PRIME SUBMODULES

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Contents

STATEMENT

This thesis is submitted in accordance with the degree of Doctor of Philosophy in the University of Glasgow. It presents the results of research undertaken by the author between October 1993 and November 1996.

Chapter 1 covers basic material concerning prime submodules and modules which satisfy the radical formula. Similar material can be found in [7], [11], [16], [17], [20], [22], [25], [27] and [33] with the exception of Theorem 1.2.28 and its corollary.

Chapters 2-5 are my own work, with the exception of 5.1 and as well as the other instances indicated within the text.

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SUMMARY

Let R be a ring. A proper submodule K of an R-module M is called *prime* if whenever $r \in R$, $m \in M$ and $rRm \subseteq K$ then $m \in K$ or $rM \subseteq K$. It is clear that prime submodules generalize the usual notion of prime ideals. The radical of a submodule N of M, denoted by $\mathrm{rad}_M(N)$ is defined to be the intersection of all prime submodules of M containing N. Now let R be a commutative ring. Let I be an ideal of R. As is well known, the radical of I, defined as the intersection of all prime ideals containing I, has the characterization $\sqrt{I} = \{r \in R : r^n \in I$, for some $n \in \mathbb{Z}^+\}$. A natural question arises, whether there is a somewhat similar characterization for the radical of a submodule, in particular, a characterization in which the knowledge of prime submodules (indeed even prime ideals) is not necessary. Under certain conditions such a characterization is provided by the concept of the envelope of a submodule.

The envelope of N, $E_M(N)$, is the collection of all $m \in M$ for which there exist $r \in R$, $a \in M$ such that $m = ra$ and $r^n a \in N$ for some positive integer n. Always $E_M(N) \subseteq \mathrm{rad}_M(N)$. We say that M satisfies the radical formula (M s.t.r.f.) if for every submodule N of M $\mathrm{rad}_M(N) =< E_M(N) >$, the submodule of M generated by $E_M(N)$. A ring R s.t.r.f. provided that every R-module s.t.r.f.. In [25] McCasland and Moore proved that a commutative ring R s.t.r.f. provided that every free R-mod ule F s.t.r.f.. Accordingly, in chapter 2, prime submodules of free modules over commutative domains are investigated.

A fundamental question in the study of prime submodules is how to describe $\mathrm{rad}_M(N)$ for a given submodule N of a module M. In the first section of chapter 3, $\mathrm{rad}_F(N)$ is described where N is is a finitely generated submodule of the free module F. In the second section the radicals of some non-finitely generated submodules of free modules are studied.

Let M_1, M_2 be R-modules such that $M_1 \oplus M_2$ s.t.r.f.. Then M_1 and M_2 both s.t.r.f.. The converse is not true in general. For example, if R is a Noetherian domain which is not Dedekind then the R-module R s.t.r.f. but the R-module $R \oplus R$ does not. But it is true in some cases and this is considered in the first section of chapter 4. For example, if R is a commutative ring and M_1, M_2 are R-modules such that M_1 s.t.r.f. and M_2 is semisimple, then $M_1 \oplus M_2$ s.t.r.f.. Also if A is a finite direct sum of cyclic Artinian R-modules, then the R-module $R \oplus A$ s.t.r.f.. The aim of the second section is to describe $E_F(N)$ in a nice way, where N is a finitely generated submodule of a free module F of finite rank.

For six different cases, results are tabulated in the following table, considering the following properties of N: "N is prime", "N is semiprime" and "the form of submodule generated by the envelope of N". This table is given for the convenience of the reader. The cases are the following:

(i) Let R be a UFD and let $a_i \in R$ $(1 \leqslant i \leqslant n)$ not all zero. Let N be the submodule $R(a_1, \ldots, a_n)$ of $F = R^{(n)}$.

(ii) Let R be a UFD, let $n \geqslant 3$ be a positive integer and a_i, $b_i \in R$ $(1 \leqslant i \leqslant n)$ such that $R = Rb_1 + \cdots + Rb_n$. Let N be the submodule $R(a_1, \ldots, a_n) + R(b_1, \ldots, b_n)$ of $F = R^{(n)}$.

(iii) Let R be a commutative ring and let a_i, $b_i \in R$ $(i = 1, 2)$ such that $R = Rb_1 + Rb_2$. Let N be the submodule $R(a_1, a_2) + R(b_1, b_2)$ of $F = R^{(2)}$.

(iv) Let R be a commutative domain, let n be a positive integer and I be an ideal of R. Let N be the submodule $I(1, \ldots, 1)$ of $F = R^{(n)}$.

(v) Let R be a UFD, let n be a positive integer and I be an ideal of R. Let N be the submodule $R(a_1, \ldots, a_n) + I(1, \ldots, 1)$ of $F = R^{(n)}$.

(vi) Let R be a domain, let n be a positive integer, let $a_{ij} \in R$ $(1 \leqslant i, j \leqslant n)$, let $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in F = R^{(n)}$ $(1 \leqslant i \leqslant n)$ and let N be the submodule $R\mathbf{a}_1 + \cdots + R\mathbf{a}_n$ of F.

	N is PRIME	N is SEMIPRIME	$< E_F(N) >$
(i)	Theorem 2.2.7	Corollary 3.1.11	Proposition 4.2.1
(ii)	Theorem 2.3.2	Corollary 4.2.5	Theorem 4.2.4
(iii)	Proposition 2.3.4	Corollary 4.2.5	Theorem 4.2.4
(iv)	Lemma 2.3.10	Corollary 4.2.8	Proposition 4.2.3
(v)	Theorem 2.3.12	Corollary 3.2.7	Theorem 3.2.5
(vi)	Proposition 2.3.9	Proposition 4.2.11	Proposition 4.2.11

In [9] Gordon and Robson proved that any ring with Krull dimension satisfies the ascending chain condition on semiprime ideals, but this result does not hold for modules in general. In particular, if R is the first Weyl algebra over a field of characteristic 0 then there are Artinian R-modules which do not satisfy the ascending chain condition on semiprime submodules. The aim of chapter 5 is to investigate when Gordon and Robson's result holds for modules. It is proved that if R is a ring which satisfies a polynomial identity then any R-module with Krull dimension satisfies the ascending chain condition on prime submodules, and, if R is left Noetherian, also the ascending chain condition on semiprime submodules.

Chapter 1

PRELIMINARIES

In this chapter we will give basic definitions and some well known results which will be needed in the following chapters. In particular we will define prime submodules, the radical of a submodule and what it means for a module to satisfy the radical formula. We will give fundamental properties as well as recent developments.

Several authors in [4], [5], [6], [7], [14], [17], [18], [19], [25] and [26] have extended the notion of prime ideals of R to prime submodules of M. Following work of McCasland and Moore [24], [25], [26] and of Jenkins and Smith [11], in a series of recent papers Man [20], [21], [22] and Man and Leung [16], have characterized which commutative Noetherian rings satisfy the radical formula (s.t.r.f.). In particular, Man showed that a commutative Noetherian domain R s.t.r.f. if and only if R is Dedekind (see Theorem 1.2.19). Theorem 1.2.27 gives Man and Leung's general result. We also prove that for a commutative (not necessarily Noetherian) domain R the polynomial ring $R[X]$ s.t.r.f. if and only if R is a field (see Theorem 1.2.28). It follows that for any commutative ring R and indeterminates X, Y the polynomial ring $R[X, Y]$ does not satisfy the radical formula.

1.1 Conventions and Basic Definitions

Let R be a ring with identity and M a unital left R-module. We shall write '$N \leqslant M$' to indicate that N is a submodule of M.

For any non-empty subset X of M, the *annihilator of X in R* will be denoted by $\text{ann}_R(X)$, or simply $\text{ann}(X)$, i.e. $\text{ann}(X) = \{r \in R : rx = 0\ (x \in X)\}$. If A is a non-empty subset of R we set $\text{ann}_M(A) = \{m \in M : am = 0\ (a \in A)\}$. Note that $\text{ann}_M(A)$ is a submodule of M if A is a right ideal of R. For any submodule N of M we shall denote $\text{ann}(M/N)$ by $(N : M)$, i.e. $(N : M) = \{r \in R : rM \subseteq N\}$ which is an ideal of R.

We define the *spectrum* of R to be the set of all prime ideals of R and denote it by $\text{Spec}(R)$.

1.1.1 Modules over a General Ring

Let R be a ring and let M be a left R-module.

Definition 1.1.1.1 *A proper submodule K of M is called prime if whenever $r \in R$, $m \in M$ and $rRm \subseteq K$ then $m \in K$ or $r \in (K : M)$. A submodule S of M is called semiprime if S is an intersection of prime submodules of M.*

It is not difficult to see that N is a prime submodule of M if and only if $(N : K) = (N : M)$ for all submodules K of M properly containing N. Clearly any prime (two sided) ideal of the ring R is a prime submodule of the left R-module R. However it is not difficult to give examples of modules which have no prime submodules. For example, if $\mathbb{Z}$ denotes the ring of rational integers then, for any prime p, as a $\mathbb{Z}$-module, the Prüfer group $\mathbb{Z}(p^\infty)$ has no prime submodules. Moreover, the zero submodule is the only prime submodule of the $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers.

Definition 1.1.1.2 *A left R-module M is called fully faithful if every non-zero submodule of M is faithful.*

Proposition 1.1.1.3 *[27, Proposition 1.1] A submodule N of a left R-module M is prime if and only if $\mathcal{P}$=(N:M) is a prime ideal of the ring R (and we say N is a $\mathcal{P}$-prime) and the left $(R/\mathcal{P})$-module M/N is fully faithful.*

Proof. ($\Rightarrow$) Suppose first that N is a prime submodule of M. Let $a, b \in R$ such that $aRb \subseteq \mathcal{P}$ then $aRbm \subseteq N$ for every $m \in M$. Since N is prime, this implies either $aM \subseteq N$ or $bm \in N$ for every $m \in M$. Thus $a \in \mathcal{P}$ or $b \in \mathcal{P}$. Hence $\mathcal{P}$ is a prime ideal. Let K be a submodule of M such that $N \subsetneqq K$. Let $(r + \mathcal{P})(K/N) = N$ i.e. $rK \subseteq N$ for some $r \in R$. This implies $r \in \mathcal{P} = (N : M)$ or $K \subseteq N$. But $K \subseteq N$ gives a contradiction. Hence $r \in \mathcal{P}$ and K/N is faithful for every submodule K of M properly containing N.

($\Leftarrow$) Now let $(N : M) = \mathcal{P}$ be a prime ideal of R and M/N be a fully faithful $(R/\mathcal{P})$-module. It is sufficient to prove that $(N : K) = (N : M)$ for every submodule K of M properly containing N. Let $r \in (N : K)$. Since M/N is a fully faithful $(R/\mathcal{P})$-module $r \in \mathcal{P}$. Thus $(N : K) \subseteq (N : M)$. Hence $(N : K) = (N : M)$. $\square$

Note: When R is a commutative domain, fully faithful modules coincide with torsion-free modules.

A prime submodule N of M is called *minimal* over a submodule K of M if, $K \subseteq N$ and there does not exist a prime submodule L of M such that $K \subseteq L \subset N$.

Lemma 1.1.1.4 *[27, Theorem 4.2] Let R be a ring, and let M be a Noetherian left R-module. Then M contains only a finite number of minimal prime submodules.*

Proof. Suppose that the result is false. Let Λ denote the collection of proper submodules N of M such that the module M/N has an infinite number of minimal prime submodules. The collection Λ is nonempty, because $0 \in \Lambda$ and, hence, has a maximal member K. Clearly, K is not a prime submodule of M. Thus, there exists a submodule L of M properly containing K and an ideal A in R such that $AL \subseteq K$ but $AM \nsubseteq K$. Hence $K \subset K + AM$. Let V be a submodule of M containing K such that V/K is a minimal prime submodule of M/K. Then $AL \subseteq K \subseteq V$. It is easy to see that, in this case V is a prime submodule of M. Hence $AM \subseteq V$ or $L \subseteq V$. This implies $V/(K + AM)$ is a minimal prime submodule of $M/(K + AM)$ or V/L is a minimal prime submodule of M/L. But by the choice of K, both the modules $M/(K + AM)$ a nd M/L have only finitely many minimal prime submodules. Thus, there are only a finite number of possibilities for the module V and, hence, also for V/K, a contradiction. □

Definition 1.1.1.5 *Given a submodule N of a module M, the prime radical $rad_M(N)$ is the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule then $rad_M(N)$ is defined to be M; in particular $rad_M(M) = M$.*

Lemma 1.1.1.6 *[11, Lemma 4] Let R be a ring and M be an R-module. If $L \subseteq N$ are submodules of M then $rad_N(L) \subseteq rad_M(L)$.*

Proof. Let P be any prime submodule of M with $L \subseteq P$. If $N \subseteq P$ then $\mathrm{rad}_N(L) \subseteq P$. If $N \nsubseteq P$ then it is easy to check that $N \cap P$ is a prime submodule of N, and hence $\mathrm{rad}_N(L) \subseteq N \cap P \subseteq P$. Thus in any case, $\mathrm{rad}_N(L) \subseteq P$. It follows that $\mathrm{rad}_N(L) \subseteq \mathrm{rad}_M(L)$. □

1.1.2 Modules over a Commutative Ring.

Throughout this subsection all rings will be commutative.

Definition 1.1.2.1 *Let R be a ring. The envelope of N, $E_M(N)$, is the collection of all $m \in M$ for which there exist $r \in R$, $a \in M$ such that $m = ra$ and $r^n a \in N$ for some positive integer n. Obviously, $E_M(M) = M$. We say that M satisfies the radical formula (M s.t.r.f.) if for every $N \leqslant M$ the radical of N is the submodule generated by its envelope, i.e. $rad_M(N) =< E_M(N) >$. A ring R satisfies the radical formula (R s.t.r.f.) provided that every R-module s.t.r.f..*

Lemma 1.1.2.2 *Let R be ring and let N be a submodule of an R-module M. Then $N \subseteq E_M(N) \subseteq< E_M(N) >\subseteq rad_M(N)$. In particular, if N is semiprime then $N = E_M(N) =< E_M(N) >= rad_M(N)$.*

Proof. It is clear that $N \subseteq E_M(N)$. Let $x \in E_M(N)$. Then $x = rm$ for some $r \in R$, $m \in M$ such that $r^k m \in N$ for some positive integer k. In this case $r^k m \in P$ for every prime submodule of M containing N. Hence $r^{k-1}m \in P$ or $rM \subseteq P$, and in any case $r^{k-1}m \in P$. By induction, it follows that $rm \in P$. Hence $rm \in \mathrm{rad}_M(N)$. Thus $E_M(N) \subseteq< E_M(N) >\subseteq \mathrm{rad}_M(N)$.

If N is semiprime then $N = \mathrm{rad}_M(N)$. Thus $N = E_M(N) =< E_M(N) >= \mathrm{rad}_M(N)$. $\square$

Note that in Lemma 1.1.2.2, $E_M(N)$ is a submodule of M in case N is a semiprime submodule of M. The following example shows that in general $E_M(N)$ is not a submodule of M.

Example 1.1.2.3 *Let M denote the free $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$ and let N denote the submodule $\mathbb{Z}(4,4) + \mathbb{Z}(9,18)$ of M. Then $E_M(N)$ is not a submodule of M.*

Proof. Note that $(2,2)$ and $(3,6)$ both belong to $E_M(N)$ because $(2,2) = 2(1,1)$, and $2^2(1,1) \in N$, $(3,6) = 3(1,2)$ and $3^2(1,2) \in N$. Suppose that $(5,8) =$

$(2,2)+(3,6) \in E_M(N)$. There exist r, a, $b \in \mathbb{Z}$ such that $(5,8) = r(a,b)$ and $r^k(a,b) \in N$ for some positive integer k. Now $5 = ra$, $8 = rb$ gives that $r = \mp 1$, so that $(a,b) \in N$, i.e. $(5,8) = x(4,4) + y(9,18)$, for some x, $y \in \mathbb{Z}$. Hence $5 = 4x + 9y$, $8 = 4x + 18y$ and $3 = 9y$, a contradiction. Thus $E_M(N)$ is not a submodule of M. $\square$

The first part of the following lemma is a generalized version of Lemma 6 in [11].

Lemma 1.1.2.4 *Let R be a ring and M be an R-module such that $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is a direct sum of submodules M_λ $(\lambda \in \Lambda)$. For each $\lambda \in \Lambda$, let N_λ be a submodule of M_λ and let $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$. Then*

(i) $rad_M(N) = \bigoplus_{\lambda \in \Lambda} rad_{M_\lambda}(N_\lambda)$,

(ii) $< E_M(N) >= \bigoplus_{\lambda \in \Lambda} < E_{M_\lambda}(N_\lambda) >$.

Proof. (i) Let K be a prime submodule of M such that $N \subseteq K$. For each $\lambda \in \Lambda$, $N_\lambda \subseteq K \cap M_\lambda$ where $K \cap M_\lambda = M_\lambda$ or $K \cap M_\lambda$ is a prime submodule of M_λ. It follows that $\mathrm{rad}_{M_\lambda} N_\lambda \subseteq K \cap M_\lambda \subseteq K$ for all $\lambda \in \Lambda$ and hence $\bigoplus_{\lambda \in \Lambda} \mathrm{rad}_{M_\lambda}(N_\lambda) \subseteq K$. Thus $\bigoplus_{\lambda \in \Lambda} \mathrm{rad}_{M_\lambda}(N_\lambda) \subseteq \mathrm{rad}_M(N)$.

Let $m \in M$ and suppose that $m \notin \bigoplus_{\lambda \in \Lambda} \mathrm{rad}_{M_\lambda}(N_\lambda)$. There exists $\mu \in \Lambda$ such that $\pi_\mu(m) \notin \mathrm{rad}_{M_\mu}(N_\mu)$, where $\pi_\mu : M \to M_\mu$ denotes the canonical projection. There exists a prime submodule P of M_μ such that $N_\mu \subseteq P$ and $\pi_\mu(m) \notin P$. If $L = P \oplus (\bigoplus_{\lambda \neq \mu} M_\lambda)$ then it is easy to check that L is a prime submodule of M, $N \subseteq L$ and $m \notin L$. Thus $m \notin \mathrm{rad}_M(N)$. Hence $\mathrm{rad}_M(N) = \bigoplus_{\lambda \in \Lambda} \mathrm{rad}_{M_\lambda}(N_\lambda)$.

(ii) Let $m \in < E_M(N) >$. Then $m = r_1 x_1 + \cdots + r_n x_n$ for some positive integer n, elements $r_i \in R$, $x_i \in M$ such that $r_i^k x_i \in N$ $(1 \leqslant i \leqslant n)$, for some positive integer k. Let $1 \leqslant i \leqslant n$. There exists a finite subset Λ' of Λ such that $x_i \in \bigoplus_{\lambda \in \Lambda'} M_\lambda$, so that $x_i = \sum_{\lambda \in \Lambda'} y_\lambda$ for some $y_\lambda \in M_\lambda$ $(\lambda \in \Lambda')$. Now

$$r_i^k x_i = \textstyle\sum_{\lambda \in \Lambda'} r_i^k y_\lambda \in N = \bigoplus_{\lambda \in \Lambda} N_\lambda.$$

Thus $r_i^k y_\lambda \in N_\lambda$ $(\lambda \in \Lambda')$. Hence $r_i y_\lambda \in E_{M_\lambda}(N_\lambda)$ $(\lambda \in \Lambda')$. Therefore

$$r_i x_i = \textstyle\sum_{\lambda \in \Lambda'} r_i y_\lambda \in \bigoplus_{\lambda \in \Lambda'} < E_{M_\lambda}(N_\lambda) > \subseteq \bigoplus_{\lambda \in \Lambda} < E_{M_\lambda}(N_\lambda) >,$$

for each $1 \leqslant i \leqslant n$. It follows that $m = r_1 x_1 + \cdots + r_n x_n \in \bigoplus_{\lambda \in \Lambda} < E_{M_\lambda}(N_\lambda) >$. Hence $< E_M(N) > \subseteq \bigoplus_{\lambda \in \Lambda} < E_{M_\lambda}(N_\lambda) >$.

Conversely, it is clear that $E_{M_\lambda}(N_\lambda) \subseteq E_M(N)$ and hence $< E_{M_\lambda}(N_\lambda) > \subseteq < E_M(N) >$ for all $\lambda \in \Lambda$. Thus $\bigoplus_{\lambda \in \Lambda} < E_{M_\lambda}(N_\lambda) > \subseteq < E_M(N) >$. It follows that $< E_M(N) > = \bigoplus_{\lambda \in \Lambda} < E_{M_\lambda}(N_\lambda) >$. $\square$

Let $\mathcal{P}$ be a prime ideal of R and $S = R \backslash \mathcal{P}$ which is a multiplicatively closed subset of R containing 1. $M_\mathcal{P} = S^{-1}M$ will denote the localisation of M at $\mathcal{P}$. Let $f : M \to M_\mathcal{P}$ be the natural map defined by $f(m) = m/1$ for all $m \in M$. For any submodule N of M, we define

$$N^e = \{\lambda \in M_\mathcal{P} : \lambda = n/s \text{ for some } n \in N \text{ and } s \in S\},$$

and we identify N^e with $N_\mathcal{P}$. For any $R_\mathcal{P}$-submodule Q of M, we define $Q^c = \{m \in M : f(m) \in Q\}$.

Lemma and Definition 1.1.2.5 *[34] Let R be a ring and I be an ideal of R. Then*

$$\sqrt{I} := \{r \in R : \textit{there exists } n \in \mathbb{N} \textit{ with } r^n \in I\}$$

is an ideal of R which contains I, and is called the radical of I and

$$\sqrt{I} = \bigcap_{\substack{\mathcal{P} \in Spec(R) \\ \mathcal{P} \supseteq I}} \mathcal{P}.$$

Proposition 1.1.2.6 *([20]) Let R be a ring and M be an R-module and $\mathcal{P}$ be a prime ideal of R. Let*

$A = \{P : P$ is a prime submodule of the R-module M with $S \cap (P : M) = \emptyset\}$,

and

$B = \{Q : Q$ *is a prime submodule of the* $R_{\mathcal{P}}$*-module* $M_{\mathcal{P}}\}$.

Then the map $P \mapsto P^e$ *is a bijective order preserving map from* A *to* B. *Its inverse map is given by* $Q \mapsto Q^c$.

Proof. Elementary. $\square$

Lemma 1.1.2.7 *[20, Corollary 2.3] Let* N *be a submodule of the* R*-module* M *and* $\mathcal{P}$, $M_{\mathcal{P}}$ *be as above. Suppose furthermore,* M *is a Noetherian* R*-module. Then* $(rad_M(N))_{\mathcal{P}} = rad_{M_{\mathcal{P}}}(N_{\mathcal{P}})$.

Proof. If $N_{\mathcal{P}} = M_{\mathcal{P}}$, then $\mathrm{rad}_{M_{\mathcal{P}}}(N_{\mathcal{P}}) = M_{\mathcal{P}} = (\mathrm{rad}_M(N))_{\mathcal{P}}$. Now suppose $M_{\mathcal{P}} \neq N_{\mathcal{P}}$. As M is a Noetherian R-module, by Lemma 1.1.1.4, there are only a finite number of minimal prime R-submodules, $P_1, \ldots, P_k$, in M containing N. Now it can easily be checked that

$$(\mathrm{rad}_M(N))_{\mathcal{P}} = (\textstyle\bigcap_{i=1}^k P_i)_{\mathcal{P}} = (\bigcap_{i=1}^k P_i)^e = \bigcap_{i=1}^k P_i^e.$$

Without loss of generality, we may assume each $P_i^e \neq M_{\mathcal{P}}$ $(1 \leqslant i \leqslant k)$. By Proposition 1.1.2.6, $P_1^e, \cdots, P_k^e$ are all the minimal prime $R_{\mathcal{P}}$-submodules of $M_{\mathcal{P}}$ which contains $N_{\mathcal{P}}$. It follows that $\mathrm{rad}_{M_{\mathcal{P}}}(N_{\mathcal{P}}) = \bigcap_{i=1}^k P_i^e$ as required. $\square$

Proposition 1.1.2.8 *[17, Proposition 2] If* N *is a proper submodule of an* R*-module* M *such that (N:M) is a maximal ideal of the commutative ring* R *then* N *is a prime submodule. In particular,* $\mathcal{M}M$ *is a prime submodule of the* R*-module* M *for every maximal ideal* $\mathcal{M}$ *of* R *such that* $\mathcal{M}M \neq M$.

Proof. Since $(N : M) = \mathcal{P}$ a maximal ideal, M/N is a vector space over the field $R/\mathcal{P}$, so a torsion-free $R/\mathcal{P}$-module. Hence N is prime by Proposition 1.1.1.3. $\square$

Proposition 1.1.2.9 *[17, Proposition 4] If N is a maximal submodule of an R-module M, then N is a prime submodule and (N:M) is a maximal ideal of R.*

Proof. N is a maximal submodule if and only if M/N is a simple R-module. Hence M/N is a cyclic R-module $R\overline{x}$ where $\overline{x} = x + M \in M/N$ and $\text{ann}_R\overline{x} = \text{ann}_R(M/N) = (N : M)$ is a maximal ideal of R by [34, Lemma 7.32]. It follows that N is prime from Proposition 1.1.2.8. □

1.2 Historical Background and Recent Developments

Lemma 1.2.1 *Let I be a proper ideal of a commutative ring R such that R s.t.r.f.. Then the ring R/I s.t.r.f..*

Proof. Let M be an (R/I)-module. Then M is an R-module and the (prime) R-submodules and (prime) (R/I)-submodules of M coincide. The result follows. □

Proposition 1.2.2 *Let n be a positive integer and let R_i $(1 \leqslant i \leqslant n)$ be commutative rings. Then the ring $R = R_1 \oplus \cdots \oplus R_n$ s.t.r.f. if and only if R_i s.t.r.f. for all $1 \leqslant i \leqslant n$.*

Proof. $(\Rightarrow)$ By Lemma 1.2.1.

$(\Leftarrow)$ Let M be an R-module. Let $M_i = R_iM$ $(1 \leqslant i \leqslant n)$. Then M_i is an R-submodule of M for each $1 \leqslant i \leqslant n$ and $M = M_1 \oplus \cdots \oplus M_n$. By Lemma 1.1.2.4,

$$\text{rad}_M(0) = \text{rad}_{M_1}(0) \oplus \cdots \oplus \text{rad}_{M_n}(0).$$

For each $1 \leqslant i \leqslant n$, the R-module M_i has the same (prime) submodules as the R_i-module M_i and hence $\text{rad}_{M_i}(0) \subseteq < E_{M_i}(0) >$. It follows that $\text{rad}_M(0) \subseteq < E_M(0) >$ by Lemma 1.1.2.4 and hence $\text{rad}_M(0) =< E_M(0) >$. $\square$

Proposition 1.2.3 *[20, Proposition 2.4] Let M be a Noetherian R-module. Then M s.t.r.f. if and only if $M_{\mathcal{M}}$ s.t.r.f. as an $R_{\mathcal{M}}$-module for every maximal ideal $\mathcal{M}$ of R.*

Proof. Let N be a submodule of M. It is not difficult to check that $< E_M(N) >_{\mathcal{P}} =< E_{M_{\mathcal{P}}}(N_{\mathcal{P}}) >$ for any prime ideal $\mathcal{P}$. The result follows from Lemma 1.1.2.7. $\square$

Definition 1.2.4 *A commutative ring R which has exactly one maximal ideal, $\mathcal{M}$ say, is said to be quasi-local. By a local ring we shall mean a commutative Noetherian ring which is quasi-local.*

Theorem 1.2.5 *[33, Theorem 1.12] Let R be a commutative Artinian ring. Then R s.t.r.f..*

Proof. Let R be a commutative Artinian ring. Then by [34, Exercise 8.50], R is isomorphic to a direct sum of Artinian local rings. By Proposition 1.2.2, we can suppose without loss of generality that R is local with unique maximal ideal $\mathcal{M}$. So $\mathcal{M}^n = 0$ for some $n > 0$ (see [3, p.90]). Thus if N is a submodule of M, $\mathcal{M}^n M \subseteq N$ which implies that $\mathcal{M}M \subseteq< E_M(N) >$. That is, $\mathcal{M} \subseteq (< E_M(N) >: M)$. Thus $< E_M(N) >$ is a prime submodule or $< E_M(N) >= M$ by Proposition 1.1.2.8. Therefore $\text{rad}_M(N) =< E_M(N) >$. $\square$

Lemma 1.2.6 *[25, Results 1.2, 1.3, 1.4] Let R be a commutative ring. Let A and A' be R-modules with $\varphi : A \to A'$ an R-module epimorphism and $B \leqslant A$ such that $B \supseteq K = ker\varphi$. Let B' be any submodule of A'. Then*

(i) if P is a prime submodule of A containing B then $\varphi(P)$ is a prime submodule of A' containing $\varphi(B)$;

(ii) if P' is a prime submodule of A' containing $\varphi(B)$ then $\varphi^{-1}(P')$ is a prime submodule of A containing B;

(iii) $\varphi(rad_A(B)) = rad_{A'}(\varphi(B))$;

(iv) $\varphi^{-1}(rad_{A'}(B')) = rad_A(\varphi^{-1}(B'))$;

(v) $\varphi(E_A(B)) = E_{A'}(\varphi(B))$;

(vi) $< \varphi^{-1}(E_{A'}(B')) >=< E_A(\varphi^{-1}(B')) >$.

Proof. (i), (ii), (iii), (iv) and (v) are routine.

(vi) Given $r \in R$, $a \in A$ and $ra \in E_A(\varphi^{-1}(B'))$ such that $r^n a \in \varphi^{-1}(B')$ for some positive integer n, then $r^n\varphi(a) \in B'$. Hence $\varphi(ra) \in E_{A'}(B')$ and thus $ra \in \varphi^{-1}(E_{A'}(B'))$. Now let $x \in \varphi^{-1}(E_{A'}(B'))$. Since $\varphi(x) \in E_{A'}(B')$, there exist $s \in R$, $a' \in A'$ and a positive integer m such that $\varphi(x) = sa'$ and $s^n a' \in B'$. Also there exists $y \in A$ such that $\varphi(y) = a'$. Thus $\varphi(s^n y) \in B'$ so that $sy \in E_A(\varphi^{-1}(B'))$. Hence $x \in< E_A(\varphi^{-1}(B')) >$ since $x - sy \in ker\varphi \subseteq$ $< E_A(\varphi^{-1}(B')) >$. □

Proposition 1.2.7 *[25, Theorem 1.5] Assume the hypothesis given in Lemma 1.2.6.*

(i) If $rad_A(B) =< E_A(B) >$ then $rad_{A'}(\varphi(B)) =< E_{A'}(\varphi(B)) >$.

(ii) If $B' \leqslant A'$ and $rad_{A'}(B') =< E_{A'}(B') >$, then $rad_A(\varphi^{-1}(B')) =$ $< E_A(\varphi^{-1}(B')) >$.

