Sharp estimates and homogenization of the control cost of the heat equation on large domains

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Abstract

We prove new bounds on the control cost for the abstract heat equation, assuming a spectral inequality or uncertainty relation for spectral projectors. In particular, we specify quantitatively how upper bounds on the control cost depend on the constants in the spectral inequality. This is then applied to the heat flow on bounded and unbounded domains modeled by a Schrödinger semigroup. This means that the heat evolution generator is allowed to contain a potential term. The observability/control set is assumed to obey an equidistribution or a thickness condition, depending on the context. Complementary lower bounds and examples show that our control cost estimates are sharp in certain asymptotic regimes. One of these is dubbed homogenization regime and corresponds to the situation that the control set becomes more and more evenly distributed throughout the domain while its density remains constant.

1 Introduction

Let us start by describing the most important example which has motivated our study of control cost estimates for the heat equation. Consider the inhomogeneous heat equation with heat generation term $-V$ on suitable domains $\Omega \subset \mathbb{R}^d$ given by

$$\dot{w} + (-\Delta + V)w = \mathbf{1}_S u, \quad w(0) = w_0 \in L^2(\Omega), \tag{1}$$

where $w, u \in L^2([0,T] \times \Omega)$, $V \in L^\infty(\Omega)$, and where the *control set* $S \subset \Omega$ is measurable with a positive measure. Hence the influence of the *control function* u is restricted to the set S. The system (1) is *null-controllable* if for every $w_0 \in L^2(\Omega)$ there exists a *control function* $u = u_{w_0} \in L^2([0,T] \times \Omega)$ such that the solution of (1) satisfies $w(T) = 0$. The *control cost in time* T is the least constant C_T such that $\|u\|_{L^2([0,T]\times\mathbb{R}^d)} \leq C_T \|w_0\|_{L^2(\mathbb{R}^d)}$ holds for all $w_0 \in L^2(\Omega)$.

The aim of this paper is to investigate sharp upper and lower bounds on the control cost in time $T > 0$ of the controlled heat equation (1), in particular, its dependence on the geometry of S. This is a natural aim since it has been shown recently that in the case $\Omega = \mathbb{R}^d$, if the system is null controllable, S *necessarily* has to satisfy certain geometric conditions, [WWZZ, EV18].

We are able to establish the optimality of our bounds in certain asymptotic regimes. A particularly appealing geometric regime is the homogenization scenario, in which the control set $S \subset \Omega$ becomes more and more evenly distributed over Ω while keeping an overall lower bound on the relative density. This corresponds to reducing local fluctuations in the density of the control set S. In such a homogenization regime we study the asymptotic behavior of the upper bound of the control cost.

Note that in the context of control theory homogenization scenarios have been studied before, see e.g. [Zua94, LZ02]. There however, as in classical homogenization theory, it is the differential operator generating the semigroup which is being homogenized, rather than the observability set.

So far, much more attention has been devoted to identifying the dependence of the control cost on the time parameter than to its geometric counterparts. In [Sei84] Seidman proved that for one-dimensional controlled heat systems the control cost in small time regime blows up at most exponentially. This result was extended to arbitrary dimension by Fursikov and Imanuilov in [FI96]. That the exponential blowup indeed occurs was established by Güichal [Güi85] for one-dimensional systems and by Miller [Mil04a] in the general case. Since then the bounds on the control cost have received a lot of attention [FZ00, Mil04b, Phu04, Mil06b, Mil06a, MZ06, TT07, DZZ08, Mil10, TT11, EZ11, LL12, Lis12, Lis15, BP17, DE, EV18, LL, Phu18]. Most of the results were obtained for bounded domains, but recently unbounded domains also became a focus of interest [BPS18, EV, WWZZ, EV18, Egi, ENT+18]. Note that there has been also interest in observability estimates if the measurement occurs only during a positive-measure subset of the time interval, see for instance [WZ17].

The most common way to obtain a bound on the control cost is a *final-state-observability estimate* (an estimate concerning the dynamics of the corresponding adjoint system) which in our context states that for all $T > 0$ there is a T-dependent constant C_{obs} such that

$$\|\mathrm{e}^{(\Delta - V)T} w\|_{L^2(\Omega)}^2 \leq C_{\mathrm{obs}}^2 \int_0^T \|\mathrm{e}^{(\Delta - V)t} w\|_{L^2(S)}^2 \mathrm{d}t \quad \text{for all } w \in L^2(\Omega).$$

The duality between null-controllability and final-state-observability implies that C_{obs} is an upper bound for the control cost. In the seminal papers [LR95, LZ98, JL99] it has been shown that one way to establish observability estimates is to prove a spectral inequality, i.e.

$$\|w\|_{L^2(\Omega)} \leq C \mathrm{e}^{d\sqrt{\lambda - \kappa}} \|w\|_{L^2(S)} \quad \text{for all } \lambda > \kappa \text{ and } w \in \operatorname{Ran} P_{-\Delta+V}(\lambda),$$

where $P_{-\Delta+V}(\lambda)$ is the projector to the spectral subspace of $-\Delta + V$ below λ, and κ is the minimal spectral point of $-\Delta + V$. This is a particularly attractive technique since the spectral inequality does not involve the time variable, i.e. it concerns only the corresponding stationary system. Consequently, quite a number of works developed abstract theorems to derive bounds on the control cost from spectral inequalities, each tailored for certain applications in mind. Among them are [Mil10, TT11, BPS18], which are also most closely related to our present paper. In spite of the variety of such earlier results, none of them is sufficient for our purposes, namely to provide sharp bounds on the control cost in several asymptotic scenarios of interest to us.

Hence, the first step to analyze heat control problems as described at the beginning of this section was to establish null-controllability of an abstract parabolic system from a suitable spectral inequality, together with an upper bound on the control cost. This is spelled out in Theorems 2.7 and 2.12. The proof of Theorem 2.7 is inspired by the direct approach of [TT11], since it turned out to be the one which can be best generalized and optimized for the geometric situations we had in mind.

On the quantitative level, the key improvement over the existing results is the dependence on our estimate on the control cost with respect to the parameters coming from the spectral inequality, cf. Remark 2.8. While Theorems 2.7 and 2.12 also cover the case when the control operator is not bounded, hence enabling its use in the case of boundary control, we do not pursue this question in the present paper.

Since our abstract theorem reduces the control cost estimate to a spectral inequality, it is also paramount for these spectral inequalities to have an explicit and – if possible – optimal dependence on parameters of interest. Recently, spectral inequalities with explicit geometry dependence on bounded and unbounded domains have been proved: in [EV18, EV] for the free heat equation controlled by a thick set, and in [NTTV] for the heat equation with potential with control supported on an equidistributed set. We combine these two spectral inequalities with our abstract theorem and obtain control cost estimates for the heat equation which are valid in a large class of (bounded and unbounded) domains and which depend on the control set S via its geometric parameters. Our bounds are uniform in the heat generation term V (they only depend on $\|V\|_\infty$) and are also uniform in Ω and S in a certain sense. The obtained estimates are much more explicit than what existed before and, together with the uniformity in Ω and S allow for the first time to study homogenization and de-homogenization limits.

The paper is divided in three parts. In Section 2 we start by proving an observability estimate from a spectral inequality, first for a non-negative operator in Theorem 2.7 and then for lower semi-bounded operators in Theorem 2.12. In the case of lower semi-bounded operators, the long time asymptotics of control cost depend on the growth bound of the corresponding semigroup. In order to better understand the upper bounds proven in Theorems 2.7 and 2.12, we compare them to lower bounds on the control of the heat equation for abstract systems and prove their sharpness in Theorem 2.13. In this section we also provide a thorough discussion of the lower and upper bounds of abstract control systems in Remark 2.16 and Table 1.

In Section 3 we then turn to system (1) and combine Theorem 2.12 with the spectral inequalities from [EV18] and [NTTV] to obtain bounds on the control cost for the free heat equation and free fractional heat equation controlled by a thick set, and a heat equation with a generation term controlled by a equidistributed set.

Finally, Section 4 is devoted to studying homogenization and de-homogenization of the control cost.

2 Abstract observability and null-controllability

For normed spaces V and W we denote by $\mathcal{L}(V, W)$ the space of bounded linear operators from V to W. Let X and U be Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_U$ and norms $\|\cdot\|$ and $\|\cdot\|_U$, respectively. Let A be a lower semibounded self-adjoint operator in X with domain $\mathcal{D}(A)$. We define $\kappa = \min \sigma(A)$ and denote by $\{P_H(\lambda) \colon \lambda \in \mathbb{R}\}$ the resolution of identity of a self-adjoint operator H. Let $\beta \in \mathbb{R}$. On X we define the scalar product

$$\langle x, y \rangle_\beta = \big\langle (I + A^2)^{\beta/2} x, (I + A^2)^{\beta/2} y \big\rangle. \tag{2}$$

For $\beta > 0$ we denote by $X_\beta \subset X$ the interpolation space obtained as the space $\mathcal{D}(I+A^2)^{\beta/2}$ equipped with the scalar product (2). For $\beta \leq 0$ we denote by $X_\beta \supset X$ the extrapolation space obtained as the completion of X with respect to the norm induced by the scalar product (2). >From now on we assume that $\beta \leq 0$ and $B \in \mathcal{L}(U, X_\beta)$. Clearly, $X_0 = X$ and the case $\beta = 0$ is of particular interest for our applications in Sections 3 and 4.

For $T > 0$, we study the abstract inhomogeneous Cauchy problem

$$\dot{w} + Aw = Bu, \quad w(0) = w_0 \in X, \tag{3}$$

where $w \in L^2([0,T], X)$ and $u \in L^2([0,T], U)$. The function u is called *control function*. The mild solution of (3) is given by

$$w(t) = \mathrm{e}^{-At} w_0 + \int_0^t \mathrm{e}^{-A(t-s)} Bu(s) \mathrm{d}s, \quad t \in [0,T]. \tag{4}$$

Note that since $\operatorname{Ran} B \subset X_\beta$, we need to give a meaning to the term $\mathrm{e}^{-A(t-s)}Bu(t)$. Therefore, we denote by $S(t)$ the semigroup in X with generator $-A$ and use the symbol $\mathrm{e}^{-A\cdot}$ for the unique extension of $S(t)$ to the space X_β. More precisely, let $U_\beta \in \mathcal{L}(X, X_\beta)$ be the isometric operator given as the unique extension of $(I+A^2)^{\beta/2} \in \mathcal{L}(X_{-\beta}, X)$. Then we have $\mathrm{e}^{-At} = U_\beta S(t) {U_\beta}^{-1}$.

Although we do not assume that B is an admissible control operator (for the definition, see, for example [TW09]), it still holds that $w(t) \in X$ for all t. This follows from

$$\mathrm{e}^{-At} X_\beta = U_\beta S(t) {U_\beta}^{-1} X_\beta = U_\beta S(t) X = U_\beta \mathcal{D}(A^\infty) \subset U_\beta X_{-\beta} = X$$

for all $t > 0$ where $\mathcal{D}(A^\infty) = \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$. Here, the equality $U_\beta S(t) X = U_\beta \mathcal{D}(A^\infty)$ follows from the fact that $S(t)$ is an analytic semigroup, cf. [Kat95, Chapter IX.1.6]. This shows $\mathrm{e}^{-A(t-s)} Bu(s) \in X$ for all $s \in [0,t)$ which implies $w(t) \in X$.

We now introduce two concepts, null-controllability and final-state-observability.

Definition 2.1. The system (3) is *null-controllable in time* $T > 0$ if for every $w_0 \in X$ there exists a control function $u = u_{w_0} \in L^2([0,T]; U)$ such that the solution (4) satisfies $w(T) = 0$. We call such a control function *null-control function in time* T. The *input map in time* T is the bounded mapping $\mathcal{B}^T \colon L^2([0,T], U) \to X$ given by $\mathcal{B}^T u = \int_0^T \mathrm{e}^{-(T-s)A} Bu(s) \mathrm{d}s$.

