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AN EXPLICIT CONDUCTOR FORMULA FOR $\mathbf{GL}_n \times \mathbf{GL}_1$

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ABSTRACT. We prove an explicit formula for the conductor of an irreducible, admissible representation of $\mathrm{GL}_n(F)$ twisted by a character of $F^\times$ where the field F is local and non-archimedean. As a consequence, we quantify the number of character twists of such a representation of fixed conductor.

1. The problem of the twisted conductor. Let F denote a non-archimedean local field of characteristic zero, and let $n \geq 2$. For an irreducible, admissible representation π of $\mathrm{GL}_n(F)$ and a quasi-character χ of $F^\times$, we can form the twist $\chi\pi = (\chi \circ \det) \otimes \pi$. Our main result, Theorem 2.6, is an explicit formula for the conductor $a(\chi\pi)$, equal to the Artin conductor, as defined in Section 3.1. This formula is given by

$$a(\chi\pi) = a(\pi) + \Delta_\chi(\pi) - \delta_\chi(\pi), \tag{1.1}$$

where $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$ are non-negative integers as defined in Theorem 2.6; they denote a dominant and a non-twist-minimal interference term, respectively. We give a detailed analysis of these terms in Section 4.2, answering questions such as, "for what number of χ is there interference present?"

As an example, computing $a(\chi\pi)$ in the limit $a(\chi) \to \infty$ is straightforward: from Proposition 2.2 and equation (2.2), we deduce that

$$a(\chi\pi) = na(\chi), \tag{1.2}$$

whenever $a(\chi) > a(\pi)$. In this case, $\Delta_\chi(\pi) = na(\chi) - a(\pi)$ and $\delta_\chi(\pi) = 0$. Bushnell and Henniart **[2]** extend (1.2) by proving the upper bound[1]

$$a(\chi\pi) \leq \max\{a(\pi),\, a(\chi)\} + (n-1)a(\chi), \tag{1.3}$$

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surrendering to a weaker bound in the region $0 \le a(\chi) \le a(\pi)$. Nevertheless, this bound is sharp in that it is attained for *some* π and χ, as in (1.2), for example.

However, in general, such examples become sparse, rendering (1.3) as rather coarse as one averages over χ with $a(\chi) \asymp a(\pi)$. In such cases, evaluating the integers $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$ exactly is of crucial importance for numerous problems in analytic number theory.

In this paper, we consider applications to studying $a(\chi\pi)$ in a quantitative fashion. For example, we count the number of χ for which $a(\chi\pi)$ is equal to a given integer (see Section 4). Such an analysis would most commonly be applied when considering $a(\chi\pi)$ on average.

Our formula may be utilized when studying the analytic behavior of automorphic L-functions. In particular, it is applicable in conjunction with the following two techniques: taking harmonic GL_1-averages and applying the functional equation for $\mathrm{GL}_n \times \mathrm{GL}_1$-$L$-functions. For example, conductors of such character twists arise in the work of Nelson, Pitale and Saha [**13**], who address the quantum unique ergodicity conjecture for holomorphic cusp forms with "powerful" level (see [**13,** Remarks 1.9, 3.16]). The current record for upper and lower bounds for the sup-norm of a Maaß-newform on GL_2 in the level aspect [**17, 18, 19**] also depends crucially on the $n = 2$ case of Theorem 2.6.

An instance where (1.1) is applied constructively is carried out in [**4**], once again, when $n = 2$. Originally, in [**1**], Brunault computed the value of ramification indices of modular parameterization maps of various elliptic curves over $\mathbb{Q}$. Whenever the newform attached to E is "twist minimal," Brunault could prove that this index was trivial (equal to 1), holding, in particular, whenever the conductor of E is square-free. This problem has now been completely solved by Saha and Corbett [**4**]. In our solution, it is the degenerate cases of (1.1), with non-trivial $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$, that give rise to the few examples of non-trivial ramification indices.

These results all concern the case $n = 2$, where the conductor formula for twists of supercuspidal representations was given by Tunnell [**21,** Proposition 3.4] in his thesis (see [**4,** Lemma 2.7] for the general case). Tunnell himself applied his formula to count isomorphism classes of supercuspidal representations of fixed odd conductor [**21,** Theorem 3.9]. He used this observation in his proof of the local

Langlands correspondence for $\mathrm{GL}_2(F)$ in the majority of cases.

Our present result is suggestive of similar applications: a bound for local Whittaker newforms (and a corresponding global sup-norm bound) in the level aspect; bounds for matrix coefficients of local representations, and estimates relating to the Voronoĭ summation problem for GL_n, to name a few.

In Section 2, we describe how irreducible, admissible representations of $\mathrm{GL}_n(F)$ are classified and then go on to give a full account of our main result. This classification assumes the least amount of necessary information in order to give a completely explicit formula. In Section 3, we give a uniform proof of our main result for the quasi-square-integrable representations (see Proposition 2.2); these representations are used as building blocks to arrive at the general case. Lastly, in Section 4, we provide a detailed analysis of the terms $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$ as found in (1.1).

2. An explicit formula for twisted conductors. Here, we give full details of the formula proposed in (1.1). We first describe the formula for quasi-square-integrable representations of $\mathrm{GL}_n(F)$, which is then used to build the result in its full generality.

