Branched 1-Manifolds and Presentations of Solenoids

by

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To my family

Abstract

The aim of the project was to study the presentations of solenoids by branched 1-manifolds. We begin by studying two properties of branched 1-manifolds which effect the presentation of solenoids, orientability and recurrence. Solenoids are shown to come in two varieties, those presented by orientable branched 1-manifolds and those presented by nonorientable branched 1-manifolds. Two methods of determining whether or not a branched 1-manifold is orientable are given. Recurrence is shown to be a necessary and sufficient condition for a branched 1-manifold to present a solenoid. We then show how the question of whether or not a branched 1-manifold is recurrent can be converted into question in graph theory for which there exist efficient algorithms.

Next we consider two special types of presentations, elementary presentations and (p,q)-block presentations, which allow us to extract algebraic invariants for the equivalence of solenoids. A slightly stronger version of a result of Williams [39] is obtained which states that any solenoid with a fixed point is equivalent to one presented by an elementary presentation. The proof is constructive and gives a method for finding elementary presentation given a presentation of a solenoid with a fixed point. Williams has shown, [39], that there is a complete invariant for the equivalence of solenoids given by an elementary presentation in terms of the shift equivalence of endomorphisms of a free group. We prove a result which shows that every solenoid can be given by a countably infinite number of equivalent (p,q)-block presentations. The (p,q)-block presentations again allow us to find invariants for the equivalence of solenoids in terms of the shift equivalence of endomorphisms of a free group.

Finally we consider further invariants for the equivalence of solenoids which are derived from the endomorphism invariants. First we examine the invariants which arise upon abelianizing the free group in question. Second we introduce new invariants which reflect some of the non-abelian character of these endomorphisms. These new non-abelian invariants are then used to solve a problem posed by Williams in [39].

Preface

This thesis is submitted to the University of Glasgow in accordance with the requirements for the degree of Doctor of Philosophy.

To my supervisor, Dr. C. Athorne, I wish to express my sincere gratitude for his help, encouragement and patience throughout my years of research. Thanks are also due to the staff and research students within the Department of mathematics for their advice and assistance.

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Contents

Chapter 1

Introduction

In this thesis we study the presentation of solenoids by branched 1-manifolds. A solenoid K_∞ is a topological space which together with a homeomorphism $h: K_\infty \to K_\infty$, called the shift map, forms a dynamical system. A solenoid K_∞ and shift map h are defined, using an inverse limit construction, by a branched 1-manifold K and a mapping $g: K \to K$ satisfying certain properties. A branched 1-manifold K and a map g defining a solenoid K_∞ and shift map h are called a presentation of the solenoid and we write $K_\infty = \{K, g\}$.

Branched 1-manifolds and solenoids play an important role in the theory of dynamical systems. The first example of the construction of a solenoid (as a hyperbolic attractor) is due to Smale [32]. In this example Smale describes how a certain mapping of the solid torus, when restricted to its hyperbolic attractor, is topologically conjugate to the solenoid presented by a circle S_1 and an expanding map $S_1 \to S_1$. More detailed examples of this construction can be found in [8] and [24].

This construction was later generalized by R. F. Williams in [38] and [39], using branched 1-manifolds, to give a model which completely describes 1-dimensional hyperbolic attractors. In particular Williams shows that if $f: M \to M$ is a diffeomorphism of a differentiable manifold M with a 1-dimensional hyperbolic attractor Λ then there exists a presentation of a solenoid $K_\infty = \{K, g\}$ with shift map h so that $f|\Lambda$ is topologically conjugate to h. Williams is also able to show that given a presentation of a solenoid $K_\infty = \{K, g\}$ with shift map h there exists a diffeomorphism $f: S^4 \to S^4$, where S^4 is the 4-sphere, which has a 1-dimensional hyperbolic attractor on which it is topologically conjugate to h. Williams has extended some of these results to n-dimensional hyperbolic attractors in [41] by considering solenoids and shift maps presented by branched

n-manifolds.

The first example of a 1-dimensional hyperbolic attractor in the two dimensional disk was given by Plykin [27]. The use of branched 1-manifolds in intrinsic in this example. A variant of this example was used in [25] to show that non-trivial hyperbolic attractors (for flows) appear in arbitrarily small perturbations of constant vector fields on tori T^n of dimensions greater than 2. As a result of this, hyperbolic attractors appear in perturbations of three or more coupled harmonic oscillators, or three or more relaxation oscillators.

Another intriguing example is that of the DA map introduced by Smale in [32]. The DA map is a modification of a two dimensional hyperbolic toral automorphism which has a 1-dimensional hyperbolic attractor. Thus the DA map restricted to its attractor is topologically conjugate to a shift map on a solenoid. In [42] Williams uses the DA example to prove that the structurally stable systems on certain 3-manifolds are not dense. While Smale's construction is based on a two dimensional torus, Williams uses a 1-dimensional hyperbolic attractor defined in terms of a solenoid.

In [28], [29], and [30], Plykin considers hyperbolic attractors of diffeomorphisms on compact manifolds with codimension 1. In these papers there are numerous examples of the construction of diffeomorphisms on compact two dimensional manifolds with 1-dimensional hyperbolic attractors. These constructions make use of presentations of solenoids by branched 1-manifolds.

Some of the questions considered in this thesis include:

1. Which branched 1-manifolds can present a solenoid?

2. If $\{K, g\}$ and $\{K', g'\}$ are presentations of solenoids K_∞ and K'_∞, respectively is there any property of the branched 1-manifolds which affect whether the shift maps h and h' can be topologically conjugate?

3. Given two solenoids K_∞ and K'_∞ with shift maps h and h' how can you determine whether or not h is topologically conjugate to h'?

We will give a complete answer to the first question. We give an affirmative answer to the second question. The third question is addressed by considering invariants for the topological conjugacy of solenoids given a presentation.

In chapter 2.) we begin with an introduction to branched 1-manifolds, finite graphs and smooth graphs. We show that every branched 1-manifold can be given the structure of a smooth graph. We define solenoids and review some properties of inverse limit systems. Finally we review some results of R.F. Williams concerning the presentations of solenoids. In particular we review a result of Williams which says that if $\{K, g\}$ and $\{K', g'\}$ are presentations of solenoids with shift maps h and h', respectively, then h is topologically conjugate to h' if and only if the maps g and g' satisfy an equivalence relation called shift equivalence.

In chapter 3.) we introduce two properties of branched 1-manifolds which have an effect on the presentation of solenoids, orientability and recurrence. They are shown to be independent of each other.

We consider orientability first. A branched 1-manifold is orientable if there is no "smooth" path on it which traverses a branch in more than one direction. On an orientable branched 1-manifold with a smooth graph structure we can give the smooth graph defined an orientation which respects its smooth structure, such an orientation is called a coherent orientation. We next consider how to determine whether or not a branched 1-manifold is orientable. In doing so we define the structure matrix of a branched 1-manifold with a given smooth graph structure. We then give necessary and sufficient conditions for a branched 1-manifold to be orientable in terms of the rank of its structure matrix. We give alternative necessary and sufficient conditions for a branched 1-manifold to be orientable in terms of the existence of a certain type of cycle on the branched 1-manifold in question. We then show that if $K_\infty = \{K, g\}$ and $K'_\infty = \{K', g'\}$ are solenoids with shift maps h and h', respectively, where h is topologically conjugate to h' then K and K' are both orientable or both nonorientable. With every nonorientable branched 1-manifold we show how we can construct an orientable double cover, which we will use to study the recurrence of nonorientable branched 1-manifolds.

Next we consider the question of recurrence. A branched 1-manifold is recurrent if there exists a closed "smooth" path on it which traverses every branch of the manifold and thus it is possible to re-trace this path infinitely often. We show that if K is an orientable branched 1-manifold with a smooth graph structure and coherent orientation then K must be strongly connected with respect to this orientation. As there exists an efficient algorithm to determine whether or not a graph with given orientation is strongly connected it can be used to determine whether or not an orientable smooth graph is recurrent. We then turn our attention to the question of recurrence in the nonorientable case. We show that if K is a nonorientable smooth graph with orientable double cover K', then K is recurrent if and only if K' is recurrent. It is therefore possible to determine whether or not a nonorientable smooth graph is reccurent by studying its double cover. We show that a nonrecurrent smooth graph will consist of a finite number of maximal reccurrent sub-branched 1-manifolds. Finally we show that a connected branched 1-manifold K is recurrent if and only if there exist a mapping $g: K \to K$ which satisfies the conditions necessary for presenting a solenoid. We note that in the higher dimensional case, branched n-manifolds, it is still an open question as to which manifolds may present a solenoid. Some examples of branched 2-manifolds which present solenoids are given in [16].

In chapter 4.) we study presentations of solenoids. In particular we study methods of moving from one presentation $\{K, g\}$ to an equivalent presentation $\{K', g'\}$ in which the smooth graph K' is simplified or to a presentation where the mapping g' has some desirable property. The methods in this chapter will be particularly useful for finding invariants for solenoids.

The first method that we consider is due to Williams [39]. We prove a slightly stronger variant of a theorem of Williams which shows how given any presentation $\{K, g\}$ where g has a fixed point we can find an equivalent presentation $\{K', g'\}$ where K' has only a single branch point which is fixed under g. Such a branched 1-manifold is called an elementary branched 1-manifold and a presentation of a solenoid by an elementary branched 1-manifold is called an elementary presentation. We then review some results of Williams which show how to find a complete algebraic invariant for the topological conjugacy of shift maps of solenoids presented by an elementary presentation. Thus these are complete invariants for solenoids $\{K, g\}$ where the map g has a fixed point. In particular given an elementary presentation $\{K, g\}$ we can associate with it an endomorphism, g_*, of a finitely generated free group. We then give a result of Williams which says that if $\{K, g\}$ and $\{K', g'\}$ are elementary presentations of with shift maps h and h' with associated endomorphisms g_* and g'_* then h is topologically conjugate to h' if and only if g_* is shift equivalent to g'_*. Here the notion of shift equivalence is defined for endomorphisms of finitely generated free groups in the obvious manner. In this manner we are able to convert the question of whether the shift maps of two solenoids are topologically conjugate into an algebraic question. The case of solenoids presented by bracheded 2-manifolds has been studied in [36] in which it is shown that in presentations $\{K, g\}$ with a special form it is possible to find an equivalent presentation with a reduced branch structure.

The second method we introduce involves the notion of a (p, q)-block presentation. In a (p, q)-block presentation $\{K, g\}$ the branched 1-manifold K is has the structure of a finite graph in which the vertex set of the graph consists of all points fixed under g^n where $p \leq n \leq q$. We show that any presentation of a solenoid may be put into a countably infinite number of equivalent (p, q)-block presentations. Given a (p, q)-block presentation $\{K, g\}$ we show how to associate with this presentation an endomorphism of a finitely generated free group, g_*. We then prove that if $\{K, g\}$ and $\{K', g'\}$ are (p, q)-block presentations with shift maps h and h' and associated endomorphisms g_* and g'_* then h topologically conjugate to h' implies that g_* is shift equivalent to g'_*. In this manner we are able to construct algebraic invariants for the topological conjugacy of solenoids.

In chapter 5.) we study algebraic invariants for the topological conjugacy of shift maps of solenoids. In general these invariants are derived from the invariants developed in the previous chapter as there is no known general method for determining when endomorphisms of finitely generated free groups are shift equivalent.

In the first section we consider invariants which arise by abelianizing the free group in question. We show that with any endomorphism ϕ of a free group we can associate a non-negative integer matrix, $\mathbf{\Phi}$. The notion of shift equivalence is then defined for non-negative integer matrices. We then prove that if ϕ and ψ are shift equivalent endomorphisms the the associated non-negative integer matrices, $\mathbf{\Phi}$ and $\mathbf{\Psi}$, are shift equivalent. The shift equivalence of non-negative integer matrices is an invariant for sub-shifts of finite type and has been extensively studied, see [40], [21] and [22]. We prove that if ϕ is an endomorphism of a free group associated with a elementary presentation or a (p, q)-block presentation then the non-negative integer matrix associated with ϕ will be in a special form called primitive. We state a result which shows that the shift equivalence of non-negative integer matrices can be reduced to shift equivalence over the integers. There exists a procedure for deciding when two non-negative integer matrices are shift equivalent [21], but it is usually quite difficult to apply. We next review two invariants which can be derived from the shift equivalence of primitive non-negative integer matrices and which are fairly easy to calculate, the Jordan form away from zero and the Bowen-Franks group. These invaraints are applied in several examples. We will find the Bowen-Franks group particularly useful when considering invariants which reflect the non-abelian nature of the endomorphisms associated with a presentation of a solenoid.

In the second section of chapter 5.) we consider invariants for the endomorphisms associated with presentations of solenoids which reflect some of their non-abelian nature. We show how we can associate with any endomorphism ϕ of a free group a matrix Φ over a group ring $\mathbb{Z}G_\phi$, where the group G_ϕ is isomorphic to the Bowen-Franks group of an integer matrix associated with ϕ. We then define the notion of shift equivalence for matrices over a group ring. We prove that if ϕ and ψ are endomorphisms with associated group ring matrices, Φ over $\mathbb{Z}G_\phi$ and Ψ over $\mathbb{Z}G_\psi$, then there will be an isomorphism $\pi\colon G_\psi \to G_\phi$ so that Φ and Ψ_π are shift equivalent. Note that here Ψ_π is the image of the matrix Ψ under the ring isomorphism $\mathbb{Z}G_\psi \to \mathbb{Z}G_\psi$ induced by the isomorphism $\pi\colon G_\psi \to G_\psi$. There is unfortunately no known method for determining when two matrices over a group ring are shift equivalent. We do however have some extra information which we are able to make use of. We prove that if $\{K_0, g_0\}$ and $\{K_m, g_m\}$ are both elementary or both (p,q)-block presentations where g_0 is shift equivalent to g_m with lag$=m$ then there will exist presentations (all elementary or all (p,q)-block) $\{K_i, g_i\}$ for $i = 1, \ldots, m-1$ so that g_{j-1} is shift equivalent to g_j for $j = 1, \ldots, m$ with lag=1. We then can prove an easy corollary of this which states that if ϕ and ψ are endomorphisms associated with elementary or (p,q)-block presentations with associated group ring matrices, Φ over $\mathbb{Z}G_\phi$ and Ψ over $\mathbb{Z}G_\psi$, then there must exist an isomorphism $\pi\colon G_\psi \to G_\phi$ so that $\text{trace}(\Phi^n) = \pi(\text{trace}(\Psi^n))$ for all $n \in \mathbb{N}$. Note that we are considering π to be the ring isomorphism induced by the group isomorphism of the same name. As the groups G_ϕ and G_ψ are abelian groups with a finite number of generating elements it is possible to list all isomorphisms $\pi\colon G_\psi \to G_\phi$ and thus calculate these invariants. We give several examples of how these invariants can be used to distinguish non-equivalent presentations of solenoids. Other algebraic invariants for the shift equivalence of solenoids have been developed in [34]. These are of quite a different flavor and involve the elementary chain of ideal of the Alexander matrix of a group associated with every solenoid with an elementary presentation.

In the third section of chapter 5.) we give the solution to a problem posed by Williams in [39]. Our solution makes use of the non-abelian invariants developed in the second section of this chapter.

In chapter 6.) we give our concluding remarks and consider possible avenues for future research.

References used in the text refer to the source from which the result was taken at the time (not necessarily the *original* source), and so many results are credited to books such as those by D. Lind and B. Marcus , or J. Hocking and G. Young. Every effort is made to give credit where credit is due to unoriginal results in this thesis. Any aberrations or omissions in this respect are unintentional, and entirely the fault of the author. The following convention is used for assigning credit; cited results are unoriginal and were obtained from the source cited, all other results are due to the author, if a result is based on the work of someone other than the author but the proof contains original work then it will not be cited but a brief explanation of the nature of the change will be given in the text.

Chapter 2

Branched 1-manifolds and Solenoids

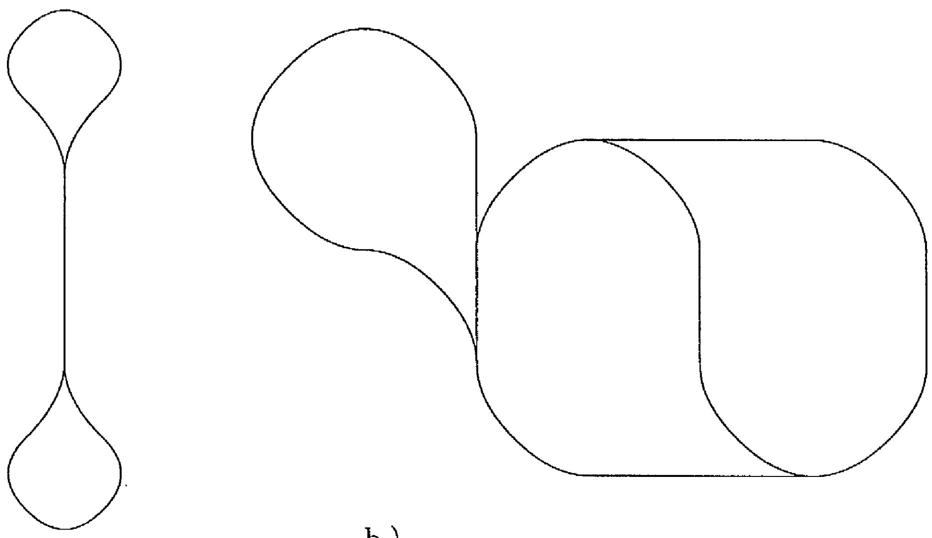
We begin by defining branched 1-manifolds and introducing some of their elementary properties. As the name might suggest branched 1-manifolds are similar to 1-manifolds except that at a finite number of points we allow branching. A differential structure is defined on a branched 1-manifold as usual.

We next review how a differentiable branched 1-manifold and a self mapping satisfying certain properties allows us to define a topological space called a solenoid using an inverse limit construction. We are then able to define a homeomorphism from the solenoid onto itself called the shift mapping. A solenoid plus a shift map forms a dynamical system. Some elementary properties concerning the presentations of solenoids are given as well as necessary and sufficient conditions for two solenoids and their shift maps to be topologically conjugate in terms of their presentations.

2.1 Branched 1-manifolds

In this section we define branched 1-manifolds and give some of their elementary properties. In doing this we will follow as closely as possible the work in [10], [24], [38], [39]. Before we can do this we will need some definitions. Fix a C^∞ function $\zeta\colon \mathbb{R} \to \mathbb{R}$ satisfying $\zeta(x) = 0$ if $x \leq 0$ and $\zeta(x) > 0$ if $x > 0$.

Definition 1 Let p, q be integers satisfying $p, q \geq 0$ and $p + q \neq 0$. Then the *local*



a.) b.)

Figure 2.1: examples of branched 1-manifolds

branched 1-manifold $Y_{p,q} \subset \mathbb{R}^2$ is given by

$$Y_{p,q} = \begin{cases} \{(x,y) \in \mathbb{R}^2 \colon y = i\zeta(x) \text{ or } y = j\zeta(-x), i = 0,\dots,p-1, j = 0,\dots,q-1\} \text{ if} \\ p,q > 0, \\ \{(x,y) \in \mathbb{R}^2 \; y = i\zeta(x), i = 0,\dots,p-1, x \geq 0\} \text{ if } q = 0, \\ \{(x,y) \in \mathbb{R}^2 \; y = i\zeta(-x), i = 0,\dots,q-1, x \leq 0\} \text{ if } p = 0. \end{cases}$$

Definition 2 A compact *branched 1-manifold* K is a compact Hausdorff topological space satisfying the following property. There is a finite subset $B \subset K$ such that for each $x \in B$ there is a neighborhood which is homeomorphic to $Y_{p,q}$ where $p \geq 2$ or $q \geq 2$. Each $x \in K - B$ has a neighborhood homeomorphic to $Y_{1,1}$, $Y_{0,1}$ or $Y_{1,0}$.

We say that a point $x \in K$ is a *point of type* (p,q) if there is a neighborhood U of x and a homeomorphism $\varphi \colon U \to Y_{p,q}$ with $\varphi(x) = 0$. The *defect* of x is defined to be the positive integer $|p-q|$. Points of type (p,q) with $p \geq 2$ or $q \geq 2$ are called *branch points.* Let B denote the set of branch points, called the *branch set* of K. The *boundary* of K, ∂K, consists of all points of type $(p,0)$ or $(0,q)$. If every point of is of type $(1,1)$ then K is a 1-manifold. Note that in [38] and [39] only points of type $(1,1)$, $(1,0)$, $(0,1)$, $(1,2)$ and $(2,1)$ are allowed. Pictures of typical branched 1-manifolds are given in figure 2.1. Note that in both branched 1-manifolds shown all vertices are of type $(2,1)$ or $(1,2)$.

Let K be a branched 1-manifold. There is a finite covering $\{U_\alpha\}$ of K by open sets such that for each open set $\{U_\alpha\}$ there is a homeomorphism $\psi_\alpha \colon U_\alpha \to Y_{p,q}$. The family

$\{(U_\alpha, \psi_\alpha)\}$ defines a C^r differential structure on K as usual by saying that a function $f: K \to \mathbb{R}$ is C^r if it is continuous and for each α, $f \circ \psi_\alpha^{-1}$ extends to a C^r function from a neighborhood V_α of $\psi_\alpha(U_\alpha)$ to $\mathbb{R}$. A branched 1-manifold has a well defined tangent bundle because the graphs of the functions in $\{i\zeta(x)\} \cup \{j\zeta(-x)\}$ have infinite order contact at any point where they meet. We can then define Riemannian metrics and C^r maps between C^r branched 1-manifolds and other manifolds as usual. Every compact C^r branched 1-manifold can be C^r embedded in $\mathbb{R}^3$ for $r \geq 0$.

Note 1 We now always assume, unless otherwise stated, given any branched 1-manifold K that it is compact and C^1 with a fixed Riemannian metric. Let $||\cdot||$ denote the induced norm on $T(K)$.

A differentiable map $f: K_1 \to K_2$ of branched 1-manifolds induces a map $Df: T(K_1) \to T(K_2)$ of their tangent bundles.

Definition 3 A differentiable map f between two branched 1-manifolds is an *immersion* if the induced map Df is a monomorphism on the tangent space at each point.

Definition 4 An *expanding map* or *expansion* of a branched 1-manifold K is a C^1 map $g: K \to K$ such that there are constants $C > 0$, $\lambda > 1$ with $||T_x(g^n)(v)|| \geq C\lambda^n ||v||$ for all $x \in K$, $n > 0$ and $v \in T_xK$.

An alternative formulation of the definition of an expansion is given in [38] as follows: the Riemanian metric determines arc length on K. A map $g: K \to K$ is an expansion if and only if g^n increases arc length by a factor of at least $C\lambda^n$. We will use these two definitions interchangeably.

2.2 Finite Graphs

In this section we define finite graphs and introduce some elementary concepts and terminology which we will use through out this thesis. We also prove a simple result showing that given a branched 1 manifold we can give it the structure of a finite graph by choosing a finite subset of it which contains the branch set. The following definition is found in [23].

Definition 5 A *finite graph*, $G(X, V)$ (or just graph for short) is a pair consisting of a Hausdorff space X and a finite subspace V (points of V are called *vertices*) such that the following conditions hold.

1. $X - V$ is the disjoint union of a finite number of open subsets $e_1, \ldots, e_k$ called *edges*. Each e_i is homeomorphic to an open interval of the real line.

2. The point set boundary, $\bar{e}_i - e_i$ consists of one or two vertices, and the pair $(\bar{e}_i, e_i)$ is homeomorphic to $(S_1, (0,1))$ if the point set boundary consists of one vertex and $([0,1], (0,1))$ if the point set boundary consists of two vertices.

A graph $G(X, V)$ is compact, since it is the union of a finite number of compact subsets (the closed edges $\bar{e}_i$ and the vertices). It may be either connected or disconnected, and it may have isolated vertices. If a vertex v belongs to the closure of an edge e_i, it is customary to say that e_i and v are *incident*. We will consistently use the notation $\partial e_i = \bar{e}_i - e_i$ to denote the boundary of the edge e_i.

A graph $G(X, V)$ will often be specified by two finite sets, V the set of vertices, E the sets whose elements are the edges of $G(X, V)$ and an incidence relation which associates with every edge one or two vertices. V and E are referred to as the *vertex set* and *edge set* respectively. We write $G(X, V) = (V, E)$, or just $G = (V, E)$ for short. We will use $I(e)$ to denote the set of vertices incident with the edge e and $I(v)$ to denote the set of edges incident with the vertex v. I is called the *incidence function*. In order to avoid confusion we use the following convention. If a graph G is written as a pair $G(X, V)$, with the G (or some other symbol) flush against the brackets, then it is being given in the sense of definition 5. If a graph G is written as a pair $G = (V, E)$, with an equal sign between the symbol G and the bracket, then it is being given as a vertex set and edge set.

Example 1 The graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and incidence relation given in the following table.

v	$I(v)$
v_1	$\{e_1, e_2, e_3\}$
v_2	$\{e_2, e_3, e_4, e_6, e_7\}$
v_3	$\{e_1, e_4, e_5\}$
v_4	$\{e_5, e_6\}$

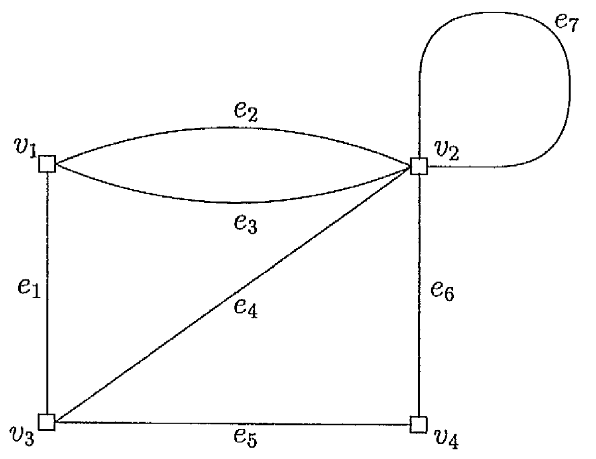


Figure 2.2: The graph G given in example 1

Figure 2.2 shows a picture of this graph. In this picture vertices are represented by small boxes and edges by lines connecting these boxes.

Let $G = (V, E)$ be a graph. G has a *vertex labeling* if the vertices $v \in V$ are distinguished from one another by names such as $v_1, v_2, \ldots, v_n$. Similarly a graph has an *edge labeling* if the edges $e \in E$ are distinguished from each other by names such as $e_1, e_2, \ldots, e_m$. If $|V| = p$ and $|E| = q$ then G is a (p, q) graph, where p is the *order* and q is the *size*, respectively of G. An edge is a *loop* or a *link* according to whether $|I(e)| = 1$ or $|I(e)| = 2$. The elements of $I(e)$ are the *ends* of e, and e is *incident* upon them and *joins* them. The ends of e are in turn incident upon e. Two distinct edges are *adjacent* if they are incident upon a common vertex. Two distinct vertices are *adjacent* on a common edge. The *degree* of a vertex v is the number of links plus twice the number of loops incident upon v, i.e., $d(v) = i + 2j$ where i and j respectively are the number of links and loops in G.

Note 2 When we want to emphasize that the underlying graph is G we write $I_G(V)$ a similar convention will be used for other functions depending on an underlying graph.

Definition 6 Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs and suppose there exist onto mappings $f\colon V_G \to V_H$ and $g\colon E_G \to E_H$ such that $I_H \circ g(e) = f \circ I_G(e)$. We say that (f, g) is a *homomorphism* from G to H, and that f and g are *vertex* and *edge homomorphisms* respectively. If f and g are bijections then (f, g) is an *isomorphism*, G

and H are then said to be *isomorphic*. It is easy to see that an isomorphism is just a relabeling of the vertices and edges of G. Clearly isomorphism defines an equivalence relation. We denote by $[G]$ the class of all graphs isomorphic to a graph G. When we write $G \cong H$ we mean that $H \in [G]$.

Definition 7 A *walk* on a graph G is an alternating sequence of vertices and edges, $W = \{v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k\}$, beginning and ending with vertices in which each edge is incident with the two vertices immediately proceeding it and following it., i.e., $v_{i-1} \in I(e_i)$ and $v_i \in I(e_i)$ for all $i \in \{1, 2, \ldots, k\}$ The walk is said to *join* the vertices v_0 and v_k. It is closed if $v_0 = v_k$ and open otherwise.

A graph is *connected* if every pair of vertices is joined by a walk. Thus G is connected if there is no partition of $V = V_1 \cup V_2$ such that no edge joins a vertex of V_1 to an vertex of V_2. This notion of connectedness clearly coincides with the topological notion when the underlying space has the structure of a finite graph.

Definition 8 A graph H is said to be a *subgraph* of a graph G if $V_H \subseteq V_G$, $E_H \subseteq E_G$ and $I_H = I_G|E_H$. A graph is said to include its subgraphs and their edges. A subgraph H is a *proper subgraph* of G if $V_H \cup E_H \subset V_G \cup E_G$ and a *spanning* subgraph if $V_H = V_G$. Note we will write $H \subseteq G$ if H is a subgraph of G.

Definition 9 An *orientation* of a graph G is a mapping $\rho: E \to V$ such that $\rho(e) \in I(e)$ for all $e \in E$.

If ρ and σ are two orientations of G and e is a loop then $\rho(e) = \sigma(e)$. For any orientation ρ of G we define its *conjugate* as the orientation ρ^* such that $\rho^*(e) \neq \rho(e)$ for each link $e \in E$. Thus $\rho^{**} = \rho$. Given an orientation ρ we say that $\rho(e)$ is the *sense* of the edge $e \in E$ under ρ and that $\rho^*(e)$ is obtained by *reversing* the sense of e. If e joins two vertices v and u, and $\rho(e) = v$ then we say that e is directed from u to v. The number of edges directed to a vertex v is called the *indegree* of v and the number of edges directed from v is called the *outdegree* of v. We call a vertex v a *source* if it is not the image of an edge under ρ, and a *sink* if $v = \rho(e)$ for all $e \in I(v)$. The set of edges directed to a vertex v is denoted $I^+_\rho(v)$, thus

$$I^+_\rho(v) = \{e \in E: \rho(e) = v\}$$

and the set of all edges directed from a vertex v is denoted I_ρ^-, thus

$$I_\rho^-(v) = \{e \in E: \rho^*(e) = v\}.$$

Suppose that $G = (V, E)$ is a finite graph. Then it is easy to give G the structure of a branched 1-manifold with branch set $B \subset V$ by choosing an atlas of coordinate neighborhoods for G. Note, however, that we can generally give G many different branched 1-manifold structures and the particular structure we get depends on how we arrange the edges at the branch points.

Note 3 When specifying a branched 1-manifold structure on G we will only specify the coordinate neighborhoods at the vertices and will operate under the assumption that any point which is not a vertex belongs to a coordinate neighborhood of type $Y_{1,1}$

In order to specify a branched 1-manifold structure on a graph G we define the notion of a switch condition on a graph . A switch condition on a graph in turn is family which consists of a switch condition for every vertex. A switch condition at a vertex then tells us how the vertices are arranged, i.e., what type of local coordinate neighborhood it has.

Definition 10 Let $G = (V, E)$ be a finite graph. A *switch condition* for a vertex $v \in V$ is a pair of subsets, $R(v)$ and $L(v)$, of the incident edges, $I(v)$, at v satisfying the following:

1. $R(v) \cup L(v) = I(v)$,

2. $R(v) \cap L(v) \subseteq \{e \in I(v): |I(e)| = 1\}$.

Definition 11 A *switch condition!graph* on a graph G is a family $S = (R(v), L(v))$ $(v \in V)$ such that $R(v)$ and $L(v)$ form a switch condition for every vertex $v \in V$.

Suppose G is a graph and S is a switch condition on G. The sets $R(v)$ and $L(v)$ are called the *right* and *left* incident edges of v respectively. At a vertex v edges belonging to $R(v)$ are said to be *opposite* the edges belonging to $L(v)$. Since only loops can belong to both $R(v)$ and $L(v)$ only loops can be opposite themselves at a vertex.

Given a switch condition $R(v), L(v)$ at a vertex v the coordinate neighborhood $\{(U_v, \varphi_v)\}$ at v is specified as follows. U_v is chosen as a small open neighborhood of v such that $U_v \cap V - \{v\} = \emptyset$. $\varphi_v: U_v \to Y_{p,q}$ where $p = \max\{|R(v)|, |L(v)|\}$ and $q = \min\{|R(v)|, |L(v)|\}$. If $\max\{|R(v)|, |L(v)|\} = |R(v)|$ and $\min\{|R(v)|, |L(v)|\} = |L(v)|$ we give the edges belonging to $R(v)$ an arbitrary ordering $\{x_0, \ldots, x_{p-1}\}$ and the edges belonging to $L(v)$

an arbitrary ordering $\{y_0, \ldots, y_{q-1}\}$ when $q \geq 1$, when $q = 0$ there will be no edges in $L(v)$. The map φ is then defined as follows. $\varphi_v(v) = (0,0)$, $\varphi|\bar{x}_i \cap U_v$ is a homeomorphism from $\bar{x}_i \cap U_v$ to $\{(x, i\zeta(x)) : x \geq 0\}$ for $i = 0, \ldots, p-1$ and, when $q \geq 1$, $\varphi_v|\bar{y}_i \cap U_v$ is a homeomorphism from $\bar{y}_i \cap U_v$ to $\{(x, j\zeta(-x)\}$ for $j = 0, \ldots q-1$. Similarly if $\max\{|R(v)|, |L(v)|\} = |L(v)|$ and $\min\{|R(v)|, |L(v)|\} = |R(v)|$ we give the edges belonging to $L(v)$ an arbitrary ordering $\{y_0, \ldots, y_{p-1}\}$ and the edges belonging to $R(v)$ an arbitrary ordering $\{x_0, \ldots, x_{q-1}\}$ when $q \geq 1$, again $R(v)$ will be empty if $q = 0$. The map φ is then defined as follows. $\varphi_v(v) = (0,0)$, $\varphi|\bar{y}_i \cap U_v$ is a homeomorphism from $\bar{y}_i \cap U_v$ to $\{(x, i\zeta(x)) : x \geq 0\}$ for $i = 0, \ldots, p-1$ and, when $q \geq 1$, $\varphi_v|\bar{x}_i \cap U_v$ is a homeomorphism from $\bar{x}_i \cap U_v$ to $\{(x, j\zeta(-x)\}$ for $j = 0, \ldots q-1$.

Note 4 When specifying a branched 1-manifold structure on a graph G using a switch condition all coordinate neighborhoods are assumed to be C^∞ compatible in the usual sense.

Definition 12 A *smooth graph* K is a graph with a designated branched 1-manifold structure. We will usually denote a smooth graph as a pair $K = (G, S)$ where G is a graph and S is a switch condition on G.

When representing a smooth graph with a diagram we will use a graph diagram where at each vertex the right incident edges are specified by using a small slash close to the vertex.

Example 2 The diagrams representing the two branched 1-manifolds of figure 2.1 as smooth graphs where the vertex set of each is chosen to be the branchset are shown in figure 2.3.

In the next theorem we show that every branched 1-manifold K can be given the structure of a graph $G = (V, E)$. In fact any branched 1-manifold can be given a countably infinite number of graph structures the only requirement being that $B \cup \partial K \subseteq V$.

Theorem 1 *Let A be a finite subset of a connected branched 1-manifold K such that the $B \cup \partial K \subseteq A$ then $G(K, A)$ is a finite graph.*

Proof. Consider $K - A$. Since A is finite and thus closed we know that $K - A$ must be open. Let F be a component of $K - A$. F must be open as it is the union of open

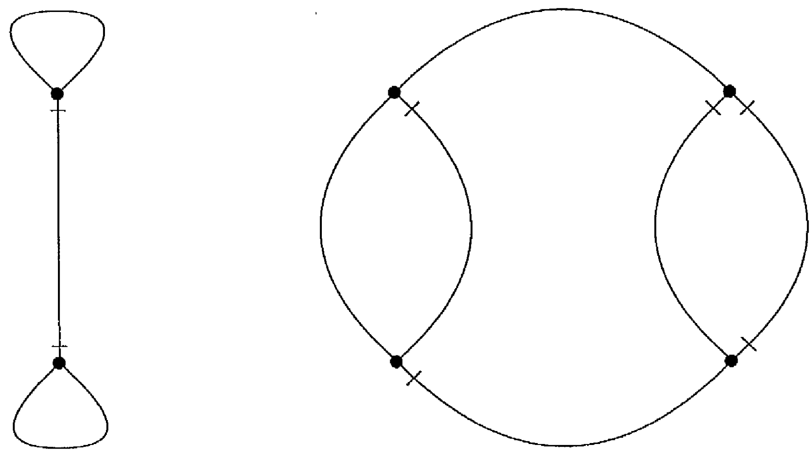


Figure 2.3: Diagrams representing the branched 1-manifolds in figure 2.1

neighborhoods homeomorphic to $Y_{1,1}$. We also know that ∂F is finite since it is a subset of A. Every point of F has a neighborhood homeomorphic to $Y_{1,1}$, i.e., the real line. Thus F is a connected 1-manifold with a countable basis and thus homeomorphic to either $(0,1)$ or S_1. F is homeomorphic to $(0,1)$ as otherwise K would be disconnected. We know this because if F is homeomorphic to S_1, then $\bar{F} = F$ and thus $\partial F = \emptyset$. As any path from a point in F to a point in $K - F$ must path through a branch point (a point in ∂F) no such path can exist. It is easy to see that ∂F must consist of one or two points and that $\bar{F}$ is homeomorphic to S_1 if ∂F consists of one point and $\bar{F}$ is homeomorphic to $(0,1)$ if ∂F consists of two points. We can see that $K - A$ consists of only finitely many components as follows. Let $A = \{a_1, \ldots a_l\}$, each a_i has a coordinate neighborhood (U_i, φ_i) such that $\varphi\colon U_i \to Y_{n(i),m(i)}$ where $\varphi_i(a_i) = (0,0)$ and we can choose the U_i so that $U_i \cap U_j = \emptyset$ for $i \neq j$. It is easy to see that $(K - A) \cap (\cup_{i=1}^{l} U_i)$ is equal to the disjoint union of $1/2\sum_{i=1}^{l}(n_i + m_i)$ sets each of which is homeomorphic to $(0,1)$ and that any component of $K - A$ must intersect two of these sets. Thus $K - A$ must consist of $1/2\sum_{i=1}^{l}(n_i + m_i)$ disjoints sets homeomorphic to $(0,1)$.

Note 5 From this point on we will always assume that a branched 1-manifold has been given the structure of a graph, i.e., that it is a smooth graph.

2.3 Solenoids

In what follows we define the generalized solenoids, their shift maps and give some of their elementary properties. Most of this material is based on the work of R.F. Williams in [38] and [39].

Definition 13 Let K be a smooth graph and $g: K \to K$ an immersion. A point $x \in K$ is *non-wandering* for g if for every neighborhood U of x there is an integer $n > 0$ such that $g^n(U) \cap U \neq \emptyset$. The set of non-wandering points is denoted by $\Omega(g)$. It is a closed g-invariant set and contains all recurrent behavior of g.

Definition 14 Let K be a smooth graph and $g: K \to K$ an immersion. Then g is a *W-mapping* if it satisfies the following conditions:

W 1. g is an expansion,

W 2. all points of K are non-wandering under g, i.e., $\Omega(g) = K$,

W 3. each point of K has a neighborhood N such that for some n , $g^n(N)$ is a 1-cell, i.e., an edge.

Note 6 Let K be a smooth graph and $g: K \to K$ a W-mapping. Condition W3 of definition 14 implies that for each branch point b of K there is a neighborhood N of b which is such that for some integer n the image of N under g^n is an edge of K. As g is an expanding immersion this is trivially true for any ordinary point.

Before proceeding to the definition of Solenoids we need to define inverse limit systems, inverse limit spaces and to review some elementary results concerning their properties. This material was found in [19] pages 91-96.

Definition 15 Let $X_0, X_1, X_2, \dots$ be a countable collection of spaces and suppose that for each $n \in \mathbb{N}$ there is a continuous mapping $f_n: X_n \to X_{n-1}$. The sequence of spaces and mappings $\{X_n, f_n\}$ is called an *inverse limit sequence* and may be represented by the following diagram,

$$\cdots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0.$$

It is easy to see that if $\{X_n, f_n\}$ is an inverse limit system and $n > m$ that there is a continuous mapping $f_{n,m}: X_n \to X_m$ formed from the composition $f_{n,m} = f_{m+1} \circ f_{m+2} \circ \cdots \circ f_{n-1} \circ f_n$.

Definition 16 Let $(x_0, x_1, \ldots, x_n, \ldots)$ be a sequence where each point x_n is a point of the space X_n and such that $x_n = f_{n+1}(x_{n+1})$ for all $n \geq 0$. Each such sequence can be identified with a point in the product space $\mathbb{P}_{n=0}^{\infty} X_n$ by considering the function $\varphi: \mathbb{Z}^+ \to \bigcup_{n=0}^{\infty} X_n$ given by $\varphi(n) = x_n$. The set of such sequences given in this way is a subset of $\mathbb{P}_{n=0}^{\infty} X_n$ and has a topology as a subspace. This topological space is called the *inverse limit space* of the sequence $\{X_n, f_n\}$. It will be denoted X_∞

The following results concerning inverse limit sequences and inverse limit spaces can be found in [19]. Theorem 2 is an existence theorem. It illustrates the fact that from any coordinate x_n toward the "front" of the sequence the coordinates of a point, i.e., x_m with $m < n$, are controlled absolutely by x_n but there is room for choice from x_n onward in the sequence.

Theorem 2 [19] *If $\{X_n, f_n\}$ is an inverse limit sequence, if each f_n is a mapping onto, and if $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$ is a set of points with, $n_1 < n_2 < \cdots < n_k < \cdots$, where $x_{n_i} \in X_{n_i}$ for $i = 1, 2, 3, \ldots$ and such that if $i < j$ then $f_{n_j, n_i}(x_{n_j}) = x_{n_i}$, then there is a point in X_∞ whose coordinate in X_{n_i} is x_{n_i} for $i = 1, 2, 3, \ldots$.*

Theorem 3 [19] *Suppose that each space X_n in the inverse limit sequence $\{X_n, f_n\}$ is a compact Hausdorff space. Then X_∞ is nonempty.*

There is a natural way to map two inverse limit sequences into each other which we will make frequent use of. Let $\{A_n, f_n\}$ and $\{B_n, g_n\}$ be two inverse limit sequences of spaces. A mapping $\Phi: \{A_n, f_n\} \to \{B_n, g_n\}$ is a collection of continuous mappings $\varphi_n: A_n \to B_n$ satisfying the condition $g_n \circ \varphi_n = \varphi_{n-1} \circ f_n$, $n \geq 1$. This condition may be given by stating that we have commutativity in the following diagram.

$$
\begin{array}{ccccccccccc}
\cdots \longrightarrow & A_n & \xrightarrow{f_n} & A_{n-1} & \xrightarrow{f_{n-1}} & A_{n-2} & \longrightarrow \cdots \longrightarrow & A_1 & \xrightarrow{f_1} & A_0 \\
& \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_{n-2}} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} \\
\cdots \longrightarrow & B_n & \xrightarrow[g_n]{} & B_{n-1} & \xrightarrow[g_{n-1}]{} & B_{n-2} & \longrightarrow \cdots \longrightarrow & B_1 & \xrightarrow[g_1]{} & B_0
\end{array}
$$

This means that we can pass from A_n to B_{n-1} in two different ways but the result is still the same. The mapping Φ induces a mapping $\varphi: A_\infty \to B_\infty$ on the inverse limit space as follows. For each point $a = (a_0, a_1, \ldots)$ we define $\varphi(a) = (\varphi_0(a_0), \varphi_1(a_1), \ldots)$. It follows immediately from the equations that $\varphi(a)$ is a point of B_∞.

Theorem 4 [19] *The mapping $\varphi: A_\infty \to B_\infty$ induced by the mapping $\Phi: \{A_n, f_n\} \to \{B_n, g_n\}$ is continuous.*

Definition 17 Let K be a branched 1-manifold and $g: K \to K$ a W-mapping. Then let K_∞ be the inverse limit space of the inverse limit sequence $\{K, g\}$, i.e., $K_i = K$ and $g_i = g$ for all i as shown in the diagram below.

$$\cdots \xrightarrow{g} K \xrightarrow{g} K \xrightarrow{g} \cdots \xrightarrow{g} K \xrightarrow{g} K$$

For a point $a = (a_0, a_1, a_2, \ldots) \in K_\infty$ let $h(a) = (g(a_0), a_0, a_1, \ldots)$ and $h^{-1}(a) = (a_1, a_2, a_3, \ldots)$. Then $h: K_\infty \to K_\infty$ is called the *shift map* and K_∞ is called the *solenoid*. We say that K_∞ is *presented* by $\{K, g\}$.

Note 7 The mapping $h: K_\infty \to K_\infty$ is a homeomorphism.

The inverse limit $T(K)_\infty$, of the tangent bundles

$$\cdots \xrightarrow{Dg} T(K) \xrightarrow{Dg} T(K) \xrightarrow{Dg} \cdots \xrightarrow{Dg} T(K) \xrightarrow{Dg} T(K)$$

is a line bundle over K_∞ and serves as a tangent bundle. The shift map h then induces a shift map Dh on $T(k)_\infty$.

Note 8 The classical solenoids [9],[35], [40] are those in which $K = S_1$ and g is an expanding map of degree=n with $n > 1$.

We now give several elementary results concerning the properties of W-mappings on branched 1-manifolds and the solenoids that they present. These were originally found in [38], [39]. They are given without proof. Throughout the remainder of this section $g: K \to K$ is an immersion of the branched 1-manifold K.

Theorem 5 [38] *If $g: K \to K$ satisfies W1 and W2 of definition 14 then K has an empty boundary.*

Note 9 If f is a mapping will use the notation Fix(f) to denote the fixed points of f.

Theorem 6 [38] *If $x \in Fix(g^n)$, then let*

$$a(x) = (x, g^{n-1}(x), g^{n-2}(x), \ldots, x, g^{n-1}(x), g^{n-2}(x), \ldots),$$

clearly $a(x) \in K_\infty$, $a(x) \in Fix(h^n)$ and there is a one to one correspondence $x \to a(x)$ between $Fix(g^n)$ and $Fix(h^n)$. If $g: K \to K$ satisfies W1 and W2 then periodic points of g are dense in K and the periodic points of h are dense in K_∞.

Two presentations $\{K, g\}$ and $\{K', g'\}$ of solenoids K_∞ , K'_∞ and their shift maps h, h' along with a map $f: K \to K'$ such that the following diagram is commutative

$$\begin{array}{ccc} K & \xrightarrow{g} & K \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ K' & \xrightarrow[g']{} & K' \end{array}$$

induce mappings $F_i: K_\infty \to K'_\infty$, for $i \in \mathbb{Z}$ by

$$F_i(x_0, x_1, \ldots) = \begin{cases} (f(x_i), f(x_{i+1}), \ldots) & \text{for } i \geq 0, \\ (g'^{-i} \circ f(x_0), g'^{-i} \circ f(x_1), \ldots) & \text{for } i \leq 0. \end{cases}$$

Definition 18 An F_i from K_∞ to K'_∞ for such an f is called *ladder map.*

These maps are interrelated as follows, $F_i = F_0 \circ h^i = h'^i \circ F_0$.

Theorem 7 *[39] The only maps between generalized solenoids which commute with their shift maps are the ladder maps*

Thus if $\theta: K_\infty \to K'_\infty$ is such that $\theta \circ h = h' \circ \theta$ then there is an $f: K \to K'$ such that $f \circ g = g' \circ f$ and an $i \in \mathbb{Z}$ such that $\theta = F_i$.

Theorem 8 *[39] A necessary and sufficient condition that the shift maps presented by $\{K, g\}$ and $\{K', g'\}$ be topologically conjugate is that there exist maps $r: K \to K'$ and $r': K' \to K$ and a positive integer m such that the following diagrams are commutative:*

$$\begin{array}{ccc} K & \xrightarrow{g} & K \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ K' & \xrightarrow[g']{} & K' \end{array} \quad \begin{array}{ccc} K & \xrightarrow{g} & K \\ {\scriptstyle r'}\uparrow & & \uparrow{\scriptstyle r'} \\ K' & \xrightarrow[g']{} & K' \end{array} \quad \begin{array}{ccc} K & \xrightarrow{g^m} & K \\ {\scriptstyle r}\downarrow & \nearrow{\scriptstyle r'} & \downarrow{\scriptstyle r} \\ K' & \xrightarrow[g'^m]{} & K' \end{array}$$

Definition 19 Two mappings g and g' are said to be *shift equivalent* if they satisfy the relations given above for some r, r' and m. We will use the notation $g \sim_s g'$ to denote shift equivalence. The integer m is called the *lag* of the shift equivalence, written lag$= m$.

It is straight-forward to verify that shift equivalence is an equivalence relation.

Definition 20 Let $g: K \to K$ be a W-mapping where $B \subset K$ is the branch set of K. g is a *W^*-mapping* if there is a finite set $A \subset K$ such that $B \subseteq A$ and $g(A) \subseteq A$.

Theorem 9 *[39] Suppose that $g: K \to K$ is a W-mapping then there is a W^*-mapping g' and a branched 1-manifold K', $g': K' \to K'$, such that $g \sim_s g'$.*

Thus every generalized solenoid can be presented by a W^*-mapping and a branched 1-manifold. We will therefore restrict our attention from here onwards to presentations of generalized solenoids given by W^*-mappings. It will also be assumed that when given a branched 1-manifold as a smooth graph $K = (G, S)$ and a W^*-mapping g from K to K the set of vertices V is chosen such that $g(V) \subseteq V$.

The following useful theorem gives a nice alternative to checking that a W-mapping g on K satisfies $W2$, i.e., that non-wandering set of is all of K.

Theorem 10 *[39] If $g: K \to K$ is an onto immersion satisfying $W1$ and $W3$ then g satisfies $W2$ if and only if there is an integer m such that g^m maps each edge e of K onto K.*

Chapter 3

Routes, Orientability and Recurrence

In this chapter we introduce two properties of smooth graphs, orientability and recurrence.

We first consider the problem of determining whether a given smooth graph is orientable or nonorientable. In doing so we introduce the structure matrix of a smooth graph and discuss methods of determining whether a smooth graph is orientable or nonorientable in terms of its structure matrix. We also devise a graph theoretic method for determining whether a given smooth graph is orientable or nonorientable.

Next we study the effect the orientability or nonorientability of a smooth graph has on the solenoids it presents. Given presentations of solenoids, $\{K, g\}$ and $\{K', g'\}$, we show that $g \sim_s g'$ implies either that K and K' are both orientable or that K and K' are both nonorientable. Thus solenoids come in two varieties (up to the topological conjugacy of their shift maps), those presented by orientable smooth graphs and those presented by nonorientable smooth graphs.

We show that every nonorientable smooth graph has an orientable double cover. This fact will be useful when we study recurrence.

We then turn to the problem of determining whether or not a smooth graph is recurrent. We show that if the smooth graph we are considering is orientable it is then possible to convert the problem of determining whether or not it is recurrent into a problem in graph theory for which efficient methods of solution exist. When the smooth graph that we are studying is nonorientable we show that it will be recurrent if and only if its orientable double cover is recurrent. Thus we can can use the same graph theoretic method to determine the recurrence of a nonorientable smooth graph by constructing its orientable double cover and applying this method to it.

Finally we study the effect the recurrence or non-recurrence of a smooth graph has on the solenoids it presents. We show that given a smooth graph K there exists a W^*-mapping on K if and only if K is recurrent.

Many of the ideas in this chapter were inspired by the work in [1].

3.1 Routes and Orientability

In this section we define what we mean by a route on a smooth graph. We define orientability, prove an elementary theorem concerning orientability and give some examples of orientable and nonorientable smooth graphs.

Throughout the rest of this chapter we will assume that $K = (G, S)$ is a connected smooth graph where $G = (V, E)$. For ease of presentation we also assume that K has no loops. If we are given a smooth graph K with loops we can apply the results of this section to it by forming the smooth graph K' from K by removing each loop e, where say $I(e) = v$, and replacing it with a walk $\{v, e', v', e'', v\}$ where $v' \notin V$ and $e', e'' \notin E$. The new switch conditions are then obtained as follows; $R_{K'}(v') = e'$, $L_{K'}(v') = e''$, if $e \in R_K(v)$ and $e \notin L_K(v)$ then $e', e'' \in R_{K'}(V)$, if $e \notin R_K(v)$ and $e \in L_K(v)$ then $e', e'' \in L_{K'}(v)$, and if $e \in R_K(v)$ and $e \in L_K(v)$ then $e' \in R_{K'}(v)$ and $e'' \in L_{K'}(v)$. The smooth graphs K and K' agree on all edges which are not loops. K and K' clearly represent the same branched 1-manifold.

Definition 21 A *route* on a smooth graph K is a walk $W = \{x_0, y_1, x_1, \ldots, x_{n-1}, y_n, x_n\}$ such that at the vertex x_i the edges y_i and y_{i+1} are opposite for all $i = 1 \ldots n-1$. A route is said to be *closed* if $x_0 = x_n$ and y_0 is opposite y_n at x_0.