Proof. It is routine to prove it by using the above results and the fact that if $0 \in S \subseteq A'$, then $\varphi^{-1}(< S >) =< \varphi^{-1}(S) >$. □

Theorem 1.2.8 *[25, Theorem 1] Let R be a commutative ring. Then R s.t.r.f. provided that any one of the following is satisfied:*

(i) every free R-module F s.t.r.f.,

(ii) every faithful R-module M s.t.r.f.,

(iii) for every R-module M, $rad_M(0) \subseteq < E_M(0) >$.

Proof. (i) and (ii) suffice by recalling that every R-module A is the image of both a free R-module and a faithful R-module. Note that if $B \leqslant A$, the preimage of B (in each case) satisfies the conditions of Lemma 1.2.6. Now we can apply Proposition 1.2.7(i).

(iii) For a given $N \leqslant M$, apply Proposition 1.2.7(ii), letting $A = M$, $A' = M/N$ and $B' = N$. $\square$

Let $\mathcal{P}$ be a prime ideal of R and suppose M is an R-module. We define $K(\mathcal{P}) = \{m \in M : cm \in \mathcal{P}M$ for some $c \in R \backslash \mathcal{P}\}$.

Next, we recall a result which was proved both in [1] and [27].

Proposition 1.2.9 *Let R be a commutative ring and M be an R-module. Let $\mathcal{P}$ be a prime ideal of R such that $K(\mathcal{P}) \neq M$. Then $K(\mathcal{P})$ is a $\mathcal{P}$-prime submodule of M and $rad_M(0) = \bigcap K(\mathcal{Q})$, where the intersection is taken over all prime ideals $\mathcal{Q}$ of R.*

Proof. Let $r \notin (K(\mathcal{P}) : M)$, $m \in M$ and $rm \in K(\mathcal{P})$. Then $r \in R\backslash\mathcal{P}$ and $rcm \in \mathcal{P}M$ for some $c \in R\backslash\mathcal{P}$. Since $rc \in R\backslash\mathcal{P}$, we have $m \in K(\mathcal{P})$. This proves that $K(\mathcal{P})$ is a prime submodule of M. Clearly $\mathcal{P} \subseteq (\mathcal{P}M : M) \subseteq (K(\mathcal{P}) : M)$. Now suppose that there is $s \in (K(\mathcal{P}) : M)$ such that $s \notin \mathcal{P}$. Then $sM \subseteq K(\mathcal{P})$. Consequently, for each $y \in M$, we have $scy \in \mathcal{P}M$ for some $c \in R\backslash\mathcal{P}$. But $sc \in R\backslash\mathcal{P}$ gives $y \in K(\mathcal{P})$. Hence $K(\mathcal{P}) = M$, a contradiction. Therefore we have $(K(\mathcal{P}) : M) = (\mathcal{P}M : M) = \mathcal{P}$ and $K(\mathcal{P})$ is a $\mathcal{P}$-prime submodule of M.

Moreover, for any prime ideal $\mathcal{Q}$ of R, if N is a $\mathcal{Q}$-prime submodule of M, then $K(\mathcal{Q}) \subseteq N$. Therefore $\mathrm{rad}_M(0) = \bigcap_{\mathcal{Q}\in Spec(R)} K(\mathcal{Q})$. $\square$

Our next aim is to prove that any Dedekind domain s.t.r.f.. In order to prove this result we require a number of lemmas.

Lemma 1.2.10 *[16, Lemma 3.3] Let R be a commutative Noetherian ring with $dimR \leqslant 1$ and M be an R-module. Then $rad_M(0) = \bigcup rad_L(0)$, where the union is taken over all finitely generated submodules L of M.*

Proof. By Lemma 1.1.1.6, $\mathrm{rad}_L(0) \subseteq \mathrm{rad}_M(0)$, for any finitely generated submodule L of M. Now let $m \in \mathrm{rad}_M(0)$. Let $\mathcal{P}_1, \dots, \mathcal{P}_n$ be all the minimal prime ideals of R. By Proposition 1.2.9 for each $1 \leqslant i \leqslant n$ there exist $c_i \in R\backslash\mathcal{P}_i$ with $c_i m \in \mathcal{P}_i M$. There are only finitely many maximal ideals of R which contains both c_i and $\mathcal{P}_i$, say $\mathcal{M}_{i1}, \cdots, \mathcal{M}_{in_i}$. By Proposition 1.2.9, $\mathrm{rad}_M(0) = \bigcap K(\mathcal{P})$, where all $\mathcal{P}$'s are prime ideals of R. Since $\dim R \leqslant 1$ it follows that $\mathrm{rad}_M(0) = [\bigcap K(\mathcal{P})] \cap [\bigcap \mathcal{M}M]$ where the intersection is taken over all the minimal prime ideals $\mathcal{P}$ and all the maximal ideals $\mathcal{M}$ of R. Hence $m \in \mathcal{M}_{ij}M$ for every $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n_i$. Together with $c_i m \in \mathcal{P}_i M$ $(1 \leqslant i \leqslant n)$, we see that there exists a finitely generated submodule L of M such that

(i) $c_i m \in \mathcal{P}_i L$ $(1 \leqslant i \leqslant n)$,

(ii) $m \in \mathcal{M}_{ij} L$ $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n_i)$.

Now let $\mathcal{M}$ be any maximal ideal of R such that $\mathcal{M} \notin \{\mathcal{M}_{ij} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n_i\}$ $(1 \leqslant i \leqslant n)$. Without loss of generality we may assume $\mathcal{P}_1 \subseteq \mathcal{M}$. Then $R = Rc_1 + \mathcal{M}$ and hence $Rm = Rc_1 m + \mathcal{M}m \subseteq \mathcal{M}L$ since $c_1 m \in \mathcal{P}_1 L \subseteq \mathcal{M}L$ and $m \in L$. By Proposition 1.2.9, $m \in \mathrm{rad}_L(0)$. Hence $\mathrm{rad}_M(0) \subseteq \bigcup \mathrm{rad}_L(0)$. $\square$

Lemma 1.2.11 *[11, Lemma 7 and Corollary] Let R be any ring and M any projective R-module. Then $rad_M(0) =< E_M(0) >$.*

Proof. We already know that $< E_M(0) > \subseteq \mathrm{rad}_M(0)$. There exists a free R-module F such that M is a direct summand of F, say $F = M \oplus A$, for some submodule A of F. There exists an index set Λ and cyclic submodules F_λ $(\lambda \in \Lambda)$ of F such that $F = \bigoplus_{\lambda\in\Lambda} F_\lambda$. By Lemma 1.1.2.4,

$$\mathrm{rad}_F(0) = \bigoplus_{\lambda\in\Lambda} \mathrm{rad}_{F_\lambda}(0) = \bigoplus_{\lambda\in\Lambda} < E_{F_\lambda}(0) > = < E_F(0) > .$$

Now let $m \in \mathrm{rad}_M(0)$. By Lemma 1.1.1.6, $m \in \mathrm{rad}_F(0)$. Then there exist n, $k \in \mathbb{N}$ and elements $r_i \in R$, $m_i \in F$ such that $r_i^k m_i = 0$ $(1 \leqslant i \leqslant n)$, and $m = r_1m_1 + \cdots + r_nm_n$. For every $1 \leqslant i \leqslant n$, there exist elements $x_i \in M$ and $a_i \in A$ such that $m_i = x_i + a_i$. Clearly, $m = r_1x_1 + \cdots + r_nx_n$, and $r_i^k x_i = 0$ $(1 \leqslant i \leqslant n)$. Thus $m \in < E_M(0) >$. Hence $\mathrm{rad}_M(0) \subseteq < E_M(0) >$. □

Now we prove that any Dedekind domain s.t.r.f..

Theorem 1.2.12 *[11, Theorem 9] Let R be a Dedekind domain and M any R-module. Then $rad_M(0) =< E_M(0) >$.*

Proof. We know $< E_M(0) > \subseteq \mathrm{rad}_M(0)$. Let $m \in \mathrm{rad}_M(0)$. By Lemma 1.2.10, $m \in \mathrm{rad}_L(0)$, for some finitely generated submodule L of M. Now $L = L_1 \oplus \cdots \oplus L_k$, for some $k \in \mathbb{N}$ and submodules L_i $(1 \leqslant i \leqslant k)$ of L such that L_i is either projective or cyclic for each $1 \leqslant i \leqslant k$ [13, Section 4]. By Lemma 1.1.2.4 and Lemma 1.2.11,

$$\begin{aligned} m \in \mathrm{rad}_{L_1}(0) \oplus \cdots \oplus \mathrm{rad}_{L_k}(0) &= < E_{L_1}(0) > \oplus \cdots \oplus < E_{L_k}(0) > \\ &\subseteq < E_L(0) > \subseteq < E_M(0) > . \end{aligned}$$

Thus $\mathrm{rad}_M(0) =< E_M(0) >$. □

We now aim to prove that any commutative Noetherian domain that s.t.r.f. is Dedekind. We begin with the following result.

Lemma 1.2.13 *[7, Lemma 4] Let R be a commutative domain and $a_1, \dots, a_n$ be elements of R, not all zero where $n \geqslant 2$. Let $F = R^{(n)}$ and*

$$K = \{(r_1, \dots, r_n) \in F : r_i a_j = r_j a_i, 1 \leqslant i, j \leqslant n\}.$$

Then K is a prime submodule of F minimal over $R(a_1, \dots, a_n)$ and $(K{:}F)=0$. Moreover $a_i K \subseteq R(a_1, \dots, a_n)$ for all $1 \leqslant i \leqslant n$.

Proof. Clearly K is a proper submodule of F. Let $r, z_i \in R$ $(1 \leqslant i \leqslant n)$ and suppose that $r(z_1, \dots, z_n) \in K$. Then $rz_i a_j = rz_j a_i$, for all i, j. If $r = 0$ then $rF \subseteq K$. If $r \neq 0$ then $z_i a_j = z_j a_i$, for all i, j, so that $(z_1, \dots, z_n) \in K$. Thus K is a prime submodule of F. Clearly $(K : F) = 0$ and $R(a_1, \dots, a_n) \subseteq K$. Suppose that $a_1 \neq 0$. Let $I = (Ra_1 : Ra_2 + \cdots + Ra_n)$. Then it can easily be checked that

$$K = I(1, a_2/a_1, \dots, a_n/a_1),$$

and hence $a_1 K \subseteq R(a_1, \dots, a_n)$. If $a_1 = 0$ then clearly $a_1 K \subseteq R(a_1, \dots, a_n)$. It follows that $a_i K \subseteq R(a_1, \dots, a_n)$, for all $1 \leqslant i \leqslant n$.

Now suppose that N is a prime submodule of F such that $R(a_1, \dots, a_n) \subseteq N \subseteq K$. There exists $1 \leqslant i \leqslant n$ such that $a_i \neq 0$ and $a_i K \subseteq R(a_1, \dots, a_n) \subseteq N$. Since $a_i \neq 0$ it follows that $a_i F \nsubseteq K$, and hence $a_i F \nsubseteq N$. Thus $K \subseteq N$ and K is minimal over $R(a_1, \dots, a_n)$. □

Theorem 1.2.14 *[20, Theorem 3.2] Let $(R, \mathcal{M})$ be a commutative Noetherian local domain of dimension 1. Suppose $F = R \oplus R$ s.t.r.f. as an R-module. Then R is a DVR (Discrete Valuation Ring).*

Proof. Choose $x \in \mathcal{M}\backslash\mathcal{M}^2$. It suffices to show that $\mathcal{M} = Rx$. As dim$R$=1 and $x \neq 0$, $\mathcal{M}$ is the only associated prime ideal of R/Rx. Hence every element of $\mathcal{M}$ is a zero divisor in R/Rx. We now show $\mathcal{M} \subseteq Rx + \mathcal{M}^2$.

Let $s \in \mathcal{M}$. By the above discussion, there exist $y \in R\backslash Rx$ and $r \in R$ with $sy = xr$. If $s \in Rx$, then $s \in Rx + \mathcal{M}^2$. Suppose $s \in \mathcal{M}\backslash Rx$. Then y is not a unit and hence $y \in \mathcal{M}\backslash Rx$. Since $x \in \mathcal{M}\backslash\mathcal{M}^2$ and $y \in \mathcal{M}\backslash Rx$, $x \notin Ry$. By Lemma 1.2.13, $K = \{(r_1, r_2) \in R\oplus R : r_1 y = r_2 x\}$ is a minimal prime submodule of $R \oplus R$ over $R(x, y)$. Let P be a prime submodule of $R \oplus R$ containing $R(x, y)$. Then $\mathcal{P}$ =ann$((R \oplus R)/P)$ is a prime ideal.

Clearly $\mathcal{P} = 0$ or $\mathcal{P} = \mathcal{M}$. If $\mathcal{P} = \mathcal{M}$, then $\mathcal{M} \oplus \mathcal{M} \subseteq P$. As $y \in \mathcal{M}\backslash Rx$ and $x \in \mathcal{M}\backslash Ry$, we have $K \subseteq P$. Suppose that $\mathcal{P} = 0$. Since $P \neq R \oplus R$, we may assume $(1, 0) \notin P$. Let $(r_1, r_2) \in P$ be given. Then $(yr_1 - r_2 x)(1, 0) = y(r_1, r_2) - r_2(x, y) \in P$. Since P is a prime submodule, it follows that $yr_1 - r_2 x \in (P : F) = 0$, i.e. $yr_1 - r_2 x = 0$. Thus $P \subseteq K$. By minimality of K, $P = K$. Hence K is the only minimal prime submodule containing $R(x, y)$. Thus $\mathrm{rad}_{R\oplus R}(R(x, y)) = K$. By hypothesis, $K =< E_{R\oplus R}(R(x, y)) >$. Clearly $(s, r) \in K$. Hence there exist $s_1, \ldots, s_k \in R\backslash\{0\}, (c_1, d_1), \ldots, (c_k, d_k) \in R \oplus R\backslash\{(0, 0)\}$, and positive integers, $n_1, \ldots, n_k$ such that

(i) $(s, r) = \sum_{i=1}^{k} s_i(c_i, d_i)$, and

(ii) $s_i^{n_i}(c_i, d_i) = f_i(x, y)$ for some $f_i \in R$, $(1 \leqslant i \leqslant k)$.

Since each $s_i \neq 0$ and R is a domain, by (ii) $c_i y = x d_i$ $(1 \leqslant i \leqslant n)$. Recall that $y \in \mathcal{M}\backslash Rx$. Consequently, each $c_i \in \mathcal{M}$. If s_i is a unit, then (ii) gives $s_i c_i \in Rx$. If $s_i \in \mathcal{M}$, then $s_i c_i \in \mathcal{M}^2$. Hence $s_i c_i \in Rx + \mathcal{M}^2$ for all $1 \leqslant i \leqslant k$. Now by (i) $s \in Rx + \mathcal{M}^2$. Therefore $\mathcal{M} = Rx + \mathcal{M}^2$. Hence $\mathcal{M}(\mathcal{M}/Rx) = \mathcal{M}/Rx$. By Nakayama's Lemma, $\mathcal{M} = Rx$. □

Theorem 1.2.15 *[20, Theorem 3.3] Suppose R is a commutative Noetherian domain of dimension 1 and $R \oplus R$ s.t.r.f. as an R-module. Then R is a Dedekind domain.*

Proof. Clear by Proposition 1.2.3 and Theorem 1.2.14. For, by [10, Theorem VIII.6.10], $R_{\mathcal{P}}$ is a DVR for every non-zero prime ideal $\mathcal{P}$ if and only if R is a Dedekind domain. □

Theorem 1.2.16 *[16, Theorem 2.2 and Corollary] Let R be a commutative Noetherian ring. Suppose $F = R \oplus R$ s.t.r.f. as an R-module, $\dim R \geqslant 1$ and $\mathcal{P}$ is a minimal prime ideal of R. Then $\mathcal{P}$ is the only $\mathcal{P}$-primary ideal of R and $R/\mathcal{P}$ is a Dedekind domain. In particular, if R is a domain then R is a Dedekind domain.*

Proof. First assume R is local with maximal ideal $\mathcal{M}$ and 0 is $\mathcal{P}$-primary. We need to show R is a DVR.

As $\dim R \geqslant 1$, $\mathcal{M} \neq \mathcal{P}$. Thus we can choose $a \in \mathcal{M} \backslash (\mathcal{M}^2 + \mathcal{P})$, by Nakayama's Lemma. If $\mathcal{M} \neq Ra$ we can choose $b \in \mathcal{M} \backslash Ra$. Consider the submodule $J(a,b)$ of $R \oplus R$ where $J = Ra + Rb$. Let L be any prime submodule of $R \oplus R$ such that $J(a,b) \subseteq L$. It follows that $(a,b) \in L$ or $JF \subseteq L$. If $JF \subseteq L$ then $(a,b) = a(1,0) + b(0,1) \in L$. In any case, $(a,b) \in L$. Hence $(a,b) \in \mathrm{rad}_{R \oplus R} J(a,b)$ so $\mathrm{rad}_{R \oplus R}(J(a,b)) = \mathrm{rad}_{R \oplus R}(R(a,b))$. As $R \oplus R$ s.t.r.f., we have $\mathrm{rad}_{R \oplus R}(R(a,b)) = < E_{R \oplus R}(J(a,b)) >$. Hence $(a,b) \in < E_{R \oplus R}(J(a,b)) >$. Then there exist positive integers $k, n_1, \ldots, n_k$ and $r_1, \ldots, r_k \in R \backslash \{0\}$, $(c_1, d_1), \cdots, (c_k, d_k) \in R \oplus R \backslash \{(0,0)\}$ such that

(i) $(a,b) = \sum_{i=1}^{k} r_i (c_i, d_i)$, and

(ii) for each $1 \leqslant i \leqslant k$, $r_i^{n_i}(c_i, d_i) = f_i(a,b)$ for some $f_i \in J$.

By (i) $a = \sum_{i=1}^{k} r_i c_i$. We are done if we can show that each $r_i c_i \in \mathcal{M}^2 + \mathcal{P}$. Let $1 \leqslant i \leqslant k$ be given.

(1) If r_i is a unit, then from (ii), we have $r_ic_i \in Ja \subseteq \mathcal{M}^2$.

(2) If $r_i \in \mathcal{P}$, then $r_ic_i \in \mathcal{M}^2 + \mathcal{P}$.

(3) If $r_i \in \mathcal{M}\backslash\mathcal{P}$ then we show $c_i \in \mathcal{M}$. Suppose not. Then from (ii), we have $r_i \in \sqrt{Ra}$ and $r_i^{n_i}(ad_i - bc_i) = 0$. Hence $b \in Ra + [\mathrm{ann}_R(r_i^{n_i}) \cap (Ra + Rb)]$ where $r_i \in \sqrt{Ra}\backslash\mathcal{P}$. Thus $b = ra + c$, for some $r \in R$ and $c \in \mathrm{ann}_R(r_i^{n_i}) \cap (Ra + Rb)$. This implies $cr_i^{n_i} = 0$. If $c \neq 0$, since $\mathcal{P}$ is the set of all zero divisors of R, $r_i^{n_i} \in \mathcal{P}$ i.e. $r_i \in \mathcal{P}$, a contradiction. If $c = 0$ then $b \in Ra$, another contradiction. Therefore $c_i \in \mathcal{M}$ and $r_ic_i \in \mathcal{M}^2 + \mathcal{P}$.

In any case, $r_ic_i \in \mathcal{M}^2 + \mathcal{P}$ for all $1 \leqslant i \leqslant k$. Hence $a \in \mathcal{M}^2 + \mathcal{P}$, but this contradicts our choice of a. Therefore $\mathcal{M} = Ra$ and hence R is a DVR.

For the general case, let I be a $\mathcal{P}$-primary ideal. By Lemma 1.2.1, $R/I \oplus R/I$ s.t.r.f. as an R/I-module. By the earlier argument, we see that $I = \mathcal{P}$ and R/I is a Dedekind domain. The result follows. □

The next result is immediate from the above theorem.

Corollary 1.2.17 *[16, Corollary 2.4] Let R be a commutative Noetherian ring. Suppose $R \oplus R$ s.t.r.f. as an R-module. Then $\dim R \leqslant 1$.*

Proposition 1.2.18 *[16, Theorem 3.4] Let R be a commutative Noetherian ring. Then the following are equivalent:*

(i) R s.t.r.f.,

(ii) $R_{\mathcal{M}}$ s.t.r.f. for any maximal ideal $\mathcal{M}$ of R,

(iii) every finitely generated $R_{\mathcal{M}}$-module s.t.r.f. for any maximal ideal $\mathcal{M}$ of R,

(iv) every finitely generated R-module s.t.r.f..

Proof. $(i) \Rightarrow (ii)$ By Proposition 1.2.3. $(ii) \Rightarrow (iii)$ Obvious. $(iii) \Rightarrow (iv)$ Follows from Proposition 1.2.3. $(iv) \Rightarrow (i)$ Follows from Corollary 1.2.17, Lemma 1.2.10 and Theorem 1.2.8(iii). □

Now the following theorem can be written:

Theorem 1.2.19 *Let R be a commutative Noetherian domain which is not a field. Then the following are equivalent:*

(i) R s.t.r.f.,

(ii) $R \oplus R$ s.t.r.f. as an R-module,

(iii) R is a Dedekind domain.

Proof. $(i) \Rightarrow (ii)$ Obvious. $(ii) \Rightarrow (iii)$ By Corollary 1.2.17 and Theorem 1.2.15. $(iii) \Rightarrow (i)$ By Theorem 1.2.12. □

The above theorem has a general form in [16]. Before we give it we require a number of lemmas.

Lemma 1.2.20 *[16, Proposition 2.5] Let R be a ring. Suppose*

(i) $R/\sqrt{0}$ s.t.r.f. as a ring and

(ii) there exist maximal ideals $\mathcal{M}_i$ and positive integers k_i $(1 \leqslant i \leqslant n)$ with

$$\sqrt{0} \cap \mathcal{M}_1^{k_1} \cap \cdots \cap \mathcal{M}_n^{k_n} = 0.$$

Then R s.t.r.f..

Proof. We can assume all the k_i's are equal to a common value k. Let M be an R-module. By Theorem 1.2.8(iii) it suffices to show $rad_M(0) \subseteq < E_M(0) >$. Clearly, $\sqrt{0}M \subseteq < E_M(0) >$. Let $m \in \mathrm{rad}_M(0)$. Then $m + \sqrt{0}M \in \mathrm{rad}_{M/\sqrt{0}M}(0)$. Since $R/\sqrt{0}$ s.t.r.f., we have

$$m + \sqrt{0}M = \sum_{i=1}^{n} r_i m_i + \sqrt{0}M$$

where $r_i \in R$, $m_i \in M$ and $r_i^{n_i} m_i \in \sqrt{0}M$ for some positive integer n_i. Hence $m = y + \sum_{i=1}^{n} r_i m_i$ for some $y \in \sqrt{0}M$. We need to show each $r_i m_i \in < E_M(0) >$.

Suppose first that $r_i \in \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_n$, then by (ii) we have $r_i^{n_i+k}m_i = r_i^k(r_i^{n_i}m_i) \in (\mathcal{M}_1^k \cap \cdots \cap \mathcal{M}_n^k)\sqrt{0}M = 0$. Hence $r_i m_i \in E_M(0)$. Now suppose that $r_i \notin \mathcal{M}_j$ for some $1 \leqslant j \leqslant n$. Without loss of generality we may assume $r_i \in \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_\ell$, and $r_i \notin \mathcal{M}_{\ell+1} \cup \cdots \cup \mathcal{M}_n$ for some $1 \leqslant \ell \leqslant n$. Then $R = Rr_i^{n_i} + \mathcal{M}_{\ell+1} \cap \cdots \cap \mathcal{M}_n$. Write $1 = sr_i^{n_i} + x$ for some $s \in R$ and $x \in \mathcal{M}_{\ell+1} \cap \cdots \cap \mathcal{M}_n$. Then $r_i m_i = sr_i^{n_i+1}m_i + r_i x m_i$. Since $r_i^{n_i}m_i \in \sqrt{0}M$, $sr_i^{n_i+1}m_i$ and $(r_i x)^{n_i}m_i$ are also in $\sqrt{0}M$. In particular, $sr_i^{n_i+1}m_i \in< E_M(0) >$. On the other hand, $r_i x \in \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_n$. By an earlier argument, $r_i x m_i \in E_M(0)$. This proves $r_i m_i \in< E_M(0) >$. □

Lemma 1.2.21 *[16, Proposition 2.6] Let R be a Noetherian ring. Suppose dimR=1 and every minimal prime ideal $\mathcal{P}$ of R is the only $\mathcal{P}$-primary ideal in R. Then condition (ii) of Lemma 1.2.20 is satisfied.*

Proof. Let $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ be all the minimal prime ideals of R, for some positive integer ℓ. Since R is Noetherian, 0 has a reduced primary decomposition which can be written as follows

$$0 = J_1 \cap \cdots \cap J_\ell \cap I_1 \cap \cdots \cap I_n$$

for some positive integers ℓ and n where each J_i is a $\mathcal{P}_i$-primary ideal and each I_j is $\mathcal{M}_j$-primary ideal for some maximal ideal $\mathcal{M}_j$. By assumption, $J_i = \mathcal{P}_i$ $(1 \leqslant i \leqslant \ell)$. Therefore, we get $0 = \sqrt{0} \cap I_1 \cap \cdots \cap I_n$. For each $1 \leqslant j$ since I_j is $\mathcal{M}_j$- primary, we have $\mathcal{M}^{k_j} \subseteq I_j$ for some large enough natural number k_j. The result follows. □

Corollary 1.2.22 *[16, Corollary 2.7] Let R be a commutative Noetherian ring. Suppose dimR=1 and there exists a unique minimal prime ideal $\mathcal{P}$ in R. Then R s.t.r.f. if and only if $R/\mathcal{P}$ is a Dedekind domain and $\mathcal{P}$ is the only $\mathcal{P}$-primary ideal in R.*

Proof. By Theorem 1.2.16, we only need to prove sufficiency. That follows from Lemmas 1.2.20 and 1.2.21. □

Lemma 1.2.23 *[22, Theorem 2.5] Suppose that R is a Noetherian ring with ideals I and J such that*

(i) $I \cap J = 0$,

(ii) $R/(I+J)$ is semisimple Artinian,

(iii) R/I s.t.r.f. and R/J is a Dedekind domain.

Then R s.t.r.f..

Proof. Let M be an R-module. We first prove that $IM \cap JM \subseteq < E_M(0) >$. Since $R/(I+J)$ is semisimple Artinian, $I + J = \mathcal{M}_1 \cdots \mathcal{M}_n$ where $\mathcal{M}_i$ are distinct maximal ideals of R ($1 \leqslant i \leqslant n$). Let $S = R \backslash (\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n)$ which is a multiplicatively closed subset of R. For short $S^{-1}M$, $S^{-1}\mathcal{A}$ will be denoted by M_S, $\mathcal{A}_S$ respectively where $\mathcal{A}$ is an ideal of R. First of all note that R_S/J_S is a principal ideal domain. $I_S \cap J_S = 0$ gives $(I_S + J_S)/J_S \cong I_S$ and hence we can identify I_S as an ideal of R_S/J_S. In this case $I_S = R_S(a/1)$, for some $a \in R$. Let $u \in IM_S \cap JM_S = aM_S \cap J_S M_S$. Then $u = (a/1)(m'/s') \in J_S M_S$, for some $m' \in M$, $s' \in S$. Thus $(a/1)^2(m'/s') \in (a/1)J_S M_S = 0$ and $u \in < E_{M_S}(0) >$. Th erefore $IM_S \cap JM_S \subseteq < E_{M_S}(0) >$. Now for a given $v \in IM \cap JM$, $v/1 \in < E_{M_S}(0) >$. Hence there exists $s \in S$ such that $sv \in < E_M(0) >$. Note that $R = Rs + I + J$. Thus we can write $1 = rs + x + y$ where $r \in R$, $x \in I$ and $y \in J$. Since $v \in IM \cap JM$ and $I \cap J = 0$, we have $v = rsv$. It follows that $IM \cap JM \subseteq < E_M(0) >$.

Note that, since R/I s.t.r.f., we have $\mathrm{rad}_{M/IM}(0) =< E_{M/IM}(0) >$. Also by Theorem 1.2.12, R/J s.t.r.f. and it follows that $\mathrm{rad}_{M/JM}(0) =< E_{M/JM}(0) >$. To prove that R s.t.r.f. it suffices to show $\mathrm{rad}_M(0) \subseteq < E_M(0) >$. Let $m \in \mathrm{rad}_M(0)$. Then $m + IM \in \mathrm{rad}_{M/IM}(0) =< E_{M/IM}(0) >$. In this case there exist $r_1, \ldots, r_k \in R$, $m_1, \ldots, m_k \in M$, and positive integers $\alpha_1, \ldots, \alpha_k$ such that $m + IM = \sum_{i=1}^{k} r_i m_i + IM$ and $r_i^{\alpha_i} m_i \in IM$ for all $1 \leqslant i \leqslant k$.

Claim: $r_i m_i \in IM + (JM \cap E_M(0))$ for $1 \leqslant i \leqslant k$.

It suffices to show the claim holds for r_1. Suppose $r_1 \in I + J$. We may assume $r_1 \in J$. Now, $r_1^{\alpha_1} m_1 \in IM$ and $I \cap J = 0$ gives $r_1^{\alpha_1+1} m_1 = 0$. It follows that $r_1 m_1 \in JM \cap E_M(0)$.

From now on, we suppose $r_1 \notin I+J$. Then $r_1 \notin \mathcal{M}_i$ for some $1 \leqslant i \leqslant n$. After renumbering the $\mathcal{M}_i$'s, we may assume $r_1 \notin \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_\ell$ and $r_1 \in \mathcal{M}_{\ell+1} \cdots \mathcal{M}_n$. Since $R = Rr_1^{\alpha_1} + \mathcal{M}_1 \cdots \mathcal{M}_\ell$, we have $r_1 = wr_1^{\alpha_1+1} + r_1 t$ for some $w \in R$ and $t \in \mathcal{M}_1 \cdots \mathcal{M}_\ell$. Note that $r_1 t \in I + J$, and so we may write $r_1 t = x_1 + y_1$ for some $x_1 \in I$, $y_1 \in J$. Now $r_1 m_1 = wr_1^{\alpha_1+1} m_1 + x_1 m_1 + y_1 m_1$. To complete the proof of the claim, it remains to show $y_1 m_1 \in JM \cap E_M(0)$. Since $I \cap J = 0$, $(r_1 t)^{\alpha_1} = x_1^{\alpha_1} + y_1^{\alpha_1}$. Recall that $r_1^{\alpha_1} m_1 \in IM$. Hence $(r_1 t)^{\alpha_1} m_1 = x_1^{\alpha_1} m_1 + y_1^{\alpha_1} m_1 \in IM$. Thus $y_1^{\alpha_1} m_1 \in IM$ and $y_1^{\alpha_1+1} m_1 = 0$. Therefore $y_1 m_1 \in JM \cap E_M(0)$. The claim has been justified.

