

Wegner estimate and disorder dependence for alloy-type Hamiltonians with bounded magnetic potential

Matthias Täufer¹ and Martin Tautenhahn²

¹*Technische Universität Dortmund, Fakultät für Mathematik, Germany*

²*Friedrich Schiller Universität Jena, Germany & Technische Universität Chemnitz, Germany*

Abstract

We consider non-ergodic magnetic random Schrödinger operators with a bounded magnetic vector potential. We prove an optimal Wegner estimate valid at all energies. The proof is an adaptation of arguments from [16], combined with a recent quantitative unique continuation estimate for eigenfunctions of elliptic operators from [1]. This generalizes Klein's result to operators with a bounded magnetic vector potential. Moreover, we study the dependence of the Wegner-constant on the disorder parameter. In particular, we show that above the model-dependent threshold $E_0(\infty) \in (0, \infty]$, it is impossible that the Wegner-constant tends to zero if the disorder increases. This result is new even for the standard (ergodic) Anderson Hamiltonian without magnetic field.

1. Introduction

We study a class of magnetic, non-ergodic random Schrödinger operators on $L^2(\mathbb{R}^d)$ of the type

$$H_\omega = H_0 + \lambda V_\omega, \quad (1)$$

where $\lambda > 0$ is the disorder parameter, $H_0 = (-i\nabla + A_0)^2 + V_0$ with a bounded electric potential $V_0 \in L^\infty(\mathbb{R}^d)$ and a bounded magnetic vector potential $A_0 \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$ satisfying $\operatorname{div}(A_0)$ bounded, and where

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u_j(x - z_j).$$

The random variables ω_j , $j \in \mathbb{Z}^d$, are independent and identically distributed with compactly supported, non-degenerate distribution, and the single-site potentials $(u_j)_{j \in \mathbb{Z}^d}$, are measurable and real-valued functions on \mathbb{R}^d satisfying

$$u_- \chi_{B(\delta_-)} \leq u_j \leq \chi_{\Lambda_{\delta_+}}$$

for some $u_- \in (0, 1]$ and $\delta_\pm > 0$, where $B(\delta_-)$ and Λ_{δ_+} denote the d -dimensional ball of radius δ_- and the d -dimensional cube of side length δ_+ , centered at 0. Furthermore, we assume that the centers z_j of the single-site potentials are in a certain sense equidistributed

in \mathbb{R}^d . Such operators are used to model quantum mechanical properties of disordered solids. While each configuration of the randomness corresponds to a particular realization of the solid, the law of the random variables models their distribution.

One distinctive feature of random operators is the phenomenon of localization, i.e. that parts of the spectrum consist only of pure point spectrum (spectral localization) or that the solutions of the Schrödinger equation stay almost surely trapped in a finite region of space for all time (dynamical localization). This is in contrast to periodic operators which exhibit only absolutely continuous spectrum. One method for proving localization is the so-called multiscale analysis introduced in [10, 9] and further developed in [31, 11, 12, 13]. The multiscale analysis is an induction argument. While the induction anchor is provided by the so-called initial-scale estimate, a so-called Wegner estimate is needed for the induction step. A Wegner estimate is an upper bound on the expected number of eigenvalues of a self-adjoint restriction $H_{\omega,L}$ of H_ω to a cube $\Lambda_L \subset \mathbb{R}^d$ of side length $L > 0$ in an energy interval $[a, b] \subset \mathbb{R}$. More precisely, a Wegner estimate is an estimate of the form

$$\mathbb{E}(\text{Tr } \chi_{[a,b]}(H_{\omega,L})) \leq C(\lambda)S(b-a)|\Lambda_L|^m, \quad (2)$$

where $C(\lambda)$ is a constant depending on λ and the various model parameters, $S(b-a)$ denotes the concentration function of the distribution of the random variables ω_j , $j \in \mathbb{Z}^d$, and $m \geq 1$. A Wegner estimate is optimal if $m = 1$.

In this note, we prove optimal Wegner estimates for the family of operators defined in (1) valid at all energies $[a, b] \subset \mathbb{R}$, see Theorems 2.1 and 2.2. Furthermore, we study the dependence of $C(\lambda)$ on the disorder parameter λ . If

$$b < E_0(\infty) := \liminf_{t \rightarrow \infty} \sigma \left(H_0 + t \sum_{j \in \mathbb{Z}^d} u_j(\cdot - j) \right) \in (0, \infty]$$

then the constant $C(\lambda)$ in the Wegner estimate (2) will tend to zero if the disorder λ tends to infinity, see Theorem 2.3. In this case, the Wegner estimate can be used to obtain the initial length scale estimate at sufficiently large disorder, and localization follows via multiscale analysis, see [31, 14]. Hence, it is natural to ask whether such a Wegner estimate (where $C(\lambda) \rightarrow 0$ if $\lambda \rightarrow \infty$) also holds above the threshold $E_0(\infty)$. Until now there have been no results in this direction, cf. [16]. We show in Theorem 2.4 that it is indeed impossible to obtain a Wegner estimate with $C(\lambda) \rightarrow 0$ if $\lambda \rightarrow \infty$ above $E_0(\infty)$. This result is even new in the absence of a magnetic field. In the case where the family H_ω is ergodic, Theorem 2.4 has an interpretation in terms of the integrated density of states, see Theorem 2.5. Theorems 2.4 and 2.5 show in particular that the spectral behaviour at large disorder changes drastically below and above the model-dependent constant $E_0(\infty) \in [0, \infty]$, see Remark 2.6.