2.1. The Langlands classification for $\mathrm{GL}_n(F)$. Let $\mathscr{A}_F(n)$ denote the set of (equivalence classes of) irreducible, admissible representations of $\mathrm{GL}_n(F)$. The natural building blocks that describe $\mathscr{A}_F(n)$ are the *quasi-square-integrable* representations; these are the $\pi \in \mathscr{A}_F(n)$ for which there exists an $\alpha \in \mathbb{R}$ such that $|\cdot|^\alpha \pi$ has square-integrable matrix coefficients on $\mathrm{GL}_n(F)$ modulo its center.

The 'Langlands classification' (due to Berstein and Zelevinsky, in this case) describes the structure of each representation in the graded ring

$$\mathscr{A}_F = \bigoplus_{n \geq 1} \mathscr{A}_F(n)$$

in terms of the subset $\mathscr{S}\mathscr{G}_F$ of quasi-square-integrable representations. By [**24,** Theorems 9.3, 9.7], we deduce an addition law $\boxplus$ on $\mathscr{S}\mathscr{G}_F$, by which $\mathscr{S}\mathscr{G}_F$ generates a free commutative monoid Λ. The classification is then the assertion that there is a bijection between $\mathscr{A}_F$ and the semi-group of non-identity elements in Λ, thus endowing $\mathscr{A}_F$ with the addition law $\boxplus$. Crucially, the maps $(\mathscr{A}_F, \boxplus) \to (\mathbb{C}, \ \cdot\ )$, given by applying L- or ε-factors, are homomorphisms of semi-groups (see **[22,** Section 2.5] for their definitions). Both expositions **[14, 22]** provide excellent background on this topic.

The upshot of this classification is that, for any $\pi \in \mathscr{A}_F(n)$, there exists a unique partition $n_1 + \cdots + n_r = n$ alongside a collection of quasi-square-integrable representations $\pi_i \in \mathscr{S}\mathscr{G}_F \cap \mathscr{A}_F(n_i)$ for $1 \leq i \leq r$ such that

$$\pi = \pi_1 \boxplus \cdots \boxplus \pi_r \tag{2.1}$$

and, for any quasi-character χ of $F^\times$, we have

$$a(\chi\pi) = a(\chi\pi_1) + \cdots + a(\chi\pi_r). \tag{2.2}$$

Equation (2.2) follows from the definition of the conductor $a(\pi)$ via the ε-factor in (3.2). Recall, too, that, for a quasi-character χ of $F^\times$, the conductor $a(\chi)$ is defined to be the least non-negative integer such that $\chi(\mathfrak{o}^\times \cap (1 + \mathfrak{p}^{a(\chi)})) = \{1\}$, where $\mathfrak{o}$ is the ring of integers of F and $\mathfrak{p} \subset \mathfrak{o}$ the unique maximal ideal.

2.2. The formula for quasi-square-integrable representations.

Definition 2.1. An irreducible, admissible representation π of $\mathrm{GL}_n(F)$ is called *twist minimal* if $a(\pi)$ is the smallest of the integers $a(\chi\pi)$ as χ varies over the quasi-characters of $F^\times$.

Recall that, for a quasi-character χ of $F^\times$, define its conductor $a(\chi)$ to be the least non-negative integer such that $\chi(U_F(a(\chi))) = \{1\}$. For quasi-square-integrable representations, the notion of twist-minimality is sufficient to give an exact formula for the conductor of their twist.

Proposition 2.2. *Let π be an irreducible, admissible, quasi-square-integrable representation of* $\mathrm{GL}_n(F)$*, and let χ be a quasi-character of $F^\times$. Then*:

$$a(\chi\pi) \leq \max\{a(\pi), na(\chi)\} \tag{2.3}$$

with equality in (2.3)*, whenever π is twist minimal or $a(\pi) \neq na(\chi)$.*

We defer our proof of Proposition 2.2 until Section 3.4.

Remark 2.3. In practice, those $\pi \in \mathscr{S}\mathscr{G}_F \cap \mathscr{A}_F(n)$ which are *not* twist minimal may be handled as follows. Tautologically, write $\pi = \mu\pi^{\min}$, where μ is a quasi-character of $F^\times$ and $\pi^{\min}$ *is* twist minimal. Then, Proposition 2.2 implies that

$$a(\chi\pi) = \max\{a(\pi^{\min}),\, na(\chi\mu)\}.$$

In particular, if $a(\pi^{\min}) < a(\pi)$, then $n \mid a(\pi)$.

We briefly mention the conductor formula of Bushnell, Henniart and Kutzko [**3,** Theorem 6.5] for $\mathrm{GL}_n \times \mathrm{GL}_m$-pairs of *supercuspidal* representations. There, they deploy the full structure theory of supercuspidal representations to prove a detailed identity relating the conductor to the respective inducing data of the given supercuspidal representations. However, this formula is difficult to apply in practice. Indeed, our own Proposition 2.2 may be derived from their work. Comparing the $m = 1$ case of [**3**] to our present result, our formula is simpler and uniformly holds on the larger set $\mathscr{S}\mathscr{G}_F$. This set contains not only the supercuspidal representations, but also, for example, the special representations, for which Proposition 2.2 recovers the known formula of Rohrlich [**16,** page 18]. Accordingly, we give an elementary proof of Proposition 2.2. This promotes our observation that the subset of twist minimal elements in $\mathscr{S}\mathscr{G}_F$ contains sufficient and necessary information to explicitly determine the conductor of any twist.

The arguments of Section 3.4 also lead to a proof of the following result on the central character.