A route on K can be interpreted as a walk on K which takes no "sharp" turns, i.e., a walk which when passing through a vertex always passes to an edge opposite the edge upon which it entered the vertex. It is easy to see that the existence of a route $W = \{x_0, y_1, x_1, \ldots, x_{n-1}, y_n, x_n\}$ implies the existence of an immersion $f\colon [0,1] \to K$, where $f(i/n) = x_i$ for $i = 0, 1, \ldots, n$ and $f(((i-1)/n, i/n)) = y_i$ for $i = 1 \ldots, n$. Similarly if the route W is closed then this implies the existence of an immersion $f\colon S_1 \to K$ where

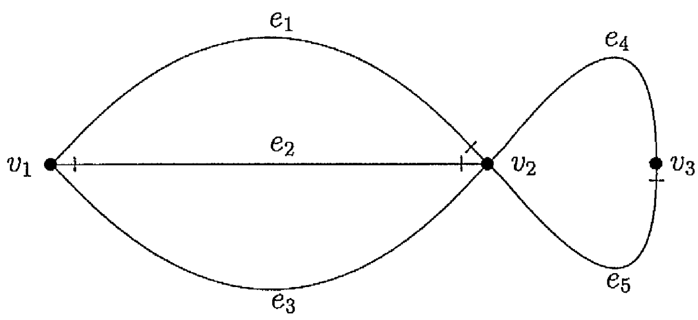


Figure 3.1: The diagram for the smooth graph K given in example 3

again $f(i/n) = x_i$ for $i = 0, 1, \ldots, n$ and $f(((i-1)/n, i/n)) = y_i$ for $i = 1 \ldots, n$. Note that here we are considering S_1 to be a circle with circumference equal to 1.

Example 3 Let K be the smooth graph where $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ and with the incidence relations and switch conditions given in the table below.

v	$R(v)$	$L(v)$
v_1	$\{e_2\}$	$\{e_1, e_3\}$
v_2	$\{e_1, e_2\}$	$\{e_3, e_4, e_5\}$
v_3	$\{e_5\}$	$\{e_4\}$

A diagram for K is given in figure 3.1. Consider $W_1 = \{v_1, e_2, v_2, e_4, v_3, e_5, v_2, e_2, v_1, e_3, v_2\}$ and $W_2 = \{v_2, e_5, v_3, e_4, v_2, e_1, v_1, e_2, v_2\}$. It is easily checked that W_1 and W_2 are routes and that W_2 is a closed route.

Definition 22 A smooth graph K is said to be *orientable* if there exists an orientation ρ on K such that at each vertex v either:

1. $I^+_\rho(v) = R(v)$ and $I^-_\rho(v) = L(v)$,

2. $I^-_\rho(v) = R(v)$ and $I^+_\rho(v) = L(v)$.

A graph which is not orientable is said to be *nonorientable.* An orientation which satisfies 1.) and 2.) is said to be a *coherent orientation.*

The definition of orientability suggests an obvious method for determining whether or not a given smooth graph is orientable, try all orientations and check whether or not one satisfies definition 22. This simple minded scheme is of course very tedious for smooth graphs of any size since the number of orientations we must check for any given smooth graph with edge set E is $2^{|E|}$. There are much more efficient ways to determine orientability which we will develop in this chapter.

Theorem 11 *Let K be a connected smooth graph. If there exists a route W on K which contains the subroutes (subsequences), $W_1 = \{v, e, v'\}$ and $W_1 = \{v', e, v\}$ for some vertices v, v', $v \neq v'$ and some edge e then K is nonorientable.*

Proof. Suppose that there exists a route on K which contains the subroutes $W_1 = \{v, e, v'\}$ and $W_2 = \{v', e, v\}$ where $v \neq v'$ and suppose further that K is orientable. Without loss of generality we can assume that the subroute W_1 occurs before the subroute W_2 in W and that there are no other subroutes of this form between W_1 and W_2. For any orientation ρ on K there are two choices for the edge e, $\rho(e) = v$ and $\rho(e) = v'$. Suppose that between W_1 and W_2 in W is the route $W_3 = \{c_0, d_1, c_1, \ldots, d_m, c_m\}$ where $c_0 = v'$ and $c_m = v'$. If e is given the orientation $\rho(e) = v$ then the orientability of K implies that d_i has must have orientation $\rho(d_i) = c_{i-1}$ for $i = 1 \ldots m$ since d_1 is opposite e and d_i is opposite d_{i+1}. Thus $\rho(d_1) = \rho^*(d_m)$, i.e., $d_1 \in I_\rho^+(v')$ and $d_m \in I_\rho^-(v')$. Also both d_1 and d_m are opposite e and thus on the same side of the vertex v' hence either $d_1, d_m \in R(v')$ or $d_1, d_m \in L(v')$. Clearly choosing the conjugate orientation for e yields $d_1 \in I_\rho^-(v')$ and $d_m \in I_\rho^+(v')$. This is a contradiction since K was chosen to be orientable. Thus we are left with two possibilities; 1.) there is no orientation ρ on K for which at every vertex v either $I_\rho^+(v) = R(v)$ and $I_\rho^-(v) = L(v)$ or $I_\rho^-(v) = R(v)$ and $I_\rho^+(v) = L(v)$, 2.) no route of this form exists. □

This result in theorem 11 can be summarized by saying that if a smooth graph has a route on it which traverses an edge in more than one direction then it is nonorientable. The contrapositive, which is also useful, states that an orientable smooth graph has no routes which traverse an edge in more than one direction. The converse to this, however, is not true. A counter example is given below. We will be able to prove a complete converse to this theorem in the special case that the smooth graph in question is recurrent.

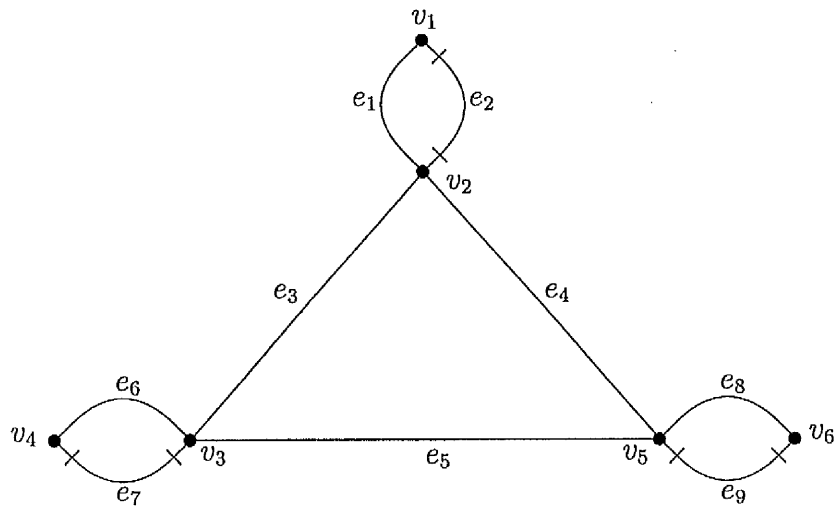


Figure 3.2: Diagram of the nonorientable smooth graph K given in example 4

Example 4 Consider the smooth graph K where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ and the incidence relations and switch conditions are given in the following table.

v	$R(v)$	$L(v)$	v	$R(v)$	$L(v)$
v_1	$\{e_2\}$	$\{e_1\}$	v_4	$\{e_7\}$	$\{e_6\}$
v_2	$\{e_2\}$	$\{e_1, e_3, e_4\}$	v_5	$\{e_9\}$	$\{e_4, e_5, e_8\}$
v_3	$\{e_7\}$	$\{e_3, e_6, e_5\}$	v_6	$\{e_9\}$	$\{e_8\}$

The diagram of K is given in figure 3.2. We can easily check by trial and error that no route traverses any edge in more that one direction and that it is impossible give K an orientation. As the chapter progresses we will develop techniques which make it checking these features more systematic. We encourage the reader to come back and apply these new techniques to this example.

Example 5 An orientable smooth graph. Consider the smooth graph K where $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and the incidence relations and switch conditions are summarized in the following table.

v	$R(v)$	$L(v)$
v_1	$\{e_2\}$	$\{e_1, e_4\}$
v_2	$\{e_4\}$	$\{e_3, e_6\}$
v_3	$\{e_2\}$	$\{e_3, e_5\}$
v_4	$\{e_5\}$	$\{e_1, e_6\}$

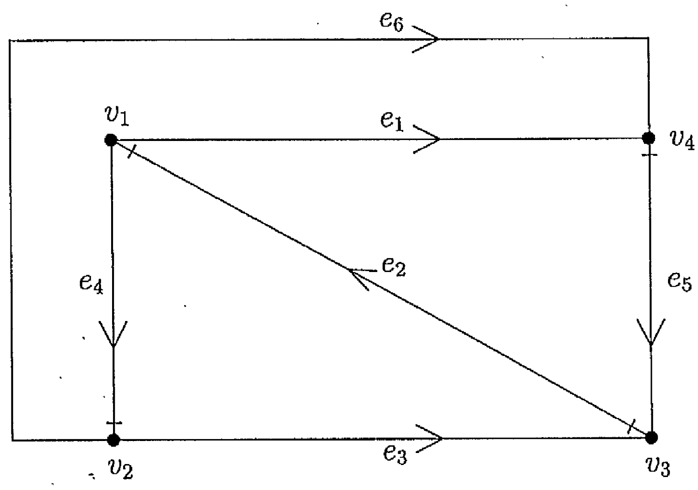


Figure 3.3: Diagram of the orientable smooth graph K given in example 5

A diagram for K is given in figure 3.3. A coherent orientation on K can be given by $\rho(e_1) = v_4$, $\rho(e_2) = v_1$, $\rho(e_3) = v_3$, $\rho(e_4) = v_2$, $\rho(e_5) = v_3$, and $\rho(e_6) = v_4$. In the diagram this orientation is represented by a small arrow head on each edge e pointing to the vertex toward which e is oriented.

Example 6 A nonorientable smooth graph. Consider the smooth graph K where $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and the incidence relations and switch conditions are summarized in the following table.

v	$R(v)$	$L(v)$
v_1	$\{e_2\}$	$\{e_1, e_4\}$
v_2	$\{e_6\}$	$\{e_3, e_4\}$
v_3	$\{e_2\}$	$\{e_3, e_5\}$
v_4	$\{e_6\}$	$\{e_1, e_5\}$

A diagram for K is given in figure 3.4. The nonorientability of K can be verified by considering the route $W = \{v_4, e_6, v_2, e_4, v_1, e_2, v_3, e_3, v_2, e_6, v_4\}$ on K. This route contains the subroutes $W_1 = \{v_4, e_6, v_2\}$ and $W_2 = \{v_2, e_6, v_4\}$ thus by theorem 11 it is nonorientable.

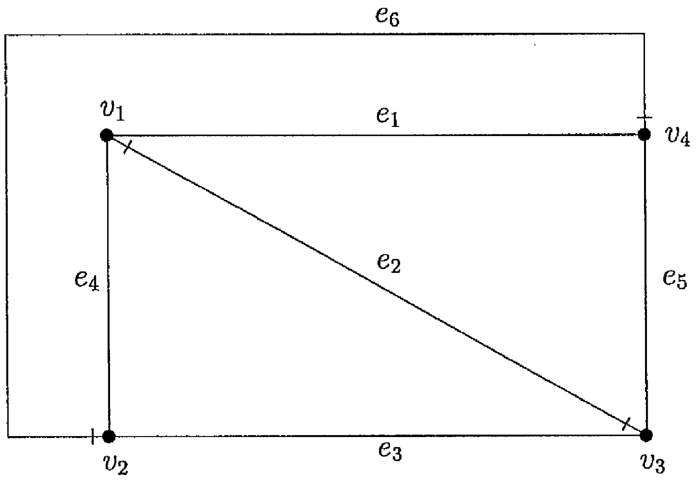


Figure 3.4: Diagram of the nonorientable smooth graph K given in example 6

3.2 The Structure Matrix of a Smooth Graph

In this section we introduce the structure matrix of a smooth graph. We prove that a smooth graph is orientable if and only if the rank of its structure matrix is one less than the number of vertices in the graph. We also show how the structure matrix can be used to find a coherent orientation on an orientable smooth graph and illustrate the relationship between the structure matrix of an orientable smooth graph and its incidence matrix.

Throughout this section we assume that K is a connected smooth graph which is given an arbitrary vertex labeling $V = \{v_1, v_2, \ldots, v_n\}$ and edge labeling $E = \{e_1, e_2, \ldots, e_m\}$.

Definition 23 The *structure matrix* $\mathbf{J}$ of a smooth graph K is the $n \times m$ matrix (J_{ij}) whose entries are

$$J_{ij} = \begin{cases} 1 & \text{if } e_j \in R(v_i) \\ -1 & \text{if } e_j \in L(v_i) \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of the structure matrix $\mathbf{J}$ of a smooth graph K several things are readily apparent. Each column of $\mathbf{J}$ contains exactly two nonzero entries which can be 1 and a -1, two 1's, or two -1's. This is because every edge is incident upon exactly two vertices. Recall we assumed that there are no loops. Each row of $\mathbf{J}$ contains at least one nonzero entry which must be 1 or -1. This is because the smooth graph K is assumed to be connected and thus every vertex must have at least one incident edge.

Example 7 The structure matrix $\mathbf{J}$ for the orientable smooth graph in example 5 is given by

$$\mathbf{J} = \begin{bmatrix} -1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Theorem 12 *A smooth graph K with structure matrix $\mathbf{J}$ is orientable if and only if there exists* $\mathbf{c}$ *such that* $\mathbf{cJ} = \mathbf{0}$ *where* $\mathbf{c} = [c_1, c_2, \ldots, c_n]$, $c_i = \pm 1$ *for* $i = 1, \ldots, n$, *and* $\mathbf{0}$ *is the* $1 \times m$ *zero matrix.*

Proof. Suppose that K is an orientable smooth graph with structure matrix $\mathbf{J}$. Let ρ be a coherent orientation on K where for each v_i either;

1. $I_\rho^+(v_i) = R(v_i)$ and $I_\rho^-(v_i) = L(v_i)$, or
2. $I_\rho^-(v_i) = R(v_i)$ and $I_\rho^+(v_i) = L(v_i)$.

Consider $\mathbf{c} = [c_1, c_2, \ldots, c_n]$ where $c_i = 1$ if v_i satisfies 1.) above and $c_i = -1$ if v_i satisfies 2.) above. The jth term $(\mathbf{cJ})_j$ of the $1 \times m$ matrix $\mathbf{cJ}$ is of the form

$$(\mathbf{cJ})_j = \sum_i^n c_i J_{ij}.$$

For any j this sum only contains two terms $c_p J_{pj}$ and $c_q J_{qj}$ where the edge e_j is incident upon the two vertices v_p and v_q. Thus we have $(\mathbf{cJ})_j = c_p J_{pj} + c_q J_{qj}$. There are three cases that we need to consider; e_j is right at v_p and left at v_q in which case $(\mathbf{cJ})_j = c_p - c_q$, e_j is right at both v_p and v_q in which case $(\mathbf{cJ})_j = c_p + c_q$, and e_j is left at both v_p and v_q in which case $(\mathbf{c}J)_j = -(c_p + c_q)$. If e_j is right at v_p and left at v_q then either $\rho(e_j) = v_p$ or $\rho(e_j) = v_q$. In the first instance we find that $c_p = c_q = 1$ in the second that $c_p = c_q = -1$. Thus we have $(\mathbf{c}J)_j = 0$. If e_j is right at both v_p and v_q then either $\rho(e_j) = v_p$ or $\rho(e_j) = v_q$. In the first instance $c_p = 1$ and $c_q = -1$ in the second instance we have $c_p = -1$ and $c_q = 1$. Thus we have $(\mathbf{cJ})_j = 0$. If e_j is left at both v_p and v_q then either $\rho(e_j) = v_p$ and $\rho(e_j) = v_q$. In the first instance we have $c_p = -1$ and $c_q = 1$ and in the second instance we have $c_p = 1$ and $c_q = -1$. Thus $(\mathbf{cJ})_j = 0$. Therefore we have shown that if K is an orientable smooth graph with structure matrix $\mathbf{J}$ then there exits $\mathbf{c} = [c_1, \ldots, c_n]$ where $c_i = \pm 1$ such that $\mathbf{cJ} = \mathbf{0}$.

Conversely suppose that K is a smooth graph with structure matrix $\mathbf{J}$ and that $\mathbf{c} = [c_1, \ldots, c_n]$ is such that $\mathbf{cJ} = \mathbf{0}$. Consider the orientation ρ on K where $\rho(e_j) = v_i$ if $c_i = 1$ and e_j is right at v_i, $\rho^*(e_j) = v_i$ if $c_i = 1$ and e_j is left at v_i, $\rho(e_j) = v_i$ if $c_i = -1$ and e_j is left at v_i, and $\rho^*(e_j) = v_i$ if $c_i = -1$ and e_j is right at v_i. Note, when defining the orientation $\rho(e)$ of an edge e in terms of its conjugate $\rho^*(e)$ we mean that $\rho(e) = b$ where b is the only element in $I(e) - \{\rho^*(e)\}$. We need to show that this orientation is well defined for the whole smooth graph K, i.e., that we are not giving any edge more than one orientation. This consists of tediously going through all possible cases. Consider the edge e_j and suppose that $I(e_j) = \{v_p, v_q\}$. There are three possibilities to consider. If e_j is right at v_p and left at v_q then we must have either $c_p = c_q = 1$ or $c_p = c_q = -1$ since $(\mathbf{cJ})_j = c_p - c_q = 0$. In the first instance we have $\rho(e_j) = v_p$ in the second instance we get $\rho(e_j) = v_q$. If e_j is right at both v_p and v_q we must have either $c_p = -c_q = 1$ or $-c_p = c_q = 1$ since $(\mathbf{cJ})_j = c_p + c_q = 0$. In the first instance we get $\rho(e_j) = v_p$ in the second instance we get $\rho(e_j) = v_q$. Lastly if e_j is left at both v_p and v_q then we must have either $c_p = -c_q = 1$ or $-c_p = c_q = 1$ since $(\mathbf{cJ})_j = -(c_p + c_q) = 0$. In the first instance $\rho(e_j) = v_q$ and in the second instance $\rho(e_j) = v_p$. We claim that the orientation ρ so defined is such that for all v_i either 1.) $I_\rho^+(v_i) = R(v_i)$ and $I_\rho^-(v_i) = L(v_i)$ or 2.) $I_\rho^-(v_i) = R(v_i)$ and $I_\rho^+(v_i) = L(v_i)$, i.e., ρ is a coherent orientation on K. For some i suppose that $c_i = 1$. If an edge e_j is right at v_i, i.e., $e_j \in R(v_i)$ we have $\rho(e_j) = v_i$ and thus $e_j \in I_\rho^+(v_i)$. If an edge e_j is left at v_i then $\rho^*(e_j) = v_i$ and thus $e_j \in I_\rho^-(v_i)$. Since $R(v_i) \cup L(v_i) = I(v_i)$ and either $e_j \in R(v)$ or $e_j \in L(v)$ but not both since K has no loops we see that 1.) above is satisfied. Suppose for some i that $c_i = -1$. If an edge e_j is right at v_i then $\rho^*(e_j) = v_i$ and thus $e_j \in I^-\rho(v_i)$. If an edge e_j is left at v_i then $\rho(e_j) = v_i$ and $e_j \in I_\rho^+(v_i)$. Again since $R(v_i) \cup L(v_i) = I(v_i)$ and either $e_j \in R(v)$ or $e_j \in L(v)$ but not both since K has no loops we see that 2.) above is satisfied. Therefore we have shown that if K is a smooth graph with structure matrix $\mathbf{J}$ and there exists a $\mathbf{c} = [c_1, \ldots, c_n]$ such that $\mathbf{cJ} = \mathbf{0}$ where $c_i = \pm 1$ then K is orientable. □

This result immediately suggests another way to test whether or not a given smooth graph K is orientable, find the structure matrix $\mathbf{J}$ of K and check to see if there exists a $\mathbf{c} = [c_1, \ldots, c_n]$, where $c_i = \pm 1$, such that $\mathbf{cJ} = 0$. This is fairly easy to do and it is lots quicker then looking at all $2^{|E|}$ possible orientations. Such a solution also gives us a way of finding a coherent orientation on K. It also suggests a relationship between the rank of the structure matrix $\mathbf{J}$ and orientability which we will investigate in the next theorem.

Example 8 In example 7 we gave the structure matrix $\mathbf{J}$ for the orientable smooth graph K given in example 5. In order to find a $\mathbf{c}$ such that $\mathbf{cJ} = \mathbf{0}$, where $c_i = \pm 1$ we can consider the equivalent set of linear equations given by $\mathbf{J}^T\mathbf{c}^T = \mathbf{0}$, where in this case $\mathbf{0}$ is the $m \times 1$ zero matrix. We may then form the augmented matrix $\mathbf{A} = [\,\mathbf{J}^T \,|\, \mathbf{0}\,]$ and perform elementary row operations on $\mathbf{A}$ until it is in reduced row-echelon form as shown below.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is then a simple matter to the read possible solutions for $\mathbf{c}$. In this case we have $c_1 = -c_4$, $c_2 = -c_4$, $c_3 = c_4$ and $c_4 = \pm 1$. Thus the two possible solutions for $\mathbf{c}$ are $[-1,-1,1,1]$ and $[1,1,-1,-1]$. These then can be used to specify the two possible coherent orientations ρ and ω on K using the procedure described in the proof of theorem 12, these are, respectively; $\rho(e_1) = v_4$, $\rho(e_2) = v_1$, $\rho(e_3) = v_3$, $\rho(e_4) = v_2$, $\rho(e_5) = v_3$, $\rho(e_6) = v_4$ and $\omega(e_1) = v_1$, $\omega(e_2) = v_3$, $\omega(e_3) = v_2$, $\omega(e_4) = v_1$, $\omega(e_5) = v_4$, $\omega(e_6) = v_2$. We then also see that $\omega = \rho^*$.

Theorem 13 *Let K be a smooth graph with $|V| = n$ and structure matrix $\mathbf{J}$. Then*

1. *K is orientable if and only if $\mathbf{J}$ has rank $n-1$,*

2. *K is nonorientable if and only if $\mathbf{J}$ has rank n.*

Proof. First we will demonstrate that K is orientable if and only if $\mathbf{J}$ has rank $n-1$. Suppose that K is an orientable smooth graph and let $\mathbf{c} = [c_1, \ldots, c_n]$ be a solution to $\mathbf{cJ} = 0$ where $c_i = \pm 1$. Consider the matrix $\mathbf{A} = \mathbf{CJ}$ where $\mathbf{C} = \mathrm{diag}(c_1, \ldots, c_n)$. It is clear that $\mathbf{A}$ and $\mathbf{J}$ must have the same rank since $\mathbf{A}$ is obtained from $\mathbf{J}$ by multiplying its rows by ± 1. Let $(\mathbf{A})_j$ denote the j-th row of $\mathbf{A}$. Since there is only one $+1$ and one -1 in each column of $\mathbf{A}$, it follows that the sum of the rows of $\mathbf{A}$ is the $1 \times m$ zero matrix and that the rank of **A** is at most $n-1$. Suppose that we have a linear relation

$$\sum_{j=1}^{n} \alpha_j (\mathbf{A})_j = \mathbf{0},$$

where not all the coefficients α_j are zero. Choose a row $(\mathbf{A})_k$ for which the coefficient $\alpha_k \neq 0$. This row will have nonzero entries in the columns corresponding to the edges incident with v_k. For each such column, there is just one other row $(\mathbf{A})_l$ with a nonzero entry in that column and in order for the given linear relation to hold we must have $\alpha_l = \alpha_k$. Thus, if $\alpha_k \neq 0$, then $\alpha_l = \alpha_k$ for all vertices v_l adjacent to v_k. Since K is connected it follows that all coefficients α_i are equal and so the given linear relation is just a multiple of

$$\sum_{j=1}^{n} (\mathbf{A})_j = \mathbf{0}.$$

Consequently the rank of **A** is $n-1$.

Conversely suppose that K is a smooth graph with structure matrix **J** and that **J** has rank $n-1$. In order to show that K is orientable we will construct a $\mathbf{c} = [c_1, \ldots, c_n]$ such that $\mathbf{cJ} = 0$, where $c_i = \pm 1$. Let $(\mathbf{J})_j$ denote the j-th row of **J**. Since **J** has rank 1, there must exist α_j, not all zero, such that the linear relation,

$$\sum_{j=1}^{n} \alpha_j (\mathbf{J})_j = \mathbf{0},$$

is satisfied. Choose a row $(\mathbf{J})_k$ for which the coefficient $\alpha_k \neq 0$. This row will have nonzero entries in the columns corresponding to the edges incident upon the vertex v_k. For each such column there is just one other row $(\mathbf{J})_l$ with nonzero entry in that column. There are two cases to consider. If the edge e_p incident with v_k and v_l is right (left) at v_k and left (right) at v_l then $\alpha_k = -\alpha_l$. If the edge e_p incident with v_k and v_l is right (left) at v_k and right (left) at v_l then $\alpha_k = \alpha_l$. Thus, if $\alpha_k \neq 0$, we must have $\alpha_l = \pm\alpha_k$ for vertices v_l adjacent to v_k. Since K is connected we also must have $\alpha_i = \pm\alpha$ for some α. We may then choose our **c** to be $\mathbf{c} = [\alpha_1/\alpha, \alpha_2/\alpha, \ldots, \alpha_n/\alpha]$. It is then easy to see that $\mathbf{cJ} = \mathbf{0}$ and thus that K is orientable.

Next we will demonstrate that K is nonorientable if and only if **J** has rank n. Suppose that K is a nonorientable smooth graph with structure matrix **J**. We know that there does not exist a **c** such that $\mathbf{cJ} = \mathbf{0}$ with $c_i = \pm 1$. We again let $(\mathbf{J})_j$ denote that j-th row of $\mathbf{J}$. Suppose there exist coefficients α_i, not all zero, such that the linear relation,

$$\sum_{j=1}^{n} \alpha_j (\mathbf{J})_j = \mathbf{0},$$

is satisfied. We know that the existence of such α_i implies the existence of a $\mathbf{c} = [\alpha_1/\alpha, \ldots, \alpha_n/\alpha]$ such that $\mathbf{cJ} = \mathbf{0}$, where $\alpha = \pm\alpha_i$. This is a contradiction since K was chosen to be nonorientable. Thus $\mathbf{J}$ has rank n.

Conversely suppose that K is a smooth graph with structure matrix $\mathbf{J}$ and that $\mathbf{J}$ has rank n. Let $(\mathbf{J})_j$ denote the j-th row of $\mathbf{J}$. Suppose there exists a $\mathbf{c}$, where $c_i = \pm 1$, such that $\mathbf{cJ} = \mathbf{0}$. This would clearly imply that the linear relation,

$$\sum_{j=1}^{n} c_i (\mathbf{J})_j = \mathbf{0},$$

is satisfied. This is a contradiction since $\mathbf{J}$ was chosen to have rank n. Thus K is nonorientable. □

This gives us a very simple method for determining whether or not a given smooth graph K is orientable. Construct the structure matrix $\mathbf{J}$ for K and calculate its rank. If we want to find a coherent orientation on K we must actually find a $\mathbf{c}$ such that $\mathbf{cJ} = \mathbf{0}$ where $c_i = \pm 1$. There are standard techniques in linear algebra for doing both of these things which are reasonably efficient.

Example 9 We can see very quickly from example 8 that the structure matrix $\mathbf{J}$ for the smooth graph K in example 5 has rank 3 and since $|V| = 4$ that K is orientable. Similarly constructing that structure matrix J for the smooth graph K in example 6 we find that J has rank 4 and since $|V| = 4$ that K is nonorientable. We also find that the structure matrix $\mathbf{J}$ of the smooth graph K in example 4 has rank 6. Thus, since $|V| = 6$, K is nonorientable.

An easy corollary of theorem 13 is that there are only two possible coherent orientations on any orientable smooth graph K. This is because the structure matrix $\mathbf{J}$ of K always has rank $|V| - 1$ and thus the system of equations given by $\mathbf{cJ} = \mathbf{0}$ depends upon only 1 parameter, say c_n, and c_n is only allowed to take two possible values ± 1. Each choice of c_n then corresponds to a choice of orientation. The two possible orientations are, also, always conjugate to each other.

We are now in a position to illustrate the relationship between the structure matrix of a smooth graph and the incidence matrix of a graph with an orientation.

Definition 24 The *incidence matrix* $\mathbf{I}$ of a graph G with orientation ρ is the $n \times m$ matrix with entries (I_{ij}) given by

$$I_{ij} = \begin{cases} 1 & \text{if } \rho(e_j) = v_i, \\ -1 & \text{if } \rho^*(e_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that K is an orientable smooth graph with structure matrix $\mathbf{J}$. Then there exists a $\mathbf{c} = [c_1 \ldots, c_n]$ such that $\mathbf{cJ} = \mathbf{0}$, where $c_i = \pm 1$. As we have seen a particular choice of $\mathbf{c}$ determines an orientation ρ on K. The matrix $\mathbf{I} = \mathbf{CJ}$, where $\mathbf{C}$ =diag$(c_1, c_2, \ldots, c_n)$, is then easily seen to have the structure of an incidence matrix and is in fact the incidence matrix for K with orientation ρ. We also see that any route $W = \{x_0, y_1, x_1, \ldots, y_k, x_k\}$ on K is a walk where either $\rho(y_i) = x_i$ for all $i = 1, \ldots, k$ or $\rho^*(y_i) = x_i$ for all $i = 1, \ldots, k$.

Definition 25 Let G be a graph with orientation ρ. An *oriented walk*, $W = \{x_0, y_1, x_1, \ldots, y_k, x_k\}$ on G is a walk where $\rho(y_i) = x_i$ for all $i = 1, \ldots, k$.

Thus on an orientable smooth graph K with a coherent orientation ρ every route W is an oriented walk with respect to either ρ or ρ^*.

The edges e of a smooth graph K are found be of three possible types; right at both incident vertices, right at one incident vertex and left at the other, or left at both incident vertices. We refer to these types as RR,RL, and LL respectively.

If $K = (G, S)$ is a smooth graph we can change the switch conditions at any vertex v by interchanging the sets of right and left edges without effecting the branched 1-manifold being represented. This is because $Y_{p,q}$ is diffeomorphic to $Y_{q,p}$ via the mapping $f(x, y) = (-x, y)$. In doing so we obtain a new smooth graph $K = (G, S')$.

Definition 26 Let $K = (G, S)$ and $K' = (G', S')$ be smooth graphs. We say that there is a smooth graph *homomorphism* (*isomorphism*) from K to K' if there exists a homomorphism (isomorphism) (f, g) from the graph G to G' which is such that for each $v \in V_G$ either:

1. $g \circ R_G(v) = R_{G'} \circ f(v)$ and $g \circ L_G(v) = L_{G'} \circ f(v)$,

2. $g \circ R_G(v) = L_{G'} \circ f(v)$ and $g \circ L_G(v) = R_{G'} \circ f(v)$.

It is easy to see that if K and K' are isomorphic smooth graphs then they are diffeomorphic as branched 1-manifolds. This then leads us to our next theorem.

Theorem 14 *An orientable smooth graph K is isomorphic to an orientable smooth graph K' whose edges are all of type RL.*

Proof. Let K be an orientable smooth graph with coherent orientation ρ, structure matrix $\mathbf{J}$, vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. Associated with ρ there is a $\mathbf{c} = [c_1, \ldots, c_n]$ where $c_i = \pm 1$. Let $\mathbf{C}$ =diag$\{c_1, \ldots, c_n\}$ and $\mathbf{A} = \mathbf{CJ}$. A is the incidence matrix of K with orientation ρ. Construct the smooth graph K' with vertex set $V' = \{v'_1, \ldots, v'_n\}$, edge set $E' = \{e'_1, \ldots, e'_m\}$. Define $f: V \to V'$ and $g: E \to E'$ such that $f(v_j) = v'_i$ and $g(e_j) = e'_j$. Define the switch conditions and incidence relations of the vertex v_i of K' to be $R_{K'}(v'_i) = g \circ R_K(v_i)$, $L_{K'}(v'_i) = g \circ L_K(v_i)$ if $c_i = 1$ and $R_{K'}(v'_i) = g \circ L_K(v_i)$, $L_{K'}(v'_i) = g \circ R_K(v_i)$ if $c_i = -1$. The matrix $\mathbf{A}$ is then clearly seen to be the structure matrix of K'. Since $\mathbf{A}$ has the structure of an incidence matrix we know that each column of $\mathbf{A}$ has exactly one $+1$ and one -1. Thus all of the edges of K' are of type RL. Also, (f, g) is an isomorphism from K to K'. □

Theorem 15 *A smooth graph K whose edges are all of type RL is orientable.*

Proof. Let K be a smooth graph with n vertices, structure matrix $\mathbf{J}$ and whose edges are all of type RL. Each column of $\mathbf{J}$ clearly contains exactly one $+1$ and one -1. Thus the sum of the rows of $\mathbf{J}$ is zero and $\mathbf{J}$ has rank $n-1$. Therefore K is orientable. □

3.3 Trees and cycles

We will now consider the question of orientability for two types of smooth graphs with boundary points; trees and cycles. In doing so we will obtain an alternative method for determining whether or not a smooth graph is orientable as well as some results which will be useful when trying to determine if a nonorientable smooth graph is recurrent.

Definition 27 A smooth graph K is a *cycle* if it consists of a closed walk where all vertices and edges are distinct. We will refer to a smooth graph which is a cycle as a *smooth cycle*.

Definition 28 A smooth graph K is a *tree* if it is non-empty, connected and contains no subgraphs which are cycles. We will refer to a smooth graph which is a tree as a *smooth tree*.

Theorem 16 *A smooth tree is orientable.*

Proof. Let K be a smooth tree with n vertices. K will then have $n-1$ edges. The structure matrix $\mathbf{J}$ for K will then be a $n \times (n-1)$ matrix and thus has rank $n-1$. Therefore K is orientable. □

Theorem 17 *Let K be a smooth cycle and N_{RR} denote the number of RR edges in K and N_{LL} denote the number of LL edges in K. Then K is orientable if and only if $N_{RR}+N_{LL}$ is even.*

Proof. Let K be a smooth cycle with n vertices and n edges. We can label the vertices $v_1, \ldots, v_n$ and the edges $e_1, \ldots, e_n$ of K so that the structure matrix $\mathbf{J}$ of K has the following form;

$$\mathbf{J} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 & b_n \\ b_1 & a_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_2 & a_3 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & & & b_{n-2} & a_{n-1} & 0 \\ 0 & \cdots & & & 0 & b_{n-1} & a_n \end{bmatrix},$$

where $a_i, b_i \in \{-1, +1\}$. For $i = 1 \ldots n-1$ let $\alpha_i = 1$ if e_i is an RL edge and $\alpha_i = -1$ if e_i is an RR or LL edge. Let R_i stand for row i of $\mathbf{J}$. By performing elementary row operations to $\mathbf{J}$; $R_2 \mapsto R_2 + \alpha_1 R_1$, $R_3 \mapsto R_3 + \alpha_2 R_2, \ldots, R_n \mapsto R_n + \alpha_{n-1} R_{n-1}$, we may convert the matrix $\mathbf{J}$ to a matrix $\mathbf{A}$ of the following form;

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & \cdots & & 0 & \sigma_1 \\ 0 & a_2 & & & 0 & \sigma_2 \\ 0 & 0 & a_3 & & 0 & \sigma_3 \\ \vdots & & & \ddots & 0 & \vdots \\ & & & & a_{n-1} & \sigma_{n-1} \\ 0 & \cdots & & & 0 & a_n + \sigma_n \end{bmatrix},$$

where $\sigma_1 = b_n$, $\sigma_2 = b_n \alpha_1$, ..., $\sigma_n = b_n \prod_{i=1}^{n-1} \alpha_i$. The matrix $\mathbf{A}$ will have rank $n-1$ if and only if $a_n + \sigma_n = 0$. Thus K will be orientable if and only if $a_n + \sigma_n = 0$. $\prod_{i=1}^{n-1} \alpha_i = 1$ if the number of RR edges plus the number of LL edges in the set $\{e_1, \ldots, e_{n-1}\}$ is even and $\prod_{i=1}^{n-1} \alpha_i = -1$ if the number of RR edges plus the number of LL edges in the set $\{e_1, \ldots, e_{n-1}\}$ is odd. Suppose that e_n is an RR or LL edge. Then

$$a_n + \sigma_n = \pm \left(1 + \prod_{i=1}^{n-1} \alpha_i\right) = 0$$

if and only if the number of RR edges plus the number of LL edges in the set $\{e_1, \ldots, e_{n-1}\}$ is odd. Thus K is orientable if and only if the number of RR edges plus the number of LL edges in K is even. Suppose that e_n is an RL edge. Then

$$a_n + \sigma_n = \pm \left(1 - \prod_{i=1}^{n-1} \alpha_i\right) = 0$$

if and only if the number of RR edges plus the number of LL edges in the set $\{e_1, \ldots, e_{n-1}\}$ is even. Thus K is orientable if and only if the number of RR edges plus the number of LL edges is even. □

In order to prove our next theorem will need the following well known result from graph theory.

Theorem 18 [4] *A non-empty graph has a spanning tree if and only if it is connected.*

Let K be a connected graph and T a spanning tree of K. Let E_T be the set of edges in T. The remaining edges belonging to $E_K - E_T$ are called the *chords* of T. For every chord $e \in E_K - E_T$ there is a unique cycle ω_e in K, see Bollobás, [3], page 37.

Theorem 19 *A connected smooth graph K is nonorientable if and only if there exists a cycle ω on K for which $N_{RR}(\omega) + N_{LL}(\omega)$ is odd, where $N_{RR}(\omega)$ is the number of RR edges in ω and $N_{LL}(\omega)$ is the number of LL edges in ω.*

Proof. First we will show that if K is a nonorientable connected smooth graph then there exists a cycle ω on K for which $N_{RR}(\omega) + N_{LL}(\omega)$ is odd. Let K be a connected nonorientable smooth graph. Then we can find a spanning tree T in K. The spanning tree T is orientable and we can find a coherent orientation ρ on T. We would like to try to extend ρ to a coherent orientation ρ' on K. Suppose that every cycle ω on K associated with a chord of T has $N_{RR}(\omega) + N_{LL}(\omega)$ an even number. Then every cycle associated with a chord of T must be orientable. We may then extend the coherent orientation ρ on T to a coherent orientation ρ' on K as follows. Define $\rho'(e) = \rho(e)$ for $e \in E_T$. For each cycle $\omega(e)$ associated with a chord, $e \in E_K - E_T$ there will be a unique choice of orientation $\rho'(e)$ for the chord e so that $\rho'|E_{\omega(e)}$ defines a coherent orientation for $\omega(e)$. Recall that the chords of T are orientable . Thus the structure matrix for $\omega(e)$ has rank $|E_{\omega(e)}| - 1 = |V_{\omega(e)}| - 1$ and a choice of coherent orientation for the edges in $E_{\omega(e)} - \{e\}$ determines a coherent orientaion on e. In this manner we define the orientation ρ' on K. The orientation ρ' is also clearly a coherent orientation since each chord e is oriented coherently at its incident vertices. Thus K is orientable which is a contradiction.

Conversely suppose that K is a smooth graph where there exists a cycle ω with $N_{RR}(\omega) + N_{LL}(\omega)$ an odd number. Suppose that K is orientable and let ρ be a coherent orientation on K. ρ would then be a coherent orientation for ω and thus ω must have $N_{RR}(\omega) + N_{LL}(\omega)$ even which is a contradiction. □.

This result gives us a new method for checking whether or not a given smooth graph K is orientable. Find a spanning tree T on K and check whether or not each cycle ω associated with a chord e of T has $N_{RR}(\omega) + N_{LL}(\omega)$ an even number. If so, then the smooth graph K will be orientable. If not then K is nonorientable.

3.4 Orientability and W^*-mappings

We now investigate the relationship between orientability and shift equivalence. We will show that solenoids come in two varieties, those presented by orientable smooth graphs and those presented by nonorientable smooth graphs.

Theorem 20 *Let K_1 and K_2 be connected smooth graphs and (f, g) a smooth graph homomorphism from K_1 onto K_2. Then there exists an onto immersion $h : K_1 \to K_2$.*

Proof. We will give a sketch of how the smooth graph homomorphism (f, g) can be used to construct an immersion h from K_1 onto to K_2. For each closed edge $\bar{\beta}$ of K_2 choose a diffeomorphism $s_\beta: \bar{\beta} \to [c_\beta, d_\beta]$ so that $Ds_\beta|_x = 1$ for all $x \in \bar{\beta}$. Each closed edge has two boundary points, the vertices incident upon β. For each edge e in K_1 we will have $g(e) = \beta$ for some edge β of K_2. Suppose that α and α' are the vertices incident upon β where $s_\beta(\alpha) = c_\beta$ and $s_\beta(\alpha') = d_\beta$. Further, suppose that the vertices incident upon e in K_1 are v and v' where v and v' are labeled so that $f(v) = \alpha$ and $f(v') = \alpha'$. We may then choose a diffeomorphism $r_e: \bar{e} \to [a_e, b_e]$ so that $Dr_e|_x = 1$ for all $x \in \bar{e}$, $r_e(v) = a_e$ and $r_e(v') = b_e$. For each interval $[a_e, b_e]$ associated with an edge e of K_1we may find a diffeomorphism $\gamma_e: [a_e, b_e] \to [c_{g(e)}, d_{g(e)}]$ so that; $\gamma_e(a_e) = c_{g(e)}$, $\gamma_e(b_e) = d_{g(e)}$,

$D\gamma_e|_{a_e} = D\gamma_e|_{b_e} = 1$ and $D\gamma_e|_x > 0$ for all $x \in (a_e, b_e)$. See appendix A for a detailed construction of such a function. We may then define the mapping $h: K_1 \to K_2$ by letting $h|\bar{e} = s_{g(e)}^{-1} \circ \gamma_e \circ r_e$ for each edge e in K_1. We must of course verify that this is indeed a well defined mapping as it could be multi-valued on the vertices of K_1. We know, however, for each closed edge $\bar{e}$ of K_1 that at the vertices v and v' incident with e we have $s_{g(e)}^{-1} \circ \gamma \circ r_e(v) = f(v)$ and $s_{g(e)}^{-1} \circ \gamma \circ r_e(v') = f(v')$. Since f is a well defined mapping on the set of vertices of K_1 so h will be well defined on the set of vertices of K_1. The mapping h is also continuous since each by definition $h|\bar{e}$ is continuous for each e in K_1, see Massey [23] page 232. Thus it only remains to be shown that h is an onto immersion. However, h is clearly onto since (f, g) is onto. Also $Dh|_x > 0$ for all $x \in K_1$ and Dh is well defined and continuous because of our choice of the r_e, γ_e and s_β. Thus all that remains to be verified is that h so defined does not bend back any of the branches at a branch point, but because of our stipulation that (f, g) be such that at every vertex v either;

1. $g \circ R_G(v) = R_{G'} \circ f(v)$ and $g \circ L_G(v) = L_{G'} \circ f(v)$ or,

2. $g \circ R_G(v) = L_{G'} \circ f(v)$ and $g \circ L_G(v) = R_{G'} \circ f(v)$,

it is easily verified that this is the case. □

Theorem 21 *A smooth graph K is orientable if and only if there is an immersion $\sigma: K \to S_1$.*

Proof. Suppose that K is an orientable smooth graph. Since K is orientable by theorem 14 there is an immersion (diffeomorphism) $\sigma_1: K \to K'$ where K' is a smooth graph with every edge of type RL. We may then refine the graph structure on K' to produce a new smooth graph K'' by picking a single point z_i in the interior of each edge e_i and calling it a vertex. The edge e_i is subdivided into two new edges w_i and u_i. For example if we have the edge e_i with $I(e_i) = \{x, y\}$, e_i right at x and left at y. Then $I(w_i) = \{x, z_i\}$ and $I(u_i) = \{z_i, y\}$. The edge w_i is right at x and left at z_i. The edge u_i is right at z_i and left at y. K'' is clearly such that every edge is of type RL. Also K' and K'' are diffeomorphic via the identity map, say, $\sigma_2: K' \to K''$. Since σ_2 is a diffeomorphism it is also an immersion. The vertex set V'' of K'' can be partitioned into two subsets V_1 and V_2. V_1 is the set of vertices in K'' which are also vertices in K'. V_2 is the set of "new" vertices from the points chosen from the interior of the edges. Similarly the edge set E'' of K'' may partitioned into two subsets E_1 and E_2 where $E_1 = \{w_i\}$ and $E_2 = \{u_i\}$. We may consider the circle, S_1, to be the smooth graph consisting of two vertices x_1, x_2 and two edges y_1, y_2 where $R(x_1) = L(x_2) = \{y_1\}$, $R(x_2) = L(x_1) = \{y_2\}$. Consider the homomorphism $(f, g): K'' \to S_1$ given by

$$f(v) = \begin{cases} x_1 & \text{if } v \in V_1 \\ x_2 & \text{if } v \in V_2 \end{cases}$$

and

$$g(e) = \begin{cases} y_1 & \text{if } e \in E_1 \\ y_2 & \text{if } e \in E_2. \end{cases}$$

We see that if $v \in V_2$ then $f(R(v)) = y_1$ and $f(L(v)) = y_2$ and if $v \in V_2$ then $f(R(v)) = y_2$ and $f(L(v)) = y_1$. Thus (f, g) is clearly a homomorphism and the conditions for theorem 20 are satisfied. Thus there exists an immersion $h: K'' \to S_1$. By composition $h \circ \sigma_2 \circ \sigma_1: K \to S_1$ will also be an immersion.

Suppose that K is a smooth graph and $\sigma: K \to S_1$ is an immersion. We need to show that K is orientable. We can give S_1 the structure of a smooth graph by considering the set of points $\sigma(V)$ to be vertices of S_1 where V is the vertex set of K. We may ensure that $|\sigma(V)| > 1$ by choosing a point in the interior of an arbitrary edge of K and calling it a vertex if necessary. Thus, without loss of generality we may assume that $|\sigma(V)| > 1$. Write $V' = \sigma(V) = \{x_1, \ldots, x_p\}$ for the set of vertices of S_1. The set of edges for S_1 will be given by $E' = \{y_1, \ldots, y_p\}$ where $R(x_1) = L(x_2) = \{y_1\}$, $R(x_2) = L(x_3) = \{y_2\}, \ldots$, $R(x_{p-1}) = L(x_p) = \{y_{p-1}\}$, $R(x_p) = L(x_1) = \{y_p\}$. S_1 is orientable and we may choose a coherent orientation ρ on S_1 as follows; $\rho(y_1) = x_2$, $\rho(y_2) = x_3$, $\ldots$, $\rho(y_{p-1}) = x_p$, $\rho(y_p) = x_1$. The immersion σ induces a homomorphism $(f, g): K \to S_1$ defined by

$$f(v) = \sigma(v) \quad \text{for } v \in V$$
$$g(e) = \sigma(e) \quad \text{for } e \in E.$$

Recall that edges are open subsets. We see that $f^{-1} \circ \rho \circ g(e)$ is a set of vertices in K, which f maps to the single vertex $\rho \circ g(e)$ in S_1. The vertex $\rho \circ g(e)$ is the sense of the image of the edge e under the orientation ρ. Since S_1 contains at least two edges $I(e) \cap f^{-1} \circ \rho \circ g(e)$ must contain only one vertex of K. Thus we may define an orientation γ on K by $\gamma(e) = I(e) \cap f^{-1} \circ \rho \circ g(e)$. We claim that γ is a coherent orientation. Let v be a vertex of K. Since σ is an immersion the homomorphism (f, g) is such that either

1. $R_{S_1} \circ f(v) = g \circ R_K(v)$ and $L_{S_1} \circ f(v) = g \circ L_K(v)$ or,

2. $R_{S_1} \circ f(v) = g \circ L_K(v)$ and $L_{S_1} \circ f(v) = g \circ R_K(v)$.

If case 1.) is true we have $I^+_\gamma(v) = R(v)$ and $I^-_\gamma(v) = L(v)$. If case 2.) is true we have $I^+_\gamma(v) = L(v)$ and $I^-_\gamma(v) = R(v)$. Thus γ defines a coherent orientation and K is orientable. □

Theorem 22 *Let $g: K \to K$ and $f: L \to L$ be W^*-mappings and let $r: K \to L$ and $s: L \to K$ be a shift equivalence between $\{K, g\}$ and $\{L, f\}$. Then r and s are immersions.*

Proof. Suppose that r is not an immersion. Then there exists a $z \in K$ such that $Dr|_z = 0$ (Recall that the tangent space at any point of a branched 1-manifold is 1-dimensional). Let m be the lag of the shift equivalence given by r and s, i.e., $s \circ r = g^m$ and $r \circ s = f^m$. Since g is an immersion so is g^m and $Dg^m|_x \neq 0$ for all $x \in K$. However, we know that if $Dr|_z = 0$ then $D(s \circ r)|_z = Ds|_{r(z)} Dr|_z = 0$, but then $D(s \circ r)|_z = Dg^m|_z = 0$ which is a contradiction. The case for s is exactly the same since shift equivalence is symmetric. □

Theorem 23 *Let $g: K \to K$ and $f: L \to L$ be W^*-mappings with $g \sim_s f$. Then K and L are both orientable or both nonorientable.*

Proof. Let $r : K \to L$ and $s: L \to K$ be the mappings giving the shift equivalence. Suppose that K is orientable and L is nonorientable. Then by theorem 21 there is an immersion $\sigma: K \to S_1$. By theorem 22 we know that s is an immersion. Since the composition of two immersions is also an immersion we have $\sigma \circ s: L \to S_1$ an immersion and thus L is orientable which is a contradiction. □

3.5 Orientable Double covers

In this section we show that we can associate with every connected nonorientable smooth graph K an orientable smooth graph M which is a double cover of K. There is a well defined mapping $f: M \to K$ which is an immersion, locally a diffeomorphism, and 2 to 1 at each point of K. We will use double covers to study nonorientable smooth graphs in later sections.

Definition 29 Let K and M be smooth graphs. We call a map $f: M \to K$ a *covering map* from M to K if it satisfies:

1. f is a continuous surjection.

2. If $x \in K$ then $f^{-1}(x)$ is a finite or countable set $\{x_1, x_1, \ldots, x_n \ldots\}$. If U is a small neighborhood of $x \in K$, then

   (a) $f^{-1}(U) = U_1 \cup U_2 \cup \ldots \cup U_n \cup \ldots$, where U_i is a small neighborhood of x_i for $i = 1, 2, \ldots$.

   (b) $U_i \cap U_j = \emptyset$ for $i \neq j$.

   (c) $f|U_i$ is a diffeomorphism onto U.

We say that M *covers* K by f, and that M is the *covering space* of K. If $|f^{-1}(x)| = 2$ for all $x \in K$ then M is a *double cover* of K.

We now wish to develop a procedure which we can use, when given any connected nonorientable smooth graph K, to construct a double cover K' which is orientable. We call such a cover an *orientable double cover* of K. The following theorem establishes the existence of a orientable double cover for every nonorientable smooth graph as well as giving a method for constructing it.

Theorem 24 *Let K be a connected nonorientable smooth graph. Then there exists a connected smooth graph K' which is an orientable double cover for K.*

Proof. Let $K = (G, S)$ be a nonorientable smooth graph where $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$, and $E = \{e_1, \ldots, e_m\}$. The orientable double cover for K is given by the smooth graph $K' = (G', S')$ where $G' = (V', E')$, $V' = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ and $E' = \{w_1, \ldots, w_m\} \cup \{z_1, \ldots, z_m\}$. The incidence relations and switch conditions for K' are assigned according to the following rules. For each edge e_i, say $I(e_i) = \{v_\alpha, v_\beta\}$, we have either;

1. e_i right at v_α and e_i left at v_β,

2. e_i right at both v_α and v_β,

3. e_i left at both v_α and v_β.

In case 1.) we assign $I(w_i) = \{x_\alpha, x_\beta\}$ with w_i right at x_α and left at x_β. Similarly $I(z_i) = \{y_\alpha, y_\beta\}$ with z_i left at y_α and right at y_β. In case 2.) we assign $I(w_i) = \{x_\alpha, y_\beta\}$

with w_i right at x_α and left at y_β. Similarly $I(z_i) = \{y_\alpha, x_\beta\}$ with z_i left at y_α and right at x_β. In case 3.) we assign $I(w_i) = \{x_\alpha, y_\beta\}$ with w_i left at x_α and right at y_β. Similarly $I(z_i) = \{y_\alpha, x_\beta\}$ with y_α right at y_α and left at y_β. We note that cases 2.) and 3.) involve an arbitrary choice as to which incident vertex is considered v_α and which v_β in K. We also note that every edge $e \in K'$ is an RL edge and thus K' is orientable.

We now need to show that K' is connected and that it is a double cover. We know that K is connected. Thus there is a closed path P on K which traverses every edge of K. We can choose P so that it contains an odd number of RR plus LL edges. This is because K is nonorientable and thus as a result of theorem 19 contains a cycle ω which contains an odd number of RR plus LL edges. Suppose that Q is a path which traverses every edge of K. We can choose Q so that it starts and ends on a vertex in ω. If ω contains an odd number of RR plus LL edges then we are done $P = Q$. If Q contains an even number of RR plus LL edges then let P be the path which traverses Q and then traverses ω. The number of RR plus LL edges in P will then be odd. We will now use P to construct a path P' on K' which traverses every edge of K' and thus show that K' is connected. We may decompose P into subpaths $P = P_1 P_2 \ldots P_q$, q odd, so that each P_i contains exactly one RR or LL edge and that this edge is the last edge traversed. We may write

$$P_i = \{\alpha_0^i, \beta_1^i, \alpha_1^i, \ldots, \alpha_{l(i)-1}^i, \beta_{l(i)}^i, \alpha_{l(i)}^i\}.$$

Note, here $\beta_{l(i)}^i$ is an RR or LL edge. Let $s_1, s_2 \colon V \to V'$ be defined by $s_1(v_i) = x_i$ and $s_2(v_i) = y_i$ for all $i = 1, \ldots, n$. Similarly define $t_1, t_2 \colon E \to E'$ be defined by $t_1(e_j) = w_j$ and $t_2(e_j) = z_j$ for all $j = 1, \ldots, m$. Consider the paths W_i and Z_i on K' given by;

$$W_i = \{s_1(\alpha_0^i), t_1(\beta_1^i), s_1(\alpha_1^i), \ldots, s_1(\alpha_{l(i)-1}^i), \delta_1^i, s_2(\alpha_{l(i)}^i)\},$$

$$Z_i = \{s_2(\alpha_0^i), t_2(\beta_1^i), s_2(\alpha_1^i), \ldots, s_2(\alpha_{l(i)-1}^i), \delta_2^i, s_1(\alpha_{l(i)}^i)\}.$$

Where δ_1^i and δ_2^i are assigned according to 1.) and 2.) below;

1. $\delta_1^i = t_1(\beta_{l(i)}^i)$ and $\delta_2^i = t_2(\beta_{l(i)}^i)$ if $I \circ t_1(\beta_{l(i)}^i) = \{s_1(\alpha_{l(i)-1}^i), s_2(\alpha_{l(i)}^i)\}$ and $I \circ t_2(\beta_{l(i)}^i) = \{s_2(\alpha_{l(i)-1}^i), s_1(\alpha_{l(i)}^i)\}$,

2. $\delta_1^i = t_2(\beta_{l(i)}^i)$ and $\delta_2^i = t_1(\beta_{l(i)}^i)$ if $I \circ t_1(\beta_{l(i)}^i) = \{s_2(\alpha_{l(i)-1}^i), s_1(\alpha_{l(i)}^i)\}$ and $I \circ t_2(\beta_{l(i)}^i) = \{s_1(\alpha_{l(i)-1}^i), s_2(\alpha_{l(i)}^i)\}$.

