mid U_p \mid B_p.$$

Further, by [5, Lemma 1.4], we know that $q_p' \mid [q, p^{2\,\mathrm{ord}_p(r_*)}]$. If $\ell \neq p$ is a prime divisor of r_*, then, recalling the notations introduced in Section 3.1,

$$f^{\chi_p \chi_\ell} = F_{\chi_p}^{\chi_\ell} - \frac{1}{\tau(\overline{\chi_\ell})} F_{\chi_p} \mid U_p \mid B_p \mid R_{\chi_\ell}.$$

It is easy to see that $F_{\chi_p}|U_p|B_p|R_{\chi_\ell} = \chi_\ell(p) F_{\chi_p}|U_p|R_{\chi_\ell}|B_p$. Also, by [2, Proposition 3.3],

$$F_{\chi_p} \mid U_p \mid R_{\chi_\ell} = \overline{\chi_\ell(p)} F_{\chi_p} \mid R_{\chi_\ell} \mid U_p = \overline{\chi_\ell(p)} \tau(\overline{\chi_\ell}) F_{\chi_p}^{\chi_\ell} \mid U_p.$$

So we finally get

$$f^{\chi_p \chi_\ell} = F_{\chi_p}^{\chi_\ell} - F_{\chi_p}^{\chi_\ell} \mid U_p \mid B_p.$$

In the same way, we apply [2, Theorem 3.2] to F_{χ_p} to deduce that there exists a $F_{\chi_p \chi_\ell} \in N_k(q_{p\ell}', (\chi_p \chi_\ell)^2)$, for some $q_{p\ell}' | [q_{p\ell}, p^{2\,\mathrm{ord}_p(r_*)} p^{2\,\mathrm{ord}_\ell(r_*)}]$ with $(q/p^{\mathrm{ord}_p(q)} \ell^{\mathrm{ord}_\ell(q)}) \mid q_{p\ell}'$ such that

$$F_{\chi_p}^{\chi_\ell} = F_{\chi_p \chi_\ell} - F_{\chi_p \chi_\ell} \mid U_\ell \mid B_\ell.$$

This implies

$$f^{\chi_p \chi_\ell} = F_{\chi_p \chi_\ell} \mid (I_2 - U_\ell \mid B_\ell)(I_2 - U_p \mid B_p).$$

where $F \mid I_2 = F$. Continuing in the same way, we obtain

$$f^\chi = F_\chi \mid \Big(\prod_{p \mid r_*} \big[I_2 - U_p \mid B_p \big] \Big) = \sum_{\ell \mid r_*} \mu(\ell) (F_\chi \mid U_\ell \mid B_\ell).$$

for some $F_\chi \in N_k(q', \chi^2)$ and some $q' \mid [q, r_*^2]$.

To prove (3.15) we first observe that

$$f^\chi = F_\chi \mid \Big(\prod_{\substack{p \mid r_* \\ F_\chi | U_p \neq 0}} \big[I_2 - U_p \mid B_p \big] \Big).$$

Now, if $p \mid r_*$ and $p \nmid q'$, then, by the definition of U_p and by $F_\chi | T_p = a_\chi(p) F_\chi$ we have

$$F_\chi \mid U_p = a_\chi(p) F_\chi - p^{k-1} \chi^2(p) F_\chi \mid B_p.$$

If, on the other hand, $p \mid r_*$ and $p \mid q'$, then $\chi(p) = 0$ and $F_\chi \mid U_p = a_\chi(p) F_\chi$. Thus

$$f^\chi = F_\chi \mid \Big(\prod_{\substack{p \mid r_*,\ p \nmid q' \\ F_\chi | U_p \neq 0}} \big[I_2 - a_\chi(p) B_p - p^{k-1} \chi^2(p) B_{p^2} \big] \prod_{\substack{p \mid (r_*, q') \\ F_\chi | U_p \neq 0}} \big[I_2 - a_\chi(p) B_p \big] \Big) \tag{3.16}$$

$$= \sum_{\ell \mid r_{*0}} \beta_{F_\chi}(\ell) F_\chi \mid B_\ell.$$

□

We can use this lemma to prove the following proposition.

Proposition 3.5. *With the notation fixed in the beginning of the section, let f be a Hecke–normalized newform $f \in N_k(q)$ and let F_χ be the newform in $N_k(q', \chi^2)$ (for some $q' \mid [q, r_*^2]$) as in Lemma 3.4. Let R'_* be the (r_*, q')-primary factor of q' and set $Q_* = \frac{R_* R'}{R'_* R r_{*0}}$.*

Then $R'/R'_* r_{*0},\ Q_* \in \mathbb{Z}$ and

$$\widetilde{f^\chi}_{R'}(z) = f^\chi \mid W_{R'}(z) = \sum_{\ell \mid r_{*0}} \beta_{F_\chi}(\ell) \ell^{-\frac{k}{2}} \left(Q_* \frac{r_{*0}}{\ell}\right)^{\frac{k}{2}} \widetilde{F_\chi}_{\frac{RR'_*}{R_*}} \left(Q_* \frac{r_{*0}}{\ell} z\right). \tag{3.17}$$

where $\widetilde{F_\chi}_{\frac{RR'_*}{R_*}} = F_\chi \mid W_{\frac{RR'_*}{R_*}}$. Further there exists $\lambda_{\frac{RR'_*}{R_*}}(F_\chi) \in \mathbb{C}$ of absolute value one such that

$$\overline{\lambda_{\frac{RR'_*}{R_*}}(F_\chi)} \widetilde{F_\chi}_{\frac{RR'_*}{R_*}} \in N_k\left(R'_* q/R_*, \overline{\chi}^2\right).$$

(The constant $\lambda_{\frac{RR'_*}{R_*}}(F_\chi)$ is an Atkin–Lehner–Li pseudo eigenvalue.*)*

Proof. We first easily see using the definitions of the invariants involved that

$$R'_* \frac{q}{R_*} = q' \tag{3.18}$$

We next prove that $R'_* r_{*0} \mid R'$. Since $R'_* \mid R'$, we only need to check that $\mathrm{ord}_p(R'_* r_{*0}) \leq \mathrm{ord}_p(R')$ for each prime $p \mid r_{*0}$. Take a prime $p \mid r_{*0}$. By definition this implies that $F_\chi \mid U_p \neq 0$, which, by [2, Corollary 3.1], is equivalent to either

- $p \nmid R'_*$, or
- $p \| R'_*$, or
- $p^2 \mid R'_*$ and $\mathrm{ord}_p(\mathrm{cond}(\chi^2)) = \mathrm{ord}_p(R'_*)$.

Recall that $p \mid r_{*0}$ implies that $p \mid r_*$ so $p \mid r$. Since $R' = [R, r^2]$, we have $p^2 \mid R'$

Now we consider each case with the prime $p \mid r_{*0}$. When $p \nmid R'_*$ then $\mathrm{ord}_p(r_{*0}) = 2$ so $\mathrm{ord}_p(R'_* r_{*0}) = 2 \leq \mathrm{ord}_p(R')$. When $\mathrm{ord}_p(R'_*) = 1$ then $\mathrm{ord}_p(r_{*0}) = 1$, so $\mathrm{ord}_p(R'_* r_{*0}) = 2 \leq \mathrm{ord}_p(R')$. When $\mathrm{ord}_p(R'_*) \geq 2$ and $\mathrm{ord}_p(\mathrm{cond}(\chi^2)) = \mathrm{ord}_p(R'_*)$, we first note that

$$\mathrm{ord}_p(R'_*) = \mathrm{ord}_p(\mathrm{cond}(\chi^2)) \leq \mathrm{ord}_p(r_*) \leq \mathrm{ord}_p(r).$$

Moreover $\mathrm{ord}_p(r_{*0}) = 1$. So we get

$$\mathrm{ord}_p(R'_* r_{*0}) \leq \mathrm{ord}_p(r) + \mathrm{ord}_p(r_{*0}) = \mathrm{ord}_p(r) + 1 \leq 2\,\mathrm{ord}_p(r) \leq \mathrm{ord}_p(R').$$

Therefore, we conclude that $R'_* r_{*0} \mid R'$.

We can use this to verify the integrality of Q_*. We have $\frac{R}{R_*} \in \mathbb{Z}$ and $\frac{R}{R_*} \mid R'$. Moreover $(R/R_*, R'_* r_{*0}) = 1$. So $Q_* = \frac{R_* R'}{R'_* R r_{*0}} = \frac{R'}{\frac{R}{R_*} R'_* r_{*0}} \in \mathbb{Z}$.