Proposition 2.4. *Let π be an irreducible, admissible, quasi-square-integrable representation of $\mathrm{GL}_n(F)$ with central character ω_π. Then*

$$a(\omega_\pi) \le \frac{a(\pi)}{n}. \tag{2.4}$$

Remark 2.5. The central character of a quasi-square-integrable representation has a relatively small conductor. In general, highly ramified central characters arise due to the components in a given $\pi_1 \boxplus \cdots \boxplus \pi_r$ for $r \ge 2$. For this reason, such representations should be handled separately, as is distinguished in this work.

2.3. The general formula. We arrive at our main result, having defined the necessary set of properties of the representations in $\mathscr{A}_F$ in order to give a complete and explicit formula for the conductor of their twists.

Theorem 2.6. *Let π be an irreducible, admissible representation of $\mathrm{GL}_n(F)$ given in terms of quasi-square-integrable representations π_i of $\mathrm{GL}_{n_i}(F)$, as described in* (2.1)*, where $n = n_1 + \cdots + n_r$ and $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$. Let χ be a quasi-character of $F^\times$. Then:*

$$a(\chi\pi) = a(\pi) + \Delta_\chi(\pi) - \delta_\chi(\pi),$$

where Δ_χ and δ_χ are semi-group homomorphisms $(\mathscr{A}_F, \boxplus) \to (\mathbb{Z}_{\geq 0}, +)$, defined by their values on the representations $\pi_i \in \mathscr{S}\mathscr{G}_F$ as follows:

$$\Delta_\chi(\pi_i) = \begin{cases} \max\{n_i a(\chi) - a(\pi_i),\, 0\} & \text{if } a(\chi) \neq a(\mu_i) \\ 0 & \text{if } a(\chi) = a(\mu_i), \end{cases}$$

and

$$\delta_\chi(\pi_i) = \begin{cases} a(\pi_i) - \max\{a(\pi_i^{\min}),\, n_i a(\chi\mu_i)\} & \text{if } a(\chi) = a(\mu_i) \\ 0 & \text{if } a(\chi) \neq a(\mu_i), \end{cases}$$

where $\pi_i^{\min}$ is twist minimal and μ_i a quasi-character of $F^\times$ such that we may write $\pi_i = \mu_i \pi_i^{\min}$.

Remark 2.7. As exhibited in the following proof, both terms $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$ are non-negative for any choice of π and χ.

Proof. Applying Proposition 2.2 to the formula in (2.2), we obtain

$$a(\chi\pi) = \sum_{i=1}^{r} \max\{a(\pi_i^{\min}),\, n_i a(\chi\mu_i)\}. \tag{2.5}$$

We now use the basic fact that, for two quasi-characters, μ and χ of $F^\times$, we have

$$a(\chi\mu) \leq \max\{a(\chi), a(\mu)\} \tag{2.6}$$

with equality in (2.6) whenever $a(\chi) \neq a(\mu)$. In particular, if $a(\chi) \neq a(\mu_i)$ for a given $1 \leq i \leq r$, then, by Proposition 2.2 and (2.6), the respective summand in (2.5) is equal to

$$\max\{a(\pi_i^{\min}),\, n_i a(\chi\mu_i)\} = \max\{a(\pi_i),\, n_i a(\chi)\}.$$

This determines the dominant term $\Delta_\chi(\pi_i)$, which is non-negative by construction. The interference term $\delta_\chi(\pi_i)$ describes the cases for which $a(\chi) = a(\mu_i)$, when the assertion that $\delta_\chi(\pi_i) \geq 0$ follows from the inequality $a(\pi_i) \geq \max\{a(\pi_i^{\min}),\, n_i a(\chi\mu_i)\}$. □

Remark 2.8. In the special case $n = 2$, we prove Theorem 2.6 [**4,** Lemma 2.7]. In general, one should understand the non-vanishing of $\delta_\chi(\pi)$ as rarely occurring, whereas $\Delta_\chi(\pi)$ describes the dominant or "usual" behavior of $a(\chi\pi)$. We make these statements explicit in a quantitative sense in subsection 4.2.

Corollary 2.9. *Let* $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ *and* χ *be as in Theorem* 2.6 *with* $\pi_i = \mu_i \pi_i^{\min}$ *for twist minimal representations* $\pi_i^{\min}$. *Define the 'totally minimal' representation* $\pi^{\mathrm{tot}} = \pi_1^{\min} \boxplus \cdots \boxplus \pi_r^{\min}$, *and let* $\Omega_\chi(\pi) = \{1 \leq i \leq r : a(\pi_i) > n_i a(\chi)\}$. *Then:*

$$a(\pi^{\mathrm{tot}}) \leq a(\chi\pi) \leq a(\pi) + a(\chi)\Bigg(n - \sum_{i \in \Omega_\chi(\pi)} n_i\Bigg). \tag{2.7}$$

Proof. The lower bound of (2.7) follows immediately from (2.2) and (2.5). On the other hand, for $i \in \Omega_\chi(\pi)$, we have $\Delta_\chi(\pi_i) = \delta_\chi(\pi_i) = 0$, by definition, noting that $\pi_i = \pi_i^{\min}$ in the case $a(\chi) = a(\mu_i)$. The upper bound now follows from using Proposition 2.2 to coarsely bound $a(\chi\pi_i) \leq a(\pi_i) + n_i a(\chi)$ for $i \notin \Omega_\chi(\pi)$. □

Proof of inequality (1.3). We recover Bushnell and Henniart's bound (1.3) using Corollary 2.9. If $a(\chi) > a(\pi)$, then $a(\chi\pi) = n a(\chi)$ by (2.5). On the other hand, if $a(\chi) \leq a(\pi)$, then (1.3) is a special case of (2.7) since we have $\Omega_\chi(\pi) \neq \varnothing$ and each $n_i \geq 1$. □

3. Conductors of twists via division algebras. In this section, we provide proofs for Propositions 2.2 and 2.4. These results uniformly apply to all quasi-square-integrable representations as is reflected in our proof. In particular, our conductor formula bypasses many of the complications occurring in the formula for supercuspidal representations given in [**3**].