It is easy to check that the composition of W_iZ_{i+1} and Z_iW_{i+1} for $i = 1, \ldots, q$, W_qZ_1 and Z_qW_1 are also well defined paths on K'. Thus we may form the path

$$P' = W_1Z_2, W_3Z_4 \ldots Z_{q-1}W_qZ_1W_2Z_3 \ldots W_{q-1}Z_q$$

which contains all edges of K'. Thus K' is connected. Lastly we need to show that there exists a covering map $h: K' \to K$ which is two to one at all points of K'. As a result of theorem 20 we know that if we find a homomorphism $(f, g): K' \to K$ which is onto and such that for all $v \in K'$ either

1. $g \circ R_K(v) = R_{K'} \circ f(v)$ and $g \circ L_K(v) = L_{K'} \circ f(v)$ or

2. $g \circ R_K(v) = R_{K'} \circ f(v)$ and $g \circ L_K(v) = L_{K'} \circ f(v)$,

there will exist an onto immersion $h: K' \to K$. This immersion is necessarily a local diffeomorphism at all points in $K' - B'$ where B' is the branch set of K'. In order to ensure that there exists an immersion h which is a local diffeomorphism at the points in B' we will need to find a homomorphism (f, g) which also satisfies a third criteria, which states that for each vertex v of K' g should be a bijection from the set of edges in K' adjacent v to the set of edges adjacent to $f(v)$ in K. The immersion h constructed according to theorem 20 will then also be a local diffeomorphism at the points in B'. Consider the homomorphism (f, g) defined by $f(x_i) = f(y_i) = v_i$ for all $i = 1, \ldots, n$ and $g(w_j) = g(z_j) = e_j$ for all $j = 1, \ldots, m$. It is clear that both f and g are both onto, but we still need to show that (f, g) is a homomorphism and that it satisfies our three criteria above. Consider the edge e_i, where say $I(e_i) = \{v_\alpha, v_\beta\}$. In cases 1.), 2.), and 3.) in the construction of K' it is easy to check that

$$I_K \circ g(w_i) = f \circ I_{K'}(w_i) = I_K \circ g(z_i) = f \circ I_{K'}(z_i) = \{v_\alpha, v_\beta\}.$$

Thus (f, g) is a homomorphism. Also if an edge w_i or z_i is right/left at $x_j \in \{x_1, \ldots, x_n\}$ then $g(w_i) = g(z_i) = e_i$ is right/left at v_j and if an edge w_i or z_i is right/left at $y_j \in \{y_1, \ldots, y_n\}$ then $g(w_i) = g(z_i) = e_i$ is left/right at v_j. Thus for all x_j we have $g \circ R_K(x_j) = R_{K'} \circ f(x_j)$ and $g \circ L_K(x_j) = L_{K'} \circ f(x_j)$ and for all y_j we have $g \circ R_K(y_j) = R_{K'} \circ f(y_j)$ and $g \circ L_K(y_j) = L_{K'} \circ f(y_j)$. The only edges which are identified under g are w_i and z_i for each i. Since w_i and z_i are never incident upon the same vertex in K' and the degrees of the vertices x_j and y_j are equal to the degree of v_j for all j we see that g must be a bijection from the adjacent edges at each vertex of K' to the adjacent edges of its image under f. Thus there exists a covering map h which is two to one at each point of K'. $\square$

Given a connected nonorientable smooth graph K the orientable double cover K' of K has several useful properties. First, given any closed path P' on K' which traverses every edge of K' the image of P' under the covering homomorphism will be a closed path on P on K which traverses every edge of K and if P' is a route then P will be a route. The second property is that given any closed path P on K which traverses every edge of K and an odd number of RR plus LL edges we can construct a path P' on K' which traverses every edge of K' in manner used in the proof of theorem 24 and, again, if P is a route then P' will be a route. The image of this path under the covering homomorphism will then be the path P traversed twice.

Example 10 Consider the nonorientable smooth graph $K = (G, S)$ where $G = (V, E)$, $V = \{v_1, v_2\}$, $E := \{e_1, e_2, e_3\}$ and the incidence relations and switch conditions are summarized in the following table.

v	$R(v)$	$L(v)$
v_1	$\{e_1\}$	$\{e_2, e_3\}$
v_2	$\{e_2\}$	$\{e_1, e_3\}$

Using the method given in theorem 24 the double cover for K is given by $K' = (G', S')$ where $G' = (E', V')$, $V' = \{x_1, x_2, y_1, y_2\}$, $E' = \{w_1, w_2, w_3, z_1, z_2, z_3\}$ and the incidence relations and switch conditions are summarized in the table below.

v	$R(v)$	$L(v)$
x_1	$\{w_1\}$	$\{w_2, w_3\}$
x_2	$\{w_2\}$	$\{w_1, z_3\}$
y_1	$\{z_2, z_3\}$	$\{z_1\}$
y_2	$\{w_3, z_1\}$	$\{z_2\}$

The smooth graph diagrams for for K and K' are given in figure 3.5.

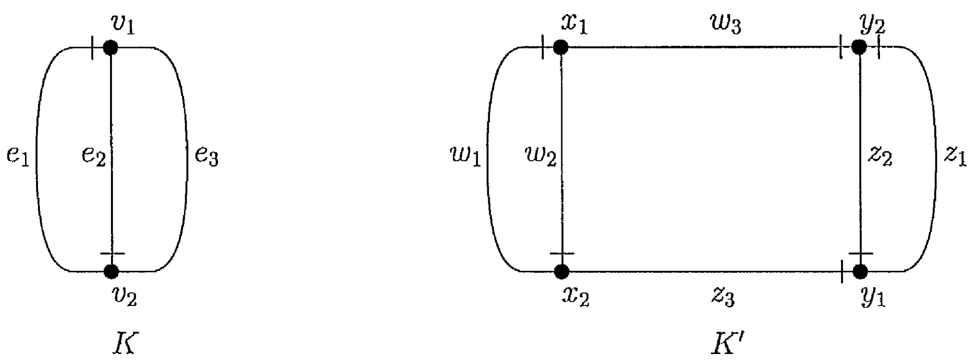


Figure 3.5: The diagrams for the smooth graphs K and K' given in example 10.

3.6 Recurrence

A second property which is important when studying W^*-mappings on smooth graphs is recurrence. In this section we define recurrence and study some elementary properties of recurrent and nonrecurrent smooth graphs. We also demonstrate how to convert the problem of determining whether or not a smooth graph is recurrent into a graph theoretic question, which can then be solved by highly efficient algorithms. Recurrence as we shall later see will turn out to be a necessary and sufficient property for the existence of a W^*-mapping on a given smooth graph.

Definition 30 A connected smooth graph K is said to be *recurrent* if there exists a closed route on K which traverses every edge in K.

In order to study this simple sounding property we will need some definitions.

Definition 31 Let K be an orientable smooth graph with coherent orientation ρ. A walk $W = \{x_0, y_1, x_1, \dots, x_{p-1}, y_p, x_p\}$ on K is said to be a *directed walk* if $\rho(y_i) = x_i$ for all $i = 1, \dots, p$.

A directed walk on a connected orientable smooth graph K is always a route on K and the set of all routes on K consists of two disjoint subsets corresponding to the directed walks under the two possible orientations on K.

Definition 32 Let K be an orientable smooth graph with coherent orientation ρ. An edge e is said to be *accessible* from an edge e' if there exists a directed walk on K which starts on e' and ends on e, i.e., a directed walk of the form $W = \{x_0, e', x_1, \dots, x_{n-1}, e, x_n\}$.

Definition 33 An orientable smooth graph K with coherent orientation ρ is said to be *strongly connected* if for any pair edges e and e' belonging to K each is accessible from each other.

Note 10 The above definitions of accessibility and strong connectedness differ slightly from the standard definition found in most texts on graph theory, for example see [6], [15], in that they are defined in terms of the edges of the graph in question as opposed to the vertices.

Our first theorem shows that for an orientable smooth graph the notions of recurrence and strong connection are equivalent. Thus an algorithm which determines whether an orientable smooth graph is strongly connected will also determine whether or not a smooth graph is recurrent.

Theorem 25 *An orientable smooth graph K with coherent orientation ρ is recurrent if and only if it is strongly connected.*

Proof. Let K be an orientable smooth graph with coherent orientation ρ. Suppose that K is recurrent. Then there exists a closed route W on K which traverses every edge of K. It is easy to see that W is a directed walk under either ρ or ρ^*. Without loss of generality we will assume that W is a directed walk under ρ. Let e and e' be two edges of K. Since W traverses every edge of K and is closed we know that e and e' occur in W and that thus there is a directed walk from e to e' and e' to e. Conversely suppose that K is strongly connected. Label the edges $e_1, \ldots, e_m$. Since K is strongly connected we can find a directed walk W_i from e_i to e_{i+1} for $i = 1, \ldots, m-1$ and a directed walk W_m from e_m to e_1, with respect to the orientation ρ. Let X_i, for $i = 1, \ldots, m$, be the sub-walk of W_i which traverses every edge that W_i traverses except the last. The composition of the directed walks $W = X_1 X_2 \ldots X_m$ will also be a directed walk. It is also a closed route. It is a route because every directed walk is a route and it is closed because at any vertex v a directed walk W must leave v from an edge opposite an edge upon which it enters it. □

We will now prove several results establishing the relationship between the recurrence of a nonorientable smooth graph K and the recurrence of its double cover K', namely that K is recurrent if and only if K' is recurrent. This will allow us to deal exclusively with orientable smooth graphs when using an algorithm to determine recurrence.

Theorem 26 *Let K be a connected nonorientable smooth graph and K' its orientable double cover. Then K is recurrent if K' is recurrent.*

Proof. Let K be a connected nonorientable smooth graph and K' its orientable double cover. Suppose that K' is recurrent. Then there exists a closed route $W' = \{x_0, y_1, x_1, \ldots, x_{n-1}, y_n, x_n\}$ on K' which traverses every edge of K'. Let $(f, g)\colon K' \to K$ be the covering homomorphism. The image $W = \{f(x_0), g(y_1), f(x_1), \ldots, f(x_{n-1}), g(y_n), f(x_n)\}$ of W'

under (f, g) is a closed route on K which traverses every edge of K. Thus K is recurrent. □

Theorem 27 *Let K be a recurrent nonorientable smooth graph. Then for every closed route $W = \{x_0, y_1, x_1, \ldots, x_{n-1}, y_n, x_n\}$ on K which traverses every edge of K there exists a vertex $v = x_i = x_j$, $i \neq j$ such that $y_i \in R(x_i)$, $y_{i+1} \in L(x_i)$, $y_j \in L(x_j)$, and $y_{j+1} \in R(x_j)$.*

Proof. Let K be a connected recurrent nonorientable smooth graph. Suppose that there exists a closed route W on K which traverses every edge of K and is such that for every vertex $v \in K$ we have either $y_i \in R(x_i)$, $y_{i+1} \in L(x_i)$ or $y_i \in L(x_i)$, $y_{i+1} \in R(x_i)$ for all $x_i = v$ in W. Clearly W cannot traverse any edge of K in more than one direction, i.e., W cannot contain both the subroutes $\{x, e, x'\}$ and $\{x', e, x\}$ for any edge e in K. Since W must contain every edge e of K we know that there must exist a subroute of W of the form $\{v', e, v\}$ and that $\{v, e, v'\}$ will not be a subroute of W, i.e., the route W always traverses e by traveling from v' to v. We may then assign an orientation ρ to K by letting $\rho(e) = v$ if $\{v', e, v\}$ is a subroute of W. It is also clear that ρ will be a coherent orientation since as W traverses a vertex v it will either always enter v upon a right edge and leave v on a left edge or it enter v upon a left edge and leave v upon a right edge. In the first instance we will have $I_\rho^+(v) = R(v)$ and $I_\rho^-(v) = L(v)$. In the second instance $I_\rho^+(v) = L(v)$ and $I_\rho^-(v) = R(v)$. This is a contradiction since K was chosen to be nonorientable. □

Theorem 28 *Let K be a recurrent nonorientable smooth graph. Then there exists a route W on K which for every edge e belonging to K contains the subpaths $\{x, e, x'\}$ and $\{x', e, x\}$, i.e., the route W traverses every edge of K in both directions.*

Proof. Let K be a connected recurrent nonorientable smooth graph. Let $W = \{x_0, y_1, x_1, \ldots, x_{p-1}, y_p, x_p\}$ be a closed route on K which traverses every edge of K. From Theorem 27 we know that there must exist a vertex $v = x_i = x_j$, $i \neq j$ such that $y_i \in R(x_i)$, $y_{i+1} \in L(x_i)$, $y_j \in L(x_j)$, and $y_{j+1} \in R(x_{j+1})$. Thus the route W enters v upon a right edge and leaves v upon a left edge as well as entering v upon a left edge and leaving v upon a right edge. Without loss of generality we will assume that $i < j$. Consider the subroutes of W; $P_1 = \{x_0, y_1, \ldots, x_i\}$, $P_2 = \{x_i, y_{i+1}, \ldots, x_j\}$ and $P_3 = \{x_j, y_{j+1}, \ldots, x_p\}$.

Let Q_1, Q_2 and Q_3 be the routes obtained by traversing P_1, P_2 and P_3, respectively, in reverse, i.e., $Q_1 = \{x_i, y_i, \ldots, x_0\}$, $Q_2 = \{x_j, y_j, \ldots, x_i\}$ and $Q_3 = \{x_p, y_p, \ldots x_j\}$. We may then form the route $P' = PP_1P_2Q_1Q_3Q_2P_3$ as a composition of the routes P, P_1, P_2, P_3, Q_1, Q_2, and Q_3. It is easily verified that P' must traverse every edge in both directions. □

We now have all the pieces to prove that in the special case that the underlying smooth graph is recurrent a converse to theorem 11 is possible. Namely, if there exists a route W on a smooth graph K which traverses an edge e in more than one direction, i.e., contains the subpaths $\{x, e, x'\}$ and $\{x', e, x\}$ then K is nonorientable. We therefore have the following result.

Theorem 29 *Let K be a recurrent smooth graph. Then there is a route W on K which contains the subpaths $\{x, e, x'\}$ and $\{x', e, x\}$ for any edge e if and only if K is nonorientable.*

Proof. This is a result of theorems 11 and 28. □

Theorem 30 *Let K be a recurrent, nonorientable smooth graph and $W = \{x_0, y_1, x_1, \ldots, x_{p-1}, y_p, x_p\}$ a route on K with $x_0 = x_p$ and either $y_1, y_p \in R(x_0)$ or $y_1, y_p \in L(x_0)$. Then if $A = \{i: y_i$ a RR or LL edge $\}$ then $|A|$ is odd.*

Proof. Let K be a connected, recurrent, nonorientable smooth graph and $W = \{x_0, y_1, x_1, \ldots, x_{p-1}, y_p, x_p\}$ a route on K with $x_0 = x_p$ and $y_1, y_p \in R(x_0)$. The case for $y_1, y_p \in L(x_0)$ is similar. Consider the smooth cycle $C = (V, E)$ defined by $V = \{v_0, \ldots, v_{p-1}\}$, $E = \{e_1, \ldots, e_p\}$, $I(e_i) = \{v_{i-1}, v_i\}$ for $i = 1, \ldots, p-1$ and $I(e_p) = \{v_{p-1}, v_0\}$. The switch conditions for C are given by; e_i is right/left at v_{i-1} if y_i is right/left at x_{i-1} and e_i is right/left at v_i if y_i is right/left at x_i. Let $A = \{i: y_i$ a RR or LL edge $\}$. There is clearly a one to one correspondence between the set of RR and LL edges of C and the elements of A. We claim that the cycle C is nonorientable and thus by theorem 17 the number of RR plus LL edges in C is odd. In order to see this consider trying to place a coherent orientation ρ on C. If we choose $\rho(e_1) = v_1$ then $\rho(e_i) = v_i$ for $i = 1, \ldots, p-1$. We then see that it is impossible to orient e_p in a coherent manner. Choosing $\rho(e_1) = v_0$ we must then have $\rho(e_i) = v_{i-1}$ for $i = 1, \ldots, p-1$. Again it is impossible to orient the edge e_p in a coherent manner. Thus C is nonorientable and the number of RR plus LL edges in C is odd. Since there is a one to one correspondence between the edges RR plus LL edges of and the elements of A we must have $|A|$ odd. □

Theorem 31 *Let K be a nonorientable smooth graph with orientable double cover K'. Then K' is recurrent if K is recurrent.*

Proof. Let $K = (G, S)$ be a connected recurrent nonorientable smooth graph, where $G = (V, E)$, $V = \{v_1, \ldots v_n\}$, and $E = \{e_1, \ldots e_m\}$. Let $K' = (G', S')$ be a double cover for K, where $G' = (V', E')$, $V' = \{x_1, \ldots x_n\} \cup \{y_1, \ldots, y_n\}$, and $E' = \{w_1, \ldots, w_m\} \cup \{z_1, \ldots, z_m\}$. The incidence relations and switch conditions for K' are assigned according to the following rules. For each edge $e_i \in K$, say $I(e_i) = \{v_\alpha, v_\beta\}$, we have either;

1. e_1 right at v_α and left at v_β,

2. e_i right at both v_α and v_β,

3. e_i left at both v_α and v_β.

In case 1.) we assign $I(w_i) = \{x_\alpha, x_\beta\}$ with w_i right at x_α and left at x_β as well as assigning $I(z_i) = \{y_\alpha, y_\beta\}$ with z_i left at y_α and right at y_β. In case 2.) we assign $I(w_i) = \{x_\alpha, y_\beta\}$ with w_i right at x_α and left at y_β as well as assigning $I(z_i) = \{y_\alpha, x_\beta\}$ with z_i left at y_α and right at x_β. In case 3.) we assign $I(w_i) = \{x_\alpha, y_\beta\}$ with w_i left at x_α and right at y_β as well as assigning $I(z_i) = \{x_\beta, x_\alpha\}$ with z_i right at x_β and right at y_α. We note that this construction is the same as that in theorem 24. Since K is recurrent and nonorientable by theorem 28 we can find a closed route $W = \{\delta_0, \gamma_1, \delta_1, \ldots, \delta_{p-1}, \gamma_p, \delta_p\}$ on K which traverses every edge of K. By theorem 27 we know that there must exist a vertex $v = \delta_i = \delta_j$ in K where $\gamma_i, \gamma_{j+1} \in R(v)$ and $\gamma_{i+1}, \gamma_j \in L(v)$. By cyclicly permuting W if necessary we can assume that $i = 0$ and $j > 0$ thus moving γ_i to γ_p , γ_{i+1} to γ_1 and setting $\delta_0 = \delta_p$ since W is a closed route. We will use W to construct a closed route W' on K' which traverses every edge of K' and thus prove that K' is recurrent. First we note that W can be decomposed into two routes $W_1 = \{\delta_0, \gamma_1, \delta_1, \ldots, \delta_j\}$ and $W_2 = \{\delta_j, \gamma_{j+1}, \delta_{j+1}, \ldots, \delta_p\}$, where $W = W_1 W_2$. Using the routes W_1 and W_2 we can construct a new route $Q = W_1 W_2 W_1$ on K as the composition of W_1 and W_2. We note that Q is not a closed route but it traverses every edge of K, starts and ends on the same vertex v and the first and last edges that it traverses are right at v. The number of times Q and W_2 traverse a RR or LL edge is odd. We also note that Q

composed with W_2 given by QW_2 is a closed route. Since the number of times Q traverses an RR or LL edge is odd we can use it to construct a route Q' on K' which traverses every edge of K' in a manner analogous to that in used in theorem 24, but Q' will not be a closed route. We can also construct a route W_2' on K' using W_2, it also will not be a closed route. Q' and W_2' will both however start and end on the same vertex and their composition $Q'W_2'$ will be a closed route on K'. We now show how this can be done. Let $s_1, s_2 \colon V \to V'$ be defined by $s_1(v_i) = x_i$ and $s_2(v_i) = y_i$. Similarly let $t_1, t_2 \colon E \to E'$ be defined by $t_1(e_j) = w_j$ and $t_2(e_j) = z_j$. We will first construct the route Q' on K'. We can decompose Q into $Q = Q_1Q_2 \ldots Q_q$ where q is odd and each Q_i contains exactly one RR or LL edge and that this edge is the last edge traversed. We write $Q_i = \{\kappa_0^i, \sigma_1^i, \kappa_1^i, \ldots, \kappa_{l(i)-1}^i, \sigma_{l(i)}^i, \kappa_{l(i)}^i\}$. Consider the routes on K';

$$M_i = \{s_1(\kappa_0^i), t_1(\sigma_1^i), s_1(\kappa_1^i), \ldots, s_1(\kappa_{l(i)-1}^i), a_1^i, s_2(\kappa_{l(i)}^i)\},$$

and

$$N_i = \{s_2(\kappa_0^i), t_2(\sigma_1^i), s_2(\kappa_1^i), \ldots, s_2(\kappa_{l(i)-1}^i), a_2^i, s_1(\kappa_{l(i)}^i)\}$$

where a_1^i and a_2^i are assigned according to 1.) and 2.) below;

1. $a_1^i = t_1(\sigma_{l(i)}^i)$ and $a_2^i = t_2(\sigma_{l(i)}^i)$ if $I \circ t_1(\sigma_{l(i)}^i) = \{s_1(\kappa_{l(i)-1}^i), s_2(\kappa_{l(i)}^i)\}$ and $I \circ t_2(\sigma_{l(i)}^i) = \{s_2(\kappa_{l(i)-1}^i), s_1(\kappa_{l(i)}^i)\}$,
2. $a_1^i = t_2(\sigma_{l(i)}^i)$ and $a_2^i = t_1(\sigma_{l(i)}^i)$ if $I \circ t_1(\sigma_{l(i)}^i) = \{s_2(\kappa_{l(i)-1}^i), s_1(\kappa_{l(i)}^i)\}$ and $I \circ t_2(\sigma_{l(i)}^i) = \{s_1(\kappa_{l(i)-1}^i), s_2(\kappa_{l(i)}^i)\}$.

It is easy to check that the composites M_iN_{i+1} and N_iM_{i+1} are also well defined routes on K'. Thus we may form the route

$$Q' = M_1N_2M_3 \ldots N_{q-1}M_qN_1M_2N_3 \ldots M_{q-1}N_q.$$

The route Q' must traverse every edge of K' since every edge of K' will belong to at least one of the N_i or M_i. We also know that M_1 starts at the same vertex as N_q ends since $s_1(\kappa_0^1) = s_1(\delta_0) = s_1(\kappa_{l(q)}^q) = s_1(\delta_j)$. Q', however, is not a closed route since σ_0^1 and a_2^q are both right at $s_1(\delta_p)$. Next we will construct the route W_2' on K'. We can decompose the route W_2 into $W_2 = L_1L_2 \ldots L_k$ where k is odd and each subroute L_i contains only one RR or LL edge which is the last edge traversed. We write $L_i = \{\mu_0^i, \nu_1^i, \mu_1^i, \ldots, \mu_{d(i)-1}^i, \nu_{d(i)}^i, \mu_{d(i)}^i\}$. Consider the routes on K';

$$O_i = \{s_1(\mu_0^i), t_1(\nu_1^i), s_1(\mu_1^i), \ldots, s_1(\mu_{d(i)-1}^i), b_1^i, s_2(\mu_{d(i)}^i)\}$$

and

$$P_i = \{s_2(\mu_0^i), t_2(\nu_1^i), s_2(\mu_1^i), \ldots, s_2(\mu_{d(i)-1}^i), b_2^i, s_1(\mu_{d(i)}^i)\}$$

where b_1^i and b_2^i are assigned according to 1.) and 2.) below;

1. $b_1^i = t_1(\nu_{d(i)}^i)$ and $b_2^i = t_2(\nu_{d(i)})$ if $I \circ t_1(\nu_{d(i)}^i) = \{s_1(\mu_{d(i)-1}), s_2(\mu_{d(i)})\}$ and $I \circ t_2(\nu_{d(i)}) = \{s_2(\mu_{d(i)-1}), s_1(\mu_{d(i)})\}$,
2. $b_1^i = t_2(\nu_{d(i)}^i)$ and $b_2^i = t_1(\nu_{d(i)})$ if $I \circ t_1(\nu_{d(i)}^i) = \{s_2(\mu_{d(i)-1}), s_1(\mu_{d(i)})\}$ and $I \circ t_2(\nu_{d(i)}) = \{s_1(\mu_{d(i)-1}), s_2(\mu_{d(i)})\}$.

Again it is easy to check that the composites O_iP_{i+1} and P_iO_{i+1} are also well defined routes on K'. Thus we may form the composite route

$$W_2' = O_1P_2O_3 \ldots P_{k-1}O_kP_1O_2P_3 \ldots O_{k-1}P_k.$$

The route W_2' starts and ends upon the same vertex as Q' since $s_1(\mu_0^1) = s_1(\delta_j) = s_1(\mu_{d(k)}^k) = s_1(\delta_p)$. However, W_2' begins and ends by traversing the edges $t_1(\nu_0^1)$ and b_2 which are left at $s_1(\delta_0)$. We may then compose Q' with W_2' to form the closed route $W' = Q'W_2'$ which traverses every edge of K'. Thus K' is recurrent. □

Before we think about algorithms for determining whether a given orientable connected smooth graph is strongly connected we will examine the structure of those that are not strongly connected. We show that there exist a finite number of nontrivial maximal strongly connected subgraphs. These maximal strongly connected subgraphs are also maximal recurrent subgraphs. These results also apply to a nonrecurrent smooth graph since we can examine its smooth orientable double cover and find its maximal recurrent subgraphs, the image of these subgraphs under the covering homomorphism will be the maximal recurrent subgraphs of the original manifold.

Definition 34 A *strongly connected component* of an orientable smooth graph K with coherent orientation ρ is a maximal strongly connected subgraph of K.

It is clear from the definition above and the previous theorems that on an orientable smooth graph K with coherent orientation ρ the strongly connected components of K are also the maximal recurrent subgraphs.

Theorem 32 *Let K be a connected orientable smooth graph with coherent orientation ρ. Then there exists a nontrivial strongly connected component L of K.*

Proof . Let K be a connected orientable smooth graph with coherent orientation ρ. It is clear that if K has a closed directed walk it must be part of a strongly connected component of K. Suppose then that there is no closed directed walk on K. Let W be a maximal directed walk on K which contains no vertex or edge more than once. The last vertex v' of this walk must clearly have $I_\rho^-(v') = \emptyset$ since otherwise it would not be maximal. This is a contradiction since it is assumed that for all vertices v of K that $I_\rho^+(v) \neq \emptyset$ and $I_\rho^-(v) \neq \emptyset$. Thus there must a directed walk on K which traverses an edge or vertex more than once and thus contains a directed sub-walk W which is closed. W is clearly nontrivial and must be part of a strongly connected component of K. □

It is also clear that any strongly connected component of an orientable connected smooth graph K with coherent orientation ρ must contain at least two edges. Recall that we are assuming that there are no loops in K. Thus if an edge e is part of a strongly connected component L there must exist a directed walk W in L from e to itself. Thus the strongly connected component must contain at least e and W and W will contain at least one edge.

We also see that a connected orientable smooth graph will contain a finite number of strongly connected components, which are also maximal recurrent subgraphs. The underlying smooth graph is recurrent if and only if it contains exactly one strongly connected component which is the entire smooth graph. These results clearly apply to nonorientable connected smooth graphs since as stated before we can examine the orientable double cover and then project the results back to the original smooth graph using the covering homomorphism.

We would like to have a method of determining whether or not a given smooth graph is recurrent and to find the maximal recurrent subgraphs if it is not. There exists an algorithm which determines whether or not a orientable smooth graph is strongly connected and which finds its strongly connected components. It is highly efficient and has a complexity of $O(\max(|V|, |E|))$. This algorithm is based on depth-first searching and was originally developed by Hopcraft and Tarjan in [20] and [33], a second treatment of this algorithm can found in Gibbons [13]. This algorithm will clearly also test for the recurrence of an orientable smooth graph since an orientable smooth graph is recurrent if and only if it is strongly connected. In order to determine whether or not a nonorientable smooth graph is recurrent we first find its orientable double cover and then determine whether it is strongly connected and thus whether or not it is recurrent. This then suffices to determine whether or not the original nonorientable smooth graph is recurrent or not in light of the fact that a nonorientable smooth graph is recurrent if and only if its orientable double cover is recurrent. Since this algorithm is well known we will not give a description here.

3.7 Recurrence and W^*-mappings

We now investigate the effect the recurrence or non-recurrence of a given smooth graph plays in determining whether or not there exists a W^*-mapping on that smooth graph. We will show that there exists a W^*-mapping on a smooth graph if and only if it is recurrent.

Theorem 33 *A smooth graph K with at least one edge is recurrent if and only if there exists an onto immersion $\sigma : S_1 \to K$.*

Proof. Suppose that there exists an immersion $\sigma: S_1 \to K$. Give S_1 the structure of a smooth graph by letting $\sigma^{-1}(V)$ be the set of vertices of S_1, where V is the vertex set of K. Write $V' = \sigma^{-1}(V) = \{x_1, \ldots, x_p\}$ by giving the vertices of S_1 an arbitrary cyclic ordering. We may then label the edge set of S_1 as $E' = \{y_1, \ldots, y_p\}$ where the incidence relations and switch conditions are given by $R(x_1) = L(x_2), R(x_2) = L(x_3), \ldots, R(x_{p-1}) = L(x_p), R(x_p) = L(x_1)$. Let $(f, g): S_1 \to K$ be the smooth graph homomorphism induced by σ, i.e., $f(x_i) = \sigma(x_i)$ and $g(y_i) = \sigma(y_i)$. Consider the walk

$$W = \{f(x_1), g(y_1), f(x_2), g(x_2), \ldots, f(x_r), g(y_r), f(x_1)\}$$

on K. Since (f, g) is a smooth graph homomorphism and W is the image under (f, g) of the closed route $\{x_1, y_1, x_2, y_2, \ldots, x_r, y_r, x_1\}$ on S_1 we know that W is a closed route on K. Also, since σ is onto we know that W must traverse every edge of K.

Suppose that K is recurrent. Then there is a closed route W on K which traverses every edge of K, $W = \{v_1, e_1, v_2, \ldots, v_p, e_p, v_1\}$. Give S_1 the structure of a smooth graph by considering a finite set of distinct points $\{x_1, x_2, \ldots, x_p\}$, labeled to reflect their cyclic order, as vertices. Write for the set of edges of S_1, $\{y_1, \ldots, y_p\}$ where the incidence relations and switch conditions are given by $R(x_1) = L(x_2), R(x_2) = L(x_3), \ldots, R(x_{p-1}) = L(x_p), R(x_p) = L(x_1)$. We may associate each vertex of S_1 with a vertex traversed by W

using the mapping f defined by $f(x_i) = v_i$. Similarly we may associate each edge of S_1 with an edge traversed by W using the mapping g defined by $g(y_i) = e_i$. The mapping $(f,g): S_1 \to K$ is then a well defined smooth graph homomorphism and since W traverses every edge of K the homomorphism (f,g) will be onto. Then by theorem 20 there exists an immersion $\sigma: S_1 \to K$. □

Theorem 34 *Let K be a connected smooth graph. Then K is recurrent if and only if there exists a W^*-mapping $g: K \to K$.*

Proof. First we will show that if there exists a W^*-mapping $g: K \to K$ then K is recurrent. Suppose that there exists a W^*-mapping $g: K \to K$. Let L be a nontrivial maximal recurrent sub-manifold of K. Since L is recurrent there exists an onto immersion $\sigma: S_1 \to L$. Let $\bar{e}$ be a closed edge (one-cell) of L. By theorem 10 there exists an integer m such that g^m maps $\bar{e}$ onto K. Since both g^m and σ are immersions their composition $g^m \circ \sigma: S_1 \to K$ is also an immersion and $g^m \circ \sigma(S_1) = K$. Thus K is recurrent.

Next we show that if K is a connected recurrent smooth graph then there exists a W^*-mapping $g: K \to K$. We will first deal with the case in which K is orientable. Let K be a connected orientable recurrent smooth graph. Let ρ be a coherent orientation on K. There exists a closed route W on K which traverses every edge of K. We may choose W so that it traverses every edge in agreement with the orientation ρ. Let v_0 be the initial and final vertex of the closed route W. Consider the mapping $g: K \to K$ where $g(v) = v_0$ for all vertices v in K and g expands each edge e along the route W. To make this more precise consider for each edge $e \in K$ a diffeomorphism $h_e: \bar{e} \to [a(e), b(e)]$ which is such that; $h_e(\rho(e)) = a(e)$, $h_e(\rho^*(e)) = b(e)$ and $DH|_x = 1$ for all $x \in \bar{e}$. Similarly we may parameterize the route W using an immersion $\pi: [c,d] \to K$ such that $D\pi|_x = 1$ for all $x \in [c,d]$. Since W traverses every edge in K we know that we must have $d-c > b(e)-a(e)$ for all e in K. We may then choose an $\epsilon > 1$ and a diffeomorphism $f_e: [a(e), b(e)] \to [c,d]$ for each edge e such that; $f_e(a(e)) = c$, $f_e(b(e)) = d$, $Df_e|_{a(e)} = Df_e|_{b(e)} = \epsilon$ and $Df_e|_x > \epsilon$ for all $x \in (a(e), b(e))$. See appendix A for a detailed construction of such a function. We may then define $g|\bar{e} = \pi \circ f_e \circ h_e$ for each closed edge $\bar{e}$.

The mapping g is well defined since the only point x on which any two f_α and f_β will both be defined will be a vertex and thus we will have $f_\alpha(x) = f_\beta(x) = v_0$. We know that g is continuous because $g|\bar{e}$ is continuous for each e in K, see Massey

[23] page 232. The mapping g is also clearly an expanding immersion since $Dg|_x = D\pi|_{f_e \circ h_e(x)} Df_e|_{h_e(x)} Dh_e|_x = D_f|_{h_e(x)} \geq \epsilon > 1$ for all $x \in K$. We also know that $\Omega(g) = K$, i.e., the nonwandering set of g is all of K, as $g(\bar{e}) = K$ and thus the conditions for theorem 10 are satisfied. Finally we need to check that there exists an integer m where for every $x \in K$ there is a neighborhood N_x such that $g^m(N_x)$ is an arc. It is clear that this is the case for any point which is not a vertex. At each vertex v every edge $e \in I_\rho^-(v)$ will have a small arc $[v, p_e]$ where $g([v, p_e]) = e_1$ and e_1 is the initial edge of the route W. Write $A_v = \bigcup_{e \in I_\rho^-(v)} [v, p_e]$. Similarly at each vertex v every edge $e \in I_\rho^+(v)$ will have a small arc $[q_e, v]$ such that $g([q_e, v]) = e_k$ and e_k is the final edge of the route W. Write $B_v = \bigcup_{e \in I_\rho^+(v)} [q_e, v]$. For each vertex v we may then let N_v be the interior of $A_v \cup B_v$. We will then necessarily have $g(N_v) = e_1 \cup \{v_0\} \cup e_k$ which is an arc. Thus the mapping g constructed as above will be an onto immersion satisfying W1,W2, and W3 and therefore a W-mapping. Since $g(V) = v_0$ it will also be a W^*-mapping.

We will now consider the case where K is a nonorientable, recurrent, connected smooth graph and show that this implies that there exists a W^*-mapping $g: K \to K$. Let K be a nonorientable, recurrent, connected smooth graph. Let T be a spanning tree for K. We may give T a coherent orientation ρ. Since K is recurrent there exists a closed route W on K which traverses every edge of K and which is such that the initial edge traversed is in T. We may also assume that W traverses its initial edge in accordance with the orientation ρ, i.e., if y_1 is the initial edge of W and v_0 is the initial vertex of W then $\rho^*(y_1) = v_0$. Let v_0 be the initial vertex traversed by W. Consider $f: T \to K$ where $f(v) = v_0$ for all vertices $v \in K$ and f expands each edge of T along the route W. Again in order to be more precise for each edge $e \in T$ consider a diffeomorphism $h_e: \bar{e} \to [a(e), b(e)]$ which is such that; $h_e(\rho^*(e)) = a(e)$, $h_e(\rho(e)) = b(e)$, $Dh_e|_x = 1$ for all $e \in \bar{e}$. We may also parameterize the route W using an immersion $\pi: [c, d] \to K$ which is such that $D\pi|_x = 1$ for all $x \in [c, d]$. We will necessarily have $d - c > b(e) - a(e)$ since W contains every edge of K. We may therefore choose an $\epsilon > 1$ and for each edge $e \in T$ a diffeomorphism $r_e: [a(e), b(e)] \to [c, d]$ such that $Dr_e|_{a(e)} = Dr_e|_{b(e)} = \epsilon$ and $Dr_e|_x > \epsilon$ for $x \in (a(e), b(e))$. We then define $f|\bar{e} = \pi \circ r_e \circ h_e$. See appendix A for a detailed construction of such a function. This is a well defined mapping from T to K for the same reasons as in the preceding paragraph. It is also continuous since $f|\bar{e}$ is continuous for each e in T.

Let $c_1, \ldots, c_p$ be the chords of T. The chords of T are found to come in three types:

1. c_i is incident upon vertices v and v' each of which is the sense under the orientation ρ of an edge in T,

2. c_i is incident upon vertices v and v' one of which is the sense under ρ of an edge in T and one of which is not the sense under ρ of an edge in T,

3. c_i is incident upon two vertices both of which are not the sense under ρ of an edge in T.

Let e_1 be the initial edge traversed by W an v_1 be the second vertex traversed, similarly let e_k be the final edge traversed by W and v_{k-1} the second to last vertex traversed. Consider the routes W_1 and W_2 on K which traverse every edge of K and start and finish on the same vertex v_0 as W. Let W_1 be such that that it starts and finishes on the same initial edge e_1 traversed by W where the first time it traverses e_1 it does so by going from v_0 to v_1 and the last time it traverses e_1 it does so by going from v_1 to v_0 . Let W_2 be such that it starts and finishes on the final edge e_k of W where the first time it traverses e_k it does so by going from v_0 to v_{k-1} and the last time it traverses e_k it does so by going from v_{k-1} to v_0. The fact that we can find such routes on K is a consequence of theorem 28.

We may construct $g: K \to K$ by letting $g|T = f$ and expanding each chord of T along one of the routes W_1, W_2 or W according to the following set of rules;

- if c_i is of type 1.) then g expands c_i along the route W_1,
- if c_i s of type 2.) then g expands c_i along the route W,
- if c_i is of type 3.) then g expands c_i along the route W_2.

To be more precise for each closed chord $\bar{c}_i$ we find a diffeomorphism $h_i: \bar{c}_i \to [a_i, b_i]$ which is such that $Dh|_x = 1$ for all $x \in \bar{c}_i$. For each of the routes W_1 and W_2 we find a parametrization γ_1 and γ_2 where $\gamma_1: [\alpha_1, \beta_1] \to K$ and $\gamma_2: [\alpha_2, \beta_2] \to K$. We also stipulate that γ_1 and γ_2 should satisfy; $D\gamma_1|_x = 1$ for all $x \in [\alpha_1, \beta_1]$ and $D\gamma_2|_x = 1$ for all $x \in [\alpha_2, \beta_2]$. For each chord c_i of type 1.) we then choose a diffeomorphism $r_i: [a_i, b_i] \to [\alpha_1, \beta_1]$ where $Dr_i|_{a_i} = Dr_i|_{b_i} = \epsilon$ and $Dr_i|_x > \epsilon$ for $x \in (a_i, b_i)$. For each chord c_i of type 2.) we then choose a diffeomorphism $r_i: [a_i, b_i] \to [c, d]$ where again $Dr_i|_{a_i} = Dr_i|_{b_i} = \epsilon$ and $Dr_i|_x > \epsilon$ for $x \in (a_i, b_i)$. Finally for each chord c_i of type 3.)

we choose a diffeomorphism r_i: $[a_i, b_i] \to [\alpha_2, \beta_2]$ where $Dr_i|_{a_i} = Dr_i|_{b_i} = \epsilon$ and $Dr_i|_x > \epsilon$ for $x \in (a_i, b_i)$. We then define for c_i of type 1.) $g|\bar{c}_i = \gamma_1 \circ r_i \circ h_i$, for c_i of type 2.) $g|\bar{c}_i = \pi \circ r_i \circ h_i$ and for c_i of type 3.) $g|\bar{c}_i = \gamma_2 \circ r_i \circ h_i$.

We now must verify that the mapping g constructed in this manner is a valid mapping as well as satisfying the conditions necessary to be a W^*-mapping. In order to verify that g is a well defined mapping it is only necessary to check that this is so on the vertices since they are the only points on which g might be multiply defined, but every vertex v in K is mapped to the vertex v_0, thus g is well defined. We again see that g is continuous since $g|\bar{e}$ is continuous for each edge in e.

In order to verify that g is an immersion we must check that Dg is a monomorphism on the tangent space a each point. It will suffice to show that Dg is well defined, continuous and not equal to zero at any point. Again the only points which may cause us trouble are the vertices since the points in the interior of each edge clearly satisfy these conditions. Let v be a vertex of K then $Dg|_v = \epsilon$, thus Dg is not equal to zero at any point and it is well defined. For any edge e in T the mappings $\pi \circ r_e \circ h_e$ are the composition of immersions and thus immersions, recall that the r_e and the h_e are such that $Dr_e|_x > 1$ for all $x \in [a(e), be)]$ and $Dh_e|_x = 1$ for all $x \in \bar{e}$. Similarly for any chord c_i the mappings $\gamma_1 \circ r_i \circ h_i$, $\gamma_2 \circ r_i \circ h_i$ and $\pi \circ r_i \circ h_i$ are all the compositions of immersions and thus immersions. If e is an edge in K which is incident with the vertex v then as we approach v along e then Dg will approach ϵ continuously by the definition of g as the composition of immersions in the interior of e.

We must also check at each vertex that we are not bending back any of the branches. Let v be a vertex. If e is an edge of T and $e \in I_\rho^-(v)$ then we can find a small arc $[v, p_e) \subset \bar{e}$ at v such that $g([v, p_e))$ maps into the initial edge of W, e_1. Similarly if e is an edge of T and $e \in I_\rho^+(v)$ then we can find a small arc $(q_e, v] \subset \bar{e}$ at v such that $g((q_e, v])$ maps into the final edge of W, e_k. If c_i is a chord which is incident with v and if c_i is of type 1.) it will be on the same side of v as the edges in $I_\rho^-(v)$ and opposite the edges in $I_\rho^+(v)$ accordingly we may find a small arc $[v, s_i) \subset \bar{c}_i$ at v such that $g([v, s_i))$ maps into the initial edge e_1 of W. If c_i is a chord incident with v and if c_i is of type 3.) then it will be on the same side of v as the edges in $I_\rho^+(v)$ and opposite the edges in $I_\rho^-(v)$, again we may find a small arc $(s_i, v] \subset \bar{c}_i$ such that $g((s_i, v])$ maps into the final edge e_k of W. If c_i is a chord incident upon v and it is of type 2.) and v is the sense of an edge in T then

c_i will be of on the same side of v as the edges in $I_\rho^-(v)$ and opposite the edges in $I_\rho^+(v)$ and we may find a small arc $[v, s_i) \subset \bar{c}_i$ at v where $g([v, s_i))$ maps into the initial edge e_1 of W. If c_i is a chord incident upon v and it is of type 2.) and v is not the sense of an edge in T then c_i is on the same side of v as the edges in $I_\rho^+(v)$ and opposite the edges in $I_\rho^-(v)$ and we may find a small arc $(s_i, v] \subset \bar{c}_i$ at v such that $g((s_i, v])$ maps into the final edge e_k of W. Thus the branches of K are not bent under g.

From the above we also see that every vertex v has a small neighborhood N_v, consisting of the union of the arcs constructed as above, such that $g(N_v) = e_1 \cup \{v_0\} \cup e_k$. Thus W3 is satisfied. Since for all $x \in K$ we have $Dg|_x \geq \epsilon > 1$ we know that g is an expansion and W1 is satisfied. Also, since $g(\bar{e}) = K$ for all edges $e \in K$ we know that the nonwandering set $\Omega(g) = K$ by theorem 10 and therefore W2 is satisfied. Lastly we know that $g(V) = \{v_0\}$ making g a W^*-mapping. □

Chapter 4

Presentations of Solenoids

In this section we discuss methods of moving from one presentation, $\{K, g\}$, of a solenoid to an equivalent presentation (up to the topological conjugacy of their shift maps), $\{K', g'\}$, in which the smooth graph K' is simplified or which is such that the W^*-mapping g' has some desirable property. The methods in this section will prove particularly useful for finding invariants for the topological conjugacy of the solenoids.

The first method that we discuss is a slightly stronger version of a method due to R.F Williams [39] and applies only to presentations where the W^*-mapping has a fixed point. It allows us to find a presentation equivalent to our original, called an elementary presentation, in which the smooth graph has only a single branch point which is fixed under the W^*-mapping. This allows us to convert the question of whether two solenoids given by elementary presentations are topologically conjugate into a purely algebraic question.

The second method we discuss involves finding a presentation $\{K', g'\}$ equivalent to the original where the W^*-mapping g' acts on a finite set of periodic points in a prescribed manner. This allows us to find algebraic invariants for the topological conjugacy of solenoids and can be applied to any solenoid.

4.1 Elementary Presentations

Given a presentation $\{K, g\}$ of a solenoid with at least one fixed point it is possible to find a shift equivalent presentation $\{K', g'\}$ in which the smooth graph K' has only a single branch point. In this section we show how this can be done. Our motivation for doing this will be to convert the question of whether two solenoids are equivalent into an algebraic problem which we can try to solve directly or use to extract invariants.

Definition 35 By an *elementary smooth graph* K we mean a smooth graph which has a single vertex b and a number of edges of which

1. some (type O) are both right and left at b,
2. some (type R) are right at b,
3. some (type L) are left at b.

Note, that an elementary smooth graph is orientable if and only if all of its edges are of type O and it is always recurrent. We will denote an orientable elementary smooth graph by $O(m)$ where m is the number of edges. A nonorientable elementary smooth graph is recurrent if and only if it possess an edge of type R and an edge of type L. We will denote a nonorientable elementary smooth graph by $N(n_1, n_2, n_3)$ where n_1 is the number of edges of type O, n_2 is the number of edges of type R, and n_3 is the number of edges of type n_3.

Our first order of business will be to demonstrate how any presentation $\{K, g\}$ of a solenoid is shift equivalent to a presentation $\{K', g'\}$ where every branched point is of type $(2, 1)$. The method we present for doing this is based on work by R. F. Williams in [38].

Theorem 35 *Let K be a smooth graph, $g: K \to K$ a W^* – mapping and b a branch point of K where there exists two arcs $[b, a]$ and $[b, c]$ at b such that $g([b, a]) = g([b, c])$ with $g(a) = g(c)$. Let $\tilde{K} = K/\sim$ with the induced differential structure and Riemannian metric, where $\sim$ is the equivalence relation given by*

$$x \sim y \text{ if and only if } \begin{cases} x = y \text{ or,} \\ x \in [b, a], y \in [b, c] \text{ and } g(x) = g(y). \end{cases}$$

Then there exists a W^*-mapping $\tilde{g}: \tilde{K} \to \tilde{K}$ such that $g \sim_s \tilde{g}$.

Proof. Let $r: K \to \tilde{K}$ be the projection map. Then $g: K \to K$ induces a mapping $\tilde{g}$ on $\tilde{K}$ given by $\tilde{g} = r \circ g \circ r^{-1}$. This is a well defined mapping since if $\{x, y\} = r^{-1}(z)$ then

$g(x) = g(y)$. Similarly we may define a mapping $s: \tilde{K} \to K$ by $s = g \circ r^{-1}$. This is again a well defined mapping for exactly the same reason as before. We then see that

$$\tilde{g} \circ r = r \circ g \circ r^{-1} \circ r = r \circ g, \; s \circ \tilde{g} = g \circ r^{-1} \circ r \circ g \circ r^{-1} = g \circ s,$$

$$s \circ r = g \circ r^{-1} \circ r = g, \text{ and } r \circ s = r \circ g \circ r^{-1} = \tilde{g}.$$

Thus we have $g \sim_s \tilde{g}$. □

Theorem 36 *Let K be a smooth graph and $g: K \to K$ a W^*-mapping. Then there exists a smooth graph K' with all branch points of type $(2,1)$ and a W^*-mapping $g': K' \to K'$ such that $g \sim_s g'$.*

Proof. If all of the branch points of K are of type $(2,1)$ then we are done. Suppose not. Write $K = K_0$ and $g = g_0$. Let B_0 be the branch set of K_0 and

$$B_0(j) = \{x \in B_0 : x \text{ of type } (p,q) \text{ with } p+q=j\}$$

for $j = 3, 4, \ldots, N_0$ where $N_0 = \max\{j: B_0(j) \neq \emptyset\}$. We will show that there exists a smooth graph K' with all its branch points of type $(2,1)$ and a W^*-mapping g' such that $g \sim_s g'$ by finding shift equivalences $g_{i-1} \sim_s g_i$ of lag=1 for $i = 1, \ldots, m = \sum_{j=1}^{N_0-3} j|B_0(N_0 - j + 1)|$ where $g_i: K_i \to K_i$, $K_i = K_{i-1}/\sim_i$ and the equivalence relations $\sim_i$ are chosen so that $K_m = K'$ has all branch points of type $(2,1)$.

There must exist a branch point $b \in B_0(N_0)$ such that there are two arcs $[b,a]$ and $[b,c]$ at b where $g_0([b,a]) = g_0([b,c])$ and $g_0(a) = g_0(c)$ a vertex, since otherwise there would not exist a neighborhood M_b of b with $g^n(M_b)$ an arc. Also since g_0 is an expansion a and c can be chosen to be points in the interior of an edge incident upon b, possibly by labeling a point of type $(1,1)$ a vertex in K_0. As in theorem 35 we let $K_1 = K_0/\sim_1$ where $\sim_1$ is the equivalence relation given by

$$x \sim_1 y \text{ if and only if } \begin{cases} x = y \text{ or}, \\ x \in [b,a], y \in [b,c] \text{ and } g(x) = g(y). \end{cases}$$

There is then a $g_1 \sim_s g_0$ of lag= 1 by theorem 35.

Let B_1 be the branch set of K_1 and

$$B_1(j) = \{x \in B_1 : x \text{ of type } (p,q) \text{ where } p+q=j\}.$$

It is easy to see that

$$|B_1(N_0)| = |B_0(N_0)| - 1,$$

$$|B_1(N_0 - 1)| = |B_0(N_0 - 1)| + 1,$$

$$|B_1(j)| = |B_0(j)| \text{ for } j = 4, \ldots, N_0 - 2,$$

$$|B_1(3)| = |B_0(3)| + 1.$$

We may now repeat this procedure for K_i and $g_i: K_i \to K_i$ to get a shift equivalence $g_i \sim_{i+1} g_{i+1}$ of lag= 1. At each stage we choose a branch point b_i where there exist arcs $[b_i, c_i]$ and $[b_i, a_i]$ at b_i such that $g_i([b_i, c_i]) = g_i([b_i, a_i])$, a_i and c_i are in the interior of an edge incident upon b_i, $g_i(a_i) = g_i(c_i)$ a vertex. The branch point b_i is chosen to belong to $B_i(N_i)$ where $B_i(j) = \{x \in B_i: x \text{ of type } (p, q) \text{ where } p+q = j\}$, $N_i = \max\{j: B_i(j) \neq \emptyset\}$, and B_i is the branch set of K_i. Again we see that

$$|B_{i+1}(N_i)| = |B_i(N_i)| - 1,$$

$$|B_{i+1}(N_i - 1)| = |B_i(N_i - 1)| + 1,$$

$$|B_{i+1}(j)| = |B_i(j)| \text{ for } j = 4, \ldots, N_i - 2,$$

$$|B_{i+1}(3)| = |B_i(3)| + 1.$$

Thus after $n_1 = |B_0(N_0)|$ steps we reach a smooth graph K_{n_1} which has $B_{n_1}(N_0) = \emptyset$. After $n_2 = 2|B_0(N_0)| + |B_0(N_0) - 1|$ steps we reach a smooth graph K_{n_2} which has $B_{n_2}(N_0) = B_{n_2}(N_0 - 1) = \emptyset$. After $n_p = \sum_{j=1}^{p} j|B_0(N_0) - j + 1|$ steps we reach a smooth graph K_{n_p} which has

$$B_{n_p}(N_0) = B_{n_p}(N_0 - 1) = \ldots = B_{n_p}(N_0 - p + 1) = \emptyset.$$

Therefore we see that after $m = \sum_{j=1}^{N_0-3} j|B_0(N_0) - j + 1|$ steps we reach a smooth graph K_m which has only branch points of type $(2, 1)$, i.e., $B_m(N_0 - j + 1) = \emptyset$ for $j = 1, \ldots, N_0 - 3$. At each stage we have a lag= 1 shift equivalence $g_{i-1} \sim_s g_i$ therefore $g_0 \sim g_m$ with lag= m. □

See example 11 for the explicit use of the techniques developed in theorem 36 above.

