Theories of Fixed Point Index and Applications

by

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To My Wife and Son
MEI SHI and TIAN LAN

Summary

This thesis is devoted to the study of theories of fixed point index for generalized and weakly inward maps of condensing type and weakly inward A-proper maps.

In Chapter 1 we recall some basic concepts such as cones, wedges, measures of non-compactness and theories of fixed point index for compact and γ-condensing self-maps. We also give some new results and provide new proofs for some known results.

In Chapter 2 we study approximatively compact sets giving examples and proving new results. The concept of an approximatively compact set is of importance in defining our index for a generalized inward map since there exists upper semicontinuous multivalued metric projections onto the approximatively compact convex set. We also introduce the concept of an M_l-set which will play an important role in defining our fixed point index for generalized inward maps of condensing type since there exists continuous single-valued metric projections onto an M_l-closed convex set. Many examples of M_l-closed convex sets are given. Weakly inward sets and weakly inward maps are studied in detail. New properties and examples on such sets and maps are given. We also introduce the new concept of generalized inward sets and generalized inward maps. The class of generalized inward maps strictly contain the class of weakly inward maps. Several necessary and sufficient conditions for a map to be generalized inward and examples of generalized inward maps are given.

In Chapter 3 we define a fixed point index for a generalized inward compact map defined on an approximatively compact convex set and obtain many new fixed point theorems and nonzero fixed point theorems. In particular, norm-type expansion and compression theorems for weakly inward continuous maps in finite dimensional Banach spaces are obtained, which have not been considered previously.

In Chapter 4 we define a fixed point index for a generalized inward maps of condensing type defined on an M_l-closed convex set and obtain many new fixed point theorems and nonzero fixed point theorems. We also apply the abstract theory to some perturbed Volterra equations.

In Chapter 5 we define a fixed point index for weakly inward A-proper maps. We obtain new fixed point theorems, nonzero fixed point theorem and results on existence of eigenvalues. We also give an application of the abstract theory to the existence of nonzero positive solutions of boundary value problems for second order differential equations.

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Contents

Preface

This thesis is a record of part of the research carried out by the author during the academic years 1994-1998. It is submitted according to the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

Almost all of the results of this thesis are the original work of the author with the exception of several results specifically mentioned in the text and attributed there to the authors concerned. Some results in Chapter 4 have been published in Trans. Amer. Math. Soc., **349** (1997), 2175-2186 and some parts of Chapter 5 have appeared in Nonlinear Analysis, **28** (1997), 315-325; The two papers are written jointly with Prof. J.R.L. Webb.

Chapter 1 contains preliminary material and several new results, Chapter 2 is mostly new and Chapters 3,4 and 5 consist of new results.

Introduction

Existence of solutions of many nonlinear equations which arise in applications can be studied in an abstract setting by considering a fixed point equation $x = Ax$. One of the powerful tools for the study of existence of solutions for the fixed point equation is a theory of fixed point index. Such theories are not only applicable to the study of existence of one or several solutions but also can be applied to treating other problems, for example, existence of positive solutions and existence of eigenvalues for equations such as $x = \lambda Ax$.

There are two classes of maps for which theories of fixed point index have been established: one is the class of condensing maps including compact maps and strict-set (or ball)-contractive maps and another is the class of maps A such that $I - A$ is an A-proper map. To mention a key requirement in defining the classical fixed point indexes, we simply outline, for example, the definition of fixed point index for a compact map. Let K be a closed convex set in a Banach space X and D a bounded open set in X such that $D_K = D \cap K \neq \emptyset$. Let $A : \overline{D}_K \to K$ be a compact map such that $x \neq Ax$ for $x \in \partial D_K$. Then one defines a fixed point index of A over D_K relative to K by the equation

$$i_K(A, D_K) = \deg(I - Ar, r^{-1}(D_K \cap B(\rho)), 0),$$

where r is a retraction from X onto K, $B(\rho) = \{x \in X : \|x\| < \rho\} \supseteq \overline{D}_K$ and $\deg(I - Ar, r^{-1}(D_K \cap B(\rho)), 0)$ is the Leray-Schauder degree. To show that the definition is reasonable, a main problem which needs to be solved is that $x \neq Ax$ must imply $x \neq Arx$ for every retraction r. In the past this has been achieved by requiring the image points of A belong to K. Therefore, a key restriction in defining the classical fixed point indices is that image points of the maps should belong to the closed convex set K. In this introduction such a map is called a self-map.

However, many fixed point theorems and nonzero fixed point theorems are known to be valid for nonself-maps, that is, they may take their values outside the closed convex set K, for example, fixed point theorems for inward or weakly inward maps obtained by Halpern and Bergman [26], Browder [7], Fan [18], Caristi [9], Reich [56], [57] [58], Webb [66] and Deimling [12]; nonzero fixed point theorems obtained by Deimling [14], [15], Deimling and Hu [16], Lan [38], and fixed point theorems for maps with weaker conditions than weakly inwardness obtained by Browder and Petryshyn [8] and Williamson [70].

Most of results mentioned above are related to maps defined on compact convex sets or to condensing maps defined in conical shells. The methods used are varied, for example, Browder's fixed point theorem [7], KKM intersection theorem [18], extension of maps [14], [16] and [38] and others. However, we note that it seems to be difficult to employ these methods to treat existence of multiple solutions and eigenvalue problems which are studied by using the classical fixed point index theories. Moreover, there have been no results related to weakly inward A-proper maps.

Therefore, it is a natural problem to ask whether the classical theories of fixed point index for condensing maps or A-proper maps can be generalized to maps which may take their values outside the closed convex sets involved.

Recently, there has been progress in extending the classical fixed point index for compact maps to weakly inward maps. Sun and Sun [63] first defined a fixed point index for a weakly inward continuous map defined on a compact convex set in a strictly convex Banach space by approximating on shrinking neighbourhoods of the compact convex set. Hu and Sun [28] defined a fixed point index for a compact map defined on a suitable closed convex set for which there is a retraction with a certain property (P). Such a retraction often exists but it is not shown in [28] that two retractions with property (P) give the same definition of index and it seems to be difficult to show this. The methods used in [63] and [28] employ the Leray-Schauder degree by considering the map Ar.

It is also worth mentioning that there are relatively few applications of these abstract theories and results for weakly inward maps to concrete equations, several applications can be found in [12].

Therefore, problems which need to be solved are as follows.

(1) Is it possible to give an unambiguous definition of index for weakly inward compact maps or more general compact maps?

(2) Is it possible to define a fixed point index for maps of condensing type which are nonself-maps?

(3) Is it possible to define a fixed point index for weakly inward A-proper maps?

(4) Is it possible to give more applications of the new results to nonlinear equations?

This thesis is devoted to the study of the four problems mentioned above.

We shall utilize the classical fixed point indices to define a new fixed point index for maps of condensing type which are nonself-maps and contain weakly inward maps. Moreover, we use the new fixed point index for weakly inward continuous maps in finite dimensional Banach spaces to define a new fixed point index for weakly inward A-proper maps. All the definitions of fixed point index we give are unambiguous. We also provide applications of new fixed point theorems and new nonzero fixed point theorems to concrete nonlinear equations.

The main idea we apply in defining a fixed point index for maps of condensing type which are nonself-maps is to pull their values into the closed convex set by suitable retractions onto the closed convex set. Roughly speaking, we shall define a fixed point index of A as the fixed point index of the map rA, where r is a suitable retraction onto the closed convex set. Since the image points of the map rA belong to the closed convex set, the classical fixed point indices can be employed. Obviously, the method is different from those mentioned above, where the Leray-Schauder degree is utilized.

In the process of defining the new fixed point index for nonself-maps of condensing type, one key problem which needs still to be solved is that $x \neq Ax$ must imply $x \neq rAx$. However, when the image points of A may be outside the closed convex sets, a main difficulty faced is that we do not know whether the fact that $x \neq Ax$ does imply $x \neq rAx$ for all retractions r. This leads us to impose suitable restrictions on A and on the closed convex set K.

We mention these restrictions in the following.

(i) The restrictions we give in defining a fixed point index for compact maps are that the closed convex set involved is an approximatively compact set and A is a generalized inward map.

The concept of an approximatively compact set was first introduced and studied by Efimov and Stechkin (see for example [62]) and then by Singer [62]. The concept was also generalized by Reich [57] to Hausdorff topological vector spaces and used by Reich [57] and Sehgal and Singh [61] to study best approximation problems and existence of fixed points for weakly inward maps in Hausdorff topological vector spaces. We employ this type of set here to assure existence of an upper semicontinuous multivalued metric projection, which enables us to apply any retraction r in our definition of fixed point index for compact maps.

The concept of a generalized inward map is a new concept which will be introduced in Chapter 2. The class of generalized inward maps strictly contains the class of weakly inward maps. Conditions very similar to our generalized inward condition have been previously used in the study of fixed points of maps, for example, in Hilbert spaces by Browder and Petryshyn [8], and for set-valued maps in locally convex spaces by Reich [58].

(ii) The restrictions we provide in defining a fixed point index for maps of condensing type are that the closed convex set involved is an M_l-set and A is a generalized inward map.

The concept of an M_l-set is a new one which will be introduced in Chapter 2. We introduce the concept of an M_l-set to assure existence of a continuous single-valued metric projection which is l-γ-contractive for some $l \in [1, \infty]$.

Because of these restrictions, our theories of fixed point index for generalized inward compact maps or generalized inward maps of condensing type can not apply to all closed convex sets in an arbitrary Banach space although they do in many special Banach spaces, for example, reflexive Banach spaces with property (H). However, they coincide with old indices when the maps are self-maps.

As an application of the new theory, we consider existence of positive solutions of the perturbed Volterra equation of the form

$$x(t) = g(t, x(t)) + \int_0^t f(s, x(s))\, ds, \quad t \in [0, 1] \tag{0.1}$$

in $L^2(0, 1)$. We allow g to take negative values but satisfy $g(t, 0) \geq 0$. Since the standard positive cone has empty interior Theorem 20.4 in [12] can not be applied to treat the above equation.

The methods we apply in defining the above new indices can not be used to define the fixed point index for weakly inward A-proper maps due to the fact that it is not clear whether rA inherits the A-properness of A, that is, if $I - A$ is A-proper, we don't know whether $I - rA$ is still A-proper. Therefore, the previous index theory for A-proper maps developed by Fitzpatrick and Petryshyn [22] can not be employed.

Fortunately, we can show that P_nA inherits the weakly inward property from A, that is, if A is weakly inward, so is P_nA. This is a key for us to develop a fixed point index for weakly inward A-proper maps. This enables us to employ the new index theory for weakly inward continuous maps in finite dimensional spaces to define a fixed point index for weakly inward A-proper maps. However, we don't know whether P_nA inherits the generalized inwardness of A. Hence, a fixed point index for a generalized inward A-proper map is not established in this thesis.

The new fixed point index for weakly inward A-proper maps coincides with the old one given in [22] when the maps are self-maps. The new theory of fixed point index for weakly inward A-proper maps is applicable to closed convex sets in a Banach space with a suitable projection scheme satisfying a mild restriction. The space is separable but need not be reflexive. It is known that many spaces which arise in applications have such a projection scheme.

As an application of the new theory we consider existence of nonzero positive solutions for the following boundary value problem of the form

$$x''(t) + f(t, x, x', x'') = 0, \quad x(0) = x(1) = 0. \tag{0.2}$$

This problem has been studied, for example, in [36], where f is a positive function. Using our theory we generalize the result obtained in [36] in several ways, in particular, we allow f to take negative values.

In Chapter 1 we recall some basic facts about the classical fixed point indices for compact maps and γ-condensing maps, which are indispensable in defining our new indices, establishing new fixed point theorems and providing applications of our new theories.

In Chapter 2 we study closed convex sets for which there exist an upper semicontinuous multivalued metric projection or a continuous single-valued metric projection and introduce the concept of a generalized inward map.

In Chapter 3 we define a fixed point index for a generalized inward map defined on an approximatively compact convex set and establish new fixed point theorems. In particular, we obtain norm-type expansion and compression theorem for weakly inward continuous maps in finite dimensional Banach spaces, which were previously thought to be impossible. We give suitable additional conditions so that the results just mentioned hold.

In Chapter 4 we define a fixed point index for generalized inward maps of condensing type and give an application of the theory to the above equation (0.1).

In Chapter 5 we define a fixed point index for weakly inward A-proper maps and give an application of the theory to the above equation (0.2).

Chapter 1

Classical Fixed Point Indices

In this chapter we first recall the concepts of cones and wedges which are indispensable to studying existence of nonzero fixed points, in particular, nonzero positive fixed points for maps. Then we recall the definitions of the classical fixed point index for compact maps, strict-γ-contractive maps and γ-condensing maps and mention their properties. Our new index theories will be developed on the basis of the classical index theories.

1.1 Wedges in Banach spaces

In this section we recall the concepts of *cone*, *wedge* and of *dual wedge* in a real Banach space and give some examples and properties, some of which are new. For further study see for example [4], [11], [12], [24], [33], [37] and [52].

Let X be a real Banach space. If S and V are subsets of X and α and β are real numbers, we define

$$\alpha S + \beta V = \{\alpha s + \beta v : s \in S, v \in V\}.$$

Let K be a subset of X. K is said to be *convex* if $\lambda K + (1-\lambda)K \subset K$ whenever $0 \leq \lambda \leq 1$. If Q is a subset of X, the smallest convex set containing Q, denoted by $\operatorname{co} Q$, is called the convex hull of Q. It is known that $\operatorname{co} Q = \{\sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and } x_i \in Q\}$.

Definition 1.1.1. A nonempty *closed* convex set K of X is said to be a *cone* if the following conditions hold.

(*i*) $\lambda K \subset K$ for all $\lambda \geq 0$.

(ii) $K \cap (-K) = \{0\}$.

The concept of a cone is closely related to the concept of a partial order in a vector space.

Definition 1.1.2. Let X be a vector space. An order relation $\leq$ in X is said to be a partial order if it satisfies

(P_1) (Reflexivity) $x \leq x$ for every $x \in X$.

(P_2) (Transitivity) for $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

(P_3) (Antisymmetry) for $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.

Definition 1.1.3. Let X be a vector space. An order relation $\leq$ in X is said to be a linear order if it satisfies

(L_1) for $x, y \in X$, if $x \leq y$, then $x + z \leq y + z$ for all $z \in X$.

(L_2) for $x, y \in X$, if $x \leq y$, then $\lambda x \leq \lambda y$ for $\lambda \geq 0$.

Now, let K be a cone in a Banach space X, one can define an order relation $\leq$ in X given by

$$x \leq y \quad \text{if and only if } y - x \in K.$$

It is easy to verify that the order relation $\leq$ is a linear partial order. Moreover, since Definition 1.1.1 always requires that a cone is closed, the linear partial order satisfies

(P_4) If $x_n \geq 0$ and $x_n \to x$, then $x \geq 0$.

Remark 1.1.4. Let $\leq$ be a linear partial order in a normed vector space X. Let $K = \{x \in X : x \geq 0\}$. Then K is a convex set in X and satisfies (i) and (ii) in Definition 1.1.1. However, K may not be closed.

Now, we introduce the concept of a normal cone.

Definition 1.1.5. A cone K is said to be normal if there exists $\beta > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \beta \|y\|$.

The following result, which can be found, for example in Theorem 1.1.1 in [24], gives a necessary and necessary condition for a cone to be normal. We give a simple and direct proof.

Theorem 1.1.6. *Let K be a cone in X. Then the following assertions are equivalent.*

(i) K is normal.

(ii) There exists $\delta > 0$ such that for all $x, y \in K$

$$\|x+y\| \geq \delta \max(\|x\|, \|y\|).$$

Proof. Assume that K is normal. Then there exists $\beta > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \beta\|y\|$. Let $x, y \in K$ and $\delta = 1/\beta$. Then $0 \leq x \leq x+y$ and $0 \leq y \leq x+y$. It follows from the normality of K that

$$\delta\|x\| \leq \|x+y\| \quad \text{and } \delta\|y\| \leq \|x+y\|.$$

This implies $\|x+y\| \geq \delta \max(\|x\|, \|y\|)$. On the other hand, assume that (ii) holds. Let $0 \leq x \leq y$ and $v = y - x$. Then $x, v \in K$. It follows from (ii) that

$$\|y\| = \|x+v\| \geq \delta \max\{\|x\|, \|v\|\} \geq \delta\|x\|.$$

Let $\beta = 1/\delta$. Then $\|x\| \leq \beta\|y\|$. □.

Obviously, if (ii) holds, then $0 < \delta \leq 1$. We call δ the *normality constant* of K.

In some situations we need to know if a cone has nonempty interior. We give some examples of cones and show whether they are normal and whether they have nonempty interior.

Example 1.1.7. Let $X = \mathbb{R}^n$. Then $\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, ..., n\}$ is a normal cone with $\delta = 1$ and has nonempty interior. In fact, $x = \{x_1, ..., x_n\}$ is an interior point of $\mathbb{R}^n_+$ if and only if $x_i > 0$ for all $i = 1, ..., n$.

Example 1.1.8. Let $X = C(\overline{\Omega})$, the Banach space of all continuous real-valued functions on $\overline{\Omega}$ with the usual maximum norm, where Ω is a bounded open set in $\mathbb{R}^n$. Let $C_+(\overline{\Omega}) = \{f \in C(\overline{\Omega}) : f(x) \geq 0 \quad \text{for all } x \in \overline{\Omega}\}$. Then $C_+(\overline{\Omega})$ is a normal cone with $\delta = 1$ and has nonempty interior. In fact, f is an interior point of $C_+(\overline{\Omega})$ if and only if $f(x) > 0$ for every $x \in \overline{\Omega}$.

Example 1.1.9. Let $X = L^p(\Omega)$ $(1 \leq p < +\infty)$, the Banach space of all (equivalence classes of) measurable, real-valued functions on Ω whose pth powers are integrable, together with the usual norm,

$$\|f\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p \, dx\Big)^{1/p},$$

where Ω is a Lebesgue measurable subset of $\mathbb{R}^n$ of positive measure. Let

$$L_+^p(\Omega) = \{f \in L^p(\Omega) : f(x) \geq 0 \quad \text{for almost all } x \in \Omega\}.$$

Then $L_+^p(\Omega)$ is a normal cone with $\delta = 1$ and has empty interior.

To show that $L_+^p(\Omega)$ has empty interior, it suffices to prove that for $\varepsilon > 0$ and $f \in L_+^p(\Omega)$, there exists $g \in L^p(\Omega) \setminus L_+^p(\Omega)$ such that $\|f - g\| < \varepsilon$. In fact, if $f = 0$, we define $g(x) := -a$ for all $x \in \overline{\Omega}$, where $0 < a < \varepsilon \,\mathrm{meas}(\Omega)^{-1/p}$. Then g satisfies the required conditions. If $f \neq 0$, it follows from a well known result on absolute continuity of Lebesgue integration that there exists $\delta > 0$ such that for any $E \subset \overline{\Omega}$ with $\mathrm{meas}(E) < \delta$,

$$\Big(\int_E f^p(x)\,dx\Big)^{1/p} < \varepsilon/2.$$

Since $f \neq 0$, we can find $E_0 \subset \overline{\Omega}$ with $\mathrm{meas}(E_0) < \delta$ such that $f(x) > 0$ for all $x \in E_0$. We define $g(x) = f(x)$ if $x \in \overline{\Omega} \setminus E_0$ and $g(x) = -f(x)$ if $x \in E_0$. Then $g \in L^p(\Omega) \setminus L_+^p(\Omega)$ and $\|f - g\| = 2(\int_{E_0} f^p(x)\,dx)^{1/p} < \varepsilon$.

Example 1.1.10. Let $X = L^\infty(\Omega)$, the Banach space of all essentially bounded, real-valued, measurable functions on Ω with the norm

$$\|f\|_{L^\infty(\Omega)} = \inf_{\Omega_0 \subset \Omega, \mathrm{meas}(\Omega_0) = 0} \big(\sup_{x \in \Omega \setminus \Omega_0} |f(x)|\big).$$

where Ω is a Lebesgue measurable subset of $\mathbb{R}^n$ of positive measure. Let

$$L_+^\infty(\Omega) = \{f \in L^\infty(\Omega) : f(x) \geq 0 \quad \text{for almost all } x \in \Omega\}.$$

Then $L_+^\infty(\Omega)$ is a normal cone with $\delta = 1$ and has nonempty interior. For example, the function $f(x) \equiv 1$ is an interior point.

Actually, we can prove the following new result.

Theorem 1.1.11. *Let $L^\infty(\Omega)$ and $L_+^\infty(\Omega)$ be as in Example 1.1.10. Then the following are equivalent.*

(1) *f is an interior point of $L_+^\infty(\Omega)$.*

(2) *There exists $\gamma > 0$ such that $f(x) \geq \gamma$ a.e. on Ω.*

Proof. Assume that f is an interior point of $L^\infty_+(\Omega)$. Then there exists $\varepsilon > 0$ such that $g \in L^\infty(\Omega)$ and $\|f - g\| < \varepsilon$ imply $g \in L^\infty_+(\Omega)$. Now, let $\gamma = \varepsilon/2$ and $g_0(x) = f(x) - \gamma$ for $x \in \overline{\Omega}$. Then $g_0 \in L^\infty(\Omega)$ and $\|f - g_0\| < \varepsilon$. Therefore, $g_0 \in L^\infty_+(\Omega)$. This implies $f(x) \geq \gamma$ a.e. on Ω. On the other hand, assume that there exists $\gamma > 0$ such that $f(x) \geq \gamma$ a.e. on Ω. Then there exists $\Omega_1 \subset \overline{\Omega}$ such that $\text{meas}(\Omega_1) = 0$ and $f(x) \geq \gamma$ for all $x \in \overline{\Omega} \setminus \Omega_1$. Let $\varepsilon = \gamma/2$ and $g \in L^\infty(\Omega)$ be such that $\|f - g\| < \varepsilon$. Then there exists $\Omega_0 \subset \overline{\Omega}$ with $\text{meas}(\Omega_0) = 0$ such that $\sup_{x \in \overline{\Omega} \setminus \Omega_0} |f(x) - g(x)| < \varepsilon$. This implies $\sup_{x \in \overline{\Omega} \setminus (\Omega_0 \cup \Omega_0)} |f(x) - g(x)| < \varepsilon$. Therefore, $g(x) > f(x) - \varepsilon \geq 0$ for all $x \in \overline{\Omega} \setminus (\Omega_0 \cup \Omega_0)$. Since $\text{meas}(\Omega_0 \cup \Omega_0) = 0$, we have $g(x) \geq 0$ a.e. on Ω. □

Example 1.1.12. Let Ω be a bounded open set in $\mathbb{R}^n$ and let $C^m(\Omega)$ be the Banach space of all m times continuously differentiable functions $u : \overline{\Omega} \to \mathbb{R}$ with the norm

$$\|u\|_m = \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha u(x)|,$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index and $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$. Then the cone $C^m_+(\overline{\Omega}) = \{u \in C^m(\overline{\Omega}) : u(x) \geq 0 \text{ for } x \in \overline{\Omega}\}$ has nonempty interior but is not normal (see Example 1.14 in [4]).

Example 1.1.13. Let Ω be a bounded open set in $\mathbb{R}^n$ and $n \in \mathbb{N}$. For $1 \leq p < \infty$ and $m \in \mathbb{N}$, let $W^{m,p}(\Omega)$ be the Sobolev space with the norm

$$\|u\|_{m,p} = \Big(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^p \, dx \Big)^{1/p},$$

where $D^\alpha u \in L^p(\Omega)$ are the generalized derivatives for $|\alpha| \leq m$. Then the cone $W^{m,p}_+ = \{u \in W^{m,p}(\Omega) : u(x) \geq 0 \text{ a.e. on } \Omega\}$ is not normal. In general, $W^{m,p}_+(\Omega)$ has empty interior. However, if $mp > n$ and $\partial\Omega$ is sufficiently smooth, then $W^{m,p}_+(\Omega)$ has nonempty interior. See Example 1.15 in [4] for details.

Remark 1.1.14. In Example 1.1.13, if $n = m = p = 2$ and $\partial\Omega$ is sufficiently smooth, then the cone $W^{2,2}_+(\Omega)$ in the Hilbert space $W^{2,2}(\Omega)$ is not normal. Hence, not every cone in a Hilbert space is normal.

Now, we introduce the concept of a minihedral cone.

Let X be a vector space and $\leq$ a partial order.

Definition 1.1.15. Let D be a subset of X. An element $y \in X$ is called the least upper bound of D if it satisfies the following two conditions.

(i) y is an upper bound of D, that is, $x \leq y$ for all $x \in D$.

(ii) $x \leq z$ for all $x \in D$ and some $z \in X$ implies $y \leq z$.

We denote the least upper bounded of D by $\sup D$. Similarly, one can define the greatest lower bound of D which is denoted by $\inf D$.

Note that since every partial order is antisymmetric, the least upper bound (or greatest lower bound) of D is unique.

Definition 1.1.16. A cone K in a Banach space X is said to be minihedral if $\sup\{x, y\}$ exists for $x, y \in X$.

Often, $\sup\{x, y\}$ is denoted by $x \vee y$ and $\inf\{x, y\}$ by $x \wedge y$.

Remark 1.1.17. A Banach space X with a minihedral cone K is a vector lattice (or a Riesz space). We refer to [2] and [3] for more details on the theory of lattices.

The following result gives sufficient and necessary conditions for a cone to be minihedral.

Proposition 1.1.18. *Let K be a cone in X. Then the following are equivalent.*

(i) *K is minihedral.*

(ii) $\sup\{x, 0\}$ *exists for $x \in X$.*

(iii) $\inf\{x, y\}$ *exists for $x, y \in X$.*

(iv) $\inf\{x, 0\}$ *exists for $x \in X$.*

Proof. Obviously, (i) implies (ii). We prove that (ii) implies (i). Let $x, y \in X$. We prove that

$$x \vee y = (x - y) \vee 0 + y.$$

In fact, since $x = (x - y) + y \leq (x - y) \vee 0 + y$ and $y \leq (x - y) \vee 0 + y$, we have

$$x \vee y \leq (x - y) \vee 0 + y.$$

Hence $(x - y) \vee 0 + y$ is a upper bound of $\{x, y\}$. Now, let $z \in X$ be such that $x \leq z$ and $y \in z$. Then $x - y \leq z - y$ and $0 \leq z - y$. This implies $(x - y) \vee 0 \leq z - y$ and $(x - y) \vee 0 + y \leq z$. Therefore, $x \vee y = (x - y) \vee 0 + y$.

Similarly, one can verify that (iii) and (iv) are equivalent. Finally, (i) and (iii) are equivalent since $\inf\{x, y\} = -\sup\{-x, -y\}$ for $x, y \in X$. □

Remark 1.1.19. The equivalence of (i) and (ii) can be found in Theorem 1.1 in [3], where it is obtained in a Riesz space. The equivalence of (i) and (iii) is mentioned in p.64 in [24] without proof.

Let K be a minihedral cone in a Banach space X. We define the positive part , the negative part , and the absolute value of an element x, respectively, by

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad \text{and} \quad |x| = x \vee (-x).$$

It is easy to verify that (i) $x = x^+ - x^-$, (ii) $x \leq x^+ \leq |x|$, (iii) $-x \leq x^- \leq |x|$, (iv) $|x| = |-x|$ and (v) $|x| \leq y$ if and only if $-y \leq x \leq y$.

We prove that the following useful result.

Proposition 1.1.20. *Let K be a minihedral cone in a Banach space X. Then*

$$|x^+ - y^+| \leq |x - y| \quad \textit{for } x, y \in X.$$

Proof. Let $x, y \in X$. Since $x = x - y + y \leq |x - y| + y \leq |x - y| + y^+$, we have $x^+ \leq |x - y| + y^+$. Similarly, we have $y^+ \leq |y - x| + x^+$. It follows that

$$-|x - y| \leq x^+ - y^+ \leq |x - y|.$$

This implies $|x^+ - y^+| \leq |x - y|$. □

Remark 1.1.21. Proposition 1.1.20 is given in Theorem 24.1 in [2]. However, our proof is more direct.

Definition 1.1.22. A Banach space X with a minihedral cone is said to a Banach lattice if its norm $\|.\|$ satisfies

(i) $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$, and

(ii) $\|x\| = \||x|\|$ for all $x \in X$.

Note that conditions (i) and (ii) are equivalent to $|x| \leq |y|$ implies $\|x\| \leq \|y\|$.

Remark 1.1.23. The above definition can be found, for example in p. 191 and Exercise 4 in [2].

A Banach lattice has an important property.

Theorem 1.1.24. *Let (X, K) be a Banach lattice. Then*

$$\|x^+ - y^+\| \leq \|x - y\| \quad \textit{for all } x, y \in X.$$

In the following we give examples of Banach lattices.

Example 1.1.25. Let $X = C(\overline{\Omega})$ and $K = C_+(\Omega)$ be as in Example 1.1.8. Then (X, K) is a Banach lattice.

Example 1.1.26. Let $X = L^p(\Omega)$ $(1 \leq p \leq +\infty)$ and $K = L^p_+(\Omega)$ be as in Examples 1.1.9 and 1.1.10. Then (X, K) is a Banach lattice for $1 \leq p \leq \infty$.

There are many other examples of Banach lattices, see for example [2].

Now, we generalize the concept of a cone to a wedge.

Definition 1.1.27. A nonempty closed convex set K of X is said to be a wedge if $\lambda K \subset K$ for all $\lambda \geq 0$.

It is easy to see that every cone is a wedge. However, the converse is false. For example, every closed subspace of X is a wedge but not a cone. In particular, a real Banach space itself is a wedge.

For every wedge K in X, one can define an order relation $\leq$ in X given by

$$x \leq y \quad \text{if and only if } y - x \in K.$$

It is easy to verify that the order relation $\leq$ is reflexive, transitive and linear; but it may not be antisymmetric.

There are wedges which are neither cones nor subspaces. We give the following new example.

Example 1.1.28. Let $X = \mathbb{R}^2$ and $K = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, y \in [0, +\infty)\}$. Then K is a wedge but is neither a cone nor a subspace of X. Moreover, K has nonempty interior. In fact, $z = (x, y)$ is an interior point of K if and only if $y > 0$.

Actually, we have a more general result: Let P be a cone in a Banach space X and W a wedge in a Banach space Y but not a cone. Then $P \times W$ is a wedge in the product space $X \times Y$ which is neither a cone nor a subspace of X.

Therefore, the class of wedges can be divided into three subclasses (see the following Proposition 1.1.29). Sometimes we need to know whether a wedge is a subspace. We collect the following simple properties, some of which were first observed and employed by the author in [39]-[41] and used in [43].

Proposition 1.1.29. *Let K be a wedge in X. Then the following statements are true.*

(i) K is a subspace of X if and only if $K = -K$.

(ii) If $K \neq -K$, then $K \setminus (-K) = K \setminus K \cap (-K) \neq \emptyset$.

(iii) One of the following three cases must occur:

(1) K is a cone, that is, $K \cap (-K) = \{0\}$;

(2) K is a subspace of X, that is, $K = -K$; or

(3) K is neither a cone nor a subspace of X, that is, $K \cap (-K) \neq \{0\}$ and $K \neq -K$.

Remark 1.1.30. K satisfies (1) and (2) if and only if $K = \{0\}$.

The following new Proposition shows that the concept of normality of a cone does not extend to a wedge which is not a cone.

Proposition 1.1.31. *Let K be a wedge in X but not a cone. Then for every $\delta > 0$ there exist $x, y \in K$ such that*

$$\|x + y\| < \delta \max(\|x\|, \|y\|).$$

Proof. Since K is a wedge but not a cone, it follows from Proposition 1.1.29 that there exists $x \in K \setminus \{0\}$ such that $-x \in K$. Let $y = -x$. Then we have

$$\|x + y\| = 0 < \delta\|x\| = \delta \max(\|x\|, \|y\|).$$

□

When $K \neq -K$, we prove the following new property for a wedge which generalizes a result in [37] from a cone to a wedge.

Theorem 1.1.32. *Let K be a wedge with $K \neq -K$. Then for every $u \in K \setminus (-K)$, there exists $\mu(u) > 0$ such that*

$$\|x + u\| \geq \mu(u)\|x\| \quad \textit{for all } x \in K. \tag{1.1}$$

Proof. The proof is by contradiction. Assume that the result is false. Then there exist $u \in K \setminus (-K)$ and $\{x_n\}_{n \in \mathbb{N}} \subset K$ such that

$$\|x_n + u\| < (1/n)\|x_n\|. \tag{1.2}$$

This implies that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. In fact, if not, there exists a subsequence $\{x_{n_k}\}$ such that $\|x_{n_k}\| \to \infty$. This implies

$$\|1 + u/\|x_{n_k}\|\| \to 1.$$

On the other hand, the above inequality (1.2) implies $\|1+u/\|x_{n_k}\|\| < 1/n$. Thus we have $\|1 + u/\|x_{n_k}\|\| \to 0$, a contradiction. The inequality (1.2) and boundedness of $\{\|x_n\|\}$ imply $x_n + u \to 0$. This implies $u \in -K$, which contradicts $u \in K \setminus (-K)$. $\square$

Theorem 1.1.32 can not be extended to the case when K is a subspace, that is, K satisfies $K = -K$. We give the following

Remark 1.1.33. Let K be a wedge with $K = -K$. Then for every $u \in K \setminus \{0\}$ and $\mu > 0$ there exists $x_0 \in K \setminus \{0\}$ such that

$$\|x_0 + u\| < \mu\|x_0\|.$$

In fact, since $K = -K$, for every $u \in K \setminus \{0\}$, one may choose $x_0 = -u$. Then x_0 satisfies the requirement.