3.1. Notation and definition of the conductor. Let π denote an irreducible, admissible representation of $\mathrm{GL}_n(F)$. Denote by $\widetilde{\pi}$ the contragredient representation and ω_π the central character of π, respectively.

3.1.1. The non-archimedean local field. We denote by $\mathfrak{o}$ the ring of integers of F; $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$; ϖ a choice of uniformizing parameter, that is, a generator of $\mathfrak{p}$; and $q = \#(\mathfrak{o}/\mathfrak{p})$. Let $|x|$ denote the absolute value of $x \in F$, normalized so that $|\varpi| = q^{-1}$ and v_F the valuation on F defined via $|x| = q^{-v_F(x)}$. We define a basis of open neighborhoods $U_F(m)$ of 1 in $U_F(0) = \mathfrak{o}^\times$ by $U_F(m) = 1 + \varpi^m\mathfrak{o}$ for $m > 0$. Let $K = \mathrm{GL}_n(\mathfrak{o})$ and, for each $m \geq 0$, let $K_1(m)$ be the subgroup of K stabilizing the row vector $(0, \ldots, 0, 1)$, from the right, modulo $\mathfrak{p}^m$.

3.1.2. The floor and ceiling functions. For $\alpha \in \mathbb{R}$, let $\lfloor \alpha \rfloor$ denote the *floor* of α, defined via $\lfloor \alpha \rfloor = m$ if and only if $m \in \mathbb{Z}$ and $m \leq \alpha < m+1$. Similarly, let $\lceil \alpha \rceil$ denote the *ceiling* of α, defined via $\lceil \alpha \rceil = m'$ if and only if $m' \in \mathbb{Z}$ and $m' - 1 < \alpha \leq m'$. Then, $\lfloor \alpha \rfloor = \lceil \alpha \rceil$ if and only if $\alpha \in \mathbb{Z}$.

3.1.3. Epsilon constants and the conductor. Here, we define the integer $a(\pi)$ as the conductor of π. Let ψ be an additive character of F, and define the exponent of ψ by $n(\psi) := \min\{m : \psi|_{\mathfrak{p}^m} = 1\}$. Godement and Jacquet proved the existence of ε-factors $\varepsilon(s, \pi, \psi) \in \mathbb{C}[q^{-s}, q^s]$ in **[7,** Theorem 3.3 (4)]. Applying the local functional equation of Godement and Jacquet twice, we obtain

$$\varepsilon(s, \pi, \psi)\varepsilon(1 - s, \widetilde{\pi}, \psi) = \omega_\pi(-1). \tag{3.1}$$

Hence, $\varepsilon(s, \pi, \psi)$ is a unit in $\mathbb{C}[q^{-s}, q^s]$, that is, a $\mathbb{C}^\times$-constant multiple of an integral power of q^{-s}. Explicitly, using **[7,** (3.3.5)] we deduce

$$\varepsilon(s, \pi, \psi) = \varepsilon(1/2, \pi, \psi)\, q^{(a(\pi) - n(\psi)n)(1/2 - s)}, \tag{3.2}$$

in which the conductor $a(\pi)$ is implicitly defined. By the local Langlands correspondence for $\mathrm{GL}_n(F)$, proven in **[8]**, the conductor $a(\pi)$ coincides with the Artin conductor of an n-dimensional Weil-Deligne representation. A fundamental property of ε-factors is that

$$\varepsilon(s, \chi\pi, \psi) = \prod_{i=1}^{r} \varepsilon(s, \chi\pi_i, \psi) \quad \text{for } \pi = \pi_1 \boxplus \cdots \boxplus \pi_r,$$

as in (2.1) (see [**7,** Theorem 3.4]). This observation proves (2.2) by applying (3.2). Moreover, if π is generic, the conductor $a(\pi)$ may be interpreted in terms of newform theory, as we now explain.

3.1.4. Conductors of generic representations and newform theory. Each representation in $\mathscr{S}\mathscr{G}_F$ is *generic*. Indeed, by showing so for the regular representation of $\mathrm{GL}_n(F)$ of the fixed central character, Jacquet showed that all discrete series representations are generic [**9,** Theorem 2.1 (3)]. By the Langlands classification, any $\pi \in \mathscr{A}_F(n)$ is generic (or "non-degenerate") if and only if π is equivalent to the (irreducible) representation parabolically induced from the external tensor product $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ of $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$ associated to $n_1 + \cdots + n_r$ (by [**24,** Theorem 9.7 (a)]). The elements of $\mathscr{S}\mathscr{G}_F$ corres-pond to those irreducible representations with $r = 1$.

Assume that $\pi \in \mathscr{A}_F(n)$ is generic. Then, the conductor $a(\pi)$ may be equivalently constructed in a language more familiar to the theory of automorphic forms: let us re-define the conductor $a(\pi)$ of π to be the least non-negative integer m such that π contains a non-zero $K_1(m)$-fixed vector.

The fundamental theorem of newform theory is that the space of $K_1(a(\pi))$-fixed vectors is one dimensional. This theorem is due to Gelfand and Každan [**6**] in the present context. The coincidence of the definitions for $a(\pi)$, given in subsections 3.1.3 and 3.1.4, is proven by Jacquet, Piatetski-Shapiro and Shalika [**11,** Theorem (5)].

