Equitable partitions of Latin-square graphs

R. A. Bailey[*†], Peter J. Cameron[*†], Alexander L. Gavrilyuk[‡§] and Sergey V. Goryainov[¶‖]

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Abstract

We study equitable partitions of Latin-square graphs, and give a complete classification of those whose quotient matrix does not have an eigenvalue -3.

Keywords: equitable partition; Latin-square graph; eigenvalue; Cayley table

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1 Introduction

In the International Workshop on Bannai–Ito Theory in Hangzhou in November 2017, the fourth author spoke about his result with the third author

[*]School of Mathematics and Statistics, University of St Andrews, St Andrews, Fife KY16 9SS, UK

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[‡]Center for Math Research and Education, Pusan National University, 2, Busandaehak-ro 63beon-gil, Geumjeong-gu, Busan, 46241, Republic of Korea; and Krasovskii Institute of Mathematics and Mechanics, S. Kovalevskaya st., 16, Yekaterinburg 620990, Russia

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[¶]Shanghai Jiao Tong University, 800 Dongchuan Rd., Minhang District, Shanghai, China; Krasovskii Institute of Mathematics and Mechanics, S. Kovalevskaya st., 16, Yekaterinburg 620990, Russia; and Chelyabinsk State University, Brat'ev Kashirinyh st., 129, Chelyabinsk, 454021, Russia

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classifying the equitable partitions of the bilinear-forms graph $\mathrm{Bil}_2(2 \times d)$ [26]. This graph can be regarded as the Latin-square graph associated with the Cayley table of the additive group of the d-dimensional vector space W over the 2-element field. An equitable partition is associated with a matrix whose spectrum is contained in that of the adjacency matrix of the graph; the result was a complete classification in the case where the eigenvalue -3 of the adjacency matrix does not occur. The parts of such a partition must be unions of rows, columns, or letters of the Cayley table, or subsquares corresponding to subspaces of W of codimension 1.

Here we present an extension of the result, to construct and classify all equitable paritions of arbitrary Latin-square graphs for which the eigenvalue -3 (the smallest of the three eigenvalues of any Latin-square graph) does not occur (see Theorem 5.4 below). Remarkably, a relatively small generalisation of the subspace construction is required; we replace this by the notion of an inflation of a "corner set" in the Cayley table of a cyclic group.

We begin in Section 2 with some preliminaries about equitable partitions, including showing that, for the main theorem, it is enough to classify sets of vertices which are parts of 2-part equitable partitions, and are minimal subject to this condition. We define Latin-square graphs in Section 3, and in Section 4 we give the inflation construction which is used to produce examples. Section 5 contains the main substance of this paper: we construct the examples, and prove that there are no more.

The paper concludes with three short sections discussing possible further directions. Sections 6–7 explore equitable partitions which do involve the eigenvalue -3. If the Latin square has order n then the other non-principal eigenvalue is $n-3$ (see Section 2): we point to work on partitions which do not involve this eigenvalue from both Statistics and Combinatorics, and observe that some examples are connected with orthogonal arrays. Finally, Section 8 generalizes the ideas from Latin squares to sets of mutually orthogonal Latin squares.

Equitable partitions of distance-regular graphs involving only the two largest eigenvalues of the graph were considered by Meyerowitz [30], who classified them for Hamming and Johnson graphs. In general, the classification of equitable partitions seems a hard problem, since it includes questions such as tight sets in polar spaces, see [4, 5, 6]. Another classification result is given in [23].

2 Equitable partitions

A partition $\Delta = \{\Delta_1, \dots, \Delta_r\}$ of the vertex set of a graph Γ is said to be *equitable* if there is an $r \times r$ matrix $M = (m_{ij})$ such that the number of vertices of Δ_j joined to a vertex $\omega \in \Delta_i$ is m_{ij}, depending on i and j but not on the choice of ω. This term is used by Godsil and Royle [25, §9.3]; Fon-Der-Flaass [20] and Krotov [28] called such partitions *perfect colourings.* We shall reserve the term *perfect* for a set which is a part of a 2-part equitable partition, see below.

The spectrum of M is contained in the spectrum of the adjacency matrix $A(\Gamma)$ of the graph Γ: indeed, the characteristic polynomial of M divides that of $A(\Gamma)$ [25, Theorem 9.3.3]. Since this result is crucial to our approach, we outline a proof. The matrix M is called the *quotient matrix* of the equitable partition. When we speak of eigenvalues of an equitable partition, we refer to eigenvalues of the corresponding quotient matrix.

We begin with some general information about equitable partitions. Let Ω be the vertex set of Γ, with $|\Omega| = N$; and let V be the N-dimensional vector space $\mathbb{R}^\Omega$, whose basis vectors correspond to the vertices of Γ. Let $A(\Gamma)$ be the adjacency matrix of Γ, and let $\mathbf{v}_i \in V$ be the characteristic vector of the part Δ_i.

From the definition of an equitable partition, we see that

$$\mathbf{v}_i A(\Gamma) = \sum_{j=1}^{r} m_{ji} \mathbf{v}_j,$$

so that the space $W = \langle \mathbf{v}_1, \dots, \mathbf{v}_r \rangle$ is invariant under the right action of $A(\Gamma)$, and the restriction of $A(\Gamma)$ to this subspace has matrix $M^\top$ relative to the given basis. (Indeed this property is equivalent to the partition being equitable.) Hence, if the (pairwise orthogonal) eigenspaces of $A(\Gamma)$ are $V_0, \dots, V_e$, then

$$W = (W \cap V_0) \oplus \cdots \oplus (W \cap V_e),$$

so the spectrum of M is contained in that of $A(\Gamma)$, and the cited result follows.

From now on, we assume that Γ is a connected regular graph with valency k. Then k is a simple eigenvalue of $A(\Gamma)$. Moreover, the quotient matrix M of an equitable partition has all row sums equal to k, so that k is an eigenvalue of M. We call k the *principal eigenvalue.*

We say that an equitable partition Δ is *μ-equitable* if its quotient matrix M has all non-principal eigenvalues equal to μ. Furthermore, we call a non-empty proper subset S of Ω a *μ-perfect set* if the partition $\{S, \Omega \setminus S\}$ is μ-equitable. Note that, if a set S is μ-perfect, then so is its complement $\Omega \setminus S$.

Proposition 2.1 *Let $\Delta = \{\Delta_1, \ldots, \Delta_r\}$ be a partition of the vertex set Ω of the regular connected graph Γ.*

(a) If Δ is μ-equitable, then each set Δ_i is μ-perfect.

(b) Conversely, if $\Delta_1, \ldots, \Delta_{r-1}$ are all μ-perfect, then Δ is μ-equitable.