Definition 36 Let K be a smooth graph, $g: K \to K$ a W^*-mapping and x_0 a point in K. The *orbit* of x_0 is the set of points $\text{orb}\,(x_0) = \{x_0, x_1, x_2, \ldots\}$ where $x_{i+1} = g^i(x_i)$, $i = 0, 1, 2, \ldots, \infty$.

Definition 37 Let K be a smooth graph, $g: K \to K$ a W^*-mapping. A point $y \in K$ is said to be *eventually periodic* if $g^m(y) = x$ for some m where x is a periodic point. The point y is referred to as being eventually periodic to orb(x).

Theorem 37 *Let K be a smooth graph, $g: K \to K$ a W^*-mapping and x_0 a periodic point of g. Then the set of points eventually periodic to orb(x) is dense in K.*

Proof. We need to show that given any point $y \in K$ and any neighborhood N_y of y there is a point $z \in N_y$ such that z is eventually periodic to orb(x). If y is eventually periodic to orb(x) then we are done. Suppose not. An easy consequence of theorem 10 is that given any arc J in K there is an integer m such that g^m maps J onto K. Thus we may choose a small arc $J_y \subset N_y$ about y. There exists an integer m such that g^m maps J_y onto K. Since $g^m|J_y$ is onto there must be a point $z \in J_y$ such that $g^m(z) = x$. Thus z is eventually periodic to orb(x). □

Theorem 38 *Let K be a smooth graph and $g: K \to K$ a W^*-mapping where every branch point of K is of type $(2,1)$. Then there is a smooth graph K' and a W^*-mapping $g': K' \to K'$ where $g \sim_s g'$ such that every branch point of K' is of type $(2,1)$ and at every point $x \in K'$ there is a neighborhood N_x of x such that $g(N_x)$ is an arc.*

We refer the reader forward to example 12 which gives an illustration of the techniques we develop in the proof of theorem 38.

Proof. If $g: K \to K$ is such that for all $x \in K$ there is a neighborhood N_x with $g(N_x)$ an arc we are finished. Suppose not.

The set of points A where there does not exist a neighborhood N_x of $x \in A$ with $g(N_x)$ an arc must be a subset of the branch set, i.e., $A \subset B$. Also we know that $g(A) \subset B$ and that $A \neq B$ since there must exist at least one branch point with a neighborhood whose image under g is an arc.

Let p be the smallest integer such that for all $x \in K$ there is a neighborhood N_x of x such that $g^p(N_x)$ is an arc. For each $i = 1, \ldots, p$ let A_i denote the set of branch points in K for which i the smallest integer such that there exists a neighborhood N_x for $x \in A_i$ with $g^i(N_x)$ an arc.

We claim that $A_i \neq \emptyset$ for each $i = 1, \ldots, p$. There exists a branch point b which belongs to A_p by definition. For each $j = 1, \ldots, p-1$ the image of b under g^j, $g^j(b)$, must not equal any of the points $b, g(b), \ldots, g^{j-1}(b)$ and $g^j(b)$ must belong to A_{p-j} since there will be a neighborhood of $g^j(b)$ which is an arc under g^{p-j}. Thus $A_i \neq \emptyset$ for $i = 2, \ldots, p-1$. It is also easy to verify that $g(A_i) \subseteq A_{i-1}$ for $i = 2, \ldots, p$.

Let $C = g(A_2)$. Label the elements of $C = \{c_1, \ldots, c_s\}$. For each $c_i \in C$ there is a neighborhood N_i where $g(N_i)$ is an arc. We may pick two points a_i and d_i in the interior of each N_i so that $g(a_i) = g(d_i)$. We may choose a_i and d_i so that they are eventually periodic to some $\operatorname{orb}(v)$ where v is a periodic point in V. This is a result of theorem 37 as well as the fact that $g(V) \subseteq V$. There will be two small arcs $[c_i, a_i]$ and $[c_i, d_i]$ at each c_i. As in theorem 35 we may form a new smooth graph $\tilde{K} = K/\sim$ where $\sim$ is the equivalence relation given by

$$x \sim y \text{ if and only if } \begin{cases} x = y \text{ or,} \\ x \in [c_i, a_i], y \in [b_i, c_i] \text{ and } g(x) = g(y) \text{ for some } i. \end{cases}$$

By theorem 35 there will exist W^*-mapping $\tilde{g}\colon \tilde{K} \to \tilde{K}$ such that $g \sim_s \tilde{g}$.

We claim the smooth graph $\tilde{K}$ has the same number of branch points as K. Under the projection map $\pi\colon K \to \tilde{K}$ we see that the image of each $x \in B - C$ is a branch point and that π is one to one and onto from $B - C$ to $\pi(B - C)$. The image under π of each c_i is an ordinary point, the image under π of each pair a_i and d_i is a branch point and the image of each $x \in K - B - \bigcup_i^s \{a_i, d_i\}$ is an ordinary point. The effect of the identification can be visualized as the collapsing of a small neighborhood around each c_i in K.

In order for $\tilde{g}$ to be a W^* mapping we must have the image of every vertex $v \in K$ a vertex in $\tilde{K}$ as well as the "new" branch points $\pi(a_i) = \pi(d_i)$ and the finite set of points $\{\pi \circ \tilde{g}^n(a_i) = \pi \circ \tilde{g}^n(d_i) \colon n = 1, \ldots, \infty\}$. Note, since a_i and d_i are eventually periodic under g that $\pi(a_i) = \pi(d_i)$ will be eventually periodic under $\tilde{g}$.

Let q be the smallest integer such that every point $x \in \tilde{K}$ has a neighborhood N_x such that $\tilde{g}(N_x)$ is an arc. We claim that $q = p - 1$. We will show that this is the case by examining the images under $\tilde{g}$ of neighborhoods of the branch points of $\tilde{K}$.

For $x \in \pi(A_1)$ we have two possibilities; $x = \pi(y)$ with $y \in C$ or $x = \pi(y)$ with $y \in A_1 - C$. In the first instance there is definitely a neighborhood N_x of x where $\tilde{g}(N_x)$ is an arc since x is not a branch point. In the second instance we know that there is a neighborhood N_y of y where $g(N_y)$ is an arc thus $\pi \circ g(N_y)$ will be an arc. $N_x = \pi(N_y)$ will be a neighborhood of x and since g and $\tilde{g}$ are commutative under π, i.e., $\pi \circ g(N_y) = \tilde{g}(N_x)$, we know that $g(N_x)$ is an arc.

For each $x \in \pi(A_2)$ there will be two points α_x and β_x such that $g(\alpha_x) = a_i$ and $g(\beta_x) = b_i$ for some i. The points α_x and β_x will be in the interior of the two edges, e_α and e_β, incident upon x and on the same side of x. We the can choose a point γ_x in the interior of the single edge opposite e_α and e_β. Consider the arcs $[\gamma_x, \alpha_x]$ and $[\gamma_x, \beta_x]$ through x. The union, $N_x = [\gamma_x, \alpha_x] \cup [\gamma_x, \beta_x]$ is a neighborhood about x. The image of N_x under g will be a neighborhood about c_i. Similarly the image of N_x under π will be a neighborhood about $\pi(x)$. The image under π of $g(N_x)$ however will be an arc since points along the two arcs $[c_i, a_i]$ and $[c_i, d_i]$ are identified under π. Thus each point $\pi(x)$ has a neighborhood $\pi(N_x)$ whose image under $\tilde{g}$ is an arc since g and $\tilde{g}$ commute with respect to π. Thus every point $x \in \pi(A_2)$ has a neighborhood N_x where $g(N_x)$ is an arc.

Since $g(A_i) \subseteq A_{i-1}$ for $i = 2, \ldots, p$ we may continue in the fashion above to show that for $x \in \pi(A_i)$ there is a neighborhood N_x such that $g^{i-1}(N_x)$ is an arc.

The only branch points in $\tilde{K}$ we have not examined yet are those of the form $\pi(a_i) = \pi(d_i)$ for some $i = 1, \ldots, s$. Each point c_i has a neighborhood N_i such that $g(c_i)$ is an arc. The points a_i and d_i where chosen to be in the interior of N_i. Thus we may choose a small neighborhood P_i about a_i and Q_i about d_i such that $g(P_i) = g(Q_i)$ and $P_i, Q_i \subset N_i$. Since $P_i, Q_i \subset N_i$ and $g(N_i)$ is an arc we know that $g(P_i) = g(Q_i)$ is an arc. Under π the union $P_i \cup Q_i$ will be a neighborhood, $W_i = \pi(P_i \cup Q_i)$, about $\pi(c_i)$. Thus since g and $\tilde{g}$ commute under π we know that $\tilde{g}(W_i) = \pi \circ g(P_i) = \pi \circ g(Q_i)$ is an arc.

Thus $q = p - 1$ where q is the smallest integer such that for all $x \in \tilde{K}$ there is a neighborhood N_x where $g^q(N_x)$ is an arc. Note that the "new" smooth graph $\tilde{K}$ has all branch point of type $(2, 1)$.

Let $\tilde{K} = K_1$ and $\tilde{g} = g_1$. If we now repeat this procedure on K_i for $i = 1, \ldots p-1$ we will produce a smooth graph K_{i+1} and W^*-mapping $g_{i+1}\colon K_{i+1} \to K_{i+1}$ where $g_i \sim_s g_{i+1}$ with lag$= 1$. Let q_i be the smallest integer such that for all $x \in K_i$ there is a neighborhood N_x where $g_i^{q_i}(N_x)$ is an arc. We will have $p = q_1 + 1 = q_2 + 2 = \cdots = q_{p-1} + (p-1)$ so that $q_{p-1} = 1$. Thus $K' = K_{p-1}$ is the desired smooth graph and $g' = g_{p-1}$is the desired W^*-mapping. □

Theorem 39 *Let K be a connected smooth graph and $g\colon K \to K$ a W^*-mapping with fixed point x_0, $A = \{x \in K\colon g(x) = x_0\}$ and $\hat{K} = K/\sim$, with the induced differential* *structure and Riemannian metric, where $\sim$ is the equivalence relation given by*

$$x \sim y \text{ if and only if } \begin{cases} x = y \text{ or,} \\ x, y \in A \cup \{x_0\}. \end{cases}$$

Then there exists a W^*-mapping $\hat{g}: \hat{K} \to \hat{K}$ such that $g \sim_s \hat{g}$.

Proof. Let $r: K \to \hat{K}$ be the projection map. Then $g: K \to K$ induces a mapping $\hat{g}$ on $\hat{K}$ given by $\hat{g} = r \circ g \circ r^{-1}$. This is clearly a well defined map since $r(x_0)$ is the only point at which r^{-1} is multivalued, but $r^{-1}(x_0) = A$ and $g(A) = x_0$ which is a single point. Similarly we may define a mapping $s: \hat{K} \to K$ by $s = g \circ r^{-1}$. The map s is well defined for the same reason as before. We then see that, as in theorem 35,

$$\tilde{g} \circ r = r \circ g \circ r^{-1} \circ r = r \circ g, \quad s \circ \tilde{g} = g \circ r^{-1} \circ r \circ g \circ r^{-1} = g \circ s,$$

$$s \circ r = g \circ r^{-1} \circ r = g, \text{ and } r \circ s = r \circ g \circ r^{-1} = \tilde{g}.$$

Thus we have $g \sim_s \tilde{g}$. □

The next theorem is based on a result by R.F. Williams in [39], but uses a slightly different method. Williams showed that if K is a smooth graph and $g: K \to K$ is a W^*-mapping then there will exist an integer m and an x_0 such that $g^m(x_0) = x_0$ and each embedded 1-sphere in K will contain a point in $g^{-1}(x_0)$. He then showed that there will exist a smooth graph K' and W^*-mapping $g': K' \to K'$ where $g' \sim_s g^m$, with shift equivalence given by $r: K \to K'$ and $s: K' \to K$, and K' such that every embedded 1-sphere in K' contains $r(x_0)$. The next theorem is similar, but it gives control over the integer m, allowing us to find a shift equivalence with $m = 1$. It differs in that it requires that we start with a mapping with a fixed point. Williams' method does not allow control over m regardless of whether or not you start with a mapping with a fixed point. We should note that our result does imply his since for any smooth graph K and W^*-mapping g there will exist an integer n such that g^n has a fixed point.

Theorem 40 *Let K be a connected smooth graph and $g: K \to K$ a W^*-mapping with fixed point x_0. Then there exists a smooth graph K' and W^*-mapping $g': K' \to K'$ such that $g \sim_s g'$, where the shift equivalence is given by the maps $r: K \to K'$ and $s: K' \to K$ and K' is such that every embedded 1-sphere in K' contains the point $r(x_0)$.*

We refer the reader forward to example 13 which gives an illustration of the techniques we develop in the proof of theorem 38.

Proof. Write $K = K_0$ and $g = g_0$. It is clear that K_0 will contain at most a finite number of embedded 1-spheres. If every one of these one spheres contains x_0 then we are done. Suppose not.

Let $A_0^0 = \{x_0\}$, $A_1^0 = \{x \in K_0 : g_0(x) = x_0\}$, and $A_i^0 = \{x \in K : g(x) \in A_{i-1}^0\}$ for $i = 1, \ldots, \infty$. We necessarily have $g(A_i^0) = A_{i-1}^0$ for $i = 1, \ldots, \infty$. Let N be the smallest integer so that every embedded 1-sphere contains a point in $\bigcup_{i=0}^N A_i^0$. We are guaranteed the existence of a finite such N by theorem 37.

Let $K_1 = K_0 / \sim_1$, with the induced differential structure and Riemannian metric, where $\sim_1$ is the equivalence relation given by

$$x \sim_1 y \text{ if and only if } \begin{cases} x = y \text{ or }, \\ x, y \in A_0^0 \cup A_1^0. \end{cases}$$

By theorem 39 there exists a W^*-mapping $g_1 : K_1 \to K_1$ where $g_1 \sim_s g_0$ with shift equivalence given by the maps $r_1 : K_1 \to K_0$ and $s_1 : K_0 \to K_1$.

Let $A_0^1 = \{r_1(A_0^0 \cup A_1^0)\}$, $A_1^1 = \{r_1(A_2^0)\}$, and $A_i^1 = \{r_1(A_{i+1}^0\}\}$ for $i = 1, \ldots, \infty$. We see that A_0^1 contains just a single point, $r(x_0)$ which is fixed under g_1. We also see that $g_1(A_i^1) = A_{i-1}^1$ for $i = 1, \ldots, \infty$.

If J is an embedded 1-sphere in K_0 which contains points of A_1^0 then $r_1(J)$ will be a finite number of embedded and immersed 1-spheres in K_1 which contain $r(x_0)$. If J is an embedded 1-sphere in K_0 which contains points in A_i^0, $i \geq 2$, and does not contain points in A_1^0 then $r_1(J)$ will be an embedded 1-sphere in K_1 which contains points in A_{i-1}^1. Thus every embedded 1-sphere in K_1 will contain a point in $\bigcup_{i=0}^{N-1} A_i^1$.

We may then continue in this fashion for $j = 1, \ldots, N-1$ where $K_{j+1} = K_j / \sim_{j+1}$ and $\sim_{j+1}$ is the equivalence relation given by

$$x \sim_{j+1} y \text{ if and only if } \begin{cases} x = y \text{ or }, \\ x, y \in A_0^j \cup A_1^j. \end{cases}$$

Then by theorem 39 there will exist a W^*-mapping $g_{j+1} : K_{j+1} \to K_{j+1}$ where $g_{j+1} \sim_s g_j$ with shift equivalence given by $r_{j+1} : K_{j+1} \to K_j$ and $s_{j+1} : K_j \to K_{j+1}$. In the above $A_0^j = \{r_j(A_0^{j-1} \cup A_1^{j-1})\}$ and $A_i^j = \{r_j(A_{i+1}^{j-1})\}$ for $i = 1, \ldots, \infty$. Again notice that A_0^j will consist of the single point $r_j \circ r_{j-1} \circ \cdots \circ r_1(x_0)$ which is fixed under g_j. K_j will be such that each embedded 1-sphere will contain a point of $\bigcup_{i=0}^{N-j} A_i^j$. When we reach $j = N$ we will have a smooth graph $K_N = K'$ which is such that every embedded 1-sphere in K' contains a point in A_0^N which consists of the single fixed point $r_N \circ r_{N-1} \circ \cdots \circ r_1(x_0)$. The mappings $r = r_N \circ r_{N-1} \circ \cdots \circ r_1$ and $s = s_N \circ s_{N-1} \circ \cdots \circ s_1$ give the shift equivalence between g_0 and g_N. □

Definition 38 If K is an elementary smooth graph (branched 1-manifold) with branch point b and $g: K \to K$ is a W^*-mapping where $g(b) = b$ then we say that $\{K, g\}$ is an *elementary presentation* of the solenoid K_∞ and shift map h.

The following theorem is based on a result of R.F Williams [39]. The proof that we give is similar to that in [39], but we have made some modifications as Williams' original proof contained an error. Given a presentation $\{K_0, g_0\}$ the technique in Williams' proof consists of finding a "new" presentation $\{K_m, g_m\}$, where $g_m \sim_s g_0$ with lag= m, by moving through a finite number of presentations, $\{K_i, g_i\}$, where $g_{i-1} \sim_s g_i$ with lag=1 for $i = 1, \ldots, m$. Williams' error was the fact in certain circumstances the map $r_i: K_{i-1} \to K_i$ which he constructed as part of the shift equivalence between K_{i-1} and K_i will not always be onto. Example 14 illustrates how this can occur. We give a "new" proof in complete detail as it gives a procedure for finding elementary presentations. We note that although the version that we give is slightly stronger than that stated in [39] this is as a result of theorem 40. The proof is essentially same except for a few modifications.

Theorem 41 *Let $g: K \to K$ be a presentation of (K_∞, h) with a fixed point x. Then there is an elementary presentation $g': K' \to K'$ such that (K'_∞, h') is topologically conjugate to (K_∞, h), i.e., $g \sim_s g'$.*

Proof. Let $K = K_0$ and $g = g_0$. From theorems 36 and 38 we know that there exists a smooth graph K_1 and W^*-mapping $g_1: K_1 \to K_1$ where $g_1 \sim_s g_0$, K_1 has all branch points of type (2,1) and every point of $x \in K_1$ is such that there exists a neighborhood N_x of x such that $g_1(N_x)$ is an arc. Let $r_1: K_0 \to K_1$ and $s_1: K_1 \to K_0$ be the mappings giving the shift equivalence. If we then perform the procedure given in theorem 40 we find a smooth graph K_2 and a W^*-mapping g_2 where $g_2 \sim_s g_1$, with the shift equivalence given by $r_2: K_1 \to K_2$ and $s_2: K_2 \to K_1$,and where K_2 is such that every 1-sphere in K_2 contains the point $r_2 \circ r_1(x_0)$. We also note that every branch point of K_2 except

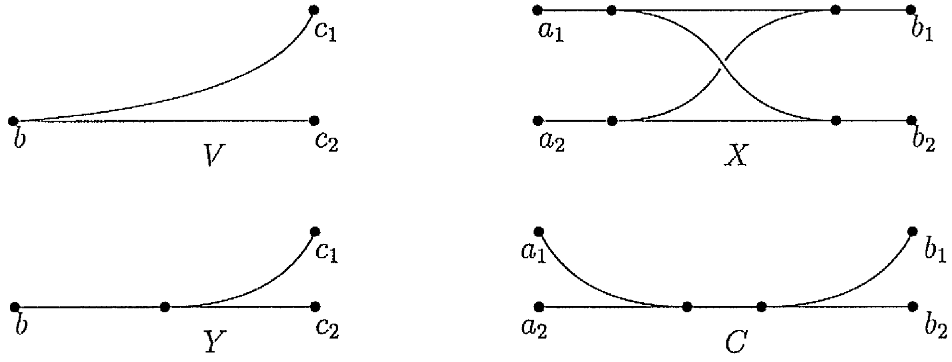


Figure 4.1:

$r_2 \circ r_1(x_0)$ will be of type (2,1) and every point in $x \in K_2$ except $r_2 \circ r_1(x_0)$ will have a neighborhood N_x such that $g_2(N_x)$ is an arc.

If K_2 has no branch points other then $b = r_2 \circ r_1(x_0)$ then we are done. Suppose not. We will find an elementary smooth graph K' and W^*-mapping $g': K' \to K'$ with $g' \sim_s g_2$ by "removing" the branch points other than $b = r_2 \circ r_1(x_0)$ using a reduction process. Let V, Y, X and C be the branched 1-manifolds and boundaries given in figure 4.1. Note that there are "natural" maps $V \to Y$ and $X \to C$. The *stem* of C is $[a_1, b_1] \cap [a_2, b_2]$. The *stem* of Y is $[b, c_1] \cap [b, c_2]$.

We proceed to remove the branch points of K_2 by a series of two types of "moves".

1. Remove a copy of C from $K_i - b$ and replace it with a copy of X.

2. Remove a copy of Y from K_i and replace it with a copy of V.

At each move we pass from $g_i: K_i \to K_i$ to a shift equivalent $g_{i+1}: K_{i+1} \to K_{i+1}$ with shift equivalence given by $r_{i+1}: K_i \to K_{i+1}$ and $s_{i+1}: K_{i+1} \to K_i$. With the exception of $r_i \circ r_{i-1} \circ \cdots \circ r_3(b)$, no branch other then those of type (2,1) are allowed. Each K_i will be such that every embedded 1-sphere will contain $r_i \circ r_{i-1} \circ \cdots \circ r_3(b)$. We call $r_i \circ r_{i-1} \circ \cdots \circ r_3(b)$ the distinguished branch point throughout and will be referred to as just b. To show that this process converges, we introduce two weight functions:

$w_1(K_i)$ is the number of stems of C-sets in $K_i - b$.

$w_2(K_i)$ is the number of stems of Y-sets in $K_i - b$.

If C_0 is a copy of C in $K_i - b$, where $C_0 = \{a_1, a_2, b_1, b_2\}$ is open in K_i, and K_{i+1} is formed by replacing C_0 with a copy X_0 of X, then $w_1(K_{i+1}) < w_1(K_i)$. We see this because if C_1' is a copy of C remaining in K_{i+1}, such that

(a) $C_1' \cap X_0 = \emptyset$. Then C_1' corresponds to $C \subset K_i$;

(b) $C_1' \cap X_0 = [a_i, b_j]$. Then C_1' corresponds to $C_1 \subset K_i$ where $C_1 \cap C_0 = [a_i, b_j]$; or

(c) $C_1' \cap X_0 = [a_1, b_1] \cup [a_1, b_2]$ (or one of the other four copies of Y in X_0), then there is a corresponding $C_1 \subset K_i$, where $C_1 \cap C_0 = [a_1, b_1] \cup [a_1, b_2]$, etc.

Finally, note that the stem of C_0 has been removed and not replaced so that $w_1(K_i) = w_1(K_{i+1}) + 1$.

If Y_0 is a copy of Y in K_i with $Y - \{b, c_1, c_2\}$ open in K_i and K_{i+1} is formed by replacing Y_0 with a copy V_0 of V, then $w_2(K_i) = w_2(K_{i+1}) + 1$. This is similar to and easier than the case for a copy of C in K_i.

Consider $g_i \colon K_i \to K_i$ a W^*-mapping which is such that every 1-sphere of K_i contains the distinguished branch point b and where every point $x \in K_i - \{b\}$ has a neighborhood N_x such that $g_i(N_x)$ is an arc. Suppose that K_i has stem copies S_1 and S_2 of C, we say that $S_1 < S_2$ provided $g_i^n(S_1) \subset S_2$ for some $n > 0$. This is a partial ordering, in particular $S_1 < S_1$ is impossible as g_i is an expansion.

Assume that $w_1(K_i) > 0$. Then there is a stem S_0 of a copy of C in $K_i - b$, minimal relative to this partial ordering. A copy C_0 of C can be chosen small enough so that if K_{i+1}' is formed as described then there is a $K_{i+1} \subseteq K_{i+1}'$ and a map $g_{i+1} \colon K_{i+1} \to K_{i+1}$ which is shift equivalent to g_i. This is done as follows. We choose C_0 small enough so that $g|C_0$ factors through an arc. This is possible as at every point of $x \in K_i$ except the distinguished b there exists a neighborhood N_x such that $g_i(N_x)$ is an arc. The natural map $X \to C$ provides the map $s_{i+1}' \colon K_{i+1}' \to K_i$. We claim there is a map $r_{i+1}' \colon K_i \to K_{i+1}'$ so that the following diagram is commutative:

$$\begin{array}{ccc} K_{i+1}' & \xrightarrow{g_{i+1}'} & K_{i+1}' \\ {\scriptstyle s_{i+1}'} \downarrow & \overset{r_{i+1}'}{\nearrow} & \downarrow {\scriptstyle s_{i+1}'} \\ K_i & \xrightarrow[g_i]{} & K_i \end{array}$$

Suppose $x \in K_i$ and consider

Case 1.) $g_i(x) \notin C_0$. Then $r_{i+1}'(x)$ is the unique point $(s_{i+1}')^{-1} \circ g_i(x)$.

Case 2.) $x \in I_1 \cap I_2$, where $g_i(I_p) = [a_p, b_p]$, $p = 1, 2$. This is impossible by the minimality of S_0.

Case 3.) $x \in I_1 \cap I_2$, where $g_i(I_p) = [a_q, b_p]$, $p = 1, 2$ for some $q = 1, 2$. Then define $r'_{i+1}|I_p$ to map I_p to $[a_q, b_p] \subset K'_{i+1}$.

Case 4.) For some p, q, x lies on one or more one-cells I, each of which maps under g_i to $[a_p, b_q] \subset C_0$. Then r'_{i+1} takes such an I to $[a_p, b_q] \subset X_0$ so that $s'_{i+1} \circ r'_{i+1} = g_i$.

Note that the maps $r'_{i+1}: K_i \to K'_{i+1}$ and $g'_{i+1}: K'_{i+1} \to K'_{i+1}$, defined above, are not necessarily onto. Thus we define

$$K_{i+1} = r'_{i+1}(K_i), \quad s_{i+1} = s'_{i+1}|r'_{i+1}(K_i),$$
$$r_{i+1} = r'_{i+1}, \text{ and } \quad g_{i+1} = g'_{i+1}|r'_{i+1}(K_i).$$

We then have $r_{i+1}: K_i \to K_{i+1}$ and $g_{i+1}: K_{i+1} \to K_{i+1}$ onto. We also then see that $g_i \sim_s g_{i+1}$. We note that since $K_{i+1} \subseteq K'_{i+1}$ we still have $w_1(K_{i+1}) < w_1(K_i)$.

Similarly, consider $g_i: K_i \to K_i$ a W^*-mapping where K_i is such that every 1-sphere in K_i contains the distinguished branched point b and at every point $x \in K_i$ except b there exists a neighborhood N_x such $g_i(N_x)$ is an arc, $w_2(K_i) = 0$, and two stem copies Y_1 and Y_2 of $Y \subset K_i - b$. We say that $Y_1 < Y_2$ provided $g_i^n(Y_1) \subset Y_2$ for some integer n. These is again a partial ordering and we can prove the following in similar fashion to the above.

Assume that $w_2(K_i) > 0$. Then there is a stem S_0 of a copy of Y in $K_i - b$ which is minimal relative to this partial ordering. A copy Y_0 of Y can be chosen small enough so that if K_{i+1} is formed as described then there is map $g_{i+1}: K_{i+1} \to K_{i+1}$ which is shift equivalent to g_i. This completes the reduction process. □

Next, we consider several examples which demonstrate the techniques developed in theorems 36, 38, 40, 41 and which are used to find an elementary presentation. We show each technique separately and concentrate on simple cases in order to keep the examples simple and to illustrate as clearly as possible the underlying technique. In general more complicated examples just involve repetition of the techniques in question.

Example 11 In this example we demonstrate how given a smooth graph K and a W^*-mapping $g: K \to K$ we can find a smooth graph K' and a W^*-mapping $g': K' \to K'$ with $g \sim_s g'$ and every branch point of K' of type (2,1). Thus we will be showing an application of the technique developed in theorem 36.

Consider the orientable smooth graph $K = (G, S)$ with $G = (V, E)$, $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ where the incidence relations and switch conditions are summarized

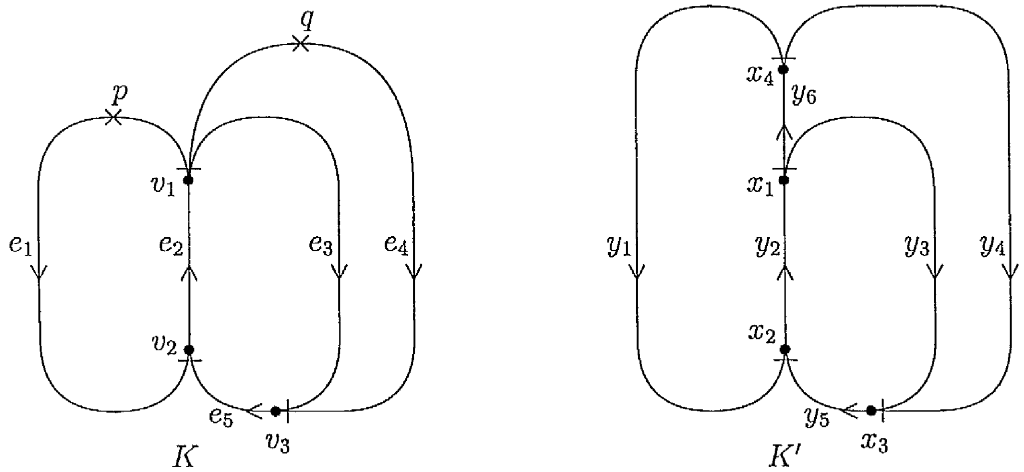


Figure 4.2: The smooth graphs K and K' given in example 11

in the table below:

v	$R(v)$	$L(v)$
v_1	$\{e_1, e_3, e_4\}$	$\{e_2\}$
v_2	$\{e_1, e_5\}$	$\{e_2\}$
v_3	$\{e_3, e_4\}$	$\{e_5\}$

A diagram showing K can be seen in figure 4.2. The orientation ρ shown in the diagram by the arrow heads is given by; $\rho(e_1) = \rho(e_5) = v_2$, $\rho(e_2) = v_1$, and $\rho(e_3) = \rho(e_4) = v_3$. Consider the W^*-mapping $g: K \to K$ given below

$$g \begin{cases} e_1 \mapsto e_1e_2e_3e_5e_2e_1 \\ e_2 \mapsto e_2e_4e_5e_2 \\ e_3 \mapsto e_4e_5e_2e_1e_2e_4 \\ e_4 \mapsto e_1e_2e_3e_5e_2e_4 \\ e_5 \mapsto e_5e_2e_1. \end{cases}$$

Since K is orientable each word $e_i \dots e_j$ above uniquely denotes a route on K and g is assumed to be an immersion which expands each edge e along the route $g(e)$. Examining K we notice that every branch point of K is of type $(2,1)$ except v_1. There exist small arcs $[v_1, p] \subset \bar{e_1}$ and $[v_1, q] \subset \bar{e_4}$ at the vertex v_1 where $g([v_1, p]) = \bar{e_1}$ and $g([v_1, q]) = \bar{e_1}$. The points x and y are shown by a small "×" in the diagram and are such that $g(p) = g(q) = v_2$. By identifying the arcs $[v_1, p]$ and $[v_1, q]$ we arrive at the smooth graph $K' = (G', S')$ with $G' = (V', E')$, $V' = \{x_1, x_2, x_3, x_4\}$, $E' = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ where the incidence relations and switch conditions are given in the table below:

x	$R(x)$	$L(x)$
x_1	$\{y_6, y_3\}$	$\{y_2\}$
x_2	$\{y_1, y_5\}$	$\{y_2\}$
x_3	$\{y_3, y_4\}$	$\{y_5\}$
x_4	$\{y_1, y_4\}$	$\{y_6\}$

The smooth graph diagram for K' is shown in figure 4.2. The orientation shown in the diagram of K' is induced by the orientation ρ on K and given by; $\gamma(y_1) = \gamma(y_5) = x_2$, $\gamma(y_2) = x_1$, $\gamma(y_6) = x_4$ and $\gamma(y_3) = \gamma(y_4) = x_3$. The projection mapping $r: K \to K'$ is such that;

$$\begin{aligned} &r(v_i) = x_i \text{ for } i = 1,2,3, \quad r(p) = r(q) = x_4, \quad r([v_1, p]) = r([v_1, q]) = y_6, \\ &r(e_j) = y_j \text{ for } j = 2,3,5, \quad r([p, v_2]) = y_1, \qquad r([q, v_3]) = y_4. \end{aligned}$$

The mapping $g': K' \to K'$ induced by g and r is such that g' fixes the vertices x_1, x_2, x_3, $g'(x_4) = x_2$ and g' expands the edges of K' along routes $y_i \ldots y_j$ as shown below;

$$g' \begin{cases} y_1 \mapsto y_2 y_3 y_5 y_2 y_6 y_1 \\ y_2 \mapsto y_2 y_6 y_4 y_5 y_2 \\ y_3 \mapsto y_6 y_4 y_5 y_2 y_6 y_1 y_2 y_6 y_4 \\ y_4 \mapsto y_2 y_3 y_5 y_2 y_6 y_4 \\ y_5 \mapsto y_5 y_2 y_6 y_1 \\ y_6 \mapsto y_6 y_1. \end{cases}$$

The mapping $s: K' \to K$ is then given as follows; $s(x_i) = v_i$ for $i = 1,2,3$, $s(x_4) = v_2$ and s expands each each edge of K' along the routes $e_i \ldots e_j$ as shown below;

$$s \begin{cases} y_1 \mapsto e_2 e_3 e_5 e_2 e_1 \\ y_2 \mapsto e_2 e_4 e_5 e_2 \\ y_3 \mapsto e_4 e_5 e_2 e_1 e_2 e_4 \\ y_4 \mapsto e_2 e_3 e_5 e_2 e_4 \\ y_5 \mapsto e_5 e_2 e_1 \\ y_6 \mapsto e_1 \end{cases}$$

We then have $g \sim_s g'$ with lag=1 via the mappings r and s. Thus $r \circ g = g' \circ r$, $g \circ s = s \circ g'$, $r \circ s = g'$, and $s \circ r = g$.

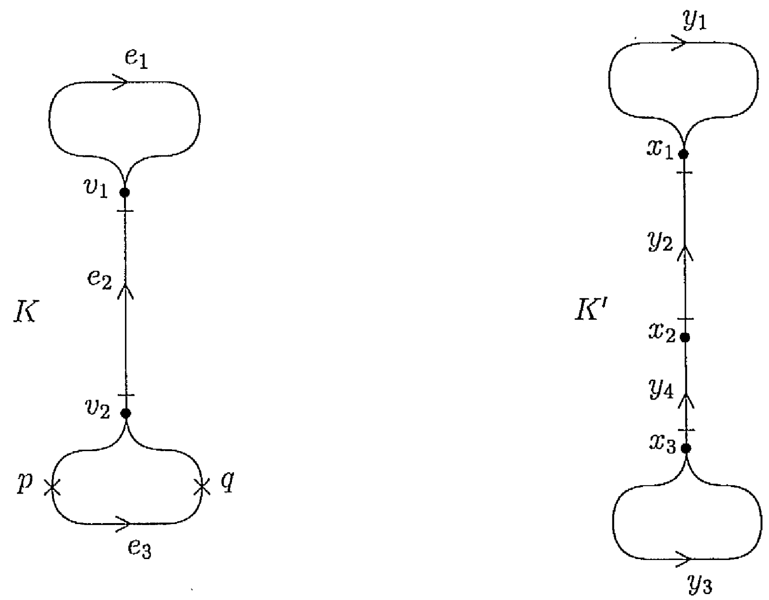


Figure 4.3: The smooth graphs K and K' given in example 12

Example 12 In this example we investigate the technique developed in theorem 38 which when given a smooth graph K where all branch points are of type (2,1) and a W^*-mapping $g: K \to K$ allows us to find a smooth graph K' and a W^*-mapping $g': K' \to K'$ where $g \sim_s g'$ and such that for all $x \in K'$ there is a neighborhood N_x of x such that $g(N_x)$ is an arc. We note that this technique is similar to that in example 11 in that both involve the "zipping" up of a small neighborhood near branch points.

Consider the nonorientable smooth graph $K = (G, S)$ where $G = (V, E)$, $V = \{v_1, v_2\}$, $E = \{e_1, e_2, e_2\}$ and incidence relations and switch conditions given in the table below:

v	$R(v)$	$L(v)$
v_1	$\{e_2\}$	$\{e_1\}$
v_2	$\{e_2\}$	$\{e_3\}$

A diagram of K is given in figure 4.3. The arbitrary orientation ρ on K shown in the figure is given by $\rho(e_1) = \rho(e_2) = v_1$ and $\rho(e_3) = v_2$. Let $g: K \to K$ be the W^*-mapping where $g(v_1) = v_2$, $g(v_2) = v_1$ and g expands each edge along the route $e_i^{n_1} \dots e_j^{n_k}$ as shown below;

$$g \begin{cases} e_1 \mapsto e_3 \\ e_2 \mapsto e_1 e_2^{-1} \\ e_3 \mapsto e_2^{-1} e_3 e_2. \end{cases}$$

Note, that in the above representation of g a positive exponent means that g expands the edge in same direction as the orientation shown in the figure and negative exponent means that g expands the edge in the opposite direction to the orientation shown in the figure. We see that there is a small neighborhood N_2 of the vertex v_2 for which $g(N_2)$is an arc, but that there is no neighborhood of v_1 whose image under g is an arc. We do see however that under g v_1 maps to v_2 thus there will be a small neighborhood N_1 of v_1 where $g^2(N_1)$ is an arc. Consider the two small arcs $[v_2, p] \subset \bar{e}_3$ and $[v_2, q] \subset \bar{e}_3$ at the vertex v_2. In the diagram p and q are represented by small "×" marks on the edge e_3. The arcs $[v_2, p]$ and $[v_2, q]$ are chosen so that $g([v_2, p]) = g([v_2, q]) = \bar{e}_2$, $g(p) = g(q) = v_2$. By identifying the arcs $[v_2, p]$ and $[v_2, q]$ we arrive at the smooth graph $K' = (G', S')$ where $g' = (V', E')$, $V' = \{x_1, x, x_3\}$, $E' = \{y_1, y_2, y_3, y_4\}$ and the incidence relations and switch conditions are given in the table below:

x	$R(x)$	$L(x)$
x_1	$\{y_2\}$	$\{y_1\}$
x_2	$\{y_2\}$	$\{y_4\}$
x_3	$\{y_4\}$	$\{y_3\}$

The smooth graph diagram for K' is shown in figure 4.3. The orientation, γ, shown in the figure is given by $\gamma(y_1) = \gamma(y_2) = x_1$, $\gamma(y_3) = x_3$, and $\gamma(y_4) = x_2$. The projection mapping $r: K \to K'$ is given by;

$$r(v_1) = x_1, \quad r(v_2) = x_2, \qquad r(p) = r(q) = x_3, \qquad\qquad r(e_1) = y_1,$$
$$r(e_2) = y_2, \quad r([p, q]) = y_3, \quad r([v_2, p]) = r([v_2, q]) = y_4.$$

The mapping $g': K' \to K'$ is such that $g(x_1) = x_2$, $g(x_2) = x_1$, $g(x_3) = x_2$, and g' expands the edges of K' over the routes $y_i^{n_1} \dots y_j^{n_k}$ as follows;

$$g' \begin{cases} y_1 \mapsto y_4^{-1} y_3 y_4 \\ y_2 \mapsto y_1 y_2^{-1} \\ y_3 \mapsto y_2^{-1} y_4^{-1} y_3 y_4 y_2 \\ y_4 \mapsto y_2. \end{cases}$$

The mapping $s: K' \to K$ is then given as follows; $s(x_1) = s(x_3) = v_2$, $s(x_2) = v_1$, and s expands the edges of K' along the routes $e_i^{n_1} \dots e_j^{n_k}$ in the manner given below;

$$s \begin{cases} y_1 \mapsto e_3 \\ y_2 \mapsto e_1 e_2^{-1} \\ y_3 \mapsto e_3 \\ y_4 \mapsto e_2. \end{cases}$$

We then have $g \sim_s g'$ with lag=1 and shift equivalence given by the mappings r and s.

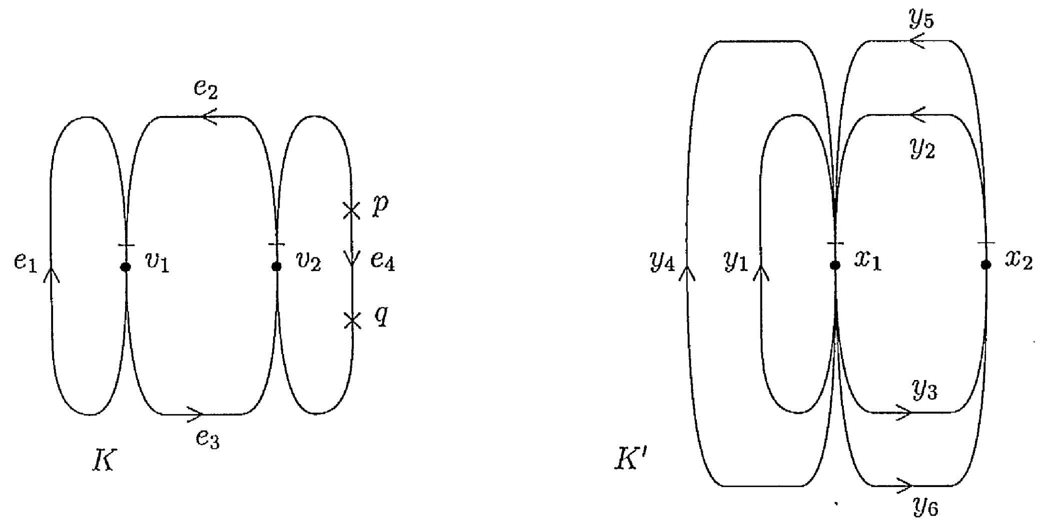


Figure 4.4: The smooth graphs K and K' given in example 13

Example 13 In this example we demonstrate the technique developed in theorem 40. Thus we will show that given a smooth graph K and a W^*-mapping $g: K \to K$ with a fixed point x_0 we can find a smooth graph K' and a W^*-mapping $g': K' \to K'$ where $g \sim_s g'$ and K' is such that every 1-sphere in K' contains x_0.

Consider the smooth graph $K = (G, S)$ where $G = (V, E)$, $V = \{v_1, v_2\}$, $E = \{e_1, e_2, e_3, e_4\}$ and the incidence relations and switch conditions are given in the table below:

v	$R(v)$	$L(v)$
v_1	$\{e_1, e_2\}$	$\{e_1, e_3\}$
v_2	$\{e_2, e_4\}$	$\{e_3, e_4\}$

A diagram for K is given in figure 4.4. K is orientable and an orientation ρ is shown in the figure with $\rho(e_1) = \rho(e_2) = v_1$ and $\rho(e_3) = \rho(e_4) = v_2$. Let $g: K \to K$ be the W^*-mapping where $g(v_1) = v_1$, $g(v_2) = v_2$ and g expands each edge of K along the routes $e_i \dots e_j$ as specified below;

$$g \begin{cases} e_1 \mapsto e_3 e_2 \\ e_2 \mapsto e_4 e_2 \\ e_3 \mapsto e_3 e_4 \\ e_4 \mapsto e_4 e_2 e_1 e_3 e_4. \end{cases}$$

We see that both v_1 and v_2 are fixed under g. We will use the vertex v_1 to find the new presentation consisting of K' and g'. Note that the embedded 1-sphere consisting of the loop e_4 does not contain v_1. The preimages of the point v_1 consist of two points p and q in the interior of the edge e_4. In the diagram p and q are represented by "×"

marks. We form the new smooth graph K' by identifying the points p, q and the vertex v_1. $K' = (G', S')$ where $G' = (V', E')$, $V = \{x_1, x_2\}$, $E' = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ and the incidence relations and switch conditions are summarized in the table below:

x	$R(x)$	$L(x)$
x_1	$\{y_1, y_2, y_4, y_5\}$	$\{y_1, y_3, y_4, y_5\}$
x_2	$\{y_2, y_5\}$	$\{y_3, y_6\}$

The smooth graph diagram for K' is shown in figure 4.4. The orientation γ shown in the figure is induced by the orientation ρ on K and is given by; $\gamma(y_1) = \gamma(y_2) = \gamma(y_4) = \gamma(y_5) = x_1$ and $\gamma(y_3) = \gamma(y_6) = x_2$. The projection mapping $r: K \to K'$ is given by;

$$r(v_i) = x_i \text{ for } i = 1, 2, \quad r([v_2, p]) = y_5, \quad r([v_2, q]) = y_6,$$
$$r(e_j) = y_j \text{ for } j = 1, 2, 3, \quad r([p, q]) = y_4.$$

The mapping $g': K' \to K'$ is then such that $g'(x_1) = x_1$, $g('x_2) = x_2$ and g' expands each edge of K' along the route $y_i \ldots y_j$ as follows;

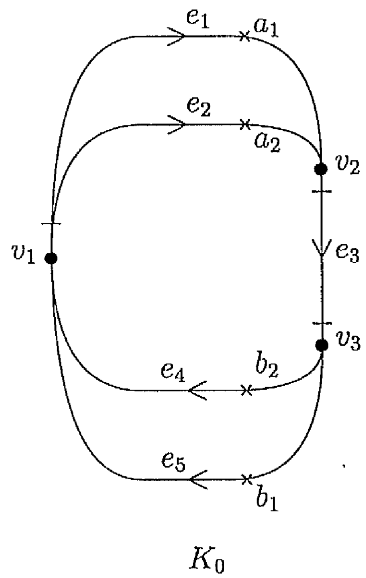
$$g \begin{cases} y_1 \mapsto y_3 y_2 \\ y_2 \mapsto y_5 y_4 y_6 y_2 \\ y_3 \mapsto y_3 y_5 y_4 y_6 \\ y_4 \mapsto y_1 \\ y_5 \mapsto y_5 y_4 y_6 y_2 \\ y_6 \mapsto y_3 y_5 y_4 y_6. \end{cases}$$

The mapping $s: K' \to K$ is given by; $s(x_1) = v_1$, $s(x_2) = v_2$ and such that s expands each edge of K' along the route $e_i \ldots e_j$ in the manner shown below;

$$s \begin{cases} y_1 \mapsto e_3 e_2 \\ y_2 \mapsto e_4 e_2 \\ y_3 \mapsto e_3 e_4 \\ y_4 \mapsto e_1 \\ y_5 \mapsto e_4 e_2 \\ y_6 \mapsto e_3 e_4 \end{cases}$$

We then have $g \sim_s g'$ with lag=1 with shift equivalence given by the mappings r and s.

Example 14 In this example we demonstrate the technique for finding an elementary presentation developed in theorem 41. We start with a smooth graph K and W^*-mapping



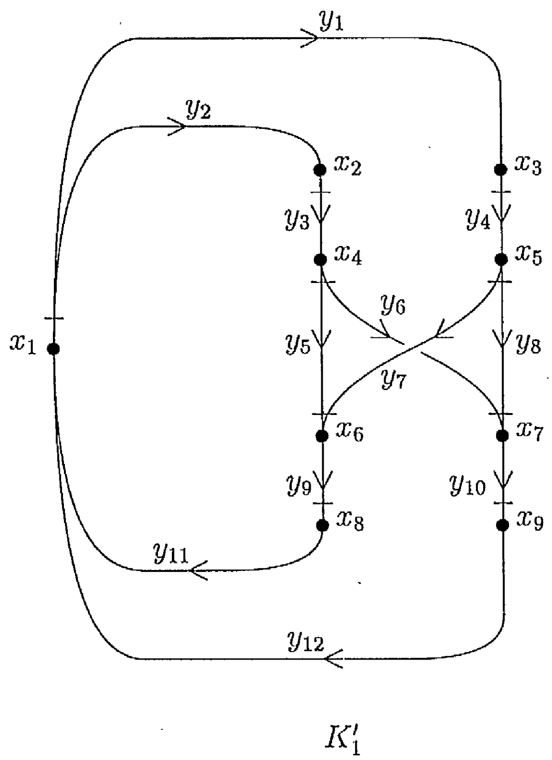


Figure 4.5: The smooth graphs K_0 and K_1' given in example 14

$g\colon K \to K$ with fixed point v_1 where K is such that every embedded 1-sphere in K contains v_1. We then show how to find an elementary smooth graph J and W^*-mapping $f\colon J \to J$ with $g \sim_s f$. The smooth graph $K = K_0 = (G_0, S_0)$ with $G_0 = (V_0, E_0)$ where $V_0 = \{v_1, v_2, v_3\}$, $E_0 = \{e_1, e_2, e_3, e_4, e_5\}$ and incidence and switch conditions summarized in the table below:

v	$R(v)$	$L(v)$
v_1	$\{e_1, e_2\}$	$\{e_4, e_5\}$
v_2	$\{e_3\}$	$\{e_1, e_2\}$
v_3	$\{e_3\}$	$\{e_4, e_5\}$

A diagram for K_0 is shown in figure 4.5. K_0 is orientable with orientation ρ, shown in the diagram, given by $\rho(e_1) = \rho(e_2) = v_2$, $\rho(e_3) = v_4$, and $\rho(e_4) = \rho(e_5) = v_1$. Let $g_0\colon K_0 \to K_0$ be the W^*-mapping where $g_0(v_i) = v_1$, for $i = 1, 2, 3$ and which expands each edge of K along the route $e_i \ldots e_j$ as shown below;

$$g_0 \begin{cases} e_1 \mapsto e_1 e_3 e_4 \\ e_2 \mapsto e_1 e_3 e_4 \\ e_3 \mapsto e_2 e_3 e_4 \\ e_4 \mapsto e_2 e_3 e_5 \\ e_5 \mapsto e_2 e_3 e_5 \end{cases}$$

It is clear from the diagram of K_0 that every 1-sphere in K_0 contains the vertex v_1. Also from the definition of g_0 we see that for every point $x \in K_0$ there is a neighborhood N_x such that $g_0(N_x)$ is a 1-cell. Thus we may begin the reduction process as given in theorem 41.

Let C, X, Y, and V be given as in figure 4.1. K_0 contains a single copy C_0 of C. Where $C_0 = [a_1, b_1] \cup [a_2, b_2]$ and $[a_1, b_1] \cap [a_2, b_2] = \bar{e}_3$. Note that $g_0(C_0)$ factors through a 1-cell. C_0 is obviously minimal under the partial ordering given in theorem 41 as it is the only copy of C in K_0. We may then form the new smooth graph K_1' from K_0 by removing C_0 from K_0 and replacing it with a copy X_0 of X. We write $K_1' = (G_1', S_1')$ with $G_1' = (V_1', E_1')$ with $V_1' = \{x_1, \ldots, x_9\}$, $E_1' = \{y_1, \ldots, y_{12}\}$ where the incidence relations and switch conditions are summarized in the table below:

x	$R(x)$	$L(x)$	x	$R(x)$	$L(x)$
x_1	$\{y_1, y_2\}$	$\{y_{11}, y_{12}\}$	x_6	$\{y_5, y_7\}$	$\{y_9\}$
x_2	$\{y_3\}$	$\{y_2\}$	x_7	$\{y_6, y_8\}$	$\{y_{10}\}$
x_3	$\{y_4\}$	$\{y_1\}$	x_8	$\{y_9\}$	$\{y_{11}\}$
x_4	$\{y_5, y_6\}$	$\{y_3\}$	x_9	$\{y_{10}\}$	$\{y_{12}\}$
x_5	$\{y_7, y_8\}$	$\{y_4\}$			

A diagram of the smooth graph K_1' is shown in figure 4.5. The orientation shown on K_1' is induced by orientation ρ on K_0. The mapping $s_1': K_1' \to K_0$ is induced by the natural map $X \to C$. We see that s_1' maps the vertices of K_1' to K_0 as follows;

$$\begin{aligned} s_1'(x_1) &= v_1, & s_1'(x_6) &= v_3, & s_1'(x_2) &= a_2, \\ s_1'(x_7) &= v_3, & s_1'(x_3) &= a_1, & s_1'(x_8) &= b_2, \\ s_1'(x_4) &= v_2, & s_1'(x_9) &= b_1, & s_1'(x_5) &= v_2. \end{aligned}$$

The map s_1' maps the edges of K_1' to K_0 as shown below;

$$s_1' \begin{cases} y_1 \mapsto [v_1, a_1] & y_5 \mapsto e_3 \quad y_9 \mapsto [v_3, b_2] \\ y_2 \mapsto [v_1, a_2] & y_6 \mapsto e_3 \quad y_{10} \mapsto [v_3, b_1] \\ y_3 \mapsto [a_2, v_2] & y_7 \mapsto e_3 \quad y_{11} \mapsto [b_2, v_1] \\ y_4 \mapsto [a_1, v_2] & y_8 \mapsto e_3 \quad y_{12} \mapsto [b_1, v_1]. \end{cases}$$

Next we need to find the map $r_1': K_0 \to K_1'$. By following the procedure given in theorem 41 we find that r_1' is such that $r_1'(v_i) = x_1$ for $i = 1, 2, 3$ and that r_1' expands each edge of K_0 along the route $y_i \dots y_j$ as follows;

$$r'_1 \begin{cases} e_1 \mapsto y_1y_4y_7y_9y_{11} \\ e_2 \mapsto y_1y_4y_7y_9y_{11} \\ e_3 \mapsto y_2y_3y_5y_9y_{11} \\ e_4 \mapsto y_2y_3y_6y_{10}y_{12} \\ e_5 \mapsto y_2y_3y_6y_{10}y_{12}. \end{cases}$$

We note that $r'_1 \colon K_0 \to K'_1$ is not onto since there does not exist an $x \in K_0$ such that $r'_1(x) \in y_8$. We will restrict our attention to the sub-manifold $K_1 = r'_1(K_0)$ and define $r_1 \colon K_0 \to K_1$ by $r_1 = r'_1$, and $s_1 \colon K_1 \to K_0$ by $s_1 = s'_1|K_1$. We then have $r_1 \colon K_0 \to K_1$ onto. The maps r_1 and s_1 are now such that $s_1 \circ r_1 = g_0$. We may then define $g_1 \colon K_1 \to K_1$ as $g_1 = r_1 \circ s_1$. The mappings are then such that the diagram below commutes

$$\begin{array}{ccc} K_1 & \xrightarrow{g_1} & K_1 \\ {\scriptstyle s_1}\downarrow & \overset{r_1}{\nearrow} & \downarrow{\scriptstyle s_1} \\ K_0 & \xrightarrow[g_0]{} & K_0 \end{array}$$

and the mappings r_1 and s_1 define a shift equivalence of lag=1 for the W^*-mappings g_0 and g_1.