Let K be a wedge with $K \neq -K$. For every $u \in K \setminus \{0\}$, we define

$$\delta(u) = \inf\{\|x + u\|/\|x\| : x \in K \setminus \{0\}\}.$$

It follows from Theorem 1.1.32 that for every $u \in K \setminus (-K)$, we have $\delta(u) > 0$. If $K \cap (-K) \neq \{0\}$, then it is easy to verify that $\delta(u) = 0$ if $u \in K \cap (-K)$ and $u \neq 0$.

Now, we define

$$\delta(K) = \sup\{\delta(u) : u \in K \setminus \{0\}\}.$$

and call it the *quasinormality constant* of K. The following result gives the estimate of the lower bound of $\delta(K)$, which is due to Dancer, Nussbaum and Stuart (see Theorem 1.1 in [11]).

Theorem 1.1.34. *Let K be a wedge with $K \neq -K$. Then $\delta(K) \in [1/2, 1]$.*

Proof. Since $K \neq -K$, there exists $x_0 \in K$ and $-x_0 \notin K$. It follows from the Hahn-Banach theorem that there exists $f \in X^*$ such that $f(x_0) > 0$ and $f(x) \geq 0$ for all $x \in K$. Let $m = \inf\{f(x) : x \in K \text{ with } \|x\| = 1\}$ and $M = \sup\{f(x) : x \in K \text{ with } \|x\| = 1\}$. Then there exist $\{u_n\} \subset K$ with $\|u_n\| = 1$ such that $f(u_n) \equiv M_n \to M$. Note that $M > 0$. For every $x \in K$, we have

$$\|x + u_n\| \geq f(x + u_n)/M \geq \|x\|(m/M + M_n/M\|x\|). \tag{1.3}$$

If $m = M$, then $\|x + u_n\| \geq \|x\|$. This implies $\delta(K) = 1$. Now assume that $m < M$. If $\|x\| \leq R$, the inequality (1.3) implies

$$\|x + u_n\| \geq \|x\|(m/M + M_n/MR). \tag{1.4}$$

On the other hand, if $\|x\| > R$, we have

$$\|x + u_n\| \geq \|x\| - 1 \geq \|x\|(1 - 1/R). \tag{1.5}$$

We choose $R > 0$ such that $(1 - 1/R) = m/M + M_n/MR$, that is,

$$R = (1 + M_n/M)(1 - m/M)^{-1}.$$

With this choice of R, inequalities (1.4) and (1.5) imply

$$\delta(K) \geq \delta(u_n) \geq 1 - (1 - m/M)(1 + M_n/M)^{-1}.$$

Then $M_n \to M$ implies

$$\delta(K) \geq 1/2 + m/2M \geq 1/2.$$

□

In [11] there is an example of a cone K with $\delta(K) = 1/2$. Hence the lower bound $\frac{1}{2}$ of $\delta(K)$ is the best possible.

In applications we will have greatest flexibility in wedges K with $\delta(K) = 1$. Note that if K is a normal cone with the normality constant δ, then its quasinormality constant $\delta(K) \geq \delta$. Hence, all the cones in Example 1.1.7-1.1.10 have the quasinormality constant 1. Also, it is easy to verify that all the cones in Examples 1.1.12-1.1.13 have the quasinormality constant 1 although they are not normal.

We mentioned earlier (see Remark 1.1.14) that not every cone in a Hilbert space is normal but it can be shown that every cone in a Hilbert space has the quasinormality constant 1.

Proposition 1.1.35. *Let K be a wedge with $K \neq -K$ in a Hilbert space. Then there exists $u \in K \setminus (-K)$ such that for all $x \in K$,*

$$\|x + u\| \geq \|x\|.$$

The above proposition can be found in [11] (see Corollary 1.3 in [11]) and we omit its proof.

In many textbooks (for example, [12] and [24]) one can find the concept of a reproducing cone. We generalize the concept to a wedge.

Definition 1.1.36. Let K be a wedge in X. K is said to be reproducing if $K - K = X$; K is said to be total if $\overline{K - K} = X$.

The following result give necessary and sufficient conditions for a wedge to be reproducing. A similar result in topological vector spaces can be found in [71] (see Proposition 1.2 in [71]).

Theorem 1.1.37. *Let K be a wedge in X. Then the following conditions are equivalent.*

(1) *K is reproducing.*

(2) *For every $x \in X$, there exists $y \in K$ such that $x \leq y$.*

(3) *For any $x, y \in X$, there exists $z \in K$ such that $x \leq z$ and $y \leq z$.*

(4) *For any $x, y \in X$, there exists $z \in X$ such that $x \leq z$ and $y \leq z$.*

Proof. Obviously, (1) and (2) are equivalent and (3) implies (4). We prove (2) implies (3) and (4) implies (2). Assume that (2) holds. Then for any $x, y \in X$, it follows from (2) that there exist $w, v \in K$ such that $x \leq w$ and $y \leq v$. Let $z = w + v$. Since K is a wedge, we have $z \in K$, $z - x = v + (w - x) \in K$ and $z - y = w + (v - y) \in K$. Therefore, (3) holds. Now, assume that (4) holds. Let $x \in X$. For x and 0, it follows from (4) that there exists $y \in X$ such that $x \leq y$ and $0 \leq y$. Hence $y \in K$ and (2) holds. □

Remark 1.1.38. By Theorem 1.1.37 we see that a wedge K in X is reproducing if and only if any two elements $x, y \in X$ have an upper bound. Therefore, every minihedral cone is reproducing.

It is known that if a cone has nonempty interior, then it is reproducing (see for example, Theorem 1.1.4 in [24]). The result can be easily extended to a wedge.

Proposition 1.1.39. *Let K be a wedge in X. If K has nonempty interior, then it is reproducing.*

Proof. Let $x_0 \in K$ be an interior point of K. Then there exists $r > 0$ such that $\{x \in X : \|x - x_0\| \leq r\} \subset K$. For every $x \in X \setminus \{0\}$, let $u = x_0 + rx/\|x\|$ and $v = x_0 - rx/\|x\|$. Then $u, v \in K$ and $x = (u - v)\|x\|/2r$. This implies $x \in K - K$. □

According to Proposition 1.1.39, we see that Examples 1.1.7, 1.1.8, 1.1.10, 1.1.12 and 1.1.28 are reproducing. Note that in Example 1.1.9, $L^p_+(\Omega)$ has empty interior but it is reproducing. In fact, for every $f \in L^p(\Omega)$, we have $|f| \in L^p_+(\Omega)$ and $|f| - f \in L^p_+(\Omega)$.

The following new example provides a cone which is not reproducing.

Example 1.1.40. Let $X = \mathbb{R}^2$ and $K = \{(x, 0) : x \geq 0\}$. Then K is a cone in X , which has empty interior and is not reproducing.

Proof. Note that $K - K = \mathbb{R} \times \{0\}$. Hence, $X \neq K - K$. This shows that K is not reproducing. □

We end this section with the study of a dual wedge.

Given a wedge K in X one can define another wedge in the dual space X^* of X which is called a dual wedge of K. The latter will be useful when one wants to know whether a map is weakly inward (see Chapter 2).

Let X be a Banach space. We denote by X^* the dual space of X, that is, the Banach space of all continuous linear functionals defined on X. Let K be a wedge in X. Let $K^* = \{f \in X^* : f(x) \geq 0 \text{ for all } x \in K\}$. Then it is easy to verify that K^* is a wedge in

X^*. In general, K^* may not be a cone even when K is a cone and $X = \mathbb{R}^n$. The following example shows this.

Example 1.1.41. Let X and K be the same as in Example 1.1.40. Then $K^* = \{(x, y) : x \geq 0, y \in \mathbb{R}\}$ is a wedge but not a cone.

The following result gives a necessary and sufficient condition for K^* to be a cone.

Proposition 1.1.42. *Let K be a wedge in X. Then the following are equivalent.*
(1) *K is total, that is, $\overline{K - K} = X$.*
(2) *K^* is a cone in X^*.*

Proof. Assume that (1) holds. Let $f \in K^* \cap (-K^*)$. We prove $f = 0$. In fact, since $f \in K^* \cap (-K^*)$, $f(x) = 0$ for all $x \in K$. This implies $f(z) = 0$ for all $z \in \overline{K - K} = X$. On the other hand, assume that K^* is a cone. If $\overline{K - K} \neq X$, then for every fixed $x_0 \in X \setminus \overline{K - K}$, there exists $f \in X^*$ such that $f(x_0) > 0$ and $f(x) = 0$ for all $x \in \overline{K - K}$. The latter implies $f \in K^* \cap (-K^*)$. Since K^* is a cone, $f(x) = 0$ for all $x \in X$ and thus, $f(x_0) = 0$, a contradiction. □

When K is a cone, the above result is mentioned in section 19.2, pp. 221 in [12] without proof.

It follows from the representation of functionals that we can often determine K^*. We list several dual wedges here.

Example 1.1.43. (1) $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$.
(2) $(C_+([0, 1]))^* = \{f : [0, 1] \to \mathbb{R} : f$ is a nonnegative function of bounded variation$\}$.
(3) $(L^p_+(\Omega))^* = L^q_+(\Omega)$ for $1 \leq p < \infty$ and $1/p + 1/q = 1$.
(4) Let K be the same as in Example 1.1.28. Then $K^* = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$.

Remark 1.1.44. Note that not every cone K in a Hilbert space satisfies $K = K^*$. For example, in Example 1.1.41, $K \neq K^*$. For examples of $K = K^*$ in Hilbert spaces, see (1) and (3) in Example 1.1.43.

Comments

In this section the following results are new.

Theorem 1.1.11, Theorem 1.1.24, Example 1.1.28, Proposition 1.1.31, Theorem 1.1.32, Remark 1.1.33, Theorem 1.1.37, Proposition 1.1.39, Example 1.1.40, Example 1.1.41, Proposition 1.1.42 and Remark 1.1.44.

The proofs of the following known results are new.

Theorem 1.1.6, the assertion that $L^p(\Omega)$ $(1 \leq p < \infty)$ has empty interior in Example 1.1.9 and Proposition 1.1.20.

1.2 The classic fixed point index for compact maps

In this section we recall the definition of the previous fixed point index for compact maps. Such an index is a generalization of the classic Leray-Schauder degree for compact vector fields defined on the closure of open subsets of some Banach space (see for example, [12], [33], [34] and others), degree theory being one of the most important tools in nonlinear functional analysis. The fixed point index can be applied to the case where degree theory is not directly applicable, for example, when the closed convex set K involved has empty interior. The theory of fixed point index is also an important tool, in particular, in the study of existence of nonzero positive fixed points for nonlinear maps. Such a theory is developed on the basis of the classic Leray-Schauder degree theory and Dugundji's extension theorem of a compact map. We refer the reader to Amann's paper [4] for a detailed study. We only outline its definition and properties here.

Definition 1.2.1. Let D be a set in a Banach space X. A map $T : D \to X$ is said to be compact if it is continuous and $\overline{T(Q)}$ (the closure) is compact in X for every bounded set $Q \subset D$.

The following result is a special case of Theorem 18.3 in [34] which is often called the Dugundji extension theorem for a compact map.

Lemma 1.2.2. *Let D be a closed subset in a Banach space X and $A : D \to X$ a compact map. Then there exists a compact map $B : X \to X$ such that $B(X) \subset \overline{\text{co}}(A(D))$ and $Bx = Ax$ for $x \in D$.*

The following extension theorem [17] (also see Section 18 in [34]) can be used in the proof of the above result. We will also utilize this result later on.

Theorem 1.2.3. *Let K be a closed convex set in a Banach space X. Then for every $\varepsilon > 0$, there exists a continuous map $r_\varepsilon : X \to K$ which satisfies $r_\varepsilon x = x$ for $x \in K$ and*

$$\|x - r_\varepsilon x\| \leq (1+\varepsilon)d(x, K) \quad \textit{for } x \in X,$$

where $d(x, K) = \inf\{\|x - y\| : y \in K\}$ is the distance from x to K.

A continuous map $r : X \to K$ is called a retraction if it satisfies $rx = x$ for every $x \in K$. Theorem 1.2.3 shows that for every closed convex set K in a Banach space X there exists a retraction from X onto K. In other words, there exists a continuous extension of the identity map I on K. We call r_ε an *ε-retraction.*

Let K be a closed convex set in a Banach space X and D a bounded open set in X. We denote by $\overline{D}_K$ and ∂D_K the closure and the boundary, respectively, of $D_K = D \cap K$ relative to K. We also use ∂B to denote the boundary of a subset B of X relative to X.

We give the following new proposition which is useful in definition of fixed point index and applications.

Proposition 1.2.4. *Let K be a closed convex set in X and D an open set in X such that $D_K \neq \emptyset$. Then*

(i) $\partial D_K \subset \partial D \cap K \subset \partial D$, where ∂D denotes the boundary of D relative to X.

(ii) $\partial D_K = \emptyset$ if and only if $D_K = K$.

Proof. We only prove the 'if' part of (*ii*). Since $\partial D_K = \emptyset$, we have

$$K = D_K \cup \partial D_K \cup (K \setminus \overline{D}_K) = D_K \cup (K \setminus \overline{D}_K).$$

As K is convex, K is connected. So one of the sets must be empty. Since $D_K \neq \emptyset$, $K \setminus \overline{D}_K$ must be empty. Consequently, we have $D_K = K$. □

Let $A : \overline{D}_K \to X$ be a map. We write $x \neq A(x)$ for all $x \in \partial D_K$ to mean that either (*i*) $\partial D_K = \emptyset$ or (*ii*) if $\partial D_K \neq \emptyset$, $x \neq A(x)$ for all $x \in \partial D_K$.

We are now in a position to give the definition of fixed point index for compact maps.

Notation: $B(\rho) = \{x \in X : \|x\| < \rho\}$.

Definition 1.2.5. Let K be a closed convex set in a Banach space X and let D be a bounded open set such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to K$ be compact. Suppose that

$x \neq A(x)$ for all $x \in \partial D_K$. Define the fixed point index by the equation

$$i_K(A, D_K) = \deg(I - Ar, r^{-1}(D_K) \cap B(\rho), 0),$$

where r is a retraction from X onto K, $B(\rho) \supset \overline{D}_K$ and $\deg(I - Ar, r^{-1}(D_K) \cap B(\rho), 0)$ is the Leray-Schauder degree.

Remark 1.2.6. It follows from Theorem 1.2.3 that there exists at least one retraction r from X onto K. Since $x \neq Ax$ for $x \in \partial D_K$, one can prove that $x \neq Arx$ for $x \in \partial(r^{-1}(D_K) \cap B(\rho))$ and thus, the Leray-Schauder degree $\deg(I - Ar, r^{-1}(D_K) \cap B(\rho), 0)$ is well-defined. It can be shown that the degree is independent of the choice of the retraction r and the radius ρ. Hence, the index $i_K(A, D_K)$ makes sense.

Remark 1.2.7. Let K be a bounded closed convex set in X. Assume that $A : K \to K$ is compact. Then the index $i_K(A, K)$ makes sense and is defined by

$$i_K(A, K) = \deg(I - Ar, B(\rho), 0),$$

where r is a retraction from X onto K, $B(\rho) \supset K$ and $\deg(I - Ar, B(\rho), 0)$ is the Leray-Schauder degree.

In fact, since K is bounded, one can choose a bounded open set D in X such that $D \supset K$. Then $D_K = K \neq \emptyset$. By (ii) of Proposition 1.2.4, $\partial D_K = \emptyset$. Let r be a retraction from X onto K. Then $r^{-1}(D_K) = r^{-1}(K) = X$. Let $\rho > 0$ be such that $B(\rho) \supset D_K = K$. Then $r^{-1}(D_K) \cap B(\rho) = X \cap B(\rho) = B(\rho)$. It follows from Definition 1.2.5 that

$$i_K(A, K) = i_K(A, D_K) = \deg(I - Ar, r^{-1}(D_K) \cap B(\rho), 0) = \deg(I - Ar, B(\rho), 0).$$

Remark 1.2.8. When K is an unbounded closed convex in X, Definition 1.2.5 does not imply that $i_K(A, K)$ makes sense since Definition 1.2.5 is restricted to a bounded domain D_K of A.

The fixed point index in Definition 1.2.5 has four main properties which, in turn, uniquely determine the index (see, for example, [4] and [24]).

Theorem 1.2.9. *Let K be a closed convex set in a Banach space X and let D be a bounded open set such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to K$ be a compact map. Suppose that $x \neq A(x)$ for all $x \in \partial D_K$. The index as defined above has the following properties.*

(P_1) *(Existence) If* $i_K(A, D_K) \neq 0$*, then* A *has a fixed point in* D_K.

(P_2) *(Normalisation) If* $u \in D_K$*, then* $i_K(\hat{u}, D_K) = 1$*, where* $\hat{u}(x) = u$ *for* $x \in \overline{D}_K$.

(P_3) *(Additivity) If* W^1, W^2 *are disjoint relatively open subsets of* D_K *such that* $x \neq A(x)$ *for* $x \in \overline{D}_K \setminus (W^1 \cup W^2)$*, then*

$$i_K(A, D_K) = i_K(A, W^1) + i_K(A, W^2)$$

(P_4) *(Homotopy property) Let* $h : [0,1] \times \overline{D}_K \to K$ *be compact such that* $x \neq h(t, x)$ *for* $x \in \partial D_K$ *and* $t \in [0,1]$*. Then*

$$i_K(h(0,.), D_K) = i_K(h(1,.), D_K).$$

In the following we give two fixed point theorems; one will be used later and the other is new.

Theorem 1.2.10. *Let* K *be a bounded closed convex set in* X *Assume that* $A : K \to K$ *is compact. Then* $i_K(A, K) = 1$*, and thus,* A *has a fixed point in* K.

Proof. Let $B(\rho) \supset K$ for some $\rho > 0$ and r be a retraction from X onto K. Then $Ar : \overline{B}(\rho) \to K$ is compact. Let $H(t, x) = tArx$ for $t \in [0,1]$ and $x \in \overline{B}(\rho)$. We prove that $x \neq tArx$ for $t \in [0,1]$ and $x \in \partial B(\rho)$. In fact, if not, there exist $t \in [0,1]$ and $x \in \partial B(\rho)$ such that $x = tArx$. Hence we obtain

$$\rho = \|x\| = t\|Arx\| \leq \|Arx\| < \rho,$$

a contradiction. It follows from the homotopy property of degree for compact maps that $\deg(I - Ar, B(\rho), 0) = \deg(I, B(\rho), 0) = 1$. By definition 1.2.5 or Remark 1.2.7 we have $i_K(A, K) = 1$ and A has a fixed point in K. □

Remark 1.2.11. The index $i_K(A, K) = 1$ was used in Lemma 14.1, p. 666 in [4], where it is stated without proof that $i_K(A, K) = 1$ can be obtained from the proof of Schauder's fixed point theorem.

Remark 1.2.12. Theorem 1.2.10 is not true for an unbounded closed convex set. For example, Let $X = \mathbb{R}$ and $K = [0, \infty)$. Let $Ax = x + 1$. Then $A : K \to K$ is compact. But A has no fixed points in K.

However, under an extra condition we can obtain a result similar to Theorem 1.2.10

Theorem 1.2.13. *Let K be an unbounded closed convex set in X and $A : K \to K$ be compact. Assume that the following condition holds.*

$$\limsup_{x \in K, \|x\| \to \infty} \|Ax\|/\|x\| < 1.$$

Then there exists $\rho_0 > 0$ such that $i_K(A, B_K(\rho)) = 1$ for all $\rho \geq \rho_0$. Hence, A has a fixed point in K.

Proof. Let $x_0 \in K$. We prove that there exists $\rho_0 > 0$ such that

$$x \neq tAx + (1-t)x_0 \quad \text{for } t \in [0,1] \quad \text{and } x \in K \quad \text{with } \|x\| \geq \rho_0.$$

In fact, if not, there exist $t_n \in [0,1]$ and $x_n \in K$ with $\|x_n\| \to \infty$ such that

$$x_n = t_n A x_n + (1-t_n)x_0.$$

This implies $\|x_n\| \leq \|Ax_n\| + \|x_0\|$. It follows from $\limsup_{x \in K, \|x\| \to \infty} \|Ax\|/\|x\| \geq 1$, which contradicts our hypothesis.

Now, using properties (P_4) and (P_2) we obtain

$$i_K(A, B_K(\rho)) = i_K(\hat{0}, B_K(\rho)) = 1 \quad \text{for all } \rho \geq \rho_0.$$

□.

Remark 1.2.14. If $A(K)$ is bounded, then the condition in Theorem 1.2.13 holds.

Comments

In this section the following are new.

Proposition 1.2.4, Remark 1.2.7, Remark 1.2.8, Remark 1.2.12 and Theorem 1.2.13.

The proof of Theorem 1.2.10 is new.

1.3 The classic fixed point index for γ-condensing maps

In this section we recall the definition of the fixed point index for γ-condensing maps. Such a theory generalizes the theory of fixed point index for compact maps mentioned in the previous section. We only show a special case of this theory. For a more general case we refer to Nussbaum's paper [48].

Let us start with the concept of measure of noncompactness in metric spaces.

Let X be a metric space with a metric d. A subset Q in X is said to be bounded if its diameter, $diam(Q)$, is bounded, that is, $diam(Q) = \sup\{d(x,y) : x, y \in Q\} < \infty$. We always denote by $B_X(x,r)$ the ball with centre x and radius r, that is, $B_X(x,r) = \{y \in X : d(x,y) < r\}$.

For a bounded set Q in X, $\gamma_X(Q)$ will stand for either the set measure of noncompactness $\alpha_X(Q)$ relative to X defined by

$$\alpha_X(Q) = \inf\{\rho > 0 : Q \quad \text{admits a finite cover by sets of diameter at most } \rho\},$$

or the ball measure $\beta_X(Q)$ defined by

$$\beta_X(Q) = \inf\{r > 0 : \quad \text{there exist } \{x_1, ..., x_n\} \subset X \quad \text{such that } Q \subset \cup_{i=1}^{n} B_X(x_i, r)\}.$$

The simplest connection between set and ball measures of noncompactness is the inequalities $\alpha_X(Q) \leq 2\beta_X(Q) \leq 2\alpha_X(Q)$. In general these inequalities are best possible and thus the two measures are different.

By these definitions it is easy to verify the following simple result.

Proposition 1.3.1. *Let K be a nonempty subset in a metric space X. Then for every bounded set $Q \subset K$,*

$$\gamma_X(Q) \leq \gamma_K(Q).$$

Actually, for measure of set-noncompactness we have the following result.

Proposition 1.3.2. *Let X be a metric space and K a nonempty subset of X. Then for every bounded subset $Q \subset K$,*

$$\alpha_K(Q) = \alpha_X(Q).$$

Proof. Let $Q \subset K$ be a bounded subset in X. Since $K \subset X$, it follows from Proposition 1.3.1 that $\alpha_X(Q) \leq \alpha_K(Q)$. On the other hand, for $\varepsilon > 0$ there exist $Q_i \subset X$, $i = 1, ..., k$ such that $diam(Q_i) \leq \alpha_X(Q) + \varepsilon$, $i = 1, ..., k$ and $Q \subset \cup_{i=1}^k Q_i$. Hence, we have $Q \subset \cup_{i=1}^k (K \cap Q_i)$ since $K \subset X$. Noting that $diam(K \cap Q_i) \leq \alpha_X(Q) + \varepsilon$. Therefore, $\alpha_K(Q) \leq \alpha_X(Q) + \varepsilon$. This implies $\alpha_K(Q) \leq \alpha_X(Q)$. □

Hence, we usually only write $\alpha(Q)$ in the following.

However, for the ball measure of noncompactness, we need suitable hypotheses to obtain the equality $\beta_K(Q) = \beta_X(Q)$.

Definition 1.3.3. Let K be a nonempty subset in a metric space X. K is said to have the ball intersection property relative to X if for every bounded subset $Q \subset K$,

$$\beta_K(Q) = \beta_X(Q).$$

Remark 1.3.4. When K is a closed unit ball in a Banach space X, the ball intersection property is introduced in [47].

Remark 1.3.5. By Proposition 1.3.1 we see that K has the ball intersection property if and only if $\beta_K(Q) \leq \beta_X(Q)$ for every bounded subset $Q \subset K$.

The following results give sufficient conditions for a set to have the ball intersection property.

Theorem 1.3.6. *Let K be a subset in a metric space X. Assume that there exists a map $r : X \to K$ such that the following conditions hold.*

(i) $\beta_K(r(Q)) \leq \beta_X(Q)$ for every bounded subset $Q \subset X$.

(ii) $rx = x$ for all $x \in K$.

Then K has the ball intersection property relative to X.

Proof. Let Q be a bounded set in K. By (i) and (ii) we have $\beta_K(Q) = \beta_K(r(Q)) \leq \beta_X(Q)$. The result follows from Remark 1.3.5. □

As an application of Theorem 1.3.6 we obtain

Theorem 1.3.7. *Let K be a subset in a metric space X. Assume that there exists a nonexpansive retraction $r : X \to K$, that is, r satisfies* (*i*) $d(rx, ry) \leq d(x, y)$ *for all* $x, y \in X$;

(*ii*) $rx = x$ *for all* $x \in K$.

Then K *has the ball intersection property relative to* X.

Proof. It is sufficient to show that r satisfies (i) of Theorem 1.3.6. Let Q be a bounded set of K and $a = \beta_X(Q)$. For $\varepsilon > 0$, there exists $\{y_1, ..., y_n\} \subset X$ such that

$$Q \subset \cup_{i=1}^{n} B_X(y_i, a + \varepsilon),$$

where $B_X(y_i, a + \varepsilon) = \{x \in X : d(y_i, x) < a + \varepsilon\}$. For every fixed $x \in Q$, there exists $i \in \{1, ..., n\}$ such that $d(y_i, x) < a + \varepsilon$. This implies

$$d(rx, ry_i) \leq d(x, y_i) < (a + \varepsilon).$$

Therefore, $rx \in B_K(ry_i, (a + \varepsilon))$ and $r(Q) \subset \cup_{i=1}^{n} B_K(ry_i, (a + \varepsilon))$. It follows from the definition of ball measure of noncompactness that $\beta_K(r(Q)) \leq (a + \varepsilon)$. This implies $\beta_K(r(Q)) \leq \beta_X(Q)$. The result follows from Theorem 1.3.6. □

Remark 1.3.8. Theorem 1.3.7 generalizes Proposition 1 in [47], where X is a Banach space and K is the closed unit ball in X.

As an application of Theorem 1.3.7 we immediately obtain the following new result.

Theorem 1.3.9. *Let* (X, K) *be a Banach lattice. Then* K *has the ball intersection property relative to* X.

Proof. Let $r(x) = x^+$ for every $x \in X$. It follows from Theorem 1.1.24 that $r : X \to K$ is a nonexpansive retraction. The result follows from Theorem 1.3.7. □

Now, we give examples of closed convex sets K which have the ball intersection property in some concrete spaces.

By using Theorem 1.3.9 and Examples 1.1.25 and 1.1.26 we obtain the following two results.

Example 1.3.10. Let $X = C(\overline{\Omega})$ and $K = C_+(\overline{\Omega})$ be as in Example 1.1.8. Then K has the ball intersection property relative to X.

When $\overline{\Omega} = [0, 1]$, the result in Example 1.3.10 was mentioned in [36] without proof.

Example 1.3.11. For each $p \in [1, \infty]$, let $X = L^p(\Omega)$ and $K = L_p^+(\Omega)$ be as in Example 1.1.9. Then K has the ball intersection property relative to X.

The following new result will be applied in Section 5.6 in Chapter 5.

Example 1.3.12. Let X and K be the same as in Example 1.3.10. Let $\rho > 0$ and $K_\rho = \{x \in K : \|x\| < \rho\}$. Then $\overline{K}_\rho$ has the ball intersection property relative to K.

Proof. We define a map $r : K \to \overline{K}_\rho$ by

$$(ru)(x) = \rho u(x)/\max\{\rho, u(x)\}.$$

Then $ru \in \overline{K}_\rho$ and thus r is well defined. It is easy to see that $ru = u$ for every $u \in \overline{K}_\rho$. Therefore, r is a retraction. Since for $u, v \in K$ and $x \in \overline{\Omega}$,

$$|(ru)(x) - (rv)(x)| \leq |u(x) - v(x)|,$$

r is nonexpansive. The result follows from Theorem 1.3.7. □

Similar to Example 1.3.12, we have the following result.

Example 1.3.13. Let $X = C(\overline{\Omega})$ be the same as in Example 1.3.10 and $\rho > 0$. Let $\overline{B}(\rho) = \{x \in X : \|x\| \leq \rho\}$. Then $\overline{B}(\rho)$ has the ball intersection property relative to X.

Proof. We define a map $r : X \to \overline{B}(\rho)$ by

$$(ru)(x) = \rho u(x)/\max\{\rho, |u(x)|\}.$$

By a similar argument to that used in Example 1.3.12 it is easy to show that r is a nonexpansive retraction. The result follows from Theorem 1.3.7. □

Remark 1.3.14. Example 1.3.13 is true if $\overline{\Omega}$ is replaced by a compact metric space. The result is mentioned in p.932 in [47] without proof.

Remark 1.3.15. There is a Theorem in [47] which shows that Example 1.3.13 is not true in $L^p(\Omega)$ when $1 \leq p < \infty$ and $p \neq 2$. We do not know whether the proof of this Theorem in [47] can carried over to Examples 1.3.12 and 1.3.13, when the space is replaced by $L^p(\Omega)$, that is, whether the results in Examples 1.3.12 and 1.3.13 are false in $L^p(\Omega)$ when $1 \leq p < \infty$ and $p \neq 2$.

Now, we use Theorem 1.3.7 to show that every closed convex subset K in a Hilbert space H has the ball intersection property relative to H.

To do this we need several well-known results which can be found, for example, in [32]. For the sake of completeness, we give their proofs.

Lemma 1.3.16. *Let K be a closed convex set in a Hilbert space X. Let $x \in X$ and $y \in K$. Then the following are equivalent.*

(1) $\|x-y\| = d(x,K)$, *where* $d(x,K) = \inf\{\|x-z\| : z \in K\}$.

(2) $(x-y, y-v) \geq 0$ *for all* $v \in K$.

Proof. Let $v \in K$. Then

$$\|x-v\|^2 = \|x-y\|^2 + \|y-v\|^2 + 2(x-y, y-v). \tag{1.6}$$

Assume that (2) holds. Then (1.6) implies $\|x-v\| \geq \|x-y\|$ for all $v \in K$ and $\|x-y\| = d(x,K)$. On the other hand, assume that (1) holds. Then (1.6) implies

$$\|y-v\|^2 + 2(x-y, y-v) \geq 0 \quad \text{for all } v \in K.$$

Let $z \in K$, $t \in (0,1)$ and $v = (1-t)y + tz$. Then $v \in K$. Putting v in the above inequality, we obtain

$$t\|y-z\|^2 + 2(x-y, y-z) \geq 0.$$

Hence, $t \to 0^+$ implies $(x-y, y-z) \geq 0$ for all $z \in K$. □

Lemma 1.3.17. *Let K be a closed convex set in a Hilbert space X. Then for each $x \in X$ there exists a unique $y \in K$ such that*

$$\|x-y\| = d(x,K).$$

Proof. Let $d = d(x,K)$ and $\{y_n\} \subset K$ with $\|x-y_n\| \to d$. Then $\{y_n\}$ is bounded and there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that $y_{n_k} \rightharpoonup y \in K$, where $\rightharpoonup$ denotes weak convergence. Since $\|.\|$ is weakly lower-semicontinuous, we have

$$\|x-y\| \leq \lim \|x-y_{n_k}\| = d.$$

This implies $\|x-y\| = d(x,K)$.

Let $y' \in K$ be such that $\|x - y'\| = d(x, K)$. By the parallelogram law we obtain

$$\|y - y'\|^2 + \|2x - (y + y')\|^2 = 2(\|x - y\|^2 + \|x - y'\|^2).$$

Since K is convex, $d(x, K) \leq \|x - (y + y')/2\|$. Thus,

$$\|y - y'\|^2 + 4d^2 \leq 2(\|x - y\|^2 + \|x - y'\|^2) = 4d^2.$$

This implies $\|y - y'\| = 0$ and $y = y'$. □

The above proof is different from that in Lemma 2.1 in [32].

Let K be a closed convex set in a Hilbert space. We define a map $r : X \to K$ by $\|x - rx\| = d(x, K)$. By Lemma 1.3.17 we see that r is well-defined. The following result shows that r is a nonexpansive retraction.