Finally, we derive a formula for

$$f^\chi \mid W_{R'} = \sum_{\ell \mid r_{*0}} \beta_{F_\chi}(\ell) F_\chi \mid B_\ell \mid W_{R'}.$$

Since, as shown above, $R'_* r_{*0} \mid R'$, we have $\ell \mid R'$ for each $\ell \mid r_{*0}$. Then, by [2, Proposition 1.5],

$$F_\chi \mid B_\ell \mid W_{R'} = \ell^{-\frac{k}{2}} F_\chi \mid W_{\frac{R'}{\ell}}.$$

Note that the $W_{R'}$-operator on the left-hand side is an operator for level $R' \frac{q}{R}$ and the $W_{\frac{R'}{\ell}}$-operator on the right-hand side is an operator for level $\frac{R'}{\ell} \frac{q}{R}$.

Set

$$W_{\frac{R'}{\ell}} = \begin{pmatrix} \frac{R'}{\ell}x_1 & x_2 \\ \frac{q}{R}\frac{R'}{\ell}x_3 & \frac{R'}{\ell}x_4 \end{pmatrix},$$

where $x_1, x_2, x_3, x_4 \in \mathbb{Z}$, $\det(W_{R'/\ell}) = R'/\ell$, $x_1 \equiv 1 \bmod q/R$ and $x_2 \equiv 1 \bmod R'/\ell$.

Since, by (3.18), $F_\chi \in N_k(R'_* q/R_*, \chi^2)$, we lower the level of $W_{\frac{R'}{\ell}}$ to $R'_* \frac{q}{R_*}$:

$$W_{\frac{R'}{\ell}} = W_{R'_* \frac{R}{R_*}} \begin{pmatrix} Q_* \frac{r_{*0}}{\ell} & \\ & 1 \end{pmatrix} \quad \text{where } W_{R'_* \frac{R}{R_*}} = \begin{pmatrix} \frac{R}{R_*} R'_* x_1 & x_2 \\ R'_* \frac{q}{R_*} x_3 & \frac{R'}{\ell} x_4 \end{pmatrix}.$$

Here $W_{\frac{RR'_*}{R_*}}$ is an operator for level $R'_* q/R_*$ and we get

$$(F_\chi \mid W_{\frac{R'}{\ell}})(z) = \left(F_\chi \mid W_{\frac{RR'_*}{R_*}} \mid \begin{pmatrix} Q_* \frac{r_{*0}}{\ell} & \\ & 1 \end{pmatrix} \right)(z) = \left(Q_* \frac{r_{*0}}{\ell} \right)^{\frac{k}{2}} \widetilde{F_\chi}_{\frac{RR'_*}{R_*}} \left(Q_* \frac{r_{*0}}{\ell} z \right).$$

This implies (3.17). Finally, by [2], there exists a constant $\lambda_{\frac{RR'_*}{R_*}}(F_\chi)$ of absolute value one, such that

$$\overline{\lambda_{\frac{RR'_*}{R_*}}(F_\chi)} \widetilde{F_\chi}_{\frac{RR'_*}{R_*}} \in N(R'_* q/R_*, \overline{\chi}^2, k).$$

□

The above lemma and proposition allow us to prove good bounds the Fourier coefficients of $\widetilde{f^\chi}_{R'}(z)$:

Proposition 3.6. *With the notations in Proposition 3.5, set*

$$\widetilde{f^\chi}_{R'}(z) = \sum_{m=1}^{\infty} b_{\chi,R'}(m) e^{2\pi i m z}. \tag{3.19}$$

Then $b_{\chi,R'}(m) = 0$ *when* $Q_* \nmid m$, *and otherwise, for* $m \in \mathbb{N}$,

$$\left| (Q_* m)^{-\frac{k-1}{2}} b_{\chi,R'}(Q_* m) \right| \ll_\epsilon \left(\frac{m}{r_{*0}} \right)^\epsilon (Q_* r_{*0})^{\frac{1}{2}} \sigma_{-1+2\epsilon}(r_{*0}), \tag{3.20}$$

for any $\epsilon > 0$.

In the above proposition $\sigma_s(n) = \sum_{d|n} d^s$ is the sum of divisors function.

Proof. Applying Proposition 3.5, we normalize the Fourier expansion of $\widetilde{F_\chi}_{\frac{RR'_*}{R_*}}$ as

$$\widetilde{F_\chi}_{\frac{RR'_*}{R_*}}(z) = \lambda_{\frac{RR'_*}{R_*}}(F_\chi) \sum_{n=1}^{\infty} \tilde{a}_{\chi, R'_* \frac{R}{R_*}}(n) e^{2\pi i n z}. \tag{3.21}$$

By [2, (1.1)], we get

$$\tilde{a}_{\chi, R'_* \frac{R}{R_*}}(p) = \begin{cases} \overline{\chi}^2(p) a_\chi(p) & \text{if } p \nmid R'_* \frac{R}{R_*} \\ \overline{a_\chi(p)} & \text{if } p \mid R'_* \frac{R}{R_*}. \end{cases} \tag{3.22}$$

We then apply (3.21) to (3.17) to get

$$\widetilde{f^\chi}_{R'}(z) = \lambda_{R'_* \frac{R}{R_*}}(F_\chi) \sum_{\ell | r_{*0}} \beta_{F_\chi}(\ell) \ell^{-\frac{k}{2}} \left(Q_* \frac{r_{*0}}{\ell} \right)^{\frac{k}{2}} \sum_{n=1}^{\infty} \tilde{a}_{\chi, R'_* \frac{R}{R_*}}(n) e^{2\pi i n Q_* \frac{r_{*0}}{\ell} z}. \tag{3.23}$$

Comparing both sides, $b_{\chi,R'}(m)=0$ when $Q_*\nmid m$. For $m\in\mathbb{N}$,

$$\begin{aligned}(3.24)\quad b_{\chi,R'}(Q_*m) &= \lambda_{R'_*\frac{R}{R_*}}(F_\chi)\sum_{\substack{\ell\mid r_{*0}\\ \frac{r_{*0}}{\ell}\mid m}}\beta_{F_\chi}(\ell)\ell^{-\frac{k}{2}}\left(Q_*\frac{r_{*0}}{\ell}\right)^{\frac{k}{2}}\tilde{a}_{\chi,R'_*\frac{R}{R_*}}\left(\frac{m}{r_{*0}/\ell}\right)\\ &= \lambda_{R'_*\frac{R}{R_*}}(F_\chi)\sum_{\ell\mid(r_{*0},m)}\beta_{F_\chi}(r_{*0}/\ell)(r_{*0}/\ell)^{-\frac{k}{2}}(Q_*\ell)^{\frac{k}{2}}\tilde{a}_{\chi,R'_*\frac{R}{R_*}}\left(\frac{m}{\ell}\right).\end{aligned}$$

Recalling (3.14), we deduce that $b_{\chi,R'}(Q_*m)$ equals

$$\lambda_{R'_*\frac{R}{R_*}}(F_\chi)\sum_{\ell\mid(r_{*0},m)}\left[\prod_{p\|r_{*0}/\ell}(-p^{-\frac{k}{2}}a_\chi(p))\prod_{p^2\|r_{*0}/\ell}(-p^{-1}\chi^2(p))\right](Q_*\ell)^{\frac{k}{2}}\tilde{a}_{\chi,R'_*\frac{R}{R_*}}\left(\frac{m}{\ell}\right).$$