By the above claim, $m = u_1 + v_1$ where $u_1 \in JM \cap < E_M(0) >$, $v_1 \in IM$. Using the above argument, we also get $m = u_2 + v_2$ where $u_2 \in IM \cap < E_M(0) >$, $v_2 \in JM$. Then $u_1 - v_2 = u_2 - v_1 \in IM \cap JM$. But we proved earlier that $IM \cap JM \subseteq < E_M(0) >$. Hence $m = (v_1 - u_2) + u_2 + u_1 \in < E_M(0) >$. Therefore $\mathrm{rad}_M(0) \subseteq < E_M(0) >$. $\square$

Lemma 1.2.24 *[16, Theorem 5.1] Let R be a commutative Noetherian ring and $I_1, \ldots, I_n$ be prime ideals in R, for some positive integer $n \geqslant 2$. Suppose that*

(i) $I_1 \cap \cdots \cap I_n = 0$,

(ii) $R = I_i + I_j$ or $R/(I_i + I_j)$ is semisimple Artinian, for any $1 \leqslant i < j \leqslant n$,

(iii) R/I_i is a Dedekind domain for all i, and

(iv) $I_{k+1} + \bigcap_{i=1}^{k} I_i = \bigcap_{i=1}^{k}(I_{k+1} + I_i)$, *for* $1 \leqslant k \leqslant n-1$.

Then R *s.t.r.f..*

Proof. We will proceed by induction on $n \geqslant 2$. The case $n = 2$ follows from Lemma 1.2.23. Suppose $n \geqslant 3$. By induction $R/(I_1 \cap \cdots I_{n-1})$ s.t.r.f.. In view of Lemma 1.2.23, it suffices to show R/L is semisimple Artinian where $L = I_n + (\bigcap_{i=1}^{n-1} I_i)$. By (iv), $L = \bigcap_{i=1}^{n-1}(I_n + I_i)$. Note that, by (ii), each $I_n + I_1, \cdots, I_n + I_{n-1}$ is a product of distinct maximal ideals of R. Since R/I_n is a De dekind domain, L is also a product of distinct maximal ideals of R. Hence R/L is semisimple Artinian. □

Notation: Let $(R, \mathcal{M})$ be a commutative Noetherian local ring. Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be all the minimal prime ideals of R and $n \geqslant 3$. We define

$$I_i = \bigcap_{\substack{k=1\\k\neq i}}^{n} \mathcal{P}_k \text{ and } I_{ij} = \bigcap_{\substack{k=1\\k\neq i\\k\neq j}}^{n} \mathcal{P}_k$$

for all $1 \leqslant i, j \leqslant n$ with $i \neq j$.

The above notation will be be fixed throughout the rest of this section.

Lemma 1.2.25 *[16, Theorem 4.1] Suppose* $n \geqslant 3$ *and* $\mathcal{M} = I_{ij} + \mathcal{P}_i$ *for all* $1 \leqslant i, j \leqslant n$ *such that* $i \neq j$. *Then the following are equivalent:*

(i) $\mathcal{M} = I_i + \mathcal{P}_i$ *for some* $1 \leqslant i \leqslant n$,

(ii) $\mathcal{M} = I_i + \mathcal{P}_i$ *for every* $i = 1, \ldots, n$,

(iii) $I_{ij} = I_i + I_j$ *for all* $1 \leqslant i, j \leqslant n$ *with* $i \neq j$,

(iv) $I_{ij} = I_i + I_j$ *for some* $1 \leqslant i, j \leqslant n$ *with* $i \neq j$.

Proof. $(ii) \Rightarrow (i)$, $(iii) \Rightarrow (iv)$ Obvious.

$(i) \Rightarrow (iii)$ Suppose $\mathcal{M} = I_i + \mathcal{P}_i$ for some $1 \leqslant i \leqslant n$. Let $1 \leqslant j \leqslant n$ with $j \neq i$ be given. By the modular law, $I_{ij} = I_{ij} \cap \mathcal{M} = I_{ij} \cap (I_i + \mathcal{P}_i) = I_i + (I_{ij} \cap \mathcal{P}_i) =$

$I_i + I_j$. Now $\mathcal{M} = I_{ij} + \mathcal{P}_j = I_i + I_j + \mathcal{P}_j = I_j + \mathcal{P}_j$, because $I_i \subseteq \mathcal{P}_j$. Now by the argument we have just given $I_{jk} = I_j + I_k$ for all $1 \leqslant j, k \leqslant n$ with $j \neq k$.

$(iii) \Rightarrow (ii)$ Let $1 \leqslant i \leqslant n$ and $j \in \{1, \dots, n\} \backslash \{i\}$. By assumption $\mathcal{M} = I_{ij} + \mathcal{P}_i$ and hence $\mathcal{M} = I_i + I_j + \mathcal{P}_i$. Since $I_j \subseteq \mathcal{P}_i$ we have $\mathcal{M} = I_i + \mathcal{P}_i$.

$(iv) \Rightarrow (i)$ Suppose $I_{ij} = I_i + I_j$ for some $1 \leqslant i, j \leqslant n$ with $i \neq j$. By hypothesis $\mathcal{M} = I_{ij} + \mathcal{P}_i$. Hence $\mathcal{M} = I_i + I_j + \mathcal{P}_i$ and $\mathcal{M} = I_i + \mathcal{P}_i$ because $I_j \subseteq \mathcal{P}_i$. $\square$

Lemma 1.2.26 *[16, Theorem 4.2] Let $(R, \mathcal{M})$ be a one dimensional Noetherian local ring and $n \geqslant 2$. If $R \oplus R$ s.t.r.f. as an R-module, then there exist $x_1, \dots, x_n \in R$ such that*

$$I_i = Rx_i + \sqrt{0},\ \mathcal{P}_i = \sum_{\substack{k=1 \\ k \neq i}}^{n} I_k;\ \text{and}\ \mathcal{M} = \mathcal{P}_i + I_i = \sum_{k=1}^{n} I_k$$

for all $1 \leqslant i \leqslant n$.

Proof. Without loss of generality we may assume $\sqrt{0} = 0$. Hence R is semiprime and $\bigcap_{i=1}^{n} \mathcal{P}_i = 0$. By Theorem 1.2.16, $R/\mathcal{P}_i$ is a DVR for all $1 \leqslant i \leqslant n$.

Let $1 \leqslant i \leqslant n$ be given. Since $R/\mathcal{P}_i$ is a DVR, we can write $\mathcal{M} = Ry + \mathcal{P}_i$ for some $y \in \mathcal{M}$. Note that $I_i \neq 0$ and $I_i \nsubseteq \mathcal{P}_i$. Hence $I_i + \mathcal{P}_i = Ry^{\ell} + \mathcal{P}_i$ for some $\ell \geqslant 1$. There exist $x_i \in I_i$ and $p_i \in \mathcal{P}_i$ such that $x_i = y^{\ell} + p_i$. We now show x_i generates I_i. Let $z \in I_i$. Then $z = ry^{\ell} + q_i$ for some $r \in R$, and $q_i \in \mathcal{P}_i$. It follows that $rx_i - z = rp_i - q_i \in I_i \cap \mathcal{P}_i = 0$. Hence $z = rx_i$. Therefore $I_i = Rx_i$.

Suppose $n = 2$. In this case $I_1 = \mathcal{P}_2$ and $I_2 = \mathcal{P}_1$. It remains to show that $\mathcal{M} = \mathcal{P}_1 + \mathcal{P}_2$. Since $R/\mathcal{P}_2$ is a DVR, we have $\mathcal{M} = Ra + \mathcal{P}_2$ where $a \in \mathcal{M} \backslash \mathcal{P}_2$. If $Ra \subseteq \mathcal{P}_1$, then $\mathcal{M} = \mathcal{P}_1 + \mathcal{P}_2$. Suppose $\mathcal{P}_1 \subset Ra$. Then $\mathcal{P}_1 = \mathcal{P}_1 a = (\mathcal{P}_1 a)a = \mathcal{P} = \cdots \subseteq \bigcap_{i=1}^{\infty} \mathcal{M}^n = 0$ by Krull's intersection Theorem, i.e. $\mathcal{P}_1 \nsubseteq Ra$. Hence we can choose $b \in \mathcal{P}_1 \backslash Ra$. Let $x \in \sqrt{Ra} \backslash (\mathcal{P}_1 + \mathcal{P}_2)$. Since $\mathcal{P}_1 \cup \mathcal{P}_2$ contains all zero divisors of R, we have $\operatorname{ann}_R x^n = 0$ for all positive integers n. By the standard argument given in the proof of The orem 1.2.16, $a \in \mathcal{M}^2 + \mathcal{P}_1 + \mathcal{P}_2$. It follows that $\mathcal{M} = \mathcal{M}^2 + \mathcal{P}_1 + \mathcal{P}_2$. By Nakayama's Lemma, we get $\mathcal{M} = \mathcal{P}_1 + \mathcal{P}_2$.

Suppose $n \geqslant 3$. For each $1 \leqslant i \leqslant n$, R/I_i is a one dimensional semiprime local ring. By Lemma 1.2.1, we also know that each R/I_i s.t.r.f. as an R/I_i-module. By applying induction to each R/I_i, we get

(i) $\mathcal{M} = \sum_{\substack{k=1 \\ k \neq i}}^{n} I_{ik}$ for all $1 \leqslant i \leqslant n$,

(ii) $\mathcal{P}_i = I_j + \sum_{\substack{k=1 \\ k \neq i}}^{n} I_{jk}$ and $I_{ij} = Rx_{ij} + I_j$ for some $x_{ij} \in \mathcal{M}$ and for all $1 \leqslant i, j \leqslant n$ with $i \neq j$.

Clearly, if i, j, k are all distinct then $I_{ik} \subseteq \mathcal{P}_j$. By this observation, (i) gives

$$\mathcal{M} = I_{ij} + \mathcal{P}_j = Rx_{ij} + I_j + \mathcal{P}_j \text{ for all } i \neq j. \tag{1.1}$$

Suppose $\mathcal{M} \neq I_n + \mathcal{P}_n$. By Lemma 1.2.25,

$$\mathcal{M} \neq I_i + \mathcal{P}_i \text{ for all } i, \tag{1.2}$$

$$I_{ij} \neq I_i + I_j \text{ for all } i \neq j. \tag{1.3}$$

By (ii) and (1.3), we get $x_{12} \in I_{12} \backslash (I_1 + I_2)$. Let $x \in (\sqrt{Rx_{12}}) \backslash (I_1 + I_2)$. Since R is semiprime, $\mathrm{ann}_R x^n = \mathrm{ann}_R x$ for any positive integer n. Note that $x \in I_{12} \backslash (\mathcal{P}_1 \cup \mathcal{P}_2)$ since $\sqrt{Rx_{12}} \subseteq I_{12}$, $I_{12} \cap \mathcal{P}_1 = I_2$ and $I_{12} \cap \mathcal{P}_2 = I_1$. Therefore , $\mathrm{ann}_R x \subset \mathcal{P}_1 \cap \mathcal{P}_2$. Clearly, $Rx_{12} + Rx_{23} \subseteq \mathcal{P}_4 \cap \mathcal{P}_5 \cap \cdots \cap \mathcal{P}_n$. Hence

$$Rx_{12} + (\mathrm{ann}_R x) \cap (Rx_{12} + Rx_{23}) \subseteq Rx_{12} + I_3.$$

Suppose $x_{23} \in Rx_{12} + (\mathrm{ann}_R x) \cap (Rx_{12} + Rx_{23})$. Then $x_{23} \in Rx_{12} + I_3$. By (1.1), $\mathcal{M} = Rx_{23} + I_3 + \mathcal{P}_3 \subseteq Rx_{12} + I_3 + \mathcal{P}_3 = \mathcal{P}_3 + I_3$, and hence $\mathcal{M} = \mathcal{P}_3 + I_3$, which contradicts (1.2). Therefore $x_{23} \notin Rx_{12} + (\mathrm{ann}_R x) \cap (Rx_{12} + Rx_{23})$. Now, by the standard argument in the proof of Theorem 1.2.16, $x_{23} \in \mathcal{M}^2 + I_1 + I_2$.

By (1.1), $\mathcal{M} = Rx_{23} + I_3 + \mathcal{P}_3 \subseteq \mathcal{M}^2 + I_1 + I_2 + I_3 + \mathcal{P}_3 = \mathcal{M}^2 + I_3 + \mathcal{P}_3$. Thus $\mathcal{M} = \mathcal{M}^2 + I_3 + \mathcal{P}_3$. By Nakayama's Lemma, we have $\mathcal{M} = I_3 + \mathcal{P}_3$, which contradicts (1.2). Therefore $\mathcal{M} = I_n + \mathcal{P}_n$. By Lemma 1.2.25 $I_{ij} = I_i + I_j$ for all $i \neq j$. The required result follows from (i) and (ii). $\square$

Theorem 1.2.27 *[16] Let R be a commutative Noetherian ring and $\mathcal{P}_1, \cdots, \mathcal{P}_n$ be all the minimal prime ideals of R. Then the following are equivalent:*

(i) R s.t.r.f.,

(ii) $R \oplus R$ s.t.r.f. as an R-module,

(iii) R is one of the following:

(a) R is Artinian, or

(b) the following conditions are satisfied:

(1) dim$R$=1 and $R/\mathcal{P}_i$ is a Dedekind domain and $\mathcal{P}_i$ is the only $\mathcal{P}_i-$ primary ideal, for every $1 \leqslant i \leqslant n$,

(2) $(\bigcap_{i=1}^{k} \mathcal{P}_i) + \mathcal{P}_{k+1} = \bigcap_{i=1}^{k}(\mathcal{P}_i + \mathcal{P}_{k+1})$, for every $1 \leqslant k \leqslant n-1$, if $n \geqslant 2$.

(3) $R = \mathcal{P}_i + \mathcal{P}_j$ or $R/(\mathcal{P}_i + \mathcal{P}_j)$ is semisimple Artinian, for every $1 \leqslant i < j \leqslant n$, if $n \geqslant 2$.

Proof. $(i) \Rightarrow (ii)$ Obvious.

$(ii) \Rightarrow (iii)$ Let $R \oplus R$ s.t.r.f. as an R-module. By Theorem 1.2.5 we can suppose R is not Artinian. Thus by Corollary 1.2.17, we may assume dim$R$=1. We may also assume $n \geqslant 2$ by Corollary 1.2.22.

By Theorem 1.2.16 (1) is satisfied. Under localization at any maximal ideal $\mathcal{M}$ of R if $\mathcal{P}_i \nsubseteq \mathcal{M}$ then $\mathcal{P}_i R_{\mathcal{M}} = R_{\mathcal{M}}$ and $\mathcal{P}_i R_{\mathcal{M}}$ remains prime otherwise. By Lemma 1.2.26, (3) holds in $R_{\mathcal{M}}$, and that both sides of the condition (2) becomes $\mathcal{M}R_{\mathcal{M}}$ if $\mathcal{M}$ contains $\mathcal{P}_{k+1}$ and $\mathcal{P}_i$ for some $1 \leqslant i \leqslant k$. Otherwise both sides will equal to $R_{\mathcal{M}}$ Hence (2) and (3) hold globally.

$(iii) \Rightarrow (i)$ Suppose (1), (2) and (3) hold. $R/\sqrt{0}$ satisfies the conditions of Lemma 1.2.24 and hence it s.t.r.f.. By Lemma 1.2.21, R satisfies (ii) in Lemma 1.2.20. Hence R s.t.r.f.. $\square$

It is not entirely clear to us which non-Noetherian rings s.t.r.f.. But at least for a polynomial ring $S[X]$ where S is a commutative domain we can say the following:

Theorem 1.2.28 *Let S be a commutative domain. Then the polynomial ring $R = S[X]$ s.t.r.f. if and only if S is a field.*

Proof. $(\Rightarrow)$ Suppose R s.t.r.f.. Then the R-module $F = R \oplus R$ s.t.r.f.. Let $0 \neq a \in S$ and let W be the ideal $\sqrt{Ra + RX}$ of R and N be the submodule $W(a, X)$ of F. First we will show that $N = E_F(N)$. Let r, s_1, s_2 belong to R such that $r^k(s_1, s_2) \in N$ for some positive integer k. There exists $w \in W$ such that $r^k(s_1, s_2) = w(a, X)$, i.e. $r^k s_1 = wa$, $r^k s_2 = wX$. It follows that $r^k s_1 X = r^k s_2 a$. If $r = 0$ then $r(s_1, s_2) \in N$. Suppose that $r \neq 0$. Then $s_1 X = s_2 a$. Since $a \neq 0$ it follows that $s_2 = Xh$ for some $h \in R$. Then $s_1 X = s_2 a = Xha$ gives $s_1 = ha$. Now $r^k(s_1, s_2) = r^k(ha, hX) = r^k h(a, X)$ and hence $r^k h \in W$. Clearly $(rh)^k \in W$ and hence $rh \in W$. Thus $r(s_1, s_2) = rh(a, X) \in N$. It follows that $E_F(N) \subseteq N$ and hence $E_F(N) = N$. Since F s.t.r.f. $N = E_F(N) =< E_F(N) >= \mathrm{rad}_F(N)$. Now let K be a prime submodule of F such that $N \subseteq K$. Then $W(a, X) \subseteq K$ gives $WF \subseteq K$ or $(a, X) \in K$. In any case $(a, X) \in K$. Thus

$$R(a, X) \subseteq \mathrm{rad}_F(N) = N = W(a, X).$$

There exists $q \in W$ such that $(a, X) = q(a, X)$. In particular, $a = qa$ so that $q = 1$. It follows that $W = R$ and hence $R = Ra + RX$. There exist $f(X)$, $g(X) \in R$ such that $1 = f(X)a + g(X)X$. Then $1 = f(0)a$ and hence a is a unit in S.

($\Leftarrow$) If S is a field then $S[X]$ is a principal ideal domain and hence a Dedekind domain. Thus $R = S[X]$ s.t.r.f. by Theorem 1.2.12. $\square$

Corollary 1.2.29 *Let R be a commutative ring. Then the polynomial ring $R[X, Y]$ does not s.t.r.f..*

Proof. Suppose $R[X, Y]$ s.t.r.f.. Let $\mathcal{P}$ be any prime ideal of R. Then the ring $(R/\mathcal{P})[X, Y] \cong R[X, Y]/\mathcal{P}[X, Y]$ s.t.r.f., by Lemma 1.2.1. Let $S = (R/\mathcal{P})[X]$. Then $S[Y] \cong (R/\mathcal{P})[X, Y]$, so s.t.r.f. but S is not a field, a contradiction. $\square$

Chapter 2

PRIME SUBMODULES OF MODULES

The aim of this chapter is to investigate prime submodules of modules over commutative domains in some special cases. For example, if R is a Dedekind domain and M is a finitely generated R-module then prime submodules of M are either certain direct summands of M or submodules N such that M/N is annihilated by a maximal ideal of R (Proposition 2.1.3). On the other hand if R is a UFD, n a positive integer and $a_1, \dots, a_n$ elements of R which are not all zero then it is shown in Theorem 2.2.7 that $R(a_1 \dots, a_n)$ is a prime submodule of the free R-module $R^{(n)}$ if and only if every common divisor of $a_1, \dots, a_n$ is a unit in R.

Again for a UFD R and $n \geqslant 3$, given $a_i, b_i \in R$ $(1 \leqslant i \leqslant n)$ such that $1 = s_1b_1 + \cdots + s_nb_n$ for some $s_i \in R$ $(1 \leqslant i \leqslant n)$, the submodule $R(a_1, \dots, a_n) + R(b_1, \dots, b_n)$ of $R^{(n)}$ is prime if and only if either $a_i = cb_i$ $(1 \leqslant i \leqslant n)$ or every common divisor of $a_i - cb_i$ $(1 \leqslant i \leqslant n)$ is a unit in R, where $c = s_1a_1 + \cdots + s_na_n$ (Theorem 2.3.2). As an application we show in Theorem 2.3.12 that if R is a UFD and I is a non-zero ideal of R then the submodule $N = R(a_1, \dots, a_n) + I(1, \dots, 1)$

of $R^{(n)}$ is prime if and only if (a) $I = R$ and every common divisor of the elements $a_i - a_1$ $(2 \leqslant i \leqslant n)$ is a unit in R or (b) $N = R(1, \dots, 1)$.

2.1 Modules over Special Rings

Proposition 2.1.1 *Let R be a 0-dimensional ring and let M be an R-module. Then a proper submodule N of M is prime if and only if $\mathcal{P}M \subseteq N$ for some prime ideal $\mathcal{P}$ of R.*

Proof. By Proposition 1.1.2.8. $\square$

A commutative domain R is called *Prüfer* if every finitely generated non-zero ideal is invertible. Given a commutative domain R it is well known that any finitely generated torsion-free R-module is projective if and only if R is a Prüfer domain (see [32, Theorem 4.22]).

Proposition 2.1.2 *Let R be a Prüfer domain and let M be a finitely generated R-module. Then a proper submodule N of M is a 0-prime submodule if and only if $M = N \oplus N'$ for some torsion-free submodule N' of M.*

Proof. Suppose first that $M = N \oplus N'$ for some torsion-free submodule N' of M. Then $M/N \cong N'$ so that M/N is torsion-free. Thus N is a 0-prime submodule of M.

Conversely, suppose that N is a 0-prime submodule of M. Then the R-module M/N is finitely generated torsion-free so that M/N is projective and hence $M = N \oplus N'$ for some submodule N'. Clearly N' is torsion-free. $\square$

Dedekind domains are precisely Noetherian Prüfer domains and have the property that every non-zero prime ideal is maximal. Combining Propositions 1.1.2.8 and 2.1.2 we have the following result.

Proposition 2.1.3 *Let R be a Dedekind domain and let M be a finitely generated R-module. Then a proper submodule N of M is prime if and only if $M = N \oplus N'$ for some torsion-free submodule N' of M or $\mathcal{P}M \subseteq N$ for some maximal ideal $\mathcal{P}$ of R.*

2.2 Cyclic Submodules of F

We now fix the following notation. Let R be a commutative domain, $n \geqslant 3$ be an integer and F be the free module $R^{(n)}$.

Lemma 2.2.1 *Let N be an m-generated submodule of F for some positive integer $m < n$. Then $(N : F) = 0$.*

Proof. Suppose that $(N : F) \neq 0$, i.e. $rF \subseteq N$ for some $0 \neq r \in R$. Let $S = R\backslash\{0\}$ and let K denote the field of fractions of R. Then the n-dimensional K-vector space $K^{(n)} \cong S^{-1}F = S^{-1}N$ and $S^{-1}N$ is generated by m elements as a vector space over the field K. Thus $n \leqslant m$, a contradiction. □

Corollary 2.2.2 *Let N be an m-generated submodule of F for some positive integer $m < n$. Then N is a prime submodule of F if and only if the R-module F/N is torsion-free.*

Proof. By Lemma 2.2.1, $(N : F) = 0$. Let F/N be a torsion-free R-module. Then N is a 0-prime submodule of F. Conversely, if N is a prime submodule of F then the module F/N is torsion-free by Proposition 1.1.1.3 and Lemma 2.2.1. □

Proposition 2.2.3 *Let $a_i \in R$ $(1 \leqslant i \leqslant n)$ such that $R = Ra_1 + \cdots + Ra_n$. Then $R(a_1, \ldots, a_n)$ is a direct summand of the free R-module $F = R^{(n)}$. Moreover $R(a_1, \ldots, a_n)$ is a 0-prime submodule of F.*

Proof. There exist $s_i \in R$ $(1 \leqslant i \leqslant n)$ such that $1 = s_1 a_1 + \cdots + s_n a_n$. Let

$$N = \{(x_1, \ldots, x_n) \in F : s_1 x_1 + \cdots + s_n x_n = 0\}.$$

Then N is a submodule of F. For any $r \in R$, $r(a_1, \ldots, a_n) \in N$ implies that $s_1 r a_1 + \cdots + s_n r a_n = 0$, i.e. $r(s_1 a_1 + \cdots + s_n a_n) = 0$, i.e $r = 0$. Hence $R(a_1, \ldots, a_n) \cap N = 0$. Moreover, for each $1 \leqslant i \leqslant n$, the element $\mathbf{e}_i$, the n-tuple in which the ith component is 1 while the others are 0, belongs to $R(a_1, \ldots, a_n) + N$. For, consider the element

$$\mathbf{e}_i - s_i(a_1, \ldots, a_n) = (-s_i a_1, \ldots, -s_i a_{i-1}, 1 - s_i a_i, -s_i a_{i+1}, \ldots, -s_i a_n)$$

and note that $\mathbf{e}_i - s_i(a_1, \ldots, a_n) \in N$ because
$s_1(-s_i a_1) + \cdots + s_{i-1}(-s_i a_{i-1}) + s_i(1 - s_i a_i) + s_{i+1}(-s_i a_{i+1}) + \cdots + s_n(-s_i a_n)$ is equal to $-s_i(s_1 a_1 + \cdots + s_n a_n) + s_i = -s_i + s_i = 0$. Thus $\mathbf{e}_i \in R(a_1, \ldots, a_n) + N$ $(1 \leqslant i \leqslant n)$. It follows that $F = R(a_1, \ldots, a_n) + N$ and hence $F = R(a_1, \ldots, a_n) \oplus N$. Since F is free it is torsion-free and the factor module $F/R(a_1, \ldots, a_n)$ is torsion-free. This implies $(R(a_1, \ldots, a_n) : F) = 0$, and hence $R(a_1, \ldots, a_n)$ is a 0-prime submodule of F. $\square$

Corollary 2.2.4 *Let $a_i \in R$ $(1 \leqslant i \leqslant n)$ such that at least one of the elements a_i $(1 \leqslant i \leqslant n)$ is a unit in R. Then $R(a_1, \ldots, a_n)$ is a prime submodule of F.*

Proof. By Proposition 2.2.3. $\square$

Corollary 2.2.5 *Let $a_i \in R$ $(1 \leqslant i \leqslant n)$ and let $\mathcal{P}$ be a prime ideal of R such that $R = Ra_1 + \cdots + Ra_n + \mathcal{P}$. Then $R(a_1, \ldots, a_n) + \mathcal{P}F$ is a $\mathcal{P}$-prime submodule of F.*

Proof. The module $F/\mathcal{P}F$ is a free module over the domain $R/\mathcal{P}$. Let $N = R(a_1, \ldots, a_n) + \mathcal{P}F$. Then $N/\mathcal{P}F = R(a_1+\mathcal{P}, \ldots, a_n+\mathcal{P})$. By Proposition 2.2.3, $N/\mathcal{P}F$ is a $\mathcal{P}$-prime submodule of the $(R/\mathcal{P})$-module $F/\mathcal{P}F$. Clearly it follows that N is a $\mathcal{P}$-prime submodule of F. □

Let $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero. By a *common divisor* of the elements a_i $(1 \leqslant i \leqslant n)$ we mean an element $d \in R$ such that $a_i = db_i$ $(1 \leqslant i \leqslant n)$ for some elements b_i $(1 \leqslant i \leqslant n)$. Clearly d is a common divisor of a_i $(1 \leqslant i \leqslant n)$ if and only if $Ra_1 + \cdots + Ra_n \subseteq Rd$. Corollary 2.2.2 has the following consequence.

Lemma 2.2.6 *Let $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero, such that $N = R(a_1, \ldots, a_n)$ is a prime submodule of $F = R^{(n)}$. Then every common divisor of a_i $(1 \leqslant i \leqslant n)$ is a unit in R.*

Proof. Let d be a common divisor of a_i $(1 \leqslant i \leqslant n)$. For each $1 \leqslant i \leqslant n$ there exists $b_i \in R$ such that $a_i = db_i$. Clearly $d \neq 0$ and $d(b_1, \ldots, b_n) = (a_1, \ldots, a_n) \in N$. By Corollary 2.2.2, $(b_1, \ldots, b_n) \in N$, i.e. $(b_1, \ldots, b_n) = r(a_1, \ldots, a_n)$ for some $r \in R$. It follows that $a_i = dra_i$ $(1 \leqslant i \leqslant n)$ and hence $dr = 1$, i.e. d is a unit in R. □

Theorem 2.2.7 *Let R be a UFD and let $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero. Then $N = R(a_1, \ldots, a_n)$ is a prime submodule of $F = R^{(n)}$ if and only if every common divisor of a_i $(1 \leqslant i \leqslant n)$ is a unit in R.*

Proof. The necessity is proved in Lemma 2.2.6.

Conversely, suppose that every common divisor of a_i $(1 \leqslant i \leqslant n)$ is a unit in R. Let $0 \neq r \in R, b_i \in R$ $(1 \leqslant i \leqslant n)$ such that $r(b_1, \ldots, b_n) \in N$, i.e. $r(b_1, \ldots, b_n) = s(a_1, \ldots, a_n)$ for some $s \in R$. Hence $rb_i = sa_i$ $(1 \leqslant i \leqslant n)$.

There exists $1 \leqslant j \leqslant n$ such that $a_j \neq 0$. Suppose that a_j is a unit in R. Then $s = rb_ja_j^{-1}$ and hence $rb_i = rb_ja_j^{-1}a_i$ giving $b_i = b_ja_j^{-1}a_i$ $(1 \leqslant i \leqslant n)$. In this case

$$(b_1, \ldots, b_n) = b_ja_j^{-1}(a_1, \ldots, a_n) \in N.$$

Now suppose that a_j is not a unit in R. Let p be any prime divisor of a_j. There exists $1 \leqslant k \leqslant n$ such that p does not divide a_k. However $rb_k = sa_k$ and $rb_j = sa_j$ together give $ra_jb_k = ra_kb_j$, so that $a_jb_k = a_kb_j$ and hence p divides b_j. Now $rb_j = sa_j$ gives $r(b_j/p) = s(a_j/p)$. Repeating this argument we conclude that a_j divides b_j, i.e. $b_j = ca_j$ for some $c \in R$. For each $1 \leqslant i \leqslant n$, $ra_ib_j = ra_jb_i$ gives $b_i = ca_i$. Hence $(b_1, \ldots, b_n) = c(a_1, \ldots, a_n) \in N$. It follows that N is a prime submodule of F. $\square$

We shall call a submodule N of F a *cyclic prime* if N is a prime submodule of F and N is a cyclic R-module.

Corollary 2.2.8 *Let R be a UFD and let N be any prime submodule of $F = R^{(n)}$ with $(N : F) = 0$. Then N is a sum of cyclic prime submodules of F.*

Proof. Let $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero, such that $(a_1, \ldots, a_n) \in N$. Let d be a greatest common divisor of the elements a_i $(1 \leqslant i \leqslant n)$. Then $a_i = db_i$ $(1 \leqslant i \leqslant n)$ for some elements b_i $(1 \leqslant i \leqslant n)$ of R. Clearly any common divisor of the elements b_i $(1 \leqslant i \leqslant n)$ is a unit in R. By Theorem 2.2.7, $R(b_1, \ldots, b_n)$ is a cyclic prime submodule of F. Moreover, $R(a_1, \ldots, a_n) \subseteq R(b_1, \ldots, b_n) \subseteq N$ by Corollary 2.2.2. The result follows. $\square$

Remark: Let F denote the free $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$ and let p be any prime in $\mathbb{Z}$. Then pF is a prime submodule of F such that $(pF : F) = (p)$ but pF is not a sum of cyclic prime submodules by Corollary 2 in [7].