Remark 2.2. Note that if the system (3) is null-controllable in time $T > 0$, then, by linearity of e^{-TA}, it is also controllable on the range of e^{-TA}. This means that for every $w_0 \in X$ and every $u_T \in \operatorname{Ran} \mathrm{e}^{-TA}$ there is a control function $u \in L^2([0,T], U)$ such that the solution of (3) satisfies $w(T) = u_T$.

Taking into account (4), a null-control function u in time T satisfies $\mathrm{e}^{-TA} u_0 + \mathcal{B}^T u = 0$. Thus, the system (3) is null-controllable in time $T > 0$ if and only if one has the relation $\operatorname{Ran} \mathcal{B}^T \supset \operatorname{Ran} \mathrm{e}^{-TA}$, which gives an alternative definition of null-controllability in terms of the controllability map.

We will always take duals of the spaces X_β with respect to the pivot space X. Hence, the dual of X_β is $X_{-\beta}$ for all $\beta \in \mathbb{R}$ and in particular $B^* \in \mathcal{L}(X_{-\beta}, U)$. In order to introduce the notion of final-state-observability, we consider the adjoint system

$$\dot{f} + Af = 0, \quad y = B^* f, \quad f(0) = f_0 \in X, \tag{5}$$

where $f \in L^2([0,T]; X)$.

Definition 2.3. The system (5) is called *final-state-observable* in time $T > 0$ if there is a constant $C_{\mathrm{obs}} > 0$ such that for all $f_0 \in X$ we have

$$\|\mathrm{e}^{-AT} f_0\|_X^2 \leq C_{\mathrm{obs}}^2 \int_0^T \|B^* \mathrm{e}^{-At} f_0\|_U^2 \mathrm{d}t. \tag{6}$$

Ineq. (6) is called *observability inequality.*

By an analogous reasoning as above, we see that $f(t) = \mathrm{e}^{-At} f_0 \in X_{-\beta}$ for all $t > 0$ whence (5) and the right hand side of (6) are well-defined. The following lemma, due to Douglas [Dou66] and Dolecki and Russell [DR77], puts these concepts into relation. For a proof we refer to [TW09, TT11].

Lemma 2.4. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be Hilbert spaces, and let $\mathcal{X}\colon \mathcal{H}_1 \to \mathcal{H}_3$, $\mathcal{Y}\colon \mathcal{H}_2 \to \mathcal{H}_3$ be bounded operators. Then, the following are equivalent:*

(a) $\operatorname{Ran} \mathcal{X} \subset \operatorname{Ran} \mathcal{Y}$*;*

(b) There is $c > 0$ such that $\|\mathcal{X}^* z\| \leq c \|\mathcal{Y}^* z\|$ for all $z \in \mathcal{H}_3$.

(c) There is a bounded operator $\mathcal{Z}\colon \mathcal{H}_1 \to \mathcal{H}_2$ satisfying $\mathcal{X} = \mathcal{Y}\mathcal{Z}$.

Moreover, in this case, one has

$$\inf\{c\colon c \text{ as in (b)}\} = \inf\{\|\mathcal{Z}\|\colon \mathcal{Z} \text{ as in (c)}\}, \tag{7}$$

and both infima are actually minima.

We note that (a) corresponds to null-controllability and (b) corresponds to final-state-observability. Lemma 2.4 provides another equivalent statement (c). It implies that there exists an operator $\mathcal{F}\colon X \to L^2([0,T],U)$ such that $-\mathrm{e}^{-AT} = \mathcal{B}^T \mathcal{F}$. Hence $\mathcal{F} w_0$ provides a null-control function in time T. Moreover, according to (7) the operator $\mathcal{F}$ can be chosen with minimal norm.

Remark 2.5. The operator $\mathcal{F}$ can even be chosen to be pointwise minimal. Let $w_0 \in X$, $T > 0$, and u be a null-control function in time T. Then the set of all null-control functions in time T is a closed affine space of the form

$$u + \operatorname{Ker} \mathcal{B}^T.$$

Let now P denote the orthogonal projection onto $\operatorname{Ker} \mathcal{B}^T$. Then we have $-\mathrm{e}^{-AT} = \mathcal{B}^T (I - P)\mathcal{F}$ and the operator $(I-P)\mathcal{F}$ does not depend on the choice of $\mathcal{F}$. Moreover, it is easy to see that for every $w_0 \in X$, the function $(I-P)\mathcal{F} w_0 \in L^2([0,T],U)$ is the unique control with minimal norm associated to the initial datum w_0. This implies in particular the second equality in (8).

Definition 2.6. Assume that the system (3) is null-controllable. We define the *control cost in time T* as

$$C_T = \sup_{\|w_0\|=1} \min\{\|u\|_{L^2([0,T],U)}\colon \mathrm{e}^{-TA} w_0 + \mathcal{B}^T u = 0\} = \min\{C_{\mathrm{obs}}\colon C_{\mathrm{obs}} \text{ satisfies (6)}\}. \tag{8}$$

Our first result concerns an observability inequality and hence null-controllability for an abstract parabolic system of the form (5). In the theorem, we assume a so-called spectral inequality, given in Ineq. (9).

Theorem 2.7. *Let $A \geq 0$ and assume that there are $d_0 > 0$, $d_1 \geq 0$ and $\gamma \in (0,1)$ such that for all $\lambda > 0$ and all $\phi \in X$ we have*

$$\|P_A(\lambda)\phi\|^2 \leq d_0 \mathrm{e}^{d_1 \lambda^\gamma} \|B^* P_A(\lambda)\phi\|_U^2. \tag{9}$$

Then for all $T > 0$ and all $\phi \in X$ we have the observability estimate

$$\left\|\mathrm{e}^{-AT}\phi\right\|^2 \leq C_{\mathrm{obs}}^2 \int_0^T \left\|B^* \mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t, \tag{10}$$

where C_{obs} satisfies

$$C_{\mathrm{obs}}^2 = \frac{C_1 d_0}{T} K_1^{C_2} \exp\left(C_3 \left(\frac{d_1 + (-\beta)^{C_4}}{T^\gamma}\right)^{\frac{1}{1-\gamma}}\right) \quad \textit{with} \quad K_1 = 2 d_0 \mathrm{e}^{-\beta} \|B\|_{\mathcal{L}(U, X_\beta)}^2 + 1.$$

Here, $C_i > 0$, $i \in \{1,2,3,4\}$, are constants depending only on γ. They are explicitly given by Eq. (24). Moreover, for all $T > 0$ the system (3) is null-controllable in time T with cost satisfying $C_T \leq C_{\mathrm{obs}}$.

Note that the right hand side in (9) is well defined since $\operatorname{Ran} P_A(\lambda) \subset X_{-\beta}$ for all $\lambda \in \mathbb{R}$. The mere statement that spectral inequalities imply observability estimates is not new. The novel aspects of Theorem 2.7 are discussed in the following remark.

Remark 2.8. There exists a huge amount of earlier approaches which transfer spectral inequalities to observability inequalities, see e.g. [LR95, Mil10, TT11, LL12, BPS18]. Some of them are formulated in a more general setting and unlike our result do not require self-adjointness of A. However, the estimates on C_{obs}^2 therein are, with respect to the dependence on d_0 and d_1, not sufficient for our purpose in Section 3. Let us explain this in more detail, and assume within this discussion that $\beta = 0$ and $\gamma = 1/2$. Our upper bound

$$C_{\mathrm{obs}}^2 = \frac{C_1 d_0}{T} K_1^{C_2} \exp\left(\frac{C_3 d_1^2}{T}\right)$$

from Theorem 2.7 features the following properties:

(i) The exponent tends to zero if $d_1 \to 0$.

(ii) The pre-factor $C_1 d_0 K_1^{C_2}/T$ does not depend on d_1 and is proportional to T^{-1}.

(iii) The estimate holds in a d_1-independent time interval (in our case $(0,\infty)$).

All three properties are paramount for the applications to homogenization and de-homogenization in Section 3. Let us stress that none of the earlier bounds we are aware of carry the features (i)–(iii) at the same time. For example, the papers [Mil10, BPS18] provide a bound of the form

$$C_{\mathrm{obs}}^2 \leq C_1 \exp\left(\frac{C_2}{T}\right), \tag{11}$$

where the dependence of the positive constants C_1 and C_2 on d_0 and d_1 can be inferred from their proof. Note that the bound (11) is missing pre-factor $1/T$. Thus, C_2 in (11) cannot be proportional to d_1^2 since for $d_1 = 0$ (full control) this contradicts the universal lower bound of order $1/T$, cf. Theorem 2.13.

In order to obtain our bound in Theorem 2.7, we improve techniques developed in [TT11]. Note that the bound given in [TT11, Theorem 1.2] already satisfies properties (i) and (iii), and carries the overall pre-factor $1/T$. However, it does not ensure that the influence of d_1 is confined only to the exponential term. Intricate parameter choices and estimates – spelled out in Lemma 2.9 – were necessary in order to achieve an estimate of the required form. Moreover, in contrast to [TT11], we do not require that the operator A has discrete spectrum, thus extending the applicability e.g. to Schrödinger operators on unbounded subsets of $\mathbb{R}^d$.

Proof of Theorem 2.7. Let $T > 0$. For $\phi \in X$, $t \in (0, T]$, and $\lambda > 0$ we use the notation

$$\begin{aligned} F(t) &= \left\|\mathrm{e}^{-At}\phi\right\|^2, & F_\lambda(t) &= \left\|\mathrm{e}^{-At}P_A(\lambda)\phi\right\|^2, & F_\lambda^\perp(t) &= \left\|\mathrm{e}^{-At}(I - P_A(\lambda))\phi\right\|^2,\\ G(t) &= \left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2, & G_\lambda(t) &= \left\|B^*\mathrm{e}^{-At}P_A(\lambda)\phi\right\|_U^2, & G_\lambda^\perp(t) &= \left\|B^*\mathrm{e}^{-At}(I - P_A(\lambda))\phi\right\|_U^2. \end{aligned}$$

Since $A \geq 0$ we have $F(t_1) \geq F(t_2)$, $F_\lambda(t_1) \geq F_\lambda(t_2)$, and $F_\lambda^\perp(t_1) \geq F_\lambda^\perp(t_2)$ if $t_1 \leq t_2$ and $\lambda > 0$. By monotonicity and our assumption (9), we obtain for all $t \in (0, T]$ and all $\lambda > 0$

$$F_\lambda(t) = \frac{2}{t}\int_{t/2}^t F_\lambda(t)\mathrm{d}\tau \leq \frac{2}{t}\int_{t/2}^t F_\lambda(\tau)\mathrm{d}\tau \leq \frac{2d_0\mathrm{e}^{d_1\lambda^\gamma}}{t}\int_{t/2}^t G_\lambda(\tau)\mathrm{d}\tau. \tag{12}$$

By spectral calculus we have

$$\begin{aligned} G_\lambda^\perp(t) &\leq \|B\|^2_{\mathcal{L}(U,X_\beta)}\|\mathrm{e}^{-At}(I - P_A(\lambda))\phi\|^2_{X_{-\beta}}\\ &= \|B\|^2_{\mathcal{L}(U,X_\beta)}\|(I + A^2)^{-\beta/2}\mathrm{e}^{-At}(I - P_A(\lambda))\phi\|^2\\ &= \|B\|^2_{\mathcal{L}(U,X_\beta)}\int_\lambda^\infty (1 + \mu^2)^{-\beta}\mathrm{e}^{-2\mu t}\mathrm{d}\|P_A(\mu)\phi\|^2. \end{aligned} \tag{13}$$