Optimal and non-optimal Wegner estimates for ergodic and non-ergodic random operators with and without magnetic field have been studied by many authors before. Let us give a brief overview here. We refer to [16] and [30] for further references. In [5], an optimal Wegner estimate at all energies is proved for the usual (ergodic) Anderson Hamiltonian under the additional assumption that a covering condition holds. In [7], the authors remove the covering condition and consider operators H_0 with non-vanishing magnetic field. More precisely, they assume that the magnetic vector potential A_0 is either periodic (which allows only for a subset of the class of periodic magnetic fields) or H_0 is the Landau Hamiltonian in two dimensions, i.e. $A_0 = B/2(-x_2, x_1)$ where B is the magnetic field strength. The

Landau Hamiltonian can be treated in spite of its unbounded vector potential A_0 since unique continuation properties for the Landau Hamiltonian are quite well understood, cf. [8]. However, this method strongly relies on the structure of the Landau Hamiltonian and breaks down for arbitrarily small perturbations of the Landau magnetic field. In the non-ergodic setting, optimal Wegner estimates have been proved, e.g., in [21] for the Landau Hamiltonian, and up to a logarithmic correction in [22] for non-magnetic Schrödinger operators. The logarithmic factor in the energy is removed in [16].

The drawback of our results is that we assume boundedness of A_0 and $\operatorname{div}(A_0)$. This assumption stems from the quantitative unique continuation result from [1]. While the long-term goal is to treat also unbounded magnetic vector potentials A_0 , we emphasize that already bounded A_0 include important physically relevant examples, cf. [17].

Our paper is organized as follows: In Section 2 we introduce the notation and state our main results. In Section 3 we quote a quantitative unique continuation estimate for solutions of elliptic equations given in [1]. In Section 4 we give the proofs of our main results, while a technical estimate is postponed to the appendix. The proof of the Wegner estimates from Theorems 2.1, 2.2 and 2.3 rely on the recent quantitative unique continuation result of [1]. By now there is a wealth of results pursuing the connection between unique continuation and spectral theory of random Schrödinger operators, see e.g. [6, 8, 2, 7, 26, 4, 21, 22, 3, 16, 28, 18, 19, 29]. While the proofs of Theorems 2.1, 2.2 and 2.3 use in particular the guiding thread of [16], they show how this approach and the recent result of [1] complement each other in an efficient way. On the contrary, the proofs of Theorem 2.4 and 2.5 are original even for the usual (ergodic) Anderson Hamiltonian with vanishing magnetic field.

2. Model and results

For $x \in \mathbb{R}^d$ or \mathbb{C}^d we denote by $|x|$ the Euclidean norm of x . For $L, r > 0$ we denote by $\Lambda_L = (-L/2, L/2)^d$ the open cube with side length L , and by $B(r) = \{y \in \mathbb{R}^d : |y| < r\}$ the open ball in \mathbb{R}^d of radius r , centered at 0. For $x \in \mathbb{R}^d$ we denote by $\Lambda_L(x) = \Lambda_L + x$ and $B(r, x) = B(r) + x$ its translates.

Let us define the class of random Schrödinger operators studied in this note. It is a generalization of the models studied in [16]. The non-random part is given by the self-adjoint magnetic Schrödinger operator

$$H_0 = (-i\nabla + A_0)^2 + V_0$$

on $L^2(\mathbb{R}^d)$ with a bounded electric potential $V_0 \in L^\infty(\mathbb{R}^d)$ and a bounded magnetic vector potential $A_0 \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that $\operatorname{div}(A_0)$ is bounded and $\inf \sigma(H_0) = 0$. Note that we can rewrite H_0 to $H_0 = -\Delta + b_0^T \nabla + c_0$ where

$$b_0(x) = -2iA_0(x) \quad \text{and} \quad c_0(x) = V_0(x) + |A_0(x)|^2 - i \operatorname{div}(A_0)(x). \quad (3)$$

In order to introduce the random part, we define the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where

$$\Omega = \prod_{k \in \mathbb{Z}^d} \mathbb{R}, \quad \mathcal{A} = \bigotimes_{k \in \mathbb{Z}^d} \mathcal{B}(\mathbb{R}), \quad \text{and} \quad \mathbb{P} = \bigotimes_{k \in \mathbb{Z}^d} \mu_k,$$

where μ_k , $k \in \mathbb{Z}^d$, are non-degenerate probability measures on \mathbb{R} with $\operatorname{supp} \mu_k \subset [0, M]$ for some $M > 0$ and all $k \in \mathbb{Z}^d$. By non-degenerate we mean that for all $L > 0$ we have

$S_L(t) \rightarrow 0$ as $t \rightarrow 0$, see below for the definition of S_L . As a consequence, the projections $\Omega \ni \omega \mapsto \omega_k, k \in \mathbb{Z}^d$ give rise to the independent sequence of random variables $(\omega_k)_{k \in \mathbb{Z}^d}$, each coordinate ω_k distributed according to the measure μ_k . We write $S_\mu(t) := \sup_{a \in \mathbb{R}} \mu([a, a+t])$ for the concentration function of a probability measure μ and set for $t \geq 0$

$$S_L(t) := \sup_{j \in \Lambda_L \cap \mathbb{Z}^d} S_{\mu_j}(t).$$

We use the symbol \mathbb{E} for the expectation with respect to the probability measure \mathbb{P} .

Let now $\delta_- \in (0, 1/2)$ and $Z = (z_j)_{j \in \mathbb{Z}^d} \subset \mathbb{R}^d$ such that

$$\forall j \in \mathbb{Z}^d: \quad B(\delta_-, z_j) \subset \Lambda_1(j).$$

For each $\omega \in \Omega$, the *crooked alloy-type potential* $V_\omega : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined by

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u_j(x - z_j),$$

where the single-site potentials $(u_j)_{j \in \mathbb{Z}^d}$, are measurable and real-valued functions on \mathbb{R}^d satisfying

$$u_- \chi_{B(\delta_-)} \leq u_j \leq \chi_{\Lambda_{\delta_+}(0)}$$

for some $u_- \in (0, 1]$ and $\delta_+ > 0$. For each $\omega \in \Omega$ and $\lambda > 0$ we define the self-adjoint operator

$$H_\omega = H_0 + \lambda V_\omega$$

in $L^2(\mathbb{R}^d)$, and call the family of operators $(H_\omega)_{\omega \in \Omega}$ the *magnetic crooked alloy-type Hamiltonian*. For $L > 0$ we denote by $H_{\omega, L}$ the restrictions of H_ω to $L^2(\Lambda_L)$ subject to Dirichlet boundary conditions. Given a Borel set $B \subset \mathbb{R}$, we denote by $P_{\omega, L}(B) := \chi_B(H_{\omega, L})$ the spectral projection onto the set B with respect to $H_{\omega, L}$.