3.2. Central simple division algebras. Let D be a division algebra over F of dimension $[D:F] = n^2$. Let $\mathrm{Nrd} = \mathrm{Nrd}_D$ denote the reduced norm on D. (See [**12,** subsection 4.1] for a pleasant construction.) Any valuation on D may be obtained via composing the reduced norm with a valuation on F (see [**20,** Theorem 1.4]); let us normalize such a choice by $v_D = v_F \circ \mathrm{Nrd}$.

3.2.1. Unit groups. Define a basis of neighborhoods of $1 \in D^\times$ by $U_D(m) = \{x \in D^\times : v_D(x-1) \geq m\}$ for $m > 0$, and let $U_D(0) = \ker(v_D)$. Note that, if $n = 1$ (so that $D = F$), we recover $U_D(m) = U_F(m)$. It is an important fact that the norm map

$$\mathrm{Nrd}\colon D^\times \longrightarrow F^\times$$

is surjective (see **[23,** page 195, Proposition 6] for instance). Upon restriction to the above neighborhoods, for each $m \geq 0$, we have $\mathrm{Nrd}(U_D(m)) = U_D(m) \cap F$.

Lemma 3.1. *For $m \geq 0$, we have the following*:

(i) $U_D(m) \cap F^\times = U_F(\lceil m/n \rceil)$;
(ii) $\mathrm{Nrd}(U_D(m)) = U_F(\lceil m/n \rceil)$.

Proof. To prove (i), note that, for all $a \in F^\times$, we have

$$v_D(a) = v_F(\mathrm{Nrd}(a)) = v_F(a^n) = n v_F(a).$$

The definition of $U_F(\lceil m/n \rceil)$ is then equivalent to that of the intersection. Now (ii) follows by applying (i) to $\mathrm{Nrd}(U_D(m)) = U_D(m) \cap F$. $\square$

3.2.2. The level of a representation of $D^\times$. If χ is a quasi-character of $F^\times$ and π' an irreducible, admissible representation of $D^\times$, analogous to the unramified case, we form the twist $\chi\pi' = (\chi \circ \mathrm{Nrd}) \otimes \pi'$. Define the *level* $l(\pi')$ of π' to be the least non-negative integer m such that $\pi'|_{U_D(m)}$ acts trivially. The notion of an ε-factor, as well as a conductor $a(\pi')$, is defined by Godement and Jacquet **[7]**, mutatis mutandis as in subsection 3.1.3.

Lemma 3.2. *Let π' be an irreducible, admissible representation of $D^\times$. The conductor $a(\pi')$ is related to the level $l(\pi)$ by the formula*

$$a(\pi') = l(\pi') + n - 1.$$

Proof. This is proven in **[12,** subsection 4.3] and explicitly stated in **[12,** (4.3.4)]. To assist with (mathematical) translation, we remark on the following: their unit groups V_j equal our $U_D(j)$ for $j \geq 0$. Fix their element $\chi \in \mathrm{Hom}(V_j/V_{j+1}, \mathbb{C}^\times)$ to be the restriction of π' to V_j where $j = l(\pi') - 1$. Then, their $c \in D$, "der Kontrolleur von χ," satisfies $v_D(c) = -a(\chi) = -a(\pi')$; it is constructed in **[12,** (4.3.1)], from where we have $v_D(c) = -n - j$, noting the non-triviality of χ on V_j. Altogether, this implies $a(\pi') = n + j = n + l(\pi') - 1$. $\square$

Lemma 3.3. *Let χ be a quasi-character of $F^\times$. Then*:

$$l(\chi \circ \mathrm{Nrd}) = n a(\chi) - n + 1.$$

Proof. By Lemma 3.1 (ii), consider χ restricted to $U_F(\lceil m/n \rceil)$ for each $m \geq 0$ as this set is equal to the image of $U_D(m)$ under Nrd. By the minimality of $a(\chi)$, the character $\chi \circ \mathrm{Nrd}$ is trivial on $U_D(m)$ whenever

$$n(a(\chi)-1) \leq m-1. \tag{3.3}$$

By the minimality of the level, we have equality in (3.3) when $m = l(\chi \circ \mathrm{Nrd})$. □

3.3. The Jacquet-Langlands correspondence for division algebras. This special case of functoriality stipulates a bijection between the following:

- the set of equivalence classes of irreducible, admissible representations of $\mathrm{GL}_n(F)$, with unitary central character, which are *square-integrable modulo center*. These are precisely the square-integrable elements of $\mathscr{SG}_F \cap \mathscr{A}_F(n)$.

- The set of equivalence classes of irreducible, admissible representations of $D^\times$ with unitary central character, where D is a central-simple F-algebra of dimension n^2.

Remark 3.4. In the above bijection, if π corresponds to π', then their central characters agree: $\omega_\pi = \omega_{\pi'}$. Moreover, $\chi\pi$ corresponds to $\chi\pi'$ for any quasi-character χ. As a consequence of the Peter-Weyl theorem, the irreducible representations of $D^\times$ are finite dimensional (since $D^\times$ is compact modulo center).

The correspondence as stated here is due to Rogawski [**15,** Theorem 5.8], where the original case $n = 2$ was famously proven by Jacquet and Langlands [**10**]. The most general statement allows one to replace $D^\times$ with $\mathrm{GL}_m(D)$, where D has dimension d^2 and m must satisfy $n = md$. This is established in [**5**] by Deligne, Kazhdan and Vignéras.