Proposition 1.3.18. *Let K be a closed convex set in a Hilbert space. Let $r : X \to K$ be defined as above. Then r is nonexpansive retraction.*

Proof. It is clear that r is a retraction. We prove that r is nonexpansive. Let $x, y \in X$. By Lemma 1.3.16 we have

$$(x - rx, rx - ry) \geq 0 \quad \text{and } (y - ry, ry - rx) \geq 0.$$

This implies

$$\begin{aligned} \|rx - ry\|^2 &= (rx - x, rx - ry) + (x - y, rx - ry) + (y - ry, rx - ry) \\ &\leq (x - y, rx - ry) \\ &\leq \|rx - ry\|\|x - y\|. \end{aligned}$$

Thus, we have $\|rx - ry\| \leq \|x - y\|$ □

By Theorem 1.3.7 and Proposition 1.3.18 we immediately obtain the following result which was mentioned in [36] without proof.

Proposition 1.3.19. *Let K be a closed convex set in a Hilbert space H. Then for each bounded set $Q \subset K$,*

$$\beta_H(Q) = \beta_K(Q).$$

Remark 1.3.20. As shown in Remark 1.1.14, not every cone in a Hilbert space is normal. Hence (H, K) may not be a Banach lattice for some cones K in a Hilbert space H. However, Proposition 1.3.19 has shown that every cone in a Hilbert space has the ball intersection property. This shows that the converse of Theorem 1.3.9 is not true.

We mentioned above that set- and ball-measures of noncompactness are different, but they share similar properties, for example they have the following four basic properties.

Lemma 1.3.21. *Let Q and S be bounded subsets in a complete metric space X. Then the following properties hold.*

(1) $\gamma_X(Q) = 0$ *if and only if Q is precompact.*

(2) $\gamma_X(\overline{Q}) = \gamma_X(Q)$, *where $\overline{Q}$ denotes the closure of Q in X.*

(3) $Q \subset D$ *implies* $\gamma_X(Q) \leq \gamma_X(D)$.

(4) $\gamma_X(Q \cup S) = \max\{\gamma_X(Q), \gamma_X(S)\}$.

It is obvious that (3) is a special case of (4). The proofs of the above properties follow easily from the definitions. In the case when X is a Banach space, the proofs of these properties can be founded in many books on Nonlinear Analysis, for example, see [12], [29], [48], and [60]. Hence, we omit the proofs of these properties.

It is known that a family of nonempty compact sets of X has a nonempty intersection if the family has the finite intersection property, that is, any finite collection of sets in the family has a nonempty intersection. In particular, a decreasing sequence of nonempty compact sets has a nonempty intersection since it has the finite intersection property. With the aid of measures of noncompactness the result can be extended to noncompact settings.

The following result obtained by Kuratowski [31] will play an important role in defining a fixed point index for γ-condensing maps. We give a simple proof for completeness (see [12], p. 53).

Lemma 1.3.22. *Let $\mathbb{N}$ be the set of positive integers. Let $\{S_n\}_{n \in \mathbb{N}}$ be a sequence of nonempty bounded closed subsets of a complete metric space X such that the following conditions hold.*

$$S_{n+1} \subset S_n \quad \textit{for } n \in \mathbb{N} \quad \textit{and} \quad \gamma_X(S_n) \to 0.$$

Then the intersection $S = \cap_{n=1}^{\infty} S_n$ is a nonempty compact set of X.

Proof. For each $n \in \mathbb{N}$, let $x_n \in S_n$ and $U_n = \{x_m : m \geq n\}$. Then we have

$$\{x_n : n \geq 1\} = \{x_1, ..., x_{n-1}\} \cup U_n.$$

It follows from the properties of γ_X-measure of noncompactness that

$$\gamma_X(\{x_n\}) = \gamma_X(U_n) \leq \gamma_X(S_n) \quad \text{for all } n \in \mathbb{N}.$$

This implies $\gamma_X\{x_n\} = 0$ and thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0 \in X$. For each $n \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $\{x_{n_k}\} \subset S_n$ for $k \geq k_0$. Since S_n is closed, we have $x_0 \in S_n$. This implies $x_0 \in \cap_{n=1}^{\infty} S_n$ and thus S is a nonempty closed set. Since $S \subset S_n$ for each $n \in \mathbb{N}$, we have $\gamma_X(S) \leq \gamma_X(S_n)$ for $n \in \mathbb{N}$ and thus $\gamma_X(S) = 0$. This shows that S is compact. □

The above result can also be extended to a family of sets which has the finite intersection property (see [27]).

Closely associated with the concept of measures of noncompactness is the notion of a k-set (or ball)-contractive map and a set (or ball)-condensing map.

Definition 1.3.23. Let X and Y be metric spaces. A continuous map $A : D \subset X \to Y$ is said to be k-(γ_X, γ_Y)-contractive if there is $k \geq 0$ such that $\gamma_Y(A(Q)) \leq k\gamma_X(Q)$ for each bounded $Q \subset D$; (γ_X, γ_Y)-condensing if $\gamma_Y(A(Q)) < \gamma_X(Q)$ for each bounded $Q \subset D$ with $\gamma(Q) \neq 0$.

As usual, when $X = Y$, we use γ_X to denote the above symbol (γ_X, γ_Y).

It is readily seen that a map is compact if and only if it is 0-(γ_X, γ_Y)-contractive. Every k-(γ_X, γ_Y)-contractive map with $k < 1$ is (γ_X, γ_Y)-condensing. When $X = Y$, there are γ_X-condensing maps that are not k-γ_X-contractive for any $k < 1$ (see [49]).

Using Proposition 1.3.2 we immediately obtain

Lemma 1.3.24. *Let K be a nonempty closed subset in a complete metric space X and $A : D \subset K \to K$ be a map. Then A is α_X-condensing if and only if A is α_K-condensing.*

Remark 1.3.25. We do not know if Lemma 1.3.24 is true for ball-condensing maps even when $D = K$.

The following result will be useful. Its proof follows directly from Definition 1.3.23 and thus, we omit it.

Lemma 1.3.26. *Let X, Y and Z be metric spaces. Let $A : D \subset X \to Y$ be k_1-(γ_X, γ_Y)-contractive and $A(D) \subset D_1 \subset Y$. Let $B : D_1 \to Z$ be k_2-(γ_Y, γ_Z)- contractive. Then $BA : D \to Z$ is k_2k_1-(γ_X, γ_Z)-contractive.*

It follows from Lemma 1.3.26 that if A is a k-(γ_X, γ_Y)-contractive map and B is 1-(γ_Y, γ_Z)-contractive and BA makes sense, then BA is also k-(γ_X, γ_Z)-contractive.

In the following we provide some k-(γ_X, γ_Y)-contractive maps.

Definition 1.3.27. Let X and Y be metric spaces. A map $A : D \subset X \to Y$ is said to be a Lipschitz map with Lipschitz constant L if A satisfies

$$d(Ax, Ay) \leq Ld(x, y) \quad \text{for } x, y \in D,$$

Lemma 1.3.28. *Let X and Y be metric spaces. Let $A : D \subset X \to Y$ be a Lipschitz map with Lispschitz constant L. Then A is L-(α_X, α_Y)-contractive.*

Proof. Let $Q \subset D$ be a bounded set in X. For $\varepsilon > 0$, there exists a finite cover $\{Q_1, ..., Q_k\}$ such that $diam(Q_i) \leq \alpha_X(Q) + \varepsilon$, $i = 1, ..., k$ and $Q \subset \cup_{i=1}^{k} Q_i$. This implies $A(Q) \subset \cup_{i=1}^{k} A(Q_i)$. By hypothesis we have $diam(A(Q_i)) \leq Ldiam(Q_i)$ for $i = 1, ..., k$. It follows that $\alpha_Y(A(Q)) \leq L(\alpha_X(Q) + \varepsilon)$. Hence, $\alpha_Y(A(Q)) \leq L\alpha_X(Q)$. □.

Let $A : D \subset X \to X$ be a Lipschitz map with the Lispschitz constant L. In general, if $D \neq X$, we can not show that A is L-(β_X, β_Y)-contractive although it is L-(α_X, α_Y)-contractive. However, if $D = X$, we can prove that it is.

Theorem 1.3.29. *Let X and Y be metric spaces. Assume that $A : X \to Y$ is a Lipschitz map with Lipschitz constant L. Then A is L-(β_X, β_Y)-contractive.*

Proof. Let $Q \subset X$ be a bounded set and $r = \beta_X(Q)$. For $\varepsilon > 0$, there exists $\{y_1, ..., y_n\} \subset X$ such that

$$Q \subset \cup_{i=1}^{n} B_X(y_i, r + \varepsilon),$$

where $B_X(y_i, r + \varepsilon) = \{x \in X : d(y_i, x) < r + \varepsilon\}$. For every fixed $x \in Q$, there exists $i \in \{1, ..., n\}$ such that $d(y_i, x) < r + \varepsilon$. This implies

$$d(Ax, Ay_i) \leq Ld(x, y_i) < L(r + \varepsilon).$$

Therefore, $Ax \in B_Y(Ay_i, L(r+\varepsilon))$ and $A(Q) \subset \cup_{i=1}^n B_Y(Ay_i, L(r+\varepsilon))$. It follows from the definition of ball measure of noncompactness that $\beta_Y(A(Q)) \leq L(r+\varepsilon)$. This implies $\beta_Y(A(Q)) \leq L\beta_X(Q)$. □

The above result can be generalized to a more general case which is also useful in applications.

Definition 1.3.30. Let X and Y be metric spaces. A map $A : \Omega \subset X \to Y$ is said to be L-semicontractive map from X to Y with $L > 0$ if there exists a map $V : X \times X \to Y$ such that the following conditions hold.

(S_1) For each fixed $x \in X$, $V(x, .) : X \to Y$ is compact.

(S_2) For each $y \in X$, the map $V(., y) : X \to Y$ is a Lipschitz map with the Lipschitz constant L.

(S_3) $A(x) = V(x, x)$ for $x \in \Omega$.

Theorem 1.3.31. *Let X and Y be metric spaces. Assume that $A : \Omega \subset X \to Y$ is a L-semicontractive map. Then A is L-(β_X, β_Y)-contractive.*

Proof. Let $Q \subset \Omega$ be a bounded set with $\beta_X(Q) = r$. For $\varepsilon > 0$, there exists $\{x_1, ..., x_n\} \subset X$ such that $Q \subset \cup_{i=1}^n B_X(x_i, r+\varepsilon)$. For each $x \in X$, $V(x, Q)$ is a precompact set in Y, so $\cup_{i=1}^n V(x_i, Q)$ is also precompact in Y. For the given $\varepsilon > 0$ there exists $\{y_1, ..., y_m\} \subset Y$ such that $\cup_{i=1}^n V(x_i, Q) \subset \cup_{j=1}^m B_Y(y_j, \varepsilon)$. Let $x \in Q$. Then we can choose $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$ such that

$$d(x, x_i) \leq r + \varepsilon \quad \text{and } d(V(x_i, x), y_j) < \varepsilon.$$

Thus we have

$$d(A(x), y_j) \leq d(V(x, x), V(x_i, x)) + d(V(x_i, x), y_j) \leq L(r+\varepsilon) + \varepsilon.$$

This implies $A(Q) \subset \cup_{j=1}^m B_Y(y_j, L(r+\varepsilon)+\varepsilon)$. Thus we have $\beta_Y(A(Q)) \leq L(r+\varepsilon)+\varepsilon$. Since ε is arbitrary small, we get $\beta_Y(A(Q)) \leq L\beta_X(Q)$. □

Remark 1.3.32. Theorem 1.3.31 generalizes Theorem 1 in [69] (also see lemma 2.3 in [55]).

Now, we show that when $X = Y$, γ_X-measure of noncompactness has more properties if the space involved is a Banach space. We collect them in the following lemma.

Lemma 1.3.33. *Let K be a closed convex set in a Banach space X. Then the measure of noncompactness relative to K has the following properties. Suppose $Q, S \subset K$. Then*

(1) $\beta_K(Q) = 0$ *if and only if Q is precompact.*

(2) $\beta_K(\overline{Q}) = \beta_K(Q)$, *where $\overline{Q}$ denotes the closure of Q relative to K.*

(3) $Q \subset D \subset K$ *implies* $\beta_K(Q) \leq \beta_K(D)$.

(4) *For each* $\lambda \in [0,1]$, $\beta_K(\lambda Q + (1-\lambda)S) \leq \lambda\beta_K(Q) + (1-\lambda)\beta_K(S)$.

(5) $\beta_K(Q \cup S) = \max\{\beta_K(Q), \beta_K(S)\}$.

(6) $\beta_K(\overline{\mathrm{co}}(Q)) = \beta_K(Q)$, *where* $\mathrm{co}(Q)$ *denotes the convex hull of Q.*

If K is a wedge, then we have

(7) $\beta_K(aQ) = a\beta_K(Q)$ *for each* $a \in [0, \infty)$.

(8) $\beta_K(Q + S) \leq \beta_K(Q) + \beta_K(S)$.

If K is a subspace of X, then we have

(9) $\beta_K(aQ) = |a|\beta_K(Q)$ *for each* $a \in \mathbb{R}$.

All of the proofs of the properties (1)-(9) follow from the definition of the relative ball measure of noncompactness β_K and are similar to those of the properties of γ-measure relative to X, which are given in usual books on Nonlinear Analysis, for example, see [12], [48] and [60]. Therefore, we only show the proof of property (6) which is not obvious.

Proof of property (6). Obviously, it suffices to show that $\beta_K(\mathrm{co}(Q)) \leq \beta_K(Q)$. Let $r = \beta_K(Q)$ and $\varepsilon > 0$. Then there exists $\{x_1, ..., x_n\} \subset K$ such that $Q \subset \cup_{i=1}^n B_K(x_i, r+\varepsilon)$. Let $C = \mathrm{co}\{x_1, ..., x_n\}$. Then $C \subset K$. Since C is compact, there exists $\{z_1, ..., z_m\} \subset C$ such that $C \subset \cup_{j=1}^m B_K(z_j, \varepsilon)$. Let $y \in \mathrm{co}\, Q$. Then there exist $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $y_i \in B_K(x_i, r+\varepsilon)$ such that $y = \sum_{i=1}^n \lambda_i y_i$. Let $z = \sum_{i=1}^n \lambda_i x_i$. Since $C \subset \cup_{j=1}^m B_K(z_j, \varepsilon)$, we have $\|z - z_j\| \leq \varepsilon$ for some z_j. Hence we obtain

$$\|y - z_j\| \leq \|y - z\| + \|z - z_j\| \leq \sum_{i=1}^n \lambda_i \|y_i - x_i\| + \varepsilon \leq r + \varepsilon + \varepsilon.$$

This implies $y \in B_K(z_j, r + 2\varepsilon)$ and thus $\mathrm{co}\, Q \subset \cup_{j=1}^m B_K(z_j, r + 2\varepsilon)$. It follows that $\beta_K(\mathrm{co}(Q)) \leq r = \beta_K(Q)$. □

By Definition 1.3.23 and Lemma 1.3.33 we obtain

Lemma 1.3.34. *Let X and Y be Banach spaces. Then*

(1) *If $A_i : D \subset X \to Y$ is k_i-(γ_X, γ_Y)-contractive for $i = 1, 2$, then $A_1 + A_2$ is $(k_1 + k_2)$-(γ_X, γ_Y)-contractive.*

It immediately follows from Lemma 1.3.34 that if A is a k-(γ_X, γ_Y)-contractive map and B is a compact map, then the sum $A + B$ is also k-(γ_X, γ_Y)-contractive.

The following result shows that condensing maps have a property of convergence.

Lemma 1.3.35. *Let D be a closed set in a Banach space X. Let $A : D \to X$ be a γ_X-condensing map and $\{x_n\} \subset D$ a bounded sequence such that $x_n - Ax_n \to w \in X$. Then $\{x_n\}$ has a convergent subsequence $x_{n_k} \to x \in D$ and $x - Ax = w$.*

Proof. Let $\{x_n\} \subset D$ be a bounded sequence and $x_n - Ax_n \to w \in X$. Then we have

$$\{x_n\} \subset \{x_n - Ax_n\} + \{Ax_n\}.$$

This implies

$$\gamma_X(\{x_n\}) \leq \gamma_X(\{x_n - Ax_n\}) + \gamma_X(\{Ax_n\}) = \gamma_X(\{Ax_n\}).$$

Since A is γ-condensing from X to X, we find $\gamma_X(\{x_n\}) = 0$. The result follows. □

The above result shows that if D is a compact set of X, then the inverse image $f^{-1}(D) = \{x \in \Omega : f(x) \in D\}$ is compact, that is f is proper.

Utilizing measure of noncompactness, one can see an essential difference between a finite dimensional space and an infinite dimensional space.

The following result is due to Nussbaum [49].

Theorem 1.3.36. *Let X be a real Banach space and $B(1) = \{x \in X : \|x\| \leq 1\}$ and $S = \{x \in X : \|x\| = 1\}$. Then we have*

(i) $\alpha_X(B(1)) = \alpha_X(S) = 0$ if X is a finite dimensional space.

(ii) $\alpha_X(B(1)) = \alpha_X(S) = 2$ if X is an infinite dimensional space.

Remark 1.3.37. Let X be a real Banach space and K a wedge in X. Let $\partial K_1 = \{x \in K : \|x\| = 1\}$. We denote by F the closure of the space generated by K. If F is a finite dimensional subspace of X, then ∂K_1 is compact. However, we don't know if the inverse of the result is true, that is, if ∂K_1 is compact, is F a finite dimensional space?

For ball-measure of noncompactness, we have

Theorem 1.3.38. *Let X be a real Banach space and $B(1) = \{x \in X : \|x\| \leq 1\}$ and $S = \{x \in X : \|x\| = 1\}$. Then we have*

(i) $\beta_X(B(1)) = \beta_X(S) = 0$ if X is a finite dimensional space.

(ii) $\beta_X(B(1)) = \beta_X(S) = 1$ if X is an infinite dimensional space.

Proof. We prove (*ii*). Let $r = \beta_X(B(1))$. Then $r \in (0, 1]$. We prove $r \geq 1$. Let $\varepsilon > 0$. Then there exist $x_1, ..., x_n \in X$ such that

$$B(1) \subset \cup_{i=1}^{n} B(x_i, r + \varepsilon). \tag{1.7}$$

Let $\rho = r + \varepsilon$. We prove that for every $i \in \{1, ..., n\}$,

$$B(x_i, \rho) \subset \cup_j^n B(x_i + \rho x_j, \rho^2). \tag{1.8}$$

In fact, let $y \in B(x_i, \rho)$ and $z = (y - x_i)/\rho$. Then $z \in B(1)$. By (1.7), there exists $j \in \{x_1, ..., x_n\}$ such that $\|x_j - z\| < \rho$. Hence, we have

$$\|y - x_i - \rho x_j\| = \|\rho z - \rho x_j\| < \rho^2.$$

This implies $B(x_i, \rho) \subset B(x_i + \rho x_j, \rho^2)$ and thus, (1.8) holds. By (1.7) and (1.8) we obtain

$$B(1) \subset \cup_{i=1}^{n} \cup_{j=1}^{n} B(x_i + \rho x_j, \rho^2).$$

It follows from definition of ball measure of noncompactness, $r \leq \rho^2 = (r + \varepsilon)^2$. Since ε is arbitrary, $r \leq r^2$ and $r \geq 1$. □

We are now in a position to mention the definition of the fixed point index for γ-condensing maps and show its properties.

We first give the definition of the fixed point index for k-γ-contractive maps with $k < 1$. We refer to [48] for more details. One can also find the definitions of fixed point index for k-set-contractive (or set-condensing) maps in [23] and [24]; for k-γ-contractive (or γ-condensing) multivalued maps from X to X in [20], [52] and [54].

Let K be a closed convex set of X and D be a bounded open set in X such that $D_K = D \cap K \neq \emptyset$. As before, we denote by $\overline{D}_K$ the closure and ∂D_K the boundary of D_K relative to K.

Definition 1.3.39. A map $A : \overline{D}_K \to K$ is a k-γ-contractive map (γ-condensing, respectively) if it is one of the following maps.

(1) A is k-α_X-contractive (α_X-condensing, respectively).

(2) A is k-β_X-contractive (β_X-condensing, respectively).

(3) A is k-β_K-contractive (β_K-condensing, respectively).

By Lemma 1.3.24 we see that A satisfies (1) if and only if A is k-α_K-contractive (α_K-condensing, respectively).

Remark 1.3.40. Previously, a k-γ-contractive (γ-condensing, respectively) map A meant that A satisfied (1) or (2) (see, for example, [20], [52] and [54]). This does not include the class of maps which satisfies (3).

As before, $x \neq A(x)$ for all $x \in \partial D_K$ means that either (i) $\partial D_K = \emptyset$ or (ii) if $\partial D_K \neq \emptyset$, $x \neq A(x)$ for all $x \in \partial D_K$.

Let $A : \overline{D}_K \to K$ be k-γ-contractive with $k < 1$ such that $x \neq Ax$ for $x \in \partial D_K$. We define

$$K_1 = \overline{\text{co}}A(\overline{D}_K) \quad \text{and } K_{n+1} = \overline{\text{co}}A(K_n \cap \overline{D}_K) \quad \text{for } n \in \mathbb{N}.$$

Note that if $\partial D_K = \emptyset$, that is, $\overline{D}_K = K$, then

$$K_1 = \overline{\text{co}}A(K) \quad \text{and } K_{n+1} = \overline{\text{co}}A(K_n) \quad \text{for } n \in \mathbb{N}.$$

The following lemma is a key towards defining the index.

Lemma 1.3.41. *Let $A : \overline{D}_K \to K$ be k-γ-contractive with $k < 1$ such that $x \neq Ax$ for $x \in \partial D_K$. Assume that $K_n \cap \overline{D}_K \neq \emptyset$ for all $n \in N$, where K_n is defined as above. Let $K_\infty = \cap_{n=1}^{\infty} K_n$. Then*

(1) K_∞ is a nonempty compact convex set.

(2) $K_\infty \cap \overline{D}_K = \cap_{n=1}^{\infty} K_n \cap \overline{D}_K$ is a nonempty compact set.

(3) There exists a compact map $B : \overline{D}_K \to K_\infty (\subset K)$ such that $Bx = Ax$ for $x \in K_\infty \cap \overline{D}_K$ and $x \neq Bx$ for $x \in \partial D_K$.

Proof. It is easy to verify the following facts:

$$K_{n+1} \subset K_n \quad \text{and } \gamma(K_{n+1}) \leq k\gamma(K_n) \leq k^{n+1}\gamma(\overline{D}_K).$$

This implies $\gamma(K_n) \to 0$ and $\gamma(K_n \cap \overline{D}_K) \to 0$. It follows from Lemma 1.3.22 that K_∞ is a nonempty compact convex set and $K_\infty \cap \overline{D}_K = \cap_{n=1}^\infty K_n \cap \overline{D}_K$ is a nonempty compact set. Therefore, A is well-defined on $K_\infty \cap \overline{D}_K$ and $A(K_\infty \cap \overline{D}_K) \subset K_\infty$. Since K_∞ is compact and A is continuous, $A : K_\infty \cap \overline{D}_K \to K_\infty$ is compact. It follows from Lemma 1.2.2 that there exists a compact map $B : \overline{D}_K \to K_\infty(\subset K)$ such that $Bx = Ax$ for $x \in K_\infty \cap \overline{D}_K$. We prove that $x \neq Bx$ for $x \in \partial D_K$. In fact, if not, $x = Bx$ for some $x \in \partial D_K$. Then $x \in K_\infty \cap \partial D_K$ and $x = Ax$, which contradicts the hypothesis that $x \neq Ax$ for all $x \in \partial D_K$. □

Definition 1.3.42. Let $A : \overline{D}_K \to K$ be k-γ-contractive with $k < 1$ such that $x \neq Ax$ for $x \in \partial D_K$. Let K_n and F be defined above. We define the fixed point index of A over D_K relative to K, denoted by $i_K(A, D_K)$, as follows.

(1) $i_K(A, D_K) = 0$ if there exists $n_0 \in \mathbb{N}$ such that $K_{n_0} \cap \overline{D}_K = \emptyset$.

(2) $i_K(A, D_K) = i_K(B, D_K)$ if $K_n \cap \overline{D}_K \neq \emptyset$ for all $n \in \mathbb{N}$,

where $B : \overline{D}_K \to K_\infty(\subset K)$ is an arbitrary map which satisfies all the conditions in (3) of Lemma 1.3.41.

Remark 1.3.43. If there exists $n_0 \in \mathbb{N}$ such that $K_{n_0} \cap \overline{D}_K = \emptyset$, then $K_n \cap \overline{D}_K = \emptyset$ for all $n \geq n_0$. If $K_n \cap \overline{D}_K \neq \emptyset$ for all $n \in \mathbb{N}$, it follows from lemma 1.3.41 that there exists at least one map $B : \overline{D}_K \to K_\infty(\subset K)$ such that $Bx = Ax$ for $x \in K_\infty \cap \overline{D}_K$ and $x \neq Bx$ for $x \in \partial D_K$. Hence the index $i_K(B, D_K)$ makes sense. If there exists another map $C : \overline{D}_K \to K_\infty(\subset K)$ such that $Cx = Ax$ for $x \in K_\infty \cap \overline{D}_K$ and $x \neq Cx$ for $x \in \partial D_K$, then $x \neq h(t, x) = tBx + (1-t)Cx$ for $t \in [0, 1]$ and $x \in \partial D_K$. It follows from the homotopy property in Theorem 1.2.9 that $i_K(B, D_K) = i_K(C, D_K)$. Therefore, $i_K(A, D_K)$ is well defined and is independent of the choice of B.

Remark 1.3.44. By Definition 1.3.39 and Lemma 1.3.24 we see that we have defined a fixed point index for k-ball-contractive maps from K to K.

One can show that the index in Definition 1.3.42 has the usual properties.

Theorem 1.3.45. *Let K be a closed convex set in a Banach space X and let D be a bounded open set such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to K$ be a k-γ-contractive map with*

$k \in [0,1)$. Suppose that $x \neq A(x)$ for all $x \in \partial D_K$. The index defined in Definition 1.3.42 has the following properties.

(P_1) *(Existence property) If $i_K(A, D_K) \neq 0$, then A has a fixed point in D_K.*

(P_2) *(Normalization) If $u \in D_K$, then $i_K(\hat{u}, D_K) = 1$, where $\hat{u}(x) \equiv u$ for $x \in \overline{D}_K$.*

(P_3) *(Additivity property) If W^1, W^2 are disjoint relatively open subsets of D_K such that $x \neq Ax$ for $x \in \overline{D}_K \setminus (W^1 \cup W^2)$, then*

$$i_K(A, D_K) = i_K(A, W^1) + i_K(A, W^2).$$

(P_4) *(Homotopy property) If $H : [0,1] \times \overline{D}_K \to K$ is continuous and such that $\gamma(H([0,1] \times Q)) \leq k\gamma(Q)$ for each $Q \subset D$ with $\gamma(Q) \neq 0$ and some $k \in (0,1)$, and if $x \neq H(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$, then*

$$i_K(H(0,\cdot), D_K) = i_K(H(1,\cdot), D_K).$$

Similar to Theorems 1.2.10 and 1.2.13 we have the following results.

Theorem 1.3.46. *Let K be a bounded closed convex set in X Assume that $A : K \to K$ is k-γ-contractive with $k < 1$. Then $i_K(A, K) = 1$, and thus, A has a fixed point in K.*

Proof. It follows from Lemma 1.3.41 and Definition 1.3.42 that

$$i_K(A, K) = i_K(B, K),$$

where $B : K \to F(\subset K)$ is defined as in Lemma 1.3.41 and is compact. By Theorem 1.2.10 $i_K(B, K) = 1$ and thus, $i_K(A, K) = 1$. □

The proof of the following result is similar to that of Theorem 1.2.13. We omit it.

Theorem 1.3.47. *Let K be a unbounded closed convex set in X and $A : K \to K$ be k-γ-contractive with $k < 1$. Assume that the following condition holds.*

$$\limsup_{x \in K, \|x\| \to \infty} \|Ax\| / \|x\| < 1.$$

Then there exists $\rho_0 > 0$ such that $i_K(A, B_K(\rho)) = 1$ for all $\rho \geq \rho_0$. Hence, A has a fixed point in K.

We generalize the above index to γ-condensing maps.

Definition 1.3.48. Let K be a closed convex set in a Banach space X and let D be a bounded open set such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to K$ be a γ-condensing map. Suppose that $x \neq Ax$ for all $x \in \partial D_K$. Define the fixed point index by the equation

$$i_K(A, D_K) = i_K(B, D_K).$$

where $B : \overline{D}_K \to K$ is a k-γ-contraction with $k \in (0,1)$ and $1-k$ sufficiently small such that $\|Ax - Bx\| \leq \tau/3$ for all $x \in \overline{D}_K$, where $\tau = \inf\{\|x - Ax\| : x \in \partial D_K\}$.

Remark 1.3.49. Since $x \neq Ax$ for $x \in \partial D_K$, it follows from Lemma 1.3.35 that $\tau > 0$. Let $x_0 \in K$. A possible choice for B is the map $B : \overline{D}_K \to K$ defined by $Bx = kAx + (1-k)x_0$. It is easy to verify that $\|Ax - Bx\| \leq \tau/3$ for all $x \in \overline{D}_K$ for $k \in (0,1)$ and $1-k$ sufficiently small, and $x \neq Bx$ for $x \in \partial D_K$. Therefore, it follows from Definition 1.3.42 that $i_K(B, D_K)$ makes sense. One can prove that the index $i_K(A, D_K)$ is independent of the choice of B. In fact, let $B_i : \overline{D}_K \to K$ be k_i-γ-contraction with $k_i \in (0,1)$ and $1-k_i$ sufficiently small such that $\|Ax - B_i(x)\| \leq \tau/3$ for all $x \in \overline{D}_K$, where $i = 1,2$. Let $H(t,x) = tB_1(x) + (1-t)B_2(x)$ for $t \in [0,1]$ and $x \in \overline{D}_K$. Then $H : [0,1] \times \overline{D}_K \to K$ is continuous and such that $\gamma(H([0,1] \times Q)) \leq k\gamma(Q)$ for each $Q \subset D$, where $k = \max\{k_1, k_2\}$. We prove that $x \neq H(t,x)$ for $t \in [0,1]$ and $x \in \partial D_K$. In fact, if not, there exists $x \in \partial D_K$ and $t \in [0,1]$ such that $x = tB_1(x) + (1-t)B_2(x)$. Then

$$\tau \leq \|x - Ax\| \leq t\|Ax - B_1x\| + (1-t)\|Ax - B_2(x)\| \leq t\tau/3 + (1-t)\tau/3 = \tau/3,$$

a contradiction. It follows from the homotopy property of Theorem 1.3.45 that

$$i_K(B_1, D_K) = i_K(B_2, D_K).$$

The fixed point index defined in Definition 1.3.48 has the usual properties. We collect these properties in the following

Theorem 1.3.50. *Let K be a closed convex set in a Banach space X and let D be a bounded open set such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to K$ be a γ-condensing map. Suppose that $x \neq A(x)$ for all $x \in \partial D_K$. The index defined in Definition 1.3.48 has the following properties.*

(P_1) *(Existence property) If* $i_K(A, D_K) \neq 0$, *then* A *has a fixed point in* D_K.

(P_2) *(Normalization) If* $u \in D_K$, *then* $i_K(\hat{u}, D_K) = 1$, *where* $\hat{u}(x) \equiv u$ *for* $x \in \overline{D}_K$.

(P_3) *(Additivity property) If* W^1, W^2 *are disjoint relatively open subsets of* D_K *such that* $x \neq Ax$ *for* $x \in \overline{D}_K \setminus (W^1 \cup W^2)$, *then*

$$i_K(A, D_K) = i_K(A, W^1) + i_K(A, W^2).$$

(P_4) *(Homotopy property) If* $H : [0,1] \times \overline{D}_K \to K$ *is continuous and such that* $\gamma(H([0,1] \times Q)) < \gamma(Q)$ *for each* $Q \subset D$ *with* $\gamma(Q) \neq 0$, *and if* $x \neq H(t,x)$ *for* $x \in \partial D_K$ *and* $t \in [0,1]$, *then*

$$i_K(H(0,\cdot), D_K) = i_K(H(1,\cdot), D_K).$$

Similar to Theorems 1.3.46 and 1.3.47 we have the following results.

Theorem 1.3.51. *Let* K *be a bounded closed convex set in* X *Assume that* $A : K \to K$ *is* γ*-condensing. Then* $i_K(A, K) = 1$, *and thus,* A *has a fixed point in* K.

Proof. Let $x_0 \in K$ and $B_n(x) = (1 - 1/n)Ax + (1/n)x_0$ for $x \in K$. It follows from Definition 1.3.48 that

$$i_K(A, K) = i_K(B_n, K)$$

for large n. By Theorem 1.3.46, $i_K(B_n, K) = 1$ for large n. Hence, $i_K(A, K) = 1$. $\square$

The proof of the following result is similar to those of Theorems 1.2.13 and 1.3.47. We omit it.

Theorem 1.3.52. *Let* K *be an unbounded closed convex set in* X *and* $A : K \to K$ *be* k*-*γ*-contractive with* $k < 1$. *Assume that the following condition holds.*

$$\limsup_{x \in K, \|x\| \to \infty} \|Ax\|/\|x\| < 1.$$

Then there exists $\rho_0 > 0$ *such that* $i_K(A, B_K(\rho)) = 1$ *for all* $\rho \geq \rho_0$. *Hence,* A *has a fixed point in* K.