Theorem 2.1. *Let $E_0 > 0$ and set*

$$\gamma_1^2 = \frac{1}{2} \delta_-^{N_2} (1 + |E_0|^{2/3} + \|b_0\|_\infty^2 + \|c_0\|_\infty^{2/3}) \quad (4)$$

where $M_d > 0$ is the constant from Theorem 3.3 and b_0, c_0 are as in Eq. (3). Then there is a constant $C_1 = C_1(d, \delta_\pm, u_-, \gamma_1, \|V_0\|_\infty, E_0)$ such that for any closed interval $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma_1$, and $\lambda > 0$, and any $L \in \mathbb{N}_{\text{odd}}$ with $L \geq 2 + \delta_+$, we have

$$\mathbb{E}(\text{Tr } P_{\omega, L}(I)) \leq C_1 \left(1 + (\lambda M)^{2 + \frac{\log d}{\log 2}} \right) S_L(\lambda^{-1}|I|) |\Lambda_L|.$$

Theorem 2.2. *Let $E_0 > 0$ and set*

$$\gamma_2 = \frac{1}{2} \delta_-^{N_2} (1 + |E_0|^{2/3} + \|b_0\|_\infty^2 + (\|c_0\|_\infty + \lambda M(2 + \delta_+)^d)^{2/3}) \quad (5)$$

where $N_2 > 0$ is the constant from Theorem 3.3 and b_0, c_0 are as in Eq. (3). Then there is a constant $C_2 = C_2(d, \delta_+, \|V_0\|_\infty)$ such that for any closed interval $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma_2$, any $\lambda > 0$, and any $L \in \mathbb{N}_{\text{odd}}$ with $L \geq 2 + \delta_+$, we have

$$\mathbb{E}(\text{Tr } P_{\omega, L}(I)) \leq C_2 (u_-^{-2} \gamma_2^{-4} (1 + E_0))^{2 + \frac{\log d}{\log 2}} S_L(\lambda^{-1}|I|) |\Lambda_L|.$$

For $t \geq 0$ we define

$$H_0(t) := H_0 + t \sum_{j \in \mathbb{Z}^d} u_j(\cdot - z_j),$$

and set

$$E_0(t) := \inf \sigma(H_0(t)) \quad \text{and} \quad E_0(\infty) := \lim_{t \rightarrow \infty} E_0(t) = \sup \{E_0(t) : t \geq 0\}.$$

Theorem 2.3. *We have $E_0(\infty) > 0$. Let $E_1 \in (0, E_0(\infty))$ and set*

$$\kappa_0 = \sup_{s > 0: E_0(s) \geq E_1} \frac{E_0(s) - E_1}{s} > 0.$$

Then for any Borel set $B \subset (-\infty, E_1]$, any $\lambda > 0$, any $L \in \mathbb{N}_{\text{odd}}$, and almost all $\omega \in \Omega$ we have

$$P_{\omega, L}(B) \left(\sum_{j \in \Lambda_L \cap \mathbb{Z}^d} u_j(\cdot - z_j) \right) P_{\omega, L}(B) \geq \kappa_0 P_{\omega, L}(B). \quad (6)$$

Moreover, for any closed interval $I \subset (-\infty, E_1]$, any $\lambda > 0$, and for any $L \in \mathbb{N}_{\text{odd}}$ with $L \geq 2 + \delta_+$, we have

$$\mathbb{E}(\text{Tr } P_{\omega, L}(I)) \leq C_3 (\kappa_0^{-2}(1 + E_1))^{2^{1 + \frac{\log d}{\log 2}}} S_L(\lambda^{-1}|I|) |\Lambda_L|, \quad (7)$$

where $C_3 > 0$ is a constant depending on d , δ_+ , $\|V_0\|_\infty$, $\|b_0\|_\infty$, and $\|c_0\|_\infty$.

The Wegner estimates from Theorems 2.1, 2.2 and 2.3 can be used as an ingredient for the multiscale analysis [10, 31, 11, 12, 13]. Let us emphasize that the multiscale analysis requires that the concentration functions S_L , $L \in \mathbb{N}$, are sufficiently regular, e.g. with a uniformly bounded density, or uniformly Hölder continuous, cf. the just mentioned references. The multiscale analysis is an induction argument to establish localization in its various manifestations (spectral, dynamical, etc.). While a Wegner estimate is required for the induction step, the so-called initial length scale estimate corresponds to the induction anchor. Hence, if the concentration functions S_L are sufficiently regular, Theorems 2.1, 2.2 and 2.3 will imply localization at energies where an appropriate initial scale estimate is satisfied. If the upper bound in the Wegner estimate becomes small at large disorder, the initial scale estimate will follow from the Wegner estimate at sufficiently large disorder as observed in [31], see also [14].

The upper bounds in Theorems 2.1 and 2.2 grow as the disorder λ increases. As discussed above, this is not sufficient to deduce an initial length-scale estimate and localization at large disorder. In contrast to that, the upper bound in Theorem 2.3 converges to 0 as the disorder parameter λ tends to ∞ . Hence, an initial scale estimate and localization at large disorder follow, albeit only for energies below $E_0(\infty)$. Note that if a covering condition

$$\sum_{j \in \mathbb{Z}^d} u_j(\cdot - z_j) \geq \varepsilon > 0$$

is satisfied, then $E_0(\infty) = \infty$ and the statement of Theorem 2.3 holds at all energies, see [5] in the case of vanishing magnetic field. In contrast, $E(\infty)$ might be finite if we do not

assume a covering condition. Since Wegner estimates with a disorder dependence as in Theorem 2.3 provide a relatively simple path to localization at large disorder, it is natural to ask if such a disorder dependence can be expected at all energies, even if no covering condition is assumed. However, so far one was not able to prove such a Wegner estimate for alloy-type models with and without magnetic field above the threshold $E(\infty)$, cf. [26, 4, 16].