3.4. The main proofs. Here, we provide a stand-alone proof of Proposition 2.2, our main result in the quasi-square integrable case. Assume the hypotheses and notation of Propositions 2.2 and 2.4, in particular, $\pi \in \mathscr{SG}_F$.

3.4.1. Proof of Proposition 2.2. The following lemma reduces the proof to the case where π is square-integrable.

Lemma 3.5. *For all quasi-characters χ with $a(\chi) = 0$, we have $a(\chi\pi) = a(\pi)$.*

Proof. Let $m \geq 0$. The space $\pi^{K_1(m)}$ of $K_1(m)$-fixed vectors in π is non-zero if and only if $(\chi\pi)^{K_1(m)} \neq \{0\}$. Since $\pi \in \mathscr{S}\mathscr{G}_F$, both π and $\chi\pi$ are generic, and so, $a(\pi) = \min\{m \geq 0 : \pi^{K_1(m)} \neq 0\} = a(\chi\pi)$. □

Henceforth, we assume π to be square-integrable. The generalized Jacquet-Langlands correspondence implies $a(\chi\pi) = a(\chi\pi')$, where π' is the irreducible, admissible, unitary representation of $D^\times$ associated to π as determined by [**15,** Theorem 5.8]. The proof of Proposition 2.2 now follows by applying Lemmas 3.2 and 3.3 to the following.

Lemma 3.6. *Let π' be an irreducible, admissible, unitary representation of $D^\times$ and χ a quasi-character of $F^\times$. Then*:

$$l(\chi\pi') \leq \max\{l(\pi'), l(\chi \circ \mathrm{Nrd})\} \tag{3.4}$$

with equality in (3.4) *whenever π' is twist minimal or $l(\pi') \neq l(\chi \circ \mathrm{Nrd})$.*

Proof. By definition, $(\chi\pi')(x) = \chi(\mathrm{Nrd}(x))\pi'(x)$ for every $x \in D^\times$. We immediately obtain (3.4) by minimality. Equality also follows in the given cases, noting that twist minimality in $a(\pi')$ is equivalent to twist minimality in $l(\pi')$ since they are linearly related (by Lemma 3.2). □

3.4.2. Proof of Proposition 2.4. Taking $m = l(\pi')$ in Lemma 3.1 (i) and using the formula of Lemma 3.2, we deduce that

$$a(\omega_\pi) \leq \left\lceil \frac{l(\pi')}{n} \right\rceil < \frac{a(\pi) - n + 1}{n} + 1 = \frac{a(\pi) + 1}{n}.$$

Thus, we infer that $na(\omega_\pi) \leq a(\pi)$, as required. □

4. Characters preserving the conductor under twisting.

The goal of this section is twofold: in subsection 4.1, we count the number of characters χ such that $a(\chi\pi)$ is equal to a given integer. Then, in subsection 4.2, we explicitly analyze the behavior of the dominant and interference terms of Theorem 2.6. These questions are motivated by their applications to analytic number theory.

4.1. Sets of twist-fixing characters.

4.1.1. Characters of a given conductor. The valuation v_F defines a split exact sequence

$$1 \longrightarrow \mathfrak{o}^\times \longrightarrow F^\times \xrightarrow{v_F} \mathbb{Z} \longrightarrow 1.$$

We thus write any quasi-character χ on $F^\times$ as $\chi(x) = \chi'(x)q^{-v_F(x)\alpha}$ for some $\alpha \in \mathbb{C}$ and a character χ' of $F^\times$ such that $\chi'(\varpi) = 1$. We denote the space of such χ' by $\mathfrak{X}$ so that the unitary dual of $\mathfrak{o}^\times$ satisfies $\widehat{\mathfrak{o}}^\times \cong \mathfrak{X}$. With interest in characters that fix the conductor under twisting, we define the following $\mathfrak{X}$-subsets:

$$\begin{aligned} \mathfrak{X}(k) &= \{\chi \in \mathfrak{X} : a(\chi) \leq k\}; \\ \mathfrak{X}'(k) &= \{\chi \in \mathfrak{X} : a(\chi) = k\} \end{aligned} \tag{4.1}$$

and

$$\mathfrak{X}'_\pi(k, j) = \{\chi \in \mathfrak{X} : a(\chi) = k \text{ and } a(\chi\pi) = j\} \tag{4.2}$$

for some $k, j \geq 0$.

Our present point of departure is to count the number of characters contained in $\mathfrak{X}'_\pi(k, j)$. We first consider the cardinalities of $\mathfrak{X}(k)$ and $\mathfrak{X}'(k)$.

Lemma 4.1. *For each $k \geq 1$, $\#\mathfrak{X}(k) = q^{k-1}(q-1)$, $\#\mathfrak{X}'(1) = q - 2$, and for $k \geq 2$, $\#\mathfrak{X}'(k) = q^{k-2}(q-1)^2$.*

Proof. Consider the subgroup series

$$\{1\} = \mathfrak{X}(0) \leq \mathfrak{X}(1) \leq \cdots \leq \mathfrak{X}(k) \leq \mathfrak{X}.$$

For $k \geq l \geq k/2 \geq 1$, we have

$$\mathfrak{X}(k)/\mathfrak{X}(l) \cong U_F(l)/U_F(k) \cong \mathfrak{o}/\mathfrak{p}^{k-l}.$$

In particular, taking $l = k - 1$ and noting $\mathfrak{X}(1) \cong (\mathfrak{o}/\mathfrak{p})^\times$, we inductively count the given cardinalities. The number $\#\mathfrak{X}'$ is obtained by subtraction. □

We remark that, in [**4,** Lemmas 2.1, 2.2] we counted the elements $\chi \in \mathfrak{X}'(k)$ for which $a(\chi\mu)$ remains fixed for a given $\mu \in \mathfrak{X}'(k)$, characterizing the existence of such elements as q becomes small. In the present work, we consider a "nonabelian" variant of this result by characterizing the set $\mathfrak{X}'_\pi(k, j)$.