Note that this justifies that $\mathrm{e}^{-At}(I - P_A(\lambda))\phi$ is indeed in $X_{-\beta}$. Recall that $\beta < 0$. Let $\Theta > 0$ to be specified later. For $\mu, t > 0$ we estimate

$$(1 + \mu^2)^{-\beta}\mathrm{e}^{-\mu t} \leq \left(1 + \left(-\frac{2\beta}{t}\right)^2\right)^{-\beta} \leq \exp\left(\frac{C_\Theta}{t^\Theta} - \beta\right), \quad C_\Theta = 2^\Theta(-\beta)^{\Theta+1}\left(\frac{2+\Theta}{\Theta}\right),$$

where the first inequality follows by maximizing with respect to μ, and the second one follows from the inequality $\ln(1 + x) \leq (2/\Theta + 1)x^{\Theta/2} + 1$ for $x \geq 0$. Hence,

$$G_\lambda^\perp(t) \leq \|B\|^2_{\mathcal{L}(U,X_\beta)}\int_\lambda^\infty \mathrm{e}^{C_\Theta/t^\Theta - \beta - \mu t}\mathrm{d}\|P_A(\mu)\phi\|^2 \leq \|B\|^2_{\mathcal{L}(U,X_\beta)}\mathrm{e}^{C_\Theta/t^\Theta - \beta - \lambda t/2}F(t/2). \tag{14}$$

Similarly we find

$$F_\lambda^\perp(t) = \int_\lambda^\infty \mathrm{e}^{-2\mu t}\mathrm{d}\|P_A(\mu)\phi\|^2 \leq \mathrm{e}^{-3\lambda t/2}\int_\lambda^\infty \mathrm{e}^{-\mu t/2}\mathrm{d}\|P_A(\mu)\phi\|^2 \leq \mathrm{e}^{-3\lambda t/2}F(t/4).$$

>From the last inequality and Ineq. (12) we obtain

$$F(t) = F_\lambda(t) + F_\lambda^\perp(t) \leq \frac{2d_0 \mathrm{e}^{d_1\lambda^\gamma}}{t} \int_{t/2}^t G_\lambda(\tau)\mathrm{d}\tau + \mathrm{e}^{-3\lambda t/2} F(t/4).$$

Since $G_\lambda(t) \leq 2(G_\lambda^\perp(t) + G(t))$ and by Ineq. (14) we obtain for all $t \in (0,T]$ and all $\lambda > 0$

$$\begin{aligned} F(t) &\leq \frac{4d_0 \mathrm{e}^{d_1\lambda^\gamma}}{t} \int_{t/2}^t (G_\lambda^\perp(\tau) + G(\tau))\mathrm{d}\tau + \mathrm{e}^{-3\lambda t/2} F(t/4) \\ &\leq \frac{4d_0 \mathrm{e}^{d_1\lambda^\gamma}}{t} \int_{t/2}^t G(\tau)\mathrm{d}\tau + \frac{4d_0 \mathrm{e}^{-\beta}\mathrm{e}^{d_1\lambda^\gamma} \|B\|^2_{\mathcal{L}(U,X_\beta)}}{t} \int_{t/2}^t \frac{F(\tau/2)}{\mathrm{e}^{\lambda\tau/2 - C_\Theta/t^\Theta}}\mathrm{d}\tau + \frac{F(t/4)}{\mathrm{e}^{3\lambda t/2}}. \end{aligned}$$

Since $F(\tau/2) \leq F(t/4)$, $\mathrm{e}^{-\lambda\tau/2} \leq \mathrm{e}^{-\lambda t/4}$, and $\mathrm{e}^{C_\Theta/\tau^\Theta} \leq \mathrm{e}^{2^\Theta C_\Theta/t^\Theta}$ for $\tau \geq t/2$, we obtain

$$\begin{aligned} F(t) &\leq \frac{4d_0 \mathrm{e}^{d_1\lambda^\gamma}}{t} \int_{t/2}^t G(\tau)\mathrm{d}\tau + \mathrm{e}^{-\lambda t/4 + 2^\Theta C_\Theta/t^\Theta} \left(2d_0 \mathrm{e}^{-\beta}\mathrm{e}^{d_1\lambda^\gamma} \|B\|^2_{\mathcal{L}(U,X_\beta)} + 1\right) F(t/4) \\ &\leq \frac{4d_0 \mathrm{e}^{d_1\lambda^\gamma}}{t} \int_{t/2}^t G(\tau)\mathrm{d}\tau + \mathrm{e}^{-\lambda t/4 + 2^\Theta C_\Theta/t^\Theta + d_1\lambda^\gamma} \left(2d_0 \mathrm{e}^{-\beta} \|B\|^2_{\mathcal{L}(U,X_\beta)} + 1\right) F(t/4). \end{aligned}$$

With the notation

$$\begin{aligned} D_1(t,\lambda) &= \frac{4d_0 \mathrm{e}^{d_1\lambda^\gamma}}{t} \int_{t/2}^t G(\tau)\mathrm{d}\tau, \quad \text{and} \\ D_2(t,\lambda) &= \mathrm{e}^{-\lambda t/4 + 2^\Theta C_\Theta/t^\Theta + d_1\lambda^\gamma} \left(2d_0 \mathrm{e}^{-\beta} \|B\|^2_{\mathcal{L}(U,X_\beta)} + 1\right) \end{aligned}$$

we can summarize that for all $t \in (0,T]$ we have

$$F(t) \leq D_1(t,\lambda) + D_2(t,\lambda) F(t/4). \tag{15}$$

This inequality can be iterated. For $k \in \mathbb{N}_0$ let $\lambda_k = \nu\alpha^k$ with $\nu > 0$ and $\alpha > 1$ to be specified later. In particular, applying Ineq. (15) with $t = T$ and $\lambda = \lambda_0$ at the first place, the term $F(4^{-1}T)$ on the right hand side can then be estimated by Ineq. (15) with $t = 4^{-1}T$ and $\lambda = \lambda_1$. This way, we obtain after two steps

$$\begin{aligned} F(T) &\leq D_1(T,\lambda_0) + D_2(T,\lambda_0)\left(D_1(4^{-1}T,\lambda_1) + D_2(4^{-1}T,\lambda_1)F(4^{-2}T)\right) \\ &= D_1(T,\lambda_0) + D_1(4^{-1}T,\lambda_1)D_2(T,\lambda_0) + D_2(T,\lambda_0)D_2(4^{-1}T,\lambda_1)F(4^{-2}T). \end{aligned}$$

After $N+1$ steps of this type we obtain

$$F(T) \leq D_1(T,\lambda_0) + \sum_{k=1}^N D_1(4^{-k}T,\lambda_k) \prod_{l=0}^{k-1} D_2(4^{-l}T,\lambda_l) + F(4^{-N-1}T) \prod_{k=0}^N D_2(4^{-k}T,\lambda_k). \tag{16}$$

In order to study the limit $N \to \infty$, we assume that $4^{\Theta+1} \leq \alpha$, $\alpha^\gamma \leq \alpha/4$, and $\nu T > 2^{\Theta+2} C_\Theta T^{-\Theta} + d_1\nu^\gamma\alpha$. This ensures that the constants

$$K_1 = 2d_0\mathrm{e}^{-\beta}\|B\|^2_{\mathcal{L}(U,X_\beta)} + 1, \quad K_2 = \nu T/4 - 2^\Theta C_\Theta/T^\Theta - d_1\nu^\gamma, \quad K_3 = \frac{K_2}{\alpha/4 - 1} - d_1\nu^\gamma \tag{17}$$

are positive. Then we have that

$$\prod_{k=0}^{N} D_2(4^{-k}T,\lambda_k) = K_1^{N+1}\prod_{k=0}^{N} \mathrm{e}^{-\nu(\alpha/4)^k T/4 + 2^{\Theta}C_\Theta 4^{\Theta k}/T^{\Theta} + d_1\nu^{\gamma}(\alpha^{\gamma})^k}$$
$$\leq K_1^{N+1}\prod_{k=0}^{N} \mathrm{e}^{(\alpha/4)^k(-\nu T/4 + 2^{\Theta}C_\Theta/T^{\Theta} + d_1\nu^{\gamma})} = K_1^{N+1}\prod_{k=0}^{N}\mathrm{e}^{-K_2(\alpha/4)^k} \tag{18}$$

Since $K_1, K_2 > 0$ and $\alpha > 4$ this tends to zero as N tends to infinity. >From Ineq. (18) and the definitions of $D_1(4^{-k}T,\lambda_k)$ and K_3, we infer that the middle term of the right hand side of Ineq. (16) obeys the upper bound

$$\sum_{k=1}^{N} D_1(4^{-k}T,\lambda_k)\prod_{l=0}^{k-1} D_2(4^{-l}T,\lambda_l)$$
$$\leq \int_0^T G(\tau)\mathrm{d}\tau \sum_{k=1}^{N}\frac{4^{k+1}d_0\exp(d_1\nu^{\gamma}(\alpha/4)^k)}{T}K_1^k\exp\left(-K_2\frac{(\alpha/4)^k-1}{\alpha/4-1}\right)$$
$$= \int_0^T G(\tau)\mathrm{d}\tau\frac{4d_0}{T}\exp\left(\frac{K_2}{\alpha/4-1}\right)\sum_{k=1}^{N}(4K_1)^k\exp\left(-K_3(\alpha/4)^k\right). \tag{19}$$

Letting N tend to infinity we obtain from Ineqs. (16), (18) and (19) that

$$\left\|\mathrm{e}^{-AT}\phi\right\|^2 \leq \tilde{C}_{\mathrm{obs}}^2\int_0^T\left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2\mathrm{d}t,$$

where

$$\tilde{C}_{\mathrm{obs}}^2 = \frac{4d_0\mathrm{e}^{d_1\nu^{\gamma}}}{T} + \frac{4d_0}{T}\exp\left(\frac{K_2}{\alpha/4-1}\right)\sum_{k=1}^{\infty}(4K_1)^k\exp\left(-K_3(\alpha/4)^k\right). \tag{20}$$

We choose Θ, α and ν as in (21) and conclude the observability inequality (10) from Lemma 2.9.

Since (10) corresponds to part (b) of Lemma 2.4 with $\mathcal{X} = \mathrm{e}^{-AT}\colon X \to X$ and $\mathcal{Y} = \mathcal{B}^T\colon L^2([0,T],U) \to X$. the system is null-controllable in time T. By the definition of C_T we have $C_T \leq C_{\mathrm{obs}}$. □

Lemma 2.9. *Let $d_0 > 0$, $d_1 \geq 0$, $\gamma \in (0,1)$, $T > 0$,*

$$\Theta = \frac{\gamma^2}{1-\gamma}, \quad \alpha = 8\cdot 4^{\frac{1}{1-\gamma}}, \quad \text{and} \quad \nu = \left(\frac{\alpha d_1}{T} + \frac{D}{T^{1-\gamma}} + \frac{E}{T}\right)^{\frac{1}{1-\gamma}}, \tag{21}$$

where

$$D = (3\alpha\ln(4K_1))^{1-\gamma}, \quad E = \left(\frac{8\cdot 2^{\Theta}C_\Theta}{D}\right)^{\frac{1-\gamma}{\gamma}}, \quad C_\Theta = 2^{\Theta}(-\beta)^{\Theta+1}\left(\frac{2+\Theta}{\Theta}\right),$$

and $K_1 = 2d_0\mathrm{e}^{-\beta}\|B\|_{\mathcal{L}(U,X_\beta)}^2 + 1$. Then we have $4^{\Theta+1} \leq \alpha$, $\alpha^{\gamma} \leq \alpha/4$, and $\nu T > 2^{\Theta+2}C_\Theta T^{-\Theta} + d_1\nu^{\gamma}\alpha$. Moreover, for all $T > 0$ the constant $\tilde{C}_{\mathrm{obs}}^2$ from (20) satisfies

$$\tilde{C}_{\mathrm{obs}}^2 \leq \frac{C_1 d_0}{T}K_1^{C_2}\exp\left(C_3\left(\frac{d_1 + (-\beta)^{C_4}}{T^{\gamma}}\right)^{\frac{1}{1-\gamma}}\right).$$

Here, $C_i > 0$, $i \in \{1,2,3,4\}$, are constants depending only on γ. They are explicitly given by Eq. (24).