Our next theorem shows that this is indeed not possible. A disorder dependence as in Theorem 2.3 holds if and only if we consider energy intervals below $E_0(\infty)$. In particular this shows that at high energies and at high disorder there is a fundamental difference between alloy-type models with and without a covering condition. This is a new result, even in the special case of vanishing magnetic potential ($A_0 = 0$) and ergodic potential (V_0 periodic, $z_j = j$, $u_j = u_0$, and $\mu_j = \mu_0$).

Theorem 2.4. *Let $E_2 \in \mathbb{R}$. The following are equivalent:*

(i) $E_2 \leq E_0(\infty)$.

(ii) *For all sufficiently large $L > 0$, and all closed intervals $I \subset (-\infty, E_2]$, we have*

$$\mathbb{E}(\mathrm{Tr} P_{\omega, L}(I)) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \infty. \quad (8)$$

For the rest of this section we assume that the family H_ω , $\omega \in \Omega$, is ergodic, i.e. V_0 and A_0 are periodic, and for all $j \in \mathbb{Z}^d$ we have $\mu_j = \mu_0$, $u_j = u_0$ and $z_j = j$. In this situation, Theorem 2.4 has an interpretation in terms of the integrated density of states (IDS). Since the main argument in this case is rather instructive, we present it here: Let us denote by $N_\lambda : \mathbb{R} \rightarrow [0, \infty)$ the IDS of the family H_ω , $\omega \in \Omega$. This is a distribution function satisfying

$$N_\lambda(E) = \lim_{L \rightarrow \infty} \frac{\mathrm{Tr} P_{\omega, L}((-\infty, E])}{|\Lambda_L|}.$$

at all continuity points of N_λ and for almost all $\omega \in \Omega$, cf. [20, 27, 15], see also [30] and the references therein. Note that the Wegner estimates from Theorems 2.1, 2.2 and 2.3 imply local Lipschitz continuity of N_λ at all $E \in \mathbb{R}$ and for all $\lambda \in (0, \infty)$, if the measure μ_0 is sufficiently regular. Moreover, if $\mathcal{U}^\infty := \mathbb{R}^d \setminus \mathrm{supp} \sum_{j \in \mathbb{Z}^d} u(\cdot - j)$ is non-empty we will denote by H_0^∞ the corresponding Dirichlet operator on \mathcal{U}^∞ , i.e. the unique self-adjoint extension of the operator $(-i\nabla + A_0)^2 + V_0$ on $C_0^\infty(\mathcal{U}^\infty) \subset L^2(\mathcal{U}^\infty)$. For its IDS we use the notation $N_{0, \infty}$.

By Floquet theory, cf. [32, 25], we obtain $N_{0, \infty}(E) > 0$ for $E > E_0(\infty)$. Furthermore,

$$N_{0, \infty}(E) \leq N_\lambda(E)$$

for all $\lambda > 0$ and all $E \in \mathbb{R}$. This follows from

$$\mu_k(H_{0, L}^\infty) \geq \mu_k(H_{\omega, L}) \quad (9)$$

where μ_k denotes the k -th eigenvalue of the corresponding operator, ordered increasingly and counting multiplicities and $H_{0, L}^\infty$ is the Dirichlet restriction of H_0^∞ to $L^2(\mathcal{U}^\infty \cap \Lambda_L)$. For the convenience of the reader, we give a proof of Ineq. (9) in the appendix. Together with Theorem 2.3, we found the following theorem.

Theorem 2.5. *Let H_ω , $\omega \in \Omega$, be ergodic. Then $\lim_{\lambda \rightarrow \infty} N_\lambda(E) \geq N_{0,\infty}(E)$ for all $E \in \mathbb{R}$ and it holds that*

- (i) $\lim_{\lambda \rightarrow \infty} N_\lambda(E) = 0$ if $E < E_0(\infty)$,
- (ii) $\lim_{\lambda \rightarrow \infty} N_\lambda(E) > 0$ if $E > E_0(\infty)$.

Remark 2.6. Theorems 2.4 and 2.5 show that the difference between the alloy-type model with and without a covering condition is not merely a technical issue. Rather, we observe that the physical behaviour of the system at large disorder fundamentally differs between the phases $E < E_0(\infty)$ and $E > E_0(\infty)$.

3. Quantitative unique continuation

In [1], the authors prove quantitative unique continuation principles for second order elliptic partial differential expressions with variable coefficients. Here, we formulate the special case where the leading term is the Laplacian. Let

$$\mathcal{H}u := -\Delta + b^T \nabla u + cu$$

with $b \in L^\infty(\mathbb{R}^d; \mathbb{C}^d)$ and $c \in L^\infty(\mathbb{R}^d; \mathbb{C})$. For $L > 0$ we denote by $\mathcal{D}(\Delta_L)$ the domain of the Laplace operator in $L^2(\Lambda_L)$ subject to Dirichlet boundary conditions. For $\Gamma \subset \mathbb{R}^d$ open and $\psi \in L^2(\Gamma)$ we denote by $\|\psi\| = \|\psi\|_\Gamma$ the usual L^2 -norm of ψ . If $\Gamma' \subset \Gamma$ we use the notation $\|\psi\|_{\Gamma'} = \|\chi_{\Gamma'} \psi\|_\Gamma$. The following theorem is a special case of Theorem 12 in [1].

Theorem 3.1. *For all $L \in \mathbb{N}_{\text{odd}}$, all measurable and bounded $V : \Lambda_L \rightarrow \mathbb{R}$, all $\psi \in \mathcal{D}(\Delta_L)$ and $\zeta \in L^2(\Lambda_L)$ satisfying $|\mathcal{H}\psi| \leq |V\psi| + |\zeta|$ almost everywhere on Λ_L , all $\delta \in (0, 1/2)$ and all sequences $X = (x_j)_{j \in \mathbb{Z}^d}$ such that $B(\delta, x_j) \subset \Lambda_1(j)$ for all $j \in \mathbb{Z}^d$, we have*

$$\|\psi\|_{S_{\delta,X}(L)}^2 + \delta^2 G^2 \|\zeta\|_{\Lambda_L}^2 \geq C_{\text{sfuc}} \|\psi\|_{\Lambda_L}^2,$$

where

$$C_{\text{sfuc}} = \delta^{N_1} (1 + \|V\|_\infty^{2/3} + \|b\|_\infty^2 + \|c\|_\infty^{2/3}) \quad \text{and} \quad S_{\delta,X}(L) = \bigcup_{j \in \mathbb{Z}^d} B(\delta, x_j) \cap \Lambda_L. \quad (10)$$

Here $N_1 \geq 1$ is a constant depending only on the dimension.