For any $m\in\mathbb{N}$, since F_χ and $\widetilde{F}_{\chi\frac{RR'_*}{R_*}}(z)$ are newforms, we have

$$|a_\chi(m)|\ll_\epsilon m^{\frac{k-1}{2}+\epsilon}\quad\text{and}\quad\left|\tilde{a}_{\chi,R'_*\frac{R}{R_*}}(m)\right|\ll_\epsilon m^{\frac{k-1}{2}+\epsilon},$$

for any $\epsilon>0$. Thus we finally get

$$\begin{aligned}\left|(Q_*m)^{-\frac{k-1}{2}}b_{\chi,R'}(Q_*m)\right| &\ll_\epsilon (Q_*m)^{-\frac{k-1}{2}}\sum_{\ell\mid(r_{*0},m)}\left[\prod_{p\|\frac{r_{*0}}{\ell}}p^{-\frac{k}{2}}p^{\frac{k-1}{2}+\epsilon}\prod_{p^2\|\frac{r_{*0}}{\ell}}p^{-1}\right]Q_*^{\frac{k}{2}}\ell^{\frac{k}{2}}\left(\frac{m}{\ell}\right)^{\frac{k-1}{2}+\epsilon}\\ &= Q_*^{\frac{1}{2}}m^\epsilon\sum_{\ell\mid(r_{*0},m)}\left[\prod_{p\|\frac{r_{*0}}{\ell}}p^{-\frac{1}{2}+\epsilon}\prod_{p^2\|\frac{r_{*0}}{\ell}}p^{-1}\right]\ell^{\frac{1}{2}-\epsilon}\\ &\le m^\epsilon r_{*0}^{\frac{1}{2}-\epsilon}Q_*^{\frac{1}{2}}\sum_{\ell\mid r_{*0}}\left[\prod_{p\|\ell}p^{-\frac{1}{2}+\epsilon}\prod_{p^2\|\ell}p^{-1}\right]\ell^{-\frac{1}{2}+\epsilon}\le\left(\frac{m}{r_{*0}}\right)^\epsilon(Q_*r_{*0})^{\frac{1}{2}}\sum_{\ell\mid r_{*0}}\ell^{-1+2\epsilon}\\ &= \left(\frac{m}{r_{*0}}\right)^\epsilon(Q_*r_{*0})^{\frac{1}{2}}\sigma_{-1+2\epsilon}(r_{*0}).\end{aligned}$$

□

3.4. **Additive twists in the special case applying to Theorem 1.2.** We now further specialize to the case of weight 2. This is the setting of our main theorem, where we consider Hecke-normalized newforms of weight 2 and level q. By Theorem 3.1 and Proposition 3.6 we have Corollary 3.7. As applications of this corollary, we then obtain an upper bound for $\left|\int_0^\infty f(a/d+iy)\,dy\right|$ (3.27) and the approximate functional equation (3.35) for $L(1,f,a/d)$.

Corollary 3.7. *Let f be a Hecke-normalized newform of weight 2 for level q. Let a, d be coprime integers and set*

$$M_d = \prod_{\substack{p|d \\ \mathrm{ord}_p(d) \geq \mathrm{ord}_p(q)}} p^{\mathrm{ord}_p(d)},$$
$$r_d = \prod_{\substack{p|d \\ \mathrm{ord}_p(d) < \mathrm{ord}_p(q)}} p^{\mathrm{ord}_p(d)},$$
$$R_d = \prod_{p|(q,r_d)} p^{\mathrm{ord}_p(q)} \prod_{p|q,\ p\nmid d} p^{\mathrm{ord}_p(q)},$$
$$R'_d = [R_d, r_d^2].$$

Further, consider $a_1 \bmod M_d$ and $a_2 \bmod r_d$ such that $a \equiv a_1 r_d + a_2 M_d \bmod d$. Then we have

$$(3.25)\quad (M_d^2 R'_d)^{s-1} \Lambda(s, f, \frac{a}{d}) = \frac{-1}{\varphi(r_d)} \sum_{\substack{n|r_d, \\ \frac{r_d}{n} \text{ square-free} \\ (n, r_d/n)=1}} \sum_{\substack{\chi \bmod n \\ \text{primitive}}} \tau(\overline{\chi}) \chi\left(a_2 \overline{\left(\frac{r_d}{n}\right)}\right) \chi^2(M_d) \Lambda(2-s, \widetilde{f^{\chi}}_{R'_d}, -\frac{\overline{R'_d a_1}}{M_d}).$$

Here $\overline{R'_d a_1}$ is the inverse of $R'_d a_1$ modulo M_d.

We also repeat the following notations for the reader's convenience. For a primitive Dirichlet character χ for $\mathrm{cond}(\chi) = r_{d*} \mid r_d$, define the invariants

$$R_{d*} = \prod_{p|r_{d*}} p^{\mathrm{ord}_p(q)},$$
$$R'_{d*} = \prod_{p|(r_{d*},q')} p^{\mathrm{ord}_p(q')},$$
$$r_{d*0} = \prod_{\substack{p|r_{d*},\ p\nmid q', \\ F_\chi|U_p \neq 0}} p^2 \prod_{\substack{p|(r_{d*},q') \\ F_\chi|U_p \neq 0}} p,$$
$$Q_{d*} = \frac{R_{d*} R'_d}{R'_{d*} R_d r_{d*0}},$$

and $b_{\chi,R'_d}(m)$ as given in Proposition 3.5 and Proposition 3.6. Then $b_{\chi,R'_d}(m) = 0$ when $Q_{d*} \nmid m$ and for $n \in \mathbb{N}$, we get

$$(3.26)\qquad \left|(Q_{d*}n)^{-\frac{1}{2}} b_{\chi,R'_d}(Q_{d*}n)\right| \ll_\epsilon \left(\frac{n}{r_{d*0}}\right)^{\epsilon} (Q_{d*} r_{d*0})^{\frac{1}{2}} \sigma_{-1+2\epsilon}(r_{*0}),$$

for any $\epsilon > 0$.

Proof. This is just a specialization of Theorem 3.1 and Proposition 3.6 to the case $k = 2$ and trivial central character ξ. The functional equation (3.4) simplifies in this case to (3.25).

Indeed, suppose that, for some $n \mid r_d$ and $e \mid \prod_{p|r_d, r\nmid n} p$, we have $r_d \neq ne$. Then $a(r_d/ne) = 0$. This is because, if $p \mid \frac{r_d}{ne} \mid r_d$, then $r^2 \mid q$ (by the definition of r_d) and thus $a(pm) = 0$, for all

$m \in \mathbb{N}$, since f is a newform. Therefore, $e = r_d/n$. Since $\prod_{p|r_d, r\nmid n} p \mid \frac{r_d}{n}$ and $e \mid \prod_{p|r_d, r\nmid n} p$, we have $e = \prod_{p|r_d, r\nmid n} p = \frac{r_d}{n}$ and hence, $(n, r_d/n) = 1$ and $\frac{r_n}{n}$ is square-free. □

As an application of this corollary we prove the following proposition which we need for the proof of our main theorem, but which is also of independent interest.

Proposition 3.8. *Let f be a Hecke-normalized newform of weight 2 for level q. Then, for each $\epsilon > 0$,*

$$\left|\int_{\infty}^{\frac{a}{d}} f(z)\, dz\right| = \left|\int_0^{\infty} f\Big(\frac{a}{d} + iy\Big)\, dy\right| \ll_\epsilon d^{\frac{1}{2}} q^{\frac{1}{4}} (qd)^\epsilon \prod_{\substack{p|d \\ \mathrm{ord}_p(d) < \mathrm{ord}_p(q)}} p^{\frac{1}{4}}. \tag{3.27}$$

Note that the product over p equals 1 if q is square-free.

Proof. We first observe that

$$M_d^2 R_d' = \frac{d^2 R_d}{(R_d, r_d^2)} = [q, d^2]. \tag{3.28}$$

The second equality holds because

$$(q, d^2) = (R_d q/R_d, M_d^2 r_d^2) = (q/R_d, M_d^2)\,(R_d, r_d^2) = \frac{q}{R_d}\,(R_d, r_d^2),$$

since $\frac{q}{R_d} | M_d$. Thus we have $(R_d, r_d^2) = (q, d^2)\frac{R_d}{q}$.

It follows from this that on the line $\Re(t) = 1 + \epsilon$,

$$(M_d^2 R_d')^{\frac{t}{2}} \Lambda(t + \frac{1}{2}, f, \frac{a}{d}) \ll_\epsilon [q, d^2]^{1/2+\epsilon} \tag{3.29}$$

because of the Stirling bound for the Gamma function.