2.3 2-Generated Submodules of F

In this section we are interested when $N = R(a_1, \dots, a_n) + R(b_1, \dots, b_n)$ is a prime submodule of $F = R^{(n)}$, where $R = Rb_1 + \cdots + Rb_n$. Consider the submodules

$$L = R(b_1, \dots, b_n) \text{ and } L' = \{(x_1, \dots, x_n) \in F : s_1x_1 + \cdots + s_nx_n = 0\}$$

of F, where $s_i \in R$ $(1 \leqslant i \leqslant n)$ and $1 = s_1b_1 + \cdots + s_nb_n$. Note first that $F = L \oplus L'$ by Proposition 2.2.3. Now $N = N \cap (L \oplus L') = L \oplus (N \cap L')$. Let $c = s_1a_1 + \cdots + s_na_n$. Then $N \cap L' \supseteq R(\mathbf{a} - c\mathbf{b})$, where $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots b_n)$, and $N = R(\mathbf{a} - c\mathbf{b}) \oplus R\mathbf{b}$, so that $N \cap L' = R(\mathbf{a} - c\mathbf{b})$.

Lemma 2.3.1 *Let R be a commutative domain and let N be a submodule of an R-module M such that the module M/N is torsion-free. Let L be a proper submodule of N. Then L is a 0-prime submodule of N if and only if L is a 0-prime submodule of M.*

Proof. Suppose first that L is a 0-prime submodule of M. Then the module M/L is torsion-free and hence the module N/L is torsion-free, i.e. L is a 0-prime submodule of N. Conversely, suppose that L is a 0-prime submodule of N. Then N/L and M/N are both torsion-free R-modules, so that M/L is torsion-free and L is a 0-prime submodule of M. □

Theorem 2.3.2 *Let R be a UFD, let $n \geqslant 3$ be a positive integer and $a_i, b_i \in R$ $(1 \leqslant i \leqslant n)$ such that $R = Rb_1 + \cdots + Rb_n$. Let $c = s_1a_1 + \cdots + s_na_n$ where $s_i \in R$ $(1 \leqslant i \leqslant n)$ and $1 = s_1b_1 + \cdots + s_nb_n$. Then $N = R(a_1, \dots, a_n) + R(b_1, \dots, b_n)$ is a prime submodule of $F = R^{(n)}$ if and only if either $a_i = cb_i$ $(1 \leqslant i \leqslant n)$ or every common divisor of $a_i - cb_i$ $(1 \leqslant i \leqslant n)$ is a unit in R.*

Proof. With the above notation, N is a prime submodule of F if and only if $N \cap L'$ is a prime submodule of L', because $F = L \oplus L'$ and $N = L \oplus (N \cap L')$ together give $F/N \cong L'/(N \cap L')$. Moreover, $(N \cap L' : L') = (N : F) = 0$ by Lemma 2.2.1. By Lemma 2.3.1, $N \cap L'$ is a prime submodule of L' if and only if $N \cap L'$ is a prime submodule of F. Now $N \cap L' = R(\mathbf{a} - c\mathbf{b})$. Thus $N \cap L'$ is a prime submodule of F if and only if $N \cap L' = 0$, i.e . $a_i = cb_i$ $(1 \leqslant i \leqslant n)$, or every common divisor of $a_i - cb_i$ $(1 \leqslant i \leqslant n)$ is a unit in R by Theorem 2.2.7. □

Remark: Note that if $N = R(a_1, \dots, a_n) + R(b_1, \dots, b_n)$ where $a_i, b_i \in R$ $(1 \leqslant i \leqslant n)$ and $R = Ra_1 + \cdots + Ra_n = Rb_1 + \cdots + Rb_n$ then in general N is not a prime submodule of F as the following example shows.

Example 2.3.3 *The submodule $N = \mathbb{Z}(2,3,5) + \mathbb{Z}(2,1,3)$ of the free $\mathbb{Z}$-module $F = \mathbb{Z}^{(3)}$ is not prime.*

Proof. Suppose that N is a prime submodule of F. The element $(4,4,8) = (2,3,5) + (2,1,3) \in N$. Thus $4(1,1,2) \in N$ and hence $(1,1,2) \in N$ by Lemma 2.2.1. It is easy to check that $(1,1,2) \neq s(2,3,5) + t(2,1,3)$ for any $s, t \in \mathbb{Z}$, a contradiction. Thus N is not prime. □

Theorem 2.3.2 deals only with the case $n \geqslant 3$. If $n = 1$ then $N = Ra_1 + Rb_1 = R$ which is not prime. We now deal with the case $n = 2$.

Proposition 2.3.4 *Let R be a commutative ring and let $a_i, b_i \in R$ $(i = 1, 2)$ such that $R = Rb_1 + Rb_2$. Then $N = R(a_1, a_2) + R(b_1, b_2)$ is a prime submodule of $F = R^{(2)}$ if and only if $R(a_1 b_2 - a_2 b_1)$ is a prime ideal of R.*

Proof. There exist elements $s_1, s_2 \in R$ such that $1 = s_1 b_1 + s_2 b_2$. Then $F = L \oplus L'$ where $L = R(b_1, b_2)$ and $L' = \{(x, y) \in F : s_1 x + s_2 y = 0\}$. Clearly $R(-s_2, s_1) \subseteq L'$. Moreover,

$$(1,0) = s_1(b_1, b_2) + (-b_2)(-s_2, s_1) \text{ and } (0,1) = s_2(b_1, b_2) + b_1(-s_2, s_1)$$

together imply $F = L + R(-s_2, s_1)$. It follows that $L' = (L \cap L') + R(-s_2, s_1) = R(-s_2, s_1)$.

As before, $N = L \oplus (N \cap L')$ and $N \cap L' = R(a_1 - cb_1, a_2 - cb_2)$ where $c = s_1a_1 + s_2a_2$. Note that $(a_1 - cb_1, a_2 - cb_2) = (a_2b_1 - b_2a_1)(-s_2, s_1)$ because

$$\begin{aligned} -s_2(a_2b_1 - b_2a_1) &= -s_2a_2b_1 + s_2b_2a_1 \\ &= -s_2a_2b_1 + (1 - s_1b_1)a_1 \\ &= a_1 - (s_1a_1 + s_2a_2)b_1 \\ &= a_1 - cb_1, \text{ and} \end{aligned}$$

$$\begin{aligned} s_1(a_2b_1 - b_2a_1) &= s_1a_2b_1 - s_1b_2a_1 \\ &= (1 - s_2b_2)a_2 - s_1b_2a_1 \\ &= a_2 - (s_1a_1 + s_2a_2)b_2 \\ &= a_2 - cb_2. \end{aligned}$$

Note also that if $r \in R$ and $r(-s_2, s_1) = 0$ then $rs_2 = 0$, $rs_1 = 0$ and hence

$$r = r1 = r(s_1b_1 + s_2b_2) = (rs_1)b_1 + (rs_2)b_2 = 0.$$

Let $d = a_1b_2 - a_2b_1$. Now $F = L \oplus L'$ and $N = L \oplus (N \cap L')$ give that

$$F/N \cong L'/(N \cap L') = R(-s_2, s_1)/Rd(-s_2, s_1) \cong R/Rd.$$

Thus N is a prime submodule of F if and only if Rd is a prime ideal of R. $\square$

In Proposition 2.3.4 it is crucial that $R = Rb_1 + Rb_2$. For, let N denote submodule $\mathbb{Z}(6,6) + \mathbb{Z}(10,10)$ of the free $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$. Then $N = \mathbb{Z}(2,2)$ and $2(1,1) \in N$, $(1,1) \notin N$, so that N is not prime (Corollary 2.2.2). However $a_1 = a_2 = 6$, $b_1 = b_2 = 10$ gives $\mathbb{Z}(a_1b_2 - a_2b_1) = 0$ which is a prime ideal of $\mathbb{Z}$.

We fix the following notation. Let n be a positive integer, let $a_{ij} \in R$ $(1 \leqslant i,j \leqslant n)$ and let $\mathbf{a}_i = (a_{i1}, \dots, a_{in}) \in R^{(n)} = F$ for all $1 \leqslant i \leqslant n$. Let $N = R\mathbf{a}_1 + \cdots + R\mathbf{a}_n$ be a proper submodule of F. Let A denote the $n \times n$ matrix (a_{ij}) over R. Proposition 2.3.4 suggests that it might be the case that N is a prime submodule of F if and only if $R(\det A)$ is a prime ideal of R, provided that

$$R = Ra_{i1} + \cdots + Ra_{in} \quad (2 \leqslant i \leqslant n).$$

The next two examples show that in fact neither of these implications is true.

Example 2.3.5 *With the above notation, $\mathbb{Z}(3,5,7) + \mathbb{Z}(0,2,1) + \mathbb{Z}(0,1,2)$ is a prime submodule of $F = \mathbb{Z}^{(3)}$ but detA=9.*

Proof. Note that $A = \begin{bmatrix} 3 & 5 & 7 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ so that clearly $\det A$=9. Moreover,

$3(1,0,0) = (3,5,7) - (0,2,1) - 3(0,1,2) \in N$,

$3(0,1,0) = 0(3,5,7) + 2(0,2,1) - (0,1,2) \in N$,

$3(0,0,1) = 0(3,5,7) - (0,2,1) + 2(0,1,2) \in N$,

and $(1,0,0) \notin N$. Thus $3F \subseteq N \neq F$. It follows that N is a prime submodule of F by Proposition 1.1.2.8. □

Example 2.3.6 *With the above notation, $\mathbb{Z}(3,5,7) + \mathbb{Z}(0,2,1) + \mathbb{Z}(0,2,1)$ is not a prime submodule of $F = \mathbb{Z}^{(3)}$ but detA=0, which is a prime ideal of $\mathbb{Z}$.*

Proof. In this case, $A = \begin{bmatrix} 3 & 5 & 7 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ and clearly $\det A$=0.

Since $N = \mathbb{Z}(3,5,7) + \mathbb{Z}(0,2,1)$, it follows that $(N : F) = 0$. Suppose that N is a prime submodule of F, i.e. the $\mathbb{Z}$-module F/N is torsion-free. Now $3(1,1,2) =$

$(3,3,6) = (3,5,7)-(0,2,1) \in N$ gives that $(1,1,2) \in N$, i.e. $(1,1,2) = a(3,5,7)+b(0,2,1)$ for some $a, b \in \mathbb{Z}$ and hence $3a = 1$, a contradiction. Thus N is not prime. □

We note the following general fact.

Proposition 2.3.7 *Let R be commutative ring, let n be a positive integer, let $a_{ij} \in R$ $(1 \leqslant i,j \leqslant n)$, let $\mathbf{a}_i = (a_{i1}, \dots, a_{in}) \in F = R^{(n)}$ $(1 \leqslant i \leqslant n)$ and let $N = R\mathbf{a}_1 + \cdots + R\mathbf{a}_n$. Let A denote the $n \times n$ matrix (a_{ij}) over R. Then*

$$R(detA) \subseteq (N : F) \subseteq \sqrt{R(detA)}$$

Proof. Let B=adj$A$, the adjugate of the matrix A. Then $(\det A)I_n = BA$, where I_n denotes the $n \times n$ identity matrix over R. Suppose that B is the $n \times n$ matrix (b_{ij}) over R. Then

$$(\det A)\mathbf{e}_i = b_{i1}\mathbf{a}_1 + \cdots + b_{in}\mathbf{a}_n \in N$$

for each $1 \leqslant i \leqslant n$. It follows that $(\det A)F \subseteq N$, i.e. $R(\det A) \subseteq (N : F)$.

Let $r \in (N : F)$. There exist elements $c_{ij} \in R$ $(1 \leqslant i,j \leqslant n)$ such that $r\mathbf{e}_i = c_{i1}\mathbf{a}_1 + \cdots + c_{in}\mathbf{a}_n$ for all $1 \leqslant i \leqslant n$. Let C denote the $n \times n$ matrix (c_{ij}) over R. Then $rI_n = CA$. Taking determinants we have

$$r^n = \det(CA) = (\det C)(\det A) \in R(\det A).$$

It follows that $(N : F) \subseteq \sqrt{R(\mathrm{detA})}$. □

Corollary 2.3.8 *With the above notation, if $R(detA)$ is a maximal ideal of R then N is a prime submodule of F.*

Proof. By Propositions 1.1.2.8 and 2.3.7. □

Next we consider what happens when $R(\det A)$ is a prime ideal of R. We have the following result.

Proposition 2.3.9 *With the notation of Proposition 2.3.7, let R be a domain and let $R(detA)$ be a non-zero prime ideal of R. Then N is a prime submodule of F.*

Proof. Let $r \in R$, $x_i \in R$ $(1 \leqslant i \leqslant n)$ such that $r(x_1, \ldots, x_n) \in N$. Then

$$r(x_1, \ldots, x_n) = s_1\mathbf{a}_1 + \cdots + s_n\mathbf{a}_n$$

for some elements $s_i \in R$ $(1 \leqslant i \leqslant n)$. In matrix notation, we have

$$r[x_1 \cdots x_n] = [s_1 \cdots s_n]A.$$

Let $B = \text{adj}A$. Then

$$r[x_1 \cdots x_n]B = [s_1 \cdots s_n]AB = d[s_1 \cdots s_n],$$

where $d = \det A$. If $B = (b_{ij})$ then

$$r(x_1b_{1j} + \cdots + x_nb_{nj}) = s_jd \in Rd$$

for all $1 \leqslant j \leqslant n$. Since Rd is prime it follows that $r \in Rd$ and hence $rF \subseteq N$ by Proposition 2.3.7, or there exist $t_j \in R$ $(1 \leqslant j \leqslant n)$ such that $x_1b_{1j} + \cdots + x_nb_{nj} = t_jd$ $(1 \leqslant j \leqslant n)$. In matrix terms, we have

$$[x_1 \cdots x_n]B = d[t_1 \cdots t_n]$$

and hence

$$[x_1 \cdots x_n]BA = d[t_1 \cdots t_n]A$$

i.e.

$$d[x_1 \cdots x_n] = d[t_1 \cdots t_n]A.$$

Since R is a domain and $d \neq 0$ it follows that $[x_1 \cdots x_n] = [t_1 \cdots t_n]A$ and hence $(x_1, \ldots, x_n) = t_1\mathbf{a}_1 + \cdots + t_n\mathbf{a}_n \in N$. It follows that N is a prime submodule of F. $\square$

Note that Example 2.3.5 shows that the converse of Proposition 2.3.9 is false in general, and Example 2.3.6 shows that in general Proposition 2.3.9 is false in case $\det A = 0$.

We now consider 2-generated submodules N of F of the form

$$N = R(a_1, \dots, a_n) + R(b, \dots, b)$$

where $b, a_i \in R$ $(1 \leqslant i \leqslant n)$. More generally, we shall consider when a submodule N of the form $R(a_1, \dots, a_n) + I(1, \dots, 1)$ is prime , where I is an ideal of R. First we prove a result which deals with the case $a_i = 0$ $(1 \leqslant i \leqslant n)$.

Lemma 2.3.10 *Let R be a commutative domain. Let I be an ideal of R. Then $I(1, \dots, 1)$ is a prime submodule of $F = R^{(n)}$ (where $n \geqslant 2$) if and only if I=0 or I=R.*

Proof. Suppose that $I = 0$. Then $I(1, \dots, 1) = 0$ and hence $I(1, \dots, 1)$ is a 0-prime submodule of F. If $I = R$ then $I(1, \dots, 1)$ is a 0-prime submodule of F since $F = I(1, \dots, 1) \oplus G$, where $G = 0 \oplus R^{(n-1)}$.

Conversely, suppose that $N = I(1, \dots, 1)$ is a prime submodule of F. Now $I(1, \dots, 1) \subseteq N$ implies that $R(1, \dots, 1) \subseteq N$, so that $N = R(1, \dots, 1)$ and hence $R = I$, or $IF \subseteq N$. Suppose that $IF \subseteq N$. Let $a \in I$. Then there exists $b \in I$ such that $a(1, 0, \dots, 0) = b(1, \dots, 1)$. Hence $a = b = 0$. It follows that $I = 0$. $\square$

We now suppose that R is a commutative domain, $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero, I is a non-zero ideal of R and $N = R(a_1, \dots, a_n) + I(1, \dots, 1)$.

Lemma 2.3.11 *Suppose that N is a prime submodule of $F = R^{(n)}$ (where $n \geqslant 2$). Then either*

(i) I=R, or

(ii) $a_1 = \cdots = a_n$ *and* $R = Ra_1 + I$.

In any case, $N = R(a_1, \ldots, a_n) + R(1, \ldots, 1)$.

Proof. Note first that $I(1, \ldots, 1) \subseteq N$ gives that $IF \subseteq N$ or $(1, \ldots, 1) \in N$. Suppose first that $IF \subseteq N$. Let $0 \neq c \in I$. Then

$$(c, 0, \ldots, 0) = c(1, 0, \ldots, 0) = r(a_1, \ldots, a_n) + s(1, \ldots, 1)$$

for some $r \in R$, $s \in I$. Since $c \neq 0$ it follows that $r \neq 0$. Then $c = ra_1 + s$, $0 = ra_i + s$ $(2 \leqslant i \leqslant n)$, and hence $0 = r(a_2 - a_i)$, for all $2 \leqslant i \leqslant n$. It follows that $a_2 = a_3 = \cdots = a_n$. By considering $(0, c, 0, \ldots, 0) \in N$, we obtain $a_1 = a_2$. Thus $a_1 = a_2 = \cdots = a_n$. But we now have

$$(c, 0, \ldots, 0) = r(a_1, \ldots, a_1) + s(1, \ldots, 1),$$

which implies $c = 0$, a contradiction. Thus $IF \nsubseteq N$. Hence $(1, \ldots, 1) \in N$, and hence

$$(1, \ldots, 1) = x(a_1, \ldots, a_n) + y(1, \ldots, 1)$$

for some $x \in R, y \in I$. If $x = 0$ then $y = 1$ and hence $I = R$. Suppose that $x \neq 0$. Then $x(a_i - a_j) = 0$ $(1 \leqslant i < j \leqslant n)$ and hence $a_i = a_j$ $(1 \leqslant i < j \leqslant n)$. Moreover, $1 = xa_1 + y \in Ra_1 + I$. Thus $R = Ra_1 + I$.

If $I = R$ then clearly $N = R(a_1, \ldots, a_n) + R(1, \ldots, 1)$. Now suppose that $a_i = a_j$ $(1 \leqslant i < j \leqslant n)$ and $1 = xa_1 + y$ (as above). Then

$$(1, \ldots, 1) = x(a_1, \ldots, a_n) + y(1, \ldots, 1) \in N.$$

Thus $N = R(a_1, \ldots, a_n) + R(1, \ldots, 1)$. □

Theorem 2.3.12 *With the above notation let* R *be a UFD and* $n \geqslant 3$. *Then* $N = R(a_1, \ldots, a_n) + I(1, \ldots, 1)$ *is a prime submodule of* F *if and only if* *(a) I=R and every common divisor of the elements $a_i - a_1$ $(2 \leqslant i \leqslant n)$ is a unit in R, or*

(b) $a_1 = \cdots = a_n$ and $R = Ra_1 + I$.

Proof. Suppose first that N is a prime submodule of F. By Lemma 2.3.11, $I = R$ or $a_1 = \cdots = a_n$ and $R = Ra_1 + I$. Suppose that $I = R$ then

$$N = R(a_1, \ldots, a_n) + R(1, \ldots, 1)$$

By Theorem 2.3.2, $a_1 = \cdots = a_n$ or every common divisor of $a_i - a_1$ $(2 \leqslant i \leqslant n)$ is a unit in R.

Conversely, if (b) holds then $N = R(1, \ldots, 1)$ and if (a) holds then $N = R(a_1, \ldots, a_n) + R(1, \ldots, 1)$ where any common factor of $a_i - a_1$ $(2 \leqslant i \leqslant n)$ is a unit. By Corollary 2.2.4 and Theorem 2.3.2, N is a prime submodule of F. □

Chapter 3

RADICALS OF SUBMODULES OF FREE MODULES

The aim of this chapter is to describe $\mathrm{rad}_M(N)$ for a given submodule N of a module M in some special cases. If $M = R$ then N is an ideal of R and $\mathrm{rad}_M(N) = \sqrt{N}$. If $M \neq R$ it has proved difficult to characterize $\mathrm{rad}_M(N)$.

Throughout this chapter all rings will be commutative with identity. We fix the following notation. Let R be a ring. Let n be a positive integer and let F be the free R-module $R^{(n)}$. Let $\mathbf{x}_i \in F$ $(1 \leqslant i \leqslant m)$, for some positive integer m. Then

$$\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \ (1 \leqslant i \leqslant m)$$

for some $x_{ij} \in R$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$. We set

$$[\mathbf{x}_1 \cdots \mathbf{x}_m] = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \in M_{m\times n}(R).$$

Thus the j^{th} row of the matrix $[\mathbf{x}_1 \cdots \mathbf{x}_m]$ consists of the components of the element $\mathbf{x}_j$ in F.

Let $A = (a_{ij}) \in M_{m\times n}(R)$. Let $t \leqslant min(m,n)$. By a $t\times t$ minor of A we mean the determinant of a $t \times t$ submatrix of A, that is a determinant of the form

$$\begin{vmatrix} a_{i(1)j(1)} & \cdots & a_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ a_{i(t)j(1)} & \cdots & a_{i(t)j(t)} \end{vmatrix}$$

where $1 \leqslant i(1) < \cdots < i(t) \leqslant m$, $1 \leqslant j(1) < \cdots < j(t) \leqslant n$. For each $1 \leqslant t \leqslant min(m,n)$, we denote by A_t the ideal of R generated by the $t \times t$ minors of A. Note that $A_1 = \sum_{j=1}^{n}\sum_{i=1}^{m} Ra_{ij} \supseteq A_2 \supseteq A_3 \supseteq \cdots \supseteq A_k$, where $k = min(m,n)$.

Let F be the free R-module $R^{(n)}$, for some positive integer n. Let $N = \sum_{i=1}^{m} R\mathbf{x}_i$ be a finitely generated submodule of F. Then $\mathbf{r} \in \text{rad}_F(N)$ if and only if $[\mathbf{r}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t \in \sqrt{[\mathbf{0}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t}$ for all $1 \leqslant t \leqslant min(m+1,n)$ (Theorem 3.1.5). As an application it is proved in Theorem 3.1.9 that if $N = \sum_{i=1}^{m} R\mathbf{x}_i + IF$ for some positive integer m and elements $\mathbf{x}_i \in F$ $(1 \leqslant i \leqslant m)$, then $\mathbf{r} \in \text{rad}_F(N)$ if and only if $[\mathbf{r}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t \in \sqrt{([\mathbf{0}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t + I)}$ for all $1 \leqslant t \leqslant min(m+1,n)$. On the other hand if R is a UFD, n a positive integer, $a_1, \ldots, a_n$ elements of R not all zero and N the submodule $R(a_1, \ldots, a_n)$ of $F = R^{(n)}$, then it is shown in Proposition 3.1.10 that $\text{rad}_F(N) = R(b_1, \ldots, b_n)$ where $b_i = (p_1 \cdots p_m a_i)/d$ $(1 \leqslant i \leqslant n)$, d is a greatest common divisor (gcd) of $a_1, \ldots, a_n$ and $p_1, \ldots, p_m$ are the pairwise non-associate prime divisors of d.

In particular, for a not necessarily finitely generated submodule N of F of the form $R(a_1, \ldots, a_n) + I(1, \ldots, 1)$ for an ideal I of R, $\text{rad}_F(N) = R(a_1, \ldots, a_n) + \sqrt{I}(1, \ldots, 1) + WF =< E_F(N) >$ if the ideal $\sum_{i=1}^{n} R(a_1 - a_i)$ is equal to R (Theorem 3.2.5).

3.1 Characterization of the Radical

In this section we describe $\text{rad}_F(N)$ where N is a finitely generated submodule of the free module F. First we make a general observation.

Let N be a proper submodule of any R-module M. Let $\mathcal{P}$ be a prime ideal of R. Then we shall denote by $K(N,\mathcal{P})$ the following subset of M:

$$K(N,\mathcal{P}) = \{m \in M : cm \in \mathcal{P}M + N, \text{ for some } c \in R\backslash\mathcal{P}\}.$$

It is clear that $K(N,\mathcal{P})$ is a submodule of M and $\mathcal{P}M + N \leqslant K(N,\mathcal{P})$.

Lemma 3.1.1 *With the above notation, $K(N,\mathcal{P}) = M$ or $K(N,\mathcal{P})$ is a prime submodule of M with $\mathcal{P} = (K(N,\mathcal{P}) : M)$.*

Proof. Suppose $K(N,\mathcal{P}) \neq M$. Apply Proposition 1.2.9 to the module M/N. $\square$

Corollary 3.1.2 *With the above notation, for any submodule N of M,*

$$rad_M(N) = \bigcap\{K(N,\mathcal{P}) : \mathcal{P} \textit{ is a prime ideal of } R\}.$$

Proof. Clear by Lemma 3.1.1 and the fact that $K(N,Q) \leqslant L$ for every prime submodule L of F containing N, where $Q = (L : M)$, a prime ideal of R. $\square$

Lemma 3.1.3 *Let R be a ring and F be the free R-module $R^{(n)}$, for some positive integer n. Let $N = \sum_{i=1}^{m} R\mathbf{x}_i$ be a finitely generated submodule of F where $m < n$. Then*

$\mathbf{r} \in rad_F(N)$ *if and only if* $[\mathbf{r}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t \in \sqrt{[\mathbf{0}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t}$ $(1 \leqslant t \leqslant m+1)$.

Proof. Suppose that $\mathbf{r} = (r_1, \ldots, r_n) \in \text{rad}_F(N)$ where $r_i \in R$ $(1 \leqslant i \leqslant n)$. Let $\mathcal{P}$ be any prime ideal of R. By Corollary 3.1.2, there exist $c \in R\backslash\mathcal{P}, s_i \in R$ $(1 \leqslant i \leqslant m)$ and $p_i \in \mathcal{P}$ $(1 \leqslant i \leqslant n)$ such that

$$\mathbf{cr} = s_1\mathbf{x}_1 + \cdots + s_m\mathbf{x}_m + \mathbf{p}$$

where $\mathbf{p} = (p_1, \ldots, p_n)$; that is, if $\mathbf{x}_i = (x_{i1}, \ldots, x_{in})$ where $x_{ij} \in R$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$ then

$$cr_i = s_1x_{1i} + s_2x_{2i} + \cdots + s_mx_{mi} + p_i \quad (1 \leqslant i \leqslant n). \tag{3.1}$$

Suppose that $1 \leqslant t \leqslant m+1$ and $[\mathbf{0}\mathbf{x}_1 \cdots \mathbf{x}_m]_t \subseteq \mathcal{P}$. Let $1 \leqslant i(1) < \cdots < i(t-1) \leqslant m, 1 \leqslant j(1) < \cdots < j(t) \leqslant n$. Let

$$X_t = \begin{vmatrix} r_{j(1)} & \cdots & r_{j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(t-1)j(1)} & \cdots & x_{i(t-1)j(t)} \end{vmatrix}$$

which is a $t \times t$ minor of $[\mathbf{r}\,\mathbf{x}_1 \cdots \mathbf{x}_m]$. Then by (3.1),

$$cX_t = \begin{vmatrix} cr_{j(1)} & \cdots & cr_{j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(t-1)j(1)} & \cdots & x_{i(t-1)j(t)} \end{vmatrix}.$$

$$\begin{aligned} &= \begin{vmatrix} \sum_{k=1}^m s_kx_{kj(1)} + p_{j(1)} & \cdots & \sum_{k=1}^m s_kx_{kj(t)} + p_{j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(t-1)j(1)} & \cdots & x_{i(t-1)j(t)} \end{vmatrix} \\ &= \sum_{k=1}^m s_k \begin{vmatrix} x_{kj(1)} & \cdots & x_{kj(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(t-1)j(1)} & \cdots & x_{i(t-1)j(t)} \end{vmatrix} + \begin{vmatrix} p_{j(1)} & \cdots & p_{j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(t-1)j(1)} & \cdots & x_{i(t-1)j(t)} \end{vmatrix} \in \mathcal{P}. \end{aligned}$$

Thus $X_t \in \mathcal{P}$. It follows that

$$[\mathbf{r}\,\mathbf{x}_1\cdots\mathbf{x}_m]_t \in \sqrt{[\mathbf{0}\,\mathbf{x}_1\cdots\mathbf{x}_m]_t}\quad .$$

Conversely suppose that $[\mathbf{r}\,\mathbf{x}_1\cdots\mathbf{x}_m]_t \in \sqrt{[\mathbf{0}\,\mathbf{x}_1\cdots\mathbf{x}_m]_t}$ for every $1 \leqslant t \leqslant m+1$. Let $\mathcal{P}$ be any prime ideal of R. It is enough to show that $\mathbf{r} \in K(N,\mathcal{P})$, by Corollary 3.1.2. If $[\mathbf{0}\,\mathbf{x}_1\cdots\mathbf{x}_m]_1 \subseteq \mathcal{P}$ then $r_i \in [\mathbf{r}\,\mathbf{x}_1\cdots\mathbf{x}_m]_1 \subseteq \mathcal{P}$ and hence $\mathbf{r} = (r_1,\ldots,r_m) \in \mathcal{P}F \subseteq K(N,\mathcal{P})$. Suppose that $[\mathbf{0}\,\mathbf{x}_1\ \cdots\mathbf{x}_m]_1 \nsubseteq \mathcal{P}$. Note that $[\mathbf{0}\,\mathbf{x}_1\ \cdots\mathbf{x}_m]_{m+1} = 0 \subseteq \mathcal{P}$. Thus there exists $1 \leqslant t \leqslant m$ such that

$$[\mathbf{0}\,\mathbf{x}_1\ \cdots\mathbf{x}_m]_t \nsubseteq \mathcal{P} \text{ but } [\mathbf{0}\,\mathbf{x}_1\cdots\mathbf{x}_m]_{t+1} \subseteq \mathcal{P}.$$

There exist $1 \leqslant i(1) < \cdots < i(t) \leqslant m$, $1 \leqslant j(1) < \cdots < j(t) \leqslant n$ such that

$$d = \begin{vmatrix} x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(t)j(1)} & \cdots & x_{i(t)j(t)} \end{vmatrix} \notin \mathcal{P}.$$

By hypothesis, for each $1 \leqslant j \leqslant n$, $\begin{vmatrix} r_j & r_{j(1)} & \cdots & r_{j(t)} \\ x_{i(1)j} & x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i(t)j} & x_{i(t)j(1)} & \cdots & x_{i(t)j(t)} \end{vmatrix} \in \mathcal{P}.$

Expanding this determinant by the first column we find that

$$dr_j + a_{i(1)}x_{i(1)j} + \cdots + a_{i(t)}x_{i(t)j} \in \mathcal{P};$$

where $a_{i(k)} = (-1)^k \begin{vmatrix} r_{j(1)} & \cdots & r_{j(t)} \\ x_{i(1)j(1)} & \cdots & x_{i(1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(k-1)j(1)} & \cdots & x_{i(k-1)j(t)} \\ x_{i(k+1)j(1)} & \cdots & x_{i(k+1)j(t)} \\ \vdots & \ddots & \vdots \\ x_{i(t)j(1)} & \cdots & x_{i(t)j(t)} \end{vmatrix}$, for each $1 \leqslant k \leqslant t$.

