PROTECTING POINTS FROM OPERATOR PENCILS

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ABSTRACT. We classify all sets of the form $\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB)$ where A and B are self-adjoint operators and B is bounded, non-negative, and non-zero. We show that these sets are exactly the complements of discrete subsets of $\mathbb{R}$, that is, of at most countable subsets of $\mathbb{R}$ that contain none of their accumulation points.

1. Introduction and main results

We study the union of spectra $\operatorname{spec}(A + tB)$, $t \in \mathbb{R}$, where A and B are self-adjoint operators in a (complex) Hilbert space, and B is bounded and non-negative. It is rather easy to cook up examples where this union is not the whole real line, that is, there can exist *protected points*, see, e.g., Example 2.4 below. The objective of this note is to classify all possible sets of such protected points. Our main result reads as follows.

Theorem 1.1. *Let A and B be self-adjoint operators on the same Hilbert space, and suppose that B is bounded, non-negative, and non-zero. Then, the set $\mathbb{R}\setminus\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A+tB)$ is discrete, that is, it is at most countable and contains none of its accumulation points. In particular, the union of these spectra is dense.*

Conversely, for every discrete $P \subset \mathbb{R}$, there exist self-adjoint operators A and B with B non-negative and bounded such that $P = \mathbb{R}\setminus\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A+tB)$.

Theorem 1.1 can be applied to obtain information on the unperturbed operator A if it is known that $\operatorname{spec}(A + tB)$ does not vary with t.

Corollary 1.2 (See also [11])**.** *Let A and B be self-adjoint operators on the same Hilbert space, and suppose that B is bounded and non-negative. If the spectrum of $A + tB$ is independent of t, that is, $\operatorname{spec}(A + tB) = \operatorname{spec}(A)$ for all $t \in \mathbb{R}$, then $B = 0$ or $\operatorname{spec}(A) = \mathbb{R}$.*

In fact, Corollary 1.2 was the starting point of our investigation and finds an application in the context of the Klein-Gordon equation in [11] featuring the matrix operator

$$\mathbf{K} = \begin{pmatrix} 0 & \sqrt{p^2+1} \\ \sqrt{p^2+1} & 2x \end{pmatrix}, \quad p = -i\frac{\mathrm{d}}{\mathrm{d}x},$$

which has the property

$$e^{-ipt}\mathbf{K}e^{ipt} = \mathbf{K} - 2t\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{1.1}$$

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Identity (1.1) resembles the definition of a *homogeneous operator*. These are operators A such that for every $t \in \mathbb{R}$ the operator $A + tI$ is unitarily equivalent to A. It is known that homogeneous operators have absolutely continuous spectrum on the whole real line [9] and a quantum mechanical instance of such a phenomenon has been produced in (7.19) of [3]. It would be interesting to study under which assumptions it is possible to weaken the notion of homegeneity in [9] to the situation of the corollary above and prove absolute continuity of the spectrum of A, and thus of $\mathbf{K}$ in (1.1).

Also recall that the *spectrum of an operator pencil* is defined as

$$\operatorname{spec}(A, B) = \{\lambda \in \mathbb{C} \colon 0 \in \operatorname{spec}(A - \lambda B)\}\,,$$

see for instance [6]. In this context, Theorem 1.1 implies that for all sets P as in the theorem there exist self-adjoint operators A and B such that $\operatorname{spec}(A - \lambda, B)$ is empty for all $\lambda \in P$.

In [4], a related situation of linear operator pencils is studied. However, rather than *protected points*, the authors investigate the set of parameters t for which a given $\lambda \in \mathbb{R}$ is in the *point spectrum* of $A + tB$. They review a result from [7] that states that this set has Lebesgue measure zero and show that sign-definiteness of B is crucial for this to hold. One easily sees that sign-definiteness of B also cannot be dropped in Theorem 1.1. In fact, for indefinite B, complete intervals can be protected, as can be seen from the following example of 2×2 matrices (cf. also Example 2.4 below):

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

With these choices we have $\operatorname{spec}(A + tB) = \{\pm\sqrt{1 + t^2}\}$, so that the complement of the union $\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB)$ agrees with $(-1, 1)$. This example is an instance of a more general result that states that a gap in the spectrum of a self-adjoint operator is preserved under an off-diagonal perturbation, see Theorem 2.1 in [1] and also Theorem 8.1 in [2]. Accordingly, also Corollary 1.2 is no longer valid for general perturbations B, that is, one can find self-adjoint operators A and $B \neq 0$ with B bounded and $\operatorname{spec}(A + tB) = \operatorname{spec}(A) \subsetneq \mathbb{R}$ for all $t \in \mathbb{R}$.