Similarly, using Corollary 3.7 we will deduce the following bound for t with $\Re(t) = -\epsilon$:

$$(M_d^2 R_d')^{\frac{t}{2}} \Lambda(t + \frac{1}{2}, f, \frac{a}{d}) \ll_\epsilon (dq)^\epsilon d q^{\frac{1}{2}} \prod_{\substack{p|d \\ \mathrm{ord}_p(d) < \mathrm{ord}_p(q)}} p^{\frac{1}{2}}. \tag{3.30}$$

This analysis is more involved, and we present most of the details. For $\Re(t) = -\epsilon$, by (3.25) and (3.19), we get

$$\begin{aligned} &(M_d^2 R_d')^{\frac{t}{2}} \Lambda(t + \frac{1}{2}, f, \frac{a}{d}) \\ &\qquad \ll_\epsilon (M_d^2 R_d')^{\frac{1+\epsilon}{2}} \frac{1}{\varphi(r_d)} \sum_{\substack{r_{d*}|r_d, \\ \frac{r_d}{r_{d*}} \text{ square-free} \\ (r_d, r_d/r_{d*})=1}} \sum_{\substack{\chi \bmod r_{d*} \\ \text{primitive}}} \sqrt{r_{d*}} \sum_{m=1}^{\infty} \frac{\left|m^{-\frac{1}{2}} b_{\chi, R_d'}(m)\right|}{m^{1+\epsilon}}. \end{aligned} \tag{3.31}$$

Note that $b_{\chi, R_d'}(m) = 0$ unless $Q_{d*} \nmid m$. Applying the bound (3.26), for any $0 < \epsilon' < \epsilon$, we get

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{\left|m^{-\frac{1}{2}} b_{\chi, R_d'}(m)\right|}{m^{1+\epsilon}} &\ll_\epsilon Q_{d*}^{-\frac{1}{2}-\epsilon} \sigma_{-1+2\epsilon'}(r_{d*0}) r_{d*0}^{\frac{1}{2}-\epsilon'} \sum_{n=1}^{\infty} \frac{n^{\epsilon'}}{n^{1+\epsilon}} \\ &\le \sigma_{-1+2\epsilon'}(r_{d*0}) r_{d*0}^{\frac{1}{2}-\epsilon'} \zeta(1 + \epsilon - \epsilon') \ll_{\epsilon, \epsilon''} r_{d*0}^{\frac{1}{2}+\epsilon''} \end{aligned}$$

since $Q_{d*} \in \mathbb{N}$. Also we have

$$r_{d*0} = \prod_{\substack{p|r_*,\ p\nmid q' \\ F_\chi|U_p\neq 0}} p^2 \prod_{\substack{p|(r_*,q') \\ F_\chi|U_p\neq 0}} p \leq \prod_{p|r_{d*}} p^2.$$

Applying this to (3.31), we get

$$(3.32)\quad (M_d^2R_d')^{\frac{t}{2}}\Lambda(t+\frac{1}{2},f,\frac{a}{d}) \ll_{\epsilon,\epsilon''} (M_d^2R_d')^{\frac{1+\epsilon}{2}}\frac{1}{\varphi(r_d)} \sum_{\substack{r_{d*}|r_d \\ \frac{r_d}{r_{d*}}\text{ square-free} \\ (r_{d*},r_d/r_{d*})=1}} \sqrt{r_{d*}}\prod_{p|r_{d*}} p^{1+2\epsilon''} \sum_{\substack{\chi \bmod r_{d*} \\ \text{primitive}}} 1$$

$$\leq (M_d^2R_d')^{\frac{1+\epsilon}{2}} \sum_{\substack{r_{d*}|r_d \\ \frac{r_d}{r_{d*}}\text{ square-free} \\ (r_{d*},r_d/r_{d*})=1}} \sqrt{r_{d*}}\prod_{p|r_{d*}} p^{1+2\epsilon''},$$

since $r_{d*} \mid r_d$ and $\sum_{\substack{\chi \bmod r_{d*} \\ \text{primitive}}} 1 \leq \varphi(r_{d*}) \leq \varphi(r_d)$. Upon setting $\frac{r_d}{r_{d*}} = \ell$, the right-hand side becomes

$$(M_d^2R_d')^{\frac{1+\epsilon}{2}} \sum_{\substack{\ell|r_d \\ \text{square-free} \\ (r_d/\ell,\ell)=1}} \sqrt{r_d/\ell}\prod_{p|r_d/\ell} p^{1+2\epsilon''} = (M_d^2R_d')^{\frac{1+\epsilon}{2}} r_d^{\frac{1}{2}}\prod_{p|r_d}p^{1+2\epsilon''} \sum_{\substack{\ell|r_d \\ \text{square-free} \\ (r_d/\ell,\ell)=1}} \ell^{-\frac{3}{2}-2\epsilon''}$$

$$\ll_{\epsilon,\epsilon''} (M_d^2R_d')^{\frac{1+\epsilon}{2}} r_d^{\frac{1}{2}}\prod_{p|r_d}p^{1+\epsilon'''}.$$

Thus

$$(3.33)\qquad (M_d^2R_d')^{\frac{t}{2}}\Lambda(t+\frac{1}{2},f,\frac{a}{d}) \ll_\epsilon [q,d^2]^{\frac{1}{2}}(dq)^\epsilon r_d^{\frac{1}{2}}\prod_{p|r_d}p,$$

as $M_d^2R_d' = [q,d^2]$. More explicitly,

$$[q,d^2]^{\frac{1}{2}} r_d^{\frac{1}{2}}\prod_{p|r_d}p = q^{\frac{1}{2}}d\frac{\prod_{p|d,\mathrm{ord}_p(d)<\mathrm{ord}_p(q)} p^{\frac{1}{2}\mathrm{ord}_p(d)+1}}{(q,d^2)^{\frac{1}{2}}}$$

and by examining the exponent of each p in the right-hand side, we see that the right-hand side is

$$\leq q^{\frac{1}{2}}d \prod_{\substack{p|d \\ \mathrm{ord}_p(d)<\frac{1}{2}\mathrm{ord}_p(q)}} p^{-\frac{1}{2}\mathrm{ord}_p(d)+1} \prod_{\substack{p|d \\ \frac{1}{2}\mathrm{ord}_p(q)\leq\mathrm{ord}_p(d)<\mathrm{ord}_p(q)}} p^{-\frac{1}{2}(\mathrm{ord}_p(q)-\mathrm{ord}_p(d))+1}.$$

When $p \mid d$ and $\mathrm{ord}_p(d) < \mathrm{ord}_p(q)$, both $\mathrm{ord}_p(d) \geq 1$ and $\mathrm{ord}_p(q) - \mathrm{ord}_p(d) \geq 1$. So we get

$$(3.34)\qquad \frac{\prod_{p|d,\mathrm{ord}_p(d)<\mathrm{ord}_p(q)} p^{\frac{1}{2}\mathrm{ord}_p(d)+1}}{(q,d^2)^{\frac{1}{2}}} \leq \prod_{\substack{p|d \\ \mathrm{ord}_p(d)<\mathrm{ord}_p(q)}} p^{\frac{1}{2}}.$$

Combining (3.34) with (3.33), we get (3.30) for $\Re(t) = -\epsilon$.
Recall that at $\Re(t) = 1+\epsilon$,

$$\Lambda(t+\frac{1}{2},f,\frac{a}{d}) \ll_\epsilon (qd)^\epsilon.$$

Similarly, by (3.30), for $\Re(t) = -\epsilon$,

$$\Lambda(t + \frac{1}{2}, f, \frac{a}{d}) \ll_\epsilon (dq)^\epsilon dq^{\frac{1}{2}} \prod_{\substack{p \mid d \\ \mathrm{ord}_p(d) < \mathrm{ord}_p(q)}} p^{\frac{1}{2}}.$$

By the Phragmén-Lindelöf convexity principle and (2.10) we deduce the proposition.

□

Finally, the functional equation of Corollary 3.7 implies the approximate functional equation (see e.g. [8, Theorem 5.3], even though the theorem is stated for an L-function with an Euler product, it applies to the case of additive twists, since the proof does not use the Euler product). This states

$$(3.35)\quad L\left(1, f, \frac{a}{d}\right) = \sum_{n \geq 1} \frac{a(n) e^{2\pi i n \frac{a}{d}}}{n} V\left(\frac{M_d R_d'^{\frac{1}{2}} X}{2\pi n}\right) - \frac{1}{\varphi(r_d)} \sum_{\substack{r_{d*} \mid r_d \\ \frac{r_d}{r_{d*}} \text{ square-free} \\ (r_{d*}, r_d/r_{d*}) = 1}} \sum_{\substack{\chi \bmod r_{d*}, \\ \text{primitive}}} \tau(\overline{\chi}) \chi(a_2 \overline{(r_d/r_{d*})}) \chi^2(M_d) \times \sum_{n=1}^{\infty} \frac{b_{\chi, R'}(Q_{d*} n) e^{-2\pi i Q_{d*} n \frac{\overline{R_d' a_1}}{M_d}}}{Q_{d*} n} V\left(\frac{M_d R_d'^{\frac{1}{2}}}{2\pi Q_{d*} n X}\right)$$

for all $X > 0$, with

$$(3.36)\qquad V(y) := \frac{1}{2\pi i} \int_{(2)} (2\pi y)^u G(u) \Gamma(u) du.$$

Here $G(u)$ is any even function which is entire and bounded in vertical strips, of arbitrary polynomial decay as $|\operatorname{Im} u| \to \infty$ and such that $G(0) = 1$.