Proof (a) Suppose that the hypotheses hold, and let $\mathbf{v}_i$ be the characteristic vector of Δ_i. Then $\mathbf{v}_i$ lies in the space $V_0 \oplus V_1$, where V_0 is the k-eigenspace (spanned by the all-1 vector $\mathbf{v}_0$) and V_1 the μ-eigenspace. Hence the span of $\mathbf{v}_0$ and $\mathbf{v}_i$ is $A(\Gamma)$-invariant, and the restriction of $A(\Gamma)$ to this subspace has eigenvalues k and μ; thus Δ_i is a μ-perfect set.

(b) Conversely suppose that $\Delta_1, \ldots, \Delta_{r-1}$ are μ-perfect. Then the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}$ and $\mathbf{v}_0$ is contained in $V_0 \oplus V_1$; since this space also contains $\mathbf{v}_r$, the conclusion follows. $\square$

The following result was stated, without proof, by Krotov, as Lemma 2 in [28]. We include the proof here for completeness.

Corollary 2.2 *Let S be a μ-perfect set, and T a non-empty proper subset of $\Omega \setminus S$. Then T is μ-perfect if and only if $S \cup T$ is μ-perfect.*

Proof The forward direction follows immediately from Proposition 2.1(b): if S and T are μ-perfect then $\{S, T, \Omega \setminus (S \cup T)\}$ is μ-equitable, and so $\Omega \setminus (S \cup T)$ (and also its complement $S \cup T$) is μ-perfect. For the converse, if $S \cup T$ is μ-perfect, then two parts of the partition $\{S, T, \Omega \setminus (S \cup T)\}$ are μ-perfect; so this partition is μ-equitable, and all its parts are μ-perfect. $\square$

Corollary 2.3 *If Δ is a μ-equitable partition then any non-trivial coarsening of Δ is μ-equitable.*

Proof All parts of Δ are μ-perfect, and so by Corollary 2.2 the same is true for any non-trivial coarsening of Δ; then Proposition 2.1(b) applies. $\square$

3 Latin-square graphs

Let Λ be a Latin square of order n. Take Ω to be the set of cells of Λ, so that $N = |\Omega| = n^2$. There are three partitions R, C and L of Ω into n parts of size n (a partition with all parts of the same size is called *uniform*). The parts of R are rows, the parts of C are columns, and the parts of L are letters.

If $\omega \in \Omega$, then $R(\omega)$, $C(\omega)$ and $L(\omega)$ denote the row, column and letter containing ω (regarded as subsets of Ω).

As in Section 2, we denote by V the n^2-dimensional vector space $\mathbb{R}^\Omega$. Let V_0 be its one-dimensional subspace of constant vectors. The characteristic vectors of all rows span an n-dimensional subspace V_R containing V_0. Columns define a similar subspace V_C, and letters a similar subspace V_L. Put $V_1 = (V_R + V_C + V_L) \cap V_0^\perp$ and $V_2 = (V_R + V_C + V_L)^\perp$, so that V is the orthogonal direct sum of V_0, V_1 and V_2.

The projection matrix for partition R is the matrix of orthogonal projection onto V_R. It replaces the entry $\mathbf{v}(\omega)$ of any vector $\mathbf{v}$ in V by the average of the entries over the row (part of R) containing ω. Projection matrices for other partitions are defined similarly.

Statisticians say that two partitions are orthogonal to each other if their projection matrices commute. An *orthogonal block structure* is defined in [1, 2, 3] to be a set of pairwise-orthogonal uniform partitions of a finite set which contains the two trivial partitions and is closed under join and meet. Thus R, C, L and the two trivial partitions form an orthogonal block structure on Ω.

The Latin square Λ defines a Latin-square graph Γ with vertex set Ω and valency $k = 3(n-1)$. Each vertex is joined to every other vertex in the same row or column or letter. Denote the adjacency matrix of Γ by A. We refer to the elements of Ω as cells or vertices, depending on the context.

Every edge of Γ is contained in n triangles, while, for every pair of vertices of Γ which are not joined by an edge, there are six vertices joined to both of them. This means that Γ is *strongly regular* and A satisfies

$$AJ = 3(n-1)J \qquad \text{and} \qquad A^2 = 3(n-1)I + nA + 6(J - A - I),$$

where I is the identity matrix of order n^2 and J is the $n^2 \times n^2$ matrix whose entries are all equal to 1: see [10, 25]. In other words, the matrices I, A and $J-A-I$ form the adjacency matrices of an association scheme of rank three: see [2]. The common eigenspaces are V_0 (of dimension 1), V_1 (of dimension

$3(n-1)$) and V_2 (of dimension $(n-1)(n-2)$). The eigenvalues of A on these three spaces are respectively $k = 3(n-1)$, $n-3$, and -3.

In the special case $n = 2$, the Latin square graph is the complete graph K_4, and the eigenspace V_2 does not occur (the formula above gives its dimension as zero).

4 Inflation

Here is a construction that we shall use several times.

Let Λ_0 be a $t \times t$ Latin square on Ω_0. Replace each occurrence of letter i by an $s \times s$ Latin square on an alphabet $\mathcal{A}_i$, where $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ if $i \neq j$, to obtain a Latin square Λ_1 of order st. There is no requirement for the t Latin squares on alphabet $\mathcal{A}_i$ to be the same, or even isomorphic.

We call Λ_1 an *s-fold inflation* of Λ_0.

Let Ω_1 be the underlying set of Λ_1. Then Ω_1 has size $(st)^2$. The inflation construction gives an orthogonal block structure on Ω_1 whose non-trivial partitions are

- rows (R), columns (C), letters (L), each with st parts of size st;
- fat rows ($\tilde{R}$), fat columns ($\tilde{C}$), fat letters ($\tilde{L}$), each with t parts of size s^2t, corresponding to the rows, columns and letters of Λ_0;
- subsquares (Q), with t^2 parts of size s^2, where Q is the infimum of every pair of $\tilde{R}$, $\tilde{C}$ and $\tilde{L}$.

Like every orthogonal block structure, this defines an association scheme on Ω_1. The partition Q is inherent in this, in the sense that its relation matrix is a sum of some of the adjacency matrices of the scheme. Therefore it defines a quotient scheme on the set of parts of Q: see [2, Chapter 10]. This quotient scheme is precisely the original Latin square Λ_0.

Theorem 4.1 *The partition Q of Ω_1 is equitable for the Latin-square graph Γ_1 defined by the Latin square Λ_1.*

Proof A vertex in a part of Q is joined to $3(s-1)$ further vertices in that part, since the induced subgraph is a Latin-square graph from a square of order s. It is joined to s vertices in each other part of Q in the same fat row, fat column, or fat letter, and to no vertex in any other part of Q. So the partition is equitable. Its quotient matrix has the form $M = 3(s-1)I + sA$, where A is the adjacency matrix of the Latin-square graph corresponding to Λ_0. The eigenvalues of A are $3(t-1)$, $t-3$ and -3; so the eigenvalues of M are $3(s-1)+3s(t-1) = 3(st-1)$, $3(s-1)+s(t-3) = st-3$, and $3(s-1)+s(-3) = -3$, the correct values for an equitable partition of a Latin-square graph with $n = st$. Moreover, their multiplicities are those of the Latin-square graph from Λ_0, namely 1, $3(t-1)$, and $(t-1)(t-2)$. $\square$

Example 1 When $t = 2$, there is a unique Latin square Λ_0 on an underlying set Ω_0 of size four. The corresponding graph Γ_0 is complete, and so all partitions of Ω_0 are equitable for it. Their s-fold inflations give equitable partitions of some Latin squares of order $2s$. In this case, the multiplicities stated in the proof of Theorem 4.1 show that all non-principal eigenvalues of the partition are $2s-3$.