2. Proofs

The first assertion of Theorem 1.1 will follow from Lemmas 2.7 and 2.8 below, the second one from Lemma 2.10. The proof of the latter is constructive. Corollary 1.2 will then follow from the first part of the theorem (cf. also Corollary 2.6 below) and the fact the spectrum of A is always closed. The core of our considerations is the following result.

Proposition 2.1. *Let A and B be as in Corollary 1.2. Then, the following are equivalent:*

(i) *0 belongs to each resolvent set $\rho(A + tB)$, $t \in \mathbb{R}$;*
(ii) *$0 \in \rho(A)$ and $BA^{-1}B = 0$.*

In this case, it holds that $0 \in \rho(A + zB)$ for all $z \in \mathbb{C}$ with

$$(A + zB)^{-1} = A^{-1} - zA^{-1}BA^{-1}. \tag{2.1}$$

Proof. Suppose that $0 \in \rho(A)$. We first observe that

$$\operatorname{spec}(BA^{-1}) \setminus \{0\} = \operatorname{spec}(B^{1/2}A^{-1}B^{1/2}) \setminus \{0\} \subset \mathbb{R}, \tag{2.2}$$

where for the last inclusion we have used that $B^{1/2}A^{-1}B^{1/2}$ is self-adjoint. Moreover, we have for all $z \in \mathbb{C} \setminus \{0\}$ that

$$A + zB = \big(I + zBA^{-1}\big)A = z\Big(\frac{1}{z}I + BA^{-1}\Big)A. \tag{2.3}$$

Hence, $0 \in \rho(A+zB)$ holds if and only if $0 \in \rho(I/z + BA^{-1})$. Identity (2.2) shows that the latter liberally holds for all $z \in \mathbb{C} \setminus \mathbb{R}$.

(i)⇒(ii). By hypothesis, we have $0 \in \rho(A + zB)$ for all $z \in \mathbb{R}$, whence, according to the considerations above, $0 \in \rho(I/z + BA^{-1})$ for all $z \in \mathbb{C}\setminus\{0\}$. This yields $\operatorname{spec}(BA^{-1}) = \{0\}$, so that $\operatorname{spec}(B^{1/2}A^{-1}B^{1/2}) = \{0\}$ by (2.2). From the self-adjointness of $B^{1/2}A^{-1}B^{1/2}$ we obtain $B^{1/2}A^{-1}B^{1/2} = 0$, and, in particular, $BA^{-1}B = 0$.

(ii)⇒(i). We have $(BA^{-1})^2 = 0$. For each $z \in \mathbb{C}$, the inverse of the operator $I + zBA^{-1}$ is therefore given by $I - zBA^{-1}$, and from (2.3) we conclude that

$$(A + zB)^{-1} = A^{-1}(I - zBA^{-1}) = A^{-1} - zA^{-1}BA^{-1}. \qquad \square$$

Remark 2.2. *Since B is bounded and nonnegative and A^{-1} is bounded, $BA^{-1}B = 0$ is in fact equivalent to nilpotence of $A^{-1}B$. Indeed, if we have $(A^{-1}B)^n = 0$ for some $n \in \mathbb{N}$, then $\operatorname{spec}((B^{1/2}A^{-1}B^{1/2})^n)$, which coincides away from 0 with $\operatorname{spec}((A^{-1}B)^n)$, must be simply $\{0\}$. This implies that $B^{1/2}A^{-1}B^{1/2} = 0$. Furthermore, $BA^{-1}B = 0$ holds if and only if the map $z \mapsto (A^{-1} - zA^{-1}BA^{-1})B$ defines a* pseudo-resolvent*; see [5] for an introduction to this notion. In this case, this map also agrees with the pseudo-resolvent $(A + zB)^{-1}B$ belonging to the operator pencil.*