4.1.2. Character twists of a given conductor. Suppose that $\pi \in \mathscr{S}\mathscr{G}_F \cap \mathscr{A}_F(n)$ so that Proposition 2.2 applies. For integers $k, j \geq 0$, if either π is twist minimal or $k \neq a(\pi)/n$, then

$$\mathfrak{X}'_\pi(k,j) = \begin{cases} \mathfrak{X}'_\pi(k) & \text{if } j = \max\{a(\pi),\, nk\} \\ \varnothing & \text{if } j \neq \max\{a(\pi),\, nk\}. \end{cases} \tag{4.3}$$

The cases considered in (4.3) are special cases of the following lemma.

Lemma 4.2. *For each $\pi \in \mathscr{S}\mathscr{G}_F \cap \mathscr{A}_F(n)$, write $\pi = \mu\pi^{\min}$ for a twist minimal representation $\pi^{\min}$. For integers $j, k \geq 0$, we have $\mathfrak{X}'_\pi(k,j) = \varnothing$ unless $a(\pi^{\min}) \leq j \leq \max\{a(\pi),\, nk\}$, in which case*

$$\#\mathfrak{X}'_\pi(k,j) \leq \#\mathfrak{X}\left(\left\lfloor \frac{j}{n} \right\rfloor\right). \tag{4.4}$$

Proof. If either π is minimal or $k \neq a(\pi)/n$, then the lemma follows by (4.3). Hence, assume that $a(\pi) = kn$ and $\pi = \mu\pi^{\min}$, where $\pi^{\min}$ is twist minimal with $a(\pi^{\min}) < a(\pi)$ and $\mu \in \mathfrak{X}'(k)$. Then, $\mathfrak{X}'_\pi(k,j) = \varnothing$ unless $a(\pi^{\min}) \leq j \leq nk$. In this case, if there exists a $\chi \in \mathfrak{X}'(k)$ such that $\max\{a(\pi^{\min}),\, na(\chi\mu)\} = j$, then there are $\#\mathfrak{X}(\lfloor j/n \rfloor)$ of them, as we must have $\chi \in \mu^{-1}\mathfrak{X}(\lfloor j/n \rfloor)$. □

More generally, Lemma 4.2 may be assembled to describe all of $\mathscr{A}_F(n)$.

Corollary 4.3. *Let $\pi \in \mathscr{A}_F(n)$. For integers $j, k \geq 0$, we have $\mathfrak{X}'_\pi(k,j) = \varnothing$ if $j > a(\pi) + nk$. Write $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ as in* (2.1)

(i) *For each $1 \leq i \leq r$, if π_i is either minimal or $a(\pi_i) \neq kn_i$, then*

$$\#\mathfrak{X}'_\pi(k,j) \leq \#\mathfrak{X}\left(\left\lfloor \frac{j}{n} \right\rfloor\right);$$

(ii) *otherwise, define the set of indices $\Psi_k(\pi) \subset \{1, \ldots, r\}$ such that $i \in \Psi_k(\pi)$ if and only if $a(\pi_i) = n_i k$ and $a(\pi_i^{\min}) < a(\pi_i)$, where $\pi_i^{\min}$ is a minimal representation satisfying $\pi_i = \mu_i \pi_i^{\min}$. Then, for any $i' \in \Psi_k(\pi)$, we have*

$$\#\mathfrak{X}'_\pi(k,j) \leq \#\mathfrak{X}\left(\left\lfloor \frac{1}{n_{i'}}\left(j - \sum_{i \notin \Psi_k(\pi)} n_i k - \sum_{i \in \Psi_k(\pi)} \{i'\} a(\pi_i^{\min})\right) \right\rfloor\right).$$

Proof. The bound $j > a(\pi) + nk$ is derived from the fact that $a(\chi\pi) \leq a(\pi) + na(\chi)$ for any $\chi \in \mathfrak{X}$ (see Corollary 2.9). Now, suppose that $a(\chi) = k$ and $a(\chi\pi) = j$. By Proposition 2.2, we have $a(\chi\pi_i) = \max\{a(\pi_i), n_i k\}$ for all $i \notin \Psi_k(\pi)$. In particular, for $\chi \in \mathfrak{X}'(k)$, we have that $a(\chi\pi) = j$ if and only if

$$j = \sum_{i \in \Psi_k(\pi)} a(\chi\pi_i) + \sum_{i \notin \Psi_k(\pi)} \max\{a(\pi_i), n_i k\}. \tag{4.5}$$

Then, if $\Psi_k(\pi) = \varnothing$ for each $\chi \in \mathfrak{X}'(k)$, as in case (i), we have $a(\chi\pi) = j$ for each χ, given (4.5) holds. Moreover, since $j \geq kn$, we obtain $\#\mathfrak{X}'_\pi(k,j) \leq \#\mathfrak{X}(\lfloor j/n \rfloor)$, as claimed. Otherwise, choose $i' \in \Psi_k(\pi)$, as in case (ii). If $a(\chi\pi) = j$, then $a(\chi\pi_{i'}) = j'$, where we define

$$j' = j - \sum_{i \neq i'} a(\chi\pi_i).$$

Then, the number of $\chi \in \mathfrak{X}'(k)$ such that $a(\chi\pi_{i'}) = j'$ is at most $\#\mathfrak{X}(\lfloor j'/n_{i'} \rfloor)$ by Lemma 4.2, whence we deduce the claim. $\square$

4.2. The leading and interference terms. Here, we detail the asymptotic behavior of $\Delta_\chi(\pi)$ and $\delta_\chi(\pi)$. Our first port of call is to describe the rarity with which the interference term satisfies $\delta_\chi(\pi) \neq 0$. The next lemma directly follows from the definition of $\delta_\chi(\pi)$ in Theorem 2.6.