In order to reduce the amount of book keeping necessary we will remove the type (1,1) vertices x_2, x_3, x_5, x_7, x_8, and x_8 from K_1 and replace them with ordinary points. We then write $K_1 = (G_1, S_1)$, with $G_1 = (V_1, E_1)$, $V_1 = \{w_1, w_2, w_3\}$, $E_1 = \{z_1, z_2, z_3, z_4, z_5\}$ and the incidence relations and switch conditions given in the table below:

w	$R(w)$	$L(w)$
w_1	$\{z_1, z_2\}$	$\{z_3, z_4\}$
w_2	$\{z_5, z_3\}$	$\{z_2\}$
w_3	$\{z_5, z_1\}$	$\{z_4\}$

A smooth graph diagram for K_1 is given in figure 4.6. The orientation shown in the diagram is induced from that on K'_1. The mapping $g_1 \colon K_1 \to K_1$ is such that $g_1(w_i) = w_1$ for $i = 1, 2, 3$ and g_1 expands each edge of K_1 along the route $z_i \dots z_j$ as given below;

$$g_1 \begin{cases} z_1 \mapsto z_1z_4z_2z_5z_4 \\ z_2 \mapsto z_1z_4 \\ z_3 \mapsto z_2z_5z_4z_2z_3 \\ z_4 \mapsto z_2z_3 \\ z_5 \mapsto z_2z_5z_4. \end{cases}$$

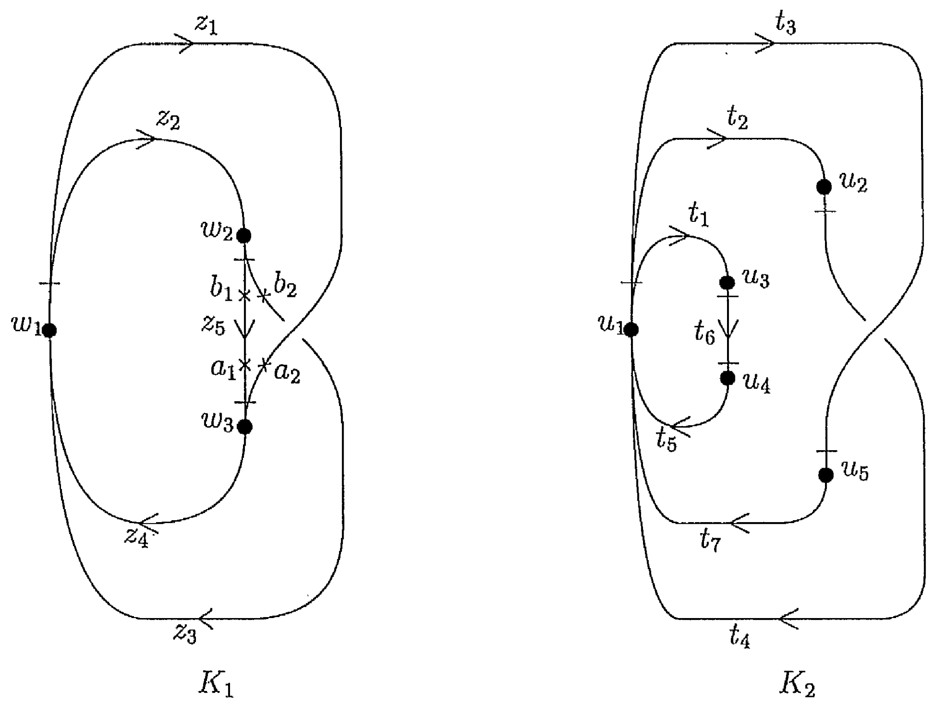


Figure 4.6: The smooth graphs K_1 and K_2 given in example 14

The smooth graph K_1 contains two copies of Y, Y_0 and Y_1, where $Y_0 = [b_1, w_1] \cup [b_2, w_1]$ and $Y_1 = [a_1, w_1] \cup [a_2, w_1]$. Here $[b_1, w_1] \cap [b_2, w_1] = z_2$ and $[a_1, w_1] \cap [a_2, w_1] = z_4$. The points a_1, a_2, b_1, b_2 are labeled in diagram 4.6 using $\times$'s. In regards to the partial ordering given in theorem 41 neither $Y_0 < Y_1$ or $Y_1 > Y_0$, thus it does not matter which we remove first. We will remove both in a single step.

We create a new smooth graph K_2 from K_1 by removing Y_0 and Y_1 and replacing them with copies V_0 and V_1 of V respectively. $K_2 = (G_2, S_2)$, with $G_2 = (V_2, E_2)$, $V_2 = \{u_1, \ldots, u_5\}$, $E_2 = \{t_1, \ldots, t_7\}$ and the incidence relations and switch conditions are summarized in the table below:

u	$R(s)$	$L(u)$
u_1	$\{t_1, t_2, t_3\}$	$\{t_4, t_5, t_7\}$
u_2	$\{t_4\}$	$\{t_2\}$
u_3	$\{t_6\}$	$\{t_1\}$
u_4	$\{t_6\}$	$\{t_5\}$
u_5	$\{t_3\}$	$\{t_7\}$

The smooth graph diagram for K_2 is given in figure 4.6. The orientation shown is again induced from that on K_1. It is easy to see that K_2 will be an elementary smooth graph if we remove all the vertices of type (1,1) and replace them with ordinary points.

The mapping $s_2: K_2 \to K_1$ induced by the natural mappings from $V_0 \mapsto Y_0$ and $V_1 \mapsto Y_1$ is such that $s_2(u_1) = w_1$, $s_2(u_2) = s_2(u_3) = w_2$, $s_2(u_4) = s_2(u_5) = w_3$ and s_2 maps the edges of K_2 to K_1 as given below;

$$s_2 \begin{cases} t_1 \mapsto z_2 & t_5 \mapsto z_4 \\ t_2 \mapsto z_2 & t_6 \mapsto z_5 \\ t_3 \mapsto z_1 & t_7 \mapsto z_4 \\ t_4 \mapsto z_3. \end{cases}$$

We must now find the mapping $r_2: K_1 \to K_2$. We will clearly have $r_2(w_i) = u_1$ for $i = 1, 2, 3$. We still need to determine the action of r_2 on the edges of K_1. By examining the image of each z_i in K_1 under g_1 as well as image of each t_i in K_2 under s_2 we find that r_2 expands each edge of K_1 along $t_i \dots t_j$ as shown below;

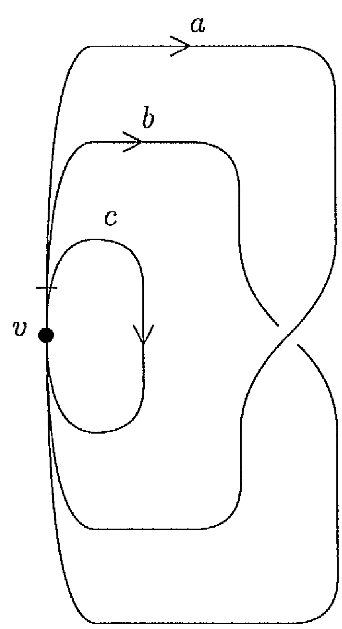
$$r_1 \begin{cases} z_1 \mapsto t_3 t_7 t_1 t_6 t_5 \\ z_2 \mapsto t_3 t_7 \\ z_3 \mapsto t_1 t_6 t_5 t_2 t_4 \\ z_4 \mapsto t_2 t_4 \\ z_5 \mapsto t_1 t_6 t_5. \end{cases}$$

The mapping $g_2: K_2 \to K_2$ is then defined by $g_2 = r_2 \circ s_2$. We see that g_2 is such that $g_2(u_i) = u_1$ for $i = 1, 2, 3, 4, 5$ and that g_2 is such that it expands each edge in K_2 along the route $t_i \dots t_j$ as given below;

$$g_2 \begin{cases} t_1 \mapsto t_3 t_7 \\ t_2 \mapsto t_3 t_7 \\ t_3 \mapsto t_3 t_7 t_1 t_6 t_5 \\ t_4 \mapsto t_1 t_6 t_5 t_2 t_4 \\ t_5 \mapsto t_2 t_4 \\ t_6 \mapsto t_1 t_6 t_5 \\ t_7 \mapsto t_2 t_4. \end{cases}$$

The mappings r_2 and s_2 then form a shift equivalence from of lag=1 of g_1 and g_2.

Putting everything together we see that $r_2 \circ r_1 \circ r_0$ and $s_2 \circ s_1 \circ s_0$ form a shift equivalence of lag=3 of g_0 and g_2. By re-labeling the type (1,1) vertices, u_2, u_3, u_4 and u_5, of K_2 as ordinary points we may write K_2 as an elementary smooth graph with three edges, say a, b, c, and a single vertex, say v (see figure 4.7). Here we are assuming that the edge



K_2

Figure 4.7: The smooth graph K_2 (relabeled) given in example 14

a is made from the edges, t_3, t_7, the edge b is made from edges t_2, t_4, and the edge c is made from the edges t_1, t_6, and t_5. The mapping g_2 will then be given by;

$$g_2 \begin{cases} a \mapsto acb \\ b \mapsto acb \\ c \mapsto acb. \end{cases}$$

In this example it is interesting to note that if we consider the circle S_1 to be an elementary smooth graph with a single edge, say p, and a single vertex, say q and then define then expanding mapping $f\colon S_1 \to S_1$ which wraps the circle around its self three times, i.e., $p \mapsto ppp$. We may then form a shift equivalence between f and g_2 as follows. Let $r\colon S_1 \to K_2$ be defined by $r(q) = v$ and r expands p along the route abc. We then define $s\colon K_2 \to S_1$ so that $s(v) = q$ and s maps each edge of K_2 to the edge p of S_1. We then see that $g_2 \sim f$. Thus g_0 is shift equivalent with lag=4 to the map f, an expanding mapping of the circle!

Theorem 42 *[39] $S(h)$ does not depend on the choice of presentation.*

Theorem 43 *[39] Let h and h' be shift maps of solenoids K_∞ and K'_∞ and assume that K_∞ and K'_∞ have elementary presentations $\{K, g\}$ and $\{K', g'\}$, respectively. Then h is topologically conjugate to h' ($g \sim_s g'$) if and only if $S(h) = S(h')$.*

Our final result in this section is a result of Williams [39] giving necessary and sufficient conditions for an endomorphism of a finitely generated non-abelian free group to realize a shift equivalence class for an elementary presentation.

Let $\alpha: F \to F$ be an endomorphism and F a finitely generated free group where the generators of F are partitioned into three subsets O (orientable), R (right-nonorientable), and L (left-nonorientable). We say that α satisfies the *Ω-condition* if for some $m > 0$, the word $\alpha^m(x)$ contains the whole alphabet $O \cup R \cup L$, for each letter x. By this we mean that for all $x \in F$ there must exist an $m > 0$ so that the word $\alpha^m(x)$ contains every generating element or its inverse. A map $\lambda: X \to X$, where X is a finite set, is said to be *eventually constant* if for some integer $m > 0$, λ^m is a constant map.

Let α and F be defined as in the preceding paragraph. For each generator of F write $x \in O^+$ and $x^{-1} \in O^-$ for $x \in O$, etc. An endomorphism α satisfies the *immersion condition* if for every $x \in O \cup R \cup L$, then $\alpha(x) = y_1 y_2 \cdots y_r$, where each y_i or its inverse is in $O \cup R \cup L$, and satisfies:

(i)

$$y_{i+1} \in \begin{cases} O^+ \cup R^\pm & \text{if } y_i \in O^+ \cup L^\pm, \\ O^- \cup L^\pm & \text{otherwise;} \end{cases}$$

(ii) α is locally orientation preserving [reversing], i.e.,

(a)

$$y_1 \in \begin{cases} O^+ \cup R^\pm & \text{if } x \in O^+ \cup R^\pm \ [O^- \cup L^\pm] \\ O^- \cup L^\pm & \text{otherwise;} \end{cases}$$

(b)

$$y_r \in \begin{cases} O^+ \cup L^\pm & \text{if } x \in O^+ \cup R^\pm \ [O^- \cup R^\pm] \\ O^- \cup R^\pm & \text{otherwise;} \end{cases}$$

Note that condition (i) says that the immersion f of the elementary branched 1-manifold K which is to induce α doesn't double back on any of the branches of K. Condition (ii.a) says that if a one-cell of K begins pointing rightward/leftward, then in the locally orientation preserving case its image under the immersion f will begin pointing rightward/leftward and in the locally orientation reversing case its image under f will begin pointing leftward/rightward. Condition (ii.b) says that if a one-cell of K ends pointing leftward/rightward then in the orientation preserving case its image under f will end pointing leftward/rightward and in the orientation reversing case its image under f will end pointing rightward/leftward.

Theorem 44 *[39] In order for a shift class $S(h)$ to have a shift map h with an elementary presentation it is necessary and sufficient that it contains an endomorphism α of a free group F on generators $O \cup R \cup L$ such that*

(a) α satisfies the immersion condition,

(b) α satisfies the Ω-condition,

(c) there are eventually constant maps $\lambda, \mu: O \cup R \cup L \to O \cup R \cup L$ such that

1. $\alpha(x)$ has the form $\lambda(x) \cdots \mu(x)$ if $x \in O$,

2. $\alpha(x)$ has the form $\lambda(x) \cdots \lambda(x)$ if $x \in R$,

3. $\alpha(x)$ has the form $\mu(x) \cdots \mu(x)$ if $x \in L$.

4.3 (p, q)-Block Presentations

In this section we introduce (p, q)-block presentations. With each (p, q)-block presentation we can associate an endomorphism of a finitely generated free group. We then show that if two presentations, say $\{K, g\}$ and $\{K', g'\}$ with associated endomorphisms g_* and g'_*, have (p, q)-block form and are shift equivalent, then their associated endomorphisms will also be shift equivalent. Thus (p, q)-block presentations allow us to find further algebraic invariants associated with presentations of solenoids. In general finding a (p, q)-block presentation shift equivalent to a given presentation of a solenoid is easier then finding a shift equivalent elementary presentation. Also, every presentation of a solenoid will be shown to be shift equivalent to a countably infinite number (p,q)-block presentations, each possibly yielding different invariants.

Definition 41 Let $\{K, g\}$ be the presentation of a solenoid with vertex set $V \neq \emptyset$. Let $p, q \in \mathbb{Z}$ where $1 \leq p \leq q$. Then $\{K, g\}$ is said to be in (p,q)-*block* form (or a (p,q)-*block presentation*) if

$$V = \bigcup_{i=p}^{q} \mathrm{Fix}\,(g^i).$$

Suppose two presentations of solenoids, $\{K, g\}$ and $\{K', g'\}$, are in (p,q)-block form and are shift equivalent with the shift equivalence given by the maps $r: K \to K'$ and $s: K' \to K$. Then it is easy to see that we must have $r(V) = V'$ and $s(V') = V$, where here V is the vertex set of K and V' is the vertex set of K'.

We will now give a result showing how to calculate the number of fixed points of g^i for each integer $i = 1, \ldots, \infty$ for a solenoid given a presentation. This result was originally shown to be true by Williams in [38] for presentations of solenoids $\{K, g\}$ where the smooth graph in question has all branch points of type (2,1) and are such that every point x has a neighborhood N_x where $g(N_x)$ is an arc. We have written a new proof of this theorem so it applies to the more general presentations of solenoids considered in this thesis.

Theorem 45 *Let $\{K, g\}$ be the presentation of a solenoid where K has vertex set V, edge set E. Let ρ be an arbitrary orientation on K. Let $\mathbf{C}$ be an $|E| \times |E|$ matrix where C_{ji} equals the number times the image $g(e_i)$ of edge e_i crosses edge e_j regardless of direction and let $\mathbf{D}$ be a $|V| \times |V|$ matrix where for each $i, j = 1, \ldots, |V|$*

$$D_{ji} = \begin{cases} 1 & \text{if } g(v_i) = v_j \text{ and } g \text{ maps right(left) edges at } v_i \text{ to right(left) edges at } v_j. \\ -1 & \text{if } g(v_i) = v_j \text{ and } g \text{ maps right(left) edges at } v_i \text{ to left(right) edges at } v_j. \\ 0 & \text{otherwise }. \end{cases}$$

Then

$$|Fix\,(g^m)| = trace\,\mathbf{C}^m - trace\,\mathbf{D}^m.$$

Proof. Consider $g: K \to K$. Let $W \subset K$ and denote by $\mathrm{Fix}\,(g^m, W)$ the set of fixed points of g^m in W. It is easy to see that

$$|\mathrm{Fix}\,(g^m)| = |\mathrm{Fix}\,(g^m, K - V)| + |\mathrm{Fix}\,(g^m, V)|.$$

We also know that

$$K - V = \bigcup_{i=1}^{|E|} e_i$$

where e_i are the edges of K. Thus we have

$$|\text{Fix}\,(g^m, K - V)| = \sum_{k=1}^{|E|} |\text{Fix}\,(g^m, e_k)|.$$

We will calculate $|\text{Fix}\,(g^m)|$ by first finding $|\text{Fix}\,(g^m, K - V)|$ and then $|\text{Fix}\,(g^m, V)|$. We will then show that their sum must be $\text{trace}\,\mathbf{C}^m - \text{trace}\,\mathbf{D}^m$.

Suppose the edge e_i maps under g^m to the route $\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_q^{n_q}$ on K where $\alpha_j \in E$ and $n_j = \pm 1$; $n_j = 1$ if e_i traverses α_j in accordance with the orientation on α_j and $n_j = -1$ otherwise. It is easy to see that $(\mathbf{C}^m)_{ii}$ will be the number of occurrences of e_i, regardless of direction (what power it is raised to), in the route $\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_q^{n_q}$.

Suppose that e_i occurs p times in the sub-route $\alpha_2^{n_2}\alpha_3^{n_3}\ldots\alpha_{q-1}^{n_{q-1}}$ then, corresponding to each occurrence of e_i, g^m will have a fixed point in e_i. If $\alpha_1 = e_i$ and $n_1 = 1$ then there will be no fixed point in e_i corresponding to this occurrence of e_i in the route, however g^m will fix an end point of e_i corresponding to this occurrence. If $\alpha_1 = e_i$ and $n_1 = -1$ then g^m will have a single fixed point in the interior of e_i corresponding to this occurrence. Similar results hold for $\alpha_q = e_i$ with $n_q = \pm 1$.

Each edge $e_i \in E$ where $g^m(e_i) = \alpha_1^{n_1}\ldots\alpha_q^{n_q}$ will be of one of the following 6 types:

1. $\alpha_1, \alpha_q \neq e_i$,
2. $\alpha_1 = e_i$, $\alpha_q \neq e_i$ or $\alpha_1 \neq e_1$, $\alpha_q = e_i$ where $n_1 = 1$ or $n_q = 1$ respectively,
3. $\alpha_1 = e_i$, $\alpha_q \neq e_i$ or $\alpha_1 \neq e_1$, $\alpha_q = e_i$ where $n_1 = -1$ or $n_q = -1$ respectively,
4. $\alpha_1 = \alpha_q = e_i$ with $n_1 = n_q = 1$,
5. $\alpha_1 = \alpha_q = e_i$ with $n_1 = -1, n_q = 1$ or $n_1 = 1$, $n_q = -1$.
6. $\alpha_1 = \alpha_q = e_i$ with $n_1 = n_q = -1$,

Let A_i be the set of edges in K with type i. We may then write $E = A_1 \cup A_2 \cup \ldots \cup A_6$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

If $e_i \in A_1$ then the number of fixed points of g^m in e_i, $|\text{Fix}\,(g^m, e_i)|$, is $(\mathbf{D}^m)_{ii}$ and none of the incident vertices of e_i is fixed.

If $e_i \in A_2$ then number of fixed points of g^m in e_i, $|\mathrm{Fix}\,(g^m, e_i)|$ is $(\mathbf{D}^m)_{ii} - 1$ and one of the incident vertices of e_i is fixed.

If $e_i \in A_3$ then the number of fixed points of g^m in e_i, $|\mathrm{Fix}\,(g^m, e_i)|$, is $(\mathbf{D}^m)_{ii}$ and none of the incident vertices of e_i is fixed.

If $e_i \in A_4$ then the number of fixed points of g^m in e_i, $|\mathrm{Fix}\,(g^m, e_i)|$, is $(\mathbf{D}^m)_{ii} - 2$ and both of the incident vertices of e_i are fixed.

If $e_i \in A_5$ then the number of fixed points of g^m in e_i, $|\mathrm{Fix}\,(g^m, e_i)|$, is $(\mathbf{D}^m)_{ii} - 1$ and one of the incident vertices of e_i is fixed.

If $e_i \in A_6$ then the number of fixed points of g^m in e_i, $|\mathrm{Fix}\,(g^m, e_i)|$, is $(\mathbf{D}^m)_{ii}$ and none of the incident vertices of e_i is fixed.

Putting these results together we see that

$$|\mathrm{Fix}\,(g^m, K - V)| = \sum_{k=1}^{|E|} |\mathrm{Fix}\,(g^m, e_k)| = \sum_{k=1}^{|E|} (\mathbf{D}^m)_{kk} - |A_2| - |A_5| - 2|A_4|.$$

We will now calculate $|\mathrm{Fix}\,(g^m, V)|$. The vertices of K which are fixed under g^m can be partitioned into two types, those whose orientation is preserved under g^m and those whose orientation is reversed under g^m. Let B_1 be the set of vertices fixed under g^m whose orientation is preserved under g^m. Let B_2 be the set of vertices fixed under g^m whose orientation is reversed under g^m. We then see that $|\mathrm{Fix}\,(g^m, V)| = |B_1| + |B_2|$. We may further refine this partition, below, by considering how the incident edges are mapped under g^m.

Let X be a finite set and $f: X \to X$ be an eventually constant map, then there will only exist one point $x \in X$ such that $f(x) = x$.

Consider a vertex v_j of V where $g(v_j) = v_j$. Suppose that the orientations of the edges incident upon v_j are preserved under g^m. We may define two maps $\lambda: R(v_j) \to R(v_j)$ and $\mu: L(v_j) \to L(v_j)$. The map λ is defined so that for $e \in R(v_j)$, where $g^m(e) = \alpha_1 \alpha_2^{n_2} \dots \alpha_q^{n_q}$, we have $\lambda(e) = \alpha_1$. The map μ is defined so that for $e \in L(v_j)$, where $g^m(e) = \alpha_1^{n_1} \dots \alpha_{q-1}^{n_{q-1}} \alpha_q$, we have $\mu(e) = \alpha_q$. Since g^m must satisfy W3 we know that maps λ and μ must both be eventually constant. Thus there can only exist one edge $e \in R(v_j)$ such that $\lambda(e) = e$ and only one edge $e \in L(v_j)$ such that $\mu(e) = e$. A similar result holds for the vertices fixed under g^m whose orientation is reversed under g^m.

We may partition the vertices of K in B_i, for $i = 1, 2$, into two classes. Let $B_{i,1} \subset B_i$ be the set of edges in B_i such that there are two $e, e' \in I(v)$ such that $g^m(e) = \alpha_1^{n_1} \dots \alpha_q^{n_q}$

with $\alpha_1 = e$ and $g^m(e') = \alpha_1^{n_1} \ldots \alpha_q^{n_q}$ with $\alpha_q = e'$. Let $B_{i,2} \subset B_i$ be the set of edges in B_i such that there is only one edge in $e \in I(e)$ such that $g^m(e) = \alpha_1 \alpha_2^{n_2} \ldots \alpha_{q-1}^{n_{q-1}} \alpha_q$ with $\alpha_1 = \alpha_q = e$. We then have $B_i = B_{i,1} \cup B_{i,2}$ where $B_{i,1} \cap B_{i,2} = \emptyset$. We may then write

$$|\text{Fix}\,(g^m, V)| = |B_{1,1}| + |B_{1,2}| + |B_{2,1}| + |B_{2,2}|.$$

Consider the matrix $\mathbf{D}^m$. If v_j is a vertex in B_1 it is easy to check that $(\mathbf{D}^m)_{jj} = 1$. If v_j is a vertex in B_2 we have $(\mathbf{D}^m)_{jj} = 1$. Thus we see that

$$\sum_{j=1}^{|V|} (\mathbf{D}^m)_{jj} = |B_1| - |B_2| = |B_{1,1}| + |B_{1,2}| - |B_{2,1}| - |B_{2,2}|.$$

Combining this with our result for $|\text{Fix}\,(g^m, V)|$ we find that

$$|\text{Fix}\,(g^m, V)| = 2|B_{1,1}| + 2|B_{1,2}| - \sum_{j=1}^{|V|} (\mathbf{D}^m)_{jj}.$$

We then see that

$$|\text{Fix}\,(g^m)| = \sum_{k=1}^{|E|} (\mathbf{C}^m)_{kk} - \sum_{j=1}^{|V|} (\mathbf{D}^m)_{jj} - |A_2| - |A_5| - 2|A_4| + 2|B_{1,1}| + 2|B_{1,2}|.$$

We claim that $2|B_{1,1}| + 2|B_{1,2}| - |A_2| - |A_5| - 2|A_4| = 0$.

For vertex $v_j \in B_{1,1}$ there are exactly two edges in A_2 and every edge in A_2 must belong to $B_{1,1}$ thus $2|B_{1,1}| = |A_2|$. For every vertex $v_j \in B_{1,2}$ there is either one edge in A_4 or two edges in A_5. We also know that every edge in $A_4 \cup A_5$ must belong to $B_{1,2}$. Thus we have $2|B_{1,2}| = 2|A_4| + |A_5|$. Therefore $2|B_{1,1}| + 2|B_{1,2}| - |A_2| - |A_5| - 2|A_4| = 0$ and the theorem is proved. □

Theorem 46 *Let K_∞ and K'_∞ be solenoids with shift maps h, and h', respectively, where h is topologically conjugate to h'. Then $|Fix(h^i)| = |Fix(h'^i)|$ for $i = 1, \ldots, \infty$. Similarly if $\{K, g\}$ is a presentation of (K_∞, h) and $\{K', g'\}$ is a presentation of (K'_∞, h') then $|Fix(g^i)| = |Fix(g'^i)|$ for $i = 1, \ldots, \infty$.*

Proof. If h is topologically conjugate to h' then there exists a homeomorphism $f \colon K_\infty \to K'_\infty$ so that $f \circ h = h' \circ f$. Recall that a homeomorphism is a continuous bijection with a continuous inverse. Suppose that $x \in \text{Fix}\,(h^m)$ for some integer $m > 0$. Then since $h^m(x) = x$ we have $h'^m \circ f(x) = f(x)$. Thus $f(x)$ will be fixed under h'^m. Since f is a bijection $s = f|\text{Fix}\,(h^m)$ will be bijection from $\text{Fix}\,(h^m)$ to $\text{Fix}\,(h'^m)$. From theorem 6 we know that if $\{K, g\}$ is a presentation of the solenoid K_∞ with shift map h then there is a 1-1 correspondence between the fixed points of h^m and g^m. Thus $|\text{Fix}\,(h^m)| = |\text{Fix}\,(h'^m)|$ and $|\text{Fix}\,(g^m)| = |\text{Fix}\,(g'^m)|$. □

Theorem 47 *Let $\{K, g\}$ be the presentation of a solenoid. Then there exists a (p, q)-block presentation $\{K', g'\}$ shift equivalent to $\{K, g\}$.*

Proof. Let $\{K_0, g_0\}$ be the presentation of a solenoid. Our first task will be to find a presentation $\{K_n, g_n\}$ shift equivalent to $\{K_0, g_0\}$ where every vertex of K_n is a periodic point under g_n. Clearly if every vertex in K_0 is periodic under g_0 then we can move on to the next step. If not then we must show how we can arrive at such a $\{K_n, g_n\}$. We do this by first showing that there will be an smallest integer $N > 0$ so that every point in $g_0^N(V_0)$ is a periodic point under g_0, where here V_0 is the vertex set of K_0. We then show how we can find a shift equivalent presentation $\{K_1, g_1\}$ so that $M = N - 1$ is the smallest integer so that all points in $g_1^M(V_1)$ are periodic points under g_1. It is then clear that if we perform this procedure N times we will arrive at a shift equivalent presentation with the desired property.

Let V_0 be the vertex set of K_0, write $A_0^0 = V_0$ and $A_i^0 = g_0^i(V_0)$ for $i = 1, \ldots, \infty$. Since g_0 is W^*-mapping we know that $g_0(V_0) \subseteq V_0$ and thus that $g_0(A_i^0) \subseteq A_i^0$ for $i = 0, \ldots, \infty$. Since A_0^0 is a finite set we know that there must exist an $N > 0$ such that for $j \geq N$ we have $g_0(A_j^0) = A_j^0 = A_N^0$. In other words N is the smallest integer so that every point in A_N^0 is a periodic point under g_0. Since every $x \in A_N^0$ is a periodic point under g_0 we know that each will have a unique pre-image $y \in A_N^0$, i.e., $g_0|A_N^0$ is a bijection.

We may partition each A_i^0 into two subsets, $A_i^0 = A_{i+1}^0 \cup C_i^0$, where $g_0(C_i^0) \subseteq A_{i+1}^0$ and $A_{i+1}^0 \cap C_i^0 = \emptyset$. The vertices in C_i^0 are vertices which are eventually periodic to the orbit of some periodic vertex in A_N^0. It is easy to see that $C_j^0 = \emptyset$ for $j \geq N$. For $0 \leq j \leq N$ the vertices in C_j^0 are eventually periodic to the orbit of some vertex in A_N^0 and "land" on this orbit after at most $N - j$ iterations of g_0.

We may define a "new" smooth graph K_1 from K_0 by identifying each $x \in C_{N-1}^0$ with the unique pre-image in A_N^0 of $g_0(x) \in A_N^0$. Thus $K_1 = K_0/\sim$ where $\sim$ is the equivalence relation given by

$$x \sim y \begin{cases} \text{if } x \in C_{N-1}^0, y \in A_N^0 \text{ and } g_0(x) = g_0(y) \text{ or }, \\ x = y. \end{cases}$$

We then define $r_0: K_0 \to K_1$ to be the projection mapping. We define the map $s_0: K_1 \to K_0$ by $s_0 = g_0 \circ r_0^{-1}$. We know that s_0 is well defined because if r_0^{-1} is multi-valued at $z \in K_1$ then $r_0^{-1}(z) = \{x_1, \ldots, x_t\} \cup \{y\}$ where $\{x_1, \ldots, x_t\} \in C_{N-1}^0$ and $y \in A_N^0$ is the unique pre-image in A_N^0 of $g_0(\{x_1, \ldots, x_t\}) \in A_N^0$. We may then define $g_1: K_1 \to K_1$ by $g_1 = r_0 \circ s_0$. We know that $g_1 \sim_s g_0$ with lag=1 since

$$g_1 \circ r_0 = r_0 \circ s_0 \circ r_0 = r_0 \circ g_0 \circ r_0^{-1} \circ r_0 = r_0 \circ g_0,$$

$$s_0 \circ g_1 = g_0 \circ r_0^{-1} \circ r_0 \circ s_0 = g_0 \circ s_0,$$

$$s_0 \circ r_0 = g_0 \circ r_0^{-1} \circ r_0 = g_0,$$

and

$$r_0 \circ s_0 = g_1.$$

We now need to show that $\{K_1, g_1\}$ is such that every point in $g_1^{N-1}(V_1)$ is periodic under g_1, where V_1 is the vertex set of K_1. We clearly have $r_0(V_0) = r_0(A_0^0) = V_1$. If we define $A_1^0 = V_1$ and $A_1^i = g_1^i(V_1)$ then clearly $r_0(A_i^0) = A_i^1$ since

$$r_0(A_i^0) = r_0 \circ g_0(A_{i-1}^0) = g_1 \circ r_0(A_{i-1}^0) = g_1(A_{i-1}^1) = A_i^1.$$

We claim that $g_1(A_{N-1}^1) = A_{N_1}^1$. We know that $A_{N-1}^0 = A_N^0 \cup C_{N-1}^0$ and $r_0(A_{N-1}^0) = r_0(A_N^0)$ by the definition of the identification. Thus $A_{N-1}^1 = A_N^1$ since

$$A_{N-1}^1 = r_0(A_{N-1}^0) = r_0(A_N^0) = A_N^1.$$

We then see that every point in $A_{N-1}^1 = g_1^{N_1}(V_1)$ is periodic under g_1.

If we repeat this procedure N times in each case moving from the presentation $\{K_{i-1}, g_{i-1}\}$ to a shift equivalent presentation $\{K_i, g_i\}$ we arrive at a presentation $\{K_N, g_N\}$ where every point in the vertex set V_N of G_N is periodic under g_N.

Our second task is to find a (p, q)-block presentation from $\{K_N, G_N\}$ by relabeling periodic ordinary points of K_N vertices. Every point in V_N is periodic under g_N thus we may partition V_N into a finite number of orbits under g_N; $V_N = O_1 \cup \cdots \cup O_j$. Each O_i has a period $n_i > 0$ under g_N, where n_i is defined to be the least integer so that for all $x \in O_i$, $g_N^{n_i}(x) = x$. Let $q = \max\{n_1, \ldots, n_i\}$ and $p = \min\{n_1, \ldots, n_i\}$. We then define K_{N+1} to be the "new" smooth graph formed from K_N by re-labeling all periodic ordinary points in K_N with period t, $p \leq t \leq q$, vertices. The mapping $g_{N+1}: K_{N+1} \to K_{N+1}$ is then defined by $g_N = g_{N+1}$ and that $g_N \sim_s g_{N+1}$ is trivial. We then have

$$V_{N+1} = \bigcup_{n=p}^{q} \operatorname{Fix}(g^n)$$

and thus that $\{K_{N+1}, g_{N+1}\}$ is a (p,q)-block presentation.

It is also easy to see that by relabeling more periodic ordinary points vertices we may give find a "new" presentation $\{K_{N+2}, g_{N+1}\}$ which is a (p', q')-block presentation for any p' and q' such that $1 \leq p' \leq p \leq q \leq q'$. □

Theorem 48 *Let $\{K, g\}$ and $\{K', g'\}$ be presentations of solenoids where $Fix(g^i) = Fix(g'^i)$ for $i = 1, \ldots, \infty$. Then $\{K, g\}$ is shift equivalent to a presentation $\{K_1, g_1\}$ and $\{K', g'\}$ is shift equivalent to a presentation $\{K_1', g_1'\}$ where $\{K_1, g_1\}$ and $\{K_1', g_1'\}$ are (p,q)-block presentations.*

Proof. From theorem 47 we know that $\{K, g\}$ is shift equivalent to a (p,q)-block presentation $\{K_0, g_0\}$ for some integers $p \geq q \geq 1$. Similarly $\{K', g'\}$ is shift equivalent to a (p', q')-block presentation $\{K_0', g_0'\}$ for some $p' \geq q' \geq 1$. Let $N = \max\{p, p'\}$ and $M = \min\{q, q'\}$. Re-label all ordinary points of K_0 with periods i, $M \leq i \leq N$ of K_0, under g_0 vertices. Similarly re-label all ordinary points of K_0' with periods i, $M \leq i \leq N$, under g_0' vertices. Denote the "new" presentations $\{K_1, g_1\}$ and $\{K_1', g_1'\}$ respectively. Both $\{K_1, g_1\}$ and $\{K_1', g_1'\}$ are clearly (N, M)-block presentations. □

We will now now show how we can associate with every (p,q)-block presentation an endomorphism of a finitely generated non-abelian free group. We also show that if $\{K, g\}$ and $\{K, g\}$ are (p,q)-block presentation of solenoids and g_* and g_*' are the associated endomorphisms then $g \sim_s g'$ implies $g_* \sim_s g_*'$. In order to do this we will need some notation. Let $A = \{a_1, \ldots, a_n\}$ be a finite set. We will denote by $\langle A \rangle$, or $\langle a_1, \ldots, a_n \rangle$ the free group (non-abelian), on the set of generators A. We will use id to denote the empty word and a_i^{-1} to denote the inverse of a_i in $\langle A \rangle$.

For every (p,q)-block presentation $\{K, g\}$ of a solenoid we will associate an endomorphism $g_*: \langle E \rangle \to \langle E \rangle$ where $\langle E \rangle$ is the free group generated by the set of edges E of K. Let $\{K, g\}$ be a (p,q)-block presentation of a solenoid. If K is orientable let ρ be a coherent orientation on K and if K is nonorientable let ρ be an arbitrary orientation on K. For every edge e in the edge set $E = \{e_1, \ldots, e_m\}$ of K choose an embedding

$\gamma_e\colon [0,1] \to \bar{e}$ so that γ_e traverses $\bar{e}$ in agreement with the orientation on e, i.e., $\gamma(0) = \rho^*(e)$ and $\gamma_e(1) = \rho(e)$. Note that if the e is a loop we assume that some path γ_e is chosen around the loop. For each edge $e \in E$ we can form the map $g \circ \gamma_e\colon [0,1] \to K$. Define $0 = t_0 < t_1 < \cdots < t_{l(e)-1} < t_{l(e)} = 1$ so that $g \circ \gamma_e((t_{i-1}, t_i)) \in E$ for $i = 1, \ldots, l(e)$. The action of the endomorphism $g_*\colon \langle E \rangle \to \langle E \rangle$ on the generating set E is then defined by $g_*(e) = \alpha_1^{n_1} \alpha_2^{n_1} \ldots \alpha_{l(e)}^{n_{l(e)}}$ where $\alpha_i = g \circ \gamma_e((t_{i-1}, t_i))$ for $i = 1, \ldots, l(e)$ and $n_i = 1$ if $g \circ \gamma_e | \alpha_i$ traverses α_i in the same "direction" as γ_{α_i} and $n_i = -1$ if $\gamma_e | \alpha_i$ traverses α_i in the opposite "direction" as γ_{α_i}. In order to make g_* an endomorphism if $\beta_1 \ldots \beta_j \in \langle E \rangle$ where $\beta_i \in E$ we define $g_*(\beta_1 \ldots \beta_j) = g_*(\beta_1) g_*(\beta_2) \ldots g_*(\beta_j)$ and $g_*(id) = id$. It is clear that g_* defined in this manner an endomorphism.

Definition 42 Let $\{K, g\}$ be a (p,q)-block presentation and let $g_*\colon \langle E \rangle \to \langle E \rangle$ be the endomorphism defined in the manner given above. Then g_* is called the *associated* endomorphism for the presentation $\{K, g\}$.

Our next theorem shows that if $\{K, g\}$ and $\{K', g'\}$ are (p,q)-block presentations with associated endomorphisms g_* and g'_* respectively, then $g \sim_s g'$ implies that $g_* \sim_s g'_*$.

Theorem 49 *Let $\{K, g\}$ and $\{K', g'\}$ be presentations of solenoids in (p,q)-block form with associated endomorphisms g_* and g'_* respectively. Then $g \sim_s g'$ implies that $g_* \sim_s g'_*$.*

Proof. Suppose that $\{K, g\}$ and $\{K', g'\}$ are (p,q)-block presentations of solenoids where $g \sim_s g'$ with lag N. Let $g_*\colon \langle E \rangle \to \langle E \rangle$ and $g'_*\colon \langle E' \rangle \to \langle E' \rangle$ be the endomorphisms associated with $\{K, g\}$ and $\{K', g'\}$ respectively. Let $r\colon K \to K'$ and $s\colon K' \to K$ be the maps giving the shift equivalence between g and g'. Since $\{K, g\}$ and $\{K', g'\}$ are (p,q)-block presentations we know that $r(V) = V'$ and $s(V') = V$ where V and V' are the vertex sets of K and K' respectively. In order to show that $g_* \sim_s g'_*$ we need to find homomorphisms $r_*\colon \langle E \rangle \to \langle E' \rangle$ and $s_*\colon \langle E' \rangle \to \langle E \rangle$ such the diagrams

$$
\begin{array}{ccc}
\langle E \rangle & \xrightarrow{g_*} & \langle E \rangle \\
{\scriptstyle r_*}\downarrow & & \downarrow{\scriptstyle r_*} \\
\langle E' \rangle & \xrightarrow[g'_*]{} & \langle E' \rangle
\end{array}
\qquad
\begin{array}{ccc}
\langle E \rangle & \xrightarrow{g_*} & \langle E \rangle \\
{\scriptstyle s_*}\uparrow & & \uparrow{\scriptstyle s_*} \\
\langle E' \rangle & \xrightarrow[g'_*]{} & \langle E' \rangle
\end{array}
\qquad
\begin{array}{ccc}
\langle E \rangle & \xrightarrow{g_*^m} & \langle E \rangle \\
{\scriptstyle r_*}\downarrow & \overset{s_*}{\nearrow} & \downarrow{\scriptstyle r_*} \\
\langle E' \rangle & \xrightarrow[g'^m_*]{} & \langle E' \rangle
\end{array}
$$

are commutative for some integer m.

We want to define the homomorphisms r_* and s_*. With each edge $e \in E$ we have an embedding $\gamma_e\colon [0,1] \to \bar{e}$ used in the definition of the endomorphism g_*. Similarly for each edge $y \in E'$ we have an embedding $\pi_y: [0,1] \to \bar{y}$ used in the definition of the endomorphism g'_*. For each edge $e \in E$ we may form the map $r \circ \gamma_e: [0,1] \to K'$. We can then find $0 = d_0 < d_1 < \cdots < d_{l(e)-1} < d_{l(e)} = 1$ so that $r \circ \gamma_e((d_{i-1}, d_i)) \in E'$ for each $i = 1, \ldots, l(e)$. We are assured that the incident vertices of e map to vertices of K' since $r(V) = V'$. We may then define the homomorphisms r_* in a manner similar that of g_*. Thus we define the action r_* on the generators of $\langle E \rangle$ by $r_*(e) = \alpha_1^{n_1} \ldots \alpha_{l(e)}^{n_{l(e)}}$ where $\alpha_i = r \circ \gamma_e((t_{i-1}, t_i))$ for $i = 1, \ldots l(e)$ and $n_i = 1$ if $r \circ \gamma_e|\alpha_i$ traverses α_i in the same "direction" as π_{α_i} and $n_i = -1$ if $r \circ \gamma_e|\alpha_i$ traverses α_i in the opposite "direction" as π_{α_i}. In order to make r_* a homomorphism from $\langle E \rangle$ to $\langle E' \rangle$ if $\beta_1\beta_2 \ldots \beta_j \in \langle E \rangle$ where $\beta_i \in E$ we define $r_*(\beta_1\beta_2 \ldots \beta_j) = r_*(\beta_1)r_*(\beta_2) \ldots r_*(\beta_j)$ and $r_*(id) = id'$. Note, here id denotes the empty word of $\langle E \rangle$ and id' the empty word of $\langle E' \rangle$. The map r_* so defined is clearly a homomorphism. The homomorphism s_* is defined in a similar manner and again because $\{K, g\}$ and $\{K', g'\}$ are both (p,q)-block presentations we are assured that $s(V') = V$.

We need to verify that r_* and s_* satisfy the usual relationships of shift equivalence. We will show that $r_* \circ g_* = g'_* \circ r_*$. We know that $r \circ g = g' \circ r$. For an edge $e \in E$ may form the maps $r \circ g \circ \gamma_e: [0,1] \to K'$ and $g' \circ r \circ \gamma_e: [0,1] \to K'$. There will exist $0 = t_0 < t_1 < \ldots < t_{s(e)-1} < t_{s(e)} = 1$ so that $r \circ g \circ \gamma_e((t_{i-1}, t_i)) \in E'$ for each $i = 1, \ldots, s(e)$. We see that $r_* \circ g_*(e) = \alpha_1^{n_1} \ldots \alpha_{s(e)}^{n_{s(e)}}$ where $\alpha_i = r \circ g \circ \gamma_e((t_{i-1}, t_i))$ for $i = 1, \ldots s(e)$ and the n_i are determined in the usual manner. Similarly $g'_* \circ r_*(e) = \delta_1^{m_1} \ldots \delta_{s(e)}^{m_{s(e)}}$ where $\delta_i = g' \circ r \circ \gamma_e((t_{i-1}, t_i))$ for $i = 1, \ldots (e)$ and the m_i are determined in the usual manner. However, since $r \circ g = g' \circ r$ we know that $r \circ g \circ \gamma_e((t_{i-1}, t_i)) = g' \circ r \circ \gamma((t_{i-1}, t_i))$ for $i = 1, \ldots, s(e)$. Thus $\alpha_i^{n_i} = \delta_i^{m_i}$ for $i = 1, \ldots, s(e)$. The proofs showing that $g_* \circ s_* = s_* \circ g'_*$, $s_* \circ r_* = g_*^N$, and $r_* \circ s_* = g_*'^N$ hold are similar. □

Note, unlike elementary presentations, given a presentation $\{K, g\}$ when we find a shift equivalent (p,q)-block presentation, $\{K', g'\}$, the smooth graph K' often has a more complicated structure then the original smooth graph K. The reason for finding a (p,q)-block presentation is generally to calculate an invariant. Thus one might be trying to determine whether or not two presentations, say $\{K_1, g_1\}$ and $\{K_2, g_2\}$ are shift equivalent. We might find shift equivalent presentations for each which are in (p,q)-block form. We would then know that the associated group endomorphism would be shift equivalent. We could then try to use these endomorphism to develop further invariants. We will do exactly this in the next chapter. While these invariants will not be able to tell us that the two presentations are shift equivalent they will sometimes allow us to say that two presentations are not shift equivalent. We finish this section by giving an example demonstrating the techniques used to find (p,q)-block presentations.

Example 15 In this example we demonstrate the techniques used in finding a (p,q)-block presentation. We start with a presentation $\{K, g\}$ which has no fixed points and six period two points. We find a $(2,2)$-block presentation shift equivalent to $\{K, g\}$.

Consider the smooth graph $K_0 = (G_0, S_0)$ with $G_0 = (V_0, E_0)$, $V_0 = \{v_1, v_2, v_3\}$, $E_0 = \{e_1, e_2, e_3, e_4, e_5\}$ where the incidence relations and switch conditions are summarized in the table below:

v	$R(v)$	$L(v)$
v_1	$\{e_2\}$	$\{e_1, e_3\}$
v_2	$\{e_2\}$	$\{e_1, e_4\}$
v_3	$\{e_4, e_5\}$	$\{e_3, e_5\}$

A smooth graph diagram for K_0 is shown in figure 4.8. K_0 is orientable and is shown in the diagram with the coherent orientation ρ where $\rho(e_1) = \rho(e_4) = v_2$, $\rho(e_2) = v_1$, and $\rho(e_3) = \rho(e_5) = v_3$.

Consider the W^*-mapping $g_0 \colon K_0 \to K_0$ where $g_0(v_1) = v_2$, $g_0(v_2) = g_0(v_3) = v_1$ and g_0 expands each edge $e \in E$ along the route $e_i \dots e_j$ as given below;

$$g_0 \begin{cases} e_1 \mapsto e_2 e_3 e_5 e_4 e_2 \\ e_2 \mapsto e_1 \\ e_3 \mapsto e_2 \\ e_4 \mapsto e_1 e_2 \\ e_5 \mapsto e_1 e_2. \end{cases}$$

We see that the vertices v_1 and v_2 are periodic under g and that v_3 is eventually periodic to the orbit $\{v_1, v_2\}$ with $g_0(v_3) = v_1$. The unique pre-image of v_1 in the period 2 orbit is v_2.

We form a new smooth graph K_1 from K_0 by identifying the vertex v_3 with the vertex v_2. The smooth graph $K_1 = \{G_1, S_1\}$ with $G_1 = (V_1, E_1)$, $V_1 = \{x_1, x_2\}$, $E_1 = \{y_1, y_2, y_3, y_4, y_5\}$ where the incidence relations and switch conditions are summarized in

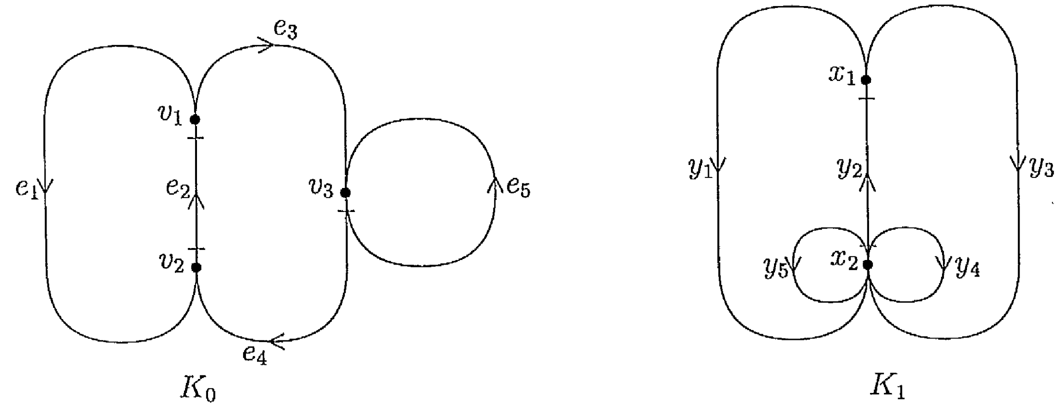


Figure 4.8: The smooth graphs K_0 and K_1 given in example 15

the table below:

x	$R(x)$	$L(x)$
x_1	$\{y_2\}$	$\{y_1, y_3\}$
x_2	$\{y_2, y_4, y_5\}$	$\{y_1, y_3, y_4, y_5\}$

The mapping $r_0: K_0 \to K_1$ is given by the projection mapping. We find that $r_0(v_1) = x_1$, $r_0(v_2) = x_2$, $r_0(v_3) = x_2$ and that r_0 maps the edges of K_0 to the edges of K_1 as shown below;

$$r_0 \begin{cases} e_1 \mapsto y_1 \\ e_2 \mapsto y_2 \\ e_3 \mapsto y_3 \\ e_4 \mapsto y_4 \\ e_5 \mapsto y_5. \end{cases}$$

The mapping $s_0: K_1 \to K_0$ is given by $g_0 \circ r_0^{-1}$. This is a well defined and continuous map since the only points which are identified under r_0 are v_2 and v_3, thus $r_0^{-1}(x_2) = \{v_2, v_3\}$. But we know that $g_0(v_2) = g_0(v_3) = v_1$. We see that $s_0(x_1) = v_2$, $s_0(x_2) = v_1$ and that s_0 maps the edges of K_1 to K_0 in the manner shown below;

$$s_0 \begin{cases} y_1 \mapsto e_2e_3e_5e_4e_2 \\ y_2 \mapsto e_1 \\ y_3 \mapsto e_2 \\ y_4 \mapsto e_1e_2 \\ y_5 \mapsto e_1e_2. \end{cases}$$

The W^*-mapping $g_1: K_1 \to K_1$ is then defined by $g_1 = r_0 \circ s_0$. Thus we see that $g_1(x_1) =$

x_2, $g_1(x_2) = x_1$ and that g_1 acts on the edges of K_1 as given below;

$$g_1 \begin{cases} y_1 \mapsto y_2y_3y_5y_4y_2 \\ y_2 \mapsto y_1 \\ y_3 \mapsto y_2 \\ y_4 \mapsto y_1y_2 \\ y_5 \mapsto y_1y_2. \end{cases}$$

We now have $g_1 \sim_s g_0$ where K_1 is such that all vertices in V_1 are periodic, with period two, under g_1.

We want to form a new smooth graph K_2 from K_1 by re-labeling all of the periodic points of period 2 in K_1 vertices. If we consider g_1^2 we see that g_1^2 acts on the edges of K_2 in the following manner;

$$g_1^2 \begin{cases} y_1 \mapsto y_1y_2y_1y_2y_1y_2y_1 \\ y_2 \mapsto y_2y_3y_5y_4y_2 \\ y_3 \mapsto y_1 \\ y_4 \mapsto y_2y_3y_5y_4y_2y_1 \\ y_5 \mapsto y_2y_3y_5y_4y_2y_1. \end{cases}$$

From this we see that there are two occurrences of y_1 in the interior of the word defined by $g_1^2(y_1)$, $y_2y_1y_2y_1y_2$, thus there will be two fixed points of g_1^2 in y_1. But since g_1 has no fixed points we know that these are both period two points. There is one occurrence of y_4 in the interior of the word defined by $g_1^2(y_4)$, $y_3y_5y_4y_2$, and one occurrence of y_5 in the interior of the word defined by $g_1^2(y_5)$, $y_3y_5y_4y_2$. Thus there will be one period two point in y_4 and one in y_5. This gives us a total of 4 ordinary points in K_1 which have period 2.