For $L > 0$ we define the differential operator $H_L : \mathcal{D}(\Delta_L) \rightarrow L^2(\Lambda_L)$ by $H_L \psi = \mathcal{H}\psi$. If

$$b = i\tilde{b} \quad \text{and} \quad c = \tilde{c} + i \operatorname{div} \tilde{b}/2 \quad (11)$$

for some bounded $\tilde{b}, \tilde{c} \in L^\infty(\mathbb{R}^d)$, then H_L is a self-adjoint operator in $L^2(\Lambda_L)$. The following theorem is a special case of Theorems 13 and 14 in [1].

Theorem 3.2. *Let (11) be satisfied. Then for all $L \in \mathbb{N}_{\text{odd}}$, all $E \in \mathbb{R}$, all $\delta \in (0, 1/2)$, all sequences $X = (x_j)_{j \in \mathbb{Z}^d}$ such that $B(\delta, x_j) \subset \Lambda_1(j)$ for all $j \in \mathbb{Z}^d$, and all $\psi \in \operatorname{Ran} \chi_{[E-\gamma, E+\gamma]}(H_L)$ with*

$$\gamma^2 = \delta^{N_2} (1 + |E|^{2/3} + \|b\|_\infty^2 + \|c\|_\infty^{2/3})$$

we have

$$\|\psi\|_{S_{\delta,X}(L)}^2 \geq \gamma^2 \|\psi\|_{\Lambda_L}^2.$$

Here $N_2 \geq 1$ is a constant depending only on the dimension and $S_{\delta,X}(L)$ is as in Eq. (10).

As a corollary we obtain

Theorem 3.3. *Let (11) be satisfied, $E_0 \in \mathbb{R}$, $\delta \in (0, 1/2)$, and*

$$\gamma^2 = \delta^{N_2} (1 + |E_0|^{2/3} + \|b\|_\infty^2 + \|c\|_\infty^{2/3}).$$

Then for all $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma$, and all sequences $X = (x_j)_{j \in \mathbb{Z}^d}$ such that $B(\delta, x_j) \subset \Lambda_1(j)$ for all $j \in \mathbb{Z}^d$, we have

$$\chi_I(H_L) W_{\delta, X}(L) \chi_I(H_L) \geq \gamma^2 \chi_I(H_L).$$

Here, $W_{\delta, X}(L)$ denotes the operator of multiplication with the characteristic function of the set $S_{\delta, X}(L)$ defined in Eq. (10).

4. Proofs

For $L \in \mathbb{N}_{\text{odd}}$, we define $U_L : \Lambda_L \rightarrow \mathbb{R}$ and $W_L : \Lambda_L \rightarrow \mathbb{R}$ by

$$U_L := \sum_{j \in \mathbb{Z}^d \cap \Lambda_L} u_j(\cdot - z_j) \quad \text{and} \quad W_L := \sum_{j \in \mathbb{Z}^d \cap \Lambda_L} \chi_{B(\delta_-, z_j)}.$$

Then $\|U_L\|_\infty \leq (2 + \delta_+)^d$ and $\|V_\omega\|_\infty \leq M \|U_L\|_\infty \leq M(2 + \delta_+)^d$ for almost all $\omega \in \Omega$. Note that

$$0 \leq W_L \leq u_- U_L, \quad W_L^2 = W_L, \quad \text{and} \quad \|W_L\|_\infty = 1.$$

4.1. Proof of Theorem 2.1

Proof of Theorem 2.1. We follow the proof of Theorem 1.4 in [16] and assume $\lambda = 1$. The general case $\lambda > 0$ follows by scaling the random variables $\omega_k \mapsto \lambda \omega_k$ which leads to $M \mapsto \lambda M$ and $S_L(|I|) \mapsto S_L(\lambda^{-1}|I|)$. Rewriting $H_\omega = -\Delta + b_0^T \nabla + c_0 + V_\omega$, where b_0, c_0 are defined in Eq. (3) and given $E_0 > 0$, we define γ_1 as in (4). Then, by Theorem 3.3, for all $L \in \mathbb{N}_{\text{odd}}$ and all intervals $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma_1$ we have

$$\chi_I(H_{0,L}) \leq \gamma_1^{-2} \chi_I(H_{0,L}) W_L \chi_I(H_{0,L}) \leq u_-^{-1} \gamma_1^{-2} \chi_I(H_{0,L}) U_L \chi_I(H_{0,L}).$$

Since $\sigma(H_0) \subset [0, \infty)$, we have $\sigma(H_{0,L}) \subset [0, \infty)$ and may assume $I \subset [0, E_0]$. Similarly to Theorem 1.4 in [16], we can now carefully follow the proof in [7], keeping in mind that the proof therein is formulated for magnetic Schrödinger operators $H_0 = (-i\nabla A_0)^2 + V_0^2$.

Note that the Combes-Thomas estimates (for magnetic Schrödinger operators) required in [7] depend only on d , δ_+ and on $\|V_0\|_\infty$, but not on the magnetic potential A_0 , see Theorem 4.6 of [23]. Hence, the constant C_1 will only depend on A_0 via γ_1 . \square

4.2. Proof of Theorem 2.2

We adapt the proof of [16, Theorem 1.5] to the magnetic setting.