The special case when $A(K)$ is bounded, is obtained in [54] (also see [53]).

Comments

The following assertions are new.

Proposition 1.3.1, Proposition 1.3.2, Theorem 1.3.6, Theorem 1.3.7, Theorem 1.3.9, Example 1.3.12, Remark 1.3.15, Lemma 1.3.24, Remark 1.3.25, Theorem 1.3.29, Theorem 1.3.31, Theorem 1.3.46, Theorem 1.3.47, Theorem 1.3.51 and Theorem 1.3.52.

The proofs of the following assertions are new. Example 1.3.10, Example 1.3.11, Example 1.3.13, Lemma 1.3.17 and Property (6) of Lemma 1.3.33.

Chapter 2

Metric Projections and Generalized Inward Maps

This Chapter is devoted to the study of closed convex sets for which there exist upper-semicontinuous multivalued metric projections or continuous single-valued metric projections and of generalized inward maps including weakly inward maps. These sets and maps will play important roles in developing our index theories.

We shall provide a sufficient condition for a closed convex set to have an upper semicontinuous multivalued metric projection, that is, if this closed convex set is an M-set defined below, then there exists an upper semicontinuous multivalued metric projection (see Theorem 2.2.4). However, it is not clear whether there is a continuous single-valued metric projection onto a closed convex M-set. We also give sufficient conditions for a closed convex set to have a continuous single-valued metric projection and examples of this type of set. A closed convex set for which there is a continuous metric projection will be called an M_∞-set. We shall give an example to show that an M_∞-set need not be an M-set. It is an interesting open question whether or not an M-set is an M_∞-set.

Weakly inward maps and generalized inward maps will be well-studied; their properties new or old and examples will be provided.

2.1 Approximatively compact sets

In this section we recall the concept of an approximatively compact set and give results on the existence of approximatively compact sets, some of which are new. The concept of approximatively compact set was first introduced and studied by N.V. Efimov and S.B. Stechkin (for example see [62]) and then by I. Singer [62]. The concept of approximatively compact sets was generalized by Reich [57] to Hausdorff topological vector spaces and used by Reich [57] and Sehgal and Singh [61] to study the best approximation problem and existence of fixed points for weakly inward maps in Hausdorff topological vector spaces. In Chapter 3 we shall develop a fixed point index for generalized inward compact maps defined on an approximatively compact convex set in Banach spaces.

Let K be a subset of a Banach space X. A sequence $\{x_n\} \subset K$ is called a *minimizing sequence* for a point $x \in X$ if

$$\lim_{n \to +\infty} \|x - x_n\| = d(x, K),$$

where $d(x, K) = \inf\{\|x - y\| : y \in K\}$.

Definition 2.1.1. A subset K of a Banach space X is said to be *approximatively compact* (for short, an M-set) relative to X if for each $x \in X$ every minimizing sequence $\{x_n\} \subset K$ for x contains a subsequence converging to an element of K.

The above definition can be found in [62] except for the terminology 'M-set'. We use the terminology 'M-set since it relates to the existence of metric projections and we shall introduce the concept of M_∞-set below and the two concepts are related.

It is obvious that a Banach space is an M-set relative to itself. Moreover, every M-set must be closed. If $K \neq X$, it is easy to verify that K is an M-set if and only if for each $x \in X \setminus K$ and sequence $\{y_n\} \subset K$ satisfying $\|x - y_n\| \to d(x, K)$ there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in K$ such that $y_{n_k} \to y$.

We now provide some new conditions which assure a set is an M-set.

Theorem 2.1.2. *Let K be a closed set in a Banach space X. Assume that the following condition holds.*

(S): Every bounded sequence $\{x_n\}$ of K contains a convergent subsequence.

Then K is an M-set relative to X.

The proof of Theorem 2.1.2 follows from Definition 2.1.1 and the condition (S).

Remark 2.1.3. The following assertions follow from Theorem 2.1.2 since the condition (S) is satisfied.

(1) Every compact set in a Banach space is an M-set.

(2) Every closed set in a finite dimensional Banach space is an M-set.

(3) Every closed set in a finite dimensional subspace of a Banach space X is an M-set relative to X. In particular, every finite dimensional subspace of a Banach space X is an M-set relative to X.

(4) Let K be a wedge in a Banach space such that $\partial K_1 = \{x \in K : \|x\| = 1\}$ is compact. Then K is an M-set.

Proof. We only show (4). Let $\{x_n\} \subset K$ be bounded. Then $\{x_n\} \subset \overline{K}_\rho = \{x \in K : \|x\| \leq \rho\}$ for some $\rho > 0$. Since $\overline{K}_\rho \subset \rho\, \mathrm{co}\{\partial K_1 \cup \{0\}\}$ and ∂K_1 is compact, so is $\overline{K}_\rho$. This implies that $\{x_n\}$ is precompact and thus has a subsequence which converges to an element in K. It follows from Theorem 2.1.2 that K is an M-set. □

The assertion (2) in Remark 2.1.3 shows that every closed convex set in a finite dimensional Banach space is an M-set. However, in an infinite dimensional Banach space, the result fails. The following new example shows this.

Example 2.1.4. Let $C[0,1]$ denote the Banach space of all continuous real-valued functions defined on $[0,1]$ with the norm $\|x\| = \max\{|x(t)| : t \in [0,1]\}$. Let $y_n(t) = t^n$ for $t \in [0,1]$ and $n \in \mathbb{N}$. Then $y_n \in C[0,1]$. Let $K = \overline{\mathrm{co}}\{y_n : n \in \mathbb{N}\}$. Then K is a closed convex subset of $C[0,1]$. It is easy to verify that $d(0,K) = 1$ and $\|0 - y_n\| = 1$ for each $n \in \mathbb{N}$. Therefore, $\|0 - y_n\| \to d(0,K)$. However, $\{y_n\}$ has no convergent subsequence. Hence K is not an M-set.

Therefore, we need to find suitable conditions on X and K to determine when K is an M-set. A nice additional condition on X is the property (H) of X which was first introduced by Fan and Glicksberg [19].

Definition 2.1.5. A Banach space is said to have the property (H) if, whenever a sequence $\{x_n\} \subset X$ satisfies the following two conditions:

(1) $x_n \rightharpoonup x$, where $\rightharpoonup$ denotes the weak convergence.

(2) $\lim_{n\to+\infty} \|x_n\| = x$,

then $\lim_{n\to+\infty} x_n = \|x\|$,

It is known that some special Banach spaces have property (H). The following definitions and properties of some special Banach spaces can be found, for example in [12], [29] and [50]. We only mention these definitions and give their properties without proofs.

Definition 2.1.6. (1) A Banach space X is uniformly convex if for every $\varepsilon \in (0,2]$ there exists $\delta(\varepsilon) > 0$ such that $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$ implies $\|(x+y)\|/2 \leq 1 - \delta(\varepsilon)$.
(2) A Banach space X is said to be locally uniformly convex if for every $\varepsilon \in (0,2]$ and $x \in X$ with $\|x\| = 1$, there exists $\delta = \delta(x,\varepsilon) > 0$ such that $\|y\| = 1$ and $\|y - x\| \geq \varepsilon$ implies $\|x+y\| \leq 2(1-\delta)$.
(3) A Banach space X is said to be k-locally uniformly convex if for every $\varepsilon > 0$ and $x \in X$ with $\|x\| = 1$, there exists $\delta = \delta(x,\varepsilon) > 0$ such that, whenever $x_1, ..., x_{k-1}$ has the following properties:

(*i*) $\|x_i\| \leq 1$,

(*ii*) $\|x - x_i\| \geq \varepsilon$,

then

$$\|x + x_1 + ..., +x_{k-1}\| \leq k(1-\delta).$$

(4) A Banach space X is said to be strictly convex if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$ implies $\|tx + (1-t)y\| < 1$ for every $t \in (0,1)$.

The relations between these special Banach spaces are collected in the following proposition.

Proposition 2.1.7. (1) *Every Hilbert space is uniformly convex.*
(2) *Every uniformly convex space is reflexive and locally uniformly convex.*
(3) *Every locally uniformly convex space is k-locally uniformly convex and strictly convex.*
(4) *Every k-locally uniformly convex space has property* (H) *(see Theorem 2.5.17, p. 56 in [29]).*

By the above proposition we see that every Hilbert space, uniformly convex space and locally uniformly convex space has property (H). However, we do not know if a strictly convex Banach space has the property (H).

The property (H) is not sufficient for a closed convex set to be an M-set. We also need to add a suitable condition to the closed convex set. Hence we introduce

Definition 2.1.8. Let K be a closed convex set in a Banach space X. K is said to have the property (W) if for any bounded sequence $\{x_n\} \subset K$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in K$ such that $x_{n_k} \rightharpoonup x$

The following lemma provides sufficient conditions for a set to have property (W).

Lemma 2.1.9. (1) *Any weakly compact convex set in a Banach space has the property* (W).

(2) *Any closed convex set in a reflexive Banach space has the property* (W).

Proof. (1) Let A be a weakly compact convex set in a Banach space X. It follows from Eberlein-Smulian Theorem (see for example Theorem 10.10 in [64]) that A is weakly closed and satisfies that for any sequence $\{x_n\} \subset A$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in A$ such that $x_{n_k} \rightharpoonup x$. Since A is weakly closed and convex, it follows from Theorem 3.12 in [59] that A is a closed convex set in $(X, \|.\|)$.

(2) The result follows from Theorem 10.6 in [64]. □

We now are in position to provide some sufficient conditions for a closed convex set in a Banach space to be an M-set.

We first have the following new result.

Theorem 2.1.10. *Let X be a Banach space with the property (H). Then any closed convex set of X with the property (W) is an M-set.*

Proof. Let K be a closed convex set in X and have the property (W). If $K = X$, the result holds. We assume that $K \neq X$. Let $x \in X \setminus K$ and let $\{y_n\} \subset K$ satisfy $\|x - y_n\| \to d(x, K)$. It is easy to verify that $\{y_n\}$ is bounded. By the property (W) of K there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in K$ such that $y_{n_k} \rightharpoonup y$. Since $\|.\|$ is weakly lower-semicontinuous, we have $\|x - y\| \leq \lim_{k\to\infty} \|x - y_{n_k}\|$ and thus, $\|x - y\| = d(x, K)$.

This implies $\|x - y_{n_k}\| \to \|x - y\|$. By the property (H) of X we have $x - y_{n_k} \to x - y$ and thus, $y_{n_k} \to y$. □

As special cases of Theorem 2.1.10 we have the following Corollaries.

Corollary 2.1.11. *Let X be a Banach space with the property (H) and K a weakly compact convex set of X. Then K is an M-set.*

This new result is a direct consequence of (1) in Remark 2.1.9.

Since a locally uniformly convex Banach space has the property (H), it follows from Corollary 2.1.11 that every weakly compact convex subset in a locally uniformly convex Banach space is an M-set.

By using the assertion (2) in Remark 2.1.9 we obtain

Corollary 2.1.12. *Let X be a reflexive Banach space with the property (H) and K a closed convex set of X. Then K is an M-set.*

The above Corollary 2.1.12 has been obtained by Singer [62]. Actually, Singer [62] has shown that a converse of Corollary 2.1.12 is true, that is, every closed convex subset of a Banach space is an M-set if and only if the space is reflexive and has the property (H). When the space X is strictly convex, the necessary and sufficient condition was first obtained by Fan and Glicksberg [19]. From Singer's result we see in every nonreflexive Banach space there exists a closed convex set that is not an M-set.

As special cases of Corollary 2.1.12 we have

Corollary 2.1.13. *Let X be a reflexive and locally uniformly convex Banach space and K a closed convex set of X. Then K is an M-set.*

Proof. Since X is locally uniformly convex Banach space, X has the property (H). The result follows from Corollary 2.1.12. □.

Remark 2.1.14. Since we don't know if a strictly convex Banach space has the property (H), we also don't know if every closed convex set in a reflexive and strictly convex Banach space is an M-set.

However, if we consider an equivalent norm in a reflexive Banach space, every closed convex set is an M-set under the new norm. To show this, we need a lemma which was obtained by Trojanski [65] (also see [50]).

Lemma 2.1.15. *Let X be a reflexive Banach space. Then there exist equivalent norms on X and X^* such that both spaces are dual to each other and are locally uniformly convex.*

By using Lemma 2.1.15 we obtain the following new and useful result.

Corollary 2.1.16. *Let $(X, \|.\|)$ be a reflexive Banach space. Then there exists an equivalent norm $\|.\|_0$ on X such that every closed convex set in $(X, \|.\|)$ is an M-set relative to $(X, \|.\|_0)$.*

Proof. It follows from Lemma 2.1.15 that there exists a equivalent norm $\|.\|_0$ on X such that $(X, \|.\|_0)$ is a reflexive and locally uniformly convex Banach space. Let K be a closed convex set in $(X, \|.\|)$, then K is a closed convex set in $(X, \|.\|_0)$. The result follows from Corollary 2.1.13. □

By using Corollary 2.1.13 we give a simple proof of the following result which was obtained in [61].

Corollary 2.1.17. *Let X be a uniformly convex Banach space and K a closed convex set of X. Then K is an M-set.*

Proof. Since X is uniformly convex, it is reflexive and locally uniformly convex. The result follows from Corollary 2.1.13. □

As an immediate corollary of Corollary 2.1.17, we have

Corollary 2.1.18. *Let X be a Hilbert space. Then any closed convex set of X is an M-set.*

The following new result provides a necessary and sufficient condition for a closed convex set in a strictly convex Banach space to be an M-set.

Theorem 2.1.19. *Let K be a closed convex set in a strictly convex Banach space X. Then the following condition are equivalent.*

(C_1) *K is an M-set.*

(C_2) *For each $x \in X \setminus K$ and sequence $\{y_n\} \subset K$ satisfying $\|x - y_n\| \to d(x, K)$ there exists $y \in K$ such that $y_n \to y$.*

Proof. It is obvious that (C_2) implies (C_1). On the other hand, assume that (C_1) holds. We prove (C_2). Let $x \in X \setminus K$ and $\{y_n\} \subset K$ satisfy $\|x - y_n\| \to d(x, K)$. Since K is an M-set, there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $y \in K$ such that $y_{n_k} \to y$. We prove $y_n \to y$. If not, there exist $a > 0$ and a subsequence $\{y_{m_j}\}$ of $\{y_n\}$ such that $\|y_{m_j} - y\| \geq a$. Since $\|x - y_{m_j}\| \to d(x, K)$ and K is an M-set, we may assume that $y_{m_j} \to y' \in K$. Hence we have $y' \neq y$. Since $\|.\|$ is lower-semicontinuous and $y, y' \in K$, we have $\|x - y\| = \|x - y'\| = d(x, K)$. Then the strict convexity of X implies $y = y'$, a contradiction. Hence $y_n \to y$ and (C_2) holds. □

Comments

In this section the following are new.

Theorem 2.1.2, Remark 2.1.3, Example 2.1.4, Definition 2.1.8, Remark 2.1.9, Theorem 2.1.10, Corollary 2.1.11, Corollary 2.1.13, Remark 2.1.14, Corollary 2.1.16 and Theorem 2.1.19.

2.2 Projections on convex sets of Banach spaces

Existence of projections (in particular, continuous metric projections) on closed convex sets of Banach spaces will play an important role in extending the classical fixed point index theories for either maps of condensing type or A-proper maps. In this section we shall investigate the existence of ε-projections (or retractions), in particular, continuous metric projections.

Definition 2.2.1. Let X be a Banach space, K a closed convex set of X and $\varepsilon > 0$. A continuous map $r : X \to K$ is said to be an ε-projection if it satisfies

$$\|x - r(x)\| \leq (1 + \varepsilon)d(x, K) \text{ for every } x \in X.$$

By the Dugundji extension theorem (see Theorem 1.2.3 or Section 18 in [34]), an ε-projection always exists for every $\varepsilon > 0$. Moreover, it is easy to see that if r_i is an ε_i-projection then $(1-t)r_1 + tr_2$ is an ε_3-projection for each $t \in [0, 1]$, where $\varepsilon_3 = \max\{\varepsilon_1, \varepsilon_2\}$.

Definition 2.2.2. Let X be a Banach space and K a closed convex set of X. A map r from X to K is called a *metric projection* if r satisfies

$$\|x - rx\| = d(x, K) \quad \text{for each } x \in X.$$

Note that Definition 2.2.2 does not require that r be continuous. It is obvious that if $K = X$, the identity map I is a unique metric projection from X to itself, where $I(x) = x$ for each $x \in X$.

If for $x \in X$ there exists $y \in K$ such that $\|x - y\| = d(x, K)$, we say that x has a nearest point y in K, that is, y is a point in K nearest to x.

Remark 2.2.3. If $K \neq X$ and $x \notin K$, then the set of nearest points in K of x is contained in ∂K, where ∂K denotes the boundary of K relative to X.

It is obvious that every continuous metric projection is an ε-projection for every $\varepsilon > 0$. Although, for every $\varepsilon > 0$, ε-projections always exist for any closed convex set K in an arbitrary Banach space X, it is not always possible to find a noncontinuous or continuous metric projection from X to K, that is, not every point $x \in X$ need have a nearest point in K. However, for some special closed convex sets of an arbitrary Banach space X, each point in X has at least one nearest point. Moreover, for some special spaces and special closed convex sets, a continuous metric projection always exists and may be a 1-set-contraction map in some special cases. Hence, we shall be particularly interested in finding conditions which assure the existence of a continuous metric projection and giving examples of these special spaces and closed convex sets.

Before we give the following result on metric projection, we recall the definition of a upper semicontinuous multivalued map.

Let X be a Banach space and we denote by 2^X the family of all subsets of X. For $B \subset X$, let

$$T^{-1}(B) = \{x \in D : T(x) \cap B \neq \emptyset\}.$$

The following definition can be found, for example in Definition 1.1, p. 4 in [13].

Definition 2.2.4. A map $T : D \subset X \to 2^X$ is said to be upper-semicontinuous if $T^{-1}(B)$ is closed in D whenever $B \subset X$ is closed in X.

The following result (ii), which is due to Singer [62], shows that if K is an M-set in a Banach space X, then each point in X has a nearest point in K. We provide a slightly different proof.

Theorem 2.2.5. *Let K be an M-set in a Banach space X. For each $x \in X$, let*

$$r(x) = \{y \in K : d(x, K) = \|x - y\|\}.$$

Then (i) $r(x)$ is a nonempty compact subset of K for each $x \in X$. Moreover, if K is convex, $r(x)$ is also convex for each $x \in X$.
(ii) $r : X \to 2^X$ is upper-semicontinuous.

Proof. (i) We first prove that $r(x)$ is compact set for each $x \in X$. In fact, let $\{y_n\} \subset r(x)$. Then $\{y_n\}$ is a minimizing sequence. Since K is an M-set, $\{y_n\}$ has a subsequence convergent to some element in K. Next, assume that K is convex, we prove that $r(x)$ is convex for each $x \in X$. Let $y_1, y_2 \in r(x)$ and $t \in [0, 1]$. Then

$$d(x, K) \leq \|x - (ty_1 + (1-t)y_2)\| \leq t\|x - y_1\| + (1-t)\|x - y_2\| \leq d(x, K).$$

This implies $d(x, K) = \|x - (ty_1 + (1-t)y_2)\|$. Finally, we prove $r(x) \neq \emptyset$ for each $x \in X$. In fact, if $x \in K$, we have $x \in r(x)$ and $r(x) \neq \emptyset$. Let $x \in X \setminus K$ and let $\{y_n\} \subset K$ satisfy $\|x - y_n\| \to d(x, K)$. Since K is an M-set, we may assume that $y_n \to y$ for some $y \in K$. This implies $\|x - y\| = d(x, K)$ and $y \in r(x)$, that is, $r(x) \neq \emptyset$.

(ii) Let B be closed in X and $\{x_n\} \subset \{x \in X : r(x) \cap B \neq \emptyset\}$ satisfy $x_n \to x_0 \in X$. We prove $x_0 \in \{x \in X : r(x) \cap B \neq \emptyset\}$. Let $y_n \in r(x_n) \cap B$ for each n. Then $\{y_n\} \subset K$ and $d(x_n, K) = \|x_n - y_n\|$. Hence $x_n \to x_0$ implies $\|x_0 - y_n\| \to d(x_0, K)$. Since K is approximately compact, we may assume that $y_n \to y$ and thus $y \in B$ and $\|x_0 - y\| = d(x_0, K)$. This implies $y \in r(x_0) \cap B$ and $x_0 \in \{x \in X : r(x) \cap B \neq \emptyset\}$. □

In general $r(x)$ defined in Theorem 2.2.5 may not a singleton even in a finite dimensional space. The following new example shows this.

Example 2.2.6. Let $C[0, 2]$ denote the Banach space of all continuous real-valued functions defined in $[0, 2]$ with the norm $\|x\| = \max\{|x(t)| : t \in [0, 2]\}$. Let X be the subspace generated by the two elements $\{1, x_0\}$, where $x_0(t) = 1 - (t-1)^2$ for each $t \in [0, 2]$. Then

X is a two dimensional subspace of $C[0,2]$. Let $K = \text{co}\{1, x_0\}$. Then for every $y \in K$, there exists $\lambda \in [0,1]$ such that $y(t) = (1-\lambda)+\lambda x_0(t) = (1-\lambda)+\lambda(2t-t^2)$ for $t \in [0,1]$. It is easy to see that $\|y\| = y(1) = 1$. Therefore, for every $y \in K$, we have $\|0-y\| = \|y\| = 1$ and thus $d(0, K) = 1$. This implies $r(0) = \{y \in K : d(0, K) = \|0 - y\|\} = K$ and thus $r(0)$ is not a singleton set.

There is an example in [62] which shows if K is not an M-set, the result (ii) in Theorem 2.2.5 may fail, that is, r may not be upper semicontinuous although $r(x)$ is a nonempty set for every $x \in X$.

It is known that an upper semicontinuous multivalued map may not have a continuous selection, that is, if g is an upper semicontinuous multivalued map, it is possible that we can not find a continuous single-valued map f such that $f(x) \in g(x)$. We employ a simple example given in [5]. Let $X = \mathbb{R}$ and a multivalued map F be defined by

$$F(x) = \begin{cases} \{-1\} & \text{if } x < 0, \\ [-1,1] & \text{if } x = 0, \\ \{1\} & \text{if } x > 0. \end{cases}$$

Then F is upper semicontinuous but it does not have any continuous selections defined in $\mathbb{R}$.

It is also known that for every lower semicontinuous multivalued map with nonempty closed convex values defined on a subset of a Banach space, there exists a continuous selection (see, for example, [46] and [13]). However, the multivalued map r given in Theorem 2.2.5 may not be lower-semicontinuous in general (see the remark at the end of section 1 in [62]). Hence, even when X is a finite dimensional Banach space, it is not clear whether there exists a continuous selection for the map r given in Theorem 2.2.5, that is, does there exist a continuous map $f : X \to K$ such that $f(x) \in r(x)$ for each $x \in X$? In other words, we don't know if there exists a continuous metric projection from X onto an M-set.

However, when X is strictly convex and K is a convex M-set, a continuous metric projection from X onto K always exists, that is, the $r(x)$ given in Theorem 2.2.5 is a singleton set for each $x \in X$ and is continuous.

Theorem 2.2.7. *Let X be a strictly convex Banach space and let K be a convex M-set. Then there exists a unique continuous metric projection from X onto K.*

Proof. For each $x \in X$, let $r(x) = \{y \in K : d(x, K) = \|x - y\|\}$. We show that r satisfies the following conditions:

(1) $r(x) \neq \emptyset$ for each $x \in X$ by Theorem 2.2.5.

(2) $r(x)$ is a singleton set for each $x \in X$.

Obviously, if $x \in K$ then $r(x) = \{x\}$. Assume that $x \in X \setminus K$, $y_1 \in K$ and $y_2 \in K$ such that $\|x - y_1\| = \|x - y_2\| = d(x, K)$. Then it is easy to verify that $\|x - (1-t)y_1 - ty_2\| = d(x, K)$ for each $t \in [0, 1]$ and this contradicts the strict convexity of X unless we have $y_1 = y_2$.

(3) r is continuous metric projection from X onto K.

Obviously, r is a metric projection from X onto K. We prove that r is continuous. Let $\{x_n\} \subset X$ and $x_n \to x$ be such that $\|x_n - r(x_n)\| = dist(x_n, K)$. This implies $\|x - r(x_n)\| \to d(x, K)$. If $x \in K$, then $r(x_n) \to x = r(x)$. If $x \in X \setminus K$, It follows from Theorem 2.1.19 that there exists $y \in K$ such that $r(x_n) \to y$. It is easy to see that $\|x - y\| = d(x, K)$ and $r(x) = y$ by (2). Hence $r(x_n) \to r(x)$. □

The continuity of r also follows directly from (ii) of Theorem 2.2.5 since a single-valued upper-semicontinuous map must be continuous.

In the following section we shall give examples which show that for a non-M-set continuous metric projections may still exist.

For a reflexive Banach space, if we consider an equivalent norm, we have the following new result whose proof follows directly from Theorem 2.2.7 and Corollary 2.1.16.

Corollary 2.2.8. *Let K be a closed convex set in a reflexive Banach space X. Then there exists a unique continuous metric projection from $(X, \|.\|_0)$ onto $(K, \|.\|_0)$, where $\|.\|_0$ is an equivalent norm on X as given in Corollary 2.1.16.*

If a continuous metric projection from a Banach space X onto a closed convex set K exists, it is easy to verify that the set of all the continuous metric projections from X onto K is a convex set. We collect the property in the following Proposition which will be used later.

Proposition 2.2.9. *Let X be a Banach space and K a closed convex set. If r_1, r_2 are continuous metric projections from X into K, then $tr_1+(1-t)r_2$ is also a metric projection for each $t \in [0,1]$.*

We end this section with several remarks and mention an open question which is related to these ideas.

Remark 2.2.10. We have shown that for an M-set K in an arbitrary Banach space X, every point in X has a nearest point in K (see Theorem 2.2.5). Moreover, if X is strictly convex and K is a convex M-set, there exists a unique continuous metric projection from X onto K.

Remark 2.2.11. It is easy to verify that if K is a closed convex set in a reflexive Banach space X, then every point in X has a nearest point in K. Moreover, if further X is strictly convex, it can be shown that there exists a unique metric projection from X onto K which, however, may not be continuous.

Remark 2.2.12. We see in the above Remarks 2.2.10 and 2.2.11 that strict convexity of X and convexity of K are used to assure uniqueness of nearest points. An inverse question is that if X is strictly convex, is the convexity of K necessary? That is, if we assume that X is a strictly convex Banach space and K is a nonempty set in X which is such that every point in X has a unique nearest point in K (that is, is a so-called Chebyschev set), whether the set K is convex. Even when X is a Hilbert space, this question has been open for a long time (see [30] and the references therein for more details, and also see Exercise 9, p.67 in [12]).

Comments

In this section the following are new.

Remark 2.2.3, Example 2.2.6, Theorem 2.2.7, Corollary 2.2.8, Proposition 2.2.9 and Remarks 2.2.10, 2.2.11 and 2.2.12.

The proof of Theorem 2.2.5 is new.

2.3 M_l-sets

In the above section we have shown that for each M-set in a strictly convex Banach space there exists a unique continuous metric projection. In this section, we shall consider closed convex sets K for which there exists a continuous metric projection r that is a k-γ-contraction and give an example to show it is possible that for a non-M-set, a continuous metric projection may exist and be a 1-set contraction.

In this section we always denote by γ the set (or ball)-measure relative to X. We define

$$\gamma(r) = \inf\{k : r \text{ is a } k\text{-}\gamma\text{-contraction }\}.$$

Definition 2.3.1. Let X be a Banach space and K a closed convex subset of X. K is said to be an M_l-set relative to X for some $1 \leq l < +\infty$ if there exists a continuous metric projection r from X to K such that $\gamma(r) = l$. K is called an M_∞-set if the metric projection is only continuous. For convenience, in this last case we write $\gamma(r) = \infty$.

Example 2.3.2. Let X be a strictly convex Banach space and let K be an M-set. Then K is a M_∞-set. The result follows from Theorem 2.2.7.

Example 2.3.3. Let X be a reflexive and locally uniformly convex Banach space and K a closed convex set of X. Then K is a M_∞-set. The result follows from Corollary 2.1.13 and Theorem 2.2.7.

By using Corollary 2.2.8 we obtain

Corollary 2.3.4. *Let X be a reflexive Banach space with the norm $\|.\|$. Then every closed convex set in $(X, \|.\|)$ is an M_∞-set relative to $(X, \|.\|_0)$, where the norm $\|.\|_0$ is the same as in Corollary 2.2.8.*

We will have greatest flexibility in the choice of mappings when K is an M_1-set. We therefore give some examples of M_1-sets.

Example 2.3.5. Let X be a Hilbert space, then any closed convex set K in X is an M_1-set.

Proof. Let K be a closed convex set in X. It follows from Proposition 1.3.18 that there exists a nonexpansive retraction $r : X \to K$ defined by

$$\|x - rx\| = d(x, K).$$

Hence r is a continuous metric projection from X onto K. Since r is nonexpansive, it follows from Lemma 1.3.28 and Theorem 1.3.29 that r is 1-set and 1-ball contractive. The result follows. □

The following Examples can be found in [42], but is essentially due to Nussbaum.

Example 2.3.6. In any Banach space X, let K be a ball. Then K is an M_1-set.

Proof. Let $x_0 \in X$, $a > 0$ and $K = \{x : \|x - x_0\| \leq a\}$. We define a map r from X onto K by

$$r(x) = \begin{cases} x & \text{if } \|x - x_0\| \leq a, \\ x_0 + \dfrac{a}{\|x - x_0\|}(x - x_0) & \text{if } \|x - x_0\| > a, \end{cases}$$

Then it is easy to verify that r is a continuous metric projection from X onto K and $r(\Omega) \subset \text{co}(x_0 \cup \Omega)$ for any bounded set of X. Hence r is a 1-γ-contraction. □

The following result shows there exists non-M-set for which there is a 1-set-contractive metric projection.

Example 2.3.7. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $X = C(\overline{\Omega})$ be the space of continuous functions endowed with the norm $\|u\| = \sup_{x \in \overline{\Omega}} |u(x)|$. Then the cone $K = \{u \in X : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\}$ is an M_1-set. Moreover, when $n = 1$ and $\Omega = [0, 1]$, K is not an M-set.

Proof. We define a map r from X onto K by

$$(ru)(x) = u^+(x) = \max\{u(x), 0\}.$$

It follows from Theorem 1.1.24 and Example 1.1.25 that r is nonexpansive and thus, continuous. Moreover, it is easy to verify that for each $v \in K$,

$$|u(x) - (ru)(x)| \leq |u(x) - v(x)| \quad \text{for each } x \in \Omega.$$

Hence r is a metric projection from X onto K. Therefore, K is an M_1-set.

We show K is not an M-set. In fact, let $x(t) = t - 1$ for $t \in [0,1]$ and $x_n(t) = t^n$ for $t \in [0,1]$. Then $x \notin K$ and $\{x_n\} \subset K$. It is easy to see that $rx \equiv 0$ for $t \in [0,1]$ and $\|x - rx\| = d(x, K) = 1$. Moreover, we have $\|x - x_n\| = 1$ for all n and $\lim_{n\to+\infty} \|x - x_n\| = d(x, K)$. However, it is known that the sequence $\{x_n\}$ has no convergent subsequence.

Remark 2.3.8. The assertion in Example 2.3.7 that K is not an M-set is new.

Example 2.3.9. Let Ω be as in Example 2.3.7, let $X = L^p(\Omega), 1 \le p \le \infty$, and let $K = \{u \in X : u(x) \ge 0 \text{ a.e.}\}$. Then K is an M_1-set.

Proof. We define a map r from X onto K by $ru = u^+$. It follows from Theorem 1.1.24 and Example 1.1.26 that r is nonexpansive. Similar to the proof in Example 2.3.7, r is a metric projection from X onto K. The result follows. □

We end this section with the following result which will be used to show that our definition of index does not depend on the choice of suitable metric projection if more than one exists.

Lemma 2.3.10. *If K is an M_l-set in a Banach space X for some $l \in [1, \infty]$ and r_1, r_2 are l-γ-contractive metric projections from X into K, then $tr_1 + (1-t)r_2$ is also an l-γ-contractive metric projection for each $t \in [0,1]$.*

Comments

The following are new.

Example 2.3.2, Example 2.3.3, Corollary 2.3.4 and Remark 2.3.8.

The other results in this section are contained in [42].

2.4 Weakly inward mappings

In this section we shall recall the concepts of inward and weakly inward sets, and inward and weakly inward maps and give some new results and examples.

Inward maps were apparently first studied by Halpern (see for example [26]). Then many results were obtained in the literature concerning inward and weakly inward maps in Halpern's sense, for both single and multivalued maps. See for example [7], [9], [14], [15], [16], [18], [25], [38], [42], [43] [56] and [57].

Let us start with the concepts of inward and weakly inward sets.