Proof. It is easy to see that $4^{\Theta+1} \le \alpha$, and $\alpha^\gamma \le \alpha/4$. For the constant K_3 from (17) we have

$$\begin{aligned} K_3 &= \frac{\nu T/4 - 2^\Theta C_\Theta / T^\Theta - d_1 \nu^\gamma \alpha/4}{(\alpha/4 - 1)} \\ &= \frac{\nu^\gamma}{\alpha - 4}\left[\left(\frac{\alpha d_1}{T} + \frac{D}{T^{1-\gamma}} + \frac{E}{T}\right) T - \frac{4\cdot 2^\Theta C_\Theta}{T^\Theta}\left(\frac{\alpha d_1}{T} + \frac{D}{T^{1-\gamma}} + \frac{E}{T}\right)^{-\frac{\gamma}{1-\gamma}} - d_1\alpha\right] \\ &\ge \frac{\nu^\gamma}{\alpha-4}\left[DT^\gamma - \frac{4\cdot 2^\Theta C_\Theta}{T^\Theta}\frac{E^{\frac{-\gamma}{1-\gamma}}}{T^{\frac{-\gamma}{1-\gamma}}}\right] = \frac{\nu^\gamma D T^\gamma}{2(\alpha-4)}. \end{aligned}$$

This shows in particular that $\nu T > 2^{\Theta+2} C_\Theta T^{-\Theta} + d_1 \nu^\gamma \alpha$. We further estimate

$$K_3 \ge \frac{\left(\frac{D}{T^{1-\gamma}}\right)^{\gamma/(1-\gamma)} D T^\gamma}{2(\alpha-4)} = \frac{D^{1/(1-\gamma)}}{2(\alpha-4)}.$$

For the constant K_2 from (17) we estimate using $\alpha \ge 8$

$$\frac{K_2}{\alpha/4 - 1} \le \nu T/2 = \frac{T}{2}\left(\frac{\alpha d_1}{T} + \frac{D}{T^{1-\gamma}} + \frac{E}{T}\right)^{\frac{1}{1-\gamma}} \le \frac{\alpha^{\frac{1}{1-\gamma}}}{2}\left(\frac{\alpha d_1 + E}{T^\gamma} + D\right)^{\frac{1}{1-\gamma}}.$$

Let us now note that for all $A > 1$, and $B > 0$ we have

$$\sum_{k=1}^\infty A^k \mathrm{e}^{-B2^k} \le \left(\frac{2\ln A}{B\mathrm{e}\ln 2}\right)^{\frac{\ln A}{\ln 2}} \frac{1}{B}, \tag{22}$$

since

$$\sum_{k=1}^\infty \mathrm{e}^{-\frac{B}{2}2^k} \le \sum_{k=1}^\infty \mathrm{e}^{-kB} = \frac{\mathrm{e}^{-B}}{1-\mathrm{e}^{-B}} = \frac{1}{\mathrm{e}^B - 1} \le \frac{1}{B}$$

and

$$\sum_{k=1}^\infty A^k \mathrm{e}^{-B2^k} \le \sup_{x\ge 1}(A^x \mathrm{e}^{-\frac{B}{2}2^x}) \sum_{k=1}^\infty \mathrm{e}^{-\frac{B}{2}2^k} = \left(\frac{2\ln A}{B\mathrm{e}\ln 2}\right)^{\frac{\ln A}{\ln 2}} \sum_{k=1}^\infty \mathrm{e}^{-\frac{B}{2}2^k}.$$

We use $\alpha \ge 8$ and apply Ineq. (22) with $A = 4K_1 > 1$, and $B = K_3$ to obtain

$$\sum_{k=1}^\infty (4K_1)^k \exp\left(-K_3(\alpha/4)^k\right) \le \sum_{k=1}^\infty (4K_1)^k \exp\left(-K_3 2^k\right) \le \left(\frac{2\ln(4K_1)}{K_3\mathrm{e}\ln 2}\right)^{\frac{\ln(4K_1)}{\ln 2}} \frac{1}{K_3}. \tag{23}$$

By the above estimate on K_3 and since $\alpha > 4$ we find

$$\frac{2\ln(4K_1)}{K_3 \mathrm{e}\ln 2} \le \frac{2}{\mathrm{e}\ln 2}\frac{\ln(4K_1)2(\alpha-4)}{D^{1/(1-\gamma)}} = \frac{2}{\mathrm{e}\ln 2}\frac{2(\alpha-4)}{3\alpha} \le 1.$$

Note that the exponent $\ln(4K_1)/\ln 2$ in (23) is positive, and that $D \geq 1$. Hence, the right hand side of (23) is bounded from above by $2(\alpha - 4)$. Using this, $\alpha \geq 8$, and $d_1\nu^\gamma \leq d_1\nu^\gamma + K_3 = K_2/(\alpha/4-1)$, we find

$$\begin{aligned}\tilde{C}_{\mathrm{obs}}^2 &= \frac{4d_0 \mathrm{e}^{d_1\nu^\gamma}}{T} + \frac{4d_0}{T}\exp\left(\frac{K_2}{\alpha/4-1}\right)\sum_{k=1}^{\infty}(4K_1)^k \exp\left(-K_3(\alpha/4)^k\right) \\ &\leq \frac{4d_0}{T}(1+K_3^{-1})\exp\left(\frac{K_2}{\alpha/4-1}\right) \\ &\leq \frac{4d_0}{T}\left(1+2(\alpha-4)\right)\exp\left(\frac{\alpha^{\frac{1}{1-\gamma}}}{2}\left(\frac{\alpha d_1 + E}{T^\gamma}+D\right)^{\frac{1}{1-\gamma}}\right),\end{aligned}$$

Since $(a+b)^x \leq 2^{x-1}(a^x+b^x)$ for $x>1$ and $a,b\geq 0$ we obtain

$$\begin{aligned}\tilde{C}_{\mathrm{obs}}^2 \leq \frac{4d_0}{T}\left(1+2(\alpha-4)\right)(4K_1)^{3\alpha^{\frac{2-\gamma}{1-\gamma}}2^{2\Theta+3}} \\ \times \exp\left(\alpha^{\frac{2}{1-\gamma}}4^{\frac{\gamma+\Theta+2}{1-\gamma}}\left(\frac{\Theta+2}{\Theta}\right)^{\frac{1}{1-\gamma}}\left(\frac{d_1+(-\beta)^{\Theta+1}}{T^\gamma}\right)^{\frac{1}{1-\gamma}}\right). \quad (24)\end{aligned}$$

□

Remark 2.10. The arguments used in the proof of Theorem 2.7 can be extended to the case where $B\colon [0,T] \to \mathcal{L}(U, X_\beta)$ is time-dependent with only minimal modifications. For the basic results about the integration theory on Hilbert spaces used here we refer to [HP57] and [DU77]. Let $\mathcal{H}_i$ for $i \in \{1,2,3\}$ be separable Hilbert spaces and $I \subset \mathbb{R}$ an interval. Then if $I \ni t \mapsto B(t) \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ and $I \ni t \mapsto A(t) \in \mathcal{L}(\mathcal{H}_2,\mathcal{H}_3)$ are measurable then the product $I \ni t \mapsto A(t)B(t) \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_3)$ is measurable as well. If additionally $I \ni t \mapsto x(t) \in \mathcal{H}_1$ is measurable, then $I \ni t \mapsto B(t)x(t) \in \mathcal{H}_2$ is measurable as well. Consequently the map $I \ni t \mapsto \|B(t)x\|^2 = \langle x, B(t)^*B(t)x\rangle \in \mathbb{R}$ is measurable too.

In what follows let us assume that the Hilbert spaces U and X_β are separable, and that $B : [0,T] \to \mathcal{L}(U, X_\beta)$ is measurable and uniformly bounded, meaning

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|B(t)\|_{\mathcal{L}(U,X_\beta)} < \infty. \quad (25)$$

Note that for any dense countable $U' \subset U \setminus \{0\}$ we have

$$\|B(t)\|_{\mathcal{L}(U,X_\beta)} = \sup_{u\in U'}\|B(t)u\|_{X_\beta}/\|u\|_U,$$

so that $t \to \|B(t)\|_{\mathcal{L}(U,X_\beta)}$ is measurable and the essential supremum (w.r.t. Lebesgue measure on $[0,T]$) makes sense. Note that for every $t \in (0,T]$, the map $[0,t] \ni s \mapsto \mathrm{e}^{-(t-s)A}$ is strongly continuous, hence measurable.

An argument analogous to the discussion at the beginning of Section 2 shows that we even have $\mathrm{e}^{-(T-s)A}B(s)u(s) \in X$ for almost all $s \in [0,T]$. In particular,

$$\mathcal{B}^T f(t) = \int_0^t \mathrm{e}^{-A(t-s)}B(s)u(s)\mathrm{d}s \in X$$

is well-defined for every $u \in L^2([0,T],U)$. For each initial state $w_0 \in X$ and every $u \in L^2([0,T],U)$, the evolution

$$w(t) = \mathrm{e}^{-At} w_0 + \mathcal{B}^T u(t), \quad t \in [0,T],$$

solves the equation

$$\dot{w}(t) + Aw(t) = B(t)u(t), \quad w(0) = w_0 \in X.$$

The following theorem is the natural generalization of Theorem 2.7 to time-dependent B:

Theorem 2.11. *Let $\beta \leq 0$, $d_0 > 0$, $d_1 \geq 0$, $T > 0$, and $\gamma \in (0,1)$. Let U, X_β be separable, $A \geq 0$, and let $B\colon [0,T] \to \mathcal{L}(U, X_\beta)$ be measurable and satisfy (25). Assume that*

$$\|P_A(\lambda)\phi\|^2 \leq d_0 \mathrm{e}^{d_1 \lambda^\gamma} \|B(t)^* P_A(\lambda)\phi\|_U^2 \quad \textit{for almost all } t \in [0,T].$$

Then for all $\phi \in X$ we have the observability estimate

$$\left\|\mathrm{e}^{-AT}\phi\right\|^2 \leq C_{\mathrm{obs}}^2 \int_0^T \left\|B^* \mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t,$$

where C_{obs} satisfies

$$C_{\mathrm{obs}}^2 = \frac{C_1 d_0}{T} K_1^{C_2} \exp\left(C_3 \left(\frac{d_1 + (-\beta)^{C_4}}{T^\gamma}\right)^{\frac{1}{1-\gamma}}\right) \quad \textit{with}$$
$$K_1 = 1 + \operatorname{ess\,sup}_{t\in[0,T]} \left(2d_0 \mathrm{e}^{-\beta} \|B(t)\|_{\mathcal{L}(U,X_\beta)}^2\right).$$

Here, $C_i > 0$, $i \in \{1,2,3,4\}$, are constants depending only on γ. They are explicitly given by Eq. (24). Moreover, the system (2.10) is null-controllable in time T with cost satisfying $C_T \leq C_{\mathrm{obs}}$.

Proof. The observability estimate is proved by following verbatim the proof of Theorem 2.7. To prove null-controllability and the control cost bound one uses part (b) of Lemma 2.4 with $\mathcal{X} = \mathrm{e}^{-AT}\colon X \to X$ and $\mathcal{Y}\colon L^2([0,T],U) \to X$, $\mathcal{Y}u = \int_0^T \mathrm{e}^{-(T-s)A} B(s)u(s)\mathrm{d}s$. Note that a priori $B(s)u(s) \in X_\beta$. However, similarly to the discussion at the beginning of Section 2, we see that $\mathrm{e}^{-(T-s)A}B(s)u(s) \in X$ for almost all $s \in [0,T]$. This shows that $\mathcal{Y}$ indeed maps into X and not into X_β. Boundedness of $\mathcal{Y}$ follows, arguing as in (13), from

$$\begin{aligned}\|\mathcal{Y}u\|^2 &= \int_0^T \|\mathrm{e}^{-(T-s)A}B(s)u(s)\|_{X_\beta}^2 \mathrm{d}s \leq \int_0^T \|B(s)u(s)\|^2 \mathrm{d}s \\ &\leq \int_0^T \|B(s)\|_{\mathcal{L}(U,X_\beta)}^2 \|u(s)\|_U^2 \mathrm{d}s \leq \operatorname*{ess\,sup}_{t\in[0,T]} \|B(t)\|_{\mathcal{L}(U,X_\beta)}^2 \|u\|_{L^2([0,T],U)}^2 < \infty. \qquad \square\end{aligned}$$

So far, we have only treated the case of non-negative A. The next theorem is an equivalent formulation of Theorem 2.7 and also treats the situation where A is not assumed to be non-negative any more but merely lower semibounded. Recall that $\min \sigma(A) = \kappa$.