Indeed, if a Latin square of order $2s$ contains a subsquare of order s, then it necessarily arises as an inflation of the order-2 square. This includes Cayley tables of groups of order $2s$ having subgroups of order s.

If Δ_0 is a partition of Ω_0 then the s-fold inflation gives a partition $\tilde{\Delta}_0$ of Ω_1 with the same number of parts.

Theorem 4.2 *Let Λ be an s-fold inflation of a Latin square Λ_0 of order t. Let Γ and Γ_0 be the Latin-square graphs defined by Λ and Λ_0 respectively.*

(a) If Δ_0 is an equitable partition for Γ_0 then $\tilde{\Delta}_0$ is equitable for Γ.

(b) If P is a $(t-3)$-perfect subset of the vertex set of Γ_0 then the union of the Q-parts corresponding to the cells in P is an $(st-3)$-perfect subset of the vertex set of Γ.

The proof is almost identical to that just given.

5 $(n-3)$-perfect sets

We return to the case where Γ is the graph defined by a Latin square of order n. Our goal is to describe the equitable partitions of Γ, especially those with all non-principal eigenvalues equal to $n-3$. The preliminary results we have given about equitable partitions show that every part of such a partition is an $(n-3)$-perfect set, and any partition all of whose parts are $(n-3)$-perfect is $(n-3)$-equitable. So our job is to describe the $(n-3)$-perfect sets.

Suppose that an equitable 2-partition has quotient matrix

$$M = \begin{bmatrix} p & b \\ c & q \end{bmatrix}.$$

Then

$$p + b = c + q = k = 3(n-1). \tag{1}$$

Furthermore, if the non-principal eigenvalue is $n-3$ then

$$p + q = k + n - 3 = 4n - 6. \tag{2}$$

Moreover, if the first part is S, then counting edges between S and its complement gives $|S|\, b = (n^2 - |S|)c$, so (since $b + c = 2n$ from Equations (1) and (2)) we have

$$2\,|S| = nc. \tag{3}$$

5.1 Construction 1

Proposition 5.1 *Any row, column or letter is an $(n-3)$-perfect set.*

Proof Let S be a row. (The other cases are similar.) The induced subgraph on S is complete, and so any vertex in S is joined to $2(n-1)$ vertices outside S; and any vertex outside S is joined to two vertices of S (one with the same column and one with the same letter). So $\{S, \Omega \setminus S\}$ is equitable, and its quotient matrix is

$$M = \begin{bmatrix} n-1 & 2(n-1) \\ 2 & 3n-5 \end{bmatrix}.$$

Thus the trace of M is $4n-6$; since it has an eigenvalue $k = 3n-3$, the other eigenvalue is $n-3$, as required. $\square$

It follows that any set which is a union of rows, or of columns, or of letters, is $(n-3)$-perfect; and hence any partition all of whose parts are of this form is equitable.

Another consequence of Corollary 2.2 is the following:

Corollary 5.2 *If an $(n-3)$-perfect set S properly contains a row T, then $S \setminus T$ is $(n-3)$-perfect; and similarly for a column or letter.*

So, in our search for the $(n-3)$-perfect sets, we may assume without loss that such a set contains no row, column, or letter. We will call such a set *slender*.

5.2 Slender sets and slices

Given a slender subset S of Ω, call a *slice* the intersection of S with any row, column or letter of Λ. Now we introduce some notation for the size of a slice. For each vertex ω in Ω, put $\rho(\omega) = |R(\omega) \cap S|$, $\kappa(\omega) = |C(\omega) \cap S|$, and $\lambda(\omega) = |L(\omega) \cap S|$. Then

$$\rho(\omega) + \kappa(\omega) + \lambda(\omega) = c \qquad \text{if } \omega \notin S. \tag{4}$$

In particular, no slice has size greater than c. Also, Equations (1)–(2) show that

$$\rho(\omega) + \kappa(\omega) + \lambda(\omega) = 3 + p = k + n - q = n + c \qquad \text{if } \omega \in S. \tag{5}$$

Equation (5) shows that if $\omega \in S$ then at least one of $\rho(\omega)$, $\kappa(\omega)$ and $\lambda(\omega)$ is greater than or equal to $(n+c)/3$.

5.3 Construction 2

Here is a construction of an equitable partition with three parts, two of which are slender sets.

Let Λ be the (back-)cyclic Latin square of order n, the Cayley table of the cyclic group Z_n of order n. We take the rows, columns, and letters to be indexed by the set $\{0, 1, \ldots, n-1\}$ of integers modulo n, so that the letter in row i and column j is $i + j$ (with addition modulo n).

Consider the partition Δ with three parts Δ_{-1}, Δ_0 and Δ_1, consisting of the cells (i, j) with $i + j < n - 1$, $i + j = n - 1$, and $i + j > n - 1$ respectively (using integer addition here).

Figure 1 shows this partition for $n = 5$ and for $n = 6$, with Δ_{-1} in bold and Δ_0 in calligraphic font. For ease of reading, the letters indexed by 0, 1, 2, etc. are shown as A, B, C, etc.

A	**B**	**C**	**D**	$\mathcal{E}$
B	**C**	**D**	$\mathcal{E}$	A
C	**D**	$\mathcal{E}$	A	B
D	$\mathcal{E}$	A	B	C
$\mathcal{E}$	A	B	C	D

A	**B**	**C**	**D**	**E**	$\mathcal{F}$
B	**C**	**D**	**E**	$\mathcal{F}$	A
C	**D**	**E**	$\mathcal{F}$	A	B
D	**E**	$\mathcal{F}$	A	B	C
E	$\mathcal{F}$	A	B	C	D
$\mathcal{F}$	A	B	C	D	E

Figure 1: Equitable partitions of cyclic Latin squares, using Construction 2

Theorem 5.3 *With the above notation, the partition Δ is equitable, with both non-principal eigenvalues equal to $n-3$.*

Proof We prove this by direct counting, see Figure 2.