Corollary 2.3. *If the equivalence in Proposition 2.1 takes place with $B \neq 0$, then $A^{-1}BA^{-1} \neq 0$ and*

$$\frac{1}{|t|\|A^{-1}BA^{-1}\| + \|A^{-1}\|} \le \operatorname{dist}\big(0, \operatorname{spec}(A+tB)\big) \le \frac{1}{|t|\|A^{-1}BA^{-1}\| - \|A^{-1}\|}$$

for $t \in \mathbb{R}$ with $|t| > \|A^{-1}\|/\|A^{-1}BA^{-1}\|$.

Proof. First note that $A^{-1}BA^{-1} \neq 0$ is a consequence of the boundedness of B and the fact that A^{-1} is bijective as a map from the whole Hilbert space to the dense subspace $\operatorname{Dom}(A)$. Now, from identity (2.1) in Proposition 2.1 we conclude that

$$|t|\|A^{-1}BA^{-1}\| - \|A^{-1}\| \le \|(A + tB)^{-1}\| \le |t|\|A^{-1}BA^{-1}\| + \|A^{-1}\|.$$

Since $\|(A + tB)^{-1}\|^{-1} = \operatorname{dist}(0, \operatorname{spec}(A + tB))$, this proves the claim. $\square$

Statement (ii) in Proposition 2.1 indicates how to construct examples where the set $\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB)$ is not the whole real line. The simplest is the following pedestrian's example of 2×2 matrices; there will be more sophisticated examples below.

Example 2.4. *Choosing*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we see that $BA^{-1}B = 0$*, whence* $\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB) \subset \mathbb{R} \setminus \{0\}$ *by Proposition 2.1. In fact,*

$$\operatorname{spec}(A + tB) = \Big\{-\frac{t}{2} \pm \sqrt{\frac{t^2}{4} + 1}\Big\}, \quad t \in \mathbb{R},$$

so that $\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB) = \mathbb{R} \setminus \{0\}$*. Note that the distance of* 0 *to the spectrum of* $A+tB$ *is given by* $\operatorname{dist}(0, \operatorname{spec}(A+tB)) = (|t|/2+\sqrt{t^2/4 + 1})^{-1}$ *and, therefore, behaves asymptotically for* $|t| \to \infty$ *exactly as predicted by Corollary 2.3.*

Remark 2.5. *The matrices of Example 2.4 appear in the context of critically damped linear systems, see for instance Example 9.3 in [10] with the nilpotent matrix*

$$A^{-1}B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Clearly, the spectral point 0 plays no particular role in the above considerations as we may replace A by $A-\lambda$ by any $\lambda \in \mathbb{R}$. Since by Corollary 2.3 the distance of every protected point to the spectrum tends to zero in the $|t| \to \infty$ limit, we deduce the following.

Corollary 2.6. *Let* A *and* B *be as in Proposition 2.1. If* $B \neq 0$*, then the union* $\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB)$ *is a dense subset of* $\mathbb{R}$.

The fact that the set $\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB)$ is dense can, in fact, be refined to the following stronger statement.

Lemma 2.7. *Let* A *and* B *be as in Corollary 1.2 with* $B \neq 0$*. Then, the set* $\mathbb{R} \setminus \bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB)$ *is at most countable.*

Proof. Pick any $y := Bx \neq 0$, and consider the function $f\colon \rho(A) \to \mathbb{R}$ defined by

$$f(z) := \langle x, B(A - z)^{-1}Bx\rangle = \langle y, (A - z)^{-1}y\rangle. \tag{2.4}$$