Theorem 2.12. *Assume that there are $d_0 > 0$, $d_1 \geq 0$ and $\gamma \in (0,1)$ such that for all $\lambda > \kappa$ and all $\phi \in X$ we have*

$$\|P_A(\lambda)\phi\|^2 \leq d_0 \mathrm{e}^{d_1(\lambda-\kappa)^\gamma} \|B^* P_A(\lambda)\phi\|_U^2.$$

Then for all $T > 0$ and all $\phi \in X$ we have the observability estimate

$$\left\|\mathrm{e}^{-AT}\phi\right\|^2 \leq C_{\mathrm{obs}}^2 \int_0^T \mathrm{e}^{-2\kappa(T-t)} \left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t, \tag{26}$$

where C_{obs} is as in Theorem 2.7. Moreover, for all $T > 0$, the system (3) is controllable in time T. Let $K_1 = 2d_0\mathrm{e}^{-\beta}\|B\|^2_{\mathcal{L}(U,X_\beta)} + 1$.

(a) If $\kappa < 0$, then the cost satisfies

$$C_T^2 \leq \inf_{t\in(0,T]} \frac{C_1 d_0}{t} K_1^{C_2} \exp\left(C_3 \left(\frac{d_1 + (-\beta)^{C_4}}{t^\gamma}\right)^{\frac{1}{1-\gamma}} - 2\kappa t\right).$$

(b) If $\kappa = 0$, then the cost satisfies

$$C_T^2 \leq \frac{C_1 d_0}{T} K_1^{C_2} \exp\left(C_3 \left(\frac{d_1 + (-\beta)^{C_4}}{T^\gamma}\right)^{\frac{1}{1-\gamma}}\right).$$

(c) If $\kappa > 0$, then the cost satisfies

$$C_T^2 \leq \inf_{t\in[0,T)} \frac{C_1 d_0}{T-t} K_1^{C_2} \exp\left(C_3 \left(\frac{d_1 + (-\beta)^{C_4}}{(T-t)^\gamma}\right)^{\frac{1}{1-\gamma}} - 2\kappa t\right).$$

Proof of Theorem 2.12. Since $P_A(\lambda) = P_{A-\kappa}(\lambda - \kappa)$, we have by assumption for all $\lambda \geq 0$ that

$$\|P_{A-\kappa}(\lambda)\phi\|^2 \leq d_0 \mathrm{e}^{d_1 \mu^\gamma} \|B^* P_{A-\kappa}(\lambda)\phi\|_U^2.$$

Since the operator $A - \kappa$ is non-negative, we obtain from Theorem 2.7 the observability estimate

$$\left\|\mathrm{e}^{-(A-\kappa)T}\phi\right\|^2 \leq C_{\mathrm{obs}}^2 \int_0^T \left\|B^*\mathrm{e}^{-(A-\kappa)t}\phi\right\|_U^2 \mathrm{d}t = C_{\mathrm{obs}}^2 \int_0^T \mathrm{e}^{2\kappa t}\left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t,$$

where C_{obs} is as in Theorem 2.7. Dividing by $\mathrm{e}^{2\kappa T}$ yields (26).

If $\kappa < 0$, we have for all $t > 0$

$$\left\|\mathrm{e}^{-At}\phi\right\|^2 \leq C_{\mathrm{obs}}^2 \mathrm{e}^{-2\kappa t} \int_0^t \left\|B^*\mathrm{e}^{-At}\phi\right\|_U^2 \mathrm{d}t.$$

Using the equivalence between observability and null-controllability as in the proof of Theorem 2.7, we conclude that the system (3) is null-controllable in time t for all $t > 0$ with cost satisfying

$$C_t^2 \leq \frac{C_1 d_0}{t} K_1^{C_2} \exp\left(C_3 \left(\frac{d_1 + (-\beta)^{C_4}}{t^\gamma}\right)^{\frac{1}{1-\gamma}} - 2\kappa t\right).$$

Note that this expression grows exponentially as t tends to infinity. However, if the system is null-controllable in time t with cost C_t, then it is also null-controllable in time $T > t$ with the same cost $C_T = C_t$. For any $t \in (0, T]$, we can choose a null-control function in time t, and apply no control in $(t, T]$. This yields the upper bound in this case.

The case $\kappa = 0$ is the statement of Theorem 2.7. If $\kappa > 0$, then we choose any $t \in (0, T)$, apply no control in $[0, t]$ and find

$$\|w(t)\| \leq \mathrm{e}^{-\kappa t}\|w_0\|.$$

Then, we apply Theorem 2.7 with initial state $w(t)$ in the time interval $[t, T]$. □

In order to investigate the sharpness of the estimates obtained above, we compare them to lower bounds. While these lower bounds are not too difficult to obtain, we provide a proof as a convenience for the reader.

Theorem 2.13. *Let $T > 0$ and assume that the system (3) is null-controllable in time T. Then*

$$C_T^2 \geq \|B\|_{\mathcal{L}(U,X_\beta)}^{-2}(1+\kappa^2)^\beta \cdot \begin{cases} \frac{1}{T} & \text{if } \kappa = 0, \\ \frac{2\kappa}{\exp(2\kappa T)-1} & \text{if } \kappa \neq 0. \end{cases}$$

Remark 2.14. If B is (a multiple of) the identity, one immediately sees from the proof that the bound in Theorem 2.13 becomes an equality. This means that Theorem 2.13 is sharp as a universal lower bound.

Corollary 2.15. *In the situation of Theorem 2.13 we have*

$$C_T^2 \geq \|B\|_{\mathcal{L}(U,X_\beta)}^{-2}(1+\kappa^2)^\beta \cdot \begin{cases} \left(\frac{1}{2T} - \kappa\right) & \text{if } \kappa < 0, \\ \frac{1}{T} & \text{if } \kappa = 0, \\ \frac{1}{T}\exp(-2\kappa T) & \text{if } \kappa > 0. \end{cases}$$

Furthermore,

$$\inf_{T>0} C_T^2 \geq \|B\|_{\mathcal{L}(U,X_\beta)}^{-2}(1+\kappa^2)^\beta \cdot \begin{cases} -2\kappa & \text{if } \kappa < 0, \\ 0 & \text{if } \kappa \geq 0. \end{cases}$$

Proof of Theorem 2.13. Since the system (3) is null-controllable in time T, an observability inequality holds and thus we have

$$\forall \phi \in X \setminus \{0\}\colon \quad \int_0^T \|B^* \mathrm{e}^{-At}\phi\|_U^2 \mathrm{d}t \neq 0.$$

Hence, by definition we have

$$C_T^2 = \sup_{\phi \in X\setminus\{0\}} \frac{\|\mathrm{e}^{-AT}\phi\|^2}{\int_0^T \|B^*\mathrm{e}^{-At}\phi\|_U^2 \mathrm{d}t}. \tag{27}$$

Let $\varepsilon > 0$ and $0 \neq \phi_0 \in P_A(\kappa+\varepsilon)$, where $\kappa = \min \sigma(A)$. By spectral calculus we find

$$\|\mathrm{e}^{-AT}\phi_0\|^2 = \int_\kappa^{\kappa+\varepsilon} \mathrm{e}^{-2\lambda T}\mathrm{d}\|P_A(\lambda)\phi_0\|^2 \geq \mathrm{e}^{-2(\kappa+\varepsilon)T}\|\phi_0\|^2$$

and

$$\begin{aligned}\int_0^T \|B^*\mathrm{e}^{-At}\phi_0\|_U^2 \mathrm{d}t &\le \|B\|^2_{\mathcal{L}(U,X_\beta)} \int_0^T \|\mathrm{e}^{-At}\phi_0\|^2_{X_{-\beta}} \mathrm{d}t \\ &= \|B\|^2_{\mathcal{L}(U,X_\beta)} \int_0^T \left(\int_\kappa^{\kappa+\varepsilon} (1+\lambda^2)^{-\beta} \mathrm{e}^{-2\lambda t} \mathrm{d}\|P_A(\lambda)\phi_0\|^2 \right) \mathrm{d}t \\ &\le \|B\|^2_{\mathcal{L}(U,X_\beta)} (1+(\kappa+\varepsilon)^2)^{-\beta} \|\phi_0\|^2 \int_0^T \mathrm{e}^{-2\kappa t} \mathrm{d}t.\end{aligned}$$

For the latter integral we obtain T if $\kappa = 0$, and $(1-\mathrm{e}^{-2\kappa T})/(2\kappa)$ if $\kappa \neq 0$. We choose $\phi = \phi_0$ in Eq. (27) and obtain an ε-dependent lower bound on C_T. The statement of the theorem follows since $\varepsilon > 0$ is arbitrary. □

In particular, we see from Corollary 2.15 that if $\kappa < 0$, then $C_\infty := \inf_{t>0} C_T$ is strictly positive. This is in contrast to the situation $\kappa \geq 0$, where $C_\infty = 0$, i.e. the control cost vanishes in the large time limit.

Remark 2.16. Let us now compare the lower bounds from Theorem 2.13 with the upper bounds from Theorems 2.7 and 2.12 in the special case $\beta = 0$. We focus on this case since in all our applications below we have $\beta = 0$.

In Table 1, we summarize the asymptotic behavior of the upper and lower bounds on the control cost in the large and small time asymptotic regime. We only keep track of the parameters T, d_1 and κ and omit multiplicative constants depending only on d_0, γ, and $\|B\|_{\mathcal{L}(U,X)}$. The parameter C stands for a constant which only depends on the parameter γ, and might change from case to case.

In the case $\kappa < 0$ or $\kappa > 0$, the upper bounds in Theorem 2.12 are given in terms of infima over $t \in (0, T]$ or $t \in (0, T)$, respectively. In order to obtain the upper bounds in the table, for $T \to 0$ we choose $t = T/2$, while in the regime $T \to \infty$ we choose $t = (-\kappa)^{-1}$ if $\kappa < 0$ and $t = T - 1$ if $\kappa > 0$. To discuss these bounds, let us first consider the case $d_1 = 0$.

		lower bound	upper bound
$\kappa < 0$	$T \to \infty$	$-\kappa$	$(-\kappa)\exp\left(C d_1^{\frac{1}{1-\gamma}} (-\kappa)^{\frac{\gamma}{1-\gamma}}\right)$
	$T \to 0$	T^{-1}	$T^{-1}\exp\left(C\left(\frac{d_1}{T^\gamma}\right)^{\frac{1}{1-\gamma}}\right)$
$\kappa = 0$	$T \to \infty$	T^{-1}	T^{-1}
	$T \to 0$	T^{-1}	$T^{-1}\exp\left(C\left(\frac{d_1}{T^\gamma}\right)^{\frac{1}{1-\gamma}}\right)$
$\kappa > 0$	$T \to \infty$	$T^{-1}\mathrm{e}^{-2\kappa T}$	$\mathrm{e}^{-2\kappa T}\exp\left(C d_1^{\frac{1}{1-\gamma}} + 2\kappa\right)$
	$T \to 0$	T^{-1}	$T^{-1}\exp\left(C\left(\frac{d_1}{T^\gamma}\right)^{\frac{1}{1-\gamma}}\right)$

Table 1: Asymptotic behavior of lower and upper bounds on C_T^2 in the case $\beta = 0$

This implies that we have full control in the sense that the control operator B is boundedly invertible. In this situation, the upper and lower bounds in Table 1 coincide except for the case when $\kappa > 0$ in the regime $T \to \infty$.