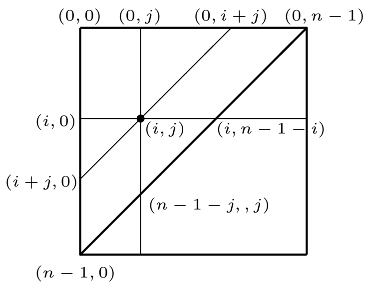

Figure 2: Counting neighbours in Theorem 5.3

Take a cell (i,j) in Δ_{-1}. Within Δ_{-1}, there are $n-i-2$ cells in the same row, $n-j-2$ in the same column, and $i+j$ with the same letter (excluding the cell (i,j) itself); so it has $2n-4$ neighbours in Δ_{-1}. The cells in Δ_0 all have letter $n-1$, which never occurs in Δ_{-1}; so (i,j) is joined to two cells in Δ_0, namely $(i,n-1-i)$ (in the same row) and $(n-1-j,j)$ (in the same column). The remaining $n-1$ neighbours are in Δ_1.

A cell $(i,n-1-i)$ in Δ_0 is joined to the other $n-1$ cells in Δ_0 (all have the same letter), and to $n-1$ cells in Δ_{-1} (of which $n-i-1$ are in the same row and i in the same column).

The other matrix coefficients follow by symmetry between Δ_{-1} and Δ_1.

Thus the partition is equitable, with quotient matrix

$$M = \begin{bmatrix} 2n-4 & 2 & n-1 \\ n-1 & n-1 & n-1 \\ n-1 & 2 & 2n-4 \end{bmatrix}.$$

This matrix has trace $5n - 9$; so its eigenvalues are $3n - 3$, $n - 3$, $n - 3$. □

In particular, the parts of the partition are $(n-3)$-perfect. The part Δ_0 is a letter, but the other two parts are obviously slender. Further partitions of this type can be found by changing the roles of rows, columns, and letters.

We will call sets of the form Δ_{-1} in this example, possibly after relabelling of rows, columns and letters, *corner sets*. Note that a corner set is disjoint from a row and a column as well as a letter, and so is a part of three different partitions of this type: as well as the one given, we have $\{\Delta_{-1}, \Delta_2, (\Delta_0 \cup \Delta_1) \setminus \Delta_2\}$, where Δ_2 is either the last row or the last column.

Theorem 4.2 now shows that if Λ is an s-fold inflation of a cyclic Latin square of order t then the inflation of the partition Δ in Theorem 5.3 is equitable with non-principal eigenvalues $n - 3$, where n is the order of Λ.

In particular, a single cell in a Latin square of order 2 is a corner set, and inflation gives subsquares of order $n/2$ in Latin squares of even order n. Hence such subsquares are $(n-3)$-perfect sets.

5.4 The main theorem

Theorem 5.4 *Let Γ be the Latin-square graph defined by a Latin square of order n, and Δ a partition of the vertex set of Γ. Then Δ is $(n-3)$-equitable if and only if each part of Δ is a disjoint union of rows, columns, letters, or inflations of corner sets.*

We know from the results of Section 2 that it is enough to describe the $(n-3)$-perfect sets, and moreover that it is enough to show the following:

Theorem 5.5 *A slender $(n-3)$-perfect set in the Latin-square graph defined by a Latin square of order n is an inflation of a corner set.*

The proof of the theorem is somewhat involved, so we begin with a summary and some comments on the notation. Let S denote a slender $(n-3)$-perfect set.

An inflation of a corner set, after suitable row and column permutations, resembles the starred region shown in Figure 3, which also shows fat rows and fat columns.

We begin by identifying the subsquare in the top right of the figure: it is the intersection of the fat row consisting of those rows meeting S in the

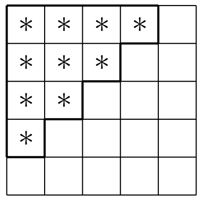

Figure 3: Sketch of the proof

maximum number of columns and the fat column consisting of those columns not meeting S. Then, inductively, we work down and to the left, identifying the subsquares on the boundary of S. The notation is introduced as the proof proceeds.

Parts (a)–(b) of the induction successively find rows whose slice sizes are strictly decreasing. These rows are numbered 1^*, 2^*, ..., by their position of occurrence in the inductive proof. Row u^* is the first row to be named after fat row $\tilde{R}_{u-1}$ is identified in part (f).

Part (c) of the induction uses row u^* and fat column $\tilde{C}_1$ (on the right of Figure 3) to define the fat letter $\tilde{L}_u$ and find some properties of it. Then part (d) uses row u^* and fat letter $\tilde{L}_1$ to define fat column $\tilde{C}_u$ in such a way that $\tilde{L}_1$ is on the back-diagonal of the square. Thus fat columns are numbered from right to left. We do not really have a viable way of numbering them that matches the reader's expectations, because we do not know at the start of the proof that, for example, the size of the subsquares divides n.

The remainder of parts (d) and (e) identify the letters in the intersection of row u^* with fat column $\tilde{C}_v$ for $v < u$.

Finally, part (f) shows that there is a fat row $\tilde{R}_u$ which contains row u^* and which has the properties necessary for an inflated square.

Once the fat letters have been assigned to the subsquares outside S, the Latin square property forces their allocation to the subsquares in S, working upwards from the penultimate fat row. However, the information gathered during the proof gives a more direct way of doing this, as we show at the end of this section.

We now embark on the details. Recall that c is the (constant) number of neighbours in S of any vertex outside S, so that Equations (1)–(5) hold. The proof makes frequent use of Equations (4) and (5).

Lemma 5.6 *If there is a slice of size z then either $z = c$ or there is a slice of size at least $(n+z)/2$.*

Proof Without loss of generality, assume that the slice of size z is contained in a row. Because S is slender, there is a vertex α in this row which is not in S. Then Equation (4) shows that $\kappa(\alpha) + \lambda(\alpha) = c - z$. If $z \neq c$ then at least one of $C(\alpha) \cap S$ and $L(\alpha) \cap S$ is not empty. Without loss of generality, there is a vertex β in $C(\alpha) \cap S$. Then Equation (5) shows that $\rho(\beta) + \lambda(\beta) = n + c - \kappa(\beta) \geq (n+c) - (c-z) = n + z$. Hence at least one of $\rho(\beta)$ and $\lambda(\beta)$ is at least $(n+z)/2$. □

Applying Lemma 5.6, if no slice has size c then for any slice we can find a strictly larger one, which is impossible. So we obtain the following corollary.

Corollary 5.7 *There is at least one slice of size c.*

By the symmetry among rows, columns and letters in a corner set, we may assume without loss of generality that the slice in this corollary is a row slice. From the corollary and the fact that S is slender, we see that $c < n$. We put $s = n - c$.

Slightly abusing notation, write ρ_i, κ_j and λ_ℓ for the size of the slice in row i, column j and letter ℓ respectively. Without loss of generality, we may assume that $\rho_1 = c$. Let L_1 be the set of s letters whose cells in row 1 are not in S; and let C_1 be the set of the s columns whose intersection with row 1 is not in S. If α is in row 1 and a column in C_1 then $\rho(\alpha) = \rho_1 = c$ and so Equation (4) shows that $\lambda(\alpha) = 0 = \kappa(\alpha)$. (See Figure 4, which also incorporates part (f) of the following theorem for the case $u = 1$.)