Definition 2.4.1. Let K be a closed convex set in a Banach space X and $x \in K$. The set

$$I_K(x) = \{x + c(z - x) : z \in K \text{ and } c \geq 0\}$$

is said to be the inward set of x relative to K. The closure of $I_K(x)$, $\overline{I}_K(x)$ is said to be the weakly inward set of x relative to K.

Geometrically the inward set $I_K(x)$ is the union of all rays beginning at x and passing through some other point of K. Hence, if x is an interior point of K, then $I_K(x) = X$. Because of the convexity of K, it is easy to verify that

$$I_K(x) = \{x + c(z - x) : z \in K \text{ and } c \geq 1\}.$$

The following two simple examples can be readily verified by Definition 2.4.1.

Example 2.4.2. Let $X = \mathbb{R}^2$ and $K = \{(x,y) \in \mathbb{R}^2 : x, y \in [0,\infty)\}$. Then for $(x,y) \in K$, we have

$$\overline{I}_K((x,y)) = \begin{cases} K & \text{if } x = y = 0 \\ \{(u,v) \in X : u \in \mathbb{R}, v \geq 0\} & \text{if } x > 0,\ y = 0 \\ \{(u,v) \in X : u \geq 0, v \in \mathbb{R}\} & \text{if } x = 0,\ y > 0 \\ X & \text{if } x > 0,\ y > 0 \end{cases}$$

Example 2.4.3. Let $X = \mathbb{R}^2$ and $K = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R} \text{ and } y \geq 0\}$. Then for $(x,y) \in K$, we have

$$\overline{I}_K((x,y)) = \begin{cases} X & \text{if } x \in \mathbb{R}, y > 0 \\ K & \text{if } x \in \mathbb{R}, y = 0 \end{cases}$$

In each case a sketch makes the assertions obvious.

We now give the properties of weakly inward sets.

Proposition 2.4.4. *(i) Let K be a closed convex set in X. Then for $x, y \in K$ and $t \in [0,1]$,*

$$tI_K(x) + (1-t)I_K(y) = I_K(tx + (1-t)y) \quad \textit{and}$$

$$t\overline{I}_K(x) + (1-t)\overline{I}_K(y) \subset \overline{I}_K(tx + (1-t)y).$$

(ii) $\overline{I}_K(x)$ *is a convex set containing* K *for each* $x \in K$.

(iii) If K *is a cone or a wedge in* X*, then* $\overline{I}_K(x)$ *is a wedge for each* $x \in K$.

Proof. We first prove that for $x, y \in K$ and $t \in [0,1]$,

$$tI_K(x) + (1-t)I_K(y) = I_K(tx + (1-t)y). \tag{1}$$

In fact, let $w \in tI_K(x) + (1-t)I_K(y)$, then $w = tu + (1-t)v$ for some $u \in I_K(x)$ and $v \in I_K(y)$. Hence there exist $c_i \geq 1$ and $z_i \in K$, $i = 1, 2$ such that

$$u = x + c_1(z_1 - x) \quad \text{and } v = y + c_2(z_2 - y).$$

Let $c = \max\{c_1, c_2\}$ and

$$z = c_1c^{-1}tz_1 + c_2c^{-1}(1-t)z_2 + (1 - c_1c^{-1})tx + (1 - c_2c^{-1})(1-t)y.$$

Since $x, y, z_1, z_2 \in K$ and K is convex, we have $z \in K$. Moreover, we have

$$w = tx + (1-t)y + c(z - (tx + (1-t)y))$$

and thus $w \in I_K(tx + (1-t)y)$.

On the other hand, let $u \in I_K(tx + (1-t)y)$. Then we have

$$u = tx + (1-t)y + c(z - (tx + (1-t)y)) = t(x + c(z - x)) + (1-t)(y + c(z - y)).$$

Since $c \geq 1$ and $z \in K$, we have $u \in tI_K(x) + (1-t)I_K(y)$.

Now, we prove (i). Let $w \in t\overline{I}_K(x) + (1-t)\overline{I}_K(y)$. Then $w = tu + (1-t)v$ for some $u \in \overline{I}_K(x)$ and $v \in \overline{I}_K(y)$. Hence there exist sequences $\{u_n\} \subset I_K(x)$ and $\{v_n\} \subset I_K(y)$ such that $u_n \to u$ and $v_n \to v$. By (1) we have $tu_n + (1-t)v_n \in I_K(tx + (1-t)y)$ and thus $w \in \overline{I}_K(tx + (1-t)y)$.

(ii) The result follows easily from the definition.

(iii) It is sufficient to prove $\lambda I_K(x) \subset I_K(x)$ for each $x \in K$.

In fact, let $\lambda y = \lambda(x + c(z - x)) \in \lambda I_K(x)$, where $c \geq 1$ and $z \in K$. Then we have

$$\lambda y = x + (1 + \lambda c - \lambda)(\lambda c(1 + \lambda c - \lambda)^{-1}z - x) \quad \text{and } 1 + \lambda c - \lambda \geq 0.$$

This implies $\lambda y \in I_K(x)$. □

The two assertions in (i) of Proposition 2.4.4 are new. The assertion (ii) is well known. (iii) is mentioned in [43] and [42] without proofs. When K is a cone, (iii) is proved in [28].

The following result is new and gives a relationship between K and $\overline{I}_K(x)$.

Proposition 2.4.5. *Let K be a closed convex set of X and $y \in X$. Then for each $x \in K$,*

$$\lim_{t \to 0^+} t^{-1} d((1-t)x + ty, K) = d(y, \overline{I}_K(x)).$$

Proof. We first prove that for $x \in K$ and $t \in (0,1]$, we have

$$t^{-1} d((1-t)x + ty, K) = d(y, (1-t^{-1})x + t^{-1}K) \tag{2}$$

In fact, for $u \in K$ we have

$$t^{-1} d((1-t)x + ty, K) \le t^{-1} \|(1-t)x + ty - u\| = \|y - (1-t^{-1})x - t^{-1}u\|.$$

This implies $t^{-1} d((1-t)x + ty, K) \le d(y, (1-t^{-1})x + t^{-1}K)$. On the other hand, for $u \in K$ we have

$$d(y, (1-t^{-1})x + t^{-1}K) \le \|y - (1-t^{-1})x - t^{-1}u\| = t^{-1}\|(1-t)x + ty - u\|.$$

This implies $t^{-1} d((1-t)x + ty, K) \ge d(y, (1-t^{-1})x + t^{-1}K)$. Consequently, (2) holds.

Since $(1-t^{-1})x + t^{-1}K \subset I_K(x)$ for each $x \in K$ and $t \in (0,1]$, we have

$$d(y, I_K(x)) \le d(y, (1-t^{-1})x + t^{-1}K) = t^{-1} d((1-t)x + ty, K).$$

On the other hand, for $\varepsilon > 0$, we have $\|y - u\| \le d(y, I_K(x)) + \varepsilon$ for some $u \in I_K(x)$. Therefore, there exist $\lambda_0 \ge 1$ and $z \in K$ such that $u = x + \lambda_0(z - x)$. Since K is convex and $x \in K$, $z(s) := sz + (1-s)x \in K$ for every $s \in (0,1]$, that is, $z - x = (z(s) - x)/s$. Hence, we have

$$u = x + (\lambda_0/s)(z(s) - x).$$

Let $t_0 = 1/\lambda_0$, $t^{-1} = \lambda_0/s$ and $u(t) = z(s)$. Then $u = x + t^{-1}(u(t) - x)$ for all $t \in (0, t_0]$. This implies $u \in (1-t^{-1})x + t^{-1}K$ for all $t \in (0, t_0]$. Hence we have

$$t^{-1} d((1-t)x + ty, K) = d(y, (1-t^{-1})x + t^{-1}K) \le \|y - u\|.$$

Therefore, for each $t \in (0, t_0]$, we have

$$d(y, I_K(x)) \le t^{-1} d((1-t)x + ty, K) \le d(y, I_K(x)) + \varepsilon.$$

This implies

$$\lim_{t \to 0^+} t^{-1} d((1-t)x + ty, K) = d(y, \overline{I}_K(x)).$$

□

Remark 2.4.6. By proposition 2.4.5 we see that this limit $\lim_{t\to 0^+} t^{-1}d((1-t)x+ty, K)$ always exists for every closed convex set in X. However, In some references, for example, [70] and [42], the lower limit was taken since they did not show the limit exists.

The following Proposition is a variant of Lemma 18.2 in [12] which gives necessary and sufficient conditions for a point $y \in X$ to belong to $\overline{I}_K(x)$. Since $\overline{I}_K(x) = X$ if x is an interior point of K, it is sufficient to consider those points in the boundary of K. We denote by ∂K the boundary of K relative to X.

Proposition 2.4.7. *Let K be a closed convex set of X with $K \neq X$ and $x \in \partial K$. Then the following conditions are equivalent.*

(*a*) $y \in \overline{I}_K(x)$.

(*b*) *For each $x^* \in X^*$ with $x^*(x) = \sup\{x^*(z) : z \in K\}$, $x^*(y) \leq x^*(x)$.*

(*c*) $\lim_{t\to 0^+} t^{-1}d((1-t)x + ty, K) = 0$.

Proof. It follows from Proposition 2.4.5 that (a) and (c) are equivalent. We prove that (a) and (b) are equivalent. We first prove that (a) implies (b). It is sufficient to show that if $y \in I_K(x)$, then (b) holds. Let $y \in I_K(x)$. Then $y = x + c(u - x)$ for some $c \geq 1$ and some $u \in K$. Let $x^* \in X^*$ with $x^*(x) = \sup\{x^*(z) : z \in K\}$. Then $x^*(u) \leq x^*(x)$ and

$$x^*(y) = x^*(x) + c(x^*(u) - x^*(x)) \leq x^*(x).$$

Hence, (b) holds. On the other hand, assume that (b) holds. If $y \notin \overline{I}_K(x)$, it follows from the separation theorem for convex sets that there exists $x^* \in X^*$ such that

$$x^*(y) > \lambda = \sup\{x^*(z) : z \in \overline{I}_K(x)\}.$$

Hence we have $x^*(y) > x^*(x)$ and

$$x^*(y) > \lambda \geq x^*(x) + c(x^*(z) - x^*(x)) \quad \text{for all } c \geq 1 \quad \text{and } z \in K.$$

Since c is arbitrary, the latter implies $x^*(z) - x^*(x) \leq 0$ for all $z \in K$, that is, $x^*(x) = \sup\{x^*(z) : z \in K\}$. We have shown $x^*(y) > x^*(x)$, which contradicts (b). □

As an application of Proposition 2.4.7 we obtain a useful property of weakly inward sets which will be used later.

Corollary 2.4.8. *Let K be a closed convex set in a Banach space X and $x \in K$. Then*

$$(1-a)x + a\overline{I}_K(x) \subset \overline{I}_K(x) \quad \textit{for every } a \geq 0.$$

Proof. If x is an interior point of K, the result obviously holds. We assume that $x \in \partial K$. Let $u = (1-a)x + aw$ with $w \in \overline{I}_K(x)$. For every $x^* \in X^*$ with $x^*(x) = \sup\{x^*(z) : z \in K\}$, we have $x^*(w) \leq x^*(x)$ by using (b) of Proposition 2.4.7. This implies

$$x^*(u) = (1-a)x^*(x) + ax^*(w) \leq x^*(x).$$

It follows from Proposition 2.4.7 that $u \in \overline{I}_K(x)$. □

Another proof of Corollary 2.4.8 will be given in Section 5.3 in Chapter 5.

When K is a wedge, the condition (b) in Proposition 2.4.7 simplifies.

Proposition 2.4.9. *Let K be a wedge of X with $K \neq X$ and $x \in \partial K$. Then the following conditions are equivalent.*

(a') $y \in \overline{I}_K(x)$.

(b') *For each $x^* \in K^*$ with $x^*(x) = 0$, $x^*(y) \geq 0$.*

Proof. It is obvious that (b') is equivalent the following condition:

(b'') for each $x^* \in X^*$ with $x^*(z) \leq 0$ for all $z \in K$ and $x^*(x) = 0$, $x^*(y) \leq 0$.

Therefore, it is sufficient to show that (b) and (b'') are equivalent. Assume that (b) holds. Let $x^* \in X^*$ with $x^*(z) \leq 0$ for all $z \in K$ and $x^*(x) = 0$. Then $x^*(x) = \sup\{x^*(z) : z \in K\}$. It follows from (b) that $x^*(y) \leq x^*(x) = 0$ and thus (b'') holds. On the other hand, assume that (b'') holds. Let $x^*(x) = \sup\{x^*(z) : z \in K\}$. Since K is a wedge and $x \in K$, we have $2x \in K$ and $x/2 \in K$. Therefore, we obtain $x^*(2x) \leq x^*(x)$ and $x^*(x/2) \leq x^*(x)$. These inequalities imply $x^*(x) = 0$ and thus, $x^*(z) \leq x^*(x) = 0$ for all $z \in K$. It follows from (b'') that $x^*(y) \leq 0 = x^*(x)$. This shows (b) holds. □

Note that if $x \in K$ and there exists $x^* \in X^*$ with $x^* \neq 0$ such that $x^*(x) = \sup\{x^*(z) : z \in K\}$, then $x \in \partial K$. Therefore, in the conditions (b), (b') and (b'') we assume $x \in \partial K$.

Using Proposition 2.4.9, we obtain the following new results.

Example 2.4.10. Let $X = C[0,1]$ and $K = C_+[0,1]$. Let $x \in K$ and define

$$E(x) = \{t \in [0,1] : x(t) = 0\}.$$

Then (1) if $E(x) = \emptyset$, that is $x(t) > 0$ for $t \in [0,1]$, or equivalently, x is an interior point of K, then $\overline{I}_K(x) = X$.

(2) If $E(x) \neq \emptyset$, that is, $x \in \partial K$, then the set $\{y \in X : y(t) \geq 0 \text{ for } t \in E(x)\} \subset \overline{I}_K(x)$, that is, if the values of y are non-negative at all points at which the values of x are zero, then y belongs to the weakly inward set $\overline{I}_K(x)$ of x.

Example 2.4.11. Let $X = L^p[0,1]$ for $1 < p < \infty$ and $K = L^p_+[0,1]$. Let m denote the Lebesgue measure on $\mathbb{R}$. Let $x \in K$ and define

$$E(x) = \{t \in [0,1] : x(t) = 0\}.$$

Then (1) if $m(E(x)) = 0$, that is, $x(t) > 0$ a.e. for $t \in [0,1]$, then $\overline{I}_K(x) = X$.

(2) If $m(E(x)) \neq 0$, then the set $\{y \in X : y(t) \geq 0 \text{ a.e for } t \in E(x)\} \subset \overline{I}_K(x)$.

We have shown the inclusion $K \subset \overline{I}_K(x)$ for every $x \in K$. Hence, we have $K \subset \cap_{x\in K}\overline{I}_K(x)$. An interesting problem is whether they are equal. In the general case we don't know whether they are equal, but we can show that the result holds under suitable conditions.

It is clear that if K is a wedge in an arbitrary Banach space then $K = \cap_{x\in K}\overline{I}_K(x)$ since $K = \overline{I}_K(0)$. It can also be shown by a direct argument that $K = \cap_{x\in K}I_K(x)$ for any closed convex set K in any Banach space X. In fact, let $u \notin K$ and $x_0 \in K$. Define $\Lambda = \{t \in [0,1] : tu + (1-t)x_0 \in K\}$. Then $0 \in \Lambda$ and thus $t_0 := \sup \Lambda$ exists. Let $y = t_0 u + (1-t_0)x_0$. Then $y \in K$ since K is closed. By the definition of t_0, we see that $\lambda u + (1-\lambda)y \notin K$ for all $\lambda \in (0,1]$ and thus $u \notin I_K(y)$. This implies $u \notin \cap_{x\in K}I_K(x)$. Hence we have $\cap_{x\in K}I_K(x) \subset K$. This, together with $K \subset \cap_{x\in K}I_K(x)$, implies $K = \cap_{x\in K}I_K(x)$.

The following result is new and generalizes Proposition 4.1 in [43] , that is, Proposition 5.3.1 in Chapter 5 below.

Proposition 2.4.12. *Let K be a closed convex set in a Banach space X such that for every $x \in X$ there exists $r(x) \in K$ such that $\|x - r(x)\| = d(x, K)$. Then we have*

$$K = \cap_{x \in K} \overline{I}_K(x).$$

Proof. We first prove that for every $x \in X$,

$$d(x, K) = d(x, \overline{I}_K(r(x))).$$

It is sufficient to show $d(x, K) \leq d(x, \overline{I}_K(r(x)))$. Let $\delta = d(x, I_K(r(x)))$. For every $\varepsilon > 0$, there exists $u \in I_K(r(x))$ such that

$$\|x - u\| \leq \delta + \varepsilon.$$

Since $u \in I_K(r(x))$, there exist $z \in K$ and $t \in (0, 1]$ such that $z = tu + (1-t)r(x)$. Hence we have

$$d(x, K) \leq \|x - z\| \leq t\|x - u\| + (1-t)\|x - r(x)\| \leq t(\delta + \varepsilon) + (1-t)d(x, K).$$

This implies $d(x, K) \leq \delta + \varepsilon$ and thus $d(x, K) \leq d(x, I_K(r(x)))$.

We now prove that $K = \cap_{x \in K} \overline{I}_K(x)$. It is sufficient to show $\cap_{x \in K} \overline{I}_K(x) \subset K$. Let $y \in \cap_{x \in K} \overline{I}_K(x)$. Then $y \in \overline{I}_K(r(y))$ and thus $d(y, \overline{I}_K(r(y)) = 0$. We have shown above that $d(y, K) = d(y, \overline{I}_K(r(y))$ and so $y \in K$. □

By Proposition 2.4.12 we see that if K is a closed convex M-set in a Banach space, or K is any closed convex set in a reflexive Banach space, then $K = \cap_{x \in K} \overline{I}_K(x)$.

Remark 2.4.13. The equality $K = \cap_{x \in K} \overline{I}_K(x)$ will be utilized in Chapter 5 below and it can be applied to determine whether a constant map defined in K is a weakly inward map. Indeed, if $K = \cap_{x \in K} \overline{I}_K(x)$, then a map $A : K \to X$ defined by $Ax = y_0$, where y_0 is a point in X, is weakly inward if and only if $y_0 \in K$. Therefore, if K is a wedge, then a constant map is weakly inward if and only if the constant belongs to the wedge K.

Following the concepts of inward and weakly inward sets, we can introduce the definitions of inward and weakly inward maps.

Definition 2.4.14. Let K be a closed convex set in a Banach space X. A map $A : D \subset K \to X$ is said to be inward (respectively, weakly inward) on D relative to K if $Ax \in I_K(x)$ (resp. $Ax \in \overline{I}_K(x)$) for $x \in D$.

By using Definition 2.4.14 and examples 2.4.2 and 2.4.3 we can easily show that the following maps are weakly inward maps.

Example 2.4.15. Let $X = \mathbb{R}^2$, $K = \{(x, y) \in \mathbb{R}^2 : x, y \in [0, \infty)\}$ and D a subset of K such that $0 \in D$. Let $f_1, f_2 : D \to \mathbb{R}$ be two maps such that the following condition holds.

(W_1) $f_1((0, y)) \geq 0$ for each $(0, y) \in D$ and $f_2((x, 0)) \geq 0$ for each $(x, 0) \in D$.

Define a map $A : D \to X$ by

$$A(x, y) = (f_1(x, y), f_2(x, y)).$$

Then it follows from Example 2.4.2 and Definition 2.4.14 that A is weakly inward on D relative to K.

Example 2.4.16. Let $X = \mathbb{R}^2$, $K = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R} \text{ and } y \geq 0\}$ and D a subset of K such that $0 \in D$. Let $f_1, f_2 : D \to \mathbb{R}$ be two maps. Assume that f_2 satisfies the following condition.

(W_2) $f_2((x, 0)) \geq 0$ for each $(x, 0) \in D$.

Define a map $A : D \to X$ by

$$A(x, y) = (f_1(x, y), f_2(x, y)).$$

Then it follows from Example 2.4.3 and Definition 2.4.14 that A is weakly inward on D relative to K.

By Propositions 2.4.7 and 2.4.9 we obtain some necessary and sufficient conditions for a map to be weakly inward.

Theorem 2.4.17. *Let K be a closed convex set of X with $K \neq X$ and $A : D \subset K \to X$ a map. Then the following conditions are equivalent.*

(*a*) *A is weakly inward on D relative to K.*

(*b*) *For each $x \in \partial K \cap D$ and each $x^* \in X^*$ with $x^*(x) = \sup\{x^*(z) : z \in K\}$, $x^*(Ax) \leq x^*(x)$.*

(*c*) *For each $x \in K$, $\lim_{t \to 0^+} t^{-1} d((1-t)x + tAx, K) = 0$.*

If K is a wedge in X with $K \neq X$, then the following conditions are equivalent.

(a') *A is weakly inward on D relative to K.*

(b') *For each $x \in \partial K \cap D$ and each $x^* \in K^*$ with $x^*(x) = 0$, $x^*(Ax) \geq 0$.*

Remark 2.4.18. Let K be a closed convex set in a Banach space X and D a subset of K. Let $A : D \to X$ be a weakly inward map. Then $A(D) \subset Y$, where Y denotes the closure of the subspace of X generated by K. Hence, if K is a *closed subspace* of X, then $A : D \to X$ is a weakly inward map if and only if $A(D) \subset K$.

We now give a new example of a weakly inward map in $\mathbb{R}^n$, which contains example 2.4.15. In the following Chapter we shall give examples of weakly inward maps in infinite dimensional Banach spaces.

Example 2.4.19. Let $I = \{1, ..., n\}$ and D be a subset in $\mathbb{R}^n$ such that $0 \in D$. Let $f_i : D \subset \mathbb{R}^n_+ \to \mathbb{R}$ be maps for each $i \in I$. We define a map $A : D \subset \mathbb{R}^n_+ \to X$ by

$$Ax = (f_1(x), ..., f_n(x)).$$

Then the following conditions are equivalent.

(1) A is a weakly inward map on D relative to $K = \mathbb{R}^n_+$.

(2) $f_i(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \geq 0$ for all $i \in I$ and $x_j \geq 0$ for $j \neq i$.

Proof. Assume that (2) holds, that is, $f_i(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \geq 0$ for all $i \in I$. Let $x = \{x_1, ..., x_n\} \in \partial K \cap D$ and $x^* = (a_1, ..., a_n) \in (\mathbb{R}^n_+)^* = K$ with $x^*(x) = 0$, that is, $a_i \geq 0$ for $i \in I$ and $\sum_{i=1}^n a_i x_i = 0$. Let $I_1 = \{i \in I : a_i = 0\}$ and $I_2 = \{i \in I : a_i \neq 0\}$. Since $x_i \geq 0$ for all $i \in I$, $\sum_{i=1}^n a_i x_i = 0$ implies $a_i x_i = 0$ for each $i \in I$ and $x_i = 0$ if $i \in I_2$. Thus we have

$$x^*(x) = \sum_{i=1}^n a_i f_i(x) = \sum_{i \in I_2} a_i f_i(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., n) \geq 0.$$

On the other hand, Assume that A is a weakly inward map. For each fixed $i \in I$ and each fixed $x = \{x_1..., 0, ..., x_n\} \in \partial K \cap D$, let $x^* = \{0, ..., 1, ..., 0\} \in (R^n_+)^*$. Then $x^*(x) = 0$. Since A is weakly inward, $x^*(Ax) \geq 0$. This implies $f_i(x_1..., 0, ..., x_n) \geq 0$. □

We end this section with the following new result.

Theorem 2.4.20. *Let X be a reflexive Banach space with the norm $\|.\|$ and K a closed convex set in $(X, \|.\|)$. Assume that $A : D \subset K \to X$ is a weakly inward compact* *map relative to K. Then there is an equivalent norm $\|.\|_0$ such that K is an M_∞-set in $(X, \|.\|_0)$ and A is a weakly inward compact map relative to $(K, \|.\|_0)$.*

Proof. The first result has been shown in Corollary 2.3.4. Note that $(\overline{I}_K(x), \|.\|) = (\overline{I}_K(x), \|.\|_0)$. Hence A is weakly inward in $(X, \|.\|)$ if and only if it is weakly inward in $(X, \|.\|_0)$. $\square$

Comments

The following are new. Example 2.4.2, Example 2.4.3, (i) in Proposition 2.4.4, Proposition 2.4.5, Example 2.4.10, Example 2.4.11, Proposition 2.4.12, Remark 2.4.13, Example 2.4.15, Example 2.4.16, Remark 2.4.18, Example 2.4.19 and theorem 2.4.20.

Corollary 2.4.8 is obtained in [43] and this proof of Corollary 2.4.8 is new.

2.5 Generalized inward maps

In this section we shall introduce two new concepts: a generalized inward set and a generalized inward map. We shall show that a generalized inward set contains (possibly strictly) a weakly inward set, and the class of generalized inward maps strictly contains that of weakly inward maps. We give some examples of generalized inward sets and generalized inward maps and provide necessary and sufficient conditions for a map to be generalized inward.

Definition 2.5.1. Let K be a closed convex set in a Banach space X and $x \in K$. Then the set

$$G_K(x) = \{y \in X \setminus K : d(y, K) < \|y - x\|\} \cup K$$

is said to be the generalized inward set of x relative to K.

The following result shows a relation between a generalized inward set $G_K(x)$ and a weakly inward set $\overline{I}_K(x)$.

Lemma 2.5.2. *Let K be a closed convex set in a Banach space X. Then*

$$\overline{I}_K(x) \subset G_K(x) \quad \textit{for each } x \in K.$$

Proof. Let $x \in K$ and $y \in \overline{I}_K(x)$. If $y \in K$, then $y \in G_K(x)$. Now we assume that $y \notin K$. Let $a \in (0,1)$, then there exists $z \in I_K(x)$ such that $z \notin K$ and $\|y-z\| \leq a\|y-x\|$. Since $z \in I_K(x) \setminus K$, there exist $t \in (0,1)$ and $u \in K$ such that $u = (1-t)z + tx \in K$. Hence, we have

$$d(y,K) \leq \|y-u\| \leq (1-t)\|y-z\| + t\|y-x\| \leq (1-t)a\|y-z\| + t\|y-x\| < \|y-x\|.$$

This implies $y \in G_K(x)$. □

The following example shows that $G_K(x)$ may strictly contain $\overline{I}_K(x)$ and may not be convex.

Example 2.5.3. Let $X = \mathbb{R}^2$ and $K = \{(x,y) : x^2 + y^2 \leq 1\}$. Then we have

$$G_K((0,-1)) = \mathbb{R}^2 \setminus \{(0,\lambda) \in \mathbb{R}^2 : \lambda < -1\}$$

and

$$\overline{I}_K((0,-1)) = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R} \text{ and } y \geq -1\}.$$

It is easy to see that $\overline{I}_K((0,-1)) \subsetneq G_K((0,-1))$ and $G_K((0,-1))$ is not convex.

In a strictly convex Banach space, we can give an equivalent version of a generalized inward set in terms of metric projections.

Lemma 2.5.4. *Let K be a closed convex set in a strictly convex Banach space. Assume that there exists a metric projection r (possibly not continuous) from X onto K. Then we have*

$$G_K(x) = \{y \in X \setminus K : x \neq r(y)\} \cup K \quad \textit{for each } x \in K.$$

Proof. For each $x \in K$, we define

$$S(x) = \{y \in X \setminus K : d(y,K) < \|y-x\|\} \quad \text{and } T(x) = \{y \in X \setminus K : x \neq r(y)\}.$$

It suffices to show that $S(x) = T(x)$ for each $x \in K$. Let $y \in S(x)$, that is, $d(y,K) < \|y-x\|$. Since r is a metric projection, we have $\|y-r(y)\| = d(y,K) < \|y-x\|$ and thus $x \neq r(y)$. This implies $y \in T(x)$. On the other hand, let $y \in T(x)$, then we have $x \neq r(y)$ and $\|y-r(y)\| = d(y,K)$. The strict convexity of X implies $d(y,K) = \|y-r(y)\| < \|y-x\|$ and thus $y \in S(x)$. □

Remark 2.5.5. Note that the set $\{y \in X \setminus K : x \neq r(y)\} = \{y \in X : x \neq r(y)\}$ since $r(u) = u$ for each $u \in K$.

As an application of Lemma 2.5.4 we obtain

Corollary 2.5.6. *Let X be a strictly convex Banach space and $K = \{x \in X : \|x\| \leq 1\}$. Then for each $x \in K$, we have*

$$G_K(x) = \begin{cases} X & \textit{if} \quad \|x\| < 1 \\ \{y \in X : y \neq sx \quad \textit{for all } s > 1\} & \textit{if} \quad \|x\| = 1 \end{cases}$$

Proof. It is obvious that $G_K(x) = X$ if $\|x\| < 1$. We now verify that when $\|x\| = 1$, the result holds. We define a map r from X onto K by

$$r(x) = \begin{cases} x & \text{if } \|x\| \leq 1 \\ \dfrac{x}{\|x\|} & \text{if } \|x\| = 1 \end{cases}$$

By Example 2.3.6 r, is a continuous metric projection from X onto K. It is easy to verify that for each $x \in X$ with $\|x\| = 1$, we have

$$\{y \in X \setminus K : x \neq r(y)\} \cup K = \{y \in X : x \neq ty \quad \text{for all } t \in (0,1)\}.$$

The result follows. □

We give the relation between a generalized inward set and the so-called normal-outward set which was introduced in [26].

Let X be a strictly convex Banach space and K be a compact convex set in X. Since K is a convex M-set (see Remark 2.1.3), it follows from Theorem 2.2.7 that there exists a unique continuous metric projection r from X onto K. For each $x \in K$, let $N(x) = \{y \in X : y \neq x \text{ and } x = r(y)\}$. Then $N(x)$ is said to be the normal-outward set of x. It is easy to verify that $N(x) = \{y \in X : x = r(y)\} \setminus K$.

As another application of Lemma 2.5.4 we have

Corollary 2.5.7. *Let X be a strictly convex Banach space and K be a compact convex set in X. Then for each $x \in K$, $G_K(x) = X \setminus N(x)$.*

Proof. Let $x \in K$. Since $G_K(x) = \{y \in X : x \neq r(y)\} \cup K$ and $N(x) = \{y \in X : x = r(y)\} \setminus K$, we have $G_K(x) = X \setminus N(x)$. □

We now define the generalized inward maps. We shall show that all weakly inward maps are generalized inward. We give an example to show that the converse does not hold, even in finite dimensional Hilbert space. Several equivalent definitions will be established in Proposition 2.5.14. Conditions very similar to our generalized inward condition have been previously used in the study of fixed points of maps, for example, in Hilbert space by Browder and Petryshyn [8], and for set-valued maps in locally convex spaces by Reich [56].

Definition 2.5.8. Let K be a closed convex set in a Banach space X. A map $A : D \subset K \to X$ is said to be generalized inward on D relative to K if $Ax \in G_K(x)$ for each $x \in D$.

Let X be a strictly convex Banach space and K is a compact convex set in X. Recall that [26] that a map $A : K \to X$ is said to be nowhere normal-outward map if $Ax \notin N(x)$ for each $x \in K$. By Corollary 2.5.7 we see that a nowhere normal-outward map and a generalized inward map coincide when the space involved is strictly convex and the closed convex set involved is compact. Therefore, the concept of a generalized inward map is a generalization of the concept of a nowhere normal-outward map from a strictly convex space to a Banach space.

By Lemma 2.5.2 we immediately obtain

Lemma 2.5.9. *If $A : D \subset K \to X$ is weakly inward on D relative to K, then A is generalized inward on D relative to K.*

The converse of Lemma 2.5.9 is false.

Example 2.5.10. Let $H = \mathbb{R}^2$, $K = \{(x,0) : x \geq 0\}$, and D the open disk of radius 5, so that $\overline{D}_K = \{(x,0) : 0 \leq x \leq 5\}$. We define a map $A : \overline{D}_K \to H$ as follows:

$$A(x,0) = \begin{cases} (x/2, x), & x \in [0,2], \\ (2x-3, 6-2x), & x \in [2,3], \\ (x/2+3/2, x/2-3/2), & x \in [3,5]. \end{cases}$$

It is readily checked from a sketch that A is generalized inward but not weakly inward. It is possible to give even simpler examples but this one also satisfies other hypotheses used later (see Corollary 4.2.3).

Example 2.5.11. Let $X = \mathbb{R}^2$ and $K = \mathbb{R} \times \{0\} = \{(x, 0) : x \in \mathbb{R}\}$. Assume that $f, g : K \to \mathbb{R}$ are continuous and satisfy the following conditions:

(G): $x \neq f(x, 0)$ if $g(x, 0) \neq 0$.

Define a map $A : K \to \mathbb{R}^2$ by $A(x, 0) = (f(x, 0), g(x, 0))$. Then A is a generalized inward map.