For every $\lambda \in \mathbb{R}$ with $B(A - \lambda)^{-1}B = 0$ we find $f(\lambda) = 0$. Hence, by Proposition 2.1, every point in $\mathbb{R} \setminus \bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB)$ must be a (real) root of f. But f can have at most countably many real roots. Indeed, we have $f'(z) = \langle y, (A - z)^{-2}y\rangle$ for all $z \in \rho(A)$, whence, in particular,

$$f'(\lambda) = \|(A - \lambda)^{-1}y\|^2 > 0 \quad \text{for} \quad \lambda \in \rho(A) \cap \mathbb{R}\,.$$

Thus, f is strictly monotone on every interval of $\rho(A)\cap\mathbb{R}$ and can therefore have at most one root there. Since $\rho(A) \cap \mathbb{R}$ is an open subset of $\mathbb{R}$, that is, an at most countable union of disjoint open intervals, this proves the claim. □

Lemma 2.8. *Let* A *and* B *be as in Corollary 1.2 with* $B \neq 0$*. If* λ *is an accumulation point of* $\mathbb{R} \setminus \bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB)$*, then* $\lambda \in \operatorname{spec}(A + tB)$ *for all* $t \in \mathbb{R}$*. In particular,* $\mathbb{R}\setminus\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A+tB)$ *contains none of its accumulation points, if any.*

Proof. Let (λ_k) be a sequence in $\mathbb{R} \setminus \bigcup_{t\in\mathbb{R}} \operatorname{spec}(A + tB)$ with $\lambda_k \to \lambda \in \mathbb{R}$ as $k \to \infty$ and $\lambda_k \neq \lambda$ for all k, and assume that $\lambda \in \rho(A + t_0B)$ for some $t_0 \in \mathbb{R}$. Set $\tilde{A} := A+t_0B$. Then, (λ_k) is a sequence in $\mathbb{R}\setminus\bigcup_{t\in\mathbb{R}} \operatorname{spec}(\tilde{A}+tB)$, so that $B(\tilde{A}-\lambda_k)^{-1}B = 0$ for all k by Proposition 2.1. Thus, the mapping $z \mapsto B(\tilde{A}-z)^{-1}B$ is analytic in λ and has zeros at every λ_k, hence vanishes in a (complex) neighbourhood of λ. Therefore, again by Proposition 2.1, a real neighbourhood of λ belongs to $\mathbb{R} \setminus \bigcup_{t\in\mathbb{R}} \operatorname{spec}(\tilde{A}+tB)$, so that the union $\bigcup_{t\in\mathbb{R}} \operatorname{spec}(\tilde{A}+tB) = \bigcup_{t\in\mathbb{R}} \operatorname{spec}(A+tB)$ is not dense in $\mathbb{R}$. The latter is a contradiction to Corollary 2.6. □

We now turn to the second assertion in Theorem 1.1. Here, let us first observe that the sets $\mathbb{R} \setminus \bigcup_{t\in\mathbb{R}} \operatorname{spec}(A+tB)$ can indeed consist of every finite or countably infinite number of points. This can easily be observed by inverting the construction in the proof of Lemma 2.7 above:

Example 2.9. *Let I be a finite or countably infinite set. On $\ell^2(I)$ consider the diagonal operator $A = \operatorname{diag}(\{\mu_k\}_{k\in I})$ with distinct real numbers μ_k, and let $y \in \ell^2(I)$ be normalized with every entry non-zero. Finally, consider the rank-one projection $B := \langle y, \cdot\rangle y \neq 0$. Clearly, $B(A-\lambda)^{-1}B = 0$ if and only if λ is a zero of the function f defined as in* (2.4) *with $x := y$. Hence, by means of Proposition 2.1, the set $\mathbb{R} \setminus \bigcup_{t\in\mathbb{R}} \operatorname{spec}(A+tB)$ consists of all real zeros of f. Now, the mapping $\rho(A)\cap\mathbb{R} \ni \lambda \mapsto f(\lambda)$ has poles in every μ_k and is strictly monotone on the intervals between them, cf. the proof of Lemma 2.7. Therefore, it must have exactly one root on every such interval. We have thus found operators A and B with any finite or countably infinite number of "protected" points.*

If we choose the μ_k in Example 2.9 to have an accumulation point, then also the protected points between them must have an accumulation point in $\mathbb{R}$. However, by Lemma 2.8, this accumulation point will not belong to the set of protected points, but rather to each spectrum $\operatorname{spec}(A+tB)$, $t \in \mathbb{R}$.