Note that d and $a_{i(k)}$ $(1 \leqslant k \leqslant t)$ are independent of j. Thus

$$dr_j + a_{i(1)}x_{i(1)j} + \cdots + a_{i(t)}x_{i(t)j} \in \mathcal{P}\ (1 \leqslant j \leqslant n),$$

i.e. $d\mathbf{r} \in R\mathbf{x}_1 + \cdots + R\mathbf{x}_m + \mathcal{P}F = N + \mathcal{P}F$, and hence $\mathbf{r} \in K(N, \mathcal{P})$. $\square$

Lemma 3.1.4 *Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2 = \{(m_1, m_2) : m_i \in M_i\ (i = 1, 2)\}$. Let N be a proper submodule of M_1. Then*

$$m \in rad_{M_1}(N) \text{ if and only if } (m, 0) \in rad_M(N \oplus 0).$$

Proof. Suppose first that $m \in \mathrm{rad}_{M_1}(N)$. Let P be a prime submodule of M such that $N \oplus 0 \subseteq P$. Let $P' = \{x \in M_1 : (x, 0) \in P\}$. It can easily be checked that $P' = M_1$ or P' is a prime submodule of M_1 and $N \subseteq P'$. Thus $m \in P'$ and hence $(m, 0) \in P$. It follows that $(m, 0) \in \mathrm{rad}_M(N \oplus 0)$.

Conversely, suppose that $(m, 0) \in \mathrm{rad}_M(N \oplus 0)$. Let Q be a prime submodule of M_1 such that $N \subseteq Q$. Then $Q \oplus M_2$ is a prime submodule of M with $N \oplus 0 \subseteq Q \oplus M_2$. Hence $(m, 0) \in Q \oplus M_2$ so that $m \in Q$. It follows that $m \in \mathrm{rad}_{M_1}(N)$. $\square$

Theorem 3.1.5 *Let R be a ring and let F be the free R-module $R^{(n)}$, for some positive integer n. Let $N = \sum_{i=1}^{m} R\mathbf{x}_i$ be a finitely generated submodule of F. Then*

$$\mathbf{r} \in rad_F(N) \text{ if and only if } [\mathbf{r}\, \mathbf{x}_1 \cdots \mathbf{x}_m]_t \in \sqrt{[\mathbf{0}\, \mathbf{x}_1 \cdots \mathbf{x}_m]_t}$$

for all $1 \leqslant t \leqslant min(m+1, n)$.

Proof. Let $k = min(m+1, n)$. Suppose first that $k = m + 1$, i.e. $m < n$. By Lemma 3.1.3, $\mathbf{r} \in \mathrm{rad}_F(N)$ if and only if $[\mathbf{r}\, \mathbf{x}_1 \cdots \mathbf{x}_m]_t \in \sqrt{[\mathbf{0}\, \mathbf{x}_1 \cdots \mathbf{x}_m]_t}$ for all $1 \leqslant t \leqslant k$.

Now suppose that $k = n$, i.e. $n \leqslant m+1$. Let $G = R^{(m+1)}$. Let $\mathbf{r} = (r_1, \ldots, r_n)$, $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{in})$ for some $r_j \in R, x_{ij} \in R$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$. By Lemma 3.1.4,

$$\mathbf{r} \in \mathrm{rad}_F(N) \text{ if and only if } (r_1, \ldots, r_n, 0, \ldots, 0) \in \mathrm{rad}_G(N'),$$

where $N' = \sum_{i=1}^{m} R(x_{i1}, \ldots, x_{in}, 0, \ldots, 0)$. Now we can apply Lemma 3.1.3 to obtain the result. □

Remark: If M is a Noetherian module over a ring R then Lemma 3.1.3 can be used to calculate $\mathrm{rad}_M(0)$ in the following manner. By replacing R by R/A, where A is the annihilator of M in R, we can suppose that M is a faithful R-module. In this case R is a Noetherian ring [34, Exercise 7.27].

Now M is a finitely generated R-module, say $M = Rm_1 + \cdots + Rm_n$ for some positive integer n and elements $m_i \in M$ $(1 \leqslant i \leqslant n)$. There exists a homomorphism

$$\begin{aligned} \varphi : F = R^{(n)} &\longrightarrow M \\ (r_1, \ldots, r_n) &\longmapsto r_1 m_1 + \cdots + r_n m_n. \end{aligned}$$

Denote $K = Ker(\varphi)$ which is a finitely generated submodule of F. Then

$$\mathrm{rad}_M(0) = \varphi(\mathrm{rad}_F(K)),$$

by Lemma 1.2.6.

In practice, the above results can be used explicitly to calculate $\mathrm{rad}_F(N)$, as we now demonstrate in a number of examples.

Example 3.1.6 *Let R be any ring, let $m < n$ be positive integers and let $A = (a_{ij})$ be an $m \times n$ matrix with entries in R such that A contains an $m \times m$ submatrix whose determinant is a unit in R. Let $\mathbf{a}_i = (a_{i1}, \cdots, a_{in}) \in F = R^{(n)}$*

$(1 \leqslant i \leqslant m)$ and let $N = R\mathbf{a}_1 + \cdots + R\mathbf{a}_m$. Then

$$\begin{aligned} rad_F(N) &= \{\mathbf{r} \in F : [\mathbf{r}\,\mathbf{a}_1 \cdots \mathbf{a}_m]_{m+1} \in \sqrt{0}\} \\ &= N + \sqrt{0}F. \end{aligned}$$

Moreover, N is a semiprime (respectively, prime) submodule of F if and only if R is a semiprime ring (respectively a domain).

Proof. For the matrix $B = [\mathbf{0}\,\mathbf{a}_1 \cdots \mathbf{a}_m]$, we have $B_m = R$ and hence $\sqrt{B_i} = R$ $(1 \leqslant i \leqslant m)$. By Lemma 3.1.3,

$$\mathbf{r} \in \mathrm{rad}_F(N) \text{ if and only if } [\mathbf{r}\,\mathbf{a}_1 \cdots \mathbf{a}_m]_{m+1} \in \sqrt{B_{m+1}} = \sqrt{0}.$$

There exist integers $1 \leqslant j(1) < \cdots < j(m) \leqslant n$ such that

$$C = \begin{bmatrix} a_{1j(1)} & \cdots & a_{1j(m)} \\ \vdots & \ddots & \vdots \\ a_{mj(1)} & \cdots & a_{mj(m)} \end{bmatrix}$$

has determinant u which is a unit in R. Then C has an inverse $D \in M_m(R)$. Consider the matrix $DA = [\mathbf{b}_1 \cdots \mathbf{b}_m]$ where

$$\mathbf{b}_i = (b_{i1}, \ldots, b_{in}) \in F \;(1 \leqslant i \leqslant m).$$

Since $A = C(DA)$ it follows that

$$N = R\mathbf{b}_1 + \cdots + R\mathbf{b}_m.$$

Note that DA contains the $m \times m$ submatrix $DC = I_m$, the $m \times m$ identity matrix. Thus $F = N \oplus L$ where L is the free submodule of F with basis consisting of the $n - m$ elements $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 as the ith component for all $i \in \{1, \ldots, n\} \backslash \{j(1), \ldots, j(m)\}$. It follows that

$$\mathrm{rad}_F(N) = N \oplus \mathrm{rad}_L(0) = N + \sqrt{0}F.$$

Moreover, $F/N \cong L$ so that N is a semiprime (respectively, prime) submodule of F if and only if 0 is a semiprime (prime) submodule of the free module L and this happens precisely when R is semiprime (a domain). $\square$

Example 3.1.7 *Let R be any ring, let m, n be positive integers and let $A = (a_{ij})$ be an $m \times n$ matrix of rank 1 with entries in R. Let $\mathbf{a}_i = (a_{i1}, \dots, a_{in}) \in F = R^{(n)}$ $(1 \leqslant i \leqslant m)$ and let $N = R\mathbf{a}_1 + \cdots + R\mathbf{a}_m$. Then $\mathbf{r} = (r_1, \dots, r_n) \in rad_F(N)$ if and only if*

(i) $r_i \in \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{n} Ra_{jk}}$ $(1 \leqslant i \leqslant n)$, and

(ii) $r_i a_{kj} - r_j a_{ki} \in \sqrt{0}$ $(1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant m)$.

Proof. Let

$$B = \begin{bmatrix} r_1 & \cdots & r_n \\ a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & \cdots & 0 \\ a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

Then $B_t = 0$ for all $3 \leqslant t \leqslant k$ and $C_t = 0$ for all $2 \leqslant t \leqslant k$, where $k = min(m+1, n)$. Now we can apply Theorem 3.1.5. $\square$

As a further application of Theorem 3.1.5, we now calculate the radical of the submodule $W(a, X)$ of Theorem 1.2.28. Let $J = Ra + RX$. Let $w \in W$. Then $w^k \in J$ for some positive integer k. If P is a prime submodule of F containing $J(a, X)$ then $w^k(a, X) \in P$ gives $w(a, X) \in P$. It follows that $\text{rad}_F(W(a, X)) = \text{rad}_F(J(a, X))$. Note that

$$J(a, X) = R(a^2, aX) + R(aX, X^2).$$

By Theorem 3.1.5, given r_1, $r_2 \in R$,

$$\begin{aligned}(r_1, r_2) \in \mathrm{rad}_F(J(a,X)) \;&\Leftrightarrow\; \begin{cases} r_1, r_2 \in W \\ aXr_1 = a^2 r_2 \text{ and } X^2 r_1 = aXr_2 \end{cases} \\ &\Leftrightarrow\; \begin{cases} r_1, r_2 \in W \\ Xr_1 = ar_2 \end{cases} \\ &\Leftrightarrow\; (r_1, r_2) \in R(a, X).\end{aligned}$$

Thus $\mathrm{rad}_F(J(a,X)) = R(a,X)$.

We can extend Theorem 3.1.5 and to do so we first prove an elementary lemma.

Lemma 3.1.8 *Let A, B, I be ideals of a ring R. Then $A \subseteq \sqrt{B+I}$ if and only if $(A+I)/I \subseteq \sqrt{(B+I)/I}$.*

Proof. Suppose first that $(A+I)/I \subseteq \sqrt{(B+I)/I}$. Let $\mathcal{P}$ be a prime ideal of R such that $B + I \subseteq \mathcal{P}$. Then $\mathcal{P}/I$ is a prime ideal of the ring R/I and $(B+I)/I \subseteq \mathcal{P}/I$. By hypothesis, $(A+I)/I \subseteq \mathcal{P}/I$ and hence $A \subseteq A + I \subseteq \mathcal{P}$. It follows that $A \subseteq \sqrt{B+I}$.

Conversely, suppose that $A \subseteq \sqrt{B+I}$. Any prime ideal of the ring R/I containing $(B+I)/I$ is of the form $\mathcal{Q}/I$ where $\mathcal{Q}$ is a prime ideal of R containing $B+I$. Now $B+I \subseteq \mathcal{Q}$ gives $A \subseteq \mathcal{Q}$ and hence $(A+I)/I \subseteq \mathcal{Q}/I$. It follows that $(A+I)/I \subseteq \sqrt{(B+I)/I}$. □

Theorem 3.1.9 *Let R be a ring and let F be the free R-module $R^{(n)}$, for some positive integer n. Let I be an ideal of R and let $N = \sum_{i=1}^{m} R\mathbf{x}_i + IF$ for some positive integer m and elements $\mathbf{x}_i \in F$ $(1 \leqslant i \leqslant m)$. Then*

$$\mathbf{r} \in rad_F(N) \textit{ if and only if } [\mathbf{r}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t \in \sqrt{([\mathbf{0}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t + I)}$$

for all $1 \leqslant t \leqslant min(m+1, n)$.

Proof. Let R^* denote the ring R/I and, for each element r in R, let r^* denote the element $r+I$ of R^*. For each element $\mathbf{f}$ in F with $\mathbf{f} = (f_1, \dots, f_n)$, let $\mathbf{f}^*$ denote the element $(f_1^*, \dots, f_n^*)$ of the free (R/I)-module $(R/I)^{(n)}$. Note that $(R/I)^{(n)} \cong F/IF$. It will be convenient to identify these two modules and denote this module by F^*. For any submodule K of F, we set $K^* = \{k^* : k \in K\}$ which is a submodule of F^*. Suppose first that $N = F$. Let $\mathbf{r} \in F$. There exist elements $b_i \in R$ $(1 \leqslant i \leqslant m)$, $\mathbf{a} = (a_1, \dots, a_n) \in IF$, where $a_i \in I$ $(1 \leqslant i \leqslant n)$ such that

$$\mathbf{r} = b_1\mathbf{x}_1 + \cdots + b_m\mathbf{x}_m + \mathbf{a}.$$

Then $[\mathbf{r}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t \in [\mathbf{0}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t + I$ for all $1 \leqslant t \leqslant min(m+1, n)$ by standard properties of determinants. The result follows in this case.

Next suppose that $N \neq F$. Let K be a prime submodule of F such that $N \subseteq K$. Then $IF \subseteq K$ and hence $K^* = K/IF$ is a prime submodule of F^* such that $N^* = R\mathbf{x}_1^* + \cdots + R\mathbf{x}_m^* \subseteq K^*$. Conversely, any prime submodule of F^* containing N^* is clearly of the form L^* for some prime submodule L of F containing N. Thus

$$(\mathrm{rad}_{F^*}(N^*)) = (\mathrm{rad}_F(N))/IF.$$

In particular, $\mathbf{r} \in \mathrm{rad}_F(N)$ if and only if $\mathbf{r}^* \in \mathrm{rad}_{F^*}(N^*)$. By Theorem 3.1.5,

$$\mathbf{r}^* \in \mathrm{rad}_{F^*}(N^*) \text{ if and only if } [\mathbf{r}^*\,\mathbf{x}_1^* \cdots \mathbf{x}_m^*]_t \in \sqrt{[\mathbf{0}\,\mathbf{x}_1^* \cdots \mathbf{x}_m^*]_t}$$

for all $1 \leqslant t \leqslant min(m+1, n)$, and by Lemma 3.1.8, this holds if and only if

$$[\mathbf{r}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t \in \sqrt{([\mathbf{0}\,\mathbf{x}_1 \cdots \mathbf{x}_m]_t + I)}$$

for all $1 \leqslant t \leqslant min(m+1, n)$. $\square$

For particular submodules, the radical can be expressed in a simple form. Recall that in Theorem 2.2.7 we proved that if R is a UFD and $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero, then $N = R(a_1, \ldots, a_n)$ is a prime submodule of $R^{(n)}$ if and only if every common divisor of a_i $(1 \leqslant i \leqslant n)$ is a unit in R.

Proposition 3.1.10 *Let R be a UFD, let n be a positive integer, let $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero, and let N be the submodule $R(a_1, \ldots, a_n)$ of $F = R^{(n)}$. Then $rad_F(N) = R(b_1, \ldots, b_n)$ where $b_i = (p_1 \cdots p_m a_i)/d$ $(1 \leqslant i \leqslant n)$, d is a greatest common divisor* (gcd) *of $a_1, \ldots, a_n$, and either d is not a unit and $p_1, \ldots, p_m$ are the pairwise non-associate prime divisors of d, or d is a unit and $p_1 = \cdots = p_m = 1$.*

Proof. Suppose that d is a gcd of a_i $(1 \leqslant i \leqslant n)$. If d is a unit in R then N is prime by Theorem 2.2.7 and hence $\text{rad}_F(N) = N = R(a_1, \ldots, a_n)$. Now suppose that d is not a unit in R. Then $d = p_1^{k_1} \cdots p_m^{k_m}$ for pairwise non-associate primes p_i $(1 \leqslant i \leqslant m)$ and positive integers k_i $(1 \leqslant i \leqslant m)$. For each $1 \leqslant i \leqslant n$ there exists $x_i \in R$ such that $a_i = dx_i$. Thus $(a_1, \ldots, a_n) = d(x_1, \ldots, x_n) = p_1^{k_1} \cdots p_m^{k_m}(x_1, \ldots, x_n)$.

Let K be any prime submodule of F such that $N = R(a_1, \ldots, a_n) \subseteq K$. Then $p_1^{k_1} \cdots p_m^{k_m} R(x_1, \ldots, x_n) \subseteq K$ and hence $p_1 \cdots p_m R(x_1, \ldots, x_n) \subseteq K$. But $p_1 \cdots p_m R(x_1, \ldots, x_n) = R(p_1 \cdots p_m x_1, \ldots, p_1 \cdots p_m x_n) = R(b_1, \ldots, b_n)$. We have proved that $R(b_1, \ldots, b_n) \subseteq \text{rad}_F(N)$. Note also that $N \subseteq R(b_1, \ldots, b_n)$.

Next we prove that

$$R(b_1, \ldots, b_n) = R(x_1, \ldots, x_n) \cap p_1 F \cap \cdots \cap p_m F.$$

Clearly $R(b_1, \ldots, b_n) \subseteq R(x_1, \ldots, x_n) \cap p_1 F \cap \cdots \cap p_m F$. Conversely, let $r \in R$ such that $r(x_1, \ldots, x_n) \in p_1 F \cap \cdots \cap p_m F$. For each $1 \leqslant i \leqslant m$, p_i divides rx_j $(1 \leqslant j \leqslant n)$ and hence p_i divides r, because $x_1, \ldots, x_n$ have no common prime divisor. Since $p_1, \ldots, p_m$ are pairwise non-associates it follows that $p_1 \cdots p_m$ divides r. Thus $r(x_1, \ldots, x_n) \in R(b_1, \ldots, b_n)$, as required.

Since $(p_iF : F) = (p_i)$ is a prime ideal of R and F/p_iF is a torsion-free $R/(p_i)$-module, by Proposition 1.1.1.3, p_iF is a prime submodule of F $(1 \leqslant i \leqslant m)$. By Theorem 2.2.7, $R(x_1, \ldots, x_n)$ is prime. Hence the proof is completed. □

A non-zero element r of a UFD R will be called *square-free* if there does not exist a prime p in R such that $r = p^2 s$ for some $s \in R$. Compare the next result with Theorem 2.2.7.

Corollary 3.1.11 *Let R be a UFD, let n be a positive integer, let $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero, and let N be the submodule $R(a_1 \ldots, a_n)$ of $F = R^{(n)}$. Then N is a semiprime submodule of F if and only if any greatest common divisor of a_i $(1 \leqslant i \leqslant n)$ is square-free.*

Proof. Let d be a greatest common divisor of a_i $(1 \leqslant i \leqslant n)$. Suppose that d is square-free. If d is a unit then N is prime by Theorem 2.2.7. Suppose that d is not a unit. Then in the notation of Proposition 3.1.10, $d = up_1 \cdots p_m$ for some unit u in R and hence $b_i = u^{-1}a_i$ $(1 \leqslant i \leqslant n)$. In this case, $N = \mathrm{rad}_F(N)$, by Proposition 3.1.10, and hence N is semiprime.

Conversely, suppose that N is semiprime. If d is a unit then square-free. Suppose that d is not a unit. Then Proposition 3.1.10 gives $N = \mathrm{rad}_F(N) = R(b_1, \ldots, b_n)$ where $b_i = (p_1 \cdots p_m a_i)/d$ $(1 \leqslant i \leqslant n)$. There exists $r \in R$ such that $(b_1, \ldots, b_n) = r(a_1, \ldots, a_n)$ and there exists $1 \leqslant j \leqslant n$ such that $a_j \neq 0$. Hence $(p_1 \cdots p_m a_j)/d = ra_j$, so that $p_1 \cdots p_m = dr$ and hence d is square-free. □

3.2 The Radicals of Particular Submodules

In the previous section we gave a description of the radical of a finitely generated submodule of a free module. In this section we shall show how to find the radical of not necessarily finitely generated submodules of free modules. Now suppose that R is a ring and F is a free R-module. We begin with a very easy case.

Proposition 3.2.1 *Let I be any ideal of R. Then*

$$rad_F(IF) = \sqrt{I}F.$$

Proof. Let $r \in \sqrt{I}$. Then $r^k \in I$ for some positive integer k. Let K be a prime submodule of F such that $IF \subseteq K$ and $x \in F$. Then $r^k x \in IF \subseteq K$ and it follows that $rx \in K$. Thus $rF \subseteq K$. This implies that $rF \subseteq \text{rad}_F(IF)$. Hence $\sqrt{I}F \subseteq \text{rad}_F(IF)$.

Conversely, note first that if $I = R$ then $\sqrt{I}F = \text{rad}_F(IF) = F$. Suppose that $I \neq R$. Note that

$$\sqrt{I}F = (\textstyle\bigcap_{\mathcal{P}\in\Omega}\mathcal{P})F = \bigcap_{\mathcal{P}\in\Omega}(\mathcal{P}F)$$

where Ω is the collection of prime ideals of R such that $I \subseteq \mathcal{P}$. Now by Proposition 1.1.1.3, $\mathcal{P}F$ is a prime submodule of F and $IF \subseteq \mathcal{P}F$ so that $\text{rad}_F(IF) \subseteq \mathcal{P}F$ for all $P \in \Omega$. Hence $\text{rad}_F(IF) \subseteq \bigcap_{\mathcal{P}\in\Omega}(\mathcal{P}F) = \sqrt{I}F$. It follows that $\text{rad}_F(IF) = \sqrt{I}F$. $\square$

Corollary 3.2.2 *Let R be a ring with prime radical W, let I be an ideal of R and let N be a direct summand of F. Then*

$$rad_F(IN) = \sqrt{I}N + WF.$$

Proof. There exists a submodule N' of F such that $F = N \oplus N'$. Note that

$$F/(\sqrt{I}N + WF) = F/(\sqrt{I}N \oplus WN') \cong (N/\sqrt{I}N) \oplus (N'/WN').$$

By Proposition 3.2.1, $\text{rad}_F(WF) = WF$. But $WF = WN \oplus WN'$. Hence $\text{rad}_{N'}(WN') = WN'$ by Lemma 1.1.2.4. Thus $\text{rad}_{N'/WN'}(0) = 0$. Similarly

$\mathrm{rad}_{N/\sqrt{I}N}(0) = 0$. Again using Lemma 1.1.2.4 we find that $\mathrm{rad}_{F/(\sqrt{I}N+WF)}(0) = 0$, i.e. $\sqrt{I}N + WF$ is a semiprime submodule of F. Since $IN \subseteq \sqrt{I}N$ it follows that $\mathrm{rad}_F(IN) \subseteq \sqrt{I}N + WF$.

Let $r \in \sqrt{I}, x \in N$. There exists a positive integer k such that $r^k \in I$. Let K be any prime submodule of F such that $IN \subseteq K$. Then $r^k x \in IN \subseteq K$ and it follows that $rx \in K$. Hence $rx \in \mathrm{rad}_F(IN)$. It follows that $\sqrt{I}N \subseteq \mathrm{rad}_F(IN)$. A similar argument shows that $WF \subseteq \mathrm{rad}_F(IN)$. Hence $\sqrt{I}N + WF \subseteq \mathrm{rad}_F(IN)$. Thus $\mathrm{rad}_F(IN) = \sqrt{I}N + WF$. □

Combining Corollary 3.2.2 and Proposition 2.2.3 we have the following result.

Corollary 3.2.3 *Let R be a ring with prime radical W, let n be a positive integer, let $a_i \in R$ $(1 \leqslant i \leqslant n)$ such that $R = Ra_1 + \cdots + Ra_n$ and let $\mathbf{a}$ be the element $(a_1, \ldots, a_n)$ of the R-module $F = R^{(n)}$. Then $rad_F(I\mathbf{a}) = \sqrt{I}\mathbf{a} + WF$ for any ideal I of R.*

Corollary 3.2.4 *With the notation of Corollary 3.2.3, the submodule $I\mathbf{a}$ is a semiprime submodule of F if and only if I is a semiprime ideal of R and $WF \subseteq I\mathbf{a}$.*

Proof. Suppose first that I is a semiprime ideal of R, i.e. $\sqrt{I} = I$, and $WF \subseteq I\mathbf{a}$. Then clearly

$$\mathrm{rad}_F(I\mathbf{a}) = I\mathbf{a} + WF = I\mathbf{a},$$

i.e. $I\mathbf{a}$ is a semiprime submodule of F.

Conversely, suppose that $I\mathbf{a}$ is a semiprime submodule of F. Then

$$I\mathbf{a} = \mathrm{rad}_F(I\mathbf{a}) = \sqrt{I}\mathbf{a} + WF,$$

so that $WF \subseteq I\mathbf{a}$. Let $x \in \sqrt{I}$. Then $x\mathbf{a} = y\mathbf{a}$ for some $y \in I$. It follows that $(x - y)\mathbf{a} = 0$, i.e. $x \in I$. Hence $\sqrt{I} = I$, i.e. I is a semiprime ideal of R. □

This brings us to the main result of this section.

Theorem 3.2.5 *Let R be a ring with prime radical W, let n be a positive integer and let $F = R^{(n)}$. Let $a_i \in R$ $(1 \leqslant i \leqslant n)$, let I be an ideal of R and let N be the submodule $R(a_1, \dots, a_n) + I(1, \dots, 1)$ of F. Let A be the ideal $\sum_{i=1}^{n} R(a_1 - a_i)$ of R. Then*

$$A rad_F(N) \subseteq R(a_1, \dots, a_n) + \sqrt{I}(1, \dots, 1) + WF \subseteq rad_F(N).$$

In particular, if $A = R$ then

$$rad_F(N) = R(a_1, \dots, a_n) + \sqrt{I}(1, \dots, 1) + WF = < E_F(N) > .$$

Proof. Clearly $R(a_1, \dots, a_n) \subseteq N \subseteq \mathrm{rad}_F(N)$. If $r \in \sqrt{I}$ then $r^k \in I$ for some positive integer k. Hence $r^k(1, \dots, 1) \in I(1, \dots, 1) \subseteq N$. It follows that $r(1, \dots, 1) \in \mathrm{rad}_F(N)$. Thus $\sqrt{I}(1, \dots, 1) \subseteq \mathrm{rad}_F(N)$. Moreover, $WF \subseteq \mathrm{rad}_F(N)$ since W is a nil ideal of R. Thus $R(a_1, \dots, a_n) + \sqrt{I}(1, \dots, 1) + WF \subseteq \mathrm{rad}_F(N)$.

Next, let $\mathbf{r} = (r_1, \dots, r_n) \in \mathrm{rad}_F(N)$ and let $2 \leqslant i \leqslant n$. We shall prove that

$$(a_1 - a_i)\mathbf{r} \in R(a_1, \dots, a_n) + \sqrt{I}(1, \dots, 1) + WF.$$

Let $\mathcal{P}$ be any prime ideal of R such that $I \subseteq \mathcal{P}$. By Corollary 3.1.2, there exists $c \in R \backslash \mathcal{P}$ such that $c\mathbf{r} \in N + \mathcal{P}F$, i.e.

$$c(r_1, \dots, r_n) = s(a_1, \dots, a_n) + t(1, \dots, 1) + (p_1, \dots, p_n)$$

for some $s \in R$, $t \in I$, $p_i \in \mathcal{P}$ $(1 \leqslant i \leqslant n)$.

Hence $cr_i - sa_i = t + p_i \in \mathcal{P}$ $(1 \leqslant i \leqslant n)$. In particular,

$$c(a_1 r_i - a_i r_1) = a_1(sa_i + t + p_i) - a_i(sa_1 + t + p_1) \in \mathcal{P}.$$

It follows that $a_1 r_i - a_i r_1 \in \mathcal{P}$ for every prime ideal $\mathcal{P}$ containing I. Hence $a_1 r_i - a_i r_1 \in \sqrt{I}$ for all $1 \leqslant i \leqslant n$. Let $1 \leqslant i \leqslant n$. Consider the element

$(w_1, \dots, w_n) = (a_1 - a_i)(r_1, \dots, r_n) - (r_1 - r_i)(a_1, \dots, a_n) - (a_1 r_i - a_i r_1)(1, \dots, 1)$. Let $1 \leqslant j \leqslant n$. Then

$$w_j = \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_i & r_j \\ a_1 & a_i & a_j \end{vmatrix}.$$

Let Q be any prime ideal of R. There exists $d \in R \backslash Q$ such that $d\mathbf{r} \in N + QF$, i.e.

$$d(r_1, \dots, r_n) = x(a_1, \dots, a_n) + y(1, \dots, 1) + (q_1, \dots, q_n)$$

for some $x \in R$, $y \in I$, $q_i \in Q$ $(1 \leqslant i \leqslant n)$. Now $dr_i = xa_i + y + q_i$ $(1 \leqslant i \leqslant n)$. Consider

$$\begin{aligned} dw_j &= \begin{vmatrix} 1 & 1 & 1 \\ dr_1 & dr_i & dr_j \\ a_1 & a_i & a_j \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ xa_1 + y + q_1 & xa_i + y + q_i & xa_j + y + q_j \\ a_1 & a_i & a_j \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 \\ q_1 & q_i & q_j \\ a_1 & a_i & a_j \end{vmatrix} \in Q. \end{aligned}$$

Thus $w_j \in Q$ for every prime ideal Q. It follows that $w_j \in W$. Hence $w_j \in W$ $(1 \leqslant j \leqslant n)$. Thus

$$\begin{gathered} (a_1 - a_i)\mathbf{r} = (r_1 - r_i)(a_1, \dots, a_n) + (a_1 r_i - a_i r_1)(1, \dots, 1) + (w_1, \dots, w_n) \in \\ R(a_1, \dots, a_n) + \sqrt{I}(1, \dots, 1) + WF, \end{gathered}$$

as required. It follows that

$$(a_1 - a_i)\mathbf{r} \in R(a_1, \dots, a_n) + \sqrt{I}(1, \dots, 1) + WF,$$

for all $1 \leqslant i \leqslant n$. Hence

$$A\mathbf{r} \subseteq R(a_1, \dots, a_n) + \sqrt{I}(1, \dots, 1) + WF,$$

for all $\mathbf{r} \in \mathrm{rad}_F(N)$, i.e.

$$A\mathrm{rad}_F(N) \subseteq R(a_1, \ldots, a_n) + \sqrt{I}(1, \ldots, 1) + WF.$$

Now suppose that $A = R$. Clearly

$$\mathrm{rad}_F(N) = R(a_1, \ldots, a_n) + \sqrt{I}(1, \ldots, 1) + WF.$$

Let $\mathbf{r} \in \mathrm{rad}_F(N)$. Then

$$\begin{aligned} \mathbf{r} &= u(a_1, \ldots, a_n) + v(1, \ldots, 1) + (z_1, \ldots, z_n) \\ &= u(a_1, \ldots, a_n) + v(1, \ldots, 1) + z_1(1, 0, \ldots, 0) + \cdots + z_n(0, \ldots, 0, 1) \end{aligned}$$

for some $u \in R$, $v \in \sqrt{I}$, $z_i \in W$ $(1 \leqslant i \leqslant n)$. There exists a positive integer m such that $v^m \in I$, $z_i^m = 0$ $(1 \leqslant i \leqslant n)$. Note that $u(a_1, \ldots, a_n) \in N$, $v^m(1, \ldots, 1) \in N$ and $z_i^m(0, \ldots, 0, 1, 0, \ldots, 0) \in N$ $(1 \leqslant i \leqslant m)$. Thus $\mathbf{r} \in\ < E_F(N) >$. It follows that $\mathrm{rad}_F(N) \subseteq < E_F(N) >$ and hence

$$\mathrm{rad}_F(N) =< E_F(N) >.$$

□

Corollary 3.2.6 *Let R be a ring with prime radical W and let $F = R^{(n)}$ for some positive integer n. Let $a_i \in R$ $(1 \leqslant i \leqslant n)$, let $b \in R$, let I be an ideal of R and let N be the submodule $R(a_1, \ldots, a_n) + I(b, \ldots, b)$ of F. Let A be the ideal $\sum_{i=1}^{n} R(a_1 - a_i)$. Then*

$$A rad_F(N) \subseteq R(a_1, \ldots, a_n) + \sqrt{I}b(1, \ldots, 1) + WF \subseteq rad_F(N).$$

In particular, if $A = R$ then

$$rad_F(N) = R(a_1, \ldots, a_n) + \sqrt{I}b(1, \ldots, 1) + WF =< E_F(N) > .$$

Proof. Clear by Theorem 3.2.5. $\square$

It is natural to ask what is the radical $\text{rad}_F(N)$ of a submodule of the form

$$N = R(a_1, \dots, a_n) + I(b_1, \dots, b_n)$$

where a_i, $b_i \in R$ $(1 \leqslant i \leqslant n)$, I is an ideal of R and $R = \sum_{i=1}^{n} R(a_1 - a_i)$. The only cases we know are the ones dealt with above.