We re-label the two period two points in y_1 as vertices x_3 and x_4. The edge y_1 then splits into 3 edges which we re-label z_1, z_2, z_3. We re-label the period two points in y_4 and y_5 as x_5 and x_6, respectively. The edges y_4 and y_5 split into two edges each, which we re-label z_4, z_5 and z_6, z_7, respectively. We call this new smooth graph K_2. The smooth graph diagram for K_2 is given in figure 4.9 The mapping $g_2\colon K_2 \to K_2$ is then such that

$$g_2(x_1) = x_2, \quad g_2(x_2) = x_2 \quad g_2(x_3) = x_6$$
$$g_2(x_4) = x_5 \quad g_2(x_5) = x_4 \quad g(x_6) = x_3,$$

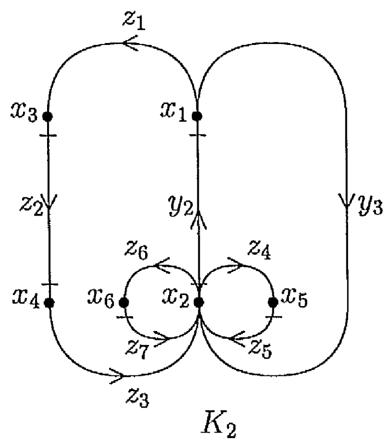


Figure 4.9: The smooth graph K_2 given in example 15

and g_2 expands each edge of K_2 along the routes in K_2 as shown below;

$$g_2 \begin{cases} z_1 \mapsto y_2 y_3 z_6 & z_2 \mapsto z_7 z_4 & z_3 \mapsto z_5 y_2 \\ z_4 \mapsto z_1 z_2 & z_5 \mapsto z_3 y_2 & z_6 \mapsto z_1 \\ z_7 \mapsto z_2 z_3 y_2 & y_2 \mapsto z_1 z_2 z_3 & y_3 \mapsto y_2. \end{cases}$$

The presentation $\{K_2, g_2\}$ is then the desired $(2,2)$-block presentation.

Chapter 5

Algebraic Invariants

In chapter 4 we demonstrated that with any presentation of a solenoid $\{K, g\}$ which is either an elementary presentation or (p, q)-block presentation we can associate an endomorphism of a finitely generated free group. We also proved that if we are given two presentations $\{K, g\}$ and $\{K', g'\}$ which are either elementary presentations or (p, q)-block presentations then $g \sim_s g'$ implies that $g_* \sim_s g'_*$ where g_* and g'_* are the associated endomorphisms. In this chapter we discuss the shift equivalence of endomorphisms of finitely presented free groups in order to develop further invariants for solenoids. In the first section we discuss invariants for endomorphisms of finitely generated free groups which arise due to the abelianization of the free group in question. Many of these invariants are found in the extensive literature concerning sub-shifts of finite type. In the second section we develop some invariants which reflect the non-abelian character of these endomorphisms. In the final section we give examples how these invariants, especially the non-abelian ones, can be used to study the shift equivalence of W^*-mappings of solenoids. In particular we will be able to solve a previously unsolved problem posed by R.F. Williams in [39]. Throughout this chapter we restrict our attention to endomorphisms induced by presentations $\{K, g\}$ where the smooth graph K is orientable and the W^*-mapping g is orientation preserving. Thus if $\phi\colon \langle a_1, \ldots, a_n\rangle \to \langle a_1, \ldots, a_n\rangle$ is such an endomorphism then for each a_i there will be no instances of a_j^{-1} for any a_j in the words $\phi(a_i)$.

5.1 Abelian Invariants

In this section we study invariants for the shift equivalence of endomorphisms of finitely generated free groups which arise by considering the abelianization of these groups. An endomorphism $\phi\colon \langle A\rangle \to \langle A\rangle$ will induce an endomorphism $\phi_0\colon \langle A\rangle_{ab} \to \langle A\rangle_{ab}$ where $\langle A\rangle_{ab}$ is the abelianization of $\langle A\rangle$. We show that if two endomorphisms ϕ and ψ are shift equivalent then the induced endomorphisms ϕ_0 and ψ_0 are also shift equivalent. The endomorphisms ϕ_0 and ψ_0 can be given a matrix representation. This then gives us a set of matrix relations for shift equivalence. These matrix relations are well known from symbolic dynamics. We will consider some of the invariants which can be derived from these matrix relations. One invariant of these matrix equations which will be particularly useful when considering non-abelian invariants is the Bowen-Franks group.

Definition 43 Let $A = \{a_1, \ldots, a_n\}$ be a finite set and $\langle A\rangle$ the free group generated by the alphabet A. The *abelianization*, $\langle A\rangle_{ab}$, of $\langle A\rangle$ is the quotient group $\langle A \mid a_i a_j = a_j a_i \text{ for all } a_i, a_j \in A\rangle$.

There is a well defined epimorphism $\gamma\colon \langle A\rangle \to \langle A\rangle_{ab}$ which is such that if $a \in \langle A\rangle$ then $\gamma(a)$ is the equivalence class of a in $\langle A\rangle_{ab}$. $\langle A\rangle_{ab}$ is generated by $\{\gamma(a_1), \ldots, \gamma(a_n)\}$. In general we will write $\gamma(a_i) = [a_i]$. Each element $\alpha \in \langle A\rangle_{ab}$ has a unique expression as $[a_1]^{m_1}[a_2]^{m_2}\ldots[a_n]^{m_n}$, where we consider $[a_i]^0 = id$. For example $\gamma(a_4a_2a_2a_3a_2) = [a_2]^3[a_3][a_4]$.

Note 11 Let F and G be two groups. If F is isomorphic to G we will write $F \cong G$.

Theorem 50 *[31] Let $A = \{a_1, \ldots, a_n\}$ and $\langle A\rangle$ be the free group generated by A. Then $\langle A\rangle_{ab}$ is isomorphic to $\mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, the direct product of n infinite cyclic groups, i.e., $\langle A\rangle_{ab} \cong \mathbb{Z}^n$.*

We will consider the direct product of n infinite cyclic groups, $\mathbb{Z}^n$, to be the group generated by the set $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$, where $\mathbf{c}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$, with the 1 in the i-th position. The binary operation for $\mathbb{Z}^n$ is then addition ,"+", defined as usual and the identity is given by $\mathbf{0} = (0, 0, \ldots, 0)^T$. The inverse of an element $\mathbf{b} = (b_1, \ldots, b_n)^T$ is then $-\mathbf{b} = (-b_1, \ldots, -b_n)^T$. An isomorphism $\xi\colon \langle A\rangle_{ab} \to \mathbb{Z}^n$ can be defined by

$$\xi([a_1]^{m_1}[a_2]^{m_2}\ldots,[a_n]^{m_n}) = m_1\mathbf{c}_1 + m_2\mathbf{c}_2 + \cdots + m_n\mathbf{c}_n,$$

where here, using a slight abuse of notation, $m_i\mathbf{c}_i = (0, \ldots, 0, m_i, 0, \ldots, 0)$.

Let $\phi\colon \langle A\rangle \to \langle A\rangle$ be an endomorphism of the finitely generated free group $\langle A\rangle$. The endomorphism ϕ induces an endomorphism $\phi_0\colon \langle A\rangle_{ab} \to \langle A\rangle_{ab}$. The action of ϕ_0 on the generators of $\langle A\rangle_{ab}$ is defined so that $\phi_0([a_i]) = \gamma \circ \phi(a_i)$. For $\alpha = \alpha_1\alpha_2\ldots\alpha_m \in \langle A\rangle_{ab}$ where $\alpha_j \in \{[a_1], \ldots, [a_n]\}$ we then define $\phi_0(\alpha) = \phi_0(\alpha_1)\phi_0(\alpha_2)\ldots\phi_0(\alpha_m)$ as well as $\phi_0(id) = [id]$. It is easily checked that ϕ_0 defined in this fashion will be an endomorphism and is such that the diagram

$$\begin{array}{ccc}
\langle A\rangle & \xrightarrow{\ \phi\ } & \langle A\rangle \\
\gamma\big\downarrow & & \big\downarrow\gamma \\
\langle A\rangle_{ab} & \xrightarrow[\ \phi_0\ ]{} & \langle A\rangle_{ab}
\end{array}$$

commutes.

Since $\langle A\rangle_{ab} \cong \mathbb{Z}^n$ the endomorphism ϕ_0 will induce an endomorphism $\phi_1\colon \mathbb{Z}^n \to \mathbb{Z}^n$, which is defined by $\phi_1 = \xi \circ \phi_0 \circ \xi^{-1}$. We then see that the diagram

$$\begin{array}{ccc}
\langle A\rangle & \xrightarrow{\ \phi\ } & \langle A\rangle \\
\gamma\big\downarrow & & \big\downarrow\gamma \\
\langle A\rangle_{ab} & \xrightarrow[\ \phi_0\ ]{} & \langle A\rangle_{ab} \\
\xi\big\downarrow & & \big\downarrow\xi \\
\mathbb{Z}^n & \xrightarrow[\ \phi_1\ ]{} & \mathbb{Z}^n
\end{array}$$

commutes.

We may give ϕ_1 a matrix representation. Let $\mathbf{\Phi}$ be the $n \times n$ matrix where Φ_{ji} equals the number of occurrences of a_j in the word $\phi(a_i)$. We claim that for any $\mathbf{c} \in \mathbb{Z}^n$ we have $\mathbf{\Phi}\mathbf{c} = \phi_1(\mathbf{c})$. Consider $\mathbf{c}_i \in \mathbb{Z}^n$. We have $\xi \circ \gamma(a_i) = \mathbf{c}_i$. Let $m(j,i)$ denote the number of occurrences of a_j in $\phi(a_i)$, then $\gamma \circ \phi(a_i) = [a_1]^{m(1,i)}[a_2]^{m(2,i)}\ldots[a_n]^{m(n,i)}$. We then see that

$$\xi \circ \gamma \circ \phi(a_i) = m(1,i)\mathbf{c}_1 + m(2,i)\mathbf{c}_2 + \cdots m(n,i)\mathbf{c}_n = \phi_1(\mathbf{c}_i).$$

For any $\mathbf{c} \in \mathbb{Z}^n$ we may write $\mathbf{c} = w_1\mathbf{c}_1 + \cdots + w_n\mathbf{c}_n$ thus we see that

$$\phi_1(\mathbf{c}) = \sum_{i=1}^{n} w_i \xi \circ \gamma \circ \phi(a_i) = \sum_{i=1}^{n} w_i \left(\sum_{j=1}^{n} m(j,i)\right) = \mathbf{\Phi}\mathbf{c}$$

since $\Phi_{ji} = m(j,i)$. It is clear from the above that the matrix $\mathbf{\Phi}$ is necessarily a non-negative integral matrix ($\mathbb{Z}^+$-matrix), i.e., each entry ϕ_{ji} is necessarily a non-negative integer.

Definition 44 Let $\phi: \langle A \rangle \to \langle A \rangle$ be an endomorphism of a finitely generated free group. Then the matrix Φ, where Φ_{ji} equals the number of of occurrences of the letter a_j in the word $\phi(a_i)$, is called the *non-negative integral matrix representation* ($\mathbb{Z}^+$-matrix representation) of the endomorphism ϕ.

Definition 45 Let Φ and Ψ be square $\mathbb{Z}^+$-matrices and $m > 0$ an integer. Φ and Ψ are said to be shift equivalent, written $\Phi \sim_s \Psi$, if there exist $\mathbb{Z}^+$-matrices $\mathbf{R}$ and $\mathbf{S}$ satisfying the four shift equivalence relations

$$\Phi\mathbf{S} = \mathbf{S}\Psi, \qquad \mathbf{R}\Phi = \Psi\mathbf{R},$$
$$\Phi^m = \mathbf{SR}, \quad \text{and } \Psi^m = \mathbf{RS}.$$

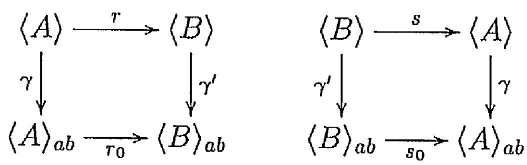
As usual the integer m is called the lag of the shift equivalence.

Matrix shift equivalence was first invented by R.F. Williams as an invariant for subshifts of finite type in [40] and has been extensively studied since. We introduce it in order to develop some further invariants for the shift equivalence of W^*-mappings on smooth graphs. We will use some of these invariants later to construct non-abelian invariants for W^*-mappings.

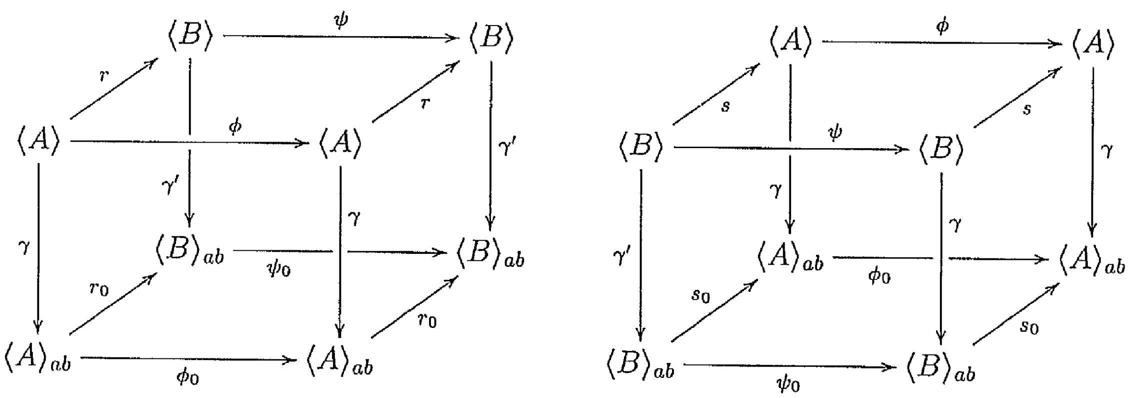
Theorem 51 *Let ϕ and ψ be endomorphisms of finitely generated free groups with $\mathbb{Z}^+$-matrix representations Φ and Ψ respectively. Then $\phi \sim_s \psi$ with lag=l implies that $\Phi \sim_s \Psi$ with lag=l.*

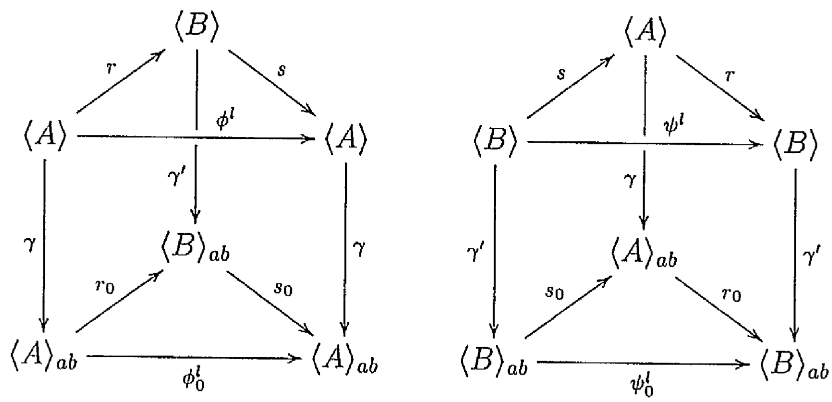
Proof. Let $\phi: \langle A \rangle \to \langle A \rangle$ and $\psi: \langle B \rangle \to \langle B \rangle$ be endomorphisms of the free groups generated by the finite sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$, respectively. Let $r: \langle A \rangle \to \langle B \rangle$ and $s: \langle B \rangle \to \langle A \rangle$ be the homomorphisms giving the shift equivalence between ϕ and ψ. We need to find $\mathbb{Z}^+$-matrices $\mathbf{R}$ and $\mathbf{S}$ which define a shift equivalence between the $\mathbb{Z}^+$-matrix representations Φ and Ψ of ϕ and ψ. Let $\mathbf{R}$ be the $m \times n$ matrix where R_{ji} is the number of occurrences of b_j in the word $r(a_i)$. Let $\mathbf{S}$ be the $n \times m$ matrix where S_{ji} is the number of occurrences of a_j in the word $s(b_i)$. We claim that the matrices $\mathbf{R}$ and $\mathbf{S}$ define a shift equivalence between Φ and Ψ.

Let $\gamma: \langle A \rangle \to \langle A \rangle_{ab}$ and $\gamma': \langle B \rangle \to \langle B \rangle_{ab}$ be the epimorphisms so that, for $a \in \langle A \rangle$, $\gamma(a)$ is the equivalence class of a in $\langle A \rangle_{ab}$, and for $b \in \langle B \rangle$, $\gamma'(b)$ is the equivalence class of b in $\langle B \rangle_{ab}$. Let $\phi_0: \langle A \rangle_{ab} \to \langle A \rangle_{ab}$ and $\psi_0: \langle B \rangle_{ab} \to \langle B \rangle_{ab}$ be the endomorphisms induced by ϕ and ψ, respectively, on the abelianizations of $\langle A \rangle$ and $\langle B \rangle$. In a similar fashion the homomorphism r will induce a homomorphism $r_0\colon \langle A\rangle_{ab} \to \langle B\rangle_{ab}$ and the homomorphism s will induce a homomorphism $s_0\colon \langle B\rangle_{ab} \to \langle A\rangle_{ab}$. The homomorphism r_0 is defined so that, for $a_i \in A$, $r_0([a_i])$ is the equivalence class of $\gamma' \circ r(a_i)$ in $\langle B\rangle_{ab}$, $r_0([id_A]) = [id_B]$, and for $\alpha = \alpha_1 \ldots \alpha_j \in \langle A\rangle_{ab}$, where $\alpha_i \in \{[a_1], \ldots, [a_n]\}$, $r_0(\alpha) = r_0(\alpha_1) \ldots r_0(\alpha_j)$. The homomorphism s_0 is defined so that; for $b_i \in B$, $s_0(b_i)$ is the equivalence class of $\gamma \circ s(b_i)$ in $\langle A\rangle_{ab}$, $s_0([id_B]) = [id_A]$, and for $\beta = \beta_1 \ldots \beta_j \in \langle B\rangle_{ab}$, where $\beta_i \in \{[b_1], \ldots, [b_m]\}$, $s_0(\beta) = s_0(\beta_1) \ldots (\beta_j)$. Note, that here we are letting id_A and id_B denote the identity for $\langle A\rangle$ and $\langle B\rangle$, respectively. It is clear that the diagrams



commute. From the diagrams





we see that the homomorphisms r_0 and s_0 define a shift equivalence between ϕ_0 and ψ_0.

Let $\{\mathbf{c_1}, \ldots, \mathbf{c_n}\}$ and $\{\mathbf{c_1}', \ldots, \mathbf{c_m}'\}$ be the sets generating $\mathbb{Z}^n$ and $\mathbb{Z}^m$, respectively. Consider the isomorphisms $\xi\colon \langle A\rangle_{ab} \to \mathbb{Z}^n$ and $\xi'\colon \langle B\rangle_{ab} \to \mathbb{Z}^m$, where

$$\xi([a_1]^{k_1} \ldots [a_n]^{k_n}) = k_1\mathbf{c_1} + \cdots + k_n\mathbf{c_n}$$

and

$$\xi'([b_1]^{w_1}\dots[b_m]^{w_m}) = w_1\mathbf{c_1}' + \dots + w_m\mathbf{c_m}'.$$

We define $\phi_1 = \xi\circ\phi_1\circ\xi^{-1}$ and $\psi_1 = \xi'\circ\psi_0\circ\xi'^{-1}$, so that ϕ_1 and ψ_1 are the endomorphisms induced on $\mathbb{Z}^n$ and $\mathbb{Z}^m$, respectively, by ϕ_0 and ψ_0. Let $r_1: \mathbb{Z}^n \to \mathbb{Z}^m$ be defined by $r_1 = \xi' \circ r_0 \circ \xi^{-1}$ and $s_1: \mathbb{Z}^m \to \mathbb{Z}^n$ be defined by $s_1 = \xi \circ s_0 \circ \xi'^{-1}$. Similar diagrams, to those for ϕ, ϕ_0 ψ, ψ_0, r, r_0, s, s_0, γ and γ', can be shown to be commutative for ϕ_0, ϕ_1, ψ_0, ψ_1, r_0, r_1, s_0, s_1, ξ, and ξ'. Thus we see that r_1 and s_1 form a shift equivalence between ϕ_1 and ψ_1.

We claim that $\mathbf{R}$ is the matrix representation for r_1 and that $\mathbf{S}$ is the matrix representation for s_1. Thus for any $\mathbf{c} \in \mathbb{Z}^n$ we should have $r_1(\mathbf{c}) = \mathbf{Rc}$ and for any $\mathbf{c}' \in \mathbb{Z}^m$ we should have $s_1(\mathbf{c}') = \mathbf{Sc}'$. Consider $\mathbf{c}_i \in \mathbb{Z}^n$. We have $\xi \circ \gamma(a_i) = \mathbf{c}_i$. We know that R_{ji} denotes the number of occurrences of b_j in $r(a_i)$, thus $\gamma' \circ r(a_i) = [b_1]^{R_{1i}}[b_2]^{R_{2i}}\dots[b_m]^{R_{mi}}$. We then see that

$$\xi' \circ \gamma' \circ r(a_i) = R_{1i}\mathbf{c}_1' + \dots + R_{mi}\mathbf{c}_m' = \sum_{j=1}^{m} R_{ji}\mathbf{c}_j'.$$

For any $\mathbf{c} \in \mathbb{Z}^n$ we may write $\mathbf{c} = w_1\mathbf{c}_1 + \dots + w_n\mathbf{c}_n$, thus we see that

$$r_1(\mathbf{c}) = \sum_{i=1}^{n} w_i\xi' \circ \gamma' \circ r(a_i) = \sum_{i=1}^{n} w_i \left(\sum_{j=1}^{m} R_{ji}\mathbf{c}_j'\right) = \mathbf{R}c.$$

Similarly, consider $\mathbf{c}_i' \in \mathbb{Z}^m$. We then have $\gamma \circ s(b_i) = \mathbf{c}_i'$. We know that S_{ji} is the number of occurrences of a_j in the word $s(b_i)$, thus $\gamma \circ s(b_i) = [a_1]^{S_{1i}}\dots[a_n]^{S_{ni}}$. We then see that

$$\xi \circ \gamma \circ (b_i) = S_{1i}\mathbf{c}_1 + \dots + S_{ni}\mathbf{c}_n.$$

For any $\mathbf{c}' \in \mathbb{Z}^m$ we may write $\mathbf{c}' = u_1\mathbf{c}_1' + \dots + u_m\mathbf{c}_m'$, thus

$$s_1(\mathbf{c}') = \sum_{i=1}^{m} u_i\xi \circ \gamma \circ s(b_i) = \sum_{i=1}^{m} u_i \left(\sum_{j=1}^{n} S_{ji}\mathbf{c}_i\right) = \mathbf{Sc}'.$$

Therefore $\mathbf{R}$ and $\mathbf{S}$ are matrix representations of r_1 and s_1, respectively. Since r_1 and s_1 form a shift equivalence between ϕ_1 and ψ_1 we know that $\mathbf{R}$ and $\mathbf{S}$ will form a shift equivalence between $\mathbf{\Phi}$ and $\mathbf{\Psi}$, the matrix representations of ϕ_1 and ψ_1. □

There are theoretical procedures to decide whether two matrices are shift equivalent [21], but they are usually very difficult to apply and would take us too far afield. We will instead focus on invariants for $\mathbb{Z}^+$-matrix shift equivalence which are easy to compute.

The two main invariants which we will develop are the Jordan form away from zero, $\mathbf{J}^\times$, and the Bowen-Franks group, **BF**. The invariance of the Jordan form away from zero is originally due to Parry and Williams [26] and was developed as an invariant for subshifts of finite type. The Bowen-Franks group is due to Bowen and Franks [5], who discovered it in their study of continuous time analogues of subshifts of finite type. Our treatment of these invariants is due to that in [22].

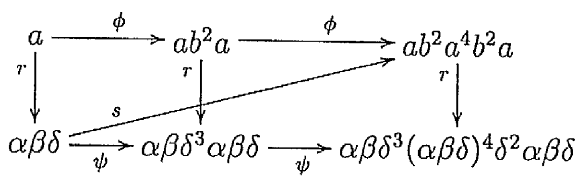
Example 16 Consider the endomorphisms $\phi\colon \langle A\rangle \to \langle A\rangle$ and $\psi\colon \langle B\rangle \to \langle B\rangle$ where $A = \{a,b\}$ and $B = \{\alpha,\beta,\delta\}$ defined by

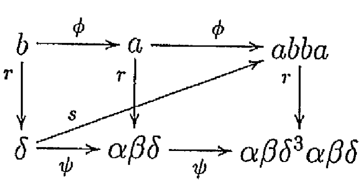
$$\begin{aligned} \phi(a) &= ab^2a, \\ \phi(b) &= a, \end{aligned} \qquad \begin{aligned} \psi(\alpha) &= \alpha\beta\delta^2, \\ \psi(\beta) &= \delta, \\ \psi(\delta) &= \alpha\beta\delta. \end{aligned}$$

The homomorphisms $r\colon \langle A\rangle \to \langle B\rangle$ and $s\colon \langle B\rangle \to \langle A\rangle$ defined by,

$$\begin{aligned} r(a) &= \alpha\beta\delta, \\ r(b) &= \delta, \end{aligned} \qquad \begin{aligned} s(\alpha) &= ab^2a^2, \\ s(\beta) &= a, \\ s(\delta) &= ab^2a, \end{aligned}$$

form a shift equivalence of lag=2 between ϕ and ψ. This can be verified by considering the action of ϕ, ψ, r, and s on the generators $\{a,b\}$ of $\langle A\rangle$ illustrated in the diagrams below:





Similar diagrams can be created to illustrate the action of these maps on the generators $\{\alpha,\beta,\delta\}$ of $\langle B\rangle$.

The $\mathbb{Z}^+$-matrix representations, $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, of ϕ and ψ, respectively, are then found to be given by

$$\boldsymbol{\Phi} = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

We may then construct a 3×2 matrix $\mathbf{R}$ from the homomorphism r and a 2×3 matrix $\mathbf{S}$ from the homomorphism s as follows;

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix}.$$

It is then easily verified that $\mathbf{R\Phi} = \mathbf{\Psi R}$ since

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}.$$

We also see that $\mathbf{\Phi S} = \mathbf{S\Psi}$ since

$$\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 6 \\ 6 & 2 & 4 \end{pmatrix}.$$

We can check that $\mathbf{\Phi}^2 = \mathbf{SR}$ since

$$\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 4 & 2 \end{pmatrix}.$$

Finally considering $\mathbf{\Psi}^2 = \mathbf{RS}$ we see that

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 3 & 1 & 2 \\ 5 & 1 & 4 \end{pmatrix}.$$

Thus $\mathbf{R}$ and $\mathbf{S}$ given a matrix shift equivalence between $\mathbf{\Phi}$ and $\mathbf{\Psi}$.

Example 17 In this example we show how matrix shift equivalence can sometimes be used to determine whether or not two endomorphisms are shift equivalent. Let $\phi\colon \langle a, b\rangle \to \langle a, b\rangle$ and $\psi\colon \langle \alpha, \beta\rangle \to \langle \alpha, \beta\rangle$ be defined by

$$\phi(a) = ab^8, \quad \psi(\alpha) = \alpha\beta^2,$$
$$\phi(b) = ab^7, \quad \psi(\beta) = \alpha\beta\alpha^3\beta^6.$$

The $\mathbb{Z}^+$-matrix representations $\mathbf{\Phi}$ and $\mathbf{\Psi}$ of ϕ and ψ respectively are then given by

$$\mathbf{\Phi} = \begin{pmatrix} 1 & 1 \\ 8 & 7 \end{pmatrix} \text{ and } \mathbf{\Psi} = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}.$$

Suppose that ϕ and ψ are shift equivalent then we know that $\mathbf{\Phi}$ and $\mathbf{\Psi}$ must be shift equivalent $\mathbb{Z}^+$-matrices. Thus we must have integral 2×2 matrices, $\mathbf{R}$ and $\mathbf{S}$, which give a shift equivalence between $\mathbf{\Phi}$ and $\mathbf{\Psi}$. Since $\mathbf{RS} = \mathbf{\Phi}^l$, for some integer $l > 0$, and $\det(\mathbf{\Phi}) = -1$, it follows that $\det(\mathbf{R}) = \pm 1$. Let

$$\mathbf{R} = \begin{pmatrix} h & i \\ j & k \end{pmatrix}.$$

The equation $\mathbf{AR} = \mathbf{RB}$ is

$$\begin{pmatrix} h+j & i+k \\ 8h+7j & 8i+7k \end{pmatrix} = \begin{pmatrix} h+2i & 4h+7i \\ j+2k & 4j+7k \end{pmatrix}.$$

Solving for j and k in terms of h and i gives $j = 2i$ and $k = 4h + 6i$, so that

$$\det(\mathbf{R}) = \det \begin{pmatrix} h & i \\ 2i & 4h+6i \end{pmatrix} = 2(h^2 + 3hi - i^2).$$

This contradicts the fact that $\det(\mathbf{R}) = \pm 1$, proving that $\mathbf{\Phi}$ is not shift equivalent to $\mathbf{\Psi}$ and thus that ϕ is not shift equivalent to ψ.

We now introduce the apparently weaker notion of shift equivalence over $\mathbb{Z}$ for integral matrices. We then show that the $\mathbb{Z}^+$-matrix representation of an endomorphism induced by either an elementary or a (p, q)-block presentation is necessarily of a special form called primitive. We then state results demonstrating that if we have two primitive $\mathbb{Z}^+$-matrices then $\mathbb{Z}^+$-matrix shift equivalence holds if and only if $\mathbb{Z}$-matrix shift equivalence holds.

Definition 46 Let $\mathbf{A}$ and $\mathbf{B}$ be integral matrices. Then $\mathbf{A}$ and $\mathbf{B}$ are *shift equivalent over* $\mathbb{Z}$, written $\mathbf{A} \sim_{\mathbb{Z}} \mathbf{B}$ (lag k), if there are rectangular integral matrices $\mathbf{R}$ and $\mathbf{S}$ satisfying the shift equivalence relations.

Definition 47 An $\mathbb{Z}^+$-matrix $\mathbf{A}$ is *irreducible* if for each ordered pair of indices (i, j), there exists some $n \geq 0$ such that $A^n_{ij} > 0$.

Theorem 52 *Let $\phi\colon \langle E\rangle \to \langle E\rangle$ be an endomorphism induced by a presentation of a solenoid, $\{K, g\}$, which is either an elementary presentation or a (p, q)-block presentation and where E is the edge set of K. Then the $\mathbb{Z}^+$-matrix representation $\mathbf{\Phi}$ of ϕ is irreducible.*

Proof. Since $g: K \to K$ is a W^*-mapping by theorem 10 we know that there will exist an integer $m > 0$ so that for every edge e_i of K we have $g^m(e_i) = K$. Let $\phi: \langle E \rangle \to \langle E \rangle$ be the endomorphism induced by g and $\mathbf{\Phi}$ the $\mathbb{Z}^+$- matrix representation of ϕ. It is then clear that for any pair of edges e_i and e_j that $\phi^m(e_i)$ will be a word in $\langle E \rangle$ which contains e_j since any parameterization of e by $[0,1]$ will, by composition, give a parametrization of $g^m(e)$ which will traverses every edge K. It is then obvious from the definition of $\mathbf{\Phi}$ that $\Phi^m_{ij} > 0$. □

Definition 48 Let **A** be a $\mathbb{Z}^+$-matrix. The *period of state* i, denoted $\text{per}(i)$ is the greatest common divisor of those integers $n \geq 1$ for which $A^n_{ii} > 0$. If no such integers exist we define $\text{per}(i) = \infty$. The *period* $\text{per}(A)$ *of the matrix* **A** is the greatest common divisor of the numbers $\text{per}(i)$ that are finite, or is ∞ is $\text{per}(i) = \infty$ for all i. A matrix is *aperiodic* if it has period 1.

Theorem 53 *Let $\phi: \langle E \rangle \to \langle E \rangle$ be an endomorphism induced by an elementary or (p,q)-block presentation $\{K, g\}$, where E is the edge set of K. Then the $\mathbb{Z}^+$-matrix representation $\mathbf{\Phi}$ of ϕ is aperiodic.*

Proof. Since $g: K \to K$ is a W^*-mapping by theorem 10 there exists an integer $m > 0$ such that for every edge e_i of K we have $g^m(e_i) = K$. Since the mapping g is onto we see that $g^n(e_i) = K$ for all $n \geq m$. Let $\phi: \langle E \rangle \to \langle E \rangle$ be the endomorphism induced by g and let $\mathbf{\Phi}$ be the $\mathbb{Z}^+$-matrix representation of ϕ. From the above properties of g and the definition of ϕ it is clear for any $e_i \in E$ and any $n \geq m$ that $\phi^n(e_i)$ will be a word in $\langle E \rangle$ which contains e_i. Thus $\Phi^n_{ii} > 0$ for all $n \geq m$ and consequently $\text{per}(i) = 1$. Since $\text{per}(i) = 1$ for all i we see that $\text{per}(\mathbf{\Phi}) = 1$. □

Definition 49 A $\mathbb{Z}^+$-matrix is *primitive* if it is both irreducible and aperiodic.

Theorem 54 *Let $\phi: \langle E \rangle \to \langle E \rangle$ be an endomorphism induced by an elementary or (p,q)-block presentation $\{K, g\}$, where E is the edge set of K. Then the $\mathbb{Z}^+$-matrix representation $\mathbf{\Phi}$ of ϕ is primitive.*

Proof. This follows from theorems 53 and 52. □

The following results can be found in [22] and are stated without proof.

Theorem 55 *[22] Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be primitive integral matrices. Then* $\mathbf{A} \sim_s \mathbf{B}$ *if and and only if* $\mathbf{A} \sim_{\mathbb{Z}} \mathbf{A}$.

Definition 50 An integral matrix $\mathbf{P}$ is said to be *invertible over* $\mathbb{Z}$ if it is nonsingular and its inverse is also integral. Two integral matrices $\mathbf{A}$ and $\mathbf{B}$ are *similar over* $\mathbb{Z}$ if there is an integral matrix $\mathbf{P}$ which is invertible over $\mathbb{Z}$ such that $\mathbf{A} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Theorem 56 *[22] Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be primitive integral matrices which are similar over* $\mathbb{Z}$. *Then* $\mathbf{A} \sim_{\mathbb{Z}} \mathbf{B}$ *with lag=1.*

Next we introduce the Jordan form away from zero, $\mathbf{J}^{\times}$, as an invariant for shift equivalence of primitive $\mathbb{Z}^+$-matrices. Of course, since two primitive $\mathbb{Z}^+$-matrices are shift equivalent over $\mathbb{Z}^+$ if and only if they are shift equivalent over $\mathbb{Z}$, the Jordan form away from zero is also an invariant for shift equivalence over $\mathbb{Z}$. Our treatment of the Jordan form away from zero is based on that found in [22].

First turn to the linear algebra of an $r \times r$ integral matrix $\mathbf{A}$. The rational field $\mathbb{Q}$ is the smallest one containing the integers and we will use it as our scalar field. When we refer to linearity, subspaces, linear combinations, etc, it will be with respect to $\mathbb{Q}$.

Let $\mathbf{A}$ be an $r \times r$ integral matrix. $\mathbf{A}$ then defines a linear transformation $\mathbf{A}: \mathbb{Q}^r \to \mathbb{Q}^r$. We will in general consider $\mathbf{A}$ as acting on the right, i.e., elements of $\mathbb{Q}^r$ will be considered to be row vectors. Also if $E \subset \mathbb{Q}^r$ then its image under $\mathbf{A}$ is denoted by $E\mathbf{A}$.

Definition 51 Let $\mathbf{A}$ be an $r \times r$ integral matrix. The *eventual range* $\mathcal{R}_A$ of A is the subspace of $\mathbb{Q}^r$ defined by

$$\mathcal{R}_A = \bigcap_{k=1}^{\infty} \mathbb{Q}^r \mathbf{A}^k.$$

The *eventual kernel* $\mathcal{K}_A$ of A is the subspace of $\mathbb{Q}^r$ defined by

$$\mathcal{K}_A = \bigcup_{k=1}^{\infty} \ker(\mathbf{A}^k),$$

where $\ker(A) = \{\mathbf{v} \in \mathbb{Q}^r : \mathbf{v}\mathbf{A} = \mathbf{0}\}$ is the kernel of $\mathbf{A}$ viewed as a linear transformation.

Let $\mathbf{A}$ be an $r \times r$ integral matrix. If $\mathbb{Q}^r\mathbf{A}^k = \mathbb{Q}^r\mathbf{A}^{k+1}$, then $\mathbf{A}$ will be invertible on $\mathbb{Q}^r\mathbf{A}^k$ and thus $\mathbb{Q}^r\mathbf{A}^k = \mathbb{Q}^r\mathbf{A}^{k+n}$ for all $n \geq 0$. If we then consider the nested sequence

$$\mathbb{Q}^r \supseteq \mathbb{Q}^r\mathbf{A} \supseteq \mathbb{Q}^r\mathbf{A}^2 \supseteq \mathbb{Q}^r\mathbf{A}^3 \ldots$$

of subspaces we see that once equality occurs in the sequence all further inclusions must be equalities as well. We also know that since proper subspaces must have strictly smaller dimension there can only be at most r strict inclusions before equality occurs. Thus

$$\mathcal{R}_A = \mathbb{Q}^r \mathbf{A}^r.$$

A similar argument shows that

$$\mathcal{K}_A = \ker(\mathbf{A}^r).$$

The eventual range of $\mathbf{A}$ is the largest subspace of $\mathbb{Q}^r$ on which $\mathbf{A}$ is invertible. Likewise the eventual kernel of $\mathbf{A}$ is the largest subspace of $\mathbb{Q}^r$ on which $\mathbf{A}$ is nilpotent. For every $\mathbf{A}$ we claim that $\mathbb{Q}^r = \mathcal{K}_A \oplus \mathcal{R}_A$. Suppose that $\mathbf{v} \in \mathcal{K}_A \cap \mathcal{R}_A$. Then $\mathbf{v}\mathbf{A}^r = \mathbf{0}$ so that $\mathbf{v} = \mathbf{0}$ since $\mathbf{A}$ is invertible on $\mathcal{R}_A$. Thus $\mathcal{R}_a \cap \mathcal{K}_A = \mathbf{0}$. Let $\mathbf{v} \in \mathbb{Q}^r$. Then $\mathbf{v}\mathbf{A}^r \in \mathcal{R}_A$. Since $\mathbf{A}$ is invertible on $\mathcal{R}_A$, there is a $\mathbf{u} \in \mathcal{R}_A$ such that $\mathbf{v}\mathbf{A}^r = \mathbf{u}\mathbf{A}^r$. Then $\mathbf{v} - \mathbf{u} \in \mathcal{K}_A$ so that $\mathbf{v} = \mathbf{u} + (\mathbf{v} - \mathbf{u}) \in \mathcal{R}_A \oplus \mathcal{K}_A$.

Definition 52 Let $\mathbf{A}$ be a square integral matrix. The *invertible part* $\mathbf{A}^\times$ of $\mathbf{A}$ is the linear transformation obtained by restricting $\mathbf{A}$ to its eventual range, i.e., $\mathbf{A}^\times : \mathcal{R}_A \to \mathcal{R}_A$ is defined by $\mathbf{A}^\times(\mathbf{v}) = \mathbf{v}\mathbf{A}$.

We will now consider the relationship between the invertible part of integral matrices and shift equivalence over $\mathbb{Z}$.

Definition 53 A linear transformation is a *linear isomorphism* if it is one-to-one and onto. We say that two linear transformations $f : V \to V$ and $g : W \to W$ are *isomorphic* if there exists a linear isomorphism $h : V \to W$ such that $h \circ f = g \circ h$.

Note that the invertible part $\mathbf{A}^\times$ of an integral matrix $\mathbf{A}$ is a linear isomorphism.

Theorem 57 *[22] Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be integral matrices where* $\mathbf{A} \sim_{\mathbb{Z}} \mathbf{B}$. *Then* $\mathbf{A}^\times$ *and* $\mathbf{B}^\times$ *are isomorphic linear transforms.*

In order to tell whether two linear transformations are isomorphic we will use their Jordan canonical forms, which we now briefly review. For a complex number λ and an integer $m \geq 1$, define the $m \times m$ *Jordan block for* λ to be

$$\mathbf{J}_m(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

A matrix **A** which has block diagonal form, say

$$\mathbf{A} = \begin{pmatrix} \mathbf{A_1} & & & \\ & \mathbf{A_2} & & \\ & & \ddots & \\ & & & \mathbf{A_k} \end{pmatrix}$$

may be written more compactly as a direct sum;

$$\mathbf{A} = \mathbf{A_1} \oplus \mathbf{A_2} \oplus \cdots \oplus \mathbf{A_k}.$$

The basic theorem on the Jordan canonical form says that every matrix is similar over the complex numbers $\mathbb{C}$ to the direct sum of Jordan blocks. Thus for every matrix **A** there are complex numbers $\lambda_1, \lambda_2, \dots, \lambda_k$, and integers $m_1, m_2, \dots, m_k$, so that **A** is similar to

$$\mathbf{J}(\mathbf{A}) = \mathbf{J}_{m_1}(\lambda_1) \oplus \mathbf{J}_{m_2}(\lambda_2) \oplus \cdots \oplus \mathbf{J}_{m_k}(\lambda_k).$$

The matrix $\mathbf{J}(\mathbf{A})$ is call the *Jordan form* of **A**. The Jordan form of a matrix is a complete invariant for the similarity of matrices over $\mathbb{C}$, thus two matrices are similar over **C** if and only if they have the same Jordan form (up to the ordering of the Jordan blocks). We also see that since the Jordan form is a complete invariant for similarity over **C** the Jordan form of a linear transform is well defined and a complete invariant for isomorphism.

If a matrix **A** has Jordan form given by

$$\mathbf{J}(\mathbf{A}) = \mathbf{J}_{m_1}(\lambda_1) \oplus \mathbf{J}_{m_2}(\lambda_2) \oplus \cdots \oplus \mathbf{J}_{m_k}(\lambda_k)$$

then the eigenvalues of **A** are the λ_j (not necessarily distinct).

Definition 54 Let **A** be an integral matrix. The *Jordan form away from zero* of **A**, written $\mathbf{J}^{\times}(\mathbf{A})$, is defined to be the Jordan form of the invertible part $\mathbf{A}^{\times}$ of **A**.

The next result is an easy consequence of theorem 57 and is found in [22].

Theorem 58 *[22] Let* A *and* B *be integral matrices where* $\mathrm{A} \sim_{\mathbb{Z}} \mathrm{B}$*. Then* $\mathrm{J}^{\times}(\mathrm{A}) = \mathrm{J}^{\times}(\mathrm{B})$.

Note, from the above result it is clear that if A and B are integral matrices and $\mathrm{A} \sim_{\mathbb{Z}} \mathrm{B}$ then the characteristic polynomials of A and B, $\mathcal{X}_{\mathrm{A}}(t)$ and $\mathcal{X}_{\mathrm{B}}(t)$ respectively, are such that $\mathcal{X}_{\mathrm{A}}(t) = t^s \mathcal{X}_{\mathrm{B}}(t)$ for some integer s.

Example 18 In this example we demonstrate how the Jordan form away from zero can sometimes be used to show that two endomorphisms are not are not shift equivalent. Let $\phi \colon \langle a, b, c\rangle \to \langle a, b, c\rangle$ and $\psi \colon \langle \alpha, \beta, \delta\rangle \to \langle \alpha, \beta, \delta\rangle$ be the endomorphisms defined by

$$\begin{aligned} \phi(a) &= abca, & \psi(\alpha) &= \delta\alpha^3\beta\delta, \\ \phi(b) &= ab^2c, & \psi(\beta) &= \beta^2\alpha\delta, \\ \phi(c) &= bc^2a, & \psi(\delta) &= \beta\delta. \end{aligned}$$

The $\mathbb{Z}^+$-matrix representations, $\mathbf{\Phi}$ and $\mathbf{\Psi}$, of ϕ and ψ are then given by

$$\mathbf{\Phi} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \text{ and } \mathbf{\Psi} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

The eventual range of both $\mathbf{\Phi}$ and $\mathbf{\Psi}$ is $\mathbb{Q}^3$. The characteristic polynomials, $\mathcal{X}_{\mathbf{\Phi}}(t)$ and $\mathcal{X}_{\mathbf{\Psi}}(t)$, of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are given by $(t-4)(t-1)^2$. However we see that $\operatorname{rank}(\mathbf{\Phi} - \mathbf{Id}) = 1$ and $\operatorname{rank}(\mathbf{\Psi} - \mathbf{Id}) = 2$ thus the Jordan forms away from zero, $\mathbf{J}^{\times}(\mathbf{\Phi})$ and $\mathbf{J}^{\times}(\mathbf{\Psi})$, of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are different and are given by

$$\mathbf{J}^{\times}(\mathbf{\Phi}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \text{ and } \mathbf{J}^{\times}(\mathbf{\Psi}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Thus ϕ and ψ are not shift equivalent.

We now consider another invariant of shift equivalence over $\mathbb{Z}$, the Bowen-Franks group. In order to do this we will need to review some results from the theory of finitely generated abelian groups.

Let $\mathbb{Z}_d$ denote the cyclic group of order d, i.e., $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$. We will let $\mathbb{Z}_0$ denote $\mathbb{Z}$. We also see that $\mathbb{Z}_1$ is the trivial group with only 1 element.

Theorem 59 *[22] Let Γ be a finitely generated abelian group. Then there exist integers $d_1, d_2, \ldots, d_k$, such that*

$$\Gamma \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k},$$

where $\cong$ denotes group isomorphism and there is a unique choice for the d_j for which $d_j \neq 1$ and d_j divides d_{j+1} for $1 \leq j \leq k-1$.

In the above theorem the d_j are called the *elementary divisors* of the group Γ. It is clear that two finitely generated abelian groups are isomorphic if and only if they have the same elementary divisors. Thus the set of elementary divisors is a complete invariant for finitely generated abelian groups.

Definition 55 Let $\mathbf{A}$ be an $r \times r$ integral matrix. The *Bowen-Franks* group of $\mathbf{A}$ is

$$\mathbf{BF}(\mathbf{A}) = \mathbb{Z}^r/\mathbb{Z}^r(\mathbf{Id} - \mathbf{A}),$$

where $\mathbb{Z}^r/(\mathbf{Id} - \mathbf{A})$ is the image of $\mathbb{Z}^r$ under the matrix $(\mathbf{Id} - \mathbf{A})$ acting on the right.

We will use the *Smith form* to " compute " the Bowen-Franks group of a given integral matrix.

Definition 56 Let $\mathbf{A}$ be an integral matrix. We define the *elementary operations* over $\mathbb{Z}$ on $\mathbf{A}$ to be:

1. Exchange two rows or two columns.

2. Multiply a row or column by -1.

3. Add an integer multiple of one column to another, or of one row to another.

It is well known that the matrices corresponding to these operations are invertible over $\mathbb{Z}$.

Theorem 60 *[22] Let $\mathbf{A}$ and $\mathbf{B}$ be integral matrices where $\mathbf{B}$ is obtained from $\mathbf{A}$ by elementary operations. Then $\mathbb{Z}/\mathbb{Z}(\mathbf{Id} - \mathbf{A}) \cong \mathbb{Z}/\mathbb{Z}(\mathbf{Id} - \mathbf{B})$.*

Theorem 61 *[22] Let* $\mathbf{A}$ *be an integral matrix. Then* $\mathbf{A}$ *can be transformed by a sequence of elementary operations into a diagonal matrix*

$$\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_k \end{pmatrix}$$

called the Smith form *where the* $d_j \geq 0$ *and* d_j *divides* d_{j+1}. *If we put* $(\mathbf{Id} - \mathbf{A})$ *into its Smith form then*

$$\mathbf{BF}(\mathbf{A}) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k}.$$

Note that by our convention, each item in the direct product with $d_j = 0$ is $\mathbb{Z}$ while those with $d_j > 0$ are finite cyclic groups.

The next theorem shows that the Bowen-Franks group is an invariant for shift equivalence over $\mathbb{Z}$. The proof is included as it gives a useful construction of isomorphism from $\mathbf{BF}(\mathbf{A})$ to $\mathbf{BF}(\mathbf{B})$ given square integral matrices $\mathbf{A}$ and $\mathbf{B}$. We will find this isomorphism useful when considering non-Abelian invariants for the shift equivalence of endomorphisms of a finitely generated free groups. Our treatment of the proof is taken from [22].

Theorem 62 *[22] Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be square integral matrices where* $\mathbf{A} \sim_{\mathbb{Z}} \mathbf{B}$. *Then* $\mathbf{BF}(\mathbf{A}) \cong \mathbf{BF}(\mathbf{B})$.

Proof. Let m denote the size of $\mathbf{A}$ and n the size of $\mathbf{B}$. Suppose that $\mathbf{R}$ and $\mathbf{S}$ give a shift equivalence of lag=l between $\mathbf{A}$ and $\mathbf{B}$. Let $\mathbf{v} \in \mathbb{Z}^m$. Then

$$\mathbf{v}(\mathbf{Id} - \mathbf{A})\mathbf{R} = \mathbf{vR} - \mathbf{vAR} = \mathbf{vR} - \mathbf{vRB} = (\mathbf{vR})(\mathbf{Id} - \mathbf{B}).$$

This shows that

$$(\mathbb{Z}^m(\mathbf{Id} - \mathbf{A}))\mathbf{R} \subseteq \mathbb{Z}^n(\mathbf{Id} - \mathbf{B}).$$

Thus $\mathbf{R}$ induces a well defined map $\hat{\mathbf{R}}$ on the quotients

$$\hat{\mathbf{R}}\colon \mathbb{Z}^m/\mathbb{Z}^m(\mathbf{Id} - \mathbf{A}) \to \mathbb{Z}^n/\mathbb{Z}^n(\mathbf{Id} - \mathbf{B}),$$

or $\hat{\mathbf{R}}\colon \mathbf{BF}(\mathbf{A}) \to \mathbf{BF}(\mathbf{B})$. Similarly, $\mathbf{S}$ induces a well defined map $\hat{\mathbf{S}}\colon \mathbf{BF}(\mathbf{B}) \to \mathbf{BF}(\mathbf{A})$. Now the map induced by $\mathbf{A}$ on the quotient $\mathbb{Z}^m/\mathbb{Z}^m(\mathbf{Id} - \mathbf{A})$ is just the identity map, since cosets of $\mathbf{v}$ and $\mathbf{vA}$ modulo $\mathbf{Z}^m(\mathbf{Id}-\mathbf{A})$ are equal. Since $\hat{\mathbf{S}}\circ\hat{\mathbf{R}}\colon \mathbf{BF}(\mathbf{A})\to\mathbf{BF}(\mathbf{A})$ is induced by $\mathbf{A}^l$, it follows that $\hat{\mathbf{S}}\circ\hat{\mathbf{R}}$ is the identity on $\mathbf{BF}(\mathbf{A})$. Similarly $\hat{\mathbf{R}}\circ\hat{\mathbf{S}}$ is the identity on $\mathbf{BF}(\mathbf{B})$. Thus $\hat{\mathbf{R}}$ is an isomorphism from $\mathbf{BF}(\mathbf{A})$ to $\mathbf{BF}(\mathbf{B})$. □

Example 19 In this example we demonstrate how the Bowen-Franks group can sometimes be used to show that two endomorphisms are not shift equivalent. Consider the endomorphisms $\phi\colon\langle a,b\rangle\to\langle a,b\rangle$ and $\psi\colon\langle\alpha,\beta\rangle\to\langle\alpha,\beta\rangle$ where

$$\begin{aligned}\phi(a)&=abab^5, &\psi(\alpha)&=\alpha^3\beta,\\ \phi(b)&=ab^2, &\psi(\beta)&=\alpha^5\beta.\end{aligned}$$

The $\mathbb{Z}^+$-matrix representations, $\mathbf{\Phi}$ and $\mathbf{\Psi}$, of ϕ and ψ are then given by

$$\mathbf{\Phi}=\begin{pmatrix}2&1\\6&2\end{pmatrix}\ \text{and}\ \mathbf{\Psi}=\begin{pmatrix}3&5\\1&1\end{pmatrix}.$$

By performing elementary operations the Smith forms, $\mathbf{\Phi}'$ and $\mathbf{\Psi}'$, of $\mathbf{Id}-\mathbf{\Phi}$ and $\mathbf{Id}-\mathbf{\Psi}$ are then found to be

$$\mathbf{\Phi}'=\begin{pmatrix}1&0\\0&8\end{pmatrix}\ \text{and}\ \mathbf{\Psi}'=\begin{pmatrix}2&0\\0&4\end{pmatrix}.$$

Thus $\mathbf{BF}(\mathbf{A})\cong\mathbb{Z}_1\oplus\mathbb{Z}_8$ and $\mathbf{BF}(\mathbf{B})\cong\mathbb{Z}_2\oplus\mathbb{Z}_8$. Therefore $\mathbf{BF}(\mathbf{A})$ is not isomorphic to $\mathbf{BF}(\mathbf{B})$ and ϕ is not shift equivalent to ψ.

Definition 57 Let $\phi\colon\langle a_1,\dots a_n\rangle\to\langle a_1,\dots,a_n\rangle$ be an endomorphism. The *Bowen-Franks* group of ϕ, written $\mathbf{BF}_\phi$, is defined to be the same as $\mathbf{BF}(\mathbf{\Phi})$, where $\mathbf{\Phi}$ is the $\mathbb{Z}^+$-matrix representation of ϕ.