The statement of the following theorem introduces further notation like L_u, C_u and R_u, analogous to the notation L_1 and C_1 already defined, to denote various sets of letters, columns or rows. In each case, we use the same notation with a $\tilde{}$ on top to denote the subset of Ω formed by the union of all letters or columns or rows in that set.

Theorem 5.8 *Assume that S is slender and $\rho_1 = c$. Let t be a positive integer. If $n > (t-1)s$ and $1 \leq u \leq t$ then the following are true. Hence $n \geq ts$.*

(a) There is no row i with $n - us < \rho_i < n - (u-1)s$.

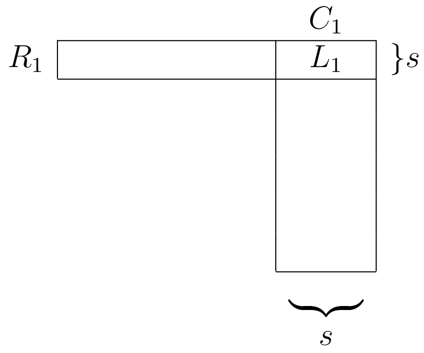

Figure 4: What we know when $u = 1$: all vertices in $\tilde{R}_1$ to the left of the subsquare marked L_1 are in S, while all those in $\tilde{C}_1$ are outside S

(b) There is a row u^* with $\rho_{u^*} = n - us$. We may label the rows so that $u^* = (u-1)s + 1$.

(c) If $u > 1$, let L_u be the set of s letters in the intersection of row u^* from part (b) with $\tilde{C}_1$. If $u \geq 1$ then every letter ℓ in L_u has $\lambda_\ell = (u-1)s$. Hence L_u and L_v are disjoint if $1 \leq v < u$.

(d) If $u > 1$, let C_u be the set of s columns where row u^* from part (b) contains letters in L_1. If $1 \leq v \leq u$ then every vertex in the intersection of $\tilde{C}_v$ with row u^* is outside S. Hence, for every column j in C_u, $\kappa_j = (u-1)s$. Moreover, every vertex in row u^* outside $\tilde{C}_1 \cup \cdots \cup \tilde{C}_u$ is in S.

(e) If $1 \leq v < u$ then the letters in the intersection of row u^* with $\tilde{C}_v$ are precisely those in L_{u-v+1}.

(f) There are precisely s rows whose slice has size $n - us$. We may label these u^*, u^*+1, ..., us without affecting the labelling of the rows in R_v for $v < u$. If R_u denotes the set of these rows, then $\tilde{R}_u \cap \tilde{C}_1$ is a Latin square on the letters in L_u. Moreover, if $1 < v \leq u$ then $\tilde{R}_u \cap \tilde{C}_v$ is a Latin square on the letters in L_{u-v+1}. Also $\tilde{R}_u \setminus (\tilde{C}_1 \cup \cdots \cup \tilde{C}_u) \subset S$.

Proof We use induction on u. First, we comment on starting the induction. Corollary 5.7 gives parts (a) and (b) for $u = 1$. The remarks following that

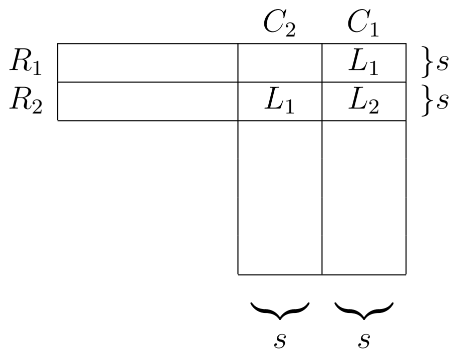

Figure 5: What we know when $u = 2$: all vertices to the left of, or above, subsquares marked L_1 are in S, while all those in, or below, those subsquares are outside S

give parts (c) and (d) for $u = 1$, and part (e) is vacuous for this case. Thus part (f) needs to be proved for $u = 1$ before the induction can proceed. This is proved by the same argument as the inductive step for part (f) below, since at this point we know that parts (a)–(e) all hold.

Parts (b), (d) and (f) for $u = 1$ give the situation summarized in Figure 4.

For higher values of u we assume parts (a)–(f) for all smaller values. For each value of u, parts (a)–(f) are proved in order, so that earlier parts may be assumed for the given value of u.

If $t \geq 2$, then parts (b) to (f) give Figure 5 for $u = 2$. Part (b) for $u = t$ gives the conclusion that $n \geq ts$.

We now turn to the proofs of parts (a) to (f).

(a) If $u = 1$ this is true because no slice has size greater than c.

If $u > 1$, assume that parts (a)–(f) are true for all v with $1 \leq v < u$. These give the situation summarized in Figure 6.

Let α be a vertex in $\tilde{C}_1 \setminus (\tilde{R}_1 \cup \cdots \cup \tilde{R}_{u-1})$. The definition of R_v for $v < u$ shows that $\rho(\alpha) \neq n - vs$ if $1 \leq v < u$. Thus part (b) for smaller values of u shows that $\rho(\alpha) < n - (u-1)s$.

Part (f) for v with $1 \leq v < u$ shows that $L(\alpha)$ is not in $L_1 \cup \cdots \cup L_{u-1}$, and so $L(\alpha)$ occurs in every row of $\tilde{R}_v \cap S$ for $1 \leq v \leq u-1$, by parts (d) and (e) for those values of v. Then part (f) for those values of v

shows that $\lambda(\alpha) \geq (u-1)s$. Moreover, $\alpha \notin S$, and so $\rho(\alpha) + \lambda(\alpha) = c$. Therefore $\rho(\alpha) \leq c - (u-1)s = n - us$. This proves part (a) for u.

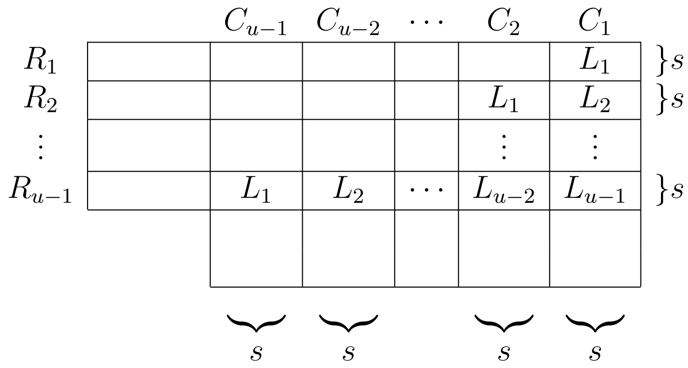

Figure 6: What we know after $u-1$ steps in the induction: all vertices to the left of, or above, subsquares marked L_1 are in S, while all those in, or below, those subsquares are outside S

(b) Since $\rho_1 = c = n - s$, this is true when $u = 1$.