Lemma 4.1. *Let $I \subset (-\infty, E_0]$ be a closed interval and $L \in \mathbb{N}_{\text{odd}}$, $L \geq 2 + \delta_+$. Suppose that there is $\kappa > 0$ such that*

$$P_{\omega,L}(I)U_L P_{\omega,L}(I) \geq \kappa P_{\omega,L}(I) \quad \text{with probability one.}$$

Then there is a constant

$$C_4 = C_4(d, \delta_+, \|V_0\|_\infty)$$

such that

$$\mathbb{E}(\text{Tr } P_{\omega,L}(L)) \leq C_4 (\kappa^{-2}(1 + E_0))^{2^{1+\frac{\log d}{\log 2}}} S_L(\lambda^{-1}|I|)|\Lambda_L|.$$

Proof of Lemma 4.1. One can follow verbatim the proof of Lemma 3.1 in [16] which partially relies on results from [7]. The latter apply to the class of magnetic Schrödinger operators as considered in this note as well. The only issue to address is the dependence of C_4 on the various parameters. In [16], Eqs. (3.8) and (3.17), constants from Combes-Thomas estimates (for non-magnetic Schrödinger operators) which depend only on d , δ_+ and on $\|V_0\|_\infty$ enter the final constant C_4 . Combes-Thomas estimates for magnetic Schrödinger operators do not depend on the magnetic potential A_0 , see Theorem 4.6 of [23]. Therefore the constant C_4 will not depend on the magnetic potential A_0 . \square

Proof of Theorem 2.2. We follow the proof of Theorem 1.5 in [16]. Given $E_0 > 0$, define γ_2 as in Eq. (5). Theorem 3.3 yields for all Λ_L with $L \in \mathbb{N}_{\text{odd}}$, all intervals $I \subset (-\infty, E_0]$ with $|I| \leq 2\gamma_2$ and almost all $\omega \in \Omega$ the estimate

$$\chi_I(H_{0,L}) \leq \gamma_2^{-2} \chi_I(H_{0,L}) W_L \chi_I(H_{0,L}) \leq u_-^{-1} \gamma_2^{-2} \chi_I(H_{0,L}) U_L \chi_I(H_{0,L}).$$

The statement of the theorem now follows from Lemma 4.1. \square

4.3. Proof of Theorem 2.3

For the proof we shall need an abstract uncertainty relation for Schrödinger operators at the bottom of the spectrum which has been developed in [4]. The following lemma is a slight generalization thereof, see Lemma 4.1 of [16].

Lemma 4.2. *Let H be a self-adjoint operator on a Hilbert space \mathcal{H} , bounded from below, and let $Y \geq 0$ be a bounded operator on \mathcal{H} . Let $H(t) = H + tY$ for $t \geq 0$, and set $E(t) = \inf \sigma(H(t))$ and $E(\infty) = \lim_{t \rightarrow \infty} E(t) = \sup_{t \geq 0} E(t)$. Suppose that $E(\infty) > E(0)$. For $E_1 \in (E(0), E(\infty))$ let*

$$\kappa = \kappa(H, Y, E_1) = \sup_{s > 0: E(s) > E_1} \frac{E(s) - E_1}{s} > 0.$$

Then for all bounded operators $V \geq 0$ on \mathcal{H} and Borel sets $B \subset (-\infty, E_1]$ we have

$$\chi_B(H + V)Y\chi_B(H + V) \geq \kappa\chi_B(H + V).$$

We recall that $H_0(t) = H_0 + t \sum_{j \in \mathbb{Z}^d} u_j(\cdot - z_j)$ for $t \geq 0$, $E_0(t) = \inf \sigma(H_0(t))$, and $E_0(\infty) = \lim_{t \rightarrow \infty} E_0(t) = \sup_{t \geq 0} E_0(t)$.

Lemma 4.3. *For all $t \geq 0$ we have*

$$E_0(t) \geq tu_- \delta_-^{N_1} \left(1 + (\|V_0\|_\infty + tu_-)^{2/3} + \|b_0\|_\infty^2 + \|c_0\|_\infty^{2/3}\right).$$

Hence,

$$E_0(\infty) \geq \sup_{t \in [0, \infty)} t \delta_-^{N_1} \left(1 + (\|V_0\|_\infty + t)^{2/3} + \|b_0\|_\infty^2 + \|c_0\|_\infty^{2/3}\right) > 0.$$

Proof. We define $\tilde{H}_0(t) := H_0 + tu_- W$ for $t \geq 0$, $\tilde{E}_0(t) = \inf \sigma(\tilde{H}_0(t))$, and $\tilde{E}_0(\infty) = \lim_{t \rightarrow \infty} \tilde{E}_0(t) = \sup_{t \geq 0} \tilde{E}_0(t)$. By assumption we have $E_0(0) = \tilde{E}_0(0) = 0$. Moreover, $E_0(\infty)$ and $\tilde{E}_0(\infty)$ are both well defined in $[0, \infty]$ by monotonicity. Since $\tilde{E}_0(t) \leq E_0(t)$ for all $t \in [0, \infty]$, it suffices to show the statement of the lemma for $\tilde{E}_0(t)$. For $L \in \mathbb{N}_{\text{odd}}$ we denote by $\tilde{H}_{0,L}(t)$ the restriction of $\tilde{H}_0(t)$ to $L^2(\Lambda_L)$ subject to Dirichlet boundary conditions with domain $\mathcal{D}(\Delta_L)$, and set $\tilde{E}_{0,L}(t) = \inf \sigma(\tilde{H}_{0,L}(t))$. Then $\tilde{E}_{0,L}(t) \geq \tilde{E}_0(t) \geq 0$ for all $t \geq 0$. Since $\|W_L\|_\infty = 1$ we have

$$\tilde{E}_{0,L}(t) \leq d \left(\frac{\pi}{L}\right)^2 + \|V_0\|_\infty + tu_-.$$

Since $\tilde{H}_{0,L}(t)$ has purely discrete spectrum, there exists $\psi(t) \in \mathcal{D}(\Delta_L)$ with $\|\psi(t)\| = 1$ such that $\tilde{H}_{0,L}(t)\psi(t) = \tilde{E}_{0,L}(t)\psi(t)$. We apply Theorem 3.1 and obtain for all $t \geq 0$

$$\begin{aligned} \langle \psi(t), W_L \psi(t) \rangle &= \|\psi\|_{S_{\delta_-, Z}(L)} \\ &\geq \delta_-^{N_1} \left(1 + (d\pi^2/L^2 + \|V_0\|_\infty + tu_-)^{2/3} + \|b_0\|_\infty^2 + \|c_0\|_\infty^{2/3}\right) \|\psi(t)\|_{\Lambda_L}^2 \end{aligned}$$

where

$$S_{\delta_-, Z}(L) = \bigcup_{j \in \mathbb{Z}^d} B(\delta_-, z_j) \cap \Lambda_L.$$