The invariants that we have developed, by considering the abelianization of the group in question, give us a useful set of tools which we can use to try and distinguish between non-shift equivalent endomorphisms. However they still leave plenty to be desired. In particular, if ϕ and ψ are two endomorphisms of finitely generated free groups $\langle E\rangle$ and $\langle E'\rangle$, it is clear that any invariant based on the abelianization of $\langle E\rangle$ and $\langle E'\rangle$ will be unable to take account of the order of the generating elements in the words $\phi(e)$, for $e\in\langle E\rangle$, and $\psi(e')$, for $e'\in\langle E'\rangle$. In the next example we give two endomorphisms which are not shift equivalent but whose $\mathbb{Z}^+$-matrix representations are shift equivalent. Thus all of the invariants developed so far must be the same for these two endomorphisms.

Example 20 Consider $\phi\colon \langle a, b\rangle \to \langle a, b\rangle$ and $\psi\colon \langle \alpha, \beta\rangle \to \langle \alpha, \beta\rangle$ where ϕ and ψ are given by

$$\phi(a) = abab, \quad \psi(\alpha) = \alpha^2\beta^2,$$
$$\phi(b) = ab, \quad \psi(\beta) = \alpha\beta.$$

The $\mathbb{Z}^+$-matrix representations $\mathbf{\Phi}$ and $\mathbf{\Psi}$ for ϕ and ψ are given by

$$\mathbf{\Phi} = \mathbf{\Psi} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}.$$

It is clear that $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are shift equivalent as $\mathbf{\Phi} = \mathbf{\Psi}$. We claim that if ϕ and ψ are induced by elementary or (p, q)-block presentations, say $\{K, g\}$ and $\{K', g'\}$, then g is not shift equivalent to g'. In the next section we will develop an invariant which will allow us to prove this claim.

5.2 Non-Abelian Invariants

In this section we develop invariants for the shift equivalence of endomorphisms of finitely generated free groups which reflect some of the non-abelian nature of the endomorphisms. In order to do this we need to introduce the notion of a group ring and Fox's free differential calculus. Our treatment of these topics is based on that found in [14], [2], and [12].

For any group G consider the set $\mathbb{Z}G$ (where $\mathbb{Z}$ denotes the ring of integers) of all formal sums of the form

$$\sum_{g\in G} n_g g$$

where $n_g \in \mathbb{Z}$ and only finitely many of the n_g are nonzero. Examples of such formal sums are $g_1 + 3g_2$ and $2g_1 - 3g_3 + g_4$ and so on. The set $\mathbb{Z}G$ has the natural structure of an abelian group where addition is given by

$$\left(\sum_{g\in G} n_g g\right) + \left(\sum_{g\in G} m_g g\right) = \sum_{g\in G} (n_g + m_g)\, g.$$

Since only finitely many of the n_g and m_g are nonzero the same holds true for $(n_g + m_g)$ and so the right hand side is an element of $\mathbb{Z}G$. If we think of G as a set, i.e., we forget about multiplication on G, then $\mathbb{Z}G$ is a free abelian group on the set G.

We get multiplication on $\mathbb{Z}G$, making it a ring, by remembering the multiplication on G. Thus

$$\left(\sum_{g\in G} n_g g\right)\left(\sum_{h\in G} m_h h\right) = \left(\sum_{g,h\in G} n_g m_h (gh)\right).$$

The above result is not quite in the right form but if we write for $k \in G$, $p_k = \sum_{g\in G} n_g m_{g^{-1}k}$, then we can partition the sum on the right hand side to get

$$\sum_{k\in G} p_k k.$$

It is well known and easy to check that p_k will be non-trivial only for a finite number of k in G. The ring $\mathbb{Z}G$ is defined in the above manner is called the *group ring* of G.

It is well known that if G is a finitely generated free abelian group then $\mathbb{Z}G$ will be a greatest common divisor (g.c.d.) domain. Recall that a g.c.d. domain is an integral domain in which any finite set of elements has a g.c.d.

We observe that if G_1 and G_2 are groups and $\eta : G_1 \to G_2$ is a homomorphism, then η induces a homomorphism from $\mathbb{Z}G_1$ to $\mathbb{Z}G_2$, which we denote by the same symbol η, where

$$\left(\sum_{g\in G_1} a_g g\right)^{\eta} = \sum_{g\in G_1} a_g g^{\eta} = \sum_{g\in G_1} a_g \eta(g).$$

Next we will need some results from the theory of " derivatives " developed by R. H. Fox in [11], [12].

Definition 58 A *derivative* on a group G is a mapping $D\colon G \to \mathbb{Z}G$ so that $D(gh) = D(g) + gD(h)$.

Theorem 63 *[14] If $D\colon G \to \mathbb{Z}$ is a derivative, then*

1. $D(g^{-1}) = -g^{-1}D(g)$,

2. $D(g^n) = (1 + g + g^2 + \cdots + g^{n-1})D(g)$ *for all* $n \in \mathbb{Z}$.

In particular we will have need for the case where the group $G = \langle x_1, \ldots, x_n\rangle$ in question is a free group on a finite number, n, generators. In this case, to each generator $x_i \in G$, there corresponds a unique derivative written

$$\frac{\partial}{\partial x_i}\colon G \to \mathbb{Z}G$$

defined by

$$\frac{\partial x_j}{\partial x_i} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where here 1 is being used for the identity of G. With $\partial/\partial x_i$ defined on the generators there is then no difficulty in extending it to all of G in the obvious manner.

For each $j = 1, \ldots, n$ there is then a well defined mapping

$$\frac{\partial}{\partial x_j} \colon \mathbb{Z}G \to \mathbb{Z}G$$

given by

$$\frac{\partial}{\partial x_j}\left(\sum_{g \in G} a_g g\right) = \sum_{g \in G} a_g \frac{\partial g}{\partial x_j},$$

where $a_g \in \mathbb{Z}$. It is known that $\frac{\partial}{\partial x_j}$ is well defined, see [2].

Example 21 Consider the free group $G = \langle a, b, c \rangle$ and the word $w = a^2bc^{-1}b \in G$. Then we see that

$$\frac{\partial w}{\partial a} = 1 + a,$$

$$\frac{\partial w}{\partial b} = a^2 + a^2bc^{-1},$$

and

$$\frac{\partial w}{\partial c} = -a^2bc^{-1}.$$

Theorem 64 *[2] Let $A = \langle a_1, \ldots, a_n \rangle$ be a finitely generated free group where w and $v_1, \ldots, v_n$ are words in A, with $w = w(a_1, a_2, \ldots, a_n)$. Then*

$$\frac{\partial}{\partial a_j}(w(v_1(a_1, \ldots, a_n)), \ldots, v_n(a_1, \ldots a_n)) = \sum_{k=1}^{n} \left(\frac{\partial w}{\partial v_k}\right)_{v_k = v_k(a_1, \ldots, a_n)} \left(\frac{\partial v_k}{\partial a_j}\right).$$

Next we introduce a new invariant for the shift equivalence of an endomorphism ϕ of a finitely generated free group. Consider the finitely generated free group $A = \langle a_1, \ldots, a_n \rangle$ and the endomorphism $\phi \colon A \to A$. It is clear from the definition of the Bowen-Franks group that when ϕ is such that for each a_i, the words $\phi(a_i)$ contain no instances of $a_1^{-1}, \ldots, a_n^{-1}$ then $\mathbf{BF}_\Phi \cong [A_\phi]$ where

$$[A_\phi] = \langle a_1, \ldots, a_n |\ \phi(a_i) = a_i,\ a_i a_j = a_j a_i \rangle.$$

It is clear that there is a natural homomorphism $\eta \colon A \to [A_\phi]$, which takes words in A to their equivalence class in $[A_\phi]$. The homomorphism η will then induce a homomorphism from $\mathbb{Z}A$ to $\mathbb{Z}[A_\phi]$, which we will also denote by η.

Definition 59 Let ϕ, A, $[A_\phi]$ and η be defined as above. The $\mathbb{Z}[A_\phi]$-*matrix representation* of ϕ is the $n \times n$ matrix $\mathbf{\Phi}$ over $\mathbb{Z}[A_\phi]$ where

$$\mathbf{\Phi} = \begin{pmatrix} \left(\frac{\partial\phi(a_1)}{\partial a_1}\right)^\eta & \left(\frac{\partial\phi(a_2)}{\partial a_1}\right)^\eta & \cdots & \left(\frac{\partial\phi(a_n)}{\partial a_1}\right)^\eta \\ \left(\frac{\partial\phi(a_1)}{\partial a_2}\right)^\eta & \left(\frac{\partial\phi(a_2)}{\partial a_2}\right)^\eta & \cdots & \left(\frac{\partial\phi(a_n)}{\partial a_2}\right)^\eta \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\partial\phi(a_1)}{\partial a_n}\right)^\eta & \left(\frac{\partial\phi(a_2)}{\partial a_n}\right)^\eta & \cdots & \left(\frac{\partial\phi(a_n)}{\partial a_n}\right)^\eta \end{pmatrix}.$$

Definition 60 Let G be a group, $\mathbb{Z}G$ the group ring over G and $m > 0$ an integer. Square matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ over $\mathbb{Z}G$ are said to be shift equivalent, written $\mathbf{\Phi} \sim_s \mathbf{\Psi}$, if there exist matrices $\mathbf{R}$ and $\mathbf{S}$ over $\mathbb{Z}G$ so that the usual shift equivalence relations hold, i.e.,

$$\mathbf{\Phi S} = \mathbf{S\Psi}, \qquad \mathbf{R\Phi} = \mathbf{\Psi R},$$
$$\mathbf{\Phi}^m = \mathbf{SR}, \quad \text{and } \mathbf{\Psi}^m = \mathbf{RS}.$$

The integer m is called the lag of the equivalence relation as usual.

Suppose that we have endomorphisms $\phi: A \to A$ and $\psi: B \to B$, where $A = \langle a_1, \ldots a_n \rangle$, $B = \langle b_1, \ldots b_m \rangle$ and $\phi \sim_s \psi$. From theorem 62 and the fact that $[A_\phi] \cong \mathbf{BF}_\phi$ and $[B_\psi] \cong \mathbf{BF}_\psi$ we know that $[A_\phi] \cong [B_\psi]$. We then clearly see that $\mathbb{Z}[A_\phi]$ and $\mathbb{Z}[B_\psi]$ are isomorphic rings, i.e., $\mathbb{Z}[A_\phi] \cong \mathbb{Z}[B_\psi]$. Note, with a slight abuse of notation, we are using $\cong$ to mean "isomorphic to" for both groups and rings. In particular any group isomorphism $\pi: [A_\phi] \to [B_\psi]$ will induce a ring isomorphism $\pi: \mathbb{Z}[A_\phi] \to \mathbb{Z}[B_\psi]$.

Suppose that G_1 and G_2 are groups and $\pi : G_2 \to G_1$ is a group isomorphism. If $\mathbf{A}$ is a matrix over $\mathbb{Z}G_2$ then we will denote by $\mathbf{A}_\pi$ the matrix over $\mathbb{Z}G_1$ where $(\mathbf{A}_\pi)_{ij} = \pi(A_{ij})$.

Theorem 65 *Let $\phi: A \to A$ and $\psi: B \to B$ be endomorphisms where $A = \langle a_1, \ldots a_n \rangle$, $B = \langle b_1, \ldots b_m \rangle$ and $\phi \sim_s \psi$ with lag=k. Let $\mathbf{\Phi}$ be the $\mathbb{Z}[A_\phi]$-matrix representation of ϕ and $\mathbf{\Psi}$ be the $\mathbb{Z}[B_\psi]$-matrix representation of ψ. Then there exists an isomorphism $\pi: [A_\phi] \to [B_\psi]$ so that $\mathbf{\Phi} \sim_s \mathbf{\Psi}_\pi$ with lag=k.*

Proof. Let $\eta: A \to [A_\phi]$ be the natural homomorphism from A to $[A_\phi]$ and $\gamma: B \to [B_\psi]$ the natural homomorphism from B to $[B_\psi]$. It is clear from the definitions that $\eta(a_i) = \eta \circ \phi(a_i)$, for $i = 1, \ldots, n$, and $\gamma(b_i) = \gamma \circ \psi(b_i)$, for $i = 1, \ldots, m$.

Let $r: A \to B$ and $s: B \to A$ be the mappings giving the shift equivalence between ϕ and ψ. Consider $\pi = \eta \circ s \circ \gamma^{-1}$. From the proof of theorem 62 we know that π is an isomorphism from $[B_\psi]$ to $[A_\phi]$. We claim that π is the isomorphism that we require. Thus we need to find matrices $\mathbf{R}$ and $\mathbf{S}$ over $\mathbb{Z}[A_\phi]$ which give a shift equivalence between $\mathbf{\Phi}$ and $\mathbf{\Psi}_\pi$.

The ij-th entry, for $i, j = 1, \ldots, n$, of the matrix $\mathbf{\Phi}$ is given by

$$\Phi_{ij} = \left(\frac{\partial\phi(a_j)}{\partial a_i}\right)^\eta.$$

Similarly the ij-th entry, for $i, j = 1, \ldots, m$, of the matrix $\mathbf{\Psi}_\pi$ is given by

$$(\Psi_\pi)_{ij} = \left(\frac{\partial\psi(b_j)}{\partial b_i}\right)^{\pi\circ\gamma}.$$

Define the $n \times m$ matrix $\mathbf{R}$ so that

$$R_{ij} = \left(\frac{\partial r(a_j)}{\partial b_i}\right)^{\pi\circ\gamma},$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Similarly define the $m \times n$ matrix $\mathbf{S}$ so that

$$S_{ij} = \left(\frac{\partial s(b_j)}{\partial a_i}\right)^\eta,$$

where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We claim that $\mathbf{R}$ and $\mathbf{S}$ give a shift equivalence between $\mathbf{\Phi}$ and $\mathbf{\Psi}_\pi$. There are four matrix equations to consider: $\mathbf{\Phi S} = \mathbf{S\Psi}_\pi$, $\mathbf{R\Phi} = \mathbf{\Psi}_\pi\mathbf{R}$, $\mathbf{\Phi}^k = \mathbf{SR}$, and $\mathbf{\Psi}_\pi^k = \mathbf{RS}$.

First will show that $\mathbf{\Phi S} = \mathbf{S\Psi}_\pi$. We know from the shift equivalence of ϕ and ψ that for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ we have

$$\left(\frac{\partial\psi\circ r(a_j)}{\partial b_i}\right)^{\pi\circ\gamma} = \left(\frac{\partial r\circ\phi(a_j)}{\partial b_i}\right)^{\pi\circ\gamma}.$$

We want to show that

$$\left(\frac{\partial\psi\circ r(a_j)}{\partial b_i}\right)^{\pi\circ\gamma} = \sum_{t=1}^{m}\left(\frac{\partial\psi(b_t)}{\partial b_i}\right)^{\pi\circ\gamma}\left(\frac{\partial r(a_j)}{\partial b_t}\right)^{\pi\circ\gamma}$$

and

$$\left(\frac{\partial r\circ\phi(a_j)}{\partial b_i}\right)^{\pi\circ\gamma} = \sum_{t=1}^{n}\left(\frac{\partial r(a_t)}{\partial b_i}\right)^{\pi\circ\gamma}\left(\frac{\partial\phi(a_j)}{\partial a_t}\right)^\eta.$$

Suppose that $r(a_j) = \alpha_1\alpha_2\ldots\alpha_p$ where $\alpha_q \in \{b_1, \ldots, b_m\}$ for $q = 1, \ldots, p$. Then

$$\begin{aligned}\left(\frac{\partial\psi\circ r(a_j)}{\partial b_i}\right)^{\pi\circ\gamma} &= \left(\frac{\partial(\psi(\alpha_1)\psi(\alpha_2)\ldots\psi(\alpha_p))}{\partial b_i}\right)^{\pi\circ\gamma}\\ &= \left[\frac{\partial\psi(\alpha_1)}{\partial b_i} + \psi(\alpha_1)\frac{\partial\psi(\alpha_2)}{\partial b_i} + \cdots + (\psi(\alpha_1)\cdots\psi(\alpha_{p-1}))\frac{\partial\psi(\alpha_p)}{\partial b_i}\right]^{\pi\circ\gamma}\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\partial\psi(\alpha_1)}{\partial b_i}\right)^{\pi\circ\gamma} + (\psi(\alpha_1))^{\pi\circ\gamma}\left(\frac{\partial\psi(\alpha_2)}{\partial b_i}\right)^{\pi\circ\gamma} + \cdots \\
&\quad + (\psi(\alpha_1)\cdots\psi(\alpha_{p-1}))^{\pi\circ\gamma}\left(\frac{\partial\psi(\alpha_p)}{\partial b_i}\right)^{\pi\circ\gamma} \\
&= \left(\frac{\partial\psi(\alpha_1)}{\partial b_i}\right)^{\pi\circ\gamma} + (\alpha_1)^{\pi\circ\gamma}\left(\frac{\partial\psi(\alpha_2)}{\partial b_i}\right)^{\pi\circ\gamma} + \cdots \\
&\quad + (\alpha_1\cdots\alpha_{p-1})^{\pi\circ\gamma}\left(\frac{\partial\psi(\alpha_p)}{\partial b_i}\right)^{\pi\circ\gamma} \\
&= \sum_{t=1}^{m} c_t\left(\frac{\partial\psi(b_t)}{b_i}\right)^{\pi\circ\gamma},
\end{aligned}
$$

using the fact that $\gamma(b_i) = \gamma\circ\psi(b_i)$, for each $b_i \in \{b_1, \ldots, b_m\}$, and thus $\pi\circ\gamma(\psi(\alpha_1)\ldots\psi(\alpha_q)) = \pi\circ\gamma(\alpha_1\alpha_2\ldots\alpha_q)$. The c_t, for each $t = 1, \ldots, m$ are then given by

$$
c_t = \left(\frac{\partial(\alpha_1\ldots\alpha_p)}{\partial b_t}\right)^{\pi\circ\gamma} = \left(\frac{\partial r(a_j)}{\partial b_t}\right)^{\pi\circ\gamma}.
$$

Thus we see that

$$
\left(\frac{\partial\psi\circ r(a_j)}{\partial b_i}\right)^{\pi\circ\gamma} = \sum_{t=1}^{m}\left(\frac{\partial\psi(b_t)}{\partial b_i}\right)^{\pi\circ\gamma}\left(\frac{\partial r(a_j)}{\partial b_t}\right)^{\pi\circ\gamma}.
$$

Next, suppose that $\phi(a_j) = \beta_1, \ldots, \beta_q$, where $\beta_i \in \{a_1, \ldots, a_n\}$ for $i = 1, \ldots, n$. Then

$$
\begin{aligned}
\left(\frac{\partial r\circ\phi(a_j)}{\partial b_i}\right)^{\pi\circ\gamma} &= \left(\frac{\partial(r(\beta_1)\cdots r(\beta_q))}{\partial b_i}\right)^{\pi\circ\gamma} \\
&= \left[\left(\frac{\partial r(\beta_1)}{\partial b_i}\right) + r(\beta_1)\left(\frac{\partial r(\beta_2)}{\partial b_i}\right) + \cdots + (r(\beta_1)\cdots r(\beta_{q-1})\left(\frac{\partial r(\beta_q)}{\partial b_i}\right)\right]^{\pi\circ\gamma}.
\end{aligned}
$$

We know, for each $a_j \in \{a_1, \ldots, a_n\}$, that

$$
\pi\circ\gamma\circ r(a_j) = \eta\circ s\circ\gamma^{-1}\circ\gamma\circ r(a_j) = \eta\circ s\circ r(a_j) = \eta\circ\phi^k(a_j) = \eta(a_j).
$$

Thus we may write

$$
\begin{aligned}
\left(\frac{\partial r\circ\phi(a_j)}{\partial b_i}\right)^{\pi\circ\gamma} &= \left(\frac{\partial r(\beta_1)}{\partial_i}\right)^{\pi\circ\gamma} + (\beta_1)^{\eta}\left(\frac{\partial r(\beta_2)}{\partial b_i}\right)^{\pi\circ\gamma} + \cdots + (\beta_1\cdots\beta_{q-1})^{\eta}\left(\frac{\partial r(\beta_q)}{\partial b_i}\right)^{\pi\circ\gamma} \\
&= \sum_{t=1}^{n} c_t\left(\frac{\partial r(a_s)}{\partial b_j}\right)^{\pi\circ\gamma}.
\end{aligned}
$$

The c_t, for $t = 1, \ldots, n$, are then given by

$$
c_t = \left(\frac{\partial(\beta_1\ldots\beta_q)}{\partial a_t}\right)^{\eta} = \left(\frac{\partial\phi(a_j)}{\partial a_t}\right)^{\eta}.
$$

Thus we see that

$$\left(\frac{\partial r \circ \phi(a_j)}{\partial b_i}\right)^{\pi\circ\gamma} = \sum_{t=1}^{n} \left(\frac{\partial r(a_t)}{\partial b_i}\right)^{\pi\circ\gamma} \left(\frac{\partial \phi(a_j)}{\partial a_t}\right)^{\eta}.$$

We therefore have $\mathbf{\Phi S} = \mathbf{S\Psi}_\pi$.

The proof the $\mathbf{R\Phi} = \mathbf{\Psi}_\pi\mathbf{R}$. is similar to that for $\mathbf{\Phi S} = \mathbf{S\Psi}_\pi$.

We will now show that $\mathbf{\Phi}^k = \mathbf{SR}$. From the shift equivalence of ϕ and ψ we know, for $i, j = 1, \ldots, n$, that

$$\left(\frac{\partial \phi^k(a_j)}{\partial a_i}\right)^{\eta} = \left(\frac{\partial s \circ r(a_j)}{\partial a_i}\right)^{\eta}.$$

We want to show that

$$\begin{aligned}
\left(\frac{\partial \phi^k(a_j)}{\partial a_i}\right)^{\eta} &= \left(\frac{\partial (\phi \circ \cdots \circ \phi(a_j))}{\partial a_i}\right)^{\eta} \\
&= \sum_{s_1, s_2, \ldots, s_{k-1}=1}^{k} \left\{ \left(\frac{\partial \phi(a_{s_{k-1}})}{\partial a_i}\right)^{\eta} \left(\frac{\partial \phi(a_{s_{k-2}})}{\partial a_{s_{k-1}}}\right)^{\eta} \cdots \left(\frac{\partial \phi(a_{s_1})}{\partial a_{s_2}}\right)^{\eta} \left(\frac{\partial \phi(a_j)}{\partial a_{s_1}}\right)^{\eta} \right\}.
\end{aligned}$$

and

$$\left(\frac{\partial s \circ r(a_j)}{\partial a_i}\right)^{\eta} = \sum_{t=1}^{m} \left(\frac{\partial s(b_t)}{\partial a_i}\right)^{\eta} \left(\frac{\partial r(a_j)}{\partial b_t}\right)^{\pi\circ\gamma}.$$

Suppose that $\phi(a_j) = \alpha_1 \ldots \alpha_p$, where $\alpha_i \in \{a_1, \ldots, a_n\}$ for $i = 1, \ldots, p$. Then

$$\begin{aligned}
\left(\frac{\partial \phi^k(a_j)}{\partial a_i}\right)^{\eta} &= \left(\frac{\partial \phi^{k-1}(\alpha_1 \cdots \alpha_p)}{\partial a_i}\right)^{\eta} \\
&= \left[\left(\frac{\partial \phi^{k-1}(\alpha_1)}{\partial a_i}\right) + \phi^{k-1}(\alpha_1)\left(\frac{\partial \phi^{k-1}(\alpha_2)}{\partial a_i}\right) + \cdots \right. \\
&\quad \left. + (\phi^{k-1}(\alpha_1)\phi^{k-1}(\alpha_2)\cdots\phi^{k-1}(\alpha_{p-1}))\left(\frac{\partial \phi^{k-1}(\alpha_p)}{\partial a_i}\right)\right]^{\eta}.
\end{aligned}$$

Using the fact that $\eta \circ \phi^{k-1}(\alpha) = \eta(\alpha)$ for all $\alpha \in \{a_1 \ldots, a_n\}$ we see that

$$\begin{aligned}
\left(\frac{\partial \phi^k(a_j)}{\partial a_i}\right)^{\eta} &= \left(\frac{\partial \phi^{k-1}(\alpha_1)}{\partial a_i}\right)^{\eta} + (\alpha_1)^{\eta}\left(\frac{\partial \phi^{k-1}(\alpha_2)}{\partial a_i}\right)^{\eta} + \cdots + (\alpha_1 \cdots \alpha_{p-1})^{\eta}\left(\frac{\partial \phi^{k-1}(\alpha_p)}{\partial a_i}\right)^{\eta} \\
&= \sum_{t=1}^{n} c_t \left(\frac{\partial \phi^{k-1}(a_t)}{\partial a_i}\right)^{\eta}.
\end{aligned}$$

The c_t, for $t = 1, \ldots, n$, are then given by

$$c_t = \left(\frac{\partial(\alpha_1 \cdots \alpha_p)}{\partial a_t}\right)^{\eta} = \left(\frac{\partial \phi(a_j)}{\partial a_t}\right)^{\eta}.$$

Thus we see that

$$\left(\frac{\partial\phi^k(a_j)}{\partial a_i}\right)^\eta = \sum_{t=1}^{n}\left(\frac{\partial\phi(a_t)}{\partial a_i}\right)^\eta\left(\frac{\partial\phi^{k-1}(a_j)}{\partial a_t}\right)^\eta.$$

Repeating this process for ϕ^{k-1}, $\phi^{k-2}, \ldots$, we find that

$$\left(\frac{\partial\phi^k(a_j)}{\partial a_i}\right)^\eta = \sum_{s_1,s_2,\ldots,s_{k-1}=1}^{k}\left\{\left(\frac{\partial\phi(a_{s_{k-1}})}{\partial a_i}\right)^\eta\left(\frac{\partial\phi(a_{s_{k-2}})}{\partial a_{s_{k-1}}}\right)^\eta\cdots\left(\frac{\partial\phi(a_{s_1})}{\partial a_{s_2}}\right)^\eta\left(\frac{\partial\phi(a_j)}{\partial a_{s_1}}\right)^\eta\right\}.$$

Suppose that $r(a_j) = \beta_1 \ldots \beta_q$, where $\beta_i \in \{b_1, \ldots, b_m\}$ for $i = 1, \ldots, q$. Then

$$\begin{aligned}\left(\frac{\partial s\circ r(a_j)}{\partial a_i}\right)^\eta &= \left(\frac{\partial s(\beta_1\cdots\beta_q)}{\partial a_i}\right)^\eta \\ &= \left[\left(\frac{\partial s(\beta_1)}{\partial a_i}\right) + s(\beta_1)\left(\frac{\partial s(\beta_2)}{\partial a_i}\right) + \cdots + (s(\beta_1)\cdots s(\beta_{q-1}))\left(\frac{\partial s(\beta_q)}{\partial a_i}\right)\right]^\eta.\end{aligned}$$

Using the fact that

$$\eta\circ s(\beta) = \eta\circ s\circ\gamma^{-1}\circ\gamma(\beta) = \pi\circ\gamma(\beta)$$

for all $\beta \in \{b_1, \ldots, b_m\}$ we see that

$$\begin{aligned}\left(\frac{\partial s\circ r(a_j)}{\partial a_i}\right)^\eta &= \left(\frac{\partial s(\beta_1)}{\partial a_i}\right)^\eta + (\beta)^{\pi\circ\gamma}\left(\frac{\partial s(\beta_2)}{\partial a_i}\right)^\eta + \cdots + (\beta_1\cdots\beta_{q-1})^{\pi\circ\gamma}\left(\frac{\partial s(\beta_q)}{\partial a_i}\right)^\eta \\ &= \sum_{t=1}^{m} c_t\left(\frac{\partial s(b_t)}{\partial a_i}\right)^\eta.\end{aligned}$$

The c_t, for $t = 1, \ldots, m$, are then given by

$$c_t = \left(\frac{\partial(\beta_1\cdots\beta_q)}{\partial b_t}\right)^{\pi\circ\gamma} = \left(\frac{\partial r(a_j)}{\partial b_t}\right)^{\pi\circ\gamma}.$$

Thus we have

$$\left(\frac{\partial s\circ r(a_j)}{\partial a_i}\right)^\eta = \sum_{t=1}^{m}\left(\frac{\partial s(b_t)}{\partial a_i}\right)^\eta\left(\frac{\partial r(a_j)}{\partial b_t}\right)^{\pi\circ\gamma}.$$

Therefore $\mathbf{\Phi}^k = \mathbf{SR}$.

The proof that $\mathbf{\Psi}_\pi^k = \mathbf{RS}$ is similar to that for $\mathbf{\Phi}^k = \mathbf{SR}$. □

Example 22 Consider the endomorphisms $\phi\colon A \to A$ and $\psi\colon B \to B$, where $A = \langle a, b\rangle$, $B = \langle\alpha, \beta\rangle$, defined by

$$\begin{aligned}\phi(a) &= abbab, & \psi(\alpha) &= \alpha\beta\alpha\alpha, \\ \phi(b) &= ab, & \psi(\beta) &= \alpha.\end{aligned}$$

Let $r: A \to B$ and $s: B \to A$ be given by

$$r(a) = \alpha\beta\alpha, \quad s(\alpha) = ab,$$
$$r(b) = \alpha, \qquad s(\beta) = b.$$

It is easy to check that r and s give a shift equivalence of lag= 1 between ϕ and ψ. The group $[A_\phi]$ is given by

$$[A_\phi] = \langle \bar{a}, \bar{b} \mid \bar{a} = \bar{a}^2\bar{b}^3, \bar{b} = \bar{a}\bar{b}, \bar{a}\bar{b} = \bar{b}\bar{a} \rangle = \langle \bar{b} \mid i = \bar{b}^3 \rangle.$$

The natural homomorphism $\eta: A \to [A_\phi]$ is then seen to be such that $\eta(a) = i$ and $\eta(b) = \bar{b}$. The group $[B_\psi]$ is given by

$$[B_\psi] = \langle \bar{\alpha}, \bar{\beta} \mid \bar{\alpha} = \bar{\alpha}^3\bar{\beta}, \bar{\beta} = \bar{\alpha}, \bar{\alpha}\bar{\beta} = \bar{\beta}\bar{\alpha} \rangle = \langle \bar{\beta} \mid i = \bar{\beta}^3 \rangle.$$

The natural homomorphism $\gamma: B \to [B_\psi]$ is seen to be such that $\gamma(\alpha) = \bar{\beta}$ and $\gamma(\beta) = \bar{\beta}$. The isomorphism $\pi: [B_\psi] \to [A_\phi]$ is given by $\pi(\bar{\beta}) = \bar{b}$. The $\mathbb{Z}[A_\phi]$-matrix representation $\mathbf{\Phi}$ of ϕ is then seen to be

$$\mathbf{\Phi} = \begin{pmatrix} 1 + abb & 1 \\ a + ab + abba & a \end{pmatrix}^\eta = \begin{pmatrix} 1 + \bar{b}^2 & 1 \\ 1 + \bar{b} + \bar{b}^2 & 1 \end{pmatrix}.$$

The $\mathbb{Z}[B_\psi]$-matrix representation $\mathbf{\Psi}$ of ψ is given by

$$\mathbf{\Psi} = \begin{pmatrix} 1 + \alpha\beta + \alpha\beta\alpha & 1 \\ \alpha & 0 \end{pmatrix}^\gamma = \begin{pmatrix} 2 + \bar{\beta}^2 & 1 \\ \bar{\beta} & 0 \end{pmatrix}.$$

Thus we see that

$$\mathbf{\Psi}_\pi = \begin{pmatrix} 2 + \bar{b}^2 & 1 \\ \bar{b} & 0 \end{pmatrix}.$$

The matrices $\mathbf{R}$ and $\mathbf{S}$ as defined in the proof of theorem 65 are found to be such that

$$\mathbf{R} = \begin{pmatrix} 1 + \bar{b}^2 & 1 \\ \bar{b} & 0 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The matrices $\mathbf{R}$ and $\mathbf{S}$ define a shift equivalence between $\mathbf{\Phi}$ and $\mathbf{\Psi}_\pi$ of lag=1, which is demonstrated below:

1. $\mathbf{\Phi S} = \mathbf{S\Psi}_\pi$ since

$$\begin{pmatrix} 1 + \bar{b}^2 & 1 \\ 1 + \bar{b} + \bar{b}^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 + \bar{b}^2 & 1 \\ 2 + \bar{b} + \bar{b}^2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 + \bar{b}^2 & 1 \\ \bar{b} & 0 \end{pmatrix}.$$

2. $\mathbf{R\Phi} = \Psi_\pi \mathbf{R}$ since

$$\begin{pmatrix} 1+\bar{b}^2 & 1 \\ \bar{b} & 0 \end{pmatrix} \begin{pmatrix} 1+\bar{b}^2 & 1 \\ 1+\bar{b}+\bar{b}^2 & 1 \end{pmatrix} = \begin{pmatrix} 2+2\bar{b}+3\bar{b}^2 & 2+\bar{b}^2 \\ 1+\bar{b} & \bar{b} \end{pmatrix}$$
$$= \begin{pmatrix} 2+\bar{b}^2 & 1 \\ \bar{b} & 0 \end{pmatrix} \begin{pmatrix} 1+\bar{b}^2 & 1 \\ \bar{b} & 0 \end{pmatrix}.$$

3. $\mathbf{\Phi} = \mathbf{SR}$ since

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\bar{b}^2 & 1 \\ \bar{b} & 0 \end{pmatrix} = \begin{pmatrix} 1+\bar{b}^2 & 1 \\ 1+\bar{b}+\bar{b}^2 & 1 \end{pmatrix}.$$

4. $\mathbf{\Psi} = \mathbf{RS}$ since

$$\begin{pmatrix} 1+\bar{b}^2 & 1 \\ \bar{b} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2+\bar{b}^2 & 1 \\ \bar{b} & 0 \end{pmatrix}.$$

Thus if we are given two endomorphisms ϕ and ψ of the free groups $A = \langle a_1, \ldots, a_n \rangle$ and $B = \langle b_1, \ldots, b_m \rangle$ we know that if ϕ and ψ are shift equivalent with lag= k then there must exist an isomorphism $\pi: B \to A$ so that $\mathbb{Z}[A_\phi]$-matrix representation, $\mathbf{\Phi}$ of ϕ is shift equivalent, with lag= k, to the image of the $\mathbb{Z}[B_\psi]$-matrix representation under π, Ψ_π. Unfortunately there is no known method of determining when two matrices over a group ring are shift equivalent. If the two endomorphisms arise from an elementary presentation or a (p,q)-block presentation of a solenoid we then have further information concerning the the matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ which we might try and use to find invariants. The following theorem by Williams [39] gives us an idea of how we might go about it.

Theorem 66 *[39] If $\{K_0, g_0\}$ and $\{K_m, g_m\}$ are presentations of solenoids and $g_0 \sim_s g_m$ with lag= m, then there exist presentations $\{K_1, g_1\}, \{K_2, g_2\}, \ldots, \{K_{m-1}, g_{m-1}\}$ so that $g_{i-1} \sim_s g_i$, with lag= 1, for $i = 1, \ldots, m$.*

Suppose we can show that starting with either two elementary presentations (or two (p,q)-block presentations) $\{K_0, g_0\}$ and $\{K_m, g_m\}$ which are shift equivalent with lag= m that there must exist elementary presentations (or (p,q)-block presentations) $\{K_1, g_1\}, \ldots, \{K_{m-1}, g_{m-1}\}$ so that $g_{i-1} \sim_s g_i$ with lag= 1 for $i = 1, \ldots, m$. Then we know if $\phi_0: A^0 \to A^0$ and $\phi_m: A^m \to A^m$ are the endomorphisms induced by g_0 and g_m that there must exist endomorphisms $\phi_1, \phi_2, \ldots, \phi_{m-1}$ so that $\phi_{i-1} \sim_s \phi_i$ with lag= 1 for $i = 1, \ldots, m$. Thus if Φ_i is the $\mathbb{Z}[A^i_{\phi_i}]$-matrix representation of ϕ_i we would then have isomorphisms $\pi_i: A^i \to A^0$ for $i = 1, \ldots, m$ so that $\Phi_0 \sim_s (\Phi_1)_{\pi_1}$ with lag= 1 and $(\Phi_{i-1})_{\pi_{i-1}} \sim_s (\Phi_i)_{\pi_i}$ with lag= 1 for $i = 2, \ldots, m$. With this in mind we prove the following modification of theorem 66. The proof is essentially the same as that in [39] the only modification being the verification that at each stage the presentations $\{K_1, g_1\}, \ldots,$ $\{K_{m-1}, g_{m-1}\}$ are either elementary presentations or (p,q)-block presentations.

Theorem 67 *Suppose $\{K_0, g_0\}$ and $\{K_m, g_m\}$ are elementary presentations (or (p,q)-block presentations) where $g_0 \sim_s g_m$ with lag= m. Then there exist elementary presentations (or (p,q)-block presentations) $\{K_1, g_1\}$, $\{K_2, g_2\}$, ..., $\{K_{m-1}, g_{m-1}\}$ so that $g_{i-1} \sim_s g_i$ with lag= 1, for $i = 1, \ldots, m$.*

Proof. Let $r: K_0 \to K_m$ and $s: K_m \to K_0$ be the mappings giving the shift equivalence between g_0 and g_m. For $x, y \in K_0$ say that $x \sim_1 y$ if and only if $g_0(x) = g_0(y)$ and $r(x) = r(y)$. Then $\sim_1$ is an equivalence relation and yields a quotient space $K_1 = K/\sim_1$, roughly as good as K_0 and K_m. Let $r_1: K_0 \to K_1$ be the quotient map. By definition both r and g_0 factor through r_1, say $g_0 = s_1 \circ r_1$ and $r = t_1 \circ r_1$. Define $g_1: K_1 \to K_1$ by $g_1 = r_1 \circ s_1$. We then have $g_0 \sim_s g_1$ with lag= 1 where the shift equivalence is given by r_1 and s_1.

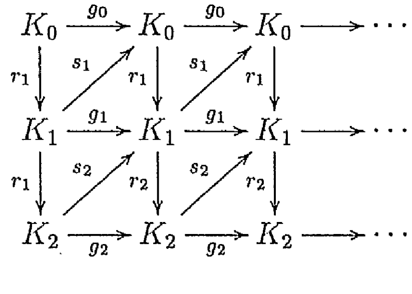
We claim that if $\{K_0, g_0\}$ and $\{K_m, g_m\}$ are elementary presentations (or (p,q)-block presentations) then $\{K_1, g_1\}$ will also be an elementary presentation (or (p,q)-block presentation), i.e., we claim that if we choose $V_1 = r_1(V_0)$, where V_0 is the vertex set of K_0, to be the vertex set of K_1 then $\{K_1, g_1\}$ will be an elementary presentation (or (p,q)-block presentation). We first note that in either case, elementary or (p,q)-block presentations, that no two vertices of K_0 will be identified under $\sim_1$. This is because in each case the vertices are periodic under g_0. Thus for every vertex of $v \in K_0$ there a unique vertex $r_1(v) \in K_1$. The only difficulty we might encounter is the existence of a branch point $b_1 \in K_1$ where $r_1^{-1}(b_1) \cap V_0 = \emptyset$, i.e., the creation of a "new" branch point as a result of the identification $\sim_1$. Suppose that $b_1 \in K_1$ is a branch point where $r_1^{-1}(b_1) \cap V_0 = \emptyset$. Then we have $r_1^{-1}(b_1) = \{x_1, \ldots, x_p\}$ where each x_j is an ordinary point of K_0. If we choose an arbitrarily small neighborhood $N \subset K_1$ of b_1 then $r_1^{-1}(N) = I_1 \cup I_2 \cup \cdots \cup I_p$ where each I_j is a small interval about x_j. Since b_1 is a branch point we must have I_i and I_j, $i \neq j$, so that either $g_0(I_i) \neq g_0(I_j)$ or $r(I_i) \neq r(I_j)$. Suppose that $g_0(I_i) \neq g_0(I_j)$, for this to hold for arbitrarily small neighborhoods N of b_1 we see that $g_0(\{x_1, \ldots, x_p\}) = z$

must be a branch point of K_0 and thus $z \in V_0$. However since the vertices of K_0 are periodic under g_0 we must have a vertex $v \in V_0$ so that $g_0(v) = z$. Thus $v \in \{x_1, \ldots, x_p\}$ and $\{x_1, \ldots, x_p\} \cap V_0 \neq \emptyset$. Suppose that $r_1(I_i) \neq r_1(I_j)$, for this to hold for arbitrarily small neighborhoods N of b_1 we must have $r(\{x_1, \ldots, x_p)\} = z'$ where z' is a branch point of K_m and thus $z' \in V_m$, where V_m is the vertex set of K_m. As r must map the vertices of K_0 to the vertices of K_m in a one-to-one and onto manner there must exist a vertex $v \in V_0$ so that $r(v) = z'$. Thus $v \in \{x_1, \ldots, x_p\}$ and $\{x_1, \ldots, x_p\} \cap V_0 \neq \emptyset$. Therefore with the choice of vertex set $v_1 = r_1(V_0)$, $\{K_1, g_1\}$ will be an elementary presentation (or (p, q)-block presentation).

Next for $x, y \in K_0$, let $x \sim_2 y$ if and only if $g_0^2(x) = g_0^2(y)$ and $r(x) = r(y)$. This is again an equivalence relation and yields a quotient space K_2. The quotient map has the for $r_2 \circ r_1 \colon K_0 \to K_2$. Also r can be factored as $r = t_2 \circ r_2 \circ r_1$. We claim that g_0^2 can be factored as $g_0^2 = s_1 \circ s_2$ where $s_2 \colon K_2 \to K_1$. Indeed, we define $s_2 = f_1 \circ r_2^{-1}$ and we need only check that s_2 is well defined. Thus let $x, y \in K_0$ determine the same point $r_2 \circ r_1(x) = r_2 \circ r_1(y)$ in K_2. It will suffice to show that $g_1 \circ r_1(x) = g_1 \circ r_1(y)$, or equivalently that $r_1 \circ g_0(x) = r_1 \circ g_0(y)$, or that $g_0(x) \sim_1 g_0(y)$. However we see that $g_0 \circ g_0(x) = g_0 \circ g_0(y)$ and $r \circ g_0(x) = g_m \circ r(x) = r \circ g_m(y) = r \circ g_0(y)$ which shows $g_0(x) \sim_1 g_0(y)$. Note that the equality $g_m \circ r(x) = g_m \circ f(y)$ follows from $r(x) = r(y)$. Again we introduce $g_2 = r_2 \circ s_2 \colon K_2 \to K_2$. We then have $g_2 \sim g_1$ with lag$= 1$.

The proof that $\{K_2, g_2\}$ with the choice of vertex set $V_2 = r_2 \circ r_2(V_0) = r_2(V_1)$ will be an elementary presentation (or $(p;q)$-block presentation) if $\{K_0, g_0\}$ and $\{K_m, g_m\}$ are elementary (or (p, q)-block presentations) is almost exactly the same as the case for $\{K_1, g_1\}$.

So far we have the diagram



Clearly this can be continued, getting presentations $\{K_3, g_3\}, \ldots, \{K_{m-1}, g_{m-1}\}, \{K_m, g_m\}$. Note that K_m can be considered to be the image $r(K_0)$ of K_0 under r because $g_0^m(x) = g_0^m(y)$ and $r(x) = r(y)$if and only if $r(x) = r(y)$. That is $r = r_m \circ r_{m-1} \circ \cdots \circ r_1$. Recall that both r and s must be onto. We also have $s = s_1 \circ s_2 \circ \cdots \circ s_m$. $\square$

With this new information we are now in a position to develop some new invariants for the shift equivalence of solenoids. In particular we will develop a new set of invariants which is computable and will allow us to distinguish in some cases between solenoids which are not shift equivalent but which have the same abelian invariants.

Theorem 68 *Let $\phi_0\colon A^0 \to A^0$ and $\phi_k\colon A^k \to A^k$ be endomorphisms induced by elementary or (p,q)-block presentations of solenoids which are shift equivalent with lag$= k$. Then*

1. *there exist endomorphisms $\phi_i\colon A^i \to A^i$ for $i = 1, \ldots, k$ so that $\phi_{i-1} \sim_s \phi_i$ with lag$= 1$,*
2. *if Φ_i is the $\mathbb{Z}[A^i_{\phi_i}]$-matrix representation of ϕ_i for $i = 0, \ldots, k$ then there exist isomorphisms $\pi_i\colon A^i \to A^0$, for $i = 1, \ldots, k$, so that*

$$\begin{array}{ll} \Phi_0 \sim_s (\Phi_1)_{\pi_1} & lag = 1, \\ (\Phi_1)_{\pi_1} \sim_s (\Phi_2)_{\pi_2} & lag = 1, \\ (\Phi_2)_{\pi_2} \sim_s (\Phi_3)_{\pi_3} & lag = 1, \\ \vdots & \vdots \\ (\Phi_{k-1})_{\pi_{k-1}} \sim_s (\Phi_k)_{\pi_k} & lag = 1, \end{array}$$

3. *there exists an isomorphism $\pi\colon A^0 \to A^k$ so that trace $(\Phi_0^n) =$ trace $((\Phi_k^n)_\pi)$ for all $n \in \mathbb{N}$.*

Proof. Item 1.) and 2.) are a direct consequence of theorem 65 and theorem 67. Because item 2.) holds we know that there must exist matrices $\mathbf{R_i}$ and $\mathbf{S_i}$, for $i = 1, \ldots, k$, so that

$$\begin{array}{ll} \Phi_0 = \mathbf{R_1 S_1}, & \mathbf{S_1 R_1} = (\Phi_1)_{\pi_1}, \\ (\Phi_1)_{\pi_1} = \mathbf{R_2 S_2}, & \mathbf{S_2 R_2} = (\Phi_2)_{\pi_2}, \\ \vdots & \vdots \\ (\Phi_{k-1})_{\pi_k} = \mathbf{R_k S_k}, & \mathbf{S_k R_k} = (\Phi_k)_{\pi_k}. \end{array}$$

As trace $(\mathbf{AB})$ = trace $(\mathbf{BA})$ for any two matrices $\mathbf{A}$ and $\mathbf{B}$ we see that

$$\text{trace } (\Phi_0) = \text{ trace } ((\Phi_1)_{\pi_1}) = \cdots = \text{ trace } ((\Phi_k)_{\pi_k}).$$

As shift equivalence of matrices $\mathbf{A}$ and $\mathbf{B}$ with lag$= j$ implies $\mathbf{A}^n$ shift equivalent to $\mathbf{B}^n$ with lag$= j$ for all $n \in \mathbb{N}$ we see that trace (Φ_0^n) = trace $((\Phi_k^n)_\pi)$ for all $n \in \mathbb{N}$. $\square$

Item 3.) of theorem 68 is particularly useful for calculating invariants for solenoids. Let $\phi\colon A \to A$ and $\psi\colon B \to B$ be endomorphisms induced by elementary presentations or (p,q)-block presentations $\{K, g\}$ and $\{K', g'\}$ respectively. Then if $g \sim_s g'$ we know that there must exist an isomorphism $\pi\colon [B_\psi] \to [A_\phi]$ so that trace $(\Phi^n) =$ trace $((\Phi_\pi)^n)$ for all $n \in \mathbb{N}$. When the groups $[B_\psi] \cong [A_\phi]$ are finite abelian groups we can find all isomorphisms $\pi\colon [B_\psi] \to [A_\phi]$. There will only be finitely many of them as they are uniquely determined by their action on the generators of $[B_\psi]$. Thus we may find the trace of Φ^n and the trace of Ψ^n for some $n \in \mathbb{N}$ and try all the isomorphisms $\pi\colon [B_\psi] \to [A_\phi]$. If there doesn't exist such an isomorphism π so that trace $(\Phi^n) =$ trace $((\Phi_\pi)^n)$ then g is not shift equivalent to g'. When the groups $[B_\psi] \cong [A_\phi]$ are not finite the trace of Φ^n and the trace Ψ^n still contain valuable information concerning the shift equivalence of g and g', for instance the orders and coefficients of the terms appearing in the traces.

Example 23 In example 20 we considered the two endomorphisms $\phi\colon A \to A$ and $\psi\colon B \to B$ with $A = \langle a, b\rangle$, $B = \langle \alpha, \beta\rangle$ and where

$$\phi(a) = abab, \quad \psi(\alpha) = \alpha^2\beta^2,$$
$$\phi(b) = ab, \qquad \psi(\beta) = \alpha\beta.$$

We demonstrated that the abelian invariants for ϕ and ψ are the same. We claimed that if ϕ and ψ were induced by elementary presentations or (p,q)-block presentations, say $\{K, g\}$ and $\{K', g'\}$ then g was not shift equivalent to g'. We will now demonstrate that this is the case. We see that

$$[A_\phi] = \langle a, b \mid a = i, b^2 = i\ \rangle = \langle b \mid b^2 = i\ \rangle,$$

and that

$$[B_\psi] = \langle \alpha, \beta \mid \alpha = i, \beta^2 = i\ \rangle = \langle \beta \mid \beta^2 = i\ \rangle.$$

Thus there is only one isomorphism $\pi\colon [B_\psi] \to [A_\phi]$ which is given by $\pi(\beta) = b$. We then see that the $\mathbb{Z}[A_\phi]$-matrix representation, Φ of ϕ is given by

$$\Phi = \begin{pmatrix} 1+ab & 1 \\ a+aba & a \end{pmatrix}^{\eta} = \begin{pmatrix} 1+b & 1 \\ 1+b & 1 \end{pmatrix}.$$

The image of the $\mathbb{Z}[B_\psi]$-matrix representation Ψ under the isomorphism π is then given by

$$\Psi_\pi = \begin{pmatrix} 1+\alpha & 1 \\ \alpha^2+\alpha^2\beta & \alpha \end{pmatrix}^{\pi\circ\gamma} = \begin{pmatrix} 2 & 1 \\ 1+b & 1 \end{pmatrix}.$$

Thus we see that trace $(\Phi) = 2 + b$ and trace $(\Psi_\pi) = 3$. Thus since π is the only isomorphism from $[A_\phi]$ to $[B_\psi]$ we see that g is not shift equivalent to g'.

Example 24 Consider the endomorphisms $\phi: A \to A$ and $\psi: B \to B$ with $A = \langle a, b, c\rangle$, $B = \langle \alpha, \beta, \delta\rangle$, where

$$\begin{aligned} \phi(a) &= abac, & \psi(\alpha) &= \alpha\beta\delta, \\ \phi(b) &= ab^5c, & \psi(\beta) &= \alpha\beta^2, \\ \phi(c) &= ac, & \psi(\delta) &= \alpha\delta^2\beta\delta^3\beta. \end{aligned}$$

Suppose that ϕ and ψ are the endomorphisms induced by two presentations $\{K, g\}$ and $\{K', g'\}$, respectively, which are either elementary presentations or (p, q)-block presentations. Is g shift equivalent to g'?

The $\mathbb{Z}^+$-matrix representation Φ_1 of ϕ is given by

$$\Phi_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The $\mathbb{Z}^+$-matrix representation, Ψ_1, of ψ is given by

$$\Psi_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix}.$$

It is easily verified that the matrices

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{pmatrix}$$

give a shift equivalence of lag$=1$ between Φ_1 and Ψ_1. Thus all of the invariants developed in section 5.1 will be the same. In particular the Bowen-Franks groups of Φ_1 and Ψ_1 are both seen to be isomorphic to $\mathbb{Z}_3$. Thus the abelian invariants for these presentations give us no information when trying to distinguish between these presentations.

The group $[A_\phi] \cong \mathbb{Z}_3$ is given by

$$\begin{aligned} [A_\phi] &= \langle a, b, c, \mid i = bac, i = ab^4c, i = a, ab = ba, ac = ca, bc = cb \rangle \\ &= \langle b, c \mid i = b^3, i = bc, bc = cb \rangle. \end{aligned}$$

From this we see that $b^3 = i$, $c = b^2$ and $a = i$. Let $\eta\colon A \to [A_\phi]$ be the natural homomorphism. The group $[B_\psi] \cong \mathbb{Z}_3$ is given by

$$[B_\psi] \;= \langle \alpha, \beta, \delta \mid i = \beta\delta, i = \alpha\beta, i = \delta^3, \alpha\beta = \beta\alpha, \alpha\delta = \delta\alpha, \beta\delta = \delta\beta \,\rangle.$$

From this we see that $i = \delta^3$, $\alpha = \delta$, and $\beta = \delta^2$. Let $\gamma\colon B \to [B_\psi]$ be the natural homomorphism. There are exactly two isomorphisms, π_1 and π_2, from $[B_\psi]$ to $[A_\phi]$ given by

$$\begin{aligned} \pi_1(i) &= i, & \pi_2(i) &= i,\\ \pi_1(\delta) &= b, & \pi_2(\delta) &= b^2,\\ \pi_1(\delta^2) &= b^2, & \pi_2(\delta^2) &= b. \end{aligned}$$

The $\mathbb{Z}[A_\phi]$-matrix representation, $\mathbf{\Phi}_2$, of ϕ is then given by

$$\mathbf{\Phi}_2 = \begin{pmatrix} 1+ab & 1 & 1 \\ a & a(1+b+b^2+b^3+b^4) & 0 \\ aba & ab^5 & a \end{pmatrix}^{\eta} = \begin{pmatrix} 1+b & 1 & 1 \\ 1 & 2+2b+b^2 & 0 \\ b & b^2 & 1 \end{pmatrix}.$$

The $\mathbb{Z}[B_\psi]$-matrix representation, $\mathbf{\Psi}_2$ of ψ is given by

$$\mathbf{\Psi}_2 = \begin{pmatrix} 1 & 1 & 1 \\ \alpha & \alpha+\alpha\beta & \alpha\delta^2+\alpha\delta^2\beta\delta^3 \\ \alpha\beta & 0 & \alpha+\alpha\delta+\alpha\delta^2\beta+\alpha\delta^2\beta\delta+\alpha\delta^2\beta\delta^2 \end{pmatrix}^{\gamma} = \begin{pmatrix} 1 & 1 & 1 \\ \delta & 1+\delta & 1+\delta^2 \\ 1 & 0 & 1+2\delta+2\delta^2 \end{pmatrix}$$

Thus we see that

$$(\mathbf{\Psi}_2)_{\pi_1} = \begin{pmatrix} 1 & 1 & 1 \\ b & 1+b & 1+b^2 \\ 1 & 0 & 1+2b+2b^2 \end{pmatrix} \quad\text{and}\quad (\mathbf{\Psi}_2)_{\pi_2} = \begin{pmatrix} 1 & 1 & 1 \\ b^2 & 1+b^2 & 1+b \\ 1 & 0 & 1+2b+2b^2 \end{pmatrix}.$$

We then see that $\operatorname{trace}(\mathbf{\Phi}_2) = 4+3b+b^2$, $\operatorname{trace}((\mathbf{\Psi}_2)_{\pi_1}) = 3+3b+2b^2$, and $\operatorname{trace}((\mathbf{\Psi}_2)_{\pi_2}) = 3+2b+3b^2$. Thus g is not shift equivalent to g'.