If $u > 1$, assume that parts (a)–(f) are true for all v with $1 \leq v < u$. Suppose that the vertex α in the proof of part (a) is chosen to minimize $\lambda(\alpha)$. If $\lambda(\alpha) > (u-1)s$ then $\rho(\alpha) < c - (u-1)s$, and so there is a vertex β in $R(\alpha) \setminus S \setminus (\tilde{L}_1 \cup \cdots \cup \tilde{L}_{u-1}) \setminus \tilde{C}_1$. Then $\kappa(\beta) > 0$ and $c = \rho(\beta)+\kappa(\beta)+\lambda(\beta) = \rho(\alpha)+\kappa(\beta)+\lambda(\beta) = c-\lambda(\alpha)+\kappa(\beta)+\lambda(\beta)$. It follows that $\lambda(\alpha) > \lambda(\beta)$. But β is not in $\tilde{L}_1 \cup \cdots \cup \tilde{L}_{u-1}$, so there is some vertex γ in $\tilde{C}_1 \setminus (\tilde{R}_1 \cup \cdots \cup \tilde{R}_{u-1})$ with $L(\gamma) = L(\beta)$. This contradicts the choice of α to minimize $\lambda(\alpha)$. It follows that $\lambda(\alpha) = (u-1)s$ and so $\rho(\alpha) = c - (u-1)s = n - us$.

The rows in $\tilde{R}_1 \cup \cdots \cup \tilde{R}_{u-1}$ are numbered 1, …, $(u-1)s$. Because α is not in that set, we may renumber the remaining rows so that α is in row $(u-1)s+1$. This proves part (b) for u.

(c) Assume that part (b) is true for u. If $\ell \in L_u$ then there is a vertex α in $\tilde{C}_1$ with $\rho(\alpha) = n - us$ and $L(\alpha) = \ell$. Since $\kappa(\alpha) = 0$ and $\rho(\alpha) + \kappa(\alpha) + \lambda(\alpha) = c$, this shows that $\lambda_\ell = \lambda(\alpha) = (u-1)s$.

If $u > 1$, assume that part (c) is true for all v with $1 \le v < u$. Then, for any such v, any letter m in L_v has $\lambda_m = (v-1)s$. Therefore L_u and L_v are disjoint. Thus part (c) is true for u.

(d) Assume that parts (b) and (c) are true for u, and that part (d) is true for all v with $1 \le v < u$. If $1 \le v < u$ and α is in $\tilde{C}_v$ and row u^* then $\rho(\alpha) = n - us$ and $\kappa(\alpha) = (v-1)s$. Since $v < u$, we have $\rho(\alpha) + \kappa(\alpha) \le n - 2s < n$. If $\alpha \in S$ then $\rho(\alpha) + \kappa(\alpha) + \lambda(\alpha) = n + c$, and so $\lambda(\alpha) > c$. This cannot happen, and so $\alpha \notin S$.

If α is in $\tilde{C}_u$ and row u^* then α is in $\tilde{L}_1$. Then $\rho(\alpha) = n - us$ and $\lambda(\alpha) = 0$. If $\alpha \in S$ then $\rho(\alpha) + \kappa(\alpha) + \lambda(\alpha) = n + c$, and so $\kappa(\alpha) = c + us > c$. This cannot happen, and so $\alpha \notin S$. Therefore $\rho(\alpha) + \kappa(\alpha) + \lambda(\alpha) = c$, which shows that $\kappa(\alpha) = (u-1)s$.

Finally, since $\rho_{u^*} = n - us$ and all vertices in the intersection of row u^* and $\tilde{C}_1 \cup \cdots \cup \tilde{C}_u$ are outside S, all the remaining vertices in row u^* must be in S. Thus part (d) is true for u.

(e) If $u > 1$, assume that parts (c), (d) and (f) are true for all v with $1 \le v < u$. If ℓ is a letter outside $L_1 \cup \cdots \cup L_{u-1}$ then part (d) shows that it occurs in S in every row in $\tilde{R}_1 \cup \cdots \cup \tilde{R}_{u-1}$, and so $\lambda_\ell \ge (u-1)s$.

If $1 \le v < u$ and α is a vertex in the intersection of row u^* with $\tilde{C}_v$, then part (d) shows that $\alpha \notin S$. Therefore $c = \rho(\alpha) + \kappa(\alpha) + \lambda(\alpha) = (n - us) + (v-1)s + \lambda(\alpha)$ and so $\lambda(\alpha) = (u-v)s$. If $v = 1$ then $u - v + 1 = u$ and by definition the letters in the intersection of row u^* with $\tilde{C}_1$ are those in L_u. If $v > 1$ then $\lambda(\alpha) \le (u-2)s$ and so $L(\alpha) \in L_1 \cup \cdots \cup L_{u-1}$. Then it follows from part (c) for integers less than u that $L(\alpha) \in L_{u-v+1}$. Hence part (e) is true for u.

(f) Assume that parts (c), (d) and (e) are true for u.

Let ℓ be a letter in L_u. This occurs in s rows of $\tilde{C}_1$, all of whose slices have size $n - us$. When $u = 1$, each letter m outside L_1 has $\lambda_m > 0$ and the argument in part (c) shows that m cannot occur in the intersection of any of these rows with $\tilde{C}_1$. For $u > 1$, parts (d) and (e) show that, for each of these rows, the letters outside $S \cup \tilde{C}_1$ are precisely those in $L_1 \cup \cdots \cup L_{u-1}$. Suppose that a letter m in L_u occurs on a vertex α in S in such a row. Then $\rho(\alpha) + \lambda(\alpha) = n - us + (u-1)s = n - s$ so $\kappa(\alpha) = n + c - n + s = n$, which is impossible because S is slender.

Hence each of these s rows intersects $\tilde{C}_1$ in a set of vertices whose letters are the set L_u.

If $u = 1$ then we are free to relabel the rows of R_1 after the first as rows, $2, \ldots, u$. When $u > 1$ then none of these rows is in $\tilde{R}_1 \cup \cdots \cup \tilde{R}_{u-1}$, so we may relabel those other than u^* as $u^* + 1, \ldots, u^* + s = us$.

If there are any more rows with slice size $n - us$ then they must contain each letter of L_u in their slice. The foregoing argument shows that this cannot happen.

Now applying the arguments in parts (d) and (e) to each row in R_u completes the proof of part (f) for u.

□

Theorem 5.8 shows that a Latin square Λ with a slender set has fat rows $\tilde{R}_u$, fat columns $\tilde{C}_v$ and fat letters $\tilde{L}_w$, all of size s, so that n must be some multiple ts of s. Parts (c), (d) and (e) of the theorem explicitly assign L_1 to each intersection $\tilde{R}_u \cap \tilde{C}_u$ on the back-diagonal, and L_{u-v+1} to each intersection $\tilde{R}_u \cap \tilde{C}_v$ below the back-diagonal. If $\tilde{R}_u \cap \tilde{C}_v$ is above the back-diagonal then $u < v$ because of the non-standard labelling of the fat columns. In this case, parts (b) and (d) show that if row i is in R_u and column j is in C_v then $\rho_i = n - us$ and $\kappa_j = (v-1)s$, so Equation (5) gives $\lambda(\alpha) = n - (v-u)s = (t - v + u)s$ if $\alpha \in \tilde{R}_u \cap \tilde{C}_v$. Then part (c) shows that the letters which occur in $\tilde{R}_u \cap \tilde{C}_v$ are precisely those in $L_{t-v+u+1}$. Relabelling fat row u as $u-1$, fat column v as $t-v$ and fat letter w as $w-2$, all modulo t, gives the back-cyclic Latin square of order t in Construction 2. Therefore Λ is an s-fold inflation of a back-cyclic Latin square of order t.