The first statement of the lemma follows since $\lim_{L \rightarrow \infty} \tilde{E}_{0,L}(t) = \tilde{E}_0(t)$. The second statement follows immediately. \square

Proof of Theorem 2.3. By Lemma 4.3 we have $E_0(\infty) > 0$ and hence $\kappa_0 > 0$. For $L \in \mathbb{N}_{\text{odd}}$ we denote by $H_{0,L}(t)$ the restriction of $H_0(t)$ to $L^2(\Lambda_L)$ subject to Dirichlet boundary conditions with domain $\mathcal{D}(\Delta_L)$, and set $E_{0,L}(t) = \inf \sigma(H_{0,L}(t))$. Using $0 \leq E_0(t) \leq E_{0,L}(t)$ we obtain

$$\kappa_{0,L} := \sup_{s > 0: E_{0,L}(s) \geq E_1} \frac{E_{0,L}(s) - E_1}{s} \geq \kappa_0 = \sup_{s > 0: E_0(s) \geq E_0(1)} \frac{E_0(s) - E_1}{s} > 0.$$

Hence, the assumptions of Lemma 4.2 are satisfied with $H = H_{0,L}$ and $Y = \sum_{j \in \mathbb{Z}^d} u_j(\cdot - z_j)$, and we obtain Ineq. (6). Ineq. (7) now follows from Ineq. (6) and Lemma 4.1. \square

4.4. Proof of Theorem 2.4

The implication (i) \Rightarrow (ii) is the statement of Theorem 2.3. In order to show the converse, we prove the contraposition: Let $E_2 > E_0(\infty)$, and $I = (-\infty, E_2]$. Note that for almost all $\omega \in \Omega$, we have $H_{\omega,L} \leq H_{\eta,L}$, where $\eta \in \Omega$, $\eta_k = M$ for all $k \in \mathbb{Z}^d$, hence

$$\mathbb{E}(\mathrm{Tr} P_{\omega,L}(I)) \geq \mathrm{Tr} P_{\eta,L}(I).$$

Since

$$\lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} E_{0,L}(t) < E_2,$$

there are $L_0 > 0$ and $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ we have

$$1 \leq \mathrm{Tr} P_{\eta,L_0}(I) \leq \mathbb{E}(\mathrm{Tr} P_{\omega,L_0}(I)).$$

Hence, (8) cannot hold.

A. Proof of Ineq. (9)

Recall that $H_{0,L}^\infty$ is the Dirichlet restriction of $H_{0,L}$ to $\mathcal{U}^\infty \cap \Lambda_L$, the complement of the support of the single-site potentials in Λ_L .

Lemma A.1. *Let $L > 0$, $\lambda > 0$, and $\mathcal{U}^\infty \cap \Lambda_L \neq \emptyset$. Then for all $\omega \in \Omega$*

$$\mu_k(H_{0,L}^\infty) \geq \mu_k(H_{\omega,L}).$$

Proof. Let $H_{t,L}$ be the operator $H_{\omega,L}$ where all the random variables ω_j are set to t . Since $\mu_k(H_{t,L}) \geq \mu_k(H_{\omega,L})$ for all $t \geq \lambda M$, it suffices to show $\mu_k(H_{0,L}^\infty) \geq \mu_k(H_{t,L})$ for some $t \geq \lambda M$. We define a family of positive and closed quadratic forms $\{\mathcal{E}_t\}_{t>0}$ on $W^{2,1}(\Lambda_L) \subset L^2(\Lambda_L)$ via

$$\mathcal{E}_t(\phi) = \langle (-i\nabla - A_0)\phi, (-i\nabla - A_0)\phi \rangle + \langle \phi, (V_0 + t \sum_j u_j(\cdot - j))\phi \rangle.$$

The form \mathcal{E}_t is the unique quadratic form associated with the self-adjoint operator $H_{t,L}$ and the family $\{\mathcal{E}_t\}_{t>0}$ is monotonously increasing in the sense of quadratic forms. By a version of Kato's monotone convergence theorem for quadratic forms, see [24, Theorem 4.1], there is a closed form \mathcal{E}_∞ which is defined as

$$\begin{aligned} \mathcal{D}(\mathcal{E}_\infty) &= \left\{ \phi \in \mathcal{D}(\mathcal{E}_1) : \sup_{t>0} \mathcal{E}_t(\phi) < \infty \right\}, \\ \mathcal{E}_\infty(\phi) &= \lim_{t \rightarrow \infty} \mathcal{E}_t(\phi) \end{aligned}$$

such that $\mathcal{E}_t \nearrow \mathcal{E}_\infty$ as $t \rightarrow \infty$ in strong resolvent sense.

There is a caveat here concerning the notion of convergence in strong resolvent sense, cf. [24]: Since the form \mathcal{E}_∞ is not densely defined on $L^2(\Lambda)$, one cannot define the corresponding resolvent in the usual manner. However, the form \mathcal{E}_∞ can be restricted to the closed subspace $\overline{\mathcal{D}(\mathcal{E}_\infty)} \subset \Lambda_L$ on which it yields a densely defined, closed form $\tilde{\mathcal{E}}_\infty$. In fact, $\tilde{\mathcal{E}}_\infty$ is the unique form corresponding to $H_{0,L}^\infty$. Let now T be the operator on $L^2(\Lambda_L)$ which

is $(H_{0,L}^\infty + 1)^{-1}$ on $\overline{\mathcal{D}(\mathcal{E}_\infty)}$ and 0 on its orthogonal complement. The nonzero eigenvalues of T are the eigenvalues of $(H_{0,L}^\infty + 1)^{-1}$. Then the notion of strong resolvent convergence of \mathcal{E}_t to \mathcal{E}_∞ means in this situation that $(H_{t,L} + 1)^{-1}\phi \rightarrow T\phi$ for all $\phi \in L^2(\Lambda_L)$.