We now give another consequence of Theorem 3.2.5.

Corollary 3.2.7 *With the notation of Theorem 3.2.5, suppose that $A = R$ and N is a proper submodule of F. Then N is a semiprime submodule of F if and only if $\sqrt{I} = I$ and $WF \subseteq N$.*

Proof. Suppose first that $\sqrt{I} = I$ and $WF \subseteq N$. Then by Theorem 3.2.5,

$$\text{rad}_F(N) = R(a_1, \dots, a_n) + I(1, \dots, 1) + WF = N.$$

Conversely, suppose that N is semiprime, i.e. $N = \text{rad}_F(N)$. By Theorem 3.2.5, $WF \subseteq N$. Let $a \in \sqrt{I}$. Again applying Theorem 3.2.5, we have

$$a(1, \dots, 1) = r(a_1, \dots, a_n) + s(1, \dots, 1)$$

for some $r \in R$, $s \in I$. Clearly $ra_i = a - s$ $(1 \leqslant i \leqslant n)$. Then $1 = s_2(a_1 - a_2) + \cdots + s_n(a_1 - a_n)$ for some $s_i \in R$ $(2 \leqslant i \leqslant n)$ and this gives that

$$r = r1 = s_2 r(a_1 - a_2) + \cdots + s_n r(a_1 - a_n) = 0.$$

It follows that $a = s \in I$. Hence $\sqrt{I} \subseteq I$, i.e. $\sqrt{I} = I$. $\square$

The following example shows that the condition $A = R$ in Corollary 3.2.7 is necessary.

Example 3.2.8 *Let $R = \mathbb{Z}$, N be the submodule $\mathbb{Z}(1,3,5) + \mathbb{Z}2(1,1,1)$ of $F = \mathbb{Z}^{(3)}$. Then $\sqrt{I} = \sqrt{\mathbb{Z}2} = \mathbb{Z}2 = I$ and $WF = 0 \subseteq N$ but N is not a semiprime submodule of F because $(0,2,4) \in rad_F(N)\backslash N$.*

Proof. By Theorem 3.1.5,

$$(r_1, r_2, r_3) \in \mathrm{rad}_F(N) \Leftrightarrow \begin{cases} 3r_1 - r_2, 5r_1 - r_3, 5r_2 - 3r_3 \in 2\mathbb{Z} \text{ and} \\ -2r_1 + 4r_2 - 2r_3 = 0. \end{cases}$$

Hence $\mathrm{rad}_F(N) = \{(a, a+2b, a+4b) : a, b \in \mathbb{Z}\}$. Thus $(0,2,4) \in \mathrm{rad}_F(N)$.

Suppose that $(0,2,4) \in N$. Then there exist $s, t \in \mathbb{Z}$ such that $(0,2,4) = s(1,3,5) + t(2,2,2)$. Hence $s = 1$ and $t = -1/2$, a contradiction. $\square$

Now, one can ask whether $A = R$ is a necessary condition for

$$\mathrm{rad}_F(N) = R(a_1, \ldots, a_n) + \sqrt{I}(1, \ldots, 1) + WF.$$

As the following example shows this is not the case.

Example 3.2.9 *Let $R = \mathbb{Z}$, $F = \mathbb{Z}^{(3)}$ and let p be a prime number. Let N be the submodule $R(p,0,p) + Rp(1,1,1)$ of F. Then $rad_F(N) = R(p,0,p) + \sqrt{Rp}(1,1,1) = R(p,0,p) + Rp(1,1,1) = N$ but $A = Rp \neq R$.*

Proof. By Theorem 3.1.5

$$(r_1, r_2, r_3) \in \mathrm{rad}_F(N) \Leftrightarrow r_1 = r_3, r_1, r_2 \in Rp,$$

so that

$$\mathrm{rad}_F(N) = \{(r, s, r) : r, s \in Rp\} = R(p,0,p) + Rp(1,1,1),$$

since $(r,s,r) = (u-v)(p,0,p) + vp(1,1,1)$, where $r = up$, $s = vp$ $(u, v \in R)$. $\square$

Chapter 4

MODULES WHICH S.T.R.F. AND ENVELOPES IN FREE MODULES

Throughout this chapter all rings will be commutative. Let M_1, M_2 be R-modules such that M_1 and M_2 both s.t.r.f.. Then $M_1 \oplus M_2$ does not have to s.t.r.f. in general. The aim of section 4.1 is to investigate when $M_1 \oplus M_2$ s.t.r.f.. For example, it is proved in Theorem 4.1.10 that if M_1 s.t.r.f. and M_2 is semisimple, then $M = M_1 \oplus M_2$ s.t.r.f.. Also it is proved in Theorem 4.1.18 that if A is a finite direct sum of cyclic Artinian R-modules, then the R-module $R \oplus A$ s.t.r.f.. An application of Theorem 4.1.18 gives that the R-module $R \oplus (R/\mathcal{M}_1^{k(1)}) \oplus \cdots \oplus (R/\mathcal{M}_n^{k(n)})$ s.t.r.f. for all positive integers $n, k(1), \ldots, k(n)$ and maximal ideals $\mathcal{M}_i$ $(1 \leqslant i \leqslant n)$ (Theorem 4.1.19).

The aim of section 4.2 is to describe $E_M(N)$ for some submodule N of an R-module M. But since $E_M(N)$ is not a submodule in general, it makes the job harder. Hence the envelope is described in some special cases. For example, if R is a UFD and F is the free R-module $R^{(n)}$ for some positive integer n and N is a cyclic submodule $R(a_1, \dots, a_n)$ of F for some elements $a_1, \dots, a_n$ of R, not all zero, then $E_F(N) = \sqrt{R}d(\frac{a_1}{d}, \dots, \frac{a_n}{d})$ where $d = \gcd(a_1, \dots, a_n)$. Corollary 4.2.2 shows that actually this coincides with $\text{rad}_F(N)$.

4.1 Modules Which Satisfy the Radical Formula

We begin this section with the following simple observation. We give the proof for completeness.

Lemma 4.1.1 *Any cyclic module s.t.r.f.. Moreover, if N is a submodule of a cyclic module M then $rad_M(N) = E_M(N)$.*

Proof. Let M be a cyclic R-module. Then $M \cong R/I$ for some ideal I of R, and without loss of generality we can suppose that $M = R/I$. Let N be any submodule of M. Then $N = J/I$ for some ideal J of R containing I. It is not difficult to check that $\text{rad}_M(N) = \sqrt{J}/I = E_M(N)$. Thus M s.t.r.f.. □

Lemma 4.1.2 *Let M be an R-module such that M s.t.r.f.. Then every homomorphic image of M s.t.r.f..*

Proof. Since M s.t.r.f., $\text{rad}_{M/N}(0) =< E_{M/N}(0) >$ for every submodule N of M. Let K be a submodule of M. Hence $\text{rad}_{(M/K)/(N/K)}(0) =< E_{(M/K)/(N/K)}(0) >$ for every submodule N containing K. Thus M/K s.t.r.f.. □

Corollary 4.1.3 *Let M_1, M_2 be R-modules such that $M_1 \oplus M_2$ s.t.r.f.. Then M_1 and M_2 both s.t.r.f..*

The converse of the above Corollary is false. For example, if R is a Noetherian domain which is not Dedekind domain then the R-module R s.t.r.f. but the R-module $R \oplus R$ does not, by Theorem 1.2.19. But it is true in some cases. Before we prove that we require a number of lemmas and propositions.

First for the sake of brevity and convenience we define the following:

Definition 4.1.4 *We will call a submodule N of an R-module M good if*

$$rad_M(N) = < E_M(N) > .$$

Note that M is a good submodule of M. Moreover every prime (or, more generally, semiprime) submodule of M is good. Note also that the module M s.t.r.f. if and only if every submodule is good.

Proposition 4.1.5 *Let R be any ring and M be an R-module such that $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is a direct sum of submodules M_λ $(\lambda \in \Lambda)$. For each $\lambda \in \Lambda$ let N_λ be a submodule of M_λ and let $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$. Then N is a good submodule of M if and only if N_λ is a good submodule of M_λ for all $\lambda \in \Lambda$.*

Proof. By Lemma 1.1.2.4. □

Lemma 4.1.6 *Let N be a submodule of an R-module M. Then N is a good submodule of M if and only if the zero submodule is a good submodule of the R-module M/N.*

Proof. By Proposition 1.2.7. □

Corollary 4.1.7 *Let M_1, M_2 be R-modules and let N_i be a submodule of M_i for i=1,2 such that $M_1/N_1 \cong M_2/N_2$. Then N_1 is a good submodule of M_1 if and only if N_2 is a good submodule of M_2.*

Proof. Suppose that N_1 is a good submodule of M_1. Then the zero submodule is a good submodule of M_1/N_1, by Lemma 4.1.6. It follows that the zero submodule is a good submodule of M_2/N_2 and hence N_2 is a good submodule of M_2, also by Lemma 4.1.6. □

Before proceeding to consider when certain direct sums s.t.r.f. we prove the following elementary result.

Lemma 4.1.8 *Let N be a direct summand of a module M and let L be a submodule of N such that L is a good submodule of M. Then L is a good submodule of N.*

Proof. Let $x \in \mathrm{rad}_N(L)$. By Lemma 1.1.1.6, $\mathrm{rad}_N(L) \subseteq \mathrm{rad}_M(L) = < E_M(L) >$ and hence $x = r_1 m_1 + \cdots + r_n m_n$ for some positive integer n and elements $r_i \in R$, $m_i \in M$ with $r_i^k m_i \in L$ $(1 \leqslant i \leqslant n)$, for some positive integer k. There exists a submodule N' of M such that $M = N \oplus N'$. For each $1 \leqslant i \leqslant n$, there exist $y_i \in N$, $z_i \in N'$ such that $m_i = y_i + z_i$. Then

$$x = r_1 m_1 + \cdots + r_n m_n = (r_1 y_1 + \cdots + r_n y_n) + (r_1 z_1 + \cdots + r_n z_n)$$

so that $x = r_1 y_1 + \cdots + r_n y_n$ and $r_i^k y_i \in L$ $(1 \leqslant i \leqslant n)$. It follows that $x \in < E_N(L) >$. Hence L is a good submodule of N. $\square$

Corollary 4.1.9 *Let M be a module such that 0 is a good submodule. Then every direct summand of M is a good submodule.*

Proof. Let N be a direct summand of M. Then $M = N \oplus N'$ for some submodule N' of M. Now 0 is a good submodule of N' by Lemma 4.1.8 and $M/N \cong N'$. By Lemma 4.1.6, N is a good submodule of M. $\square$

Theorem 4.1.10 *Let M be an R-module with submodules M_1 and M_2 such that M_1 s.t.r.f., M_2 is semisimple and $M = M_1 \oplus M_2$. Then M s.t.r.f..*

Proof. Let $\pi_2 : M \to M_2$ denote the canonical projection. Let N be any submodule of M. Then $\pi_2(N)$ is a submodule of M_2 and hence $M_2 = L \oplus \pi_2(N)$, for some submodule L. Now $M = M_1 \oplus L \oplus \pi_2(N)$ implies that $M = (M_1 \oplus L) + N$ and hence $M/N \cong (M_1 \oplus L)/((M_1 \oplus L) \cap N)$. By Corollary 4.1.7, to prove that N is a good submodule of M it is sufficient to prove that $(M_1 \oplus L) \cap N$ is a good submodule of $M_1 \oplus L$. Let $\pi : M_1 \oplus L \to L$ denote the canonical projection. Then $\pi((M_1 \oplus L) \cap N) \subseteq L \cap \pi_2(N) = 0$, so that $(M_1 \oplus L) \cap N \subseteq M_1$. By hypothesis $(M_1 \oplus L) \cap N$ is a good submodule of M_1. Since 0 is a semiprime submodule of L it follows that 0 is a good submodule of L. By Proposition 4.1.5, $(M_1 \oplus L) \cap N$ is a good submodule of $M_1 \oplus L$. Hence N is a good submodule of M. It follows that M s.t.r.f.. □

Corollary 4.1.11 *Let M be any semisimple R-module. Then the R-module $R \oplus M$ s.t.r.f..*

Proof. The R-module R s.t.r.f.. Apply Theorem 4.1.10. □

In particular, Theorem 4.1.10 gives that every semisimple module s.t.r.f.. This fact is clear, however, because if N is a proper submodule of a semisimple module M then N is an intersection of maximal submodules of M and every maximal submodule of M is prime. Thus every proper submodule of M is semiprime and so is good.

Lemma 4.1.12 *Let N be a submodule of an R-module M and let $\mathcal{M}$ be a maximal ideal of R. Then $rad_M(\mathcal{M}^k N) = \mathcal{M}M \cap rad_M(N)$ for any positive integer k.*

Proof. Let P be any prime submodule of M such that $\mathcal{M}^k N \subseteq P$. Then $\mathcal{M}M \subseteq P$ or $N \subseteq P$, i.e. $\mathcal{M}M \cap \mathrm{rad}_M(N) \subseteq P$. Thus

$$\mathcal{M}M \cap \mathrm{rad}_M(N) \subseteq \mathrm{rad}_M(\mathcal{M}^k N).$$

Conversely, $\mathcal{M}^k N \subseteq \mathcal{M}M$ and $\mathcal{M}M = M$ or $\mathcal{M}M$ is a prime submodule of M. Thus $\mathrm{rad}_M(\mathcal{M}^k N) \subseteq \mathcal{M}M$. Also clearly $\mathrm{rad}_M(\mathcal{M}^k N) \subseteq \mathrm{rad}_M(N)$. □

Lemma 4.1.13 *Let M be an R-module and let $\mathcal{M}$ be a maximal ideal of R such that for each $x \in M$ there exists a positive integer k such that $\mathcal{M}^k x = 0$. Then the zero submodule of M is good.*

Proof. Let $y \in \text{rad}_M(0)$. Then $\mathcal{M}M = M$ or $\mathcal{M}M$ is a prime submodule of M. In any case $y \in \mathcal{M}M$. There exist a positive integer n and elements $r_i \in \mathcal{M}$, $y_i \in M$ $(1 \leqslant i \leqslant n)$ such that $y = r_1y_1 + \cdots + r_ny_n$. For each $1 \leqslant i \leqslant n$ there exists a positive integer $k(i)$ such that $\mathcal{M}^{k(i)}y_i = 0$. Let $k = \max\{k(i) : 1 \leqslant i \leqslant n\}$. Then $r_i^k y_i = 0$ $(1 \leqslant i \leqslant n)$. It follows that 0 is a good submodule. □

Corollary 4.1.14 *Let M be an R-module and let $\mathcal{M}$ be a maximal ideal of R such that for each $x \in M$ there exists a positive integer k such that $\mathcal{M}^k x = 0$. Then M s.t.r.f..*

Proof. Let N be any submodule of M. Applying Lemma 4.1.13 to the R-module M/N, we see that the zero submodule of M/N is good. By Lemma 4.1.6, N is a good submodule of M. It follows that M s.t.r.f.. □

Lemma 4.1.15 *Let $L \subseteq N$ be submodules of an R-module M such that L is a good submodule of M and $\text{rad}_M(N) = \text{rad}_M(L)$. Then N is a good submodule of M.*

Proof. Let $m \in \text{rad}_M(N)$. Then $m \in \text{rad}_M(L)$. There exist positive integers n, k and elements $r_i \in R$, $m_i \in M$ such that $m = r_1m_1 + \cdots + r_nm_n$ and $r_i^k m_i \in L \subseteq N$ $(1 \leqslant i \leqslant n)$. It follows that N is good. □

Lemma 4.1.16 *Let R be a quasi-local ring with unique maximal ideal $\mathcal{M}$ and let k be a positive integer. Then the R-module $R \oplus (R/\mathcal{M}^k)$ s.t.r.f..*

Proof. Let $M = R \oplus (R/\mathcal{M}^k)$ and let $\pi_1 : M \to R$ and $\pi_2 : M \to R/\mathcal{M}^k$ denote the canonical projections. Let N be any submodule of M. If $\pi_1(N) = R$ then $M = N + (0 \oplus R/\mathcal{M}^k)$ so that $M/N \cong (0 \oplus R/\mathcal{M}^k)/(N \cap (0 \oplus R/\mathcal{M}^k))$ which is a homomorphic image of $R/\mathcal{M}^k$ and hence also of R. By Lemma 4.1.2, M/N s.t.r.f. and hence N is good by Lemma 4.1.6. If $\pi_2(N) = R/\mathcal{M}^k$ then a similar argument shows that M/N is a homomorphic image of R. Thus again N is good.

Now suppose that $\pi_1(N) \neq R$ and $\pi_2(N) \neq R/\mathcal{M}^k$. Thus $\pi_1(N) \subseteq \mathcal{M}$ and $\pi_2(N) \subseteq \mathcal{M}/\mathcal{M}^k$. Hence $N \subseteq \pi_1(N) \oplus \pi_2(N) \subseteq \mathcal{M} \oplus (\mathcal{M}/\mathcal{M}^k) = \mathcal{M}M$. Clearly $\mathcal{M}^k N \subseteq R \oplus 0$ so that $\mathcal{M}^k N$ is good in $R \oplus 0$. Since $R/\mathcal{M}^k$ is cyclic it follows that $R/\mathcal{M}^k$ s.t.r.f. and hence the zero submodule is good. By Proposition 4.1.5, $\mathcal{M}^k N$ is a good submodule of M. Since $N \subseteq \mathcal{M}M$ and $\mathcal{M}M$ is a prime submodule of M it follows that $\mathrm{rad}_M(N) \subseteq \mathcal{M}M$ and, by Lemma 4.1.12, $\mathrm{rad}_M(\mathcal{M}^k N) = \mathrm{rad}_M(N)$. Thus N is good by Lemma 4.1.15. $\square$

Lemma 4.1.17 *Let R be a quasi-local ring with unique maximal ideal $\mathcal{M}$ and let $n, k(1), \ldots, k(n)$ be positive integers. Then the R-module $R \oplus (R/\mathcal{M}^{k(1)}) \oplus \cdots \oplus (R/\mathcal{M}^{k(n)})$ s.t.r.f..*

Proof. Let $M = R \oplus (R/\mathcal{M}^{k(1)}) \oplus \cdots \oplus (R/\mathcal{M}^{k(n)})$. Let $M_0 = R$, and let $M_i = R/\mathcal{M}^{k(i)}$ $(1 \leqslant i \leqslant n)$, so that $M = M_0 \oplus M_1 \oplus \cdots \oplus M_n$. For each $0 \leqslant i \leqslant n$, let $\pi_i : M \to M_i$ denote the canonical projection. We prove the result by induction on n. If $n = 1$ then the result is proved by Lemma 4.1.16. Suppose that $n > 1$.

Let N be any submodule of M. If $\pi_0(N) = M_0$ then the proof of Lemma 4.1.16 shows that M/N is a homomorphic image of the R-module $M_1 \oplus \cdots \oplus M_n$. By Corollary 4.1.14 and Lemma 4.1.2, M/N s.t.r.f. and by Lemma 4.1.6 the submodule N is good. If $\pi_i(N) = M_i$ for some $1 \leqslant i \leqslant n$ then the proof of Lemma 4.1.16 shows that M/N is a homomorphic image of the R-module

$M' = M_0 \oplus M_1 \oplus \cdots \oplus M_{i-1} \oplus M_{i+1} \oplus \cdots \oplus M_n$. By induction on n, M' s.t.r.f. and hence N is good by Lemma 4.1.6.

Now suppose that $\pi_i(N) \neq M_i$ for all $0 \leqslant i \leqslant n$. Then

$$N \subseteq \bigoplus_{i=0}^{n} \pi_i(N) \subseteq \mathcal{M}M.$$

By the proof of Lemma 4.1.16 it follows that N is good. Hence M s.t.r.f.. □

Theorem 4.1.18 *Let A be a finite direct sum of cyclic Artinian R-modules. Then the R-module $R \oplus A$ s.t.r.f..*

Proof. By Exercise 8.49 in [34], it is sufficient to prove the result when R is a quasi-local ring with unique maximal ideal $\mathcal{M}$. Since A is a finite direct sum of cyclic Artinian submodules, we can write A in the form $Ra_1 \oplus \cdots \oplus Ra_n$. Note that for every a_i $(1 \leqslant i \leqslant n)$, $Ra_i \cong R/\text{ann}(a_i)$ as R-modules, by Lemma 7.24 in [34]. Thus the ring $R/\text{ann}(a_i)$ is Artinian and hence Noetherian for all $1 \leqslant i \leqslant n$. Therefore A is a Noetherian module and has a finite composition length. There exists a positive integer k such that $\mathcal{M}^k A = 0$. By Lemmas 4.1.2 and 4.1.17, $R \oplus A$ s.t.r.f.. □

The same argument proves the next result.

Theorem 4.1.19 *The R-module $R \oplus (R/\mathcal{M}_1^{k(1)}) \oplus \cdots \oplus (R/\mathcal{M}_n^{k(n)})$ s.t.r.f. for all positive integers $n, k(1), \ldots, k(n)$ and maximal ideals $\mathcal{M}_i$ $(1 \leqslant i \leqslant n)$ (not necessarily distinct).*

Theorem 4.1.20 *Let R be a one dimensional Noetherian domain. Then*

(i) the R-module $R \oplus R$ s.t.r.f. if and only if R is a Dedekind domain,

(ii) the R-module $R \oplus (R/\mathcal{A})$ s.t.r.f. for every non-zero ideal $\mathcal{A}$ of R.

Proof. (i) By Theorem 1.2.19.

(ii) Let $\mathcal{A}$ be any non-zero ideal of R. Since R is one dimensional it follows that the ring $R/\mathcal{A}$ is Artinian. Thus the R-module $R/\mathcal{A}$ is cyclic Artinian. Now we can apply Theorem 4.1.18. □

Note that if S is a Noetherian domain which is not Dedekind and R is the polynomial ring $S[X]$ then the R-module $R \oplus (R/RX)$ does not s.t.r.f.. For if $R \oplus (R/RX)$ s.t.r.f. then so too does its homomorphic image $(R/RX) \oplus (R/RX)$. In this case the S-module $S \oplus S$ s.t.r.f. and hence S is a Dedekind domain by Theorem 1.2.19, a contradiction.

The same argument gives the following result.

Lemma 4.1.21 *Let R be a Noetherian ring and let $\mathcal{P}$ be a non-maximal prime ideal of R such that the R-module $R \oplus (R/\mathcal{P})$ s.t.r.f.. Then the domain $R/\mathcal{P}$ is Dedekind.*

The converse of Lemma 4.1.21 is false. We now give an example of a two-dimensional local Noetherian domain R and a prime ideal $\mathcal{P}$ of R such that the ring $R/\mathcal{P}$ is a PID (hence Dedekind) but the R-module $R \oplus (R/\mathcal{P})$ does not s.t.r.f..

Example 4.1.22 *Let F be a field and let $R = F[[X, Y]]$, the ring of formal power series in indeterminates X, Y over F. Then R is Noetherian local domain with unique maximal ideal $\mathcal{M} = RX + RY$. Let M denote the R-module $R \oplus (R/RY)$. Then M does not s.t.r.f..*

Proof. Let $\overline{R} = R/RY$. Then $\overline{R} \cong F[[X]]$ which is a PID. For each $r \in R$, let $\overline{r}$ denote the element $r + RY$ of $\overline{R}$. Let N be the submodule $\mathcal{M}(Y, \overline{X})$ of M. We shall show that $\text{rad}_M(N) = RY \oplus R\overline{X}$. Let P be a prime submodule of M such that $\mathcal{M}(Y, \overline{X}) \subseteq P$. Then $\mathcal{M}M \subseteq P$ or $(Y, \overline{X}) \in P$. In any case,

$(Y,\overline{X}) \in P$. Thus, $L = R(Y,\overline{X}) \subseteq \text{rad}_M(N)$. Since $N \subseteq L$ it follows that $\text{rad}_M(N) = \text{rad}_M(L)$.

Let $\mathcal{Q}$ be a prime ideal of R such that $Y \notin \mathcal{Q}$ and let Q be any $\mathcal{Q}$-prime submodule of M (i.e. $(Q : M) = \mathcal{Q}$). Then $\mathcal{Q}M = \mathcal{Q}\oplus((\mathcal{Q}+RY)/RY) \subseteq Q$ and also $Y(0\oplus(R/RY)) = 0 \subseteq Q$ implies that $0\oplus(R/RY) \subseteq Q$. Thus $\mathcal{Q}\oplus(R/RY) \subseteq Q$. Since $M/(\mathcal{Q}\oplus(R/RY)) \cong R/\mathcal{Q}$ it follows that $Q = \mathcal{Q}\oplus(R/RY)$. Thus $L \nsubseteq Q$ since $Y \notin \mathcal{Q}$.

Let $\mathcal{Q}$ be a prime ideal of R such that $Y \in \mathcal{Q}$. Then $\mathcal{Q} = RY$ or $\mathcal{Q} = \mathcal{M}$ because RY is prime and $R/RY \cong F[[X]]$. Thus $\text{rad}_M(L) = K(L, RY) \cap K(L, \mathcal{M})$ by Corollary 3.1.2. Since $L \subseteq \mathcal{M}M = \mathcal{M}\oplus(\mathcal{M}/RY)$, we have $K(L,\mathcal{M}) = \mathcal{M}M$.

Let a, $b \in R$ such that $(a,\overline{b}) \in K(L, RY)$. Note that for all $c \in R\backslash RY$ there exist $0 \neq f(X) \in F[[X]]$ and $r \in R$ such that $c = f(X) + rY$. Moreover, $f(X) = X^k u$ for some integer $k \geqslant 0$ and unit u in $F[[X]]$. Thus we can suppose that $X^k(a,\overline{b}) \in L + (RY)M = L + YM = R(Y,\overline{X}) + (RY \oplus 0) = RY \oplus R\overline{X}$, i.e. $X^k a \in RY$ and $X^k\overline{b} \in R\overline{X}$, i.e. $a \in RY$. Thus $K(L, RY) \subseteq RY \oplus \overline{R}$. But $X(RY \oplus \overline{R}) \subseteq RY \oplus R\overline{X} = L + (RY)M$ gives that $RY \oplus \overline{R} \subseteq K(L, RY)$. Thus $K(L, RY) = RY \oplus \overline{R}$. Now

$$\begin{aligned}\text{rad}_M(N) = \text{rad}_M(L) &= K(L,RY) \cap K(L,\mathcal{M}) \\ &= (RY \oplus \overline{R}) \cap \mathcal{M}M \\ &= RY \oplus R\overline{X}.\end{aligned}$$

Now $(Y,0) = Y(1,0)$ and $Y^2(1,0) = Y(Y,\overline{X}) \in N$. Thus $(Y,0) \in< E_M(N) >$. Suppose that $(0,\overline{X}) \in< E_M(N) >$, i.e. $(0,\overline{X}) = r_1(s_1,\overline{t}_1) + \cdots + r_n(s_n,\overline{t}_n)$ where $r_i^k(s_i,\overline{t}_n) \in N$, for some positive integers n, k and elements $r_i, s_i, t_i \in R$ $(1 \leqslant i \leqslant n)$. Suppose that $r, s, t \in R, m \in \mathcal{M}$ and $r^k(s,\overline{t}) = m(Y,\overline{X})$ for some positive integer k. Then $r^k s = mY$ and $r^k\overline{t} = m\overline{X}$. If r is a unit then $(s,\overline{t}) \in N$. Suppose $r \in \mathcal{M}$. Then $ms\overline{X} = r^k s\overline{t} = mY\overline{t} = \overline{0}$ so that $ms \in RY$, since R/RY

is a domain. If $m \in RY$ then $r^k\bar{t} = 0$ gives $r\bar{t} = \bar{0}$. Suppose $m \notin RY$. Then $s = Ys_1$ gives $r^kYs_1 = mY$ so that $m = r^ks_1$, hence $r^k\bar{t} = r^ks_1\overline{X}$. Either $r \in RY$ and $r\bar{t} = \bar{0}$ or $r \notin RY$ and $\bar{t} = s_1\overline{X}$ gives $r\bar{t} \in \mathcal{M}s_1\overline{X} \subseteq \mathcal{M}\overline{X}$. Thus in any case $r\bar{t} \in \mathcal{M}\overline{X}$. It follows that $r_i\bar{t}_i \in \mathcal{M}\overline{X}$ $(1 \leqslant i \leqslant n)$ and hence $\overline{X} = r_1\bar{t}_1 + \cdots + r_n\bar{t}_n = u\overline{X}$ for some $u \in \mathcal{M}$. Then $(1-u)\overline{X} = 0$ so $\overline{X} = 0$, i.e. $X \in RY$, a contradiction. Thus $(0, \overline{X}) \notin < E_M(N) >$. It follows that M does not s.t.r.f..