Let us now assume $d_1 > 0$. The lower bounds cannot be improved by Theorem 2.13. In the large time regime, the upper and lower bounds exhibit qualitatively the same asymptotic behavior except for the case when $\kappa > 0$. The different asymptotic behavior which we observe in the small time regime cannot be avoided. In fact, there exist examples where the exponential blowup of the type $\exp(CT^{-\gamma/(1-\gamma)})$ indeed occurs, see e.g. [FZ00, Mil04a]. They consider the controlled heat equation with control in a subset of the domain, see Section 3 for details on the controlled heat equation. Note that this example corresponds to $\gamma = 1/2$. This shows in particular that the upper bounds in Table 1 are sharp in this regime.

Remark 2.17. Let $X = U$, $\beta = 0$ and $B = I$. In this case one can explicitly construct null-control functions in time $T > 0$. We give two examples. The first one is given by

$$u_1(t) = \int_\kappa^\infty f_T(\lambda) \mathrm{d}P_A(\lambda) w_0, \quad \text{where} \quad f_T(\lambda) = \begin{cases} -T^{-1} & \text{if } \lambda = 0, \\ \frac{-\lambda}{\mathrm{e}^{\lambda T}-1} & \text{if } \lambda \neq 0. \end{cases}$$

The second one is given by

$$u_2(t) = \int_\kappa^\infty \mathrm{e}^{\lambda t} g_T(\lambda) \mathrm{d}P_A(\lambda) w_0, \quad \text{where} \quad g_T(\lambda) = \begin{cases} -T^{-1} & \text{if } \lambda = 0, \\ \frac{-2\lambda}{\mathrm{e}^{2\lambda T}-1} & \text{if } \lambda \neq 0. \end{cases}$$

The fact that u_1 and u_2 are null-control functions in time T follows from the Duhamel formula (4) and spectral calculus. Note that u_1 is time-independent while u_2 is time-dependent. Moreover, it follows that $C_T \leq \|u_i\|_{L^2([0,T];X)}$, $i \in \{1, 2\}$. We estimate

$$C_T^2 \leq \|u_1\|^2_{L^2([0,T];U)} \leq \begin{cases} T^{-1} & \text{if } \kappa = 0, \\ T \left| \frac{\kappa}{\mathrm{e}^{\kappa T}-1} \right|^2 & \text{if } \kappa \neq 0 \end{cases} \tag{28}$$

and

$$C_T^2 \leq \|u_2\|^2_{L^2([0,T];X)} \leq \begin{cases} T^{-1} & \text{if } \kappa = 0, \\ \frac{2\kappa}{\mathrm{e}^{2\kappa T}-1} & \text{if } \kappa \neq 0. \end{cases} \tag{29}$$

Since the upper bound in (29) coincides with the lower bound in Theorem 2.13, we conclude that u_2 is the (unique) null-control function in time T with minimal norm. Furthermore, the inequalities in (29) are actually equalities.

If $\kappa = 0$ the bounds in (28) and (29) coincide. Hence, in this case, the optimal null-control function in time T is a time-independent function.

We also see that for certain choices of T and κ, there is a constant-in-time null-control function in time T with norm which is close to the optimal one. This is related to the so-called turnpike property, see, for example [TZZ18].

3 Spectral inequalities and explicit cost for the controlled heat equation

In this section, we apply the results from Section 2 to the controlled heat equation with heat generation term on bounded and unbounded domains. More precisely, our setting is as follows.

Let $d \in \mathbb{N}$, $\alpha_i, \beta_i \in \mathbb{R} \cup \{\pm\infty\}$ with $\beta_i - \alpha_i > 0$, and

$$\Omega = \bigtimes_{i=1}^{d} (\alpha_i, \beta_i). \tag{30}$$

We denote by $-\Delta$ the self-adjoint Laplace operator in $L^2(\Omega)$ with Dirichlet, Neumann or periodic boundary conditions. Here we allow for periodic boundary conditions only if $\Omega = \Lambda_L = (-L/2, L/2)^d$ for some $L > 0$. Moreover, let $V \in L^\infty(\Omega)$ be real-valued, and define the self-adjoint operator H_Ω in $L^2(\Omega)$ by

$$H_\Omega = -\Delta + V.$$

In $L^2(\Omega)$ we consider the controlled heat equation with heat generation term $(-V)$

$$\dot{w} + H_\Omega w = \mathbf{1}_{S\cap\Omega} u, \quad w(0,\cdot) = w_0 \in L^2(\Omega), \tag{31}$$

where $T > 0$, $w, u \in L^2([0,T] \times \Omega)$, and where S is non-empty and measurable, usually given by a (ρ, a)-thick set or a (G, δ)-equidistributed set, see below for definitions. Note that we simultaneously treat bounded and unbounded domains such as $\mathbb{R}^d$, half-spaces, infinite strips, or hypercubes.

Theorems 2.7 and 2.12 translate spectral inequalities into null-controllability of the corresponding controlled Cauchy problem with explicit estimates on the control cost in all times. We will now apply them to the case $X = U = L^2(\Omega)$, $A = H_\Omega = -\Delta + V$, and $B = \mathbf{1}_{S\cap\Omega}$. In this setting we have in particular $\beta = 0$, and the spectral inequality reads

$$\forall \lambda > \kappa \ \forall \phi \in L^2(\Omega) \colon \quad \|P_{H_\Omega}(\lambda)\phi\|^2_{L^2(\Omega)} \leq d_0 \mathrm{e}^{d_1(\lambda-\kappa)^\gamma} \|\mathbf{1}_{S\cap\Omega} P_{H_\Omega}(\lambda)\phi\|^2_{L^2(\Omega)}. \tag{32}$$

We start by defining two geometric situations for the subset $S \subset \Omega$ where (32) is satisfied, cf. Fig. 1. For a measurable set $M \subset \mathbb{R}^d$ we denote by $|M|$ its Lebesgue measure, and for $x \in \mathbb{R}^d$ and $\rho > 0$ we denote by $B(x,r) = \{y \in \mathbb{R}^d \colon |x-y| < r\}$ the ball of radius r centered at x.

Definition 3.1 (Equidistributed set)**.** Let $G, \delta > 0$. We say that a set $S \subset \mathbb{R}^d$ is (G,δ)-*equidistributed* if S is measurable, and

$$\forall j \in (G\mathbb{Z})^d \ \exists z_j \in \Lambda_G + j \colon \quad B(z_j, \delta) \subset S \cap (\Lambda_G + j).$$

Definition 3.2 (Thick set)**.** Let $\rho \in (0,1]$ and $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ with $a_j > 0$ for $j \in \{1, \ldots, d\}$. We say that a set $S \subset \mathbb{R}^d$ is (ρ, a)-*thick* if S is measurable, and for each parallelepiped

$$P = \bigtimes_{j=1}^{d} \left[x_j - \frac{a_j}{2}, x_j + \frac{a_j}{2}\right] \quad \text{with} \quad x_j \in \mathbb{R} \quad \text{for} \quad j \in \{1, \ldots, d\}$$

we have

$$|S \cap P| \geq \rho \, |P| \, .$$

Note that every (G,δ)-equidistributed set is (ρ, a)-thick for some ρ and a but there exist (ρ, a)-thick sets which are not (G, δ)-equidistributed for any G and δ.

Now, we cite two spectral inequalities.

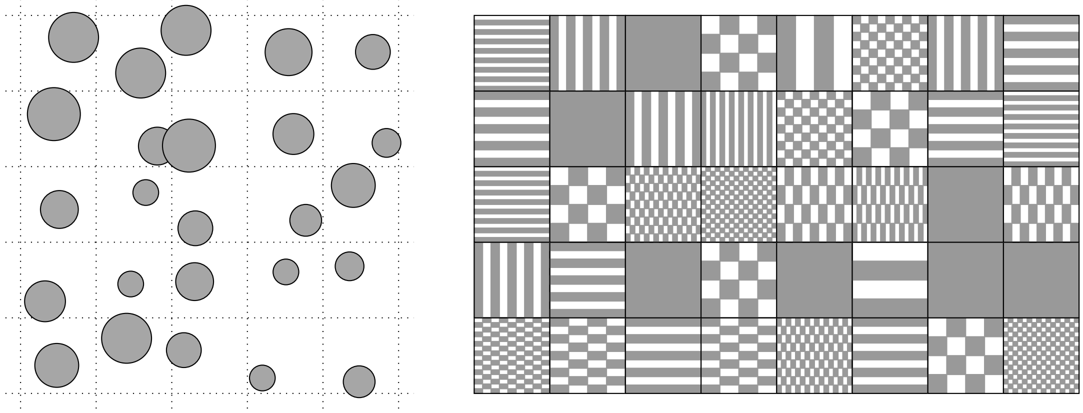

Figure 1: Illustrations of an equidistributed set (left) and a thick set (right).

Theorem 3.3 ([NTTV18, NTTV]). *Let $G, \delta > 0$, $\alpha_i - \beta_i \geq G$, $\Lambda_G \subset \Omega$, $S \subset \mathbb{R}^d$ be (δ, G)-equidistributed, $V \in L^\infty(\Omega)$ real-valued, and $\lambda \in \mathbb{R}$. Then we have for all $\phi \in L^2(\Omega)$*

$$\|P_{H_\Omega}(\lambda)\phi\|^2_{L^2(S\cap\Omega)} \geq C_{\mathrm{si}} \|P_{H_\Omega}(\lambda)\phi\|^2_{L^2(\Omega)},$$

where

$$C_{\mathrm{si}} = \sup_{\kappa\in\mathbb{R}} \left(\frac{\delta}{G}\right)^{N\left(1+G^{4/3}\|V-\kappa\|_\infty^{2/3}+G\sqrt{(\lambda-\kappa)_+}\right)},$$

$t_+ := \max\{0, t\}$ for $t \in \mathbb{R}$, and where $N > 0$ is a constant depending only on the dimension. In particular, we have for all $\lambda > \kappa$

$$C_{\mathrm{si}} \geq d_0 \mathrm{e}^{d_1(\lambda-\kappa)^{1/2}} \quad \text{with} \quad d_0 = \left(\frac{\delta}{G}\right)^{N(1+G^{4/3}\|V-\kappa\|_\infty^{2/3})} \quad \text{and} \quad d_1 = NG \ln\left(\frac{\delta}{G}\right).$$

The following result was proven in the $\mathbb{R}^d$ case in [Kov00, Kov01] and adapted to cubes and some other geometries in [EV, EV18, Egi]. Such estimates are often called Logvinenko-Sereda Theorems. We do not expound the history of this topic but refer the reader e.g. to the survey [ENT+18].

Theorem 3.4 ([Kov00, EV, EV18]). *Let $V = 0$, and $\Omega = \mathbb{R}^d$ or $\Omega = \Lambda_L$ for some $L > 0$. Let further $S \subset \mathbb{R}^d$ be a (ρ, a)-thick set. If $\Omega = \Lambda_L$ we assume that $a_j \leq L$ for all $j \in \{1, \ldots, d\}$. Then we have for all $\phi \in L^2(\Omega)$*

$$\|P_{H_\Omega}(\lambda)\phi\|^2_{L^2(\Omega)} \leq d_0 \mathrm{e}^{d_1\lambda^{1/2}} \|P_{H_\Omega}(\lambda)\phi\|^2_{L^2(S\cap\Omega)},$$

where

$$d_0 = \left(\frac{C^d}{\rho}\right)^{Cd} \quad \text{and} \quad d_1 = C|a|_1 \ln\left(\frac{C^d}{\rho}\right).$$

Here, C is a universal positive constant.