This implies that the k -th eigenvalue (counted from above) of $(H_{t,L} + 1)^{-1}$ converges from above to the k -th eigenvalue (counted from above) of $(H_{0,L}^\infty + 1)^{-1}$. \square

Acknowledgments

It is our privilege and honor to thank our mentor Ivan Veselić for the idea to pursue this research direction. Furthermore, we gratefully acknowledge helpful discussions with Peter Stollmann.

References

- [1] D. I. Borisov, M. Tautenhahn, and I. Veselić, *Scale-free quantitative unique continuation and equidistribution estimates for solutions of elliptic differential equations*, to appear in J. Math. Phys, arXiv:1512.06347 [math.AP].
- [2] J. Bourgain and C. E. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math. **161** (2005), no. 2, 389–426.
- [3] J. Bourgain and A. Klein, *Bounds on the density of states for Schrödinger operators*, Invent. Math. **194** (2013), no. 1, 41–72.
- [4] A. Boutet de Monvel, D. Lenz, and P. Stollmann, *An uncertainty principle, Wegner estimates and localization near fluctuation boundaries*, Math. Z. **269** (2011), no. 3, 663–670.
- [5] J.-M. Combes and P. D. Hislop, *Localization for some continuous, random Hamiltonians in d -dimensions*, J. Funct. Anal. **124** (1994), no. 1, 149–180.
- [6] J.-M. Combes, P. D. Hislop, and F. Klopp, *Hölder continuity of the integrated density of states for some random operators at all energies*, Int. Math. Res. Not. **2003** (2003), no. 4, 179–209.
- [7] J.-M. Combes, P. D. Hislop, and F. Klopp, , *An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators*, Duke Math. J. **140** (2007), no. 3, 469–498.
- [8] J.-M. Combes, P. D. Hislop, F. Klopp, and G. Raikov, *Global continuity of the integrated density of states for random Landau Hamiltonians*, Commun. Part. Diff. Eq. **29** (2004), no. 7-8, 1187–1213.
- [9] J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, *Constructive proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **101** (1985), no. 1, 21–46.
- [10] J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Commun. Math. Phys. **88** (1983), no. 2, 151–184.

- [11] F. Germinet and A. Klein, *Bootstrap multiscale analysis and localization in random media*, Commun. Math. Phys. **222** (2001), no. 2, 415–448.
- [12] F. Germinet and A. Klein, *Explicit finite volume criteria for localization in continuous random media and applications*, Geom. Funct. Anal. **13** (2003), no. 6, 1201–1238.
- [13] F. Germinet and A. Klein, *New characterizations of the region of complete localization for random Schrödinger operators*, J. Stat. Phys. **122** (2006), no. 1, 73–94.
- [14] W. Kirsch, *An invitation to random Schrödinger operators*, Random Schrödinger operators, Panor. Synthèses, vol. 25, Soc. Math. France, Paris, 2008, with an appendix by Frédéric Klopp, pp. 1–119.
- [15] W. Kirsch and F. Martinelli, *On the density of states of Schrödinger operators with a random potential*, J. Phys. A: Math. Gen. **15** (1982), no. 7, 2139–2156.
- [16] A. Klein, *Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators*, Commun. Math. Phys. **323** (2013), no. 3, 1229–1246.
- [17] A. Krakovsky, *Electronic band structure in a periodic magnetic field*, Phys. Rev. B **53** (1996), no. 13, 8469–8472.
- [18] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić, *Scale-free uncertainty principles and Wegner estimates for random breather potentials*, C. R. Math. **353** (2015), no. 10, 919–923.
- [19] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić, *Scale-free unique continuation principle, eigenvalue lifting and Wegner estimates for random Schrödinger operators*, to appear in Anal. PDE, arXiv:1609.01953 [math.AP].
- [20] L. A. Pastur, *The Schrödinger equation with random potential*, Teoret. Mat. Fiz. **6** (1971), no. 3, 415–424.
- [21] C. Rojas-Molina, *Characterization of the Anderson metal-insulator transition for non ergodic operators and application*, Ann. Henri Poincaré **13** (2012), no. 7, 1575–1611.
- [22] C. Rojas-Molina and I. Veselić, *Scale-free unique continuation estimates and applications to random Schrödinger operators*, Commun. Math. Phys. **320** (2013), no. 1, 245–274.
- [23] Z. Shen, *An improved Combes-Thomas estimate of magnetic Schrödinger operators*, Ark. Mat. **52** (2014), no. 2, 383–414.
- [24] B. Simon, *A canonical decomposition for quadratic forms with applications to monotone convergence theorems*, J. Funct. Anal. **28** (1978), no. 3, 377–385.
- [25] J. Sjöstrand, *Microlocal analysis for the periodic magnetic Schrödinger equation and related questions*, Microlocal analysis and applications (L. Cattabriga and L. G. Rodino, eds.), Lecture Notes in Mathematics, vol. 1495, Springer, Berlin, 1991, pp. 237–332.

- [26] P. Stollmann, *From uncertainty principles to Wegner estimates*, Math. Phys. Anal. Geom. **13** (2010), no. 2, 145–157.
- [27] M. A. Subin, *Spectral theory and the index of elliptic operators with almost-periodic coefficients*, Uspekhi Mat. Nauk **34** (1979), no. 2, 95–135.
- [28] M. Täufer and I. Veselić, *Conditional Wegner estimate for the standard random breather potential*, J. Stat. Phys. **161** (2015), no. 4, 902–914.
- [29] M. Täufer and I. Veselić, *Wegner estimate for Landau-breather Hamiltonians*, J. Math. Phys. **57** (2016), no. 7, 072102 (8 pages).
- [30] I. Veselić, *Existence and regularity properties of the integrated density of states of random Schrödinger operators*, Lecture Notes in Mathematics, vol. 1917, Springer, Berlin, 2008.
- [31] H. von Dreifus and A. Klein, *A new proof of localization in the Anderson tight binding model*, Commun. Math. Phys. **124** (1989), no. 2, 285–299.
- [32] J. Zak, *Magnetic translation group*, Phys. Rev. **134** (1964), no. 6A, A1602–A1606.