Note that $L = R(Y, \overline{X})$ is good because $(Y, 0) = Y(1,0)$ and $Y^2(1,0) = Y(Y, \overline{X}) \in L$, $(0, \overline{X}) = (Y, \overline{X}) - (Y, 0) = 1(Y, \overline{X}) + (-Y)(1,0)$ where $1^2(Y, \overline{X}) \in L$, $(-Y)^2(1,0) = Y(Y, \overline{X}) \in L$. $\square$

4.2 The Envelopes in Free Modules

Let F be a free module of finite rank and let N be a finitely generated submodule of F. One can ask if we can describe $E_F(N)$ in some nice way. This will be the aim of this section.

It seems sensible to begin with the case of a UFD R and a cyclic submodule N.

Proposition 4.2.1 *Let R be a UFD and let F be the free R-module $R^{(n)}$ for some positive integer n. Let $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero, and let N be the submodule $R(a_1, \ldots, a_n)$ of F. Then $E_F(N) = \sqrt{Rd}(\frac{a_1}{d}, \ldots, \frac{a_n}{d})$ where $d = \gcd(a_1, \ldots, a_n)$.*

Proof. If $n = 1$ then $F = R$, $N = Ra_1$, $d = a_1$ and $E_F(N) = \sqrt{Rd} = \sqrt{Rd}(\frac{a_1}{d})$. Now let $n \geqslant 2$ and $r \in \sqrt{Rd}$. Then $r^k = sd$ for some $s \in R$. Hence

$$r^k(\tfrac{a_1}{d}, \ldots, \tfrac{a_n}{d}) = s(a_1, \ldots, a_n) \in N$$

and it follows that $r(\frac{a_1}{d}, \ldots, \frac{a_n}{d}) \in E_F(N)$. Thus

$$\sqrt{Rd}(\tfrac{a_1}{d}, \dots, \tfrac{a_n}{d}) \subseteq E_F(N).$$

Conversely let $t, x_i \in R$ $(1 \leqslant i \leqslant n)$ such that

$$t^m(x_1, \dots, x_n) = w(a_1, \dots, a_n)$$

for some positive integer m and $w \in R$. If $t = 0$ then

$$t(x_1, \dots, x_n) = (0, \dots, 0) = 0(\tfrac{a_1}{d}, \dots, \tfrac{a_n}{d}) \in \sqrt{Rd}(\tfrac{a_1}{d}, \dots, \tfrac{a_n}{d}).$$

Suppose that $t \neq 0$. Now $t^m x_i = wa_i$ $(1 \leqslant i \leqslant n)$ so that

$$t^m(x_i a_j - x_j a_i) = 0$$

and hence $x_i a_j = x_j a_i$ $(1 \leqslant i < j \leqslant n)$, because $t^m \neq 0$. Let $b_i = \frac{a_i}{d}$ $(1 \leqslant i \leqslant n)$. Then

$$x_i b_j = x_j b_i \ (1 \leqslant i < j \leqslant n).$$

There exists $1 \leqslant i \leqslant n$ such that $a_i \neq 0$ and hence $b_i \neq 0$. Consider the equations $x_i b_j = x_j b_i$ $(1 \leqslant j \leqslant n)$. Let p be any prime which divides b_i. Because b_i $(1 \leqslant i \leqslant n)$ are coprime, there exists $1 \leqslant j \leqslant n$ such that $p \nmid b_j$. Then $x_i b_j = x_j b_i$ gives p divides x_i. Now consider the equations

$$(\frac{x_i}{p})b_j = x_j(\frac{b_i}{p}) \quad (1 \leqslant j \leqslant n).$$

Repeating this argument we find that b_i divides x_i, i.e. $x_i = yb_i$ for some $y \in R$. For each $1 \leqslant j \leqslant n$, $x_j b_i = x_i b_j = y b_j b_i$ which gives that $x_j = yb_j$. Hence

$$(x_1, \dots, x_n) = y(b_1, \dots, b_n).$$

Now $t(x_1, \dots, x_n) = ty(b_1, \dots, b_n)$ and

$$t^m y(b_1, \dots, b_n) = w(a_1, \dots, a_n) = wd(b_1, \dots, b_n).$$

In particular, $t^m y = wd$. Hence $(ty)^m \in Rd$ and $ty \in \sqrt{Rd}$. Thus

$$t(x_1, \dots, x_n) = ty(b_1, \dots, b_n) \in \sqrt{Rd}(b_1, \dots, b_n) = \sqrt{Rd}(\tfrac{a_1}{d}, \dots, \tfrac{a_n}{d}).$$

It follows that $E_F(N) \subseteq \sqrt{Rd}(\frac{a_1}{d}, \dots, \frac{a_n}{d})$. □

Corollary 4.2.2 *Let R be a UFD and let F be the free R-module $R^{(n)}$, for some positive integer n. Then every cyclic submodule N of F is good. Moreover,*

$$rad_F(N) = E_F(N).$$

Proof. Let $a_i \in F$ $(1 \leqslant i \leqslant n)$ and let $N = R(a_1, \dots, a_n)$. If $a_i = 0$ $(1 \leqslant i \leqslant n)$ then $\text{rad}_F(N) = 0$ and hence $\text{rad}_F(N) = E_F(N)$. Suppose that $a_i \neq 0$ for some $1 \leqslant i \leqslant n$. Let $(x_1, \dots, x_n) \in \text{rad}_F(N)$. Then

$$x_i \in \sqrt{Ra_1 + \cdots + Ra_n}$$

and $x_i a_j = x_j a_i$ for all $1 \leqslant i, j \leqslant n$, by Theorem 3.1.5.

Let d denote the gcd of $a_1, \dots, a_n$ and $b_i = \frac{a_i}{d}$ $(1 \leqslant i \leqslant n)$. Then $x_i b_j = x_j b_i$ for all $1 \leqslant i < j \leqslant n$. By the argument given in the proof of Proposition 4.2.1 we have, $(x_1, \dots, x_n) = w(b_1, \dots, b_n)$ for some $w \in R$. Let p be any prime divisor of d. Then $a_i \in Rd \subseteq Rp$ $(1 \leqslant i \leqslant n)$ gives $Ra_1 + \cdots + Ra_n \subseteq Rp$ and hence $\sqrt{Ra_1 + \cdots + Ra_n} \subseteq Rp$, because Rp is a prime ideal. Thus $x_i \in Rp$ for all $1 \leqslant i \leqslant n$. But $x_i = wb_i$ $(1 \leqslant i \leqslant n)$ so that p divides wb_i for all $1 \leqslant i \leqslant n$. Since the elements b_i $(1 \leqslant i \leqslant n)$ are coprime, there exists $1 \leqslant j \leqslant n$ such that p does not divide b_j and hence p divides w. Thus $w \in Rp$ for every prime divisor p of d and it follows that $w \in \sqrt{Rd}$. Thus

$$(x_1, \dots, x_n) \in \sqrt{Rd}(b_1, \dots, b_n) \subseteq E_F(N)$$

by Proposition 4.2.1. Therefore $\text{rad}_F(N) \subseteq E_F(N)$. But it is well known that $E_F(N) \subseteq \text{rad}_F(N)$, and so the result is proved. □

Proposition 4.2.3 *Let R be a UFD, let F be the free R-module $R^{(n)}$, let $a_i \in R$ $(1 \leqslant i \leqslant n)$, not all zero, let $\mathcal{B}$ be an ideal of R and let N be the submodule $\mathcal{B}(a_1, \ldots, a_n)$ of F. Then $E_F(N) = \sqrt{\mathcal{B}d}(b_1, \ldots, b_n)$, where $b_i = \frac{a_i}{d}$ and $d = \gcd(a_1, \ldots, a_n)$.*

Proof. Let $x \in \sqrt{\mathcal{B}d}$; then $x^k = bd$, for some $b \in \mathcal{B}$ and positive integer k. Hence

$$x^k(\tfrac{a_1}{d}, \ldots, \tfrac{a_n}{d}) = b(a_1, \ldots, a_n) \in N \cdot$$

and it follows that $x(\frac{a_1}{d}, \ldots, \frac{a_n}{d}) \in E_F(N)$. Hence $\sqrt{\mathcal{B}d}(b_1, \ldots, b_n) \subseteq E_F(N)$. Let $r, x_i \in R$ $(1 \leqslant i \leqslant n)$ such that $r^m(x_1, \ldots, x_n) = b(a_1, \ldots, a_n)$ for some $b \in \mathcal{B}$. If $r = 0$ then $r(x_1, \ldots, x_n) = (0, \ldots, 0) = 0(b_1, \ldots, b_n) \in \sqrt{\mathcal{B}d}(b_1, \ldots, b_n)$. Suppose that $r \neq 0$. Now $r^m x_i = ba_i$, $1 \leqslant i \leqslant n$. Thus $r^m(x_i a_j - x_j a_i) = 0$ $(1 \leqslant i, j \leqslant n)$. Since $r^m \neq 0$, $x_i a_j = x_j a_i$ $(1 \leqslant i < j \leqslant n)$. Thus $x_i b_j = x_j b_i$ $(1 \leqslant i < j \leqslant n)$. By the argument given in the proof of Proposition 4.2.1, we find that $(x_1, \ldots, x_n) = y(b_1, \ldots, b_n)$. Thus $r(x_1, \ldots, x_n) = ry(b_1, \ldots, b_n)$. Hence

$$r^m y(b_1, \ldots, b_n) = b(a_1, \ldots, a_n) = bd(b_1, \ldots, b_n).$$

Thus $r^m y = bd$ and $(ry)^m \in \mathcal{B}d$. Therefore $ry \in \sqrt{\mathcal{B}d}$ and $ry(b_1, \ldots, b_n) \in \sqrt{\mathcal{B}d}(b_1, \ldots, b_n)$. $\square$

Next we show that if R is a UFD then certain 2-generated submodules of free R-modules of finite rank are good.

Theorem 4.2.4 *Let R be a UFD, let $n \geqslant 3$ be a positive integer and a_i, $b_i \in R$ $(1 \leqslant i \leqslant n)$ such that $R = Rb_1 + \cdots + Rb_n$. Let $c = s_1 a_1 + \cdots + s_n a_n$ where $s_i \in R$ $(1 \leqslant i \leqslant n)$ and $1 = s_1 b_1 + \cdots + s_n b_n$. Let d be any* gcd *of the elements $a_i - cb_i$ $(1 \leqslant i \leqslant n)$ if $a_j - cb_j \neq 0$ for some $1 \leqslant j \leqslant n$, and otherwise let $d = 1$. Let N denote the submodule $R(a_1, \ldots, a_n) + R(b_1, \ldots, b_n)$ of $F = R^{(n)}$. Then $rad_F(N) = < E_F(N) > = R(b_1, \cdots, b_n) + R(f_1, \ldots, f_n)$ where*

$f_i = (p_1 \cdots p_m)(a_i - cb_i)/d$ $(1 \leqslant i \leqslant n)$ and either d is not a unit and $p_1, \ldots, p_m$ are the pairwise non-associate prime divisors of d, or d is a unit and $p_1 = \cdots = p_m = 1$. In particular, N is a good submodule of F.

Proof. Suppose first that $a_i - cb_i = 0$ $(1 \leqslant i \leqslant n)$. Then $N = R(b_1, \ldots, b_n)$ which is a direct summand of F and hence is prime by Proposition 2.2.3. The result follows in this case.

Let $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ and $L = R\mathbf{b}$. Then $F = L \oplus L'$ where L' is the submodule $\{(x_1 \ldots, x_n) \in F : s_1x_1 + \cdots + s_nx_n = 0\}$, by Proposition 2.2.3. It follows that $N = N \cap F = L \oplus (N \cap L')$. By Lemma 1.1.2.4, $\text{rad}_F(N) = \text{rad}_L(L) \oplus \text{rad}_{L'}(N \cap L') = L \oplus \text{rad}_{L'}(N \cap L')$. Moreover, by Lemma 3.1.4,

$$\text{rad}_{L'}(N \cap L') = L' \cap \text{rad}_F(N \cap L').$$

By the remarks at the beginning of section 2.3, $N \cap L' = R(\mathbf{a} - c\mathbf{b})$ and by Proposition 3.1.10,

$$\text{rad}_F(N \cap L') = \begin{cases} N \cap L' & \text{if } d \text{ is a unit} \\ R(f_1, \ldots, f_n) & \text{otherwise} \end{cases}$$

where d is a gcd of $a_i - cb_i$ $(1 \leqslant i \leqslant n)$, and, in case d is not a unit, $f_i = (p_1 \cdots p_m)(a_i - cb_i)/d$ where $p_1, \ldots, p_m$ are the pairwise non-associate prime divisors of d. Thus $\text{rad}_F(N) = R(b_1, \ldots, b_n) + R(f_1, \ldots, f_n)$, as required.

If d is a unit in R then

$$\text{rad}_F(N) = R\mathbf{b} + R(\mathbf{a} - c\mathbf{b}) = R\mathbf{a} + R\mathbf{b} = N \subseteq < E_F(N) >,$$

so that $\text{rad}_F(N) =< E_F(N) >$. Suppose that d is not a unit in R. Then $(p_1 \cdots p_m)^k = sd$ for some positive integer k and element $s \in R$. Therefore $(p_1 \cdots p_m)^{k-1}(f_1, \ldots, f_n) = s(\mathbf{a} - c\mathbf{b}) \in N$. It follows that

$$(f_1, \ldots, f_n) = (p_1 \cdots p_m)((a_1 - cb_1)/d, \ldots, (a_n - cb_n)/d) \in E_F(N)$$

and hence $\mathrm{rad}_F(N) \subseteq \langle E_F(N) \rangle$. Thus $\mathrm{rad}_F(N) = \langle E_F(N) \rangle$ and N is a good submodule of F. $\square$

Compare the following result with Theorem 2.3.2.

Corollary 4.2.5 *With the notation of Theorem 4.2.4, $N = R(a_1, \ldots, a_n) + R(b_1, \ldots, b_n)$ is a semiprime submodule of F if and only if either $a_i = cb_i$ $(1 \leqslant i \leqslant n)$ or every common divisor of $a_i - cb_i$ $(1 \leqslant i \leqslant n)$ is square-free.*

Proof. Suppose first that $a_i = cb_i$ $(1 \leqslant i \leqslant n)$. Then N is a direct summand of F, by the proof of Theorem 4.2.4, and hence a prime submodule of F. Suppose that $a_j - cb_j \neq 0$ for some $1 \leqslant j \leqslant n$ and d is a greatest common divisor of $a_i - cb_i$ $(1 \leqslant i \leqslant n)$ where d is square-free. By Theorem 4.2.4,

$$\begin{aligned} \mathrm{rad}_F(N) &= R(b_1, \ldots, b_n) + R(f_1, \ldots, f_n) \\ &= R\mathbf{b} + R(\mathbf{a} - c\mathbf{b}) \\ &= R\mathbf{a} + R\mathbf{b} = N. \end{aligned}$$

Thus N is a semiprime submodule of F.

Conversely, suppose that N is a semiprime submodule of F. Suppose that $a_i \neq cb_i$ for some $1 \leqslant i \leqslant n$. By the proof of Theorem 4.2.4, $N \cap L' = \mathrm{rad}_F(N \cap L')$, and hence

$$R(a_1 - cb_1, \ldots, a_n - cb_n) = \frac{p_1 \cdots p_m}{d}(a_1 - cb_1, \ldots, a_n - cb_n)R$$

(in the notation of Theorem 4.2.4). Since $a_i - cb_i \neq 0$ it follows that $d = up_1 \cdots p_m$ for some unit u, i.e. d is square-free. $\square$

Next we give an example to show that in Theorem 4.2.4 the condition $R = Rb_1 + \cdots + Rb_n$ is necessary.

Example 4.2.6 *Let R be the UFD $\mathbb{Z}[X]$ and let N be the submodule $R(4, 2X) + R(2X, X^2)$ of the free R-module $F = R^{(2)}$. Then N is not a good submodule of F.*

Proof. Let J denote the (maximal) ideal $R2 + RX$. Then $N = J(2, X)$. We saw on page 52 that $\text{rad}_F(N) = R(2, X)$. On the other hand, in Theorem 1.2.28 we proved that $< E_F(N) > \subseteq E_F(\sqrt{J}(2, X)) = \sqrt{J}(2, X) \neq R(2, X)$. Thus N is not a good submodule of F. $\square$

Proposition 4.2.7 *Let R be a domain and let I be an ideal of R. Let $F = R^{(n)}$ for some positive integer n and let N be the submodule $I(1, \ldots, 1)$ of F. Then $rad_F(N) = \sqrt{I}(1, \ldots, 1) = E_F(N)$.*

Proof. Let K be any prime submodule of F such that $N \subseteq K$. Then $I(1, \ldots, 1) \subseteq K$ so that $IF \subseteq K$ or $R(1, \ldots, 1) \subseteq K$. By Proposition 2.2.3, $R(1, \ldots, 1)$ is a prime submodule of F. Hence

$$\text{rad}_F(N) = R(1, \ldots, 1) \cap \text{rad}_F(IF) = R(1, \ldots, 1) \cap \sqrt{I}F$$

by Proposition 3.2.1. Hence

$$\text{rad}_F(N) = \sqrt{I}(1, \ldots, 1).$$

Let $x \in \sqrt{I}(1, \ldots, 1)$; then $x = s(1, \ldots, 1)$ for some $s \in \sqrt{I}$. Now $s^m \in I$ for some positive integer m and hence $s^m(1, \ldots, 1) \in N$, i.e. $x \in E_F(N)$. It follows that $\text{rad}_F(N) \subseteq E_F(N)$ and hence $\text{rad}_F(N) = E_F(N)$. $\square$

Corollary 4.2.8 *Let R be a domain and let I be an ideal of R. Then the submodule $I(1, \ldots, 1)$ of the free R-module $F = R^{(n)}$ is semiprime if and only if $\sqrt{I} = I$.*

Proof. By Proposition 4.2.7. $\square$

The situation for 2-generated submodules is more complicated. Let R be a UFD and let F be the free R-module $R^{(2)}$. Let $a_{ij} \in R$ $(1 \leqslant i, j \leqslant 2)$ and let

N be the submodule $R(a_{11}, a_{12}) + R(a_{21}, a_{22})$ of F. Suppose first that $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Let d denote the gcd of the elements a_{ij} $(1 \leqslant i, j \leqslant n)$. Let $b_{ij} = \frac{a_{ij}}{d}$ $(1 \leqslant i, j \leqslant n)$ and let X denote the set of elements (r_1, r_2) in F such that

$$R\begin{vmatrix} r_1 & r_2 \\ b_{11} & b_{12} \end{vmatrix} + R\begin{vmatrix} r_1 & r_2 \\ b_{21} & b_{22} \end{vmatrix} \subseteq R\Delta, \text{ where } 0 \neq \Delta = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \frac{1}{d^2}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Proposition 4.2.9 *With the above notation,*

$$E_F(N) = \{r(s_1, s_2) : r, s_1, s_2 \in R, r^k = td \textit{ and } (ts_1, ts_2) \in X \textit{ for some positive integer } k \textit{ and some } t \in R\}.$$

Proof. Let $r, u, v \in R$ where $r^k(u, v) \in N$ for some positive integer k. Let e denote the gcd of u and v. Thus

$$r(u, v) = re(\tfrac{u}{e}, \tfrac{v}{e}) \text{ and } (re)^k(\tfrac{u}{e}, \tfrac{v}{e}) \in N.$$

Hence without loss of generality u and v are coprime. There exist $x, y \in R$ such that

$$\begin{aligned} r^k(u, v) &= x(a_{11}, a_{12}) + y(a_{21}, a_{22}) \\ &= xd(b_{11}, b_{12}) + yd(b_{21}, b_{22}) \end{aligned}$$

i.e.

$$r^k u = d(xb_{11} + yb_{21}) \text{ and } r^k v = d(xb_{12} + yb_{22})$$

Thus d divides both $r^k u$ and $r^k v$. Since u and v are coprime it follows that d divides r^k, i.e. $r^k \in Rd$ and hence $r^k = td$, where $t \in R$. Then

$$tu = xb_{11} + yb_{21} \text{ and } tv = xb_{12} + yb_{22}.$$

Thus $tub_{22} - tvb_{21} = x\Delta$, i.e.

$$t\begin{vmatrix} u & v \\ b_{21} & b_{22} \end{vmatrix} = x\Delta \text{ and similarly } t\begin{vmatrix} u & v \\ b_{11} & b_{12} \end{vmatrix} = -y\Delta.$$

Thus r, u, v have the required properties.

Conversely suppose $f \in F$, where $f = r(s_1, s_2)$, $r^k = td$ for some $k \geqslant 1, t \in R$ and $t(s_1, s_2) \in X$. Then

$$\begin{vmatrix} ts_1 & ts_2 \\ b_{11} & b_{12} \end{vmatrix} = x\Delta \text{ and } \begin{vmatrix} ts_1 & ts_2 \\ b_{21} & b_{22} \end{vmatrix} = y\Delta$$

for some $x, y \in R$. Thus

$$ts_1b_{12} - ts_2b_{11} = x\Delta, \text{ and } ts_1b_{22} - ts_2b_{21} = y\Delta.$$

This implies that

$$ts_1(b_{12}b_{21} - b_{22}b_{11}) = (xb_{21} - yb_{11})\Delta \text{ and } ts_2(b_{12}b_{21} - b_{11}b_{22}) = (xb_{22} - yb_{12})\Delta.$$

Since $\Delta \neq 0$, we have $t(s_1, s_2) = y(b_{11}, b_{12}) - x(b_{21}, b_{22})$. Now

$$r^k(s_1, s_2) = dt(s_1, s_2) = y(a_{11}, a_{12}) - x(a_{21}, a_{22}) \in N.$$

Thus $f \in E_F(N)$ and the result is proved. □

Now we consider elements $a_{ij} \in F$ $(1 \leqslant i, j \leqslant 2)$, not all zero, such that $a_{11}a_{22} - a_{12}a_{21} = 0$. Then there exist coprime elements b, c in R (possibly $b = 0$ or $c = 0$ but not both) such that $(a_{11}, a_{12}) = u(b, c)$ and $(a_{21}, a_{22}) = v(b, c)$ for some $u, v \in R$. Thus

$$\begin{aligned} R(a_{11}, a_{12}) + R(a_{21}, a_{22}) &= Ru(b, c) + Rv(b, c) \\ &= (Ru + Rv)(b, c) \end{aligned}$$

and so we can deal with the case in Proposition 4.2.3.

Dauns [4], [5] defines a submodule N of M to be semiprime if $N = E_M(N)$. Recall that when N is semiprime $\mathrm{rad}_F(N) = E_F(N)$ is proved in Lemma 1.1.2.2. In fact the converse of Lemma 1.1.2.2 is false as the following result shows. The result is based on an example in [11].

Proposition 4.2.10 *Let S be a domain, let R=$S[X]$ and let F be the free R-module $R^{(2)}$. Let $0 \neq a \in S$, let $W = \sqrt{Ra + RX}$ and let N denote the submodule $W(a,X)$ of F. Then $N = E_F(N)$. Moreover, N is semiprime if and only if a is a unit in S.*

Proof. $N = E_F(N)$ by the proof of Theorem 1.2.28. If N is semiprime then $N = \mathrm{rad}_F(N) = E_F(N) =< E_F(N) >$. Again by the proof of Theorem 1.2.28, a is a unit.

Suppose that a is a unit in S. Then $W = R$ and $N = R(a, X) = R(1, a^{-1}X)$ which is a direct summand and hence a 0-prime submodule of F, by Proposition 2.2.3. Thus N is semiprime. $\square$

Let R be a domain and let $F = R^{(n)}$ for some positive integer $n \geqslant 2$. Let $\mathbf{a}_i = (a_{i1}, \ldots, a_{in})$ $(1 \leqslant i \leqslant n)$ and let N denote the submodule $R\mathbf{a}_1 + \cdots + R\mathbf{a}_n$ of F.

Proposition 4.2.11 *With the above notation, let A denote the $n \times n$ matrix (a_{ij}) over R and let $\Delta = detA$. Suppose that $R\Delta$ is a non-zero semiprime ideal of R. Then $N = E_F(N)$.*

Proof. Let r, s_i $(1 \leqslant i \leqslant n)$ be elements of R such that $r^k(s_1, \ldots, s_n) \in N$ for some positive integer k. There exist elements $x_i \in R$ $(1 \leqslant i \leqslant n)$ such that

$$r^k(s_1, \ldots, s_n) = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n.$$

We can write this equation in matrix notation as follows:

$$r^k[s_1 \cdots s_n] = [x_1 \cdots x_n]A.$$

Let $\mathrm{adj}A$ denote the adjugate of A and recall that $A(\mathrm{adj}A) = (\mathrm{adj}A)A = \Delta I_n$, where I_n is the $n \times n$ identity matrix. Then

$$r^k[s_1 \cdots s_n]\mathrm{adj}A = [x_1 \cdots x_n]\Delta I_n = [(\Delta x_1) \cdots (\Delta x_n)].$$

Let $[t_1 \cdots t_n] = [s_1 \cdots s_n]\mathrm{adj}A$. For each $1 \leqslant i \leqslant n$, $r^k t_i = \Delta x_i$ gives $(rt_i)^k \in R\Delta$ and hence $rt_i = \Delta z_i$ for some $z_i \in R$. Thus

$$r[s_1 \cdots s_n]\mathrm{adj}A = r[t_1 \cdots t_n] = \Delta[z_1 \cdots z_n].$$

Now

$$r[s_1 \cdots s_n](\mathrm{adj}A)A = \Delta[z_1 \cdots z_n]A$$

gives $r\Delta[s_1 \cdots s_n] = \Delta[z_1 \cdots z_n]A$, so that $r[s_1 \cdots s_n] = [z_1 \cdots z_n]A$. In other words,

$$r(s_1, \ldots, s_n) = z_1\mathbf{a}_1 + \cdots + z_n\mathbf{a}_n \in N.$$

It follows that $E_F(N) = N$. $\square$

Note that in Proposition 4.2.11 the condition $R\Delta$ is non-zero is necessary, as the following example shows.

Example 4.2.12 *Let N denote the submodule $\mathbb{Z}(4,0,4) + \mathbb{Z}(0,4,4) + \mathbb{Z}(4,4,8)$ of $F = \mathbb{Z}^{(3)}$. Then $E_F(N) =< E_F(N) >= rad_F(N) \neq N$. But $\mathbb{Z}\Delta = 0$ which is a semiprime ideal of $\mathbb{Z}$.*

Proof. By Theorem 3.1.5,

$$(r_1, r_2, r_3) \in \mathrm{rad}_F(N) \Leftrightarrow \begin{bmatrix} r_1 & r_2 & r_3 \\ 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}_t \in \sqrt{\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}_t} \quad \text{where } 1 \leqslant t \leqslant 3.$$

Hence

$$\mathrm{rad}_F(N) = \{(2a, 2b, 2(a+b)) : a, b \in \mathbb{Z}\}.$$

For any element $x = (2a, 2b, 2(a+b)) \in \mathrm{rad}_F(N)$, note that $2^2(a, b, a+b) = a(4,0,4) + b(0,4,4) \in N$. Hence $x \in E_F(N)$. On the other hand, for example $(2,2,4) \in \mathrm{rad}_F(N)$, but $(2,2,4) \notin N$. Therefore

$$E_F(N) = < E_F(N) > = \mathrm{rad}_F(N) \neq N.$$

□

Proposition 4.2.13 *With the notation of Proposition 4.2.11 with $R\Delta$ is non-zero semiprime, suppose that R is a one-dimensional Noetherian domain and $\Delta \neq 0$. Then N is a semiprime submodule of F.*

Proof. By Proposition 2.3.7, $R\Delta \subseteq (N : F)$. But the $R/R\Delta$-module F/N is semisimple and hence N is an intersection of maximal submodules, i.e. N is semiprime. □

Definition 4.2.14 *Let R be a commutative ring and M be any R-module. A submodule Q of M is called primary if whenever $r \in R$, $m \in M$ and $rm \in Q$ then $m \in Q$ or $r^k \in (Q : M)$ for some positive integer k.*

Proposition 4.2.15 *Let R be a commutative ring. Let M be any R-module and Q be a $\mathcal{P}$-primary submodule of M. Then*

$$< E_M(Q) > = Q + \mathcal{P}M.$$

Proof. Note first that for any submodule N of M,

$$\sqrt{(N : M)} \subseteq (< E_M(N) >: M).$$

Thus $Q + \mathcal{P}M \subseteq < E_M(Q) >$.

Conversely, suppose $rm \in E_M(Q)$. Then there exists a positive integer k such that $r^k m \in Q$. Since Q is $\mathcal{P}$-primary this implies that either $m \in Q$ or $r^k \in \sqrt{(Q : M)} = \mathcal{P}$. If $m \in Q$ then $rm \in Q$. If $r^k \in \sqrt{(Q : M)} = \mathcal{P}$ then $r \in \mathcal{P}$ and hence $rm \in \mathcal{P}M$. In any case $rm \in Q + \mathcal{P}M$. Thus $E_M(Q) \subseteq Q + \mathcal{P}M$. Therefore

$$< E_M(Q) > = Q + \mathcal{P}M.$$

□

Chapter 5

CHAIN CONDITIONS IN MODULES WITH KRULL DIMENSION

In this chapter rings are not assumed to be commutative. Gordon and Robson proved that any ring with Krull dimension satisfies the ascending chain condition (ACC) on semiprime ideals (see Theorem 5.1.9). But this result does not hold for modules in general. In particular it is proved in Theorem 5.2.6 that if R is the first Weyl algebra over a field of characteristic 0 then there are Artinian R-modules which do not satisfy the ACC on semiprime submodules. The aim of this chapter is to investigate when Gordon and Robson's result holds for modules. For example, if R is a PI-ring then any R-module with Krull dimension satisfies the ACC on prime submodules (see Theorem 5.2.11), and if R is left Noetherian, also the ACC on semiprime submodules (see Theorem 5.3.2).

5.1 On Krull Dimension

Let R be a ring and M be an R-module. The Krull dimension of M will be denoted by $k(M)$. The Krull dimension of a ring R is defined to be the Krull dimension of the left R-module R and will be denoted by $k(R)$.

In this section we will give some relevant properties of Krull dimension which will be used later. For the definition and other basic properties of Krull dimension see [8], [9] and [29].