Proof. For every $(u, v) \in X$, let $r(u, v) = (u, 0)$. Then r is the unique metric projection from X onto K. It follows from Lemma 2.5.4 that for every $(x, 0) \in K$,

$$G_K(x, 0) = \{(u, v) \in X \setminus K : u \neq x \quad \text{and } v \in \mathbb{R}\} \cup \{(x, 0)\}.$$

Hence, $X \setminus G_K(x, 0) = \{(x, v) \in \mathbb{R}^2 : v \neq 0\}$. Now, assume that A is not generalized inward. Then there exists $(x, 0) \in K$ such that $A(x, 0) \in X \setminus G_K(x, 0)$. Therefore, we have $x = f(x, 0)$ and $g(x, 0) \neq 0$, which contradicts the hypothesis (G). □

Remark 2.5.12. In Example 2.5.11 we see that, although K is a proper subspace of X, $A(K) \not\subset K$. This will allow us to develop a fixed point index for such a kind of map which maps a subspace into a larger space. However, as shown in Remark 2.4.18, it is not possible for a weakly inward map to carry a subspace into a larger space.

By using Corollary 2.5.6 we can show that the generalized inward condition and the Leray-Schauder condition are equivalent on a *ball* in a strictly convex space.

Lemma 2.5.13. *Let X be a strictly convex Banach space and $K = \{x \in X : \|x\| \leq 1\}$. Let $A : D \subset K \to X$. Then the following conditions are equivalent.*

(1) *A is a generalized inward map on D relative K.*

(2) *A satisfies the Leray Schauder condition:*

(LS) $x \neq tAx$ for all $t \in (0, 1)$ and all $x \in D$ with $\|x\| = 1$.

In the following we give some necessary and sufficient conditions for a map to be generalized inward.

Proposition 2.5.14. *Let K be a closed convex set in a Banach space X and $A : D \subset K \to X$. Then the following conditions are equivalent.*

(G) A is generalized inward on D relative to K.

(G-I): $d(Ax, K) < \|x - Ax\|$ *for* $x \in D$ *with* $Ax \notin K$,

(H_1) *For every* $x \in D$ *with* $Ax \notin K$*, there exists* $y \in I_K(x)$ *such that* $\|Ax - y\| < \|x - Ax\|$;

(H_2) $d(Ax, I_K(x)) < \|x - Ax\|$ *for* $x \in D$ *with* $Ax \notin K$*;*

(H_3) $d(Ax, \overline{I}_K(x)) < \|x - Ax\|$ *for* $x \in D$ *with* $Ax \notin K$*;*

(H_4) *For every* $x \in D$ *with* $Ax \notin K$,

$$d((1-t)x + tAx, K) < t\|x - Ax\| \quad \textit{for all } t \in (0,1);$$

(H_5) *For every* $x \in D$ *with* $Ax \notin K$*, there exists* $t \in (0,1)$ *such that*

$$d((1-t)x + tAx, K) < t\|x - Ax\|;$$

(H_6) $\liminf_{t \to 0^+} t^{-1} d((1-t)x + tAx, K) < \|x - Ax\|$ *for* $x \in D$ *with* $Ax \notin K$.

Proof. It is obvious that (G) and G-I are equivalent and by Proposition 2.4.5 (H_3) and (H_6) are equivalent. We prove that G-I and H_1-H_3 are equivalent. Obviously H_1 is equivalent to each of H_2 and H_3. It is easy to see that G-I implies H_1. Conversely, if H_1 holds and also $y \in K$, then $d(Ax, K) \le \|Ax - y\| < \|x - Ax\|$ and thus $(G$-$I)$ holds. If $y \notin K$ there exist $t \in (0,1)$ and $z \in K$ such that $z = ty + (1-t)x$. Hence by H_1 we have

$$d(Ax, K) \le \|Ax - z\| \le t\|Ax - y\| + (1-t)\|Ax - x\| < \|x - Ax\|.$$

Next we prove that G-I and H_4-H_5 are equivalent. Firstly, H_4 implies H_5, because if $\|(1-t)x + tAx - y\| < t\|x - Ax\|$, then

$$\|(1-st)x + stAx - [(1-s)x + sy]\| < st\|x - Ax\|, \quad \text{for } 0 \le s \le 1.$$

Now we prove that G-I implies H_4 and that H_5 implies G-I. Assume that G-I holds. Since K is convex, $(1-t)x + ty \in K$ for each $y \in K$ and $t \in (0,1)$ and thus we have

$$d((1-t)x + tAx, K) \le \|(1-t)x + tAx - (1-t)x - ty\| = t\|y - Ax\|,$$

Consequently, we obtain $t^{-1} d((1-t)x + tAx, K) \le d(Ax, K)$. Thus H_4 holds.

Now assume that H_5 holds. There exist $t \in (0,1)$ and $u \in K$ such that

$$\|(1-t)x + tAx - u\| < t\|x - Ax\|.$$

Hence we have

$$\begin{aligned} d(Ax, K) \le \|Ax - u\| &\le \|Ax - (1-t)x - tAx\| + \|(1-t)x + tAx - u\| \\ &< (1-t)\|x - Ax\| + t\|x - Ax\| = \|x - Ax\|, \end{aligned}$$

that is, G-I holds. □

When $D = K$ much of this Proposition is known. H_6 has been studied by Cramer and Ray amongst others, see for example [70]. Also Williamson [70] shows G-I is equivalent to H_6 (and other conditions) in Hilbert spaces. If $D = K$ is a ball then it is easy to see that G-I implies the Leray Schauder condition

(LS) $Ax \neq \lambda x$ for all $\lambda > 1$ and all $x \in \partial D$.

We have shown in Lemma 2.5.13 that the converse also holds if the space is strictly convex.

Comments

All the results of this section are new, and the following results are contained in [42].

Lemma 2.5.9, Example 2.5.10, Lemma 2.5.13 and part of Proposition 2.5.14.

Chapter 3

A fixed point index for generalized inward compact maps

In this chapter we shall develop a new fixed point index for a generalized inward compact map defined on a convex M-set. New fixed point theorems and nonzero fixed point theorems are obtained. In particular, we obtain nonzero fixed point theorems including norm-type expansion and compression theorems for continuous weakly inward maps in a finite dimensional Banach space. In general, norm-type expansion and compression theorems for a continuous map defined in a finite dimensional Banach space do not hold, but they are true if the map is defined in a wedge which is not a finite dimensional space and the image points of the map are contained in the wedge. However, we shall see that this norm-type expansion and compression theorem for a weakly inward map does not hold unless the image points of the map are contained in the wedge. Hence, an interesting problem is whether it is true under suitable additional conditions? We indeed find such an additional condition which is optimal since we shall give an example to show the result does not hold without it.

All the results of this Chapter are new.

3.1 Definition of a fixed point index for generalized inward compact maps

In this section we shall develop a fixed point index for a generalized inward map defined on a convex M-set of a Banach space by employing ε-projections.

The concept of ε-projection has been given in Definition 2.2.1. We recall it here for convenience. Let X be a Banach space, K a closed convex set of X and $\varepsilon > 0$. Recall that a continuous map $r : X \to K$ is said to be an *ε-projection* if it satisfies

$$\|x - rx\| \leq (1+\varepsilon)d(x, K) \quad \text{for every } x \in X.$$

We start with the following Lemma which allows us to define the fixed point index and show that it does not depend on the ε-projection provided ε is sufficiently small.

Lemma 3.1.1. *Let K be a convex M-set in a Banach space X and let Ω be a closed, bounded subset of K. Let $h : [0,1] \times \Omega \to X$ be compact and such that $h(t, \cdot)$ is generalized inward on Ω for each $t \in [0,1]$. Then, if $x \neq h(t,x)$ for all $x \in \Omega$ and all $t \in [0,1]$, there exists $\varepsilon_0 > 0$ such that $x \neq r_\varepsilon(h(t,x))$ for all $\varepsilon \leq \varepsilon_0$, for all $x \in \Omega$, and $t \in [0,1]$.*

Proof. If this is false, there are sequences $\varepsilon_n \to 0$, $x_n \in \Omega$, $t_n \in [0,1]$ such that

$$r_{\varepsilon_n}(h(t_n, x_n)) = x_n.$$

Thus we have

$$\|x_n - h(t_n, x_n)\| = \|r_{\varepsilon_n}(h(t_n, x_n)) - h(t_n, x_n)\| \leq (1+\varepsilon_n)d(h(t_n, x_n), K).$$

Since h is compact, we may assume that $h(t_n, x_n) \to y_0$. Then we have $\|y_0 - x_n\| \to d(y_0, K)$. Since K is an M-set, by passing to subsequences we may suppose that $x_n \to x_0 \in \Omega$ and $t_n \to t_0 \in [0,1]$. Then the continuity of h implies $h(x_n, t_n) \to h(t_0, x_0)$. Hence we have

$$\|x_0 - h(t_0, x_0)\| \leq d(h(t_0, x_0), K).$$

Since $h(t_0, .)$ is generalized inward, we have $h(t_0, x_0) \in K$ and $x_0 = h(t_0, x_0)$, a contradiction which proves the result. □

Lemma 3.1.3 generalizes Lemma 3.1 in [43] in the following ways: X may not be a finite dimensional space and $h(t, .)$ may not be weakly inward map.

Remark 3.1.2. In the proof of Lemma 3.1.3 we see that a key to proving that $x \neq h(t,x)$ implies $x \neq r_\varepsilon(h(t,x))$ is that the set K is a closed convex M-set. We don't know whether $x \neq h(t,x)$ implies $x \neq r_\varepsilon(h(t,x))$ for a general closed convex set K. Therefore, this restricts our definition of index to a closed convex M-set instead of a general closed convex set.

As a special case of Lemma 3.1.1 we obtain a key result towards defining our index.

Lemma 3.1.3. *Let K be a convex M-set in a Banach space X and let Ω be a bounded closed subset of K. Let $A : \Omega \to X$ be a generalized inward compact map such that $x \neq Ax$ for every $x \in \Omega$. Then there exists $\varepsilon_0 > 0$ such that $x \neq r_\varepsilon Ax$ for all $\varepsilon \leq \varepsilon_0$ and for all $x \in \Omega$.*

Proof. Let $h(t,x) = Ax$ for $x \in \Omega$ and $t \in [0,1]$. Then the result follows by Lemma 3.1.3. □

We now are in position to define a fixed point index for a generalized inward compact map.

Let D be an open set in X. We denote by $\overline{D}_K$ and ∂D_K the closure and the boundary, respectively, of $D_K = D \cap K$ relative to K.

Definition 3.1.4. Let K be a convex M-set in a Banach space X and let D be a bounded open set such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to X$ be compact and such that A is a generalized inward map on ∂D_K. Suppose that $x \neq A(x)$ for all $x \in \partial D_K$. Define the fixed point index by the equation

$$i_K(A, D_K) = i_K(r_\varepsilon A, D_K),$$

for ε sufficiently small, where $i_K(r_\varepsilon A, D_K)$ is the fixed point index as defined in Section 1.2 in Chapter 1 (also see Amann [4]).

By Lemma 3.1.3, with $\Omega = \partial D_K$, we see that $i_K(r_\varepsilon A, D_K)$ is well defined. Also by considering the homotopy $h(x,t) = tr_{\varepsilon_1}A(x) + (1-t)r_{\varepsilon_2}A(x)$, we see that it is independent of the ε-retraction for ε sufficiently small. Therefore, the index defined in Definition 3.1.4 makes sense.

Remark 3.1.5. (i) If $A : \overline{D}_K \to K$ then our definition coincides with the standard one (see Section 2.2 in Chapter 2) since $r_\varepsilon A = A$ in that case. However, they do not contain each other because Definition 3.1.4 requires that the set K be a closed convex M-set while the standard one applies to any closed convex set.

(ii) Since any closed convex subset in a finite dimensional Banach space is a convex M-set, our index generalizes that index defined by Lan and Webb [43] in the following ways:

(a) X may be an infinite dimensional Banach space.

(b) A need not be weakly inward.

(iii) If there exists a continuous metric projection r_0 from X onto K, that is, K is an M_∞-set, our definition coincides with that index given by Lan and Webb [42] since the index equals $i_K(r_0 A, D_K)$. However, they do not contain each other since, as has been shown, an M_∞-set may not be an M-set and it is not clear whether an M-set is an M_∞-set.

(iv) Since a compact convex set in an arbitrary Banach space X is a convex M-set, our index applies to any compact convex set in X. When X is a real strictly convex, an index was defined in [63] for weakly inward maps in a different way, but it does not apply when X is not strictly convex.

(v) Another index was given in [28] for weakly inward compact maps when there exists a retraction with property (P). A retraction with property (P) need not be a metric projection (see the retraction given in part (a) of Lemma 2.3 in [28]). However, it is not clear whether the index defined in [28] is independent of the choice of the retractions satisfying property (P). A fixed retraction must be used in [28]. Also, we do not know whether for each M-convex set K there exists a retraction onto K with property (P). Therefore, we do not know whether the index given in [28] can apply to any M-convex set.

The fixed point index in Definition 3.1.4 has most of the usual properties of fixed point index. Although we have defined the index for a map A which only requires A to be generalized inward on ∂D_K, we need A to be generalized inward map on $\overline{D}_K$ to ensure that a nonzero index implies the existence of fixed points.

Theorem 3.1.6. *Let K be a convex M-set in a Banach space X and let D be a bounded open set such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to X$ be a generalized inward compact map. Suppose that $x \neq A(x)$ for all $x \in \partial D_K$. The index as defined above has the following properties.*

(P_1) *(Existence) If $i_K(A, D_K) \neq 0$, then A has a fixed point in D_K.*

(P_2) *(Normalisation) If $u \in D_K$, then $i_K(\hat{u}, D_K) = 1$, where $\hat{u}(x) = u$ for $x \in \overline{D}_K$.*

(P_3) *(Additivity) If W^1, W^2 are disjoint relatively open subsets of D_K such that $x \neq A(x)$ for $x \in \overline{D}_K \setminus (W^1 \cup W^2)$, then*

$$i_K(A, D_K) = i_K(A, W^1) + i_K(A, W^2)$$

(P_4) *(Homotopy property) Let $h : [0,1] \times \overline{D}_K \to X$ be compact and such that $h(t,.) : \partial D_K \to X$ is generalized inward for each $t \in [0,1]$. If $x \neq h(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$, then*

$$i_K(h(0,.), D_K) = i_K(h(1,.), D_K).$$

Proof. (P_1) If $i_K(A, D_K) \neq 0$ then $i_K(r_\varepsilon A, D_K) \neq 0$ for every ε sufficiently small. Hence, by the usual existence property, $r_\varepsilon A$ has a fixed point which yields a fixed point of A by Lemma 3.1.1.

(P_2) is obvious.

(P_3) follows easily from the corresponding property for $i_K(r_\varepsilon A, D_K)$.

(P_4) follows from the usual homotopy property on utilizing Lemma 3.1.1. □

Remark 3.1.7. There is another way to define the index. Let K be an M-closed convex set in a Banach space X. We have shown in Theorem 2.2.5 that there exists an upper semicontinuous metric projection r from X onto K with nonempty compact convex values. It can be shown that rA is a compact multivalued map. Hence, one can define the fixed point index of A as the fixed point index for the multivalued map rA. The index of multivalued compact maps $A : \overline{D}_K \to 2^K$ has been well studied, for example see [20] and the related references therein.

3.2 Fixed point theorems

In this section we shall obtain some new fixed point and nonzero fixed point theorems by using the results in section 3.1.

Theorem 3.2.1. *Let K be a bounded convex M-set in a Banach space X and $A : K \to X$ be a generalized inward compact map. Then $i_K(A, K) = 1$, and thus A has a fixed point in K.*

Proof. The result follows from Definition 3.1.4 and Theorem 1.3.52. □

As a special case of Theorem 3.2.1, we immediately obtain the following Corollary.

Corollary 3.2.2. *Let K be a compact convex set in a Banach space X and $A : K \to X$ a continuous generalized inward map. Then A has a fixed point in K.*

Corollary 3.2.2 generalizes Lemma 18.4 in [12] to a generalized inward map. Moreover, our proof is different that of Lemma 18.4 in [12].

Theorem 3.2.3. *Let K be an unbounded convex M-set in a Banach space X and $A : K \to X$ be a generalized inward compact map such that $A(K)$ is bounded. Then there exists $\rho_0 > 0$ such that $i_K(A, B_K(\rho)) = 1$ for all $\rho \geq \rho_0$, where $B(\rho) = \{x \in X : \|x\| < \rho\}$. Therefore, A has a fixed point in K.*

Proof. We prove that there exists $M > 1$ such that for sufficiently small $\varepsilon > 0$, $\|r_\varepsilon Ax\| \leq M$ for $x \in K$. In fact, if not, for each $n \in \mathbb{N}$ there exist $\frac{1}{n}$-projection $r_{\frac{1}{n}}$ on K and $x_n \in K$ such that $\|r_{\frac{1}{n}} Ax_n\| \to +\infty$. Hence, we have

$$\|r_{\frac{1}{n}} Ax_n\| - \|Ax_n\| \leq \|Ax_n - r_{\frac{1}{n}} Ax_n\| \leq (1 + \frac{1}{n}) d(Ax_n, K).$$

Since $\overline{A(K)}$ is compact, we may assume that $Ax_n \to y_0$. This implies

$$\limsup \|r_{\frac{1}{n}} Ax_n\| \leq \|y_0\| + d(y_0, K) < \infty,$$

a contradiction. Hence, there exist $\rho_0 > 0$ such that

$$\overline{A(K) \cup r_\varepsilon A(K)} \subset B(\rho_0) \quad \text{for } \rho \geq \rho_0 \quad \text{and sufficiently small } \varepsilon.$$

It follows from Definition 3.1.4 that for each $\rho \geq \rho_0$,

$$i_K(A, B_K(\rho)) = i_K(r_\varepsilon A, B_K(\rho)) = 1.$$

□

The following example shows that in Theorem 3.2.3, the condition that $A(K)$ is bounded can not be dropped.

Example 3.2.4. Let $X = \mathbb{R}^2$ and $K = \mathbb{R} \times \{0\}$. Define a continuous map $A : K \to X$ by

$$A(x, 0) = (x + 1, x^2 + 1).$$

By Example 2.5.11 we see that A is a generalized inward map. It is easy to verify that $A(K)$ is unbounded and A has no fixed points in K.

When A is weakly inward, we have the following result whose proof is similar to those of Theorems 1.2.13 and 1.3.52. We omit its proof.

Theorem 3.2.5. *Let K be a unbounded convex M-set in a Banach space X and $A : K \to X$ be a weakly inward compact map. Assume that the following condition holds.*

$$\limsup_{x \in K, \|x\| \to \infty} \|Ax\|/\|x\| < 1.$$

Then there exists $\rho_0 > 0$ such that $i_K(A, B_K(\rho)) = 1$ for all $\rho \geq \rho_0$, where $B(\rho) = \{x \in X : \|x\| < \rho\}$. Therefore, A has a fixed point in K.

Theorem 3.2.6. *Let K be a convex M-set in a Banach space X and D a bounded open set such that $D_K \neq \emptyset$ and $\overline{D}_K$ is a bounded M-set. Assume that $A : \overline{D}_K \to X$ is a generalized inward compact map such that*

(1) there exists $x_0 \in D_K$ such that $tA + (1-t)\hat{x}_0$ is generalized inward on ∂D_K for each $t \in [0, 1)$;

(LS) $x \neq tA(x) + (1-t)x_0$ for all $x \in \partial D_K$ and $t \in [0, 1)$,

Then A has a fixed point in $\overline{D}_K$, and if $x \neq A(x)$ for $x \in \partial D_K$, then $i_K(A, D_K) = 1$.

Proof. Assume without loss of generality that $x \neq A(x)$ for $x \in \partial D_K$. Let $h(t, x) = tAx + (1-t)x_0$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. By hypothesis (LS), $x \neq h(t, x)$ for $x \in \partial D_K$

and $t \in [0,1]$. It is also clear that $h : [0,1] \times \overline{D}_K \to X$ is compact and by hypothesis (1) $h(t,.) : \partial D_K \to X$ is a generalized inward map for each $t \in [0,1]$. Consequently it follows from Theorem 3.1.6 that $i_K(A, D_K) = i_K(\hat{x_0}, D_K) = 1$. □

In general, if A is a generalized inward map, tA may not be a generalized inward map for $t \in [0,1]$. However, in some special cases, tA is a generalized inward map for $t \in [0,1]$ provided A is a generalized inward map satisfying some extra conditions (see Corollary 4.2.3 in Chapter 4).

As a special case of Theorem 3.2.6, we have

Corollary 3.2.7. *Let K be a convex M-set in a Banach space X and D a bounded open set such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a weakly inward compact map such that*

(LS) there exists $x_0 \in D_K$ such that $x \neq tA(x) + (1-t)x_0$ for all $x \in \partial D_K$ and $t \in [0,1)$,

Then A has a fixed point in $\overline{D}_K$, and if $x \neq A(x)$ for $x \in \partial D_K$, then $i_K(A, D_K) = 1$.

Remark 3.2.8. The following norm-type boundary condition implies (LS).

$(B_1) : \|Ax\| < \|x\| + \|x - Ax\|$ for each $x \in \partial D_K$ with $\|Ax\| > \|x\|$.

When X is a Hilbert space, (B_1) is equivalent to the following condition:

$(B_2) : (x, Ax) < \|x\| \, \|Ax\|$ for each $x \in \partial D_K$ with $\|Ax\| > \|x\|$.

This condition is then equivalent to (LS) because of the known criterion for equality in the Schwarz inequality.

The following condition (B_3) implies (B_2).

(B_3) $\|Ax\|^2 \leq \|x\|^2 + \|x - Ax\|^2$ for each $x \in \partial D_K$ with $\|Ax\| > \|x\|$.

3.3 Nonzero fixed point theorems for weakly inward compact maps

In this section we discuss the existence of nonzero fixed points for weakly inward compact maps by employing the usual conditions used in the classical index theory.

Lemma 3.3.1. *Let K be an M-wedge in a Banach space X and D a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a weakly inward compact map such* *that $x \neq Ax$ for $x \in \partial D_K$ and the following condition holds.*

(*E*) *There exists $e \in K \setminus \{0\}$ such that $x \neq Ax + \lambda e$ for $x \in \partial D_K$ and $\lambda > 0$.*

Then $i_K(A, D_K) = 0$.

Proof. Assume, for a contradiction, that $i_K(A, D_K) \neq \{0\}$. For $\lambda > 0$, let $h(t,x) = Ax + t\lambda e$ for $x \in \overline{D}_K$ and $t \in [0,1]$. By hypothesis, $x \neq h(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$. As $\overline{I}_K(x)$ is a wedge containing K we see that $h(t,.) : \overline{D}_K \to X$ is weakly inward for each $t \in [0,1]$. It follows from (P_4) that we have $i_K(A, D_K) = i_K(A + \lambda\hat{e}, D_K) \neq \{0\}$. Hence for each $n \in \mathbb{N}$, there exists $x_n \in \overline{D}_K$ such that $x_n = Ax_n + ne$. As $\overline{D}_K$ and $A(\overline{D}_K)$ are bounded this implies $e = 0$, a contradiction. □

Theorem 3.3.2. *Let K be an M-wedge in X and let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Let $A : \overline{D}_K \to X$ be a weakly inward compact map. Suppose the following conditions are satisfied.*

(*LS*) *$x \neq tAx$ for $x \in \partial D^1_K$ and $0 \leq t < 1$.*

(*E*) *There exists $e \in K \setminus \{0\}$ such that $x \neq Ax + \lambda e$ for $x \in \partial D_K$ and $\lambda > 0$.*

Then A has a fixed point in $\overline{D}_K \setminus D^1_K$. The same conclusion remains valid if (LS) holds on ∂D_K and (E) holds on ∂D^1_K.

Proof. Suppose that A has no fixed point in $\partial D_K \cup \partial D^1_K$. It follows from Corollary 3.2.7 and Lemma 3.3.1 that $i_K(A, D^1_K) = 1$ and $i_K(A, D_K) = 0$. By the additivity property (P_3) of index we have

$$i_K(A, D_K \setminus \overline{D^1_K}) = i_K(A, D_K) - i_K(A, D^1_K) = -1$$

and thus A has a fixed point in $\overline{D}_K \setminus D^1_K$. The proof is exactly similar if the hypotheses are interchanged. □

Now, we generalize Lemma 3.3.1 by relaxing the condition (*E*). We consider two cases: one is when $\partial K_1 = \{x \in K : \|x\| = 1\}$ is compact and the other is when ∂K_1 is not compact.

We first consider the case when ∂K_1 is compact.

Lemma 3.3.3. *Let K be a wedge in a Banach space X such that $\partial K_1 = \{x \in K : \|x\| = 1\}$ is compact. Let D be a bounded open set such that $D_K \neq \emptyset$. Let $A, B : \overline{D}_K \to X$ be weakly inward continuous maps. Assume that the following conditions hold.*

(A_1) $x \neq Ax + \lambda Bx$ *for* $x \in \partial D_K$ *and* $\lambda \geq 0$.

(A_2) $Bx \neq 0$ *for* $x \in \partial D_K$.

(A_3) *There exists* $e \in \partial K_1$ *such that* $Bx \neq -\delta e$ *for* $x \in \partial D_K$ *and* $\delta > 0$.

Then $i_K(A, D_K) = 0$.

Proof. By the assertion 4 in Remark 2.1.3 K is an M-wedge. Since ∂K_1 is compact and $\overline{D}_K$ is bounded closed, $\overline{D}_K$ is compact. This, together with continuity of A and B, implies A and B are compact. We prove that there exists $\lambda_0 > 1$ such that the following holds:

$$x \neq Ax + \lambda_0 Bx + \beta e \quad \text{for } x \in \partial D_K \quad \text{and } \beta \geq 0. \tag{3.1}$$

In fact, if not, there exist $\{\lambda_n\} \subset (1, +\infty)$ with $\lambda_n \to \infty$, $\{\beta_n\} \subset [0, +\infty)$ and $\{x_n\} \subset \partial D_K$ such that $x_n = Ax_n + \lambda_n Bx_n + \beta_n e$. Hence, $Bx_n + \lambda_n^{-1}\beta_n e \to 0$ and as $\{Bx_n\}$ is bounded we may assume that $\lambda_n^{-1}\beta_n \to \delta_0 \in [0, +\infty)$. Since ∂K_1 is compact, we may assume that $x_n \to x_0 \in \partial D_K$. Therefore, we have $Bx_0 = -\delta_0 e$. By hypothesis (A_2) and (A_3), we have $\delta_0 < 0$, which contradicts the fact that $\delta_0 \in [0, \infty)$.

Let $\lambda_0 > 1$ satisfy (3.1). Note that $A + \lambda_0 B$ is weakly inward. It follows from (3.1) and Lemma 3.3.1 that $i_K(A + \lambda_0 B, D_K) = 0$. Let $h(t, x) = Ax + t\lambda_0 Bx$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. By hypothesis (A_1), we have $x \neq h(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$. It follows from the homotopy of index that $i_K(A, D_K) = i_K(A + \lambda_0 B, D_K) = 0$. □

We give some special cases of the condition (A_3) in the following remark.

Remark 3.3.4. (1) In Lemma 3.3.3, if K is a cone, that is, $K \cap (-K) = \{0\}$ and $B(\partial D_K) \subset K$, then the condition ($A_3$) is satisfied automatically. In fact, since $K \neq -K$, it follows from Proposition 1.1.29 that $K \setminus (-K) = K \setminus K \cap (-K) \neq \emptyset$. Let $e \in K \setminus (-K)$ with $\|e\| = 1$. Then is easy to verify that $Bx \neq -\delta e$ for $x \in \partial D_K$ and $\delta > 0$.

(2) In Lemma 3.3.3, if K is neither a cone nor a subspace of X, that is, $K \cap (-K) \neq \{0\}$ and $K \neq -K$, then the following condition implies (A_3).

(A_4) $\overline{B(\partial D_K)} \cap (-K \setminus K) = \emptyset$.

In fact, let $e \in K \setminus (-K)$ with $\|e\| = 1$. Then we see that the set $\{-te : t > 0\} \subset -K \setminus K$. Hence, ($A_3$) follows from hypothesis (A_4).

An important special case of Lemma 3.3.3 is the following Corollary which has not been considered previously.

Corollary 3.3.5. *Let K be a wedge in a finite dimensional Banach space X with $K \neq -K$. Let D be a bounded open set such that $D_K \neq \emptyset$. Let $A, B : \overline{D}_K \to X$ be weakly inward continuous maps. Assume that (A_1), (A_2) and (A_3) of Lemma 3.3.3 hold. Then $i_K(A, D_K) = 0$.*

From Remark 3.3.4 we see that in Corollary 3.3.5, if $B(\partial D_K) \subset K$, then the condition (A_3) is satisfied automatically. However, the following examples show that in Corollary 3.3.5, if the condition $B(\partial D_K) \subset K$ fails and B does not satisfy the condition (A_3), then the result of Corollary 3.3.5 may not be true whether K is a cone or not.

The first example is when K is a cone.

Example 3.3.6. Let $X = \mathbb{R}^2$ and $K = \{(x, y) \in \mathbb{R}^2 : x, y \in [0, \infty)\}$. Let $D = \{z \in X : \|z\| < 1\}$. Define maps $A, B : \overline{D}_K \to X$ by

$$A((x, y)) = B((x, y)) = (-x, -y).$$

By Example 2.4.15 we see that A and B are weakly inward continuous maps. It is also easy to verify that A and B satisfy the conditions (A_1) and (A_2) in Corollary 3.3.5 and $\partial D_K = -\partial D_K$. The latter implies $B(\partial D_K) \subsetneq K$ and B does not satisfy the condition (A_3). We show that the result of Corollary 3.3.5 is false. Indeed, it is easy to see that $z \neq Az$ for $z \in \partial D_K$ and $\|Az\| = \|z\|$ for $z \in \partial D_K$. Hence, A satisfies the (LS) condition in Corollary 3.2.7 with $x_0 = 0$. It follows from Corollary 3.2.7 that $i_K(A, D_K) = 1$.

The following example is when K is not a cone.

Example 3.3.7. Let $X = \mathbb{R}^2$ and $K = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R} \text{ and } y \in [0, \infty\}$. Let $D = \{z \in X : \|z\| < 1\}$. Define maps $A, B : \overline{D}_K \to X$ by

$$A((x, y)) = B((x, y)) = (-x, -y).$$

By a similar method to that in Example 3.3.6 we can show that A and B satisfy all the conditions of Corollary 3.3.5 except the condition (A_3), and we have $i_K(A, D_K) = 1$.

Another important special case of Lemma 3.3.3 is the following Corollary which also has not been considered previously, that is the case when $K = -K$, or equivalently, K is a subspace of a finite dimensional space X. Since the range of a weakly inward map relative to the subspace K must be contained in K (see Remark 2.4.18), we may assume without loss of generality that $K = X$.

Corollary 3.3.8. *Let X be a finite dimensional Banach space and D a bounded open set of X. Let $A, B : \overline{D} \to X$ be continuous maps such that the following conditions hold.*

(A_1) $x \neq Ax + \lambda Bx$ *for* $x \in \partial D$ *and* $\lambda \geq 0$.

(A_2) $Bx \neq 0$ *for* $x \in \partial D$.

(A_3) *There exists* $e \in X$ *with* $\|e\| = 1$ *such that* $Bx \neq -\delta e$ *for* $x \in \partial D$ *and* $\delta > 0$.

Then $i_X(A, D) = 0$.

The following example shows the condition (A_3) can not be dropped even when $B = A$.

Example 3.3.9. Let $X = \mathbb{R}^2$ and $D = \{x \in X : \|x\| < 1\}$. Let $\theta_0 \in (0, 2\pi)$ be fixed. Define a continuous map $A : \overline{D} \to X$ by

$$A(r\cos\theta, r\sin\theta) = (r\cos(\theta + \theta_0), r\sin(\theta + \theta_0)),$$

where $r \in [0, 1]$ and $\theta \in [0, 2\pi)$. It is easy to verify that for each $x \in \partial D$, A satisfies (i) $\|Ax\| = \|x\|$; (ii) $x \neq Ax$; and (iii) $A(\partial D) = \partial D$, where $\partial D = \{x \in X : \|x\| = 1\}$. In Lemma 3.3.3, let $A = B$. We see that (i) and (ii) imply (A_1) and (A_2) respectively. However, (iii) shows that (A_3) fails. On the other hand, (i) and (ii) imply that A satisfies $x \neq tAx$ for $x \in \partial D$ and $t \in [0, 1]$. It follows from Corollary 3.2.7 that $i_X(A, D) = 1$. This shows that A does not satisfy (A_3) and its index is one.

By the same method as before we immediately obtain the following result.

Theorem 3.3.10. *Let K be a wedge in X such that ∂K_1 is compact and let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Let $A : \overline{D}_K \to X$ be a weakly inward continuous map. Suppose the following conditions are satisfied.*

(LS) $x \neq tAx$ *for* $x \in \partial D^1_K$ *and* $0 < t < 1$.

There exists a weakly inward continuous map $B : \overline{D}_K \to X$ *such that*

(A_1') $x \neq Ax + \lambda Bx$ *for* $x \in \partial D_K$ *and* $\lambda \geq 0$.

(A_2') $Bx \neq 0$ *for* $x \in \partial D_K$.

(A_3') *There exists* $e \in \partial K_1$ *such that* $Bx \neq -\delta e$ *for* $x \in \partial D_K$ *and* $\delta > 0$.