Suppose that $\phi\colon A \to A$ and $\psi\colon B \to B$ are two endomorphisms which are induced by presentations $\{K, g\}$ and $\{K', g'\}$ which are either elementary presentations or (p,q)-block presentations. It is clear that if $g \sim_s g'$ then $g^m \sim_s (g')^m$ for all $m \in \mathbb{N}$. In general $[A_{\phi^m}]$ will not be isomorphic to $[A_{\phi^n}]$ for $m \neq n$. Thus we may try using powers of the endomorphisms ϕ and ψ to generate more invariants. For example suppose that $g \sim_s g'$ and $m \in \mathbb{N}$. Let $\mathbf{\Phi}_m$ be the $\mathbb{Z}[A_{\phi^m}]$-matrix representation of Φ^m and $\mathbf{\Psi}_m$ the

$\mathbb{Z}[B_{\psi^m}]$-matrix representation of ψ^m. We then know that there must exist an isomorphism $\pi: [B_{\psi^m}] \to [A_{\phi^m}]$ so that trace$(\mathbf{\Phi}_m) =$ trace$((\mathbf{\Psi}_m)_\pi)$. The next example shows how such further invariants can be useful.

Example 25 Consider the endomorphisms $\phi: A \to A$ and $\psi: B \to B$, with $A = \langle a, b \rangle$ and $B = \langle \alpha, \beta \rangle$, where

$$\phi(a) = ab^2, \quad \psi(\alpha) = \alpha\beta,$$
$$\phi(b) = a^3b, \quad \psi(\beta) = \alpha^6\beta.$$

Suppose that ϕ and ψ are induced by presentations $\{K, g\}$ and $\{K', g'\}$, respectively, and that these presentations are either both elementary presentations or (p,q)-block presentations. Is g shift equivalent to g'?

The $\mathbb{Z}^+$-matrix representation of ϕ is

$$\mathbf{\Phi} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$$

and the $\mathbb{Z}^+$-matrix representation of ψ is

$$\mathbf{\Psi} = \begin{pmatrix} 1 & 6 \\ 1 & 1 \end{pmatrix}.$$

It is easily verified that the matrices

$$\mathbf{R} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} 8 & 3 \\ 1 & 16 \end{pmatrix}$$

give a shift equivalence of lag$=3$ between $\mathbf{\Phi}$ and $\mathbf{\Psi}$. Thus the abelian invariants are of no use in trying to distinguish between g and g'. A simple calculation shows that Bowen-Franks groups of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are isomorphic to $\mathbb{Z}_6$.

The group $[A_\phi]$ is given by

$$[A_\Phi] = \langle a, b \mid a = ab^2, b = a^3b, ab = ba, \rangle.$$

A simple calculation shows that $a^3 = i$ and $b^2 = i$. Let $\eta: A \to [A_\phi]$ be the natural homomorphism. The group $[B_\psi]$ is given by

$$[B_\psi] = \langle \alpha, \beta \mid \alpha = \alpha\beta, \beta = \alpha^6\beta, \alpha\beta = \beta\alpha \rangle.$$

We see that $\beta = i$ and $\alpha^6 = i$. Let $\gamma: B \to [B_\psi]$ be the natural homomorphism.

The $\mathbb{Z}[A_\phi]$-matrix representation of ϕ is given by

$$\Phi_1 = \begin{pmatrix} 1 & 1+a+a^2 \\ a+ab & a^3 \end{pmatrix}^\eta = \begin{pmatrix} 1 & 1+a+a^2 \\ a+ab & 1 \end{pmatrix}.$$

The $\mathbb{Z}[B_\psi]$-matrix representation of ψ is given by

$$\Psi_1 = \begin{pmatrix} 1 & 1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5 \\ \alpha & \alpha^6 \end{pmatrix}^\gamma = \begin{pmatrix} 1 & 1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5 \\ \alpha & 1 \end{pmatrix}.$$

Consider the isomorphism $\pi\colon [A_\phi] \to [B_\Psi]$ given by

$$\begin{aligned} \pi(i) &= i, & \pi(a^2) &= \alpha^4, & \pi(ab) &= \alpha^5, \\ \pi(a) &= \alpha^2, & \pi(b) &= \alpha^3, & \pi(a^2b) &= \alpha. \end{aligned}$$

The image of Φ_1 under π is given by

$$(\Phi_1)_\pi = \begin{pmatrix} 1 & 1+\alpha^2+\alpha^4 \\ \alpha^2+\alpha^5 & 1 \end{pmatrix}.$$

It is clear that trace($(\Phi_1)_\pi$) =trace(Ψ_1) = 2. Further a short calculation shows that

$$\text{trace}((\Phi_1)_\pi^n) = 2\left\{\text{trace}((\Phi_1)_\pi^{n-1}) + \alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5\right\},$$

and

$$\text{trace}(\Psi_1^n) = 2\left\{\text{trace}(\Psi_1^{n-1}) + \alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5\right\}.$$

Thus we see that trace($(\Phi_1)_\pi^n$) = trace(Ψ_1^n) for all $n \in \mathbb{N}$. This is again no use in distinguishing between g and g'.

We might next consider the $\mathbb{Z}[A_{\phi^2}]$-matrix representation of ϕ^2 and the $\mathbb{Z}[B_{\psi^2}]$-matrix representation of ψ^2. The group $[A_{\phi^2}]$ is given by

$$[A_{\phi^2}] = \langle a, b \mid a = a^7b^4, b = a^6b^7, ab = ba \rangle.$$

A short calculation shows that $[A_{\phi^2}] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$ and that $a^6 = i$ and that $b^2 = i$. Let $\eta'\colon A \to [A_{\phi^2}]$ be the natural homomorphism. The group $[B_{\psi^2}]$ is given by

$$[B_{\psi^2}] = \langle \alpha, \beta \mid \alpha = \alpha^7\beta^2, \beta = \alpha^{12}\beta^7, \alpha\beta = \beta\alpha \rangle.$$

Again a short calculation shows that $[B_{\psi^2}] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$ and that $\alpha^6 = i$ and $\beta^2 = i$. Let $\gamma'\colon B \to [B_{\psi^2}]$ be the natural homomorphism.

The $\mathbb{Z}[A_{\phi^2}]$-matrix representation of ϕ^2 is given by

$$\Phi_2 = \begin{pmatrix} 1+a+a^2+a^3+a^4b+a^5b+b & 1+a+a^2+a^3+a^4+a^5 \\ a+a^4+2ab & 1+a+a^2+a^3+ab+a^2b+a^3b \end{pmatrix}.$$

The $\mathbb{Z}[B_{\psi^2}]$-matrix representation of ψ^2 is given by

$$\Psi_2 = \begin{pmatrix} 1+\alpha\beta+\alpha^2\beta+\alpha^3\beta+\alpha^4\beta+\alpha^5\beta+\beta \\ \alpha+\alpha\beta \end{pmatrix.}$$

$$\begin{matrix} 2+\alpha+2\alpha^2+\alpha^3+2\alpha^4+\alpha^5+\alpha\beta+\alpha^3\beta+\alpha^5\beta \\ 1+\alpha+\alpha^3+\alpha^5+\alpha^2\beta+\alpha^4\beta+b \end{matrix}\Bigg).$$

We then see that

$$\text{trace}(\Phi_2) = 2(1+a+a^2+a^3)+ab+a^2b+a^3b+a^4b+a^5b+b,$$

and

$$\text{trace}(\Psi_2) = 2(1+\alpha^2\beta+\alpha^4\beta+b)+\alpha+\alpha^3+\alpha^5+\alpha\beta+\alpha^3\beta+\alpha^5\beta.$$

Suppose that we have an isomorphism $\pi\colon [B_{\psi^2}] \to [A_{\phi^2}]$. Using the fact that $\text{trace}((\Psi_2)_\pi) = \pi \circ \text{trace}(\Psi_2)$ we see that

$$\text{trace}((\Psi_2)_\pi) = 2[1+\pi(\alpha^2\beta)+\pi(\alpha^4\beta)+\pi(b)]+\pi(\alpha)+\pi(\alpha^3)+\pi(\alpha^5)+\pi(\alpha\beta)+\pi(\alpha^3\beta)+\pi(\alpha^5\beta).$$

The terms in $\text{trace}(\Phi_2)$ with coefficient=2 are 1, a, a^2, a^3 and have orders 0, 6, 3, 2, respectively, in $[A_{\phi^2}]$. The terms with coefficient=2 in $\text{trace}(\Phi_2)$ are 1, $\alpha^2\beta$, $\alpha^4\beta$, β, and have orders 0, 6, 6, and 2, respectively, in $[B_{\psi^2}]$. The image of the terms with coefficient=2 in $\text{trace}(\Psi_2)$ under an isomorphism π again must be terms with coefficient=2. Since an isomorphism must preserve the order of elements it is clear that there can not exist an isomorphisms π so that $\text{trace}(\Phi_2) = \text{trace}((\Psi_2)_\pi)$. Thus g is not shift equivalent to g'.

5.3 Applications

In this section we give the solution to a problem posed by Williams in [39]. Consider the orientable elementary smooth graph with only two edges, $O(2)$. Classify, up to shift equivalence, all elementary presentations of the form $\{O(2), g\}$, where the mapping g is orientation preserving, and which induce endomorphisms whose characteristic polynomial upon abelianization is given by $t^2 - 3t - 2$. It is clear from theorems 43 and 44 that this is equivalent to classifying all orientation preserving endomorphisms of the group $A = \langle a, b \rangle$ whose characteristic polynomials, upon abelianization, is $t^2 - 3t - 2$ and which satisfy conditions (a), (b), and (c) of theorem 44. Note, an endomorphism $g: A \to A$ is said to be *orientation preserving* if there are no instances of a^{-1} or b^{-1} in the words $g(a)$ and $g(b)$.

We can write down all 2×2 $\mathbb{Z}^+$-matrices with characteristic polynomial $t^2 - 3t - 2$;

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}.$$

With each of these matrices we can write down all orientation preserving endomorphisms $g: A \to A$ whose $\mathbb{Z}^+$-matrix representation is given by this matrix and which satisfy conditions (a), (b), and (c) of theorem 44:

Matrix	Associated endomorphisms				
$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$	$a \mapsto a^2ba$ $b \mapsto a^2$	$a \mapsto aba^2$ $b \mapsto a^2$			
$\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$	$a \mapsto a^2b^2a$ $b \mapsto a$	$a \mapsto ababa$ $b \mapsto a$	$a \mapsto ab^2a^2$ $b \mapsto a$		
$\begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix}$	$a \mapsto a^2b^4$ $b \mapsto ab$	$a \mapsto abab^3$ $b \mapsto ab$	$a \mapsto ab^2ab^2$ $b \mapsto ab$	$a \mapsto ab^3ab$ $b \mapsto ab$	$a \mapsto bab^3a$ $b \mapsto ba$
	$a \mapsto b^2ab^2a$ $b \mapsto ba$	$a \mapsto b^3aba$ $b \mapsto ba$	$a \mapsto b^4a^2$ $b \mapsto ba$		
$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$	$a \mapsto b^2$ $b \mapsto bab^2$	$a \mapsto b^2$ $b \mapsto b^2ab$			
$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$	$a \mapsto ab^2$ $b \mapsto a^2b^2$	$a \mapsto ab^2$ $b \mapsto abab$	$a \mapsto bab$ $b \mapsto ba^2b$	$a \mapsto b^2a$ $b \mapsto baba$	$a \mapsto b^2a$ $b \mapsto b^2a^2$
$\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$	$a \mapsto b$ $b \mapsto ba^2b^2$	$a \mapsto b$ $b \mapsto babab$	$a \mapsto b$ $b \mapsto b^2a^2b$		

$\begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$	$a \mapsto ab$ $b \mapsto a^4b^2$	$a \mapsto ab$ $b \mapsto a^3bab$	$a \mapsto ab$ $b \mapsto a^2ba^2b$	$a \mapsto ab$ $b \mapsto aba^3b$	$a \mapsto ba$ $b \mapsto ba^3ba$
	$a \mapsto ba$ $b \mapsto ba^2ba^2$	$a \mapsto ba$ $b \mapsto baba^3$	$a \mapsto ba$ $b \mapsto b^2a^4$		
$\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}$	$a \mapsto ab^4$ $b \mapsto ab^2$	$a \mapsto bab^3$ $b \mapsto bab$	$a \mapsto b^2ab^2$ $b \mapsto bab$	$a \mapsto b^3ab$ $b \mapsto bab$	$a \mapsto b^4a$ $b \mapsto b^2a$
$\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$	$a \mapsto a^2b^2$ $b \mapsto a^2b$	$a \mapsto abab$ $b \mapsto a^2b$	$a \mapsto ab^2a$ $b \mapsto aba$	$a \mapsto baba$ $b \mapsto ba^2$	$a \mapsto b^2a^2$ $b \mapsto ba^2$
$\begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$	$a \mapsto a^2b$ $b \mapsto a^4b$	$a \mapsto aba$ $b \mapsto a^3ba$	$a \mapsto aba$ $b \mapsto a^2ba^2$	$a \mapsto aba$ $b \mapsto aba^3$	$a \mapsto ba^2$ $b \mapsto ba^4$

Thus there are 46 maps in total, let V denote the set of all such endomorphisms. For any endomorphism $g \in V$ we may try to find maps $g' \in V$ shift equivalent to g with lag=1 in the following manner. Suppose $\alpha, \beta \in A$ and that $g(a) = w_1(\alpha, \beta)$ and $g(b) = w_2(\alpha, \beta)$, where here $w_1(\alpha, \beta) = u_1 \dots u_n$, $w_2(\alpha, \beta) = v_1 \dots v_m$ and $u_i, v_j \in \{\alpha, \beta\}$. Consider $r: A \to A$ where $r(a) = w_1(a, b)$, $r(b) = w_2(a, b)$. Note, here $w_i(a, b)$ is the word $w_i(\alpha, \beta)$ with each instance of α replaced by a and each instance of β replaced by b. Next consider the mapping $s: A \to A$ where $s(a) = \alpha$ and $s(b) = \beta$. Define $g': A \to A$ by $g' = r \circ s$. We will then have $g \sim_s g'$ with lag$= 1$. In order to verify this it is clear that it is only necessary to check that $s \circ r = g$, but we see that

$$s \circ r(a) = s(w_1(a, b)) = w_1(\alpha, \beta) = g(a)$$

and

$$s \circ r(b) = s(w_2(a, b)) = w_2(\alpha, \beta) = g(b).$$

Example 26 Consider $g: A \to A$ where g is given by

$$\begin{aligned} a &\mapsto bab^3 = (ba)(b)(b)(b) = w_1(\alpha, \beta), \\ b &\mapsto bab = (ba)(b) = w_2(\alpha, \beta), \end{aligned}$$

and $\alpha, \beta \in A$ are such that $\alpha = ba$ and $\beta = b$. We then have $r: A \to A$ given by

$$r \begin{cases} a \mapsto ab^3, \\ b \mapsto ab, \end{cases}$$

and $s\colon A \to A$ given by

$$s \begin{cases} a \mapsto ba, \\ b \mapsto b. \end{cases}$$

The map $g'\colon A \to A$, defined by $g' = r \circ s$, is found to be

$$g' \begin{cases} a \mapsto abab^3, \\ b \mapsto ab. \end{cases}$$

It is easy to check that r and s define a shift equivalence of lag= 1 between g and g'.

It is time consuming but for each map $g \in V$ we may construct a set N_g of maps $g' \in V$ shift equivalent to g with lag= 1 by considering all factorizations into two words $\alpha, \beta \in A$ of $g(a)$ and $g(b)$. We may then construct a graph $\Gamma = (V, E)$ where for each (unordered) pair of maps g and g' in V there is an edge $e \in E$ if $g' \in N_g$. Note $g' \in N_g$ implies that $g \in N_{g'}$.

Suppose that two maps g and g' belong to the same connected components of Γ, it is then clear that $g \sim_s g'$ as we can find a path (of shift equivalences) from g to g' in Γ. The graph Γ is found to consist of four connected components, say Γ_1, Γ_2, Γ_3, and Γ_4, the details of this very tedious process are suppressed. We note that this fact was known to Williams in [39]. A computer program (for details see appendix B) written in the Maple programming language was used to find the connected components Γ_i of Γ.

The connected component Γ_1 of Γ consists of the following 28 endomorphisms:

$a \mapsto b^2$ $b \mapsto bab^2$	$a \mapsto b^2$ $b \mapsto b^2ab$	$a \mapsto b$ $b \mapsto ba^2b^2$	$a \mapsto b$ $b \mapsto b^2a^2b$	$a \mapsto ab^4$ $b \mapsto ab^2$
$a \mapsto bab^3$ $b \mapsto bab$	$a \mapsto b^3ab$ $b \mapsto bab$	$a \mapsto b^4a$ $b \mapsto b^2a$	$a \mapsto ab$ $b \mapsto a^3bab$	$a \mapsto ab$ $b \mapsto aba^3b$
$a \mapsto ba$ $b \mapsto ba^3ba$	$a \mapsto ba$ $b \mapsto baba^3$	$a \mapsto ab^2$ $b \mapsto abab$	$a \mapsto b^2a$ $b \mapsto baba$	$a \mapsto abab^3$ $b \mapsto ab$
$a \mapsto ab^3ab$ $b \mapsto ab$	$a \mapsto bab^3a$ $b \mapsto ba$	$a \mapsto b^3aba$ $b \mapsto ba$	$a \mapsto abab$ $b \mapsto a^2b$	$a \mapsto baba$ $b \mapsto ba^2$
$a \mapsto a^2b$ $b \mapsto a^4b$	$a \mapsto aba$ $b \mapsto a^3ba$	$a \mapsto aba$ $b \mapsto a^ba^3$	$a \mapsto ba^2$ $b \mapsto ba^4$	$a \mapsto a^2b^2a$ $b \mapsto a$
$a \mapsto ab^2a^2$ $b \mapsto a$	$a \mapsto a^2ba$ $b \mapsto a^2$	$a \mapsto aba^2$ $b \mapsto a^2$		

The connected component Γ_2 of Γ consists of the following 2 endomorphisms:

$a \mapsto bab$ $b \mapsto ba^2b$	$a \mapsto ab^2a$ $b \mapsto aba$

The connected component Γ_3 of Γ consists of the following 12 endomorphisms;

$a \mapsto b$ $b \mapsto babab$	$a \mapsto b^2ab^2$ $b \mapsto bab$	$a \mapsto ab$ $b \mapsto a^2ba^2b$	$a \mapsto ba$ $b \mapsto ba^2ba^2$	$a \mapsto ab^2$ $b \mapsto a^2b^2$
$a \mapsto b^2a$ $b \mapsto b^2a^2$	$a \mapsto ab^2ab^2$ $b \mapsto ab$	$a \mapsto b^2ab^2a$ $b \mapsto ba$	$a \mapsto a^2b^2$ $b \mapsto a^2b$	$a \mapsto b^2a^2$ $b \mapsto ba^2$
$a \mapsto aba$ $b \mapsto a^2ba^2$	$a \mapsto ababa$ $b \mapsto a$			

The connected component Γ_4 of Γ consists of the following 4 endomorphisms;

$a \mapsto ab$ $b \mapsto a^4b^2$	$a \mapsto ba$ $b \mapsto b^2a^4$	$a \mapsto a^2b^4$ $b \mapsto ab$	$a \mapsto b^4a^2$ $b \mapsto ba$

We know that all of the endomorphisms in each connected component Γ_i of Γ are shift equivalent to each other. We don't know whether the endomorphisms in component Γ_i are shift equivalent to the endomorphisms in component Γ_j for $i \neq j$. We might try to distinguish them by considering invariants. It takes a little time but it is possible to show that all 10 of the 2×2 matrices with characteristic polynomial $t^2 - 3t - 2$ are shift equivalent. Thus the abelian invariants will be of no use in distinguishing the connected components of Γ. From each of the connected components of Γ we will pick a representative endomorphism and find some non-abelian invariants.

From component Γ_1 of Γ we choose the endomorphism g_1 given by

$$g_1 \begin{cases} a \mapsto ab^2a^2, \\ b \mapsto a, \end{cases}$$

as a representative. The group $[A_{g_1}]$ is given by

$$[A_{g_1}] = \langle a, b \mid a = a^3b^2, b = a, ab = ba \rangle.$$

We find that $[A_{g_1}] \cong \mathbb{Z}_4$ and that $a = b$ and $i = a^4$. Let η_1 be the natural homomorphism from A to $[A_{g_1}]$. The $\mathbb{Z}[A_{g_1}]$-matrix representation of g_1 is given by

$$\mathrm{G}_1 = \begin{pmatrix} 1 + ab^2 + ab^2a & 1 \\ a + ab & 0 \end{pmatrix}^{\eta_1} = \begin{pmatrix} 2 + a^3 & 1 \\ a + a^2 & 0 \end{pmatrix}.$$

Thus we see that the trace($\mathbf{G}_1$) $= 2 + a^3$.

From component Γ_2 of Γ we choose endomorphism g_2, given by

$$g_1 \begin{cases} a \mapsto ab^2a, \\ b \mapsto aba, \end{cases}$$

as a representative. The group $[A_{g_2}]$ is given by

$$[A_{g_2}] = \langle a, b \mid a = a^2b^2, b = a^2b, ab = ba \,\rangle.$$

We see that $[A_{g_2}] \cong \mathbb{Z}_4$ and that $a^2 = i$, $a = b^2$, and $b^4 = i$. Let η_2 be the natural homomorphism from A to $[A_{g_2}]$. The $\mathbb{Z}[A_{g_2}]$-matrix representation of g_2 is given by

$$\mathbf{G}_2 = \begin{pmatrix} 1+ab^2 & 1+ab \\ a+ab & a \end{pmatrix}^{\eta_2} = \begin{pmatrix} 2 & 1+b^3 \\ b^2+b^3 & b^2 \end{pmatrix}.$$

Thus we see that trace($\mathbf{G}_2$) $= 2 + b^2$.

From component Γ_3 of Γ we choose endomorphism g_3, given by

$$g_3 \begin{cases} a \mapsto ababa, \\ b \mapsto a, \end{cases}$$

as a representative. The group $[A_{g_3}]$ is given by

$$[A_{g_3}] = \langle a, b \mid a = a^3b^2, b = a, ab = ba \,\rangle.$$

We see that $[A_{g_3}] \cong \mathbb{Z}_4$ and that $a_4 = i$ and $b = a$. Let η_2 be the natural homomorphism from A to $[A_{g_3}]$. The $\mathbb{Z}[A_{g_3}]$-matrix representation of g_3 is given by

$$\mathbf{G}_3 = \begin{pmatrix} 1+ab+abab & 1 \\ a+aba & 0 \end{pmatrix}^{\eta_3} = \begin{pmatrix} 2+a^2 & 1 \\ a+a^3 & 0 \end{pmatrix}.$$

Thus we see that trace($\mathbf{G}_3$) $= 2 + a^2$.

From the component Γ_4 of Γ we choose endomorphism g_4, given by

$$\begin{cases} a \mapsto ab, \\ b \mapsto a^4b^2, \end{cases}$$

as a representative. We see that $[A_{g_4}] \cong \mathbb{Z}_4$ and that $a^4 = i$ and $b = i$. Let η_4 be the natural homomorphism from A to $[A_{g_4}]$. The $\mathbb{Z}[A_{g_4}]$-matrix representation of g_4 is given by

$$\mathbf{G}_4 = \begin{pmatrix} 1 & 1+a+a^2+a^3 \\ a & a^4+a^4b \end{pmatrix}^{\eta_4} = \begin{pmatrix} 1 & 1+a+a^2+a^3 \\ a & 2 \end{pmatrix}.$$

Thus we see that trace($\mathbf{G}_4$) = 3.

From this we see that there is no endomorphism in $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ which is shift equivalent to one in Γ_1. We know this because there is no isomorphism π from $[A_{g_1}]$ to $[A_{g_i}]$ for $i = 2, 3, 4$ so that $\pi(\text{trace}(\mathbf{G}_1)) = \text{trace}(\mathbf{G}_i)$.

We also see that there is no endomorphism in $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ which is shift equivalent to one in Γ_4. We know this because there can be no isomorphism π from $[A_{g_4}]$ to $[A_{g_i}]$, for $i = 1, 2, 3$, so that $\pi(\text{trace}(\mathbf{G}_4)) = \text{trace}(\mathbf{G}_i)$.

We therefore see that Γ_1 and Γ_4 are distinct classes up to shift equivalence. We don't know, however, whether the maps in Γ_2 are shift equivalent to the maps in Γ_3 as the isomorphism $\pi\colon [A_{g_2}] \to [A_{g_3}]$ where $\pi(b) = a$ is such that $\pi(\text{trace}(\mathbf{G}_2)) = \text{trace}(\mathbf{G}_3)$. We might try considering higher powers of g_2 and g_3 in order to find further invariants. Doing so, say for g_2^2 and g_3^2, we find that there is an isomorphism $\pi\colon [A_{g_2^2}] \to [A_{g_3^2}]$ so that $\pi(\text{trace}(\mathbf{G}_2')) = \text{trace}(\mathbf{G}_3')$, where $\mathbf{G}_2'$ is the $\mathbb{Z}[A_{g_2^2}]$-matrix representation of g_2 and $\mathbb{Z}[A_{g_3^2}]$-matrix representation of g_3. This leads us to suspect that the endomorphisms in Γ_2 are shift equivalent to the endomorphisms in Γ_3. In fact we will be able to show that they are.

In order to do this we will consider the representative endomorphism g_2 of Γ_2. The endomorphism g_2 induces a W^*-mapping ϕ_2 of the elementary smooth graph $O(2)$. We note that ϕ_2 has two fixed points. One of these fixed points is of course the branch point, y, the other, x, is in the interior of a branch. One idea we might try is to find an elementary presentation about the other fixed point. By this we mean mappings r and s from $O(2)$ to $O(2)$ so that $r(x)$ is a branch point, and $s \circ r = \phi_2$. We could then define $\phi_2'\colon O(2) \to O(2)$ by $\phi' = s \circ r$ and we would have $\phi_2 \sim_s \phi_2'$ with lag= 1.

Let $O(2)$ be the orientatable elementary smooth graph, with coherent orientation ρ and two edges a and b. Let $\phi_2\colon O(2) \to O(2)$ be such that ϕ_2 wraps a along the route $abba$ and wraps b alone the route aba. ϕ_2 is the W^*-mapping induced by the endomorphism g_2. We give $O(2)$ a new smooth graph structure by adding 3 ordinary points to the vertex set. These points are the fixed point x in the branch b and the two pre-images of x in the the branch a, which we label x_1 and x_2 in agreement with the orientation on a. The three "new" edges formed from a are labeled a_1, a_2, and a_3, in agreement with the orientation on $O(2)$, i.e., $\rho(a_1) = x_1$, $\rho(a_2) = x_2$, and $\rho(a_3) = y$. The two "new" edges formed from b are labeled b_1 and b_2, again in agreement with the orientation on $O(2)$, thus $\rho(b_1) = x$

and $\rho(b_2) = y$. The map ϕ_2 in terms of this new smooth graph structure on $O(2)$ is such that $\phi_2(x_1) = \phi_2(x_2) = \phi_2(x) = x$, $\phi_2(y) = y$ and ϕ_2 expands the edges of $O(2)$ along routes of $O(2)$ as shown below;

$$\phi_2 \begin{cases} a_1 \mapsto a_1 a_2 a_3 b_1, \\ a_2 \mapsto b_2 b_1, \\ a_3 \mapsto b_2 a_1 a_2 a_3, \\ b_1 \mapsto a_1 a_2 a_3 b_1, \\ b_1 \mapsto b_2 a_1 a_2 a_3. \end{cases}$$

Consider the smooth graph $O(2)' = O(2)/\sim$, with the induced differential structure and Riemannian metric, where $\sim$ is the equivalence relation given by

$$x \sim y \text{ if and only if } \begin{cases} \phi_2(x) = \phi_2(y), \; x \in \bar{a_1}, \text{ and } y \in \bar{b_1}, \\ \phi_2(x) = \phi_2(y), \; x \in \bar{a_3}, \text{ and } y \in \bar{b_2}. \\ x = y \end{cases}$$

It is easy to check that $O(2)'$ is diffeomorphic to $O(2)$. See figure 5.1 for an illustration. Let $r: O(2) \to O(2)'$ be the projection mapping. We may give $O(2)'$ a smooth graph structure by considering the points $r(x)$ and $r(y)$ to be vertices in $O(2)'$. Note that $v = r(x)$ is the branch point of $O(2)'$ and that $w = r(y)$ is an ordinary point. We then see that $O(2)'$ has three edges $\alpha_2 = r(a_1) = r(b_1)$, $\alpha_1 = r(a_3) = r(b_3)$, and $\beta = r(a_2)$. The coherent orientation ρ' on $O(2)'$ induced from ρ on $O(2)$ is such that $\rho(\alpha_1) = r(y)$, $\rho(\alpha_2) = r(x)$ and $\rho(\beta) = r(x)$.

We may then define $\phi_2' = r \circ \phi_2 \circ r^{-1}$. This is clearly a well defined mapping where $\phi_2'(v) = v$, $\phi_2'(w) = w$ and is such that ϕ_2' expands the edges of $O(2)'$ along the routes of $O(2)'$ as shown below;

$$\phi_2' \begin{cases} \alpha_1 \mapsto \alpha_1 \alpha_2 \beta \alpha_1, \\ \alpha_2 \mapsto \alpha_2 \beta \alpha_1 \alpha_2, \\ \beta \mapsto \alpha_1 \alpha_2. \end{cases}$$

If we then define $s = g \circ r^{-1}$, well defined for the same reason as ϕ_2', we will then clearly have $\phi_2 \sim_s \phi_2'$ with lag$= 1$.

We may then give $O(2)'$ a "new" smooth graph structure by remove the ordinary point $w = r(y)$ from the vertex set. The route consisting of the two edges α_1 and α_2 then becomes a single edge, α. The mapping ϕ_2' in terms of this new smooth graph structure

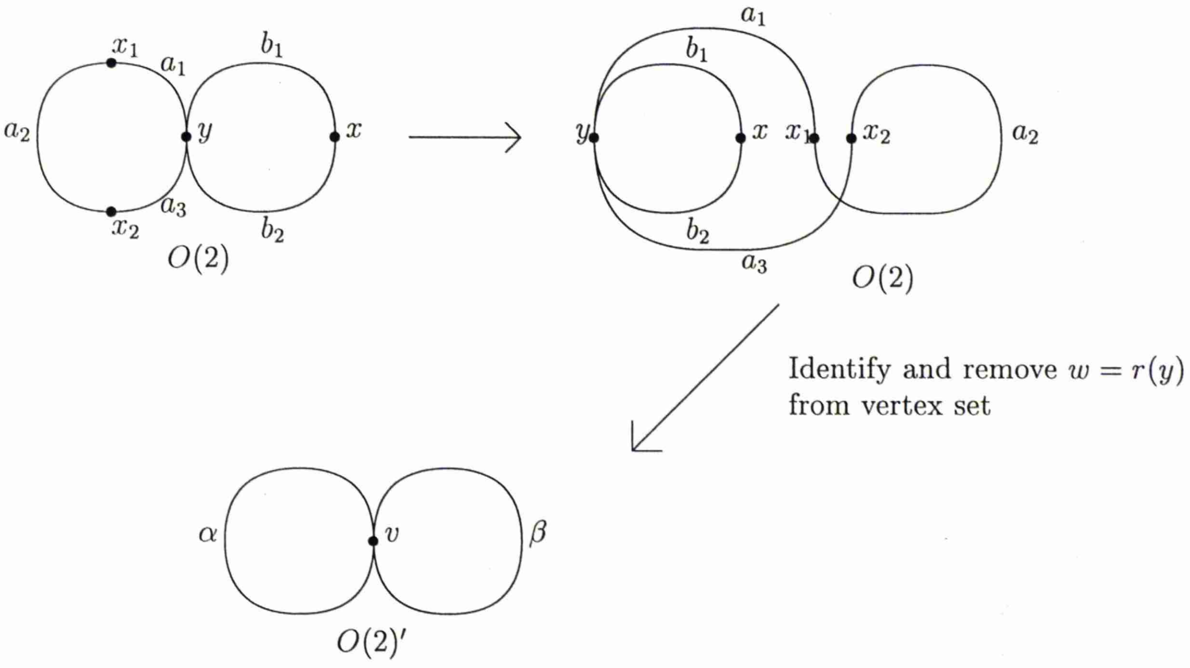


Figure 5.1: The smooth graphs $O(2)$ and $O(2)'$.

is such that ϕ_2' expands the edges α and β along the routes of $O(2)'$ as shown below;

$$\phi_2' \begin{cases} \alpha \mapsto \alpha\beta\alpha\beta\alpha, \\ \beta \mapsto \alpha. \end{cases}$$

It is clear, that ϕ_2' induces the endomorphism g_2 belonging to Γ_3. Thus g_2 and g_3 are shift equivalent by theorem 43 and the endomorphisms in Γ_2 are shift equivalent to the endomorphisms in Γ_3. We should point out, however, that there is no known example of shift equivalence between g_2 and g_3 or between any endomorphism in Γ_2 and one in Γ_3.

Finally we may state the solution to the problem posed by Williams. The elementary presentations $\{O(2), g\}$ which induce orientation preserving endomorphisms with characteristic polynomial $t^2 - 3t - 2$ form three equivalence classes up to shift equivalence. Those that induce endomorphisms in Γ_1, those that induce endomorphisms in $\Gamma_2 \cup \Gamma_3$ and those that induce endomorphisms in Γ_4.

Chapter 6

Concluding Remarks

In this thesis the presentation of solenoids by branched 1-manifolds has been studied.

We have shown that solenoids come in two varieties (up to the topological conjugacy of thier shift maps), those presented by orientable branched 1-manifolds and those presented by nonorientable branched 1-manifolds. Recurrence was shown to be a necessary and sufficient condition for a branched 1-manifold to present a solenoid.

Two special types of presentation were considered, elementary presentations and (p, q)-block presentations. Given a presentation methods for finding an equivalent elementary presentation (in the case that there exits a fixed point) or an equivalent (p, q)-block presentation were given. We demonstrated how to calculate algebraic invariants given an elementary or (p, q)-block presentation. These invariants are in terms of the shift equivalence of endomorphisms of finitely generated free groups.

We then considered invariants derived from the endomorphism invariants as it is not known how to determine if endomorphisms of finitely generated free groups are shift equivalent. First we examined some invariants which arise when the free group in question is abelianized. We then constructed some new invariants which reflect the non-abelian nature of the endomorphisms. The non-abelian invariants were then used to solve a problem posed by Williams in [39]

There is still considerable scope for future research into the presentation of solenoids by branched 1-manifolds. In particular given two presentations of solenoids $\{K, g\}$ and $\{K', g'\}$ it is still not known how to determine whether or not g is shift equivalent to g' and thus whether or not the shift map h of K_∞ is topologically conjugate to the shift map h' of K'_∞.

We close this chapter by giving four possible problems for future research.

Problem 1.) Given a presentation $\{K, g\}$ and a branched 1-manifold K' is it possible to decide whether or not there exists a W^*-mapping $g': K' \to K'$ so that $g \sim_s g'$ and thus whether K' can present a solenoid equivalent to K? It is clearly not always possible to find such a g'. For example, let $n, m \in \mathbb{Z}$ with $n > m$ and consider a presentation $\{O(n), f\}$ where the W^*-mapping f induces an endomorphism $f_*: \langle a_1, \ldots, a_n \rangle \to \langle a_1, \ldots, a_n \rangle$ which upon abelianization has characteristic polynomial $\chi(t) = t^n + k_1 t^{n-1} + \cdots + k_{n-1}t + k_n$. Suppose further that not all of the k_i for $n \leq i \leq m+1$ are zero. It is then clear that there can exist no W^*-mapping, g, on $O(m)$ which is shift equivalent to f as it would have to induce an endomorphism $g_*: \langle b_1, \ldots, b_m \rangle \to \langle b_1, \ldots, b_m \rangle$ which upon abelianization has characteristic polynomial $t^s \chi(t)$ for some $s \in \mathbb{Z}$. In the other direction we see that in certain circumstances it is some times possible to state sufficient conditions. For example, let $\{O(1), f\}$ be the presentation of a solenoid where f induces the endomorphism f_* which upon abelianization has characteristic polynomial $t^2 + d$. Then on any orientable elementary smooth graph $O(m)$ where $d > m$ there exists a W^*-mapping, g, with $g \sim_s f$. For we have $f_*: \langle a \rangle \to \langle a \rangle$ with $f_*(a) = a^d$. Define $r: \langle a \rangle \to \langle \alpha_1, \ldots, \alpha_m \rangle$ by $r(a) = \alpha_1 \alpha_2 \ldots \alpha_m$ and define $s: \langle \alpha_1, \ldots, \alpha_m \rangle \to \langle a \rangle$ by

$$
\begin{aligned}
s(\alpha_i) &= a \qquad\qquad \text{for } 1 \leq i \leq m-1, \\
s(\alpha_m) &= a^{d-m+1}.
\end{aligned}
$$

If we then define $g_* = r \circ s$ it is clear that $f_* \sim_s g_*$. It is easy to check that g_* so defined satisfies conditions 1.), 2.), and 3.) of theorem 44 thus there will be a W^*-mapping $g: O(m) \to O(m)$ which induces g_*. It is the author's belief that results of this nature exist in much greater generality.

Problem 2.) Consider the set, $G(n, m)$, of all maps, g, from $O(2) \to O(2)$ that induce endomorphisms which upon abelianization have characteristic polynomials $t^2 - nt + m$. For which n, m can we classify the maps in $G(n, m)$ up to shift equivalence? This is a generalization of the Problem posed by Williams (n=3,m=-2) in [39] and which was solved in section 5.3. We have successfully carried out this classification for several other choices of n and m, but it is not know how far this can be done.

Problem 3.) Let Φ and Ψ be square matrices over the group ring $\mathbb{Z}G$ where G is a finitely generated abelian group. Does there exist a procedure for deciding when Φ is shift equivalent to Ψ? Such a procedure is known to exist for integer and non-negative integer matrices, [21].

Problem 4.) Let $\mathbf{\Phi}_0$ and $\mathbf{\Phi}_k$ be square matrices over the group ring $\mathbb{Z}G$ where G is a finitely generated abelian group. Suppose futher that there exist matrices $\mathbf{R}_1, \mathbf{S}_1, \ldots,$ $\mathbf{R}k\mathbf{S}_k$ over $\mathbb{Z}G$ so that;

$$\begin{aligned} \mathbf{\Phi}_0 &= \mathbf{R}_1\mathbf{S}_1, & \mathbf{S}_1\mathbf{R}_1 &= \mathbf{\Phi}_1, \\ \mathbf{\Phi}_1 &= \mathbf{R}_2\mathbf{S}_2, & \mathbf{S}_2\mathbf{R}_2 &= \mathbf{\Phi}_2, \\ &\vdots & &\vdots \\ \mathbf{\Phi}_{k-1} &= \mathbf{R}_k\mathbf{S}_k, & \mathbf{S}_k\mathbf{R}_k &= \mathbf{\Phi}_k. \end{aligned}$$

We showed that this implies that $\operatorname{trace}(\mathbf{\Phi}_0^n) = \operatorname{trace}(\mathbf{\Phi}_k^n)$ for all $n \in \mathbb{N}$. What other invariants can be derived from this set of relations? Invariants of this sort could be useful for determining whether or not two solenoids given by elementary or compatible (p, q)-block presentations are equivalent.

Appendix A

Some functions on the real line

In this appendix we detail the construction of the functions used in theorems 20 and 34. In order to do this we will first show how to construct a C^∞ "bump" function $\lambda: \mathbb{R} \to \mathbb{R}$.

Let $a, b, c, d \in \mathbb{R}$ with $b > a > 0$ and $d > c > 0$. We start with the C^∞ map $\alpha: \mathbb{R} \to \mathbb{R}$, where

$$\alpha(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x} & \text{if } x > 0. \end{cases}$$

We then define $\beta: \mathbb{R} \to \mathbb{R}$, choosing δ so that $0 < \delta < (b-a)/2$, by

$$\beta(x) = \alpha(x - \delta)\alpha((b-a)/2 - x).$$

Next define $\gamma: \mathbb{R} \to \mathbb{R}$ by

$$\gamma(x) = \frac{\int_x^{\frac{b-a}{2}} \beta(s)ds}{\int_\delta^{\frac{b-a}{2}} \beta(s)ds}.$$

Finally we arrive at our "bump" function $\lambda: \mathbb{R} \to \mathbb{R}$ by defining $\lambda(x) = \gamma(|x|)$. It is well known, see [18] , that the function λ defined in this manner will be C^∞ and that it satisfies the following properties;

1. $\lambda(x) = 1$ if $|x| \leq \delta$,

2. $0 < \lambda(x) < 1$ if $\delta < |x| < (b-a)/2$,

3. $\lambda(x) = 0$ if $|x| \geq (b-a)/2$.

We see that λ is symmetric about the point $x = 0$ and that in this sense it can be said to be "centered" at $x = 0$. We would like to find another "bump" function, π, which is a copy of λ with its center shifted to the point $x = (a+b)/2$. Such a function is clearly given by

$$\pi(x) = \lambda(x - \frac{1}{2}(a+b)).$$

Again it is clear that π is C^∞ and that it satisfies;

1. $\pi(x) = 1$ if $a + \delta \leq x \leq b - \delta$,

2. $0 < \pi(x) < 1$ if $a < x < a + \delta$ or $b - \delta < x < b$,

3. $\pi(x) = 0$ if $x \leq a$ or $x \geq b$.

Consider

$$A = \int_{-\infty}^{\infty} \pi(x)dx = \int_a^b \pi(x)dx.$$

It is clear that A must be finite and that $(b-a) - 2\delta < A < (b-a)$.

We will now construct the C^∞ function $f: [a,b] \to [c,d]$ used in theorem 34. The function f that we are looking for was stipulated to satisfy the following properties for some $\epsilon > 1$ of our choice;

1. $f(a) = c$ and $f(b) = d$,

2. $f'(a) = f'(b) = \epsilon$,

3. $f'(x) > \epsilon$ for $x \in (a,b)$.

Note, that here we also assume that $d - c > b - a$.

Let

$$k = \frac{(d-c) - \epsilon(b-a)}{A}.$$

Choose ϵ so that $1 < \epsilon < (d-c)/(b-a)$. This is possible since $d - c > b - a$. With this choice of ϵ we see that $k > 0$.

Define

$$f(x) = \int_a^x \{\epsilon + k\pi(s)\}\, ds + c + a - \epsilon a.$$

It is clear that f defined in this manner will be C^∞. We claim that f satisfies the the three properties given above. Integrating we see that

$$f(x) = \epsilon(x-a) + c + k\int_a^x \pi(s)ds.$$

Thus we have

$$f(a) = c$$

and

$$f(b) = \epsilon(b-a) + c + \frac{(d-c) - \epsilon(b-a)}{A} A = d.$$

As $f'(x) = \epsilon + k\pi(x)$ we see that $f'(a) = \epsilon$, $f'(b) = \epsilon$, and $f'(x) > \epsilon$, for $x \in (a,b)$. Thus f is the desired function.

In theorem 20 we stipulated the existence of a function $g\colon [a,b] \to [c,d]$ which is such that;

1. $g(a) = c$ and $g(b) = d$,

2. $g'(a) = g'(b) = 1$,

3. $g'(x) > 0$ for all $x \in (a,b)$.

Note that in this case there is no assumption being made as to whether $d - c \geq b - a$ or $b - a \geq d - c$.

Let

$$k = \frac{(d-c) - (b-a)}{A}.$$

Note that in this case k will not necessarily be positive.

Define

$$g(x) = \int_a^x \{1 + k\pi(x)\} ds + c$$

where π is the "bump" function constructed earlier with $\delta > 0$ chosen so that $\delta < (d-c)/2$ and $\delta < (b-a)/2$. We claim that g is the desired function.

Integrating we see that

$$g(x) = x - a + c + k \int_a^x \pi(s) ds.$$

It follows that $g(a) = c$ and $g(b) = d$.

As $g'(x) = 1 + k\pi(x)$ we see that $g'(a) = 1$ and $g'(b) = 1$. If $(d-c) - (b-a) \geq 0$ then we will have $k \geq 0$ and thus $g'(x) > 0$ for all $x \in (a,b)$.

Suppose that $(d-c) - (b-a) < 0$. By choice we know that $2\delta < (d-c)$ and thus that $-2\delta > -(d-c)$. By adding $(b-a)$ to both sides we find that

$$(b-a) - 2\delta > (b-a) - (d-c).$$

Using the facts that

$$A > (b - a) - 2\delta$$

and

$$(b - a) - (d - c) > \{(b - a) - (d - c)\}\pi(x),$$

for $x \in (a, b)$, we see that

$$A > \{(b - a) - (d - c)\}\pi(x).$$

This then implies that

$$1 + \frac{\{(d - c) - (b - a)\}}{A}\pi(x) = 1 + k\pi(x) = g'(x) > 0.$$

Thus g is the desired function.

Appendix B

Maple routines

In this appendix we give a computer program, comps, written in the Maple V3 programing language which were used in section 5.3 to help solve the the problem posed by Williams in [39]. The program comps is written to do the following:

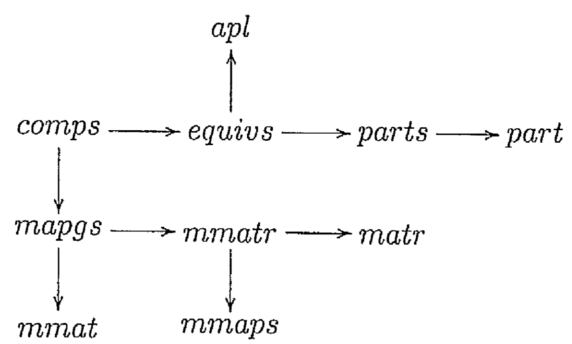
1. Given $n, m \in \mathbb{Z}$ find the set, V, of all endomorphisms $g\colon \langle a, b\rangle \to \langle a, b\rangle$ whose characteristic polynomials upon abelianization are given by $t^2 - nt + m$ and which satisfy conditions (a), (b), and (c) of theorem 44.

2. For each map $g \in V$ find *a* set N_g of maps $g' \in V$ shift equivalent to g with lag= 1 by considering all factorizations into two words $\alpha, \beta \in A$ of the words $g(a)$ and $g(b)$.

3. Construct the graph $\Gamma = (V, E)$ where for each (unordered) pair of maps g and g' in V there is an edge $e \in E$ if $g' \in N_g$. Note $g' \in N_g$ implies that $g \in N'_g$.

4. Find the connected components of Γ.

The main program comps calls 9 subroutines; equivs, apl, parts, part, mapgs, mmat, mmatr, mmaps, and matr as shown in the diagram below.



Given two integers, n and m, as arguments comps(n,m) returns a list

$$\{\text{component}_1, \text{component}_2, \ldots, \text{component}_k\},$$

where each component_i is a connected component of the graph Γ. Each component_i is itself a list

$$\{\text{map}_{i,1}, \text{map}_{i,2}, \ldots, \text{map}_{i,p(i)}\}$$

where the $\text{map}_{i,j}$ are the vertices of component_i. The $\text{map}_{i,j}$ are representations of the endomorphisms in V and are given by a list containing two sublists

$$[\,[a_1, a_2, \ldots, a_s],\ [b_1, b_2, \ldots, b_t]\,]$$

where the $a_i, b_j \in \{a, b\}$. Such a list represents the endomorphism ϕ: $\langle a, b\rangle \to \langle a, b\rangle$ where

$$\phi(a) = a_1 a_2 \cdots a_s,$$
$$\phi(b) = b_1 b_2 \cdots b_t.$$

The program comps requires the Maple V3 package linalg. Details about the Maple V3 programing language can be found in [7].

```
    comps := proc(n,m)
        local S1,S2,i,j,T,Tinv,Vrts,Edgs,G,A,B;
        S1 := mapgs(n,m);
        S2 := [seq(equivs(op(S1[i])),i = 1 .. nops(S1))];
        T := table([seq(i = S1[i],i = 1 .. nops(S1))]);
        Tinv := table([seq(S1[i] = i,i = 1 .. nops(S1))]);
        Vrts := {seq(i,i = 1 .. nops(S1))};
        Edgs := {};
        for i to nops(S1) do
                for j to nops(S2[i]) do
                        if not S1[i] = S2[i][j] then
                                Edgs := {{Tinv[S1[i]],Tinv[S2[i][j]]},op(Edgs)};
                        fi;
                od;
        od;
        G := networks[graph](Vrts,Edgs);
        A := networks[components](G);
        B := {};
        for i to nops(A) do
                B := {op(B),{seq(T[op(j,op(i,A))],j = 1 .. nops(op(i,A)))}};
        od;
        B;
end;
equivs := proc(L1,L2)
        local L,A,LL,i,j,c1,c2,b1,b2;
        L := parts(L1,L2);
        A := [seq(convert({op(L[i][1])} union {op(L[i][2])},list),i = 1 .. nops(L))];
        LL := [];
        for i to nops(L) do
```

```
            c1 := [];
            c2 := [];
            b1 := [];
            b2 := [];
            for j to nops(L[i][1]) do
                    if L[i][1][j] = A[i][1] then
                            c1 := [op(c1),a];
                            b1 := [op(b1),b];
                    else
                            c1 := [op(c1),b];
                            b1 := [op(b1),a];
                    fi;
            od;
            for j to nops(L[i][2]) do
                    if L[i][2][j] = A[i][1] then
                            c2 := [op(c2),a];
                            b2 := [op(b2),b];
                    else
                            c2 := [op(c2),b];
                            b2 := [op(b2),a];
                    fi;
            od;
            LL := [op(LL),[apl(A[i][1],[c1,c2]),apl(A[i][2],[c1,c2])],
                    [apl(A[i][2],[b1,b2]),apl(A[i][1],[b1,b2])]];
        od;
        c1 := subs(b = a,a = b,L2);
        c2 := subs(b = a,a = b,L1);
        b1 := [seq(c1[nops(c1)-i],i = 0 .. nops(c1)-1)];
        b2 := [seq(c2[nops(c2)-i],i = 0 .. nops(c2)-1)];
        LL := [op(LL), [[seq(L1[nops(L1)-i],i = 0 .. nops(L1)-1)],
            [seq(L2[nops(L2)-i],i = 0 .. nops(L2)-1)]], [b1,b2]];
        LL;
end;
apl := proc(x,L)
        local i,S;
        S := [];
        for i to nops(x) do
            if x[i] = a then
                    S := [op(S),op(L[1])];
            else
                    S := [op(S),op(L[2])];
            fi;
        od;
        S;
end;
parts := proc(S1,S2)
        local L1,L2,i,j,L;
        L := [];
        L1 := part(S1);
        L2 := part(S2);
        for i to nops(L1) do
            for j to nops(L2) do
                    if nops({op(L1[i])} union {op(L2[j])}) = 2 then
                            L := [op(L),[L1[i],L2[j]]];
                    fi;
            od;
        od;
        L;
end;
part := proc(S)
        local L,LL,i,j;
```

```
        LL := combinat[powerset]([seq(i,i = 1 .. nops(S)-1)]);
        L := [[S]];
        for i to nops(LL) do
                if not (nops(LL[i]) = 0 or nops(LL[i]) = nops(S)-1) then
                        L := [op(L),[[S[1 .. LL[i][1]]], seq([S[LL[i][j]+1 .. LL[i][j+1]]],
                                j = 1 .. nops(LL[i])-1), [S[LL[i][nops(LL[i])]+1 .. nops(S)]]]];
                fi;
        od;
        L;
end;
mapgs := proc(n,m)
        local i,L,LL;
        L := [];
        LL := [op(mmat(n,m))];
        for i to nops(LL) do
                L := [op(L),op(mmatr(LL[i]))];
        od;
        ;L
end;
mmatr := proc(A)
        mmaps(matr(A));
end;
matr := proc(A)
        local L1,L2,i;
        L1 := [seq(a,i = 1 .. A[1,1]),seq(b,i = 1 .. A[2,1])];
        L2 := [seq(a,i = 1 .. A[1,2]),seq(b,i = 1 .. A[2,2])];
        L1,L2;
end;
mmaps := proc(L1,L2)
        local L,i,j,S1,S2;
        L := [];
        S1 := combinat[permute](L1);
        S2 := combinat[permute](L2);
        for i to nops(S1) do
                for j to nops(S2) do
                        if S1[i][1] = S2[j][1] and S1[i][nops(L1)] = S2[j][nops(L2)] then
                                L := [op(L),[S1[i],S2[j]]];
                        fi;
                od;
        od;
        L;
end;
mmat := proc(N,M)
        local i,j,L,temp;
        L := {};
        for i from 0 to N do
                temp := [op(numtheory[divisors](i*(N-i)-M))];
                for j to nops(temp) do
                        L := L union matrix([[i,temp[j]], [(i*(N-i)-M)/temp[j],N-i]]);
                od;
        od;
        L;
end;
```

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