Lemma 4.4 (Absence of interference)**.** *Let π be an irreducible, admissible representation of $\mathrm{GL}_n(F)$ written, as in* (2.1)*, in terms of irreducible, quasi-square-integrable representations, $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$. Recall that $\pi_i \in \mathscr{S}\mathscr{G}_F$ is a representation of $\mathrm{GL}_{n_i}(F)$ for $1 \leq i \leq r$. Let χ be a quasi-character of $F^\times$.*

(i) *We have $\delta_\chi(\pi) = 0$ if $n_i \nmid a(\pi_i)$ for each $1 \leq i \leq r$.*

(ii) *Suppose that $n_i \mid a(\pi_i)$ for some $1 \leq i \leq r$. Then, $\delta_\chi(\pi_i) = 0$, whenever $a(\pi_i) \neq n_i a(\chi)$.*

(iii) *Suppose that $a(\pi_i) = n_i a(\chi)$ for some $1 \leq i \leq r$. Then, $\delta_\chi(\pi_i) = 0$ if and only if $a(\chi\mu_i) = a(\chi)$, where $\pi_i = \mu_i \pi_i^{\min}$ is written as the μ_i-twist of a minimal representation $\pi_i^{\min}$.*

Proof. Recall that $\delta_\chi(\pi_i) = a(\pi_i) - \max\{a(\pi_i^{\min}), n_i a(\chi\mu_i)\}$ for $a(\chi) = a(\mu_i)$, and vanishes otherwise. If $n_i \nmid a(\pi_i)$, then $a(\pi_i) = a(\pi_i^{\min}) \geq$

$n_i a(\mu_i) \geq n_i a(\chi\mu_i)$ for each $1 \leq i \leq r$. This proves (i). For (ii), we let $n_i \mid a(\pi)$. If $a(\pi_i^{\min}) = a(\pi_i)$, we argue as in (i). Else, $a(\pi_i) = n_i a(\mu_i) = n_i a(\chi)$ when $\delta_\chi(\pi_i) \neq 0$, as claimed. The vanishing of $\delta_\chi(\pi_i)$ in (iii) is characterized by the condition $n_i a(\chi) = \max\{a(\pi_i^{\min}), n_i a(\chi\mu_i)\}$ for $a(\chi) = a(\mu_i)$. If $n_i a(\chi) = a(\pi_i^{\min})$, we again argue as in (i), forcing the remaining condition $a(\chi\mu_i) = a(\chi)$. □

Corollary 4.5 (Dominant behavior)**.** *In each case of Lemma* 4.4 *for which* χ *and* $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ *satisfy* $\delta_\chi(\pi) = 0$*, we have the "dominant" conductor formula*

$$a(\chi\pi) = \sum_{i=1}^{r} \max\{a(\pi_i),\, n_i a(\chi)\}. \tag{4.6}$$

Our final task is to quantify the rarity of $\delta_\chi(\pi) = 0$, as in Lemma 4.4.

Lemma 4.6 (Regularity of interference)**.** *Let* $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ *as in* (2.1)*. Suppose that* $\chi \in \mathfrak{X}$ *and that, for some* $1 \leq i \leq r$*, we have* $\delta_\chi(\pi_i) \neq 0$*. Write* $\pi_i = \mu_i \pi_i^{\min}$ *as per Lemma* 4.4 (iii)*. Then, for each* $0 < j \leq a(\pi_i) - a(\pi_i^{\min})$ *satisfying* $j \equiv a(\pi_i) \bmod n_i$*, there are precisely*

$$\#\mathfrak{X}\left(\frac{a(\pi_i) - j}{n}\right) \tag{4.7}$$

characters $\chi \in \mathfrak{X}$ *such that* $\delta_\chi(\pi_i) = a(\pi_i) - j$*. The number of* $\chi \in \mathfrak{X}(a(\pi_i)/n)$ *satisfying* $\delta_\chi(\pi_i) = a(\pi_i)$ *is*

$$(q-2) \times \#\mathfrak{X}\left(\frac{a(\pi_i)}{n} - 1\right). \tag{4.8}$$

Proof. The number in (4.7) is determined by the necessity that

$$\chi \in \mu_i^{-1}\mathfrak{X}\left(\frac{a(\pi_i) - j}{n}\right).$$

Similarly, we count up to the number in (4.8) by observing that $\chi \in \mathfrak{X}(a(\pi_i)/n_i)$, but χ is not an element of $\mathfrak{X}((a(\pi_i)/n_i) - 1)$ nor $\mu_i^{-1}\mathfrak{X}((a(\pi_i)/n_i) - 1)$. □

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ENDNOTES

1. Inequality (1.3) is a special case of both [**2,** Theorem 1] and our main result, Theorem 2.6. (See also Corollary 2.9 for a more precise inequality.)

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