Remark 3.5. In the case where $\Omega = \mathbb{R}^d$ Theorem 3.4 has been proven in [Kov00] under the assumption that the Fourier transform $\mathcal{F}\phi$ of ϕ satisfies

$$\operatorname{supp}(\mathcal{F}\phi) \subset \bigtimes_{j=1}^{d} \left[x_j - \frac{b_j}{2}, x_j + \frac{b_j}{2}\right] \quad \text{for some} \quad x_j \in \mathbb{R} \text{ and } b_j > 0, \quad j \in \{1, \ldots, d\}. \tag{33}$$

Here, $\mathcal{F}$ denotes the standard Fourier transformation on $L^2(\mathbb{R}^d)$. In the case where $\Omega = \Lambda_L$ and periodic boundary conditions. Theorem 3.4 has been proven in [EV] under assumption (33). Here, the Fourier transform $\mathcal{F}\phi$ of $\phi \in L^2(\Lambda_L)$ is given by

$$\mathcal{F}\phi : \left(\frac{2\pi}{L}\mathbb{Z}\right)^d \to \mathbb{C}, \quad (\mathcal{F}\phi)(k) = \frac{1}{L^d} \int_{\Lambda_L} \phi(x) \mathrm{e}^{-\mathrm{i}(x \cdot k)} \mathrm{d}x.$$

In both cases, one can show that functions $\phi \in \operatorname{Ran} P_{H_\Omega}(\lambda)$ as considered in Theorem 3.4 satisfy Assumption (33) with $x_j = 0$, and $b_j = 2\sqrt{\lambda}$. This has been carried out in Section 5 of [EV18]. This statement remains true if $\Omega = \Lambda_L$ and Dirichlet or Neumann boundary conditions are imposed, see again Section 5 of [EV18].

Theorem 3.4 immediately implies the following

Theorem 3.6. *In the situation of Theorem 3.4, we have for all $\lambda \geq 0$, all $\theta > 0$ and all $\phi \in L^2(\Omega)$ that*

$$\|P_{(-\Delta)^\theta}(\lambda)\phi\|_{L^2(\Omega)}^2 \leq d_0 \mathrm{e}^{d_1 \lambda^{1/(2\theta)}} \|P_{(-\Delta)^\theta}(\lambda)\phi\|_{L^2(S\cap\Omega)}^2 \tag{34}$$

where

$$d_0 = \left(\frac{\rho}{C^d}\right)^{Cd} \quad \textit{and} \quad d_1 = C|a|_1 \ln\left(\frac{\rho}{C^d}\right),$$

and C is a universal positive constant.

Proof. We estimate, using the transformation formula for spectral measures, cf. [Sch12, Prop. 4.24], and Theorem 3.4

$$\begin{aligned}\|P_{(-\Delta)^\theta}(\lambda)\phi\|_{L^2(\Omega)}^2 = \|P_{(-\Delta)}(\lambda^{1/\theta})\phi\|_{L^2(\Omega)}^2 &\leq d_0 \mathrm{e}^{d_1 \lambda^{1/(2\theta)}} \|P_{(-\Delta)}(\lambda^{1/\theta})\phi\|_{L^2(S\cap\Omega)}^2 \\ &= d_0 \mathrm{e}^{d_1 \lambda^{1/(2\theta)}} \|P_{(-\Delta)^\theta}(\lambda)\phi\|_{L^2(S\cap\Omega)}^2.\end{aligned}$$ □

Note that the exponent $1/2\theta$ in (34) is smaller than one if $\theta > 1/2$. Now we combine the spectral inequalities from Theorems 3.9, 3.7, and 3.6 with Theorem 2.7 and immediately deduce the following explicit estimates on the control cost.

Theorem 3.7 (Negative Laplacian with control on thick sets)**.** *Let $\Omega = \mathbb{R}^d$, or $\Omega = \Lambda_L$ for some $L > 0$. Let further $S \subset \mathbb{R}^d$ be a (ρ, a)-thick set. If $\Omega = \Lambda_L$ we assume that $a_j \leq L$ for all $j \in \{1, \ldots, d\}$. Then for all $\phi \in L^2(\Omega)$, and all $T > 0$ we have*

$$\|\mathrm{e}^{\Delta T}\|_{L^2(\Omega)}^2 \leq C_{\mathrm{obs}}^2 \int_0^T \|\mathrm{e}^{\Delta t}\phi\|_{L^2(S\cap\Omega)}^2 \mathrm{d}t,$$

where

$$C_{\mathrm{obs}}^2 = \frac{C_1}{T} \rho^{-C_2 d} \exp\left(\frac{C_3 |a|_1^2 \ln^2(C_4^d/\rho)}{T}\right).$$

Here, C_1, C_2, C_3, and C_4 are universal positive constants. Moreover, for all $T > 0$ the system (31) with $V = 0$ is null-controllable in time T, and the cost satisfies $C_T \leq C_{\mathrm{obs}}$.

Theorem 3.8 (Fractional negative Laplacian with control on thick sets)**.** *Let $\Omega = \mathbb{R}^d$, or $\Omega = \Lambda_L$ for some $L > 0$ and let $\theta > 1/2$. Let further $S \subset \mathbb{R}^d$ be a (ρ, a)-thick set. If $\Omega = \Lambda_L$ we assume that $a_j \leq L$ for all $j \in \{1, \ldots, d\}$. Then for all $\phi \in L^2(\Omega)$, and all $T > 0$ we have*

$$\|\mathrm{e}^{-(-\Delta)^\theta T}\|_{L^2(\Omega)}^2 \leq C_{\mathrm{obs}}^2 \int_0^T \|\mathrm{e}^{-(-\Delta)^\theta t}\phi\|_{L^2(S\cap\Omega)}^2 \mathrm{d}t,$$

where

$$C_{\mathrm{obs}}^2 = \frac{C_1}{T}\rho^{-C_2 d} \exp\left(\frac{C_3\bigl(|a|_1 \ln(C_4^d/\rho)\bigr)^{\frac{2\theta}{2\theta-1}}}{T^{\frac{1}{2\theta-1}}}\right)$$

Here, C_1, C_2, C_3, and C_4 are universal positive constants. Moreover, for all $T > 0$ the system

$$\dot{w} + (-\Delta)^\theta w = \mathbf{1}_{S\cap\Omega} u, \quad w(0,\cdot) = w_0 \in L^2(\Omega), \tag{35}$$

is null-controllable in time T, and the cost satisfies $C_T \leq C_{\mathrm{obs}}$.

Theorem 3.9 (Schrödinger operator with control on equidistributed sets)**.** *Let $G, \delta > 0$, $\Omega \subset \mathbb{R}^d$ be as in (30) with $\alpha_i - \beta_i \geq G$ for all $i \in \{1, \ldots, d\}$, $\Lambda_G \subset \Omega$, $S \subset \mathbb{R}^d$ be (δ, G)-equidistributed, and $V \in L^\infty(\Omega)$ real-valued. Then for all $\phi \in L^2(\Omega)$, and all $T > 0$ we have*

$$\|\mathrm{e}^{-H_\Omega T}\|_{L^2(\Omega)}^2 \leq C_{\mathrm{obs}}^2 \int_0^T \|\mathrm{e}^{-H_\Omega t}\phi\|_{L^2(S\cap\Omega)}^2 \mathrm{d}t,$$

where

$$C_{\mathrm{obs}}^2 = \left(\frac{\delta}{G}\right)^{-C_2(1+G^{4/3}\|V-\kappa\|_\infty^{2/3})} \inf_{t\in(0,T]} \frac{C_1}{t} \exp\left(\frac{C_3 G^2 \ln^2(\delta/G)}{t} - 2\kappa t\right) \qquad \text{if } \kappa < 0,$$

$$C_{\mathrm{obs}}^2 = \left(\frac{\delta}{G}\right)^{-C_2(1+G^{4/3}\|V\|_\infty^{2/3})} \frac{C_1}{T} \exp\left(\frac{C_3 G^2 \ln^2(\delta/G)}{T}\right) \qquad \text{if } \kappa = 0,$$

$$C_{\mathrm{obs}}^2 = \left(\frac{\delta}{G}\right)^{-C_2(1+G^{4/3}\|V-\kappa\|_\infty^{2/3})} \inf_{t\in[0,T)} \frac{C_1}{T-t} \exp\left(\frac{C_3 G^2 \ln^2(\delta/G)}{T-t} - 2\kappa t\right) \qquad \text{if } \kappa > 0.$$

Here, C_1, C_2, and C_3 are positive constants depending only on the dimension. Moreover, for all $T > 0$ the system (31) is null-controllable in time T, and the cost satisfies $C_T \leq C_{\mathrm{obs}}$.

Remark 3.10. Note that the constants in Theorems 3.7, 3.8 and 3.9 are uniform in $\Omega \subset \mathbb{R}^d$, and depend on V only via its L^∞-norm.

Remark 3.11. >From talk announcements on the internet and by private communication [Moy] we have learned that G. Lebeau and I. Moyano are working on problems related to ours. In particular this concerns Schrödinger operators $H = -\Delta + V$, where V is an analytic function and satisfies certain additional regularity conditions. (In fact, they allow Δ to be an Laplace-Beltrami operator with certain analytic metrics on $\mathbb{R}^d$.) According to G. Lebeau and I. Moyano, for a thick set $S \subset \mathbb{R}^d$, any $\lambda, E \in \mathbb{R}$ and $\phi \in L^2(\mathbb{R}^d)$ one has

$$\|P_H(\lambda)\phi\|_{L^2(\mathbb{R}^d)}^2 \leq d_0 \mathrm{e}^{d_1\sqrt{\lambda_+}} \|P_H(\lambda)\phi\|_{L^2(S)}^2. \tag{36}$$

Here d_0, d_1 are constants which depend on the properties of the potential V and the thick set S. This opens up the possibility to derive observability and control cost estimates for controls on thick sets and Schrödinger semigroups with analytic potentials.

If $\min \sigma(A) = 0$, our Theorem 2.7 and the bound (36) of Lebeau and Moyano imply the observability estimate

$$\|\mathrm{e}^{-H_\Omega T}\|_{L^2(\Omega)}^2 \leq C_{\mathrm{obs}}^2 \int_0^T \|\mathrm{e}^{-H_\Omega t}\phi\|_{L^2(S\cap\Omega)}^2 \mathrm{d}t, \quad C_{\mathrm{obs}}^2 \leq \frac{C_1 d_0}{T}(2d_0+1)^{C_2} \exp\left(C_3 \frac{d_1^2}{T}\right),$$

and consequently the corresponding upper bound on the control cost.

Remark 3.12. In Theorems 3.7, 3.8, and 3.9, the asymptotic behavior of the upper bound on C_T in the limit $T \to 0$ and $T \to \infty$ is optimal as discussed in Remark 2.16, see also Table 1. We also note that the term $\|V\|_\infty^{2/3}$ in the bound in Theorem 3.9 is optimal, at least in even space dimensions, see [DZZ08].

Furthermore, the dependence of the rate of the exponential term on the parameter G in Theorem 3.9 is optimal. This follows by considering the special case $V = 0$ and comparing it to the following lower bound on the control cost in terms of the geometry given in [Mil04a]. For the heat equation on smooth, connected manifolds Ω with control operator $B = \chi_S$ for an open $S \subset \Omega$ Miller proves that the control cost C_T in time T satisfies

$$\sup_{x\in\Omega, \overline{B}(x,\rho)\subset\Omega\setminus\overline{S}} \rho^2/4 \leq \liminf_{T\to 0} T \ln C_T. \tag{37}$$

Theorem 3.9 on the other hand implies

$$\limsup_{T\to 0} T \ln C_T \leq C_3 G^2 \ln^2(\delta/G). \tag{38}$$

Thus, we complement the lower bound in (37) by an upper bound. Now, let Ω be as in the theorem and choose a particular (G,δ)-equidistributed set S of the form

$$S = \bigcup_{j\in(G\mathbb{Z})^d} B(z_j,\delta) \cap \Omega.$$

Then the complement of $\overline{S}$ in Ω contains a ball of radius

$$\rho = \frac{1}{2}\left(\frac{G}{2} - \delta\right) = G\frac{1-2\delta/G}{4} \quad \text{whence} \quad G \leq \frac{4}{1-2\delta/G} \sup_{\overline{B}(\rho)\subset\Omega\setminus\overline{S}} \rho.$$

Combining this with (37) and (38), we find

$$\begin{aligned}\sup_{x\in\Omega, \overline{B}(x,\rho)\subset\Omega\setminus\overline{S}} \rho^2/4 &\leq \liminf_{T\to 0} T \ln C_T \\ &\leq \limsup_{T\to 0} T \ln C_T \leq C_3 \left(\frac{4\ln(\delta/G)}{1-2\delta/G}\right)^2 \sup_{x\in\Omega, \overline{B}(x,\rho)\subset\Omega\setminus\overline{S}} \rho^2.\end{aligned}$$

Varying the parameters G and δ, and taking the limit $G \to 0$ or $G \to \infty$, respectively, while keeping δ/G constant, this reasoning shows that the factor G^2 in the exponential term in Theorem 3.9 is optimal. Another way to interpret this is that in the exponential, the geometric parameters G is in the same relation to the time T as the order of space and time derivatives in the underlying heat equation.