The method of Example 2.9 does not allow us to directly choose the protected points. This can be achieved by the following construction, which concludes the proof of Theorem 1.1:

Let $P \subset \mathbb{R}$ be a finite or countably infinite discrete set. Choose an orthonormal basis $(\psi_\lambda)_{\lambda\in P}$ of $\mathcal{H} = \ell^2(P)$, and consider the (not necessarily bounded) self-adjoint operator K on $\mathcal{H}$ with

$$K\psi_\lambda = \lambda\psi_\lambda \quad \text{for all} \quad \lambda \in P.$$

Clearly, every $\lambda \in P$ is an isolated eigenvalue of multiplicity one. Thus, we have $\operatorname{spec}(K) = \overline{P}$, and the operator K has simple spectrum (see, e.g., [8, Section 5.4] for a definition). In particular, by [8, Proposition 5.20] there exists a *cyclic vector* $v \in \mathcal{H}$, which means that $\{v, Kv, K^2v, \dots\}$ spans a dense subspace of $\mathcal{H}$.

We now add one extra dimension and define self-adjoint operators A and B on the Hilbert space $\tilde{\mathcal{H}} := \mathcal{H} \oplus \mathbb{C}$ by

$$A := \begin{pmatrix} K & v \\ \langle v, \cdot\rangle & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.5}$$

Lemma 2.10. *For A and B as in* (2.5) *we have $\bigcup_{t\in\mathbb{R}} \operatorname{spec}(A+tB) = \mathbb{R}\setminus P$.*

Proof. For each $t \in \mathbb{R}$ the operator $A+tB$ is a finite rank perturbation of $K \oplus 0$, so that its spectrum consists of the essential spectrum of K, that is, the accumulation points of P, and isolated eigenvalues of finite multiplicity.

Let $\lambda \in P$. We need to show that $\lambda \in \rho(A+tB)$ for all $t \in \mathbb{R}$. Here, it suffices to see that each $A+tB-\lambda$ has trivial kernel since λ is not in the essential spectrum of $A+tB$. To this end, let $x \oplus \alpha \in \operatorname{Ker}(A+tB-\lambda)$, that is,

$$\begin{pmatrix} (K-\lambda)x + \alpha v \\ \langle v, x\rangle + (t-\lambda)\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.6}$$

We expand $v = \sum_\lambda \beta_\lambda \psi_\lambda$ in the basis $(\psi_\lambda)_\lambda$ and note that cyclicity of v forces all β_λ to be non-zero. Now, by (2.6) we have $(K-\lambda)x = -\alpha v$. Since $(K-\lambda)x$ is orthogonal to ψ_λ and $\beta_\lambda \neq 0$, this implies that $\alpha = 0$. Hence, $(K-\lambda)x = 0$ and, therefore, $x = \langle \psi_\lambda, x\rangle \psi_\lambda$ is a multiple of ψ_λ. It then follows from (2.6) that $0 = \langle v, x\rangle = \overline{\beta_\lambda}\langle \psi_\lambda, x\rangle$, which yields that also $x = 0$.

Conversely, let $\lambda \in \mathbb{R} \setminus \bigcup_{t\in\mathbb{R}} \operatorname{spec}(A+tB)$. Since, in particular, $\lambda \in \rho(A)$, there is $x \oplus \alpha \in \tilde{\mathcal{H}} \setminus \{0\}$ with $(A-\lambda)(x\oplus\alpha) = 0 \oplus 1$. Now, by Proposition 2.1 we have $B(A-\lambda)^{-1}B = 0$ and, therefore, $0 = B(A-\lambda)^{-1}B(0\oplus 1) = 0 \oplus \alpha$. This implies that $\alpha = 0$ and, in turn, $(K-\lambda)x = 0$ with $x \neq 0$. Hence, λ is an eigenvalue of K, that is, $\lambda \in P$. □

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