4. The asymptotics of $A_h^{\pm}(M)$ as $M \to \infty$.

Recall the description for $\alpha_{n,M}(t)$ given in (2.14)

$$\alpha_{n,M}(t) = \frac{1}{M} \sum_{d \mid M} \sum_{\substack{a \bmod d \\ (a,d)=1}} e^{-2\pi i n \frac{a}{d}} L\big(f, t, \frac{a}{d}\big).$$

Our aim is to analyze the asymptotics of

$$A_h^{\pm}(M) = \sum_{n \in \mathbb{Z}} \widehat{h}(n) (\alpha_{-n,M}(1) \pm \alpha_{n,M}(1)),$$

as $M \to \infty$.

We first prove the formula (4.12) for $\alpha_{n,M}(1)$. To accomplish this, we first apply the approximate functional equation given in (3.35). Then we estimate error terms by applying Weil's bound for Kloosterman sums and making use of our explicitly described terms.

For $d \mid M$, we recall the notations M_d, r_d and R_d given in Corollary 3.7:

$$d = M_d r_d, \quad (M_d, r_d) = 1, \quad r_d \mid R_d, \quad R_d \mid q \quad \text{and} \quad (q/R_d, R_d) = 1.$$

Moreover $r_d < R_d$ unless $r_d = R_d = 1$. Also $R'_d = [R_d, r_d^2]$, $M_d^2 R'_d = [q, d^2]$. For any divisor r_{d*} of r_d, we have $R'_{d*} \mid [R_{d*}, r_{d*}^2]$ and r_{d*0} as described in Corollary 3.7. Recall that $Q_{d*} = \frac{R_{d*}R'_d}{R'_{d*}R_d r_{d*0}}$. and it is proved in Lemma 3.4 that $Q_{d*} \in \mathbb{N}$. Finally, $a_1 \bmod M_d$ and $a_2 \bmod r_d$ are such that $a \equiv a_1 r_d + a_2 M_d \bmod d$.

We apply (3.35) to each $L(t, f, \frac{a}{d})$, with $X = X_d$, and substitute into (2.14) with $t = 1$:

$$\begin{aligned}\alpha_{n,M}(1) = &\frac{1}{M}\sum_{d|M}\sum_{\substack{a \bmod d\\(a,d)=1}} e^{-2\pi i n\frac{a}{d}}\sum_{\ell\geq 1}\frac{a(\ell)e^{2\pi i\ell\frac{a}{d}}}{\ell}V\left(\frac{M_d R_d'^{\frac{1}{2}}X_d}{2\pi\ell}\right)\\ &-\frac{1}{M}\sum_{d|M}\sum_{\substack{a \bmod d\\(a,d)=1}} e^{-2\pi i n\frac{a}{d}}\frac{1}{\varphi(r_d)}\sum_{\substack{r_{d*}|r_d\\ \frac{r_d}{r_{d*}}\text{ square-free}\\(r_{d*},r_d/r_{d*})=1}}\sum_{\substack{\chi \bmod r_{d*},\\ \text{primitive}}}\tau(\overline{\chi})\chi(a_2\overline{(r_d/r_{d*})})\chi^2(M_d)\\ &\times\sum_{\ell=1}^{\infty}\frac{b_{\chi,R'}\left(Q_{d*}\ell\right)e^{-2\pi i Q_{d*}\ell\frac{\overline{R'_d a_1}}{M_d}}}{Q_{d*}\ell}V\left(\frac{M_d R_d'^{\frac{1}{2}}}{2\pi Q_{d*}\ell X_d}\right).\end{aligned}\tag{4.1}$$

Here $\overline{R'_d a_1}R'_d a_1 \equiv 1 \bmod M_d$. Set $X_d = \frac{X}{M_d R_d'^{\frac{1}{2}}}$, with X independent of d. Since

$$\sum_{d|M}\sum_{\substack{a \bmod d\\(a,d)=1}} e^{2\pi i\frac{a}{d}(-n+\ell)} = \begin{cases} M, & \text{if } n\equiv \ell \bmod M\\ 0, & \text{otherwise}\end{cases}$$

and

$$\begin{aligned}&\sum_{\substack{a \bmod d\\(a,d)=1}} e^{-2\pi i n\frac{a}{d}}\chi(a_2\overline{(r_d/r_{d*})})e^{-2\pi i Q_{d*}\ell\frac{\overline{R'_d a_1}}{M_d}}\\ &=\sum_{\substack{a_1 \bmod M_d\\(a_1,M_d)=1}} e^{-2\pi i n\frac{a_1}{M_d}}e^{-2\pi i Q_{d*}\ell\frac{\overline{R'_d a_1}}{M_d}}\sum_{\substack{a_2 \bmod r_d\\(a_2,r_d)=1}}\chi(a_2\overline{(r_d/r_{d*})})e^{-2\pi i n\frac{a_2}{r_d}}\\ &= S\left(n, \ell Q_{d*}\overline{R'_d}; M_d\right)\chi(-\overline{r_d/r_{d*}})c_{\tilde{\chi}_{r_d}}(n),\end{aligned}\tag{4.2}$$

we get

$$\begin{aligned}\alpha_{n,M}(1) = &\sum_{\substack{\ell\geq 1\\ \ell\equiv n \bmod M}}\frac{a(\ell)}{\ell}V\left(\frac{X}{2\pi\ell}\right)\\ &-\frac{1}{M}\sum_{d|M}\frac{1}{\varphi(r_d)}\sum_{\substack{r_{d*}|r_d\\ \frac{r_d}{r_{d*}}\text{ square-free}\\(r_{d*},r_d/r_{d*})=1}}\sum_{\substack{\chi \bmod r_{d*},\\ \text{primitive}}}\tau(\overline{\chi})\chi(-\overline{(r_d/r_{d*})}M_d^2)c_{\tilde{\chi}_{r_d}}(n)\\ &\times\sum_{\ell=1}^{\infty}\frac{b_{\chi,R'}\left(Q_{d*}\ell\right)}{Q_{d*}\ell}S(n, \ell Q_{d*}\overline{R'_d}; M_d)V\left(\frac{M_d^2 R'_d}{2\pi Q_{d*}\ell X}\right).\end{aligned}\tag{4.3}$$

Here $\tilde{\chi}_{r_d}$ is a Dirichlet character modulo r_d, which is induced from the primitive character $\chi \bmod r_{d*}$.

For the last sum of (4.3) we use Weil's bound for Kloosterman sums, which implies, as $(R'_d, M_d) = 1$ and $(Q_{d*}, M_d) = 1$,

$$\left|S(n, \ell Q_{d*}\overline{R'_d}; M_d)\right| \leq (n, \ell, M_d)^{\frac{1}{2}} M_d^{\frac{1}{2}} \sigma_0(M_d) \tag{4.4}$$

By applying (4.4) and (3.20), we get

$$\begin{aligned}\left|\sum_{\ell=1}^{\infty} \frac{b_{\chi,R'}(Q_{d*}\ell)}{Q_{d*}\ell} S(n, \ell Q_{d*}\overline{R'_d}; M_d) V\left(\frac{M_d^2 R'_d}{2\pi Q_{d*}\ell X}\right)\right| \\ \ll_\epsilon M_d^{\frac{1}{2}} \sigma_0(M_d) r_{d*0}^{\frac{1}{2}-\epsilon} \sigma_{-1+2\epsilon}(r_{d*0}) \sum_{\ell=1}^{\infty} \frac{(n, \ell, M_d)^{\frac{1}{2}}}{\ell^{\frac{1}{2}-\epsilon}} V\left(\frac{M_d^2 R'_d}{2\pi Q_{d*}\ell X}\right)\end{aligned} \tag{4.5}$$

for any $\epsilon > 0$. Since $(n, \ell, M_d) \mid (n, M_d)$, we get

$$\leq M_d^{\frac{1}{2}} \sigma_0(M_d) r_{d*0}^{\frac{1}{2}-\epsilon} \sigma_{-1+2\epsilon}(r_{d*0}) \sum_{g|(n,M_d)} g^\epsilon \sum_{\ell=1}^{\infty} \frac{1}{\ell^{\frac{1}{2}-\epsilon}} V\left(\frac{M_d^2 R'_d}{2\pi Q_{d*}\ell g X}\right).$$