Then A *has a fixed point in* $\overline{D}_K \setminus D_K^1$. *The same conclusion remains valid if* (LS) *holds on* ∂D_K *and* (A_1')-(A_3') *hold on* ∂D_K^1.

As an application of Theorem 3.3.10, we give an example.

Example 3.3.11. Let $X = \mathbb{R}^n$ and $K = \mathbb{R}_+^n$. Let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D_K^1} \subset D_K$. Let $I = \{1, ..., n\}$ and for each $i \in I$, $f_i : \overline{D}_K \to \mathbb{R}$ be a continuous map. Assume that the following conditions hold.

(F_1) $f_i(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \geq 0$ for $i \in I$.

(F_2) $f_n(x) > x_n$ for all $x \in \partial D_K$.

(F_3) $f_n(x) < x_n$ for $x \in \partial D_K^1$.

Define a map $A : \overline{D}_K \to X$ by

$$Ax = (f_1(x), ..., f_n(x)).$$

Then A has a fixed point in $\overline{D}_K \setminus D_K^1$.

Proof. It follows from the condition (F_1) and Example 2.4.19 that A is a weakly inward map. It is easy to verify that (F_2) implies conditions (A_1')-(A_3') in Theorem 3.3.10 with $B = A$ and $e = (0, ..., 0, 1)$ holds and (F_3) implies A satisfies the (LS) condition on ∂D_K^1. The result follows from Theorem 3.3.10. □

As a special case of Theorem 3.3.10, we obtain a norm-type wedge compression and expansion result.

Corollary 3.3.12. *Let* K *be a wedge in* X *such that* ∂K_1 *is compact and let* D^1, D *be bounded open sets in* X *such that* $0 \in D^1$ *and* $\overline{D_K^1} \subset D_K$. *Let* $A : \overline{D}_K \to X$ *be a weakly inward continuous map. Suppose the following conditions are satisfied.*

(i) $\|Ax\| \leq \|x\|$ *for* $x \in \partial D_K^1$.

(ii) $\|x\| \leq \|Ax\|$ *for* $x \in \partial D_K$.

(iii) *There exists* $e \in \partial K_1$ *such that* $Ax \neq -\delta e$ *for* $x \in \partial D_K$ *and* $\delta > 0$.

Then A *has a fixed point in* $\overline{D}_K \setminus D_K^1$. *The same conclusion remains valid if* (i) *holds on* ∂D_K *and* (ii) *and* (iii) *hold on* ∂D_K^1.

As an application of Corollary 3.3.12, we give an example on existence of nonzero fixed points.

Example 3.3.13. Let $X = \mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ and $D = \{z = (x,y) \in X : \|z\| < 2\}$ and $D^1 = \{z = (x,y) \in X : \|z\| < 1\}$. Define a continuous map $A : \overline{D} \to X$ by

$$A(x,y) = (x, |y|),$$

Then it is easy to verify that $\|Az\| = \|z\| = 1$ for $z \in \partial D^1$ and $\|Az\| = \|z\| = 2$ for $z \in \partial D$. Also, we see easily that $Az \neq -\delta(0,1)$ for $z \in \partial D$ and $\delta > 0$. It follows from Corollary 3.3.12 that A has a fixed point in $\overline{D} \setminus D^1$.

Now, we consider the case when ∂K_1 is not compact. Note that in this case X must be an infinite dimensional Banach space.

Lemma 3.3.14. *Let K be an M-wedge in an infinite dimensional Banach space X such that $\partial K_1 = \{x \in K : \|x\| = 1\}$ is not compact. Let D be a bounded open set such that $D_K \neq \emptyset$. Let $A, B : \overline{D}_K \to X$ be weakly inward compact maps. Assume that the following conditions hold.*

(B_1) $x \neq Ax + \lambda Bx$ *for* $x \in \partial D_K$ *and* $\lambda \geq 0$.

(B_2) $\alpha = \inf\{\|Bx\| : x \in \partial D_K\} > 0$.

Then $i_K(A, D_K) = 0$.

Proof. We first prove that there exist $e \in \partial K_1$ such that

$$\{te : t > 0\} \cap -\overline{B(\partial D_K)} = \emptyset. \tag{3.2}$$

In fact, if (3.2) is false, for each $x \in \partial K_1$, there exists $t_x > 0$ such that $t_x x \in -\overline{B(\partial D_K)}$. Thus the set $Q = \{t_x x : \|x\| = 1\}$ is relatively compact and hence the set $\overline{\text{co}}(Q \cup \{0\})$ is compact. By hypothesis (A_2), we have $t_x \geq \alpha$ so that the set $\{x \in K : \|x\| = \alpha\}$ is contained in the compact set $\overline{\text{co}}(Q \cup \{0\})$ and thus is compact, a contradiction.

Choose a fixed $e \in \partial K_1$ which satisfies (3.2). We assert that there exists $\lambda_0 > 1$ such that the following result holds.

$$x \neq Ax + \lambda_0 Bx + \beta e \quad \text{for } x \in \partial D_K \quad \text{and } \beta \geq 0. \tag{3.3}$$

In fact, if not, there exist $\{\lambda_n\} \subset (1,+\infty)$, $\{\beta_n\} \subset [0,+\infty)$ and $\{x_n\} \subset \partial D_K$ such that $x_n = Ax_n + \lambda_n Bx_n + \beta_n e$. Hence, $Bx_n + \lambda_n^{-1}\beta_n e \to 0$ and as $\{Bx_n\}$ is bounded we may assume that $\lambda_n^{-1}\beta_n \to b \in [0,+\infty)$. By hypothesis (A_2), $b > 0$. It follows that $Bx_n \to -be$ and $be \in -\overline{B(\partial D_K)}$. The latter contradicts (3.2).

Let $\lambda_0 > 1$ which satisfies (3.3). Note that $A + \lambda_0 B$ is weakly inward. It follows from (3.3) and Lemma 3.3.1 that $i_K(A+\lambda_0 B, D_K) = 0$. Let $h(t,x) = Ax + t\lambda_0 Bx$ for $x \in \overline{D}_K$ and $t \in [0,1]$. By hypothesis (A_1), we have $x \neq h(t,x)$ for $x \in \partial(D_K)$ and $t \in [0,1]$. It follows from the homotopy of index that $i_K(A, D_K) = i_K(A + \lambda_0 C, D_K) = 0$. □

Theorem 3.3.15. *Let K be an M-wedge in an infinite dimensional Banach space X such that ∂K_1 is not compact and let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Let $A : \overline{D}_K \to X$ be a weakly inward compact map. Suppose the following conditions are satisfied.*

(LS) *$x \neq tAx$ for $x \in \partial D^1_K$ and $0 \leq t < 1$.*

There exists a weakly inward compact map $B : \overline{D}_K \to X$ such that

(B_1) *$x \neq Ax + \lambda Bx$ for $x \in \partial D_K$ and $\lambda \geq 0$.*

(B_2) *$\alpha = \inf\{\|Bx\| : x \in \partial D_K\} > 0$.*

Then A has a fixed point in $\overline{D}_K \setminus D^1_K$. The same conclusion remains valid if (LS) holds on ∂D_K and (B_1) and (B_2) hold on ∂D^1_K.

Proof. Suppose that A has no fixed point in $\partial D_K \cup \partial D^1_K$. It follows from Corollary 3.2.7 and Lemma 3.3.14 that $i_K(A, D^1_K) = 1$ and $i_K(A, D_K) = 0$. By the additivity property (P_3) of index we have

$$i_K(A, D_K \setminus \overline{D^1_K}) = i_K(A, D_K) - i_K(A, D^1_K) = -1$$

and thus A has a fixed point in $\overline{D}_K \setminus D^1_K$. The proof is exactly similar if the hypotheses are interchanged. □

As a special case of Theorem 3.3.15, we obtain a norm-type wedge compression and expansion result.

Corollary 3.3.16. *Let K be an M-wedge in an infinite dimensional Banach space X such that ∂K_1 is not compact and let D^1, D be bounded open sets in X such that $0 \in D^1$*

and $\overline{D_K^1} \subset D_K$. *Let* $A : \overline{D}_K \to X$ *be weakly inward compact map. Suppose the following conditions are satisfied.*

(i) $\|Ax\| \leq \|x\|$ *for* $x \in \partial D_K^1$.

(ii) $\|x\| \leq \|Ax\|$ *for* $x \in \partial D_K$.

(iii) $\alpha = \inf\{\|Ax\| : x \in \partial D_K\} > 0$.

Then A *has a fixed point in* $\overline{D}_K \setminus D_K^1$. *The same conclusion remains valid if (i) holds on* ∂D_K *and (ii) and (iii) hold on* ∂D_K^1.

Chapter 4

A fixed point index for generalized inward maps of condensing type

In Chapter 3 we have employed the ε-projections to define an index. We have seen that the method can only be applied to treat compact maps because we don't know whether the ε-projections involved are k-γ-contractive or γ-condensing. However, we have shown in Section 2.3 of Chapter 2 that there are closed convex M_1-sets for which there are continuous 1-γ-contractive metric projections. Therefore, we can employ continuous metric projections to define a fixed point index which allows us to consider generalized inward condensing maps. This Chapter is joint work with Prof. J. R. L. Webb and has been published in Trans. Amer. Math. Soc. (see [42]).

All of the results of this Chapter except for Lemma 4.4.1 and Remark 4.1.4 were published in Trans. Amer. Math. Soc., **349** (1997), 2175-2186.

4.1 Definition of a fixed point index for generalized inward maps of condensing type

In this section we shall develop a fixed point index for generalized inward maps of condensing type defined on a closed convex M_l-set. We always denote by γ the set (or ball)-measure relative to X.

We begin with two key Lemmas.

The first result is an important step in developing the index theory for generalized inward maps, it shows that rA has the same fixed points as A.

Lemma 4.1.1. *Let K be a closed convex set in a Banach space X and suppose $A : \Omega \subset K \to X$ is a generalized inward map on Ω relative to K such that $x \neq Ax$ for $x \in \Omega$. If r is a metric projection from X to K, then $x \neq rAx$ for all $x \in \Omega$.*

Proof. The proof is by contradiction. Assume that there exists $x \in \Omega$ such that $x = rAx$. If $Ax \in K$, then $x = rAx = Ax$, a contradiction. If $Ax \notin K$, then

$$\|Ax - x\| = \|Ax - rAx\| = d(Ax, K) < \|x - Ax\|$$

since r is a metric projection and A is generalized inward, another contradiction. $\square$

Notation: In the following we require rA to be γ-condensing. If $\gamma(A) = k$ and $\gamma(r) = l$ this is so if $kl < 1$. We will write $kl < 1$ to mean either the case just mentioned or $l = 1$ and A is condensing or A is compact and r is continuous.

Lemma 4.1.2. *Let K be a closed convex M_l-set in a Banach space X for some $l \in [1, \infty]$ and let D be a bounded open set in X such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to X$ be a k-γ-contractive map with $kl < 1$ such that A is generalized inward on ∂D_K relative to K. Suppose that $x \neq Ax$ for all $x \in \partial D_K$. Then for any l-γ-contractive metric projections r_1, r_2 from X to K, we have $i_K(r_1 A, D_K) = i_K(r_2 A, D_K)$.*

Proof. Let $H(t,x) = tr_1Ax + (1-t)r_2Ax$ for $x \in \overline{D}_K$ and $t \in [0,1]$. Obviously $H : [0,1] \times \overline{D}_K \to K$ is continuous and such that $\gamma(H([0,1] \times Q) < \gamma(Q)$ for each $Q \subset D$ with $\gamma(Q) \neq 0$. By Lemmas 2.3.10 and 4.1.1 we have $x \neq tr_1Ax + (1-t)r_2Ax$ for $x \in \partial D_K$ and $t \in [0,1]$. It follows from (P_4) of Theorem 1.3.50 in Chapter 1 that $i_K(r_1A, D_K) = i_K(r_2A, D_K)$. $\square$

We now define the fixed point index for generalized inward maps of γ-condensing type.

Definition 4.1.3. Let K be a closed convex M_l-set in a Banach space X for some $l \in [1, \infty]$ and D a bounded open set in X such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to X$ be k-γ-contractive map with $kl < 1$ such that A is a generalized inward on ∂D_K relative to K

and $x \neq Ax$ for all $x \in \partial D_K$. Then we define the fixed point index of A over D_K with respect to K as follows:

$$i_K(A, D_K) = i_K(rA, D_K)$$

where $r : X \to K$ is any l-γ-contractive metric projection and $i_K(rA, D_K)$ is the fixed point index for condensing mappings defined in section 1.3 (also see [48]).

By Lemmas 4.1.1 and 4.1.2 we see that $i_K(A, D_K)$ makes sense and is independent of the choice of l-γ-contractive metric projection. Note also that the new index coincides with the usual one when $A(\overline{D}_K) \subseteq K$.

In Definition 4.1.3, when K be an M_∞-closed convex set, we have defined an index for generalized inward compact map. Comparing this case with Definition 3.1.4, we see that when K is an M and M_∞-closed convex set, the two indexes coincide.

Remark 4.1.4. It is possible to consider the measure of noncompactness β_K in Definition 4.1.3. For example, assume that $A : \overline{D}_K \to X$ is a (β_K, β_X)-condensing, and suppose $r : X \to K$ is 1-(β_X, β_K)-contractive and is a metric projection from X to K. Then $rA : \overline{D}_K \to K$ is β_K-condensing. One may define $i_K(A, D_K) = i_K(rA, D_K)$. However, rA is then β_X-condensing since the existence of r implies that K has the ball intersection property (see Theorem 1.3.6).

The fixed point index in definition 4.1.3 has most of the usual properties of fixed point index. As in Chapter 3, although we can define the index for a map A that is generalized inward on ∂D_K, we need A to be generalized inward on $\overline{D}_K$ to ensure that a nonzero index implies the existence of fixed points.

Theorem 4.1.5. *Let K be a closed convex M_l-set in a Banach space X for some $l \in [1, \infty]$ and D a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a generalized inward k-γ-contractive map with $kl < 1$ and $x \neq Ax$ for $x \in \partial D_K$. Then the index satisfies the following properties.*

(P_1) *(Existence property) If $i_K(A, D_K) \neq 0$, then A has a fixed point in D_K.*

(P_2) *(Normalization) If $u \in D_K$, then $i_K(\hat{u}, D_K) = 1$, where $\hat{u}(x) = u$ for $x \in \overline{D}_K$.*

(P_3) *(Additivity property) If W^1, W^2 are disjoint relatively open subsets of D_K such that*

$x \neq Ax$ for $x \in \overline{D}_K \setminus (W^1 \cup W^2)$, then

$$i_K(A, D_K) = i_K(A, W^1) + i_K(A, W^2).$$

(P_4) (Homotopy property) If $H : [0,1] \times \overline{D}_K \to X$ is continuous and such that for each $t \in [0,1]$, $H(t,.) : \partial D_K \to X$ is a generalized inward map and either
(a) when $l \neq 1$, there exist $k \geq 0$ such that $kl < 1$ and $\gamma(H([0,1] \times Q)) \leq k\gamma(Q)$ for each $Q \subset \overline{D}_K$, or
(b) when $l = 1$, $\gamma(H([0,1] \times Q)) < \gamma(Q)$ for each $Q \subset \overline{D}_K$ with $\gamma(Q) \neq 0$.
Then, if $x \neq H(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$,

$$i_K(H(0,.), D_K) = i_K(H(1,.), D_K).$$

Proof. (P_1) If $i_K(A, D_K) \neq 0$, then we have $i_K(rA, D_K) \neq 0$ by Definition 4.1.3. The earlier version of (P_1) implies that rA has a fixed point in D_K and, by Lemma 4.1.1, A has too.
(P_2) is obvious.
(P_3) follows easily from the definition and use of Lemma 4.1.1.
(P_4) If $x \neq H(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$, by Lemma 4.1.1 we have $x \neq rH(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$. The homotopy property for condensing maps implies the result. □

4.2 Fixed point theorems for generalized inward k-γ-contractive maps

In this section we shall obtain some new fixed point theorems by using the fixed point index developed in the previous section.

Theorem 4.2.1. *Let K be a closed convex M_l-set in a Banach space X for some $l \in [1,\infty]$ and let $A : K \to X$ be a k-γ-contractive generalized inward map with $kl < 1$ and such that $A(K)$ is bounded. Then there is $\rho_0 > 0$ such that $i_K(A, B_K(\rho)) = 1$ for all $\rho \geq \rho_0$, where $B(\rho) = \{x \in X : \|x\| < \rho\}$. Hence, A has a fixed point in K.*

Proof. Let r be an l-γ-contractive metric projection from X to K. Since $A(K)$ is bounded, $rA(K)$ is bounded too. Let $\rho_0 > 0$ be such that $\overline{A(K) \cup rA(K)} \subset B(\rho_0)$. Then, for each $\rho \geq \rho_0$, $\overline{rA(K)} \subset B_K(\rho)$ and $x \neq Ax$ for $x \in \partial B_K(\rho)$. By Definition 4.1.3 and property (P_5) we have $i_K(A, B_K(\rho)) = i_K(rA, B_K(\rho)) = 1$. □

Note that in Theorem 4.2.1, if K is a bounded convex M_l-set, then $A(K)$ is bounded and $B_K(\rho) = K$ for large ρ. Hence, in this case, $i_K(A, K) = 1$.

Theorem 4.2.1 extends Theorem 18.3 in [12].

Theorem 4.2.2. *Let K be a closed convex M_l-set in a Banach space X for some $l \in [1, \infty]$ and D a bounded open set such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a k-γ-contractive generalized inward map with $kl < 1$ such that*

(i) there exists $x_0 \in D_K$ such that $tA(x) + (1-t)x_0$ is a generalized inward map on ∂D_K for each $t \in (0, 1)$, and

(LS) $x \neq tA(x) + (1-t)x_0$ for all $x \in \partial D_K$ and $t \in (0, 1)$.

Then A has a fixed point in $\overline{D}_K$, and if $x \neq A(x)$ for $x \in \partial D_K$, then $i_K(A, D_K) = 1$.

Proof. Assume without loss of generality that $x \neq A(x)$ for $x \in \partial D_K$. Let $H(t, x) = tAx + (1-t)x_0$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. By hypothesis $H(t, .) : \partial D_K \to X$ is a generalized inward map for each $t \in [0, 1]$ and $x \neq H(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$. Hence, from Theorem 4.1.5, $i_K(A, D_K) = i_K(\hat{x}_0, D_K) = 1$. □

The special case of Theorem 4.2.2 when A is weakly inward is an improvement of Theorem 1 in [14].

Note that if $0 \in D_K$ and A is generalized inward on ∂D_K it does not follow in general that tA is generalized inward on ∂D_K for $t \in [0, 1]$, though it does for weakly inward maps. This limits the use of homotopy arguments. The next result gives hypotheses that ensure that tA is generalized inward.

Corollary 4.2.3. *Let K be a closed convex set in a Hilbert space H and D a bounded open set such that $0 \in D_K$. Assume that $A : \overline{D}_K \to H$ is a γ-condensing generalized inward map such that the following conditions hold:*

(ii) $(x, Ax) \leq \|x\|^2$ for $x \in \partial D_K$ with $Ax \notin K$;

(LS) $x \neq tAx$ for $x \in \partial D_K$ and $t \in (0, 1)$.

Then A has a fixed point in $\overline{D}_K$.

Proof. It is sufficient to prove that tA is a generalized inward map on ∂D_K for each $t \in (0,1)$. In fact, for any fixed $x \in \partial D_K$ and $t \in (0,1)$ such that $tAx \notin K$, we have $Ax \notin K$ since $0 \in K$ and K is convex. Since $(x, Ax) \leq \|x\|^2$ and $0 < t < 1$, we have $2t(1-t)(x, Ax) \leq (1-t^2)\|x\|^2$. This implies

$$t^2\|x - Ax\|^2 = t^2\|x\|^2 + t^2\|Ax\|^2 - 2t^2(x, Ax) \leq \|x\|^2 + t^2\|Ax\|^2 - 2t(x, Ax) \leq \|x - tAx\|^2.$$

Thus we have $t\|x - Ax\| \leq \|x - tAx\|$. Since A is generalized inward on ∂D_K, there exists $y \in K$ such that $\|Ax - y\| < \|x - Ax\|$. Hence

$$d(tAx, K) \leq \|tAx - ty\| < t\|x - Ax\| \leq \|x - tAx\|.$$

□

Note that, if in Corollary 4.2.3, (2) holds for all $x \in \partial D_K$ then (LS) holds too. Further if D is a ball and $K = \overline{D}$ and $A : K \to H$ is generalized inward then (LS) holds.

Example 2.5.10 given earlier satisfies all of the conditions in Corollary 4.2.3 but is not weakly inward.

4.3 Nonzero fixed point theorems for weakly inward maps

In this section we discuss the existence of nonzero fixed points for weakly inward maps. As mentioned earlier (see Theorem 2.4.20), in a reflexive Banach space, it is often possible to change to an equivalent locally uniformly convex norm, the weakly inward property is preserved. Therefore, for compact maps, for most of the results of this section one can change to such an equivalent norm if necessary. Hence, when the map involved is compact, all the results obtained in this section apply to any closed convex set in a reflexive Banach space.

Theorem 4.3.1. *Let K be an M_l-wedge in a Banach space X for some $l \in [1, \infty]$ and D a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a k-γ-contractive map with $kl < 1$ and A is weakly inward on $\overline{D}_K$ relative to K. If there exists $e \in K \setminus \{0\}$ such that $x \neq Ax + \lambda e$ for $x \in \partial D_K$ and $\lambda \geq 0$, then $i_K(A, D_K) = 0$.*

Proof. Assume, for a contradiction argument, that $i_K(A, D_K) \neq 0$ and let $\lambda > 0$. Let $H(t, x) = Ax + t\lambda e$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. By hypothesis, $x \neq H(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$. It is obvious that H satisfies (P_4) in Theorem 4.1.5 when $l > 1$ and when $l = 1$. Also $H(t, .) : \overline{D}_K \to X$ is weakly inward on $\overline{D}_K$ for each $t \in [0, 1]$. By Theorem 4.1.5 we have $i_K(A, D_K)= i_K(A + \lambda\hat{e}, D_K) \neq 0$. Hence for each $n \in N$, there exists x_n in the bounded set $\overline{D}_K$ such that $x_n = Ax_n + ne$; this is impossible since $e \neq 0$. □

We use Theorem 4.3.1 to obtain nonzero solutions when K is a wedge.

Theorem 4.3.2. *Let K be an M_l-wedge in a Banach space X for some $l \in [1, \infty]$ and D^1, D bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Suppose $A : \overline{D}_K \to X$ is k-γ-contractive map with $kl < 1$ and A is weakly inward on $\overline{D}_K$. Suppose the following conditions hold.*

(LS) $x \neq tAx$ for $x \in \partial D^1_K$ and $t \in (0, 1)$.

(E) there exists $e \in K \setminus \{0\}$ such that $x \neq Ax + \lambda e$ for $x \in \partial D_K$ and $\lambda > 0$.

Then A has a fixed point in $\overline{D}_K \setminus D^1_K$.

The same assertion is valid if (LS) holds on D_K while (E) holds on D^1_K.

Proof. We may assume that A has no fixed point in $\partial D_K \cup \partial D^1_K$. It follows from Theorem 3.1 that $i_K(A, D^1_K) = 1$ and from Theorem 4.3.1 that $i_K(A, D_K) = 0$. By the additivity property of index we have

$$i_K(A, D_K \setminus \overline{D^1_K}) = i_K(A, D_K) - i_K(A, D^1_K) = 0 - 1 = -1$$

and thus A has a fixed point in $\overline{D}_K \setminus D^1_K$.

The result can be proved similarly when the hypotheses are interchanged. □

Theorem 4.3.2 extends Theorem 3.3.15 above and improves Theorem 2 in [14] and Theorem 3 in [20].

Theorem 4.3.3. *Let K, D^1, D and A be as in Theorem 4.3.2 and suppose there exists a weakly inward compact map $C : \overline{D}_K \to X$ such that the following conditions hold.*

(LS) $x \neq tAx$ *for* $x \in \partial D^1_K$ *and* $t \in (0,1)$.

(i) $\partial K_1 = \{x \in K : \|x\| = 1\}$ *is not compact.*

(ii) $\alpha := \inf\{\|Cx\| : x \in \partial D_K\} > 0$.

(iii) $x \neq Ax + \lambda Cx$ *for* $x \in \partial D_K$ *and* $\lambda > 0$.

Then A has a fixed point in $\overline{D}_K \setminus D^1_K$.

The same assertion is valid if we assume that (LS) holds on D_K while (ii) and (iii) hold on D^1_K.

Proof. We may assume that A has no fixed point in $\partial D_K \cup \partial D^1_K$. As in Lemma 3.3.14, there exist $e \in \partial K_1$ and $\lambda_0 > 0$ such that

$$x \neq Ax + \lambda_0 Cx + \beta e \quad \text{for } x \in \partial D_K \quad \text{and } \beta \geq 0.$$

Note that $A + \lambda_0 C$ is weakly inward as $\overline{I}_K(x)$ is a wedge. By Theorem 4.3.1 we have $i_K(A + \lambda_0 C, D_K) = 0$. Let $H(t,x) = Ax + t\lambda_0 Cx$ for $x \in \overline{D}_K$ and $t \in [0,1]$. By hypothesis, we have $x \neq H(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$. By (P_4) in Theorem 4.1.5 we get $i_K(A, D_K) = 0$. It follows from (P_3) that $i_K(A, D_K \setminus \overline{D^1_K}) = -1$ and thus A has a fixed point in $D_K \setminus \overline{D^1_K}$. □

Theorem 4.3.3 improves Theorem 1 in [38]. Even the special case of $A = C$ in Theorem 4.3.3 improves Theorem 3 in [14].

Finally in this section we give conditions that assure the existence of at least two nonzero fixed points in K.

Theorem 4.3.4. *Let K, D, D^1 be as in Theorem 4.3.2 and let $A : K \to X$ be a k-γ-contractive weakly inward map with $kl < 1$ and $A(K)$ bounded. Suppose that (LS) holds on ∂D^1_K and (E) holds on ∂D_K. Then A has at least two nonzero fixed points in K.*

Proof. Since $A(K)$ is bounded, by Theorem 4.2.1 there exists a bounded open set D^2 such that $\overline{D}_K \subset D^2_K$ and $i_K(A, D^2_K) = 1$. If A has no fixed point on ∂D_K, then by (E) and the additivity property we have $i_K(A, D^2_K \setminus \overline{D}_K) = i_K(A, D^2_K) - i_K(A, D_K) = 1 - 0 = 1$. Hence A has a fixed point x_1 in $D^2_K \setminus D_K$. If A has a fixed point in ∂D^1_K, then the conclusion holds. If $x \neq Ax$ for $x \in \partial D_K^1$, by the proof of Theorem 4.3.2 we have $i_K(A, D_K \setminus \overline{D_K^1}) = -1$ and thus A has a fixed point x_2 in $D_K \setminus \overline{D_K^1}$. □

By an argument similar to that of Theorem 4.3.4, and using the proof of Theorem 4.3.3 we obtain

Theorem 4.3.5. *Under the hypotheses of Theorem 4.3.3 suppose that $A : K \to X$ is weakly inward on all of K and with $A(K)$ bounded. Then A has at least two nonzero fixed points in K.*

4.4 Application

In this section we consider the perturbed Volterra integral equation

$$x(t) = g(t, x(t)) + \int_0^t f(s, x(s))\, ds, \quad t \in [0, 1]. \tag{4.1}$$

We need the following inequality of Gronwall type (see for example Lemma 8.1 in [10] for more general case).

Lemma 4.4.1. *Let $f \in L^1[0,1]$ and $f(t) \geq 0$ a.e on $[0,1]$. Assume that f satisfies the following condition.*

(G) There exist $C > 0$ and $M > 0$ such that

$$f(t) \leq C + M \int_0^t f(s)\, ds \quad a.e.\ on\ [0,1].$$

Then $f(t) \leq Ce^{Mt}$ a.e. on $[0,1]$.

Proof. Let $g(t) = C + M\int_0^t f(s)\, ds$ for $t \in [0,1]$. Then $g'(t) = Mf(t) \leq Mg(t)$ a.e on $[0,1]$. Hence

$$(g(t)e^{-Mt})' = g'(t)e^{-Mt} - Mg(t)e^{-Mt} \leq 0 \quad \text{a.e. on } [0,1].$$

Integrating the above inequality, we obtain $g(t)e^{-Mt} - g(0) \leq 0$ and $g(t) \leq g(0)e^{Mt} = Ce^{Mt}$ for $t \in [0,1]$. The result follows. □

Now, we give our main result in this section.

We make the following hypotheses:

(C_1) $g : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and there exists $L \in (0,1)$ such that

$$|g(t,x) - g(t,y)| \le L|x-y| \ \text{ for } t \in [0,1] \ \text{ and } x, y \in \mathbb{R}^+.$$

(C_2) $g(t,0) \ge 0$ for $t \in [0,1]$.

(C_3) $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ satisfies Carathéodory conditions and there exists $b > 0$ such that

$$|f(t,x)| \le b(1+x) \ \text{ for } t \in [0,1] \ \text{ and } x \in \mathbb{R}^+.$$

(C_4) For every $x \in L^2[0,1]$ with $x(t) \ge 0$ a.e. on $[0,1]$ there exists $M_x \ge 0$ such that

$$\int_0^t f(s, x(s))\, ds \ge -M_x\, x(t) \ \text{ for } t \in [0,1].$$

Remark 4.4.2. (C_4) holds if $f(t,x) \ge 0$ for $x \ge 0$. If we used the standard theory of maps with values in a cone then in place of (C_2) we would have to assume that $g(t,x) \ge 0$ for all $x \ge 0$.

Theorem 4.4.3. *Assume that* (C_1)-(C_4) *hold. Then equation* (4.1) *has a solution x in $L^2[0,1]$ such that $x(t) \ge 0$ for a.e. $t \in [0,1]$.*

Proof. Let $X = L^2[0,1]$ and $K = \{x \in X : x(t) \ge 0 \ \text{ a.e. on } [0,1]\}$. Then K is a cone in X. We define G, F and $A : K \to X$ by

$$Gx(t) = g(t, x(t)), \quad Fx(t) = \int_0^t f(s, x(s))\, ds \quad \text{and} \quad A = G + F.$$

(C_1) implies that $|g(t,x)| \le M + Lx$ for $x \in K$, where M is a bound for $|g(t,0)|$, so that G is continuous and maps bounded sets in K into bounded sets in X. Also (C_1) implies that G is an L-set-contraction. Let $x \in K$ be such that $x^*(x) = 0$ for some $x^* \in K^*$. Then x^* can be identified with an L^2 function, $x^*(t) \ge 0$ a.e. and $\int_0^1 x(t)\, x^*(t)\, dt = 0$. Thus $x^*(t) = 0$ a.e. on the set $\{t : x(t) \ne 0\}$. Therefore, we have

$$x^*(Gx) = \int_0^1 g(s, x(s)) x^*(s)\, ds = \int_{\{s : x(s) = 0\}} g(s,0) x^*(s)\, ds \ge 0.$$

It follows from Theorem 2.4.17 that G is weakly inward.

From (C_3), F is compact. For any $x \in K$, by (C_4) we have $Fx + M_x\, x \in K$. Therefore F is weakly inward since $-x \in I_K(x)$ and $\overline{I}_K(x)$ is a wedge. Hence A is a weakly inward L-set-contraction.

We assert that the set $\{x \in K : x = \lambda Ax,\ 0 \leq \lambda \leq 1\}$ is bounded. Assume that $x \in K$ is such that $x = \lambda Ax$ for some $\lambda \in (0, 1]$. By using (C_1), (C_3) and Lemma 4.4.1 we obtain $\|x\| \leq m$ for a suitable constant m, independent of λ, proving our assertion. By Theorem 5.4.4, A has a fixed point in K. □

Remark 4.4.4. Note that K has empty interior so Theorem 20.4 in [12] cannot be used to treat the above problem. Also, since $A(K)$ may be unbounded, Theorem 3.1 (hence Theorem 18.3 in [12]) cannot be used either.

Chapter 5

A fixed point index for weakly inward A-proper maps

In Chapters 3 and 4 we have obtained theories of the fixed point index for compact maps and maps of condensing type. These theories require that the closed convex sets K satisfy suitable conditions, for example, K is an M-convex set or M_l-closed convex set. If X is reflexive with the property (H), then the index theory for compact maps can be applied to all closed convex sets since any closed convex set is an M-convex set (see Corollary 2.1.12 and the following remark). If X is reflexive and the compact map involved is weakly inward, then this theory can be applied to all closed convex set in this reflexive Banach space since one can obtain an equivalent norm which is such that every closed convex set is an M-convex set under the new norm (see Corollary 2.1.16) and the weakly inward property of the map is unchanged (see Theorem 2.4.20). However, this theory is not applicable to all closed convex sets in any Banach space because if every closed convex set in a Banach space is a M-set, then the space must be reflexive and have property (H) (see Corollary 2.1.12 and the following remark). In the same way, the theory developed in Chapter 4 also is not applicable in every Banach space since not every closed convex set is an M_l-set.