Since every (G, δ)-equidistributed set is (ρ, a)-thick for some ρ and a, the above example also applies to the upper bound in Theorem 3.7.

We summarize that the term $\exp(-C/T)$ in the control cost estimates in Theorems 3.7 and Theorem 3.9 is characteristic for the controlled heat equation while the term T^{-1} is characteristic for control problems and always occurs.

4 Homogenization and de-homogenization

We now introduce homogenization and de-homogenization asymptotics of the control cost. To our knowledge, this is a novel concept. The bounds of Theorems 3.7 and 3.9 are the first ones which allow to pursue such an investigation. The abstract control cost estimate in Theorem 2.7 as well as the spectral inequalities, proved in [NTTV18, EV18] and spelled out in Theorems 3.3 and 3.4 are crucial ingredients here. In fact, the insufficiency of earlier bounds on the control cost was the main motivation for proving the upper bounds on the control cost in Section 2.

We will first treat homogenization. This means that the control set $S \subset \Omega$ becomes more and more evenly distributed over the space while keeping an overall lower bound on the relative density. This corresponds to reducing local fluctuations in the density of the control set S. In the case of (G, δ)-equidistributed sets this corresponds to G and δ simultaneously tending to 0 while their ratio remains constant. In the case of (ρ, a)-thick sets, this corresponds to a tending to zero while ρ is kept constant. We refer to Figure 2 for an illustration in the case of (G, δ)-equidistributed sets. The first example shows that

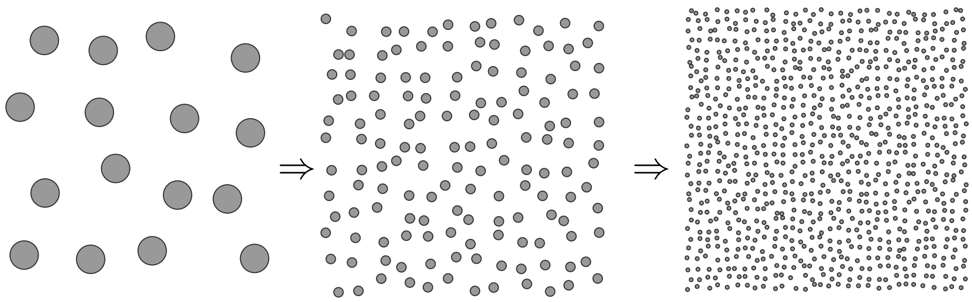

Figure 2: Illustration of homogenization in the case of (G, δ)-equidistributed sets

homogenization counteracts the exponential singularity of the control cost in the small time regime.

Example 4.1. We consider the controlled heat equation (31) and assume that $V = 0$ and $\Omega = \mathbb{R}^d$ or $\Omega = \Lambda_L$. We fix a density $\rho \in (0, 1)$.

For every $a \in \mathbb{R}^d$ with positive entries (and with $a_i \leq L$ if $\Omega = \Lambda_L$), and every (ρ, a)-thick set S_a, Theorem 3.7 implies that system (31) with $S = S_a$ is null-controllable in every time $T > 0$ with

$$C_T^2 \leq \frac{C_1}{T} \rho^{-C_2 d} \exp\left(\frac{C_3 |a|_1^2 \ln^2(C_4^d/\rho)}{T}\right).$$

Now, recall that homogenization means that we let the vector a tend to zero while keeping ρ constant. In the homogenization limit $a \to 0$, the upper bound tends to

$$\frac{C_1}{T}\rho^{-C_2 d}. \tag{39}$$

We see that in the limit $a \to 0$, the exponential singularity at $T = 0$, which is characteristic for controlled heat equation, vanishes. We conclude that homogenization counteracts this $\exp(C/T)$ singularity. Moreover, we note that the control cost is always bounded from below by the cost of a system with full control. By Theorem 2.13 we conclude that $C_T^2 \geq 1/T$ for all $a \in \mathbb{R}^d$ with positive entries. Hence, the term $1/T$ cannot vanish in the homogenization limit. Note that the expression in (39) coincides (up to constants) with the control cost of the system with full control and $\kappa = 0$ considered in Remark 2.17.

The next example shows that also in the presence of a potential, homogenization annihilates the exponential singularity at small times. Furthermore, the effect of potentials on the control cost disappears in the homogenization regime – up to the effect of the potential on κ.

Example 4.2. We consider the controlled heat equation with bounded and real-valued potential V as in (31). Note that $\kappa \geq -\|V\|_\infty$. For all $G, \delta > 0$ and all (G, δ)-equidistributed sets $S_{G,\delta}$ such that $\Lambda_G \subset \Omega$, Theorem 3.9 implies that the system (31) with $S = S_{G,\delta}$ is null-controllable in every time $T > 0$ with

$$C_T^2 \leq \left(\frac{\delta}{G}\right)^{-C_2(1+G^{4/3}\|V-\kappa\|_\infty^{2/3})} \inf_{t\in(0,T]} \frac{C_1}{t} \exp\left(\frac{C_3 G^2 \ln^2(\delta/G)}{t} - 2\kappa t\right) \qquad \text{if } \kappa < 0,$$

$$C_T^2 \leq \left(\frac{\delta}{G}\right)^{-C_2(1+G^{4/3}\|V\|_\infty^{2/3})} \frac{C_1}{T} \exp\left(\frac{C_3 G^2 \ln^2(\delta/G)}{T}\right) \qquad \text{if } \kappa = 0,$$

$$C_T^2 \leq \left(\frac{\delta}{G}\right)^{-C_2(1+G^{4/3}\|V-\kappa\|_\infty^{2/3})} \inf_{t\in[0,T)} \frac{C_1}{T-t} \exp\left(\frac{C_3 G^2 \ln^2(\delta/G)}{T-t} - 2\kappa t\right) \qquad \text{if } \kappa > 0.$$

Homogenization now means sending G and δ to zero while keeping δ/G constant. In this limit, the upper bounds tend to

$$\left(\frac{\delta}{G}\right)^{-C_2} \inf_{t\in(0,T]} \frac{C_1}{t} \exp(-2\kappa t), \quad \left(\frac{\delta}{G}\right)^{-C_2} \frac{C_1}{T}, \quad \text{and} \quad \left(\frac{\delta}{G}\right)^{-C_2} \inf_{t\in[0,T)} \frac{C_1}{T-t} \exp(-2\kappa t),$$

corresponding to the cases $\kappa > 0$, $\kappa = 0$, and $\kappa > 0$ where we used monotonicity in order to interchange limits and infima. It is straightforward to see that we recover the upper bounds from Table 2.16 with $d_1 = 0$. This shows that in the homogenization limit, the limit of the upper bounds on the control cost coincides with the control cost of the system with full control from Remark 2.17.

Moreover we see that the influence of the potential V on the control cost is annihilated up to the effect of the potential on κ.

Now, we treat the complementary regime which we call de-homogenization. Let $\Omega = \mathbb{R}^d$. For (G, δ)-equidistributed sets, de-homogenization means sending G and δ simultaneously to ∞ while G/δ remains constant. In the context of (ρ, a)-thick sets this means that all coordinates of a tend to ∞ while ρ is constant.

Even though the overall relative density of the control set remains, de-homogenization allows for larger and larger void areas between the components of S where no control is applied, see Figure 3. It is unsurprising that for fixed time T, our upper bound on the control cost will in general increase since the diffusive nature of the heat equation makes it harder for components of the control set to exert control in larger and larger areas where there is no or only little control. In particular, the considerations in [Mil04a], see also Remark 3.12, show that for fixed time, the control cost estimate must tend to ∞ in the de-homogenization limit. However, since C_T is non-increasing in time, one can ask if it is possible to keep

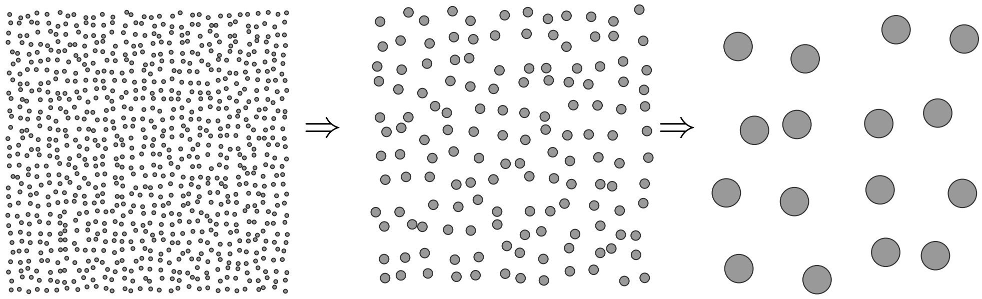

Figure 3: Illustration of de-homogenization in the case of (G, δ)-equidistributed sets

the control cost constant by simultaneously letting T tend to ∞ in the de-homogenization regime. The following example positively answers this question and provides a rate between the required time and the order of de-homogenization.

Example 4.3. We consider the controlled fractional heat equation (35) and assume that $\Omega = \mathbb{R}^d$. We fix a density $\rho \in (0, 1)$.

For every $T > 0$ and every $a \in \mathbb{R}^d$ with positive entries, and every (ρ, a)-thick set S_a, Theorem 3.8 implies that system (35) with $S = S_a$ is null-controllable in every time $T > 0$ with

$$C_T^2 \leq \frac{C_1}{T} \rho^{-C_2 d} \exp\left(\frac{C_3 \big(|a|_1 \ln(C_4^d/\rho) \big)^{\frac{2\theta}{2\theta - 1}}}{T^{\frac{1}{2\theta-1}}} \right).$$

There are three model parameters here: the parameters ρ and a, describing the geometry of the control set, and the time T. Since we already chose ρ and a, the only remaining way to accommodate for the increase in our upper bound when a tends to infinity is to modify the remaining parameter T by choosing

$$T \sim |a|_1^{2\theta}$$

(due to the $1/T$ term in front of the exponential, we can even allow for a small logarithmic correction to this relation). We have recovered the relation between time and space derivatives $\dot{w} = -(-\Delta)^\theta w$ from the underlying fractional heat equation. This is an indication that our estimates on the control cost with respect to time and space parameters are close to being optimal.

Example 4.4. We consider the controlled heat equation with bounded and real-valued potential V as in (31), and assume that $\Omega = \mathbb{R}^d$ and $\kappa > 0$. For all $G, \delta > 0$ and all (G,δ)-equidistributed sets $S_{G,\delta}$, Theorem 3.9 implies that the system (31) with $S = S_a$ is null-controllable in every time $T > 0$ with

$$C_T^2 \leq \left(\frac{\delta}{G}\right)^{-C_2(1+G^{4/3}\|V-\kappa\|_\infty^{2/3})} \frac{2C_1}{T} \exp\left(\frac{2C_3 G^2 \ln^2(\delta/G)}{T} - \kappa T\right)$$

As in Example 4.3, the increase of the upper bound in the de-homogenization limit can be accommodated by choosing $T \sim G^{4/3}$. Note that this exponent $G^{4/3}$ is related to the counterexample in [DZZ08]. However, there are special cases, such as a constant, positive potential V in which choosing $T \sim G$ is sufficient to compensate the increase of the control cost in the de-homogenization regime.

In Example 4.4 we assumed $\kappa > 0$. In the case where $\kappa \leq 0$ this argument does not work anymore. However, if $\kappa < 0$, we know from Theorem 2.13 and Corollary 2.15 that $C_\infty = \inf_{T>0} C_T > 0$ for every choice of G and δ. It is an interesting question whether C_∞ tends to infinity (and if yes at which rate) or remains bounded in the de-homogenization limit. It seems that this is not accessible with the techniques presented in this paper.

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