If $y < M^{-\epsilon'}$, one easily checks, by moving the line of integration in (3.36) to the right, that $V(y) \ll M^{-K_{\epsilon'}}$, for arbitrarily large K_ϵ. Consequently,

$$\begin{aligned}\sum_{g|(n,M_d)} g^\epsilon \sum_{\ell=1}^{\infty} \frac{1}{\ell^{\frac{1}{2}-\epsilon}} V\left(\frac{M_d^2 R'_d}{2\pi Q_{d*}\ell g X}\right) \ll \sum_{g|(n,M_d)} g^\epsilon \sum_{\ell \ll M^{\epsilon'} \frac{M_d^2 R'_d}{2\pi Q_{d*} g X}} \frac{1}{\ell^{\frac{1}{2}-\epsilon}} \\ \ll \sum_{g|(n,M_d)} g^\epsilon M^{\epsilon'} \left(\frac{M_d^2 R'_d}{2\pi Q_{d*} g X}\right)^{\frac{1}{2}+\epsilon} \leq \sum_{g|(n,M_d)} g^\epsilon M^{\epsilon'} \left(\frac{M_d^2 R'_d}{2\pi g X}\right)^{\frac{1}{2}+\epsilon}\end{aligned} \tag{4.6}$$

since $Q_{d*} \geq 1$ (See Proposition 3.6). Applying this to (4.5), we get

$$\begin{aligned}\left|\sum_{\ell=1}^{\infty} \frac{b_{\chi,R'}(Q_{d*}\ell)}{Q_{d*}\ell} S(n, \ell Q_{d*}\overline{R'_d}; M_d) V\left(\frac{M_d^2 R'_d}{2\pi Q_{d*}\ell X}\right)\right| \\ \ll_\epsilon M_d^{\frac{1}{2}} \sigma_0(M_d) r_{d*0}^{\frac{1}{2}-\epsilon} \sigma_{-1+2\epsilon}(r_{d*0}) \sum_{g|(n,M_d)} g^\epsilon M^{\epsilon'} \left(\frac{M_d^2 R'_d}{2\pi g X}\right)^{\frac{1}{2}+\epsilon} \\ \ll X^{-\frac{1}{2}-\epsilon} M^{\epsilon'} M_d^{\frac{3}{2}+2\epsilon} \sigma_0(M_d) {R'_d}^{\frac{1}{2}+\epsilon} \prod_{p|r_{d*}} p\end{aligned} \tag{4.7}$$

By definition of R'_d, we have

$$R'_d = [R_d, r_d^2] = R_d r_d \frac{r_d}{(R_d, r_d^2)} = R_d r_d \frac{1}{(R_d/r_d, r_d)} \leq R_d r_d.$$

The third equality holds since $r_d \mid R_d$. Also recall that $d = M_d r_d$. Therefore, referring to (4.7), we have

$$\left|\sum_{\ell=1}^{\infty} \frac{b_{\chi,R'}(Q_{d*}\ell)}{Q_{d*}\ell} S(n, \ell Q_{d*}\overline{R'_d}; M_d) V\left(\frac{M_d^2 R'_d}{2\pi Q_{d*}\ell X}\right)\right| \ll_\epsilon X^{-\frac{1}{2}-\epsilon} M^{\epsilon'} d^{\frac{3}{2}+2\epsilon} \sigma_0(M_d) r_d^{-1-\epsilon} R_d^{\frac{1}{2}+\epsilon} \prod_{p \mid r_{d*}} p. \tag{4.8}$$

Note (see (3.9)) that $c_{\tilde{\chi}_{r_d}}(n) = 0$ if $\frac{r_d}{r_{d*}r_{d0}} \nmid n$. Here $r_{d0} = \prod_{p \mid r_d, p \nmid r_{d*}} p = \frac{r_d}{r_{d*}}$ since r_d/r_{d*} is square-free and $(r_{d*}, r_d/r_{d*}) = 1$. So $\frac{r_d}{r_{d*}r_{d0}} = 1$ and we get

$$c_{\tilde{\chi}_{r_d}}(n) = \tau(\chi)\overline{\chi}(n)\mu\left((r_d/r_{d*}, n)\right)\varphi\left((r_d/r_{d*}, n)\right).$$

Thus

$$\left|c_{\tilde{\chi}_{r_d}}(n)\right| \le \sqrt{r_{d*}}\varphi((n, r_d/r_{d*})) \le \frac{r_d}{\sqrt{r_{d*}}}. \tag{4.9}$$

By applying (4.8) and (4.9) to the second summation in (4.3), we get

$$\begin{aligned}
&\frac{1}{M}\sum_{d \mid M}\frac{1}{\varphi(r_d)} \sum_{\substack{r_{d*} \mid r_d \\ \frac{r_d}{r_{d*}} \text{ square-free} \\ (r_{d*}, r_d/r_{d*})=1}} \sum_{\substack{\chi \bmod r_{d*} \\ \text{primitive}}} \tau(\overline{\chi})\chi(-\overline{(r_d/r_{d*})M_d^2}) c_{\tilde{\chi}_{r_d}}(n) \\
&\qquad \times \sum_{\ell=1}^{\infty} \frac{b_{\chi,R'}(Q_{d*}\ell)}{Q_{d*}\ell} S(n, \ell Q_{d*}\overline{R'_d}; M_d) V\left(\frac{M_d^2 R'_d}{2\pi Q_{d*}\ell X}\right) \\
&\qquad \ll \frac{X^{-\frac{1}{2}-\epsilon}M^{\epsilon'}}{M} \sum_{d \mid M} \sigma_0(M_d) d^{\frac{3}{2}+2\epsilon} r_d^{-\epsilon} R_d^{\frac{1}{2}+\epsilon} \sum_{\substack{\ell \mid r_d, \\ \text{square-free} \\ (r_d/\ell, \ell)=1}} \prod_{p \mid r_d/\ell} p \\
&\qquad \ll_\epsilon \frac{X^{-\frac{1}{2}-\epsilon}M^{\epsilon'}}{M} \sum_{d \mid M} \sigma_0(M_d) d^{\frac{3}{2}+2\epsilon} r_d^{-\epsilon} R_d^{\frac{1}{2}+\epsilon} \prod_{p \mid r_d} p.
\end{aligned} \tag{4.10}$$

Since $d \mid M$, we see that the right-hand side is

$$\le \frac{M^{\frac{1}{2}+2\epsilon+\epsilon'}\sigma_0(M)}{X^{\frac{1}{2}+\epsilon}} \sum_{d \mid M} \prod_{\substack{p \mid q, \\ p \nmid d}} p^{\left(\frac{1}{2}+\epsilon\right)\operatorname{ord}_p(q)} \prod_{p \mid r_d} p^{1+\left(\frac{1}{2}+\epsilon\right)\operatorname{ord}_p(q)}.$$

For each $d \mid M$,

$$\begin{aligned}
\prod_{\substack{p \mid q, \\ p \nmid d}} p^{\left(\frac{1}{2}+\epsilon\right)\operatorname{ord}_p(q)} \prod_{p \mid r_d} p^{1+\left(\frac{1}{2}+\epsilon\right)\operatorname{ord}_p(q)} &= q^{\frac{1}{2}+\epsilon} \prod_{p \mid (d,q)} p^{-\left(\frac{1}{2}+\epsilon\right)\operatorname{ord}_p(q)} \prod_{p \mid r_d} p^{1+\left(\frac{1}{2}+\epsilon\right)\operatorname{ord}_p(q)} \\
&\le q^{\frac{1}{2}+\epsilon} \prod_{\substack{p \mid d \\ \operatorname{ord}_p(d) < \operatorname{ord}_p(q)}} p \le q^{\frac{1}{2}+\epsilon} \prod_{\substack{p \mid (q,M) \\ p^2 \mid q}} p.
\end{aligned} \tag{4.11}$$

The condition of the last product implies that the term is 1 when q is square-free or $(q, M) = 1$. We thus have

$$\alpha_{n,M}(1) = \sum_{\ell \equiv n \bmod M} \frac{a(\ell)}{\ell} V\left(\frac{X}{2\pi\ell}\right) + \mathcal{O}\Bigg(M^{\frac{1}{2}+2\epsilon+\epsilon'}\sigma_0(M)^2 X^{-\frac{1}{2}-\epsilon} q^{\frac{1}{2}+\epsilon} \prod_{\substack{p|(q,M)\\ p^2|q}} p\Bigg). \tag{4.12}$$

This allows us to prove the following lemma.

Lemma 4.1. *Let $M > 1$. For each $X > 0$, we have*

$$\sum_{n\in\mathbb{Z}} \hat{h}(n)\alpha_{\pm n,M}(1) = \sum_{n\in\mathbb{Z}} \hat{h}(n) \sum_{m\equiv \pm n \bmod M} \frac{a(m)}{m} V\left(\frac{X}{2\pi m}\right) + \mathcal{O}\Bigg(X^{-\frac{1}{2}-\epsilon} M^{\frac{1}{2}+\epsilon} q^{\frac{1}{2}+\epsilon} \prod_{\substack{p|(q,M)\\ p^2|q}} p\Bigg), \tag{4.13}$$

for any $\epsilon > 0$.