Definition 5.1.1 *An element c in R is called regular (or a non-zero-divisor) provided $cr \neq 0$ and $rc \neq 0$ for every non-zero element r in R. If I is a proper ideal of R then $\mathcal{C}(I)$ will denote the set of elements c in R such that $c+I$ is a regular element in the ring R/I. Clearly $c \in \mathcal{C}(I)$ if and only if for any $r \in R, cr \in I$ or $rc \in I$ implies $r \in I$.*

Proposition 5.1.2 *[29, 6.3.5 Proposition] A semiprime ring with Krull dimension is a left Goldie ring.*

Lemma 5.1.3 *[8, Ex.13F] Let R be a ring with Krull dimension. If $\mathcal{P}$ is a prime ideal of R, and I is an ideal with $I \supset \mathcal{P}$ then $k(R/I) < k(R/\mathcal{P})$.*

Proof. The non-zero ideal $I/\mathcal{P}$ of $R/\mathcal{P}$ is essential in the prime right Goldie ring $R/\mathcal{P}$. So $I/\mathcal{P}$ contains a regular element $c+\mathcal{P}$ in $R/\mathcal{P}$. Since in the chain $\{(c+\mathcal{P})^n(R/\mathcal{P}) : n \text{ is a positive integer }\}$ the factors are all isomorphic to $(R/\mathcal{P})/((cR+\mathcal{P})/\mathcal{P})$, we have

$$k(R/I) \leqslant k((R/\mathcal{P})/((cR+\mathcal{P})/\mathcal{P})) < k(R/\mathcal{P}).$$

□

Theorem 5.1.4 *[9, Theorem 7.1] Any ring R with Krull dimension has the ascending chain condition (ACC) on prime ideals.*

Proof. Suppose $\mathcal{P}_1 \subset \mathcal{P}_2$ are prime ideals of the ring R. By Lemma 5.1.3, $k(R/\mathcal{P}_2) < k(R/\mathcal{P}_1)$. Therefore an ascending chain of primes in R, $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots$, gives a decreasing sequence of ordinals, $k(R/\mathcal{P}_1) > k(R/\mathcal{P}_2) > \cdots$, which is not possible. □

Lemma 5.1.5 *[9, Proposition 1.4] A module with Krull dimension has finite uniform dimension.*

Proof. Suppose the result is false. Amongst the modules for which it fails, choose one, M, of minimal Krull dimension, α say. Clearly $\alpha \geqslant 0$. Suppose that $M \supseteq \bigoplus_{i=1}^{\infty} A_i$ for non-zero submodules A_i. For each non-negative integer n set $M_n = \bigoplus_{j=1}^{\infty} A_{(2^n j)}$ and consider the infinite chain $M_0 \supset M_1 \supset M_2 \supset \cdots$. Each factor M_i/M_{i+1} is an infinite direct sum and yet has Krull dimension less than or equal to α. By minimality of α, $k(M_i/M_{i+1}) = \alpha$. Hence, by the definition of Krull dimension, $k(M) > \alpha$, a contradiction. □

The following lemma is needed to prove Theorem 5.1.8.

Lemma 5.1.6 ***(König's unendlichkeitslemma)****[15, Chapter VI] Let $S_1, S_2, \ldots$ be an infinite sequence of disjoint non-empty finite sets and $\prec$ be a relation in $S_1 \cup S_2 \cup \cdots$ such that whenever n is a positive integer and $x \in S_{n+1}$, there exists a $y \in S_n$ such that $y \prec x$. Then there exists an infinite sequence $x_1, x_2, x_3, \ldots$ such that $x_n \in S_n$ $(n = 1, 2, \ldots)$ and $x_1 \prec x_2 \prec x_3 \prec \ldots$.*

Proposition 5.1.7 *[9, Proposition 7.3] In a ring R with Krull dimension there are only finitely many prime ideals minimal over any ideal. In particular each semiprime ideal is a finite intersection of prime ideals.*

Proof. Let I be an ideal of R and S be the intersection of all prime ideals of R containing I. Then since R/S has Krull dimension, R/S is a semiprime left

Goldie ring, by Proposition 5.1.2. Therefore, as is well known, the zero ideal of R/S is a finite intersection of primes of R/S. Hence there are only finitely many minimal primes over S and S is their intersection. $\square$

Theorem 5.1.8 *[9, Theorem 7.7] Any ring with ascending chain condition (ACC) on prime ideals has ACC on finite intersections of prime ideals.*

Proof. Let R be the ring and $S_0 \subset S_1 \subset S_2 \subset \cdots$ an infinite strictly ascending chain of ideals, each being a finite intersection of primes. Let $S_i^\sharp$ denote the set of primes of R minimal over S_i. From the assumption on S_i, it follows that the set $S_i^\sharp$ must be finite and $S_i = \bigcap_{\mathcal{P} \in S_i^\sharp} \mathcal{P}$.

The aim is to apply Lemma 5.1.6 to a suitable directed graph G, producing an infinite ascending chain of primes. The vertex set of G is $V = \bigcup_{i=0}^{\infty} S_i^\sharp$. The set V is clearly infinite. An edge in G is an ordered pair $(\mathcal{P}, \mathcal{Q})$ where $\mathcal{P} \subset \mathcal{Q}$ and $\mathcal{P} \in S_i^\sharp$, $\mathcal{Q} \in S_{i+1}^\sharp$ for some i. The index i is uniquely determined when it exists; for if $\mathcal{P} \in S_j^\sharp$ and $j > i$ then $S_{i+1} \subseteq S_j \subseteq \mathcal{P} \subset \mathcal{Q}$, contradicting the description of $\mathcal{Q}$. This same argument shows that every vertex has finite index. Also note that G has no closed paths.

Consider the finite paths from some vertex in $S_0^\sharp$ to a vertex $\mathcal{P}$. Since the set $\bigcup_{j \leqslant i} S_j^\sharp$ is finite for any fixed i, it follows that there is a longest such path; say it has length n. Then we call n the height of $\mathcal{P}$. If $\mathcal{P} \notin S_0^\sharp$ then the set $\{\mathcal{Q} \in V : (\mathcal{Q}, \mathcal{P}) \text{ is an edge}\}$ has finite cardinality greater than 0. An easy induction now shows that there are only finitely many vertices of height n. Hence Lemma 5.1.6 asserts the existence of an infinite path, which is similar to the existence of an infinite strictly ascending chain of primes. $\square$

The following theorem is the result of Theorem 5.1.8.

Theorem 5.1.9 *[9, Theorem 7.6] A ring with Krull dimension has the ACC for semiprime ideals.*

5.2 Prime Submodules

Recall that for any submodule N of M $\text{ann}(M/N)$ is denoted by $(N : M)$, i.e. $(N : M) = \{r \in R : rM \subseteq N\}$. Thus a proper submodule N of M is prime if and only if $(N : M) = (N : L)$ for any submodule L of M properly containing N.

Before we extend Gordon and Robson's result which was given in Theorem 5.1.9, we note the following.

Lemma 5.2.1 *Let R be any simple ring. Then the following statements are equivalent for an R-module M.*
(i) M is Noetherian.
(ii) M satisfies ACC on semiprime submodules.
(iii) M satisfies ACC on prime submodules.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ Clear.

$(iii) \Rightarrow (i)$ It is easy to check that every proper submodule of M is prime. Thus (iii) implies (i). □

Let R be any ring. An R-module M will be called *uniserial* if M has a unique finite composition series. The next two lemmas are presumably well known but we give their proofs for convenience.

Lemma 5.2.2 *Let R be any ring. Let M be an R-module with a maximal submodule N and a simple submodule $S \subsetneq N$ such that N and M/S are both uniserial. Then M is uniserial.*

Proof. Let $0 = N_0 \subset S = N_1 \subset \cdots \subset N_k = N$ be the unique composition series of N. Then $0 = N_1/S \subset N_2/S \subset \cdots \subset N_k/S \subset M/S$ is the unique composition series of M/S. Clearly M has finite composition length. Let L be any non-zero submodule of M. Suppose that $L \cap S = 0$. Then $L \cap N = 0$ since S is essential in N. Thus $L \nsubseteq N$ and hence $M = L + S$. In this case, $N = (N \cap L) + S = S$, a contradiction. Thus $L \cap S \neq 0$, so that $S \subseteq L$ and hence $L/S = M/S$ or $L/S = N_i/S$, i.e. $L = M$ or $L = N_i$ for some $1 \leqslant i \leqslant k$. Therefore M is uniserial. $\square$

Lemma 5.2.3 *Let R be any ring. Let M be an R-module such that there exists a chain of submodules $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq \bigcup_{n \geqslant 1} M_n = M$ with M_n/M_{n-1} simple and M_n uniserial for all $n \geqslant 1$. Let N be any proper submodule of M. Then $N = M_n$ for some $n \geqslant 0$.*

Proof. There exists a positive integer k such that $M_k \nsubseteq N$. Let n be the least integer such that $M_n \nsubseteq N$. Then $n \geqslant 1$. Thus $M_{n-1} \subseteq N$ so that $M_{n-1} \subseteq N \cap M_n \subset M_n$. Since M_n/M_{n-1} is simple it follows that $N \cap M_n = M_{n-1}$. Suppose that $N \cap M_s \neq M_{n-1}$ for some $s > n$, and choose s as small as possible. Then $N \cap M_{s-1} = M_{n-1}$ gives $N \cap M_s \nsubseteq M_{s-1}$. Because M_s/M_{s-1} is simple and M_s is uniserial, we have $N \cap M_s = M_s$. Thus $M_n \subseteq M_s \subseteq N$, a contradiction. It follows that $N \cap M_s = M_{n-1}$ for all $s > n$ and

$$N = N \cap M = N \cap \left(\bigcup_{s \geqslant 1} M_s\right) = \bigcup_{s \geqslant 1} (N \cap M_s) = M_{n-1}.$$

$\square$

Let k be any field of characteristic 0. Then $A_1(k)$ denotes the first Weyl algebra consisting of polynomials over k in indeterminates x, y subject to $xy - yx = 1$ (see [8], [29]).

Lemma 5.2.4 *Let k be a field of characteristic 0 and let $R = A_1(k)$. Let $p \in k[y]$. Then the R-module $R/R(x-p)$ is simple.*

Proof. Let L be a left ideal of R properly containing $R(x-p)$. Since $R = k[y] + k[y]x + k[y]x^2 + \cdots$ it follows that $R = k[y] + R(x-p)$. Thus there exists an element $0 \neq f(y) \in L$. Now

$$f'(y) = xf(y) - f(y)x = (x-p)f(y) - f(y)(x-p) \in L.$$

Repeating this argument we obtain $L \cap k \neq 0$, i.e. $L = R$. Thus $R(x-p)$ is a maximal left ideal of R and hence $R/R(x-p)$ is a simple R-module. □

The next result is due to McConnell and Robson [28]. We give here an elementary proof.

Lemma 5.2.5 *Let k be a field of characteristic 0 and let $R = A_1(k)$. Let p, q be distinct members of $k[y]$. Then the R-module $R/R(x-p)(x-q)$ is uniserial if and only if $p - q \notin k$.*

Proof. Suppose first that $p - q \in k$. Then

$$\begin{aligned} (x-p)(x-q) &= ((x-q)-(p-q))(x-q) \\ &= (x-q)^2 - (p-q)(x-q) \\ &= (x-q)^2 - (x-q)(p-q) \\ &= (x-q)((x-q)-(p-q)) \\ &= (x-q)(x-p). \end{aligned}$$

In this case,

$$R/R(x-p)(x-q) = (R(x-p)/R(x-p)(x-q)) \oplus (R(x-q)/R(x-p)(x-q)).$$

Thus $R/R(x-p)(x-q)$ is not uniserial.

Conversely, suppose that $p - q \notin k$. Note that

$$(x-q)y - y(x-q) = (xy - yx) + (yq - qy) = 1,$$

so that we can make the change of variable $x \mapsto x - q$ and suppose without loss of generality that $q = 0$. Note that in this case $p \notin k$. By Lemma 5.2.4, Rx and $R(x-p)$ are both maximal left ideals of R. Moreover, $Rx/R(x-p)x \cong R/R(x-p)$. Thus the R-module $R/R(x-p)x$ has length 2. Suppose there exists a left ideal L such that

$$R = Rx + L \text{ and } Rx \cap L = R(x-p)x. \tag{5.1}$$

Then $1 - fx \in L$ for some $f \in R$.

We now claim that

$$x^n \in k[y]x + R(x-p)x \tag{5.2}$$

for all positive integers n. Note that $x = 1x + 0x$ and $x^2 = px + (x-p)x$ so that (5.2) holds for $n = 1, 2$. Suppose that $m \geqslant 2$ is a positive integer such that (5.2) holds for $1 \leqslant n \leqslant m$. Consider

$$\begin{aligned} x^{m+1} &= x^{m-1}[(x^2 - px) + px] \\ &= x^{m-1}(x^2 - px) + x^{m-1}px \\ &= x^{m-1}(x^2 - px) + (px^{m-1} + a_0 + a_1x + \cdots + a_{m-2}x^{m-2})x \end{aligned}$$

for some $a_i \in k[y]$ $(1 \leqslant i \leqslant m-2)$. Thus

$$x^{m+1} \in R(x-p)x + k[y]x^m + k[y]x^{m-1} + \cdots + k[y]x \subseteq R(x-p)x + k[y]x$$

by the induction hypothesis. Hence (5.2) holds for all positive integers n.

Combining (5.1) and (5.2) gives $g \in k[y]$ such that $1 - gx \in L$. Now $x(1-gx) \in L$ so that $x - (gx + g')x \in L$, i.e. $(1-g')x - gx^2 \in L$, where g' is the derivative of g in $k[y]$. Now

$$(1 - g' - gp)x = (1 - g')x - gx^2 + g(x^2 - px) \in Rx \cap L$$

and hence $(1-g'-gp)x \in R(x-p)x$. This implies that $1-g'-gp \in R(x-p)\cap k[y] = \{0\}$. Since $p \notin k$ it follows that $1 - g' - gp \neq 0$, a contradiction. Thus there does not exist a left ideal L of R satisfying (5.1). This proves that $R/R(x-p)x$ is a uniserial R-module, as required. $\square$

Theorem 5.2.6 *Let k be a field of characteristic 0 and let $R = A_1(k)$. Let $\{p_n : n \geqslant 1\}$ be any collection of elements of the polynomial ring $k[y]$ such that $p_m - p_n \notin k$ for all $1 \leqslant n < m < \infty$. For each positive integer n let B_n denote the submodule $R(x-p_n)^{-1}\cdots(x-p_1)^{-1}$ of Q, the quotient division ring of R, let $B = \bigcup_{n\geqslant 1} B_n$, and let M=B/R. Then*

(i) R is a simple Noetherian domain,

(ii) $0 \subsetneq B_1/R \subset B_2/R \subset \cdots \subsetneq \bigcup_{n\geqslant 1} B_n/R = M$ are all the submodules of M,

(iii) M is Artinian, and

(iv) M does not satisfy ACC on prime submodules.

Proof. It is well known that R is a simple Noetherian domain (see [8, Corollaries 1.13 and 1.15] or [29, 1.3.5]). Clearly

$$R = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq \bigcup_{n\geqslant 1} B_n = B.$$

Moreover, $B_1/R = R(x-p_1)^{-1}/R \cong R/R(x-p_1)$ which is a simple R-module by Lemma 5.2.4, and for any $n \geqslant 2$,

$$\begin{aligned} B_n/B_{n-1} &= R(x-p_n)^{-1}\cdots(x-p_1)^{-1}/R(x-p_{n-1})^{-1}\cdots(x-p_1)^{-1} \\ &\cong R/R(x-p_n), \end{aligned}$$

which is simple. Now for each $n \geqslant 1$,

$$B_n/R \cong R/R(x-p_1)\cdots(x-p_n).$$

If $n=2$, $R/R(x-p_1)(x-p_2)$ is a uniserial R-module by Lemma 5.2.5. If $n \geqslant 3$ then $B_n/B_1 \cong R/R(x-p_2)\cdots(x-p_n)$ which is uniserial by induction on n and B_{n-1}/R is also uniserial by induction on n. By Lemma 5.2.2, B_n/R is uniserial for all $n \geqslant 1$. Now Lemma 5.2.3 gives (ii). Clearly (iii) follows and by Lemma 5.2.1 so too does (iv). $\square$

Contrast Theorem 5.2.6 with the following result.

Theorem 5.2.7 *Let R be a ring such that every left primitive homomorphic image is (left) Artinian. Let M be an Artinian R-module. Then M satisfies ACC on semiprime submodules.*

Proof. If M does not contain any prime submodules then the result is true vacuously. Now suppose that M contains a prime submodule. Let Φ be the set of all submodules of M which can be expressed as an intersection of a finite number of prime submodules. By the minimal condition, Φ has a minimal member K, say. There exist prime submodules $K_1, \ldots, K_n$ such that

$$K = K_1 \cap \cdots \cap K_n.$$

Let L be any prime submodule of M. Then

$$K = K_1 \cap \cdots \cap K_n \supseteq L \cap K_1 \cap \cdots \cap K_n \in \Phi.$$

By the minimality of K we have $K = L \cap K_1 \cap \cdots \cap K_n$. Hence $K \subseteq L$. Thus K is contained in any semiprime submodule of M.

Consider K_1. Now $K_1 \neq M$ and hence there exists a submodule U of the Artinian module M, containing K_1, such that U/K_1 is simple. Let $\mathcal{P}$ =ann(U/K_1).

By hypothesis, the ring $R/\mathcal{P}$ is simple Artinian. But $\mathcal{P}(M/K_1) = 0$, because K_1 is prime, and hence M/K_1 is semisimple. Thus M/K_i is semisimple for all $1 \leqslant i \leqslant n$. Being Artinian, M/K_i is Noetherian for all $1 \leqslant i \leqslant n$. Hence M/K is Noetherian. It foll ows that M satisfies ACC on semiprime submodules. $\square$

Recall that if R is a ring which satisfies a polynomial identity, i.e. a PI-ring for short, then every left primitive image of R is Artinian [29, 13.3.8]. For the definition and basic properties of PI-rings see [29]. In particular, note that if $\mathcal{P}$ is a prime ideal of a PI-ring R then the ring $R/\mathcal{P}$ is (left) Goldie [29, 13.6.6]. Our next aim is to show that if R is a PI-ring then any R-module M with arbitrary Krull dimension satisfies ACC on prime submodules.

Definition 5.2.8 *Let R be a prime left Goldie ring. Let M be a left R-module. Then the singular submodule of M is given by*

$$Z(M) = \{m \in M : cm = 0 \textit{ for some } c \in \mathcal{C}(0)\}.$$

M is called a torsion module if M=$Z(M)$, and M is called torsion-free if $Z(M)$=0.

Definition 5.2.9 *A proper submodule N of M is called strongly prime if $\mathcal{P} = (N : M)$ is a prime ideal of R such that the ring $R/\mathcal{P}$ is (prime) left Goldie and the left $(R/\mathcal{P})$-module M/N is torsion-free.*

Lemma 5.2.10 *(See [27, Proposition 2.1 and Corollary 2.8]). For any ring R, any strongly prime submodule of an R-module is prime. Moreover, the converse holds if R is a PI-ring.*

This brings us to the main result of this section.

Theorem 5.2.11 *Let R be a PI-ring and let M be an R-module with Krull dimension. Then M satisfies ACC on prime submodules.*

Proof. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be any ascending chain of prime submodules of M. For each $i \geqslant 1$ let $\mathcal{P}_i = (K_i : M)$, so that $\mathcal{P}_i$ is a prime ideal of R and M/K_i is a torsion-free module over the prime Goldie ring $R/\mathcal{P}_i$. Without loss of generality, $K_1 = 0$ and $\mathcal{P}_1 = 0$.

Suppose that $k(M) = \alpha$, for some ordinal $\alpha \geqslant -1$. We prove that M has ACC on prime submodules by induction on α. If $\alpha = -1$ then $M = 0$ and there is nothing to prove.

Now suppose that $\alpha \geqslant 0$ and that the result holds for R-modules of Krull dimension less than α. Note that $0 = \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \subseteq \cdots$ is an ascending chain of prime ideals of R. Suppose that $\mathcal{P}_t \neq 0$ for some $t \geqslant 2$. By [29, 13.6.4] $\mathcal{P}_t$ contains a non-zero central (and hence regular) element c. Now

$$M \supseteq cM \supseteq c^2M \supseteq \cdots$$

is a descending chain of submodules of M and hence $k(c^sM/c^{s+1}M) < \alpha$ for some $s \geqslant 1$. Note that because M is torsion-free (Lemma 5.2.10), $c^sM/c^{s+1}M \cong M/cM$ and hence $k(M/cM) < \alpha$. But $cM \leqslant K_t$, so that $k(M/K_t) < \alpha$. Now

$$0 = K_t/K_t \subseteq K_{t+1}/K_t \subseteq K_{t+2}/K_t \subseteq \cdots$$

is an ascending chain of primes in M/K_t. By induction on α,

$$K_n/K_t = K_{n+1}/K_t = K_{n+2}/K_t = \cdots,$$

and hence $K_n = K_{n+1} = K_{n+2} = \cdots$ for some $n \geqslant t$.

Otherwise, $\mathcal{P}_i = 0$ $(i \geqslant 1)$. Thus M/K_i is a torsion-free R-module for all $i \geqslant 1$ by Lemma 5.2.10. Now $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ is an ascending chain of submodules of a module M with finite uniform dimension by Lemma 5.1.5, and hence there exists $q \geqslant 1$ such that K_i is essential in K_{i+1} for all $i \geqslant q$. But this implies that K_{i+1}/K_i is torsion and hence $K_i = K_{i+1}$ for all $i \geqslant q$, i.e.

$K_q = K_{q+1} = K_{q+2} = \cdots$. Therefore M satisfies the ACC on prime submodules $\square$

Modifying the proof of Theorem 5.2.11 somewhat we have the next result.

Theorem 5.2.12 *Let R be a ring which satisfies ACC on prime ideals and let M be an R-module with Krull dimension. Then M satisfies ACC on strongly prime submodules.*

Proof. Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$ be any ascending chain of strongly prime submodules of M. With the notation of the proof of Theorem 5.2.11, $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \subseteq \cdots$ is an ascending chain of prime ideals of R. By hypothesis, $\mathcal{P}_t = \mathcal{P}_{t+1} = \mathcal{P}_{t+2} = \cdots$ for some positive integer t. There exists $s \geqslant t$ such that K_i is essential in K_{i+1} for all $i \geqslant s$. But M/K_i is torsion-free as a module over the prime left Goldie ring $R/\mathcal{P}_i$. Thus $K_i = K_{i+1}$ for all $i \geqslant s$. $\square$

Corollary 5.2.13 *Let R be a ring with left Krull dimension and let M be an R-module with Krull dimension. Then M satisfies ACC on strongly prime submodules.*

Proof. By Theorems 5.1.4 and 5.2.12. $\square$

In particular, if we take $M = R$ in Corollary 5.2.13 we have the following result:

Corollary 5.2.14 *Let R be a ring with left Krull dimension. Then R satisfies ACC on strongly prime left ideals.*

If R is a ring with left Krull dimension and $\mathcal{P}$ is a prime ideal of R then $R/\mathcal{P}$ is a left Goldie ring (Proposition 5.1.2) and the left $(R/\mathcal{P})$-module $R/\mathcal{P}$ is torsion-free. Thus every prime ideal of R is a strongly prime left ideal of R. Thus Corollary 5.2.14 generalizes Theorem 5.1.4. We do not know if rings with Krull dimension satisfy ACC on prime left ideals.

5.3 Semiprime Submodules

In this section we shall be concerned with when a module with Krull dimension satisfies ACC on semiprime submodules. Nagata [30, Proposition 34 Corollary] (see also [12, Theorem 87]) proved that a ring R which satisfies ACC on semiprime ideals has the property that every non-zero homomorphic image has only a finite number of minimal prime ideals, equivalently every semiprime ideal of R is a finite intersection of prime ideals. If R is a general ring and M an R-module such that every non-zero homomorphic image has only a finite number of minimal prime submodules then every semiprime submodule of M is a finite intersection of prime submodules by [27, p.1059]. We do not know if the converse is true in general, but it is true in the following special case.

Theorem 5.3.1 *Let R be any ring. Then the following statements are equivalent for an R-module M.*

(i) M satisfies ACC on semiprime submodules.

(ii) (a) M satisfies ACC on prime submodules, and

(b) every non-zero homomorphic image of M has only a finite number of minimal prime submodules.

(iii) (a) M satisfies ACC on prime submodules, and

(b) every semiprime submodule of M is a finite intersection of prime submodules.

Proof. $(i) \Rightarrow (ii)$ Clearly M satisfies $(ii)(a)$. Suppose that $(ii)(b)$ does not hold. There exists a proper submodule N of M such that M/N has an infinite number of minimal prime submodules. Then

$$\mathrm{rad}(N) = \bigcap\{K : K \text{ is a prime submodule of } M \text{ and } N \subseteq K\}$$

is a semiprime submodule of M and $M/\mathrm{rad}(N)$ has an infinite number of minimal prime submodules.

Let S be a semiprime submodule of M chosen maximal such that M/S has an infinite number of minimal prime submodules. Then S is not prime. There exist $r \in R$ and a submodule L of M such that $S \subsetneqq L$, $rL \subseteq S$ and $rM \not\subseteq S$. By the choice of S, the modules $M/\mathrm{rad}(L)$ and $M/\mathrm{rad}(RrM+S)$ both have only a finite number of minimal prime submodules. Let K be a prime submodule of M with $S \subseteq K$ such that K/S is a minimal prime submodule of M/S. Then $rL \subseteq K$ so that $L \subseteq K$ or $RrM + S \subseteq K$. Thus $\mathrm{rad}(L) \subseteq K$ or $\mathrm{rad}(RrM + S) \subseteq K$. If $\mathrm{rad}(L) \subseteq K$ then $K/\mathrm{rad}(L)$ is one of the finite number of minimal prime submodules of the module $M/\mathrm{rad}(L)$. Similarly if $\mathrm{rad}(RrM + S) \subseteq K$ then $K/\mathrm{rad}(RrM + S)$ is one of the finite number of minimal prime submodules of $M/\mathrm{rad}(RrM + S)$. It follows that M/S has only a finite number of minimal prime submodules, a contradiction. Thus M satisfies $(ii)(b)$.

$(ii) \Rightarrow (iii)$ Let $K \subseteq N$ be submodules of M. Then it is easy to check that N is a prime submodule of M if and only if N/K is a prime submodule of M/K. Now suppose S is a semiprime submodule of M. Now M/S has only a finite number of minimal prime submodules $S_1/S, \ldots, S_n/S$ for some positive integer n where $S \subseteq S_i \subseteq M$ $(1 \leqslant i \leqslant n)$. Then S_i is a prime submodule of M for all $1 \leqslant i \leqslant n$ and $S = \bigcap_i^n S_i$.

$(iii) \Rightarrow (i)$ By the proof of Theorem 5.1.8. $\square$

We have been unable to settle for a general PI-ring R whether every R-module with Krull dimension satisfies ACC on semiprime submodules. We have the following special case.

Theorem 5.3.2 *Let R be a left Noetherian PI-ring and let M be an R-module with Krull dimension. Then M satisfies ACC on semiprime submodules.*

Proof. Suppose that the result is false. Let $\alpha \geqslant -1$ be the least ordinal such that there exists a left Noetherian PI-ring R with $k(R) = \alpha$ and an R-module

M with Krull dimension but M does not satisfy ACC on semiprime submodules. Clearly $\alpha \geqslant 0$. By Theorems 5.2.11 and 5.3.1, we can suppose without loss of generality that M contains an infinite number of minimal prime submodules.

Since R is left Noetherian, there exist a positive integer s and prime ideals $\mathcal{T}_i$ $(1 \leqslant i \leqslant s)$ such that $\mathcal{T}_1 \cdots \mathcal{T}_s = 0$ [29, 2.2.17]. If K is a minimal prime submodule of M then $(\mathcal{T}_1 \cdots \mathcal{T}_s)M \subseteq K$ gives $\mathcal{T}_i M \subseteq K$ and $K/\mathcal{T}_i M$ is a minimal prime submodule of $M/\mathcal{T}_i M$ for some $1 \leqslant i \leqslant s$. There exists $1 \leqslant j \leqslant s$ such that $M/T_j M$ has an infinite number of minimal prime submodules. Hence we can pass to the ring R/T_j and suppose without loss of generality that R is a prime ring.

Let $Z = Z(M)$. Then Z is a prime submodule of M (Lemma 5.2.10). Clearly $Z \neq 0$. There exist a positive integer n and uniform submodules U_i $(1 \leqslant i \leqslant n)$ of Z such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of Z. For each $1 \leqslant i \leqslant n$, let $\mathcal{P}_i = \mathrm{ass}U_i = \{r \in R : rV = 0$ for some non-zero submodule V of $U_i\}$. Note that $\mathcal{P}_i$ is a non-zero prime ideal of R for each $1 \leqslant i \leqslant n$ by [8, Lemma 4.22] and [29, 13.6.6]. By [29, 13.6.4] there exist a non-zero central element c of R such that $c \in \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_n$. Now $\mathrm{ann}_Z(c)$ is an essential submodule of Z and hence $c^t Z = 0$ for some positive integer t, by [29, 4.2.2 and 4.2.6].

Let K be a minimal prime submodule of M. If $Z \subseteq K$ then $K = Z$. Suppose that $Z \nsubseteq K$. Then $c^t Z = 0 \subseteq K$ gives $cM \subseteq K$ and K/cM is a minimal prime submodule of the (R/Rc)-module M/cM. But $k(R/Rc) < k(R) = \alpha$ [9, Corollary 7.2] so that, by the choice of α, M/cM has only a finite number of minimal prime submodules by Theorem 5.3.1. This contradiction proves the result. $\square$

Another special case is the following result.

Theorem 5.3.3 *Let R be a PI-ring with Krull dimension and let M be a finitely generated R-module with Krull dimension. Then M satisfies ACC on semiprime* *submodules.*

Proof. We follow the proof of Theorem 5.3.2. By Proposition 5.1.7 we can suppose without loss of generality that R is a prime ring. Let $Z = Z(M)$. By Zorn's Lemma, there exists a submodule W of M maximal with respect to $Z \cap W = 0$. Then $M/(Z \oplus W)$ is torsion and hence M/W is torsion. Because M is finitely generated, there exists a non-zero central element c such that $cM \subseteq W$. Then $cZ \subseteq Z \cap W = 0$. The result now follows by the proof of Theorem 5.3.2. □

Corollary 5.3.4 *Let R be a commutative ring and let M be a finitely generated R-module with Krull dimension. Then M satisfies ACC on semiprime submodules.*

Proof. Without loss of generality M is faithful. Now $M = Rm_1 + \cdots + Rm_k$ for some positive integer k and elements $m_i \in M$ $(1 \leqslant i \leqslant k)$. Define $\theta : R \to M^{(k)}$ by $\theta(r) = (rm_1, \ldots, rm_k)$ for all $r \in R$. Then θ is an R-monomorphism and hence the ring R has Krull dimension. The result now follows by Theorem 5.3.3. □

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