The elementary abelian 2-group has no cyclic quotient of order greater than 2, and so the only inflation of a corner set which occurs in its Cayley table is a subsquare corresponding to a subgroup of index 2. Thus we recover the result of Gavrilyuk and Goryainov [26] which was the starting point.

6 -3-perfect sets

Let Δ_1 be a non-empty proper subset of the set Ω of vertices of a Latin-square graph, where $|\Omega| = n^2$ and $|\Delta_1| = m$. Let Δ_2 be the complement of Δ_1, so that $|\Delta_2| = n^2 - m$. The *contrast* between Δ_1 and Δ_2 is defined to be any non-zero multiple of the vector $\mathbf{z}$ which takes the value $n^2 - m$ on each element of Δ_1 and the value $-m$ on each element of Δ_2. Now Δ_1 and Δ_2 are -3-perfect sets if and only if this contrast is in V_2, which happens if and only if the entries in $\mathbf{z}$ sum to zero on each row, column and letter. This means that the partition Δ is *strictly orthogonal* to each of R, C and L: see [3, p. 8]. In the special case that $|\Delta_1| = |\Delta_2|$, this means that $\{R, C, L, \Delta\}$ forms an *orthogonal array of strength two* on Ω.

If Δ_1 is any *transversal* for Λ (a set of cells meeting each row, column and letter just once) then Δ satisfies this condition. More generally, Δ satisfies this condition if Δ_1 meets each row, column and letter of Λ in a constant number ℓ of cells. Such a subset was called an *ℓ-plex* by Wanless in [32]. Freeman called 2-plexes and 3-plexes *duplexes* and *triplexes* respectively in [21, 22]. Thus a -3-perfect set is precisely a *plex*, that is, an ℓ-plex for some value of ℓ. A partition of Ω is -3-equitable if and only if its parts are plexes. In [16, 17, 18], Finney called such a partition an *orthogonal partition* of the Latin square Λ, and found such partitions for Latin squares of orders 4, 5 and 6. Thus, we have the following:

Theorem 6.1 *A -3-perfect subset of the Latin square graph Γ of Λ is the set of cells of a plex in Λ. A -3-equitable partition of Γ arises from a partition of the set of cells into plexes, that is, an orthogonal partition of Λ.*

Egan and Wanless called such a partition *indivisible* in [15] if no proper refinement is -3-equitable. Just as our Section 5 finds minimal $(n-3)$-perfect sets, some minimal -3-perfect sets are found in [9, 14].

The remainder of this section gives a few examples of -3-perfect sets and -3-equitable partitions.

Example 2 Figure 7(a) shows a Graeco-Latin square of order 4. Let Λ be the Latin square defined by the Latin letters. Let Δ_1 be the union of Greek letters α and β. Then Δ_1 and its complement give the equitable partition for Γ shown in Figure 7(b).

When $\ell > 1$, it may be possible to find an ℓ-plex which is not a union of disjoint transversals. Then Δ_1 is -3-perfect, and gives an equitable partition of Γ. Figure 8 shows an example. It is not isomorphic to the one in Figure 7(b), even though both have $n = 4$, $k = 9$, $p = 3 = q$ and $c = b = 6$.

In a similar way we can give partitions with more than two parts, where the parts are not unions of transversals.

A	α	B	β	C	γ	D	δ
B	δ	A	γ	D	β	C	α
C	β	D	α	A	δ	B	γ
D	γ	C	δ	B	α	A	β

(a)

A	**B**	C	D
B	A	**D**	**C**
C	**D**	A	B
D	C	**B**	**A**

(b)

Figure 7: The Greek letters α and β in the Graeco-Latin square of order 4 in (a) give the equitable partition shown in (b), where the elements of Δ_1 are shown in bold

A	**B**	C	D
B	C	**D**	**A**
C	**D**	A	B
D	A	**B**	**C**

Figure 8: An equitable partition of the cyclic Latin square of order 4, which has no transversal (the elements of Δ_1 are shown in bold)

Example 3 Figure 9 shows the Latin square of order 7 defined by the Steiner triple system of order 7. The three different fonts show a -3-equitable partition with parts of sizes 7, 14 and 28. This is strictly orthogonal to each of the partitions into rows, columns and letters.

Example 4 If Δ is uniform and strictly orthogonal to each of R, C and L but the size of the parts of Δ is not n, then $\{R, C, L, \Delta\}$ is a *mixed* orthogonal array. These are discussed in [27, Chapter 9], whose Table 9.25 gives many examples with $n = 6$ in which Δ has three parts of size twelve.

One of these examples is shown in Figure 10. The natural order from [27] is used, but the underlying Latin square is isotopic to the Cayley table of a cyclic group.

Table 12.7 of [27] shows that Finney gave more examples for these numbers in [19], and that examples with $n = 10$ and parts of Δ having size 20 were given in [29, 33].

So the problem of -3-perfect sets in Latin-square graphs is equivalent to that of indecomposable plexes, on which work is ongoing.

$\mathcal{A}$	$\mathbf{C}$	$\mathbf{B}$	E	D	G	F
C	$\mathcal{B}$	A	$\mathbf{G}$	F	E	$\mathbf{D}$
B	A	$\mathcal{C}$	F	$\mathbf{G}$	D	$\mathbf{E}$
E	G	$\mathbf{F}$	$\mathcal{D}$	A	$\mathbf{C}$	D
$\mathbf{D}$	F	G	$\mathbf{A}$	$\mathcal{E}$	B	C
G	$\mathbf{E}$	D	C	$\mathbf{B}$	$\mathcal{F}$	A
$\mathbf{F}$	D	E	B	C	$\mathbf{A}$	$\mathcal{G}$

Figure 9: A Latin square of order 7 with a −3-equitable partition into three parts, one of size 7 (calligraphic letters), one of size 14 (bold), and one of size 28

$\mathbf{A}$	$\mathbf{B}$	C	$\mathcal{F}$	$\mathcal{D}$	E
$\mathcal{B}$	$\mathcal{C}$	$\mathbf{A}$	D	E	$\mathbf{F}$
C	A	$\mathcal{B}$	$\mathbf{E}$	$\mathbf{F}$	$\mathcal{D}$
$\mathbf{D}$	$\mathbf{E}$	F	$\mathcal{C}$	$\mathcal{A}$	B
$\mathcal{E}$	$\mathcal{F}$	$\mathbf{D}$	A	B	$\mathbf{C}$
F	D	$\mathcal{E}$	$\mathbf{B}$	$\mathbf{C}$	$\mathcal{A}$

Figure 10: A mixed orthogonal array, giving a uniform −3-equitable partition of a Latin square of order 6 into three parts (indicated by bold, calligraphic and normal fonts)

7 Mixed equitable partitions

We content ourselves with two examples of equitable partitions of Latin-square graphs where both non-principal eigenvalues occur.