Proof. Replacing ℓ by m in (4.12), we get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \hat{h}(n)\alpha_{\pm n,M}(1) = \sum_{n\in\mathbb{Z}} \hat{h}(n) &\sum_{m\equiv \pm n \bmod M} \frac{a(m)}{m} V\left(\frac{X}{2\pi m}\right) \\ &+ \mathcal{O}\Bigg(M^{\frac{1}{2}+2\epsilon+\epsilon'}\sigma_0(M)^2 X^{-\frac{1}{2}-\epsilon} q^{\frac{1}{2}+\epsilon} \prod_{\substack{p|(q,M)\\ p^2|q}} p \Bigg(\sum_{n\in\mathbb{Z}} |\widehat{h}(n)|\Bigg)\Bigg).\end{aligned} \tag{4.14}$$

Now, for $h = h_\delta$ with $\delta = \delta_M > M^{-1+\eta}$, for some $0 < \eta < 1$, (2.6) implies that

$$\sum_{\substack{n\in\mathbb{Z}\\ \delta(|n|+1)>(|n|+1)^{1-\eta}M^{1-\eta}}} |\hat{h}_\delta(n)| \ll_K \sum_{\substack{n\in\mathbb{Z}\\ \delta(|n|+1)>(|n|+1)^{1-\eta}M^{1-\eta}}} (1+|n|)^{-1}(\delta(|n|+1))^{-K},$$

for arbitrary K. Choosing $K = K'/(1-\eta)$, with $K' \gg 1$, we see that this portion of the sum is $\ll M^{-K'}$, for arbitrary K'.

Taking the remaining portion of the sum,

$$\sum_{\substack{n\in\mathbb{Z}\\ \delta(|n|+1)\le(|n|+1)^{1-\eta}M^{1-\eta}}} |\hat{h}_\delta(n)| \ll \sum_{\substack{n\in\mathbb{Z}\\ |n|\ll M^{2/\eta-2}}} |\hat{h}_\delta(n)| \ll M^{\epsilon''},$$

as for $n \neq 0$, $|\hat{h}_\delta(n)| \le 1/|n|$. Thus the error term of (4.14) is

$$\mathcal{O}_\epsilon\Bigg(X^{-\frac{1}{2}-\epsilon} M^{\frac{1}{2}+\epsilon} q^{\frac{1}{2}+\epsilon} \prod_{\substack{p|(q,M)\\ p^2|q}} p\Bigg),$$

with a new $\epsilon > 0$. □

The next proposition gives us an estimate for the first term of the right-hand side of (4.13).

Lemma 4.2. *For $h = h_\delta$ with $\delta = \delta_M > M^{-1+\eta}$, for some fixed $\eta > 0$, we have,*

$$\sum_{n\in\mathbb{Z}} \hat{h}_\delta(n) \sum_{m\equiv \pm n \bmod M} \frac{a(m)}{m} V\left(\frac{X}{2\pi m}\right) = \sum_{n\ge 1} \hat{h}(\pm n)\frac{a(n)}{n} + \mathcal{O}\left(X^{-\frac{1}{2}+\epsilon}\right) + \mathcal{O}\left(X^{\frac{1}{2}+\epsilon}M^{-1+\epsilon}\right).$$

Proof. Referring to (4.13), we consider

$$\sum_{n\in\mathbb{Z}}\hat{h}(n)\sum_{m\equiv\pm n \bmod M}\frac{a(m)}{m}V\left(\frac{X}{2\pi m}\right) = \sum_{n\geq 1}\hat{h}(\pm n)\frac{a(n)}{n}V\left(\frac{X}{2\pi n}\right)+\sum_{n\in\mathbb{Z}}\sum_{\substack{m\equiv\pm n \bmod M\\ m\neq\pm n}}\frac{a(m)}{m}V\left(\frac{X}{2\pi m}\right).$$

We first consider the diagonal terms with $m=\pm n$ in the sum. Upon moving the line of integration of the integral

$$V(y)=\frac{1}{2\pi i}\int_{(2)}(2\pi y)^u G(u)\Gamma(u)du,$$

(see (3.36)) we get

$$\sum_{n\geq 1}\hat{h}(\pm n)\frac{a(n)}{n}V\left(\frac{X}{2\pi n}\right) = \sum_{n\geq 1}\hat{h}(\pm n)\frac{a(n)}{n}+\sum_{n\geq 1}\hat{h}(\pm n)\frac{a(n)}{n}\frac{1}{2\pi i}\int_{(-\frac{1}{2}+\epsilon)}\left(\frac{X}{2\pi n}\right)^u G(u)\frac{\Gamma(u+1)}{(2\pi)^u}\frac{du}{u}.$$

Since the second sum is $\ll \sum_{n\geq 1}|\hat{h}(\pm n)|n^{-\epsilon'}X^{-1/2+\epsilon}$, inequality (2.6) implies that the sum converges and we have

$$\sum_{n\geq 1}\hat{h}(\pm n)\frac{a(n)}{n}+\mathcal{O}_\epsilon\left(X^{-1/2+\epsilon}\right). \tag{4.15}$$

Now we are left with the off-diagonal, namely the terms $n\neq\pm m$. Note that the length of the sum over m is $\mathcal{O}_\epsilon\big(X^{1+\epsilon}\big)$ by the rapid decay of V. We separate into two cases: $|n|\leq M/2$ and $|n|>M/2$.

For the latter, (2.6) implies

$$\sum_{\substack{n\in\mathbb{Z}\\ |n|>M/2}}\hat{h}(n)\sum_{\substack{m\equiv\pm n \bmod M\\ m\neq\pm n}}\frac{a(m)}{m}V\left(\frac{X}{2\pi m}\right)\ll\sum_{\substack{n\in\mathbb{Z}\\ |n|>M/2}}\frac{(\delta_M(1+|n|))^{-K}}{1+|n|}\sum_{\substack{m\equiv\pm n \bmod M\\ 0<m\ll X^{1+\epsilon}, m\neq\pm n}}\frac{1}{m^{1/2-\epsilon}}$$
$$\ll\delta_M^{-K}\sum_{\substack{n\in\mathbb{Z}\\ |n|>M/2}}(1+|n|)^{-K-1}\sum_{0<m\ll X^{1+\epsilon}}\frac{1}{m^{1/2-\epsilon}}\ll_{K,\epsilon}\delta_M^{-K}M^{-K}X^{1/2+\epsilon}.$$

Since, $\delta_M>1/M^{1-\eta}$, that is, $\delta_M M>M^\eta$, by assuming X will be less than some fixed power of Mq, we get $\mathcal{O}\left((qM)^{-K}\right)$ with $K>1$ arbitrarily large.

For the former case we note that as $m\neq\pm n$, the congruence relation modulo M forces $m>M/2$. We then calculate using $K=0$ in (2.6), and recalling that the contribution from

$m > X^{1+\epsilon}$ is smaller than $(Mq)^{-K}$ for arbitrary $K \gg 1$, we get

$$\sum_{\substack{n\in\mathbb{Z},\\ |n|\le M/2}} \hat{h}(n) \sum_{\substack{m\equiv \pm n \bmod M\\ m\neq \pm n}} \frac{a(m)}{m} V\left(\frac{X}{2\pi m}\right) \ll \sum_{|n|\le M/2} \frac{1}{|n|+1} \sum_{0<|\ell|\ll X^{1+\epsilon}/M} \frac{a(\pm n + M\ell)}{(\pm n + M\ell)}$$
$$\ll \sum_{|n|\le M/2} \frac{1}{|n|+1} \sum_{\substack{0<|\ell|\ll X^{1+\epsilon}/M\\ (\pm n + M\ell)>0}} \frac{M^{-1/2+\epsilon}}{(\frac{\pm n}{M}+\ell)^{1/2-\epsilon}} \ll_\epsilon X^{1/2+\epsilon} M^{-1+\epsilon}. \tag{4.16}$$

Combining (4.15) with (4.16) yields the proposition. □

We can combine Lemma 4.2 with Lemma 4.1 to get the asymptotics of $\sum \hat{h}(n)\alpha_{\pm n,M}(1)$. To this end, we will compare the error terms produced in Lemmas 4.1 and 4.2 to determine a value of X that gives the optimal bound. Setting the error terms from (4.13) and (4.16) equal, we get

$$X^{-\frac12} M^{\frac12} q^{\frac12} \prod_{\substack{p|(q,M)\\ p^2|q}} p = X^{\frac12} M^{-1}.$$

This gives us

$$X = M^{\frac32} q^{\frac12} \prod_{\substack{p|(q,M)\\ p^2|q}} p.$$

Thus the error from these two contributions is

$$\mathcal{O}_\epsilon\left(X^{\frac12} M^{-1} (Mq)^\epsilon\right) = \mathcal{O}_\epsilon\Bigg((Mq)^\epsilon M^{-\frac14} q^{\frac14} \prod_{\substack{p|(q,M)\\ p^2|q}} p^{\frac12}\Bigg).$$

The remaining error (from (4.15)) is dominated by these terms since $X^{-1/2+\epsilon} \ll X^{-1/2+\epsilon} M$. From this, together with Lemmas 4.1 and 4.2, we deduce the following

Proposition 4.3. *Let $M > 1$. For $h = h_\delta$ with $\delta = \delta_M > M^{-1+\eta}$ for some fixed $0 < \eta < 1$, we have,*

$$\sum_{n\in\mathbb{Z}} \hat{h}_\delta(n)\alpha_{\pm n,M}(1) = \sum_{n\ge1} \hat{h}_\delta(\pm n)\frac{a(n)}{n} + \mathcal{O}_\epsilon\Bigg((Mq)^\epsilon M^{-\frac14} q^{\frac14} \prod_{\substack{p|(q,M)\\ p^2|q}} p^{\frac12}\Bigg).$$

Therefore we obtain

$$A_h^\pm(M) = \sum_{n\ge1} (\widehat{h}_\delta(-n) \pm \widehat{h}_\delta(n))\frac{a(n)}{n} + \mathcal{O}_\epsilon\Bigg((Mq)^\epsilon M^{-\frac14} q^{\frac14} \prod_{\substack{p|(q,M)\\ p^2|q}} p^{\frac12}\Bigg).$$

In the next section we study the sum

$$\sum_{n\ge1} (\widehat{h}_\delta(-n) \pm \widehat{h}_\delta(n))\frac{a(n)}{n},$$

choose δ_M for the final error term and conclude the proof of Theorem 1.2.

5. Proof of Theorem 1.2

For fixed x we consider $h = h_\delta$. Combining (2.7) and Proposition 4.3 we deduce

$$\frac{1}{2}A_h^{\pm}(M) = \frac{1}{M}\sum_{0\le a\le M}\left\langle\frac{a}{M}\right\rangle^{\pm} h_\delta(\frac{a}{M}) = \sum_{n\ge1}(\widehat{h}_\delta(-n)\pm\widehat{h}_\delta(n))\frac{a(n)}{n} + \mathcal{O}\Big((Mq)^{\epsilon}M^{-\frac14}q^{\frac14}\prod_{\substack{p|(q,M)\\ p^2|q}}p^{\frac12}\Big). \tag{5.1}$$

Lemma 5.1. *For $h = h_\delta$ with $\delta = \delta_M$, we have*

$$\sum_{n\ge1}\hat{h}_\delta(n)\frac{a(n)}{n} = \sum_{n\ge1}\frac{1-e^{-2\pi inx}}{2\pi in}\frac{a(n)}{n} + \mathcal{O}_\epsilon\left(\delta_M^{\frac12-\epsilon}\right).$$

Proof. We have

$$\left|\sum_{n\ge1}\hat{h}_\delta(n)\frac{a(n)}{n} - \sum_{n\ge1}\frac{1-e^{-2\pi inx}}{2\pi in}\frac{a(n)}{n}\right| \le \left|\sum_{n>\delta_M^{-1}}\hat{h}_\delta(n)\frac{a(n)}{n}\right| + \left|\sum_{n>\delta_M^{-1}}\frac{1-e^{-2\pi inx}}{2\pi in}\frac{a(n)}{n}\right| + \sum_{n=1}^{\delta_M^{-1}}\left|\hat{h}_\delta(n) - \frac{1-e^{-2\pi inx}}{2\pi in}\right|\frac{a(n)}{n}. \tag{5.2}$$

Because of (2.6), we have

$$\left|\sum_{n>\delta_M^{-1}}\hat{h}_\delta(n)\frac{a(n)}{n}\right| \ll \sum_{n>\delta_M^{-1}}\frac{1}{n^{\frac32-\epsilon}} \ll_\epsilon \delta_M^{\frac12-\epsilon}.$$

Since $\frac{1-e^{-2\pi inx}}{2\pi in}$ is likewise $\ll n^{-1}$, the same bound holds for the second sum in the right-hand side of (5.2).

For the last sum of (5.2), we observe that, because of (2.5), we have

$$\hat{h}_\delta(n) = \frac{1-e^{-2\pi inx}}{2\pi in} + \frac{1}{2\pi in}\int_{-\frac12}^{\frac12}\phi(t)\left((e^{2\pi i\delta_M n(1-t)}-1) - e^{-2\pi inx}(e^{-2\pi in\delta_M(1+t)}-1)\right)dt = \frac{1-e^{-2\pi inx}}{2\pi in} + \frac{1}{2\pi in}\mathcal{O}\left(n\delta_M\right) \tag{5.3}$$

because $e^{2\pi i\delta_M n(1-t)} = 1 + \mathcal{O}\left(n\delta_M\right)$ (since $n\delta_M \le 1$). Thus

$$\sum_{n=1}^{\delta_M^{-1}}\left|\hat{h}_\delta(n) - \frac{1-e^{-2\pi inx}}{2\pi in}\right|\frac{a(n)}{n} \ll_\epsilon \delta_M\sum_{n=1}^{\lceil\delta_M^{-1}\rceil}\frac{1}{n^{\frac12-\epsilon}} \ll_\epsilon \delta_M\delta_M^{-\frac12-\epsilon} = \delta_M^{\frac12-\epsilon}.$$

□

Similarly, we can prove that

$$\sum_{n\ge1}\hat{h}_\delta(-n)\frac{a(n)}{n} = \sum_{n\ge1}\frac{1-e^{2\pi inx}}{-2\pi in}\frac{a(n)}{n} + \mathcal{O}_\epsilon(\delta_M^{\frac12-\epsilon}).$$

Entering this and Lemma 5.1 into (5.1), we derive the main terms of Theorem 1.2.

To determine the error term we note that the error terms we have obtained from our analysis are $\mathcal{O}_\epsilon\big(\delta_M^{\frac{1}{2}-\epsilon}\big)$ from the above,

$$\delta_M M^{\frac{1}{2}} q^{\frac{1}{4}} (qM)^\epsilon \prod_{\substack{p|M \\ \mathrm{ord}_p(M)<\mathrm{ord}_p(q)}} p^{\frac{1}{4}} \leq \mathcal{O}_\epsilon\left(\delta_M M^{\frac{1}{2}} q^{\frac{1}{4}} (qM)^\epsilon \prod_{\substack{p|(q,M) \\ p^2|q}} p^{\frac{1}{4}}\right)$$

from Lemma 2.1, and

$$\mathcal{O}_\epsilon\left((Mq)^\epsilon M^{-\frac{1}{4}} q^{\frac{1}{4}} \prod_{\substack{p|(q,M) \\ p^2|q}} p^{\frac{1}{2}}\right)$$

from Proposition 4.3.

Setting

$$\delta_M M^{\frac{1}{2}} q^{\frac{1}{4}} (qM)^\epsilon \prod_{\substack{p|(q,M) \\ p^2|q}} p^{\frac{1}{4}} = (Mq)^\epsilon M^{-\frac{1}{4}} q^{\frac{1}{4}} \prod_{\substack{p|(q,M) \\ p^2|q}} p^{\frac{1}{2}}$$

gives us

$$\delta_M^{\frac{1}{2}} = (Mq)^\epsilon M^{-\frac{3}{8}} \prod_{\substack{p|(q,M), \\ p^2|q}} p^{\frac{1}{8}}$$

So certainly $\delta_M^{\frac{1}{2}}$ is smaller than other error terms in (5.1):

$$\delta_M^{\frac{1}{2}} < M^{-\frac{1}{4}} q^{\frac{1}{4}} \prod_{\substack{p|(q,M) \\ p^2|q}} p^{\frac{1}{2}}.$$

Thus the final error is

$$\mathcal{O}_\epsilon\left((Mq)^\epsilon M^{-\frac{1}{4}} q^{\frac{1}{4}} \prod_{\substack{p|(q,M) \\ p^2|q}} p^{\frac{1}{2}}\right).$$

This completes the proof of Theorem 1.2.

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