Example 5 As in any strongly regular graph, the distance partition with respect to a vertex α (whose classes are $\{\alpha\}$, the vertices adjacent to α, and the rest) is equitable: all three eigenvalues occur [25, §4.5].

Example 6 In Theorem 4.1, we observed that, if Λ is an s-fold inflation of a Latin square Λ_0 of order t, then the partition of Λ into subsquares is equitable, and has t^2 parts. We saw that all three eigenvalues occur if and only if $t > 2$.

8 Mutually orthogonal Latin squares

If $\Lambda_1, \ldots, \Lambda_{m-2}$ are mutually orthogonal Latin squares of order n, then we can form a graph whose vertices are the cells, two vertices being joined if they lie in the same row or column or have the same letter in one of the squares. This graph is strongly regular with valency $m(n-1)$ and other eigenvalues $n-m$ and $-m$. This raises the possibility of determining the $(n-m)$-perfect sets in this graph, as a generalisation of the main theorem of this paper. However, we expect that this will be more difficult.

Example 7 The set of cells given by any row, column, or letter in any of the squares Λ_i is $(n-m)$-perfect.

Example 8 There is an inflation construction for sets of $m-2$ mutually orthogonal Latin squares, as follows. We are given $m-2$ MOLS of order t, say $\Lambda_1, \ldots, \Lambda_{m-2}$. Let s be a number for which $m-2$ MOLS of order s exist. Choose $(m-2)t$ pairwise disjoint alphabets $\mathcal{A}_{kl}$, for $k = 1, \ldots, m-2$ and $l = 1, \ldots, t$. Each square of the inflation will be partitioned into $t \times t$ subsquares each of size $s \times s$. For the (i,j) subsquare, we choose $m-2$ MOLS of order s, where the alphabet for the kth square is $\mathcal{A}_{kl}$, where l is the symbol in position (i,j) in Λ_k. If $t = m-1$, then each $s \times s$ array is the set of cells of an $(n-m)$-perfect set, where $n = st$. Hence any union of such sets is $(n-m)$-perfect.

Figure 11 shows a pair of orthogonal Latin squares of order 3, and a 3-fold inflation. For convenience we have used the same orthogonal squares (apart from choice of alphabet) in each of the 3×3 positions except the bottom right.

11	23	32
22	31	13
33	12	21

$a\alpha$	$b\gamma$	$c\beta$	$d\eta$	$e\iota$	$f\theta$	$g\delta$	$h\zeta$	$i\epsilon$
$b\beta$	$c\alpha$	$a\gamma$	$e\theta$	$f\eta$	$d\iota$	$h\epsilon$	$i\delta$	$g\zeta$
$c\gamma$	$a\beta$	$b\alpha$	$f\iota$	$d\theta$	$e\eta$	$i\zeta$	$g\epsilon$	$h\delta$
$d\delta$	$e\zeta$	$f\epsilon$	$g\alpha$	$h\gamma$	$i\beta$	$a\eta$	$b\iota$	$c\theta$
$e\epsilon$	$f\delta$	$d\zeta$	$h\beta$	$i\alpha$	$g\gamma$	$b\theta$	$c\eta$	$a\iota$
$f\zeta$	$d\epsilon$	$e\delta$	$i\gamma$	$g\beta$	$h\alpha$	$c\iota$	$a\theta$	$b\eta$
$g\eta$	$h\iota$	$i\theta$	$a\delta$	$b\zeta$	$c\epsilon$	$d\alpha$	$f\beta$	$e\gamma$
$h\theta$	$i\eta$	$g\iota$	$b\epsilon$	$c\delta$	$a\zeta$	$e\beta$	$d\gamma$	$f\alpha$
$i\iota$	$g\theta$	$h\eta$	$c\zeta$	$a\epsilon$	$b\delta$	$f\gamma$	$e\alpha$	$d\beta$

Figure 11: An inflation of a pair of mutually orthogonal Latin squares of order 3, partitioned into nine 5-perfect sets

Example 9 Recall that a Cameron–Liebler line class $\mathcal{L}$ is a set of lines of the 3-dimensional projective geometry $\mathrm{PG}(3,q)$ such that every spread of $\mathrm{PG}(3,q)$ shares the same number, say x, of lines with $\mathcal{L}$. It follows from [31, Theorem 1(vii)] that every line ℓ of $\mathrm{PG}(3,q)$ intersects $(q+1)x+q^2-1$ lines of $\mathcal{L}$ if $\ell \in \mathcal{L}$, and $(q+1)x$ lines if $\ell \notin \mathcal{L}$. Thus, $\mathcal{L}$ is a (q^2-1)-perfect set of a strongly regular graph Γ (known as the Grassmann graph $J_q(4,2)$) defined on the set of lines of $\mathrm{PG}(3,q)$ with two distinct lines being adjacent if they intersect in a point.

For a vertex ℓ, let $\Gamma_i(\ell)$, denote the set of vertices of Γ at distance i from ℓ, for $i \in \{1,2\}$. It is also shown in [31, Theorem 1(viii)] that

$$|\Gamma_1(\ell) \cap \Gamma_1(\ell') \cap \mathcal{L}| = x + q\,|\{\ell,\ell'\} \cap \mathcal{L}|$$

holds for any two non-adjacent vertices ℓ, ℓ' of Γ. Since $\ell' \in \Gamma_2(\ell)$ and

$$|\Gamma_1(\ell') \cap \Gamma_2(\ell) \cap \mathcal{L}| = |\Gamma_1(\ell') \cap \mathcal{L}| - |\Gamma_1(\ell) \cap \Gamma_1(\ell') \cap \mathcal{L}|$$

holds, one can see that, for given line ℓ, $|\Gamma_1(\ell') \cap \Gamma_2(\ell) \cap \mathcal{L}|$ takes on only two values, which depend on whether $\ell' \in \mathcal{L}$, and thus $\Gamma_2(\ell) \cap \mathcal{L}$ is a perfect set of the graph Σ induced on $\Gamma_2(\ell)$. A straightforward calculation shows that it is a $(q^2 - q - 1)$-perfect set, while the graph Σ, which is the bilinear forms graph $\mathrm{Bil}_q(2 \times 2)$ (see, for example, [7, Chapter 9.5.A]), can be viewed as a graph of $q - 1$ mutually orthogonal Latin squares of order q^2.

Many non-trivial (and non-isomorphic) examples of Cameron–Liebler line classes have been found recently, see [8, 11, 12, 13, 24], which thus give rise to various examples of perfect sets in the corresponding graphs of mutually orthogonal Latin squares.

We do not know of any analogue of a corner set in this situation.

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