In this Chapter we shall develop a theory of fixed point index for weakly inward A-proper maps. We shall see that this theory can be applied to nearly all closed convex sets in a Banach space with a suitable projection scheme. A Banach space with a projection scheme must be separable but need not be reflexive, for example, the space of continuous functions defined in $[0,1]$ has a suitable scheme. Hence, this theory is not applicable in all Banach spaces, but includes all spaces with a Schauder basis and spaces which arise in the applications have such a projection scheme. Therefore, to require the space has a projection scheme is a mild restriction.

In general, the methods given in Chapters 3 and 4 can not be carried over to establish the index theory for weakly inward A-proper maps since it is not clear whether rA inherits the A-properness of A, that is, if $I - A$ is A-proper, we don't know whether $I - rA$ is still A-proper. Therefore, the previous index theory for A-proper cone maps developed in [22] can not be employed.

Fortunately, we can show that $P_n A$ inherits the weakly inwardness property from A, that is, if A is weakly inward, so is $P_n A$. This enables us to employ the index theory for weakly inward continuous maps in finite dimensional spaces developed in Chapter 3 to establish an index theory for weakly inward A-proper maps.

5.1 Preliminaries

Let X be a Banach space and suppose that there exist a sequence of finite-dimensional subspaces $X_n \subset X$ and a sequence of continuous linear projections $P_n : X \to X_n$ such that $P_n x \to x$ for each $x \in X$. X will be called a Banach space with projection scheme $\Gamma = \{X_n, P_n\}$. Obviously, such a space is separable since $X = \overline{\cup_{n\in N} X_n}$ and we have $\sup_{n\in N} \|P_n\| = c < \infty$ by the uniform boundedness principle (see for example, Proposition 7.7 in [12]. X may also be called a π_c-space.

We shall give examples of Banach spaces and corresponding projection schemes.

Recall that a sequence $\{x_n\}$ in a Banach space X is called a Schauder basis of X if for every $x \in X$ there exists a unique sequence of scalars $\{a_n\}$ so that $x = \sum_{i=1}^{\infty} a_i x_i$. We refer to [44] and [45] for the detailed study of bases.

Let X be a Banach space with a Schauder basis $\{x_n\}$. Let $X_n = span\{x_1, ..., x_n\}$, where $spanB$ denotes the subspace generated by a set B. For each $n \in \mathbb{N}$, define a projection $P_n : X \to X_n$ by

$$P_n(\sum_{i=1}^{\infty} a_i x_i) = \sum_{i=1}^{n} a_i x_i.$$

Then $P_n : X \to X_n$ is a bounded linear operator and $\sup_n \|P_n\| < \infty$ (see [44]).

The projections $\{P_n\}$ are called the natural projections associated with the basis $\{x_n\}$; the number $\sup_n \|P_n\|$ is called the basis constant of $\{x_n\}$. A basis whose basis constant is 1 is called a monotone basis. These facts show that every Banach space with a Schauder basis has a projection scheme and every Banach space with a monotone basis is a π_1-space. There are many spaces which are π_1-spaces, for example c_0, l_p, $1 \leq p < \infty$, $C[0,1]$ and $L^p[0,1]$ for $1 \leq p < \infty$ (see [45]).

Let K be a closed convex set in a Banach space X with a projection scheme $\Gamma = \{X_n, P_n\}$. One is interested in this case when $P_n K \subset K_n = K \cap X_n$ for all $n \in N$ because it is a necessary requirement for developing a theory of fixed point index for A-proper maps, which will be shown later. In general, we don't know whether $P_n K \subset K_n$ for every closed convex set K, but it is a mild restriction and is true for cones and wedges which often arise in applications. We give two examples here.

Example 5.1.1. Let $X = C[0,1]$ and $K = C_+[0,1]$. Then there exist a projection scheme $\Gamma = \{X_n, P_n\}$ such that X is a π_1-space and $P_n K \subset K_n$ for all $n \in N$.

Proof. We define a sequence $\{x_n\}$ of X by

$x_0(t) = \chi_{[0,1]}(t)$,

$x_1(t) = t\chi_{[0,1]}(t)$,

$x_2(t) = x_1(2t) + \chi_{[0,1]}(2t-1) - x_1(2t-1)$,

$x_{2^n+i}(t) = x_2(2^n t - i + 1)$, $i = 1, 2, ..., n = 1, 2, ...$,

where χ_S denotes the characteristic function of a set S, that is, $\chi_S(t) = 1$ if $t \in S$ and $\chi_S(t) = 0$ if $t \notin S$ (see Theorem 5, p.49, [45]).

Let $X_n = span\{x_0, ..., x_{2^n}\}$ for $n \in N$. Then $\{X_n\}$ is a sequence of finite dimensional subspaces of X. For every $x \in X$ let $y_n \in X$ for $n \in \mathbb{N}$ be defined by

$$y_n(t) = x(t) \quad \text{for } t = 0, 1/2^n, 2/2^n, 3/2^n, ..., 1,$$

interpolating linearly between these points. Then $y_n \in X_n$ and $y_n \to x$ in X.

For every $n \in N$ we define a map $P_n : X \to X_n$ by $P_n(x) = y_n$. Then $\{P_n\}$ are the natural projections associated with the basis $\{x_n\}$ and $\sup_n \|P_n\| \leq 1$. Hence X is a π_1-space with the projection scheme $\Gamma = \{X_n, P_n\}$.

We now prove that $P_n K \subset K_n$ for all $n \in \mathbb{N}$. Indeed, let $x \in K$. Then it is easy to see that $y_n(t) \geq 0$ for $t \in [0,1]$ and $n \in \mathbb{N}$, where y_n is defined above. Since $y_n \in X_n$, we have $y_n \in K \cap X_n$ and $P_n K \subset K_n$. □

We shall show that the same holds in $L^p[0,1]$. First, we recall a well-known result.

Lemma 5.1.2. $C[0,1]$ *is dense in* $L^p[0,1]$ *for* $1 \leq p < \infty$.

The above result can be found in Theorem 3.14 in [59] or Theorem 2.13 in [1]. Therefore, we omit the proof of Lemma 5.1.2.

Next, we give the following lemma.

Lemma 5.1.3. *Let* $X = L^p[0,1]$ *for* $1 \leq p < \infty$ *and* $K = L^p_+[0,1]$. *Then*

$$\overline{K \cap C[0,1]} = K,$$

that is, $K \cap C[0,1]$ *is dense in* K.

Proof. It is clear that $\overline{K \cap C[0,1]} \subset K$. We prove the reverse inclusion. Let $f \in K$ and $\varepsilon > 0$. It follows from Lemma 5.1.2 that there exists $g \in C[0,1]$ such that

$$\|f - g\|_{L^p[0,1]} < \varepsilon.$$

This implies $\|f - |g|\|_{L^p[0,1]} < \varepsilon$. Since $g \in C[0,1]$, $|g| \in K \cap C[0,1]$. Therefore, $f \in \overline{K \cap C[0,1]}$ and $K \subset \overline{K \cap C[0,1]}$. □

Example 5.1.4. Let $X = L^p[0,1]$ for $1 \leq p < \infty$ and $K = L^p_+[0,1]$. Then there exist a projection scheme $\Gamma = \{X_n, P_n\}$ such that X is a π_1-space and $P_n K \subset K_n$ for all $n \in \mathbb{N}$.

Proof. It is shown in Theorem 6, p.49, [45] that the Haar system, which is given by

$x_1(t) = \chi_{[0,1]}(t)$,

$x_{2^i+j}(t) = 2^{i/2}[\chi_{[0,1]}(2^{i+1}t - 2j + 2) - \chi_{(0,1]}(2^{i+1}t - 2j + 1)]$ for $j = 1, ..., 2^i, i = 0, 1, 2, ...$

is a monotone basis for X.

We now prove that $P_n K \subset K_n$ for all $n \in \mathbb{N}$. Let $X_n = span\{x_j : j \leq 2^n\}$ and P_n be the natural projection scheme associated with the basis $\{x_n\}$. From the proof of Theorem 6, p.49, in [45] we see that for every $x \in C[0,1]$,

$$P_n x(t) = y_n(t) \quad \text{for } t = j/2^n, j = 1, 2, ..., 2^n.$$

Hence we have $P_n(K \cap C[0,1]) \subset K_n \cap C[0,1]$. It follows from Lemma 5.1.3 and the continuity of P_n that $P_n K \subset K_n$ for all $n \in \mathbb{N}$. □

One can show that a more general result holds when $X = L^p(\Omega)$, $K = L^p_+(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded open set in $\mathbb{R}^n$, and the projection scheme considered is the one given in [35]. The result has been employed in [36].

Closely associated with π_c-spaces is the concept of A-proper maps. The latter has proved of importance in theory and applications. We recall the definition and refer to [55] for a good account of the theory and many applications. It is known that if one is interested in the existence of fixed points the weaker concept of A-proper at 0 often suffices [55], [66].

Definition 5.1.5. A map $T : D \subset X \to X$ is said to be approximation-proper (A-proper, for short) at a point $q \in X$ with respect to Γ if $T_n \equiv P_n T|_{D \cap X_n} : D \cap X_n \to X_n$ is continuous for each $n \in \mathbb{N}$ and whenever $\{x_{n_k} : x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $T_{n_k} x_{n_k} \to q$, then $\{x_{n_k}\}$ has a subsequence which converges to some $x \in D$ with $Tx = q$. T is said to be A-proper on a set K if it is A-proper at all points of K. A-proper alone means A-proper on X.

Closely related to the A-proper class are the P_γ-compact maps.

Definition 5.1.6. For a fixed $\gamma \geq 0$, a map $A : D \subset X \to X$ is said to be P_γ-compact at a point $q \in X$ with respect to Γ if $\lambda I - A$ is A-proper at q for each λ dominating γ (that is $\lambda \geq \gamma$ if $\gamma > 0$ and $\lambda > 0$ if $\gamma = 0$). A is said to be P_γ-compact on a set K if it is P_γ-compact at all points of K.

The above definition can be found in [36] or [42].

In the following we give some examples of A-proper and P_γ-compact maps.

To show that a k-ball contractive map with $k \in [0,1)$ is P_γ-compact for every $\gamma \in (k,1)$, we slightly generalize a special case of a known result due to Nussbaum (see for example, Lemma 1 in [68]).

Lemma 5.1.7. *Let K be a subset in a π_1-space X which satisfies $P_nK \subset K$. Assume that $\{y_n\}$ is a sequence in K. Then for an arbitrary subsequence $\{y_{n_k}\}$, written for simplicity $\{y_k\}$, we have*

$$\beta_K(\{P_k y_k\}) \leq \beta_K(\{y_k\}).$$

Proof. Let $r = \beta_K(\{y_k\})$ and $\epsilon > 0$. Then there exist $\{x_1, ..., x_m\} \subset K$ such that

$$\{y_k\} \subset \cup_{i=1}^m B_K(x_i, r+\epsilon).$$

Since $P_n x \to x$ for each $x \in X$, there exists $k_0 \in \mathbb{N}$ such that

$$\|P_k x_i - x_i\| < \epsilon \quad \text{for all } k \geq k_0 \quad \text{and } i \in \{1, ..., m\}.$$

Hence for $k \geq k_0$, if $y_k \in B_K(x_i, r+\epsilon)$ we have

$$\|P_k y_k - x_i\| \leq \|P_k y_k - P_k x_i\| + \|P_k x_i - x_i\| \leq r + 2\epsilon.$$

Thus $P_k y_k \in B_K(x_i, r+2\epsilon)$. This implies

$$\{P_k y_k\}_{k \geq k_0} \subset \cup_{i=1}^m B(x_i, r+2\epsilon).$$

Hence, $\beta_K(\{P_k y_k\}_{k\geq k_0}) \leq r + 2\epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\beta_K(\{P_k y_k\}_{k\geq k_0}) \leq r$ and thus, $\beta_K(\{P_k y_k\}) \leq \beta_K(\{y_k\})$. □

We also need the following new result.

Lemma 5.1.8. *Let K be a subset in a Banach space X and $\{x_k\}$ and $\{y_k\}$ be two sequences in K. Assume that $x_k - y_k \to 0$. Then*

$$\beta_K(\{x_k\}) = \beta_K(\{y_k\}).$$

Proof. Let $r = \beta_K(\{x_k\})$ and $\varepsilon > 0$. Then there exist $z_1, ..., z_m \in K$ such that $\{x_k\} \subset \cup_{j=1}^m B_K(z_j, r+\varepsilon)$. Since $x_k - y_k \to 0$, there exists $k_0 > 0$ such that $\|x_k - y_k\| < \varepsilon$ for $k \geq k_0$. For every $k \geq k_0$, there exists $j(k) \in \{1, ..., m\}$ such that $\|x_k - z_{j(k)}\| \leq r + \varepsilon$. This implies

$$\|y_k - z_{j(k)}\| \leq \|y_k - x_k\| + \|x_k - z_{j(k)}\| \leq \varepsilon + (r+\varepsilon).$$

Hence, we have

$$y_k \in B_K(z_{j(k)}, r + 2\varepsilon) \subset \cup_{j=1}^{m} B_K(z_j, r + 2\varepsilon).$$

This implies $\{y_k\}_{k \geq k_0} \subset \cup_{j=1}^{m} B_K(z_j, r + 2\varepsilon)$ and thus $\beta_K(\{y_k\}) \leq r + 2\varepsilon$. Therefore, $\beta_K(y_k) \leq \beta_K(\{x_k\})$. Similarly, we have $\beta_K(\{x_k\}) \leq \beta_K(\{y_k\})$. □

Remark 5.1.9. We do not know if $\beta_K(\{x_k\}) = \beta_X(\{y_k\})$ if $\{x_k\} \subset K$ and $\{y_k\} \subset X$ such that $x_k - y_k \to 0$.

Remark 5.1.10. By Proposition 1.3.2 and properties of set-measure of noncompactness, it is easy to verify that in Lemma 5.1.8 if the condition $x_k - y_k \to 0$ is replaced by the weak condition: $x_k - y_k \to q \in X$, then Lemma 5.1.8 is true for the set-measure.

Proposition 5.1.11. *Let K be a subset in a π_1-space X such that $P_n K \subset K$ and D a closed set in K. Assume that $A : D \subset K \to K$ is a continuous β_K-condensing map. Then $I - A$ is A-proper at 0.*

Proof. Let $\{x_{n_k} : x_{n_k} \in D \cap K_{n_k}\}$ be bounded and $x_{n_k} - P_{n_k} A x_{n_k} \to 0$. It follows from Lemmas 5.1.8 and 5.1.7 that

$$\beta_K(\{x_{n_k}\}) = \beta_K(\{P_{n_k} A x_{n_k}\}) \leq \beta_K(\{A x_{n_k}\}).$$

Since A is condensing, $\beta_K(\{x_{n_k}\}) = 0$ and $\{x_{n_k}\}$ is precompact. This implies $\{x_{n_k}\}$ has a subsequence which converges to some $x \in D$. By continuity of A it follows that $x - Ax = 0$. □

As special cases of Proposition 5.1.11 we have

Proposition 5.1.12. *Let K be a wedge in a π_1-space X and D a closed set in K. Assume that $A : D \subset K \to K$ is a continuous k-β_K-contractive map with $k \in [0, 1)$. Then for every $\lambda > k$, $\lambda I - A$ is A-proper on K. In particular, A is P_γ-compact on K for every $\gamma \in (k, 1)$.*

Proof. Let $\mu \in [0, 1/k)$. Then $\mu A : D \subset K \to K$ is μk-β_K-contractive with $\mu k < 1$. It follows from Proposition 5.1.11 that $I - \mu A$ is A-proper at 0. Since K is a wedge, $I - \mu A - q$ is A-proper at 0 for every $q \in K$. Hence, $I - \mu A$ is A-proper at q for every $q \in K$. This implies that $\lambda I - A$ is A-proper on K for every $\lambda > k$. □

Following the above proof, we see that in Proposition 5.1.12 if A is a continuous and β_K-condensing map. Then A is P_1-compact on K.

As an application of Proposition 5.1.12 we have

Corollary 5.1.13. *Let K be a closed subset in a π_1-space X. Assume that K has the ball intersection property relative to X. Let D be a closed set in K and $k \in [0, 1)$. Assume that $A : D \to X$ is a continuous map such that*

$$\beta_X(A(Q)) \leq k\beta_K(Q) \quad \textit{for each bounded set } Q \subset D.$$

Then A is P_γ-compact for every $\gamma \in (k, 1)$.

Remark 5.1.14. If f is a strict contraction defined on a closed convex set $K \neq X$ it is not known whether $I - f$ is A-proper but $I - f$ is A-proper at 0 when f is also weakly inward [66]. Also it is possible to show that accretive operators defined on a closed convex subset K of X are A-proper on K when $I - A$ is weakly inward, see [66].

Comments

Lemma 5.1.3, Lemma 5.1.7, Lemma 5.1.8, Proposition 5.1.11, Proposition 5.1.12 and Corollary 5.1.13 are new.

The proofs of the following results are new.

Example 5.1.1, Example 5.1.4

5.2 A fixed point index for a weakly inward continuous map in a finite dimensional Banach space

In Chapter 3 we have defined an index for a generalized inward compact maps. In this section we quote a special case of the fixed point index given in Definition 3.1.4 for convenience of applications. Throughout this section X will denote a finite dimensional Banach space.

Let K be a closed convex set in X and D an open set in X. We denote by $\overline{D}_K$ and ∂D_K the closure and the boundary, respectively, of $D_K = D \cap K$ relative to K.

Definition 5.2.1. Let D be a bounded open set such that $D_K \neq \emptyset$. Let $f : \overline{D}_K \to X$ be continuous and weakly inward on $\overline{D}_K$ relative to K and suppose that $f(x) \neq x$ for all $x \in \partial D_K$. Define the fixed point index by the equation

$$i_K(f, D_K) = i_K(r_\varepsilon f, D_K),$$

for ε sufficiently small, where r_ε is an ε-projection from X onto K and $i_K(r_\varepsilon f, D_K)$ is the fixed point index as defined in Section 1.2 of Chapter 1 (or as in Amann [4]).

Note that K is an M-set and f is a generalized inward compact map, so that Definition 5.2.1 is a special case of Definition 3.1.4.

By using Theorems 3.1.6 and 3.2.3 we obtain

Theorem 5.2.2. *$f : \overline{D}_K \to X$ be continuous and weakly inward on $\overline{D}_K$ and such that $x \neq f(x)$ for all $x \in \partial D_K$. The index as defined above has the following properties.*

(P_1) (Existence) If $i_K(f, D_K) \neq 0$, then f has a fixed point in D_K.

(P_2) (Normalisation) If $u \in D_K$, then $i_K(\hat{u}, D_K) = 1$, where $\hat{u}(x) = u$ for $x \in \overline{D}_K$.

(P_3) (Additivity) If W^1, W^2 are disjoint relatively open subsets of D_K such that $x \neq f(x)$ for $x \in \overline{D}_K \setminus (W^1 \cup W^2)$, then

$$i_K(f, D_K) = i_K(f, W^1) + i_K(f, W^2)$$

(P_4) (Homotopy property) Let $h : [0,1] \times \overline{D}_K \to X$ be continuous and such that $h(t,.) : \overline{D}_K \to X$ is weakly inward for each $t \in [0,1]$. If $x \neq h(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$, then

$$i_K(h(0,.), D_K) = i_K(h(1,.), D_K).$$

(P_5) If K is bounded closed convex set and $f : K \to X$ is continuous weakly inward map, then $i_K(f, K) = 1$.

5.3 The fixed point index for weakly inward A-proper maps

From now on X denotes an infinite dimensional Banach space with a projection scheme Γ, K stands for a closed, convex set and D a bounded open set in X. As before, $\overline{D}_K$

and ∂D_K denote the closure and the boundary, respectively, of $D_K = D \cap K$ relative to K, (where we suppose $D \cap K \neq \emptyset$). Let $A : \overline{D}_K \to X$ be a weakly inward map (relative to K). We wish to show that $A_n = P_n A : \overline{D}_{K_n} \to X_n$ is weakly inward (relative to $K_n = K \cap X_n$). It is evident that this is so if

$$P_n \overline{I}_K(x) \subseteq \overline{I}_{K_n}(x) \text{ for every } x \in K_n.$$

We will show that this happens *if and only if* $P_n K \subseteq K$. This assumption is the standard one in the theory of A-proper maps involving cones and is usually easily verified in applications. It is of interest that it cannot be weakened in our setting.

We begin with a result which has a geometrical flavour. A more general result has been given in Proposition 2.4.12. We provide another simple proof of the result here.

Proposition 5.3.1. *Let X be a finite dimensional space and let K be a closed convex subset of X. Then $K = \bigcap_{x \in K} \overline{I}_K(x)$.*

Proof. Denote the set $\bigcap_{x \in K} \overline{I}_K(x)$ by K_1, and suppose there is $x_1 \in K_1 \setminus K$. As X is finite dimensional we may, and do, change to an equivalent Hilbert space norm. Let $x_0 \in K$ be the point of K nearest to x_1 relative to the new metric induced by the inner product $(.,.)$. It follows from Lemma 1.3.16 (also see for example Theorem 2.3 in [32], or Proposition 9.2 in [12]), that

$$(x_1 - x_0, x_0 - k) \geq 0 \quad \text{for all } k \in K.$$

We assert that

$$(x_1 - x_0, x_0 - w) \geq 0 \quad \text{for all } w \in I_K(x_0).$$

Indeed, if $w \in I_K(x_0)$ then $w = (1-a)x_0 + ak,\ a \geq 1,\ k \in K$, so that $x_0 - w = a(x_0 - k)$, which proves the assertion. By continuity it follows that the same inequality holds for all $w \in \overline{I}_K(x_0)$. But $x_1 \in \overline{I}_K(x_0)$ so that

$$-\|x_1 - x_0\|^2 = (x_1 - x_0, x_0 - x_1) \geq 0,$$

which gives the impossible equality $x_1 = x_0$. □

Lemma 5.3.2. *$P_n K \subseteq \overline{I}_{K_n}(x)$ for all $x \in K_n$ if and only if $P_n \overline{I}_K(x) \subseteq \overline{I}_{K_n}(x)$ for all $x \in K_n$.*

Proof. Since $K \subseteq \overline{I}_K(x)$ for every $x \in K$, the 'if' statement is obvious. Conversely, assume that $P_n K \subseteq \overline{I}_{K_n}(x)$ for all $x \in K_n$. As P_n is continuous it suffices to prove that $P_n I_K(x) \subseteq \overline{I}_{K_n}(x)$ for every $x \in K_n$. Let $x \in K_n$ and $y \in I_K(x)$, so that $y = (1-a)x + aw$, where $a \geq 0$ and $w \in K$. Thus, $P_n y = (1-a)x + aP_n w$. Since $P_n w \in P_n K \subseteq \overline{I}_{K_n}(x)$, by hypothesis, applying Lemma 5.3.1 we obtain $P_n y \in \overline{I}_{K_n}(x)$. □

Proposition 5.3.3. *$P_n \overline{I}_K(x) \subseteq \overline{I}_{K_n}(x)$ for all $x \in K_n$ if and only if $P_n K \subseteq K$.*

Proof. From Lemma 5.3.2 we have only to show that $P_n K \subseteq K_n$ if and only if $P_n K \subseteq \overline{I}_{K_n}(x)$ for all $x \in K_n$. However this is immediate using Proposition 5.3.1 □

For convenience we give another simple proof of Corollary 2.4.8 here.

Lemma 5.3.4. *For $w \in \overline{I}_K(x)$,*

$$(1-a)x + aw \in \overline{I}_K(x) \quad \text{for every} \quad a \geq 0.$$

Proof. We have $w = \lim_{n\to\infty} w_n$ where $w_n = (1-c_n)x + c_n y_n$, $y_n \in K$, $c_n \geq 0$. Then $(1-a)x + aw_n = (1-ac_n)x + ac_n y_n \in I_K(x)$ so that $(1-a)x + aw \in \overline{I}_K(x)$.

From now on it will be assumed, even if not explicitly mentioned, that $P_n K \subseteq K$. We observe that this implies $P_n K = K_n$ since P_n is a projection.

We now show that $A_n = P_n A$ inherits useful properties from A when $I - A$ is A-proper.

Lemma 5.3.5. *Suppose that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to X$ be such that $I - A$ is A-proper at 0 and that $x \neq Ax$ for $x \in \partial D_K$. Then there exists $n_0 \in \mathbb{N}$ such that $x \neq A_n x$ for $x \in \partial D_{K_n}$ and $n \geq n_0$.*

Proof. If not, there exists a sequence, written for simplicity $\{x_n : x_n \in \partial D_{K_n}\}$ such that $x_n - A_n x_n = 0$. Since $I - A$ is A-proper at 0 there is a subsequence $x_{n_k} \to x$ and $x = Ax$. Since, for each n, $\partial D_{K_n} \subset \partial D_K$ and the latter set is closed we have $x \in \partial D_K$, so our hypothesis has been contradicted. □

Definition 5.3.6. Let K be a closed convex set in a Banach space X with projection scheme $\Gamma = \{X_n, P_n\}$ and D a bounded open set in X such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to X$ be a weakly inward map where $I - A$ is A-proper at 0 and such that $x \neq Ax$ for $x \in \partial D_K$. Write A_n for the map $P_n A$ restricted to $\overline{D}_{K_n}$. Let $\mathbb{Z}'$ denote the set of all integers together with $\{+\infty\}$ and $\{-\infty\}$. We define the fixed point index of A over D_K with respect to K as follows:

$$i_K(A, D_K) = \{m \in \mathbb{Z}' : \text{ there exists an infinite sequence } n_k \to \infty \text{ such that } i_{K_{n_k}}(A_{n_k}, D_{K_{n_k}}) \to m\},$$

where $i_{K_n}(A_n, D_{K_n})$ is the fixed point index for weakly inward maps as defined in the previous section.

Remark 5.3.7. By Lemma 5.3.5 there exists $n_0 \in \mathbb{N}$ such that $x \neq A_n x$ for $x \in \partial D_{K_n}$ and $n \geq n_0$. Also by Proposition 5.3.3 it follows that A_n is weakly inward relative to K_n for each $n \in \mathbb{N}$. Hence $i_{K_n}(A_n, D_{K_n})$ makes sense and the index is well-defined. The index is a generalization of the index defined in [22] and coincides with the old definition if A has its image in K. Note that, in general the index is a set not a single integer, a situation shared by the degree of A-proper maps.

The fixed point index for weakly inward maps A where $I - A$ is A-proper at 0, as introduced above, has the usual properties which we now give. First recall that $G : [0,1] \times \Omega \to X$ is called an *A-proper homotopy* (at q) if $P_n G$ is continuous and whenever $\{x_n \in \Omega \cap X_n\}$ is a bounded sequence and $\{t_n \in [0,1]\}$ are such that $P_n G(t_n, x_n) \to q$ (for a subsequence) then $\{x_n\}$ and $\{t_n\}$ have convergent subsequences with limits x, t such that $G(t, x) = q$.

Theorem 5.3.8. *Let $A : \overline{D}_K \to X$ be a weakly inward map where $I - A$ is A-proper at* 0 *and such that $x \neq Ax$ for $x \in \partial D_K$. Then the index defined above has the following properties.*

(P_1) (Existence) If $i_K(A, D_K) \neq \{0\}$, then A has a fixed point in D_K.

(P_2) (Normalisation) If $u \in D_K$, then $i_K(\hat{u}, D_K) = \{1\}$, where $\hat{u}(x) = u$ for $x \in \overline{D}_K$.

(P_3) (Additivity) If W^1, W^2 are disjoint relatively open subsets of D_K such that $x \neq Ax$ for $x \in \overline{D}_K \setminus (W^1 \cup W^2)$, then

$$i_K(A, D_K) \subset i_K(A, W^1) + i_K(A, W^2)$$

with equality holding if either $i_K(A, W^1)$ or $i_K(A, W^2)$ is a singleton.

(P_4) (Homotopy property) Let $H : [0,1] \times \overline{D}_K \to X$ be such that $H(t, .) : \overline{D}_K \to X$ is weakly inward for each $t \in [0,1]$ and $x - H(t,x)$ is an A-proper homotopy (at 0). If

$x \neq H(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$, then

$$i_K(H(0,.), D_K) = i_K(H(1,.), D_K).$$

Proof. Much of this is routine using the A-proper property and the finite dimensional results proved above. We refer to the discussion of degree theory in the monograph [55] for the standard type of arguments used . We illustrate the methods and prove some necessary parts only.

(P_1) Since $i_{K_n}(A_n, D_{K_n}) \neq 0$ for infinitely many n, by the finite dimensional result there exist corresponding $x_n \in D_{K_n}$ such that $x_n = A_n x_n$. Exactly as in Lemma 5.3.5, this gives $x_{n_k} \to x$ and $x = Ax$. Thus $x \in \overline{D}_K$. As $x \neq Ax$ for all $x \in \partial D_K$ the result is shown.

(P_4) The definition of A-proper homotopy shows that $x \neq P_n H(t,x)$ for all $x \in \partial D_K$, $t \in [0,1]$ for n sufficiently large. Therefore $P_n H$ is an appropriate homotopy in X_n and so the two sets of finite dimensional indices coincide for sufficiently large n. □

Comments

All the results of this section were published in [43].

5.4 Fixed point theorems

In this section we shall obtain some new fixed point and nonzero fixed point theorems by using the fixed point index defined above. Recall that it is tacitly assumed that the closed convex set K is such that $P_n K \subset K$ for every n.

We first prove that (P_5) carries over to the infinite dimensional setting.

Theorem 5.4.1. *Let K be a bounded closed convex set and suppose $A : K \to X$ is weakly inward on K where $I - A$ is A-proper at 0. Then $i_K(A, K) = 1$, and hence A has a fixed point in K.*

Proof. The result follows from Definition 5.3.6 and Property (P_5). □

Remark 5.4.2. Note that in Theorem 5.4.1, $A(K)$ need not be bounded.

Theorem 5.4.3. *Let K be an unbounded closed convex set and suppose $A : K \to X$ is weakly inward on K where $I - A$ is A-proper at 0. Suppose also that $A(K)$ is bounded. Then there exists $\rho_0 > 0$ such that $i_K(A, B_K(\rho)) = \{1\}$ for all $\rho \geq \rho_0$, and hence A has a fixed point in K.*

Proof. Since $P_n x \to x$ as $n \to \infty$ there exists $c < \infty$ such that $\|P_n\| \leq c$ for all n. By (P_5) of Theorem 3.1.6, for each n there exists $\rho_n > 0$ such that $i_{K_n}(A_n, B_{K_n}(\rho)) = 1$ for all $\rho \geq \rho_n$. Now ρ_n is uniformly bounded by some ρ_0, because ρ_n is a bound for $A_n K_n$ and $\|P_n Ax\| \leq c\|Ax\|$. This proves the result. □

This result extends the basic fixed point theorem for weakly inward compact maps [26] where it is assumed that K is a bounded set. It is closely related to a result of Lafferriere for P_1-compact maps, see property (P) of [36].

In the next result we employ the well-known Leray-Schauder boundary condition.

Theorem 5.4.4. *Let D be a bounded open set such that $0 \in D_K$. Assume that $A : \overline{D}_K \to X$ is weakly inward and P_1-compact at 0 and satisfies*
(LS) $x \neq tA(x)$ for all $x \in \partial D_K$ and $t \in [0, 1)$.
Then A has a fixed point in $\overline{D}_K$. Furthermore, if $x \neq A(x)$ for $x \in \partial D_K$, then $i_K(A, D_K) = \{1\}$.

Proof. Without loss of generality we may suppose that $x \neq A(x)$ for $x \in \partial D_K$. Let $H(t, x) = tAx$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. By hypothesis, $x \neq H(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$. $H(t, .) : \overline{D}_K \to X$ is weakly inward for each $t \in [0, 1]$ and $I - H$ is an A-proper homotopy because $I - tA$ is A-proper at 0 for every $t \geq 0$. It follows from Theorem 5.3.8 that

$$i_K(A, D_K) = i_K(\hat{0}, D_K) = \{1\}.$$

□

Remark 5.4.5. Each of the following hypotheses implies (LS) when $0 \in D_K$,

$(*)$ $\|Ax\| < \|x\| + \|x - Ax\|$ for each $x \in \partial(D_K)$ with $\|Ax\| > \|x\|$;

$(**)$ $\|Ax\|^2 \leq \|x\|^2 + \|x - Ax\|^2$ for each $x \in \partial(D_K)$ with $\|Ax\| > \|x\|$.

Now we discuss the existence of nonzero fixed points for weakly inward maps by giving some conditions which ensure the fixed point index is $\{0\}$. The first employs a well-known hypothesis, which, as previously, we denote by (E).

Lemma 5.4.6. *Let K be a wedge in X and D a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is weakly inward, $I - A$ is A-proper on K, and $A(\overline{D}_K)$ is bounded. Suppose that $x \neq Ax$ for $x \in \partial D_K$ and*
(E) there exists $e \in K \setminus \{0\}$ such that $x \neq Ax + \lambda e$ for $x \in \partial D_K$ and $\lambda > 0$.
Then $i_K(A, D_K) = \{0\}$.

Proof. Assume, for a contradiction, that $i_K(A, D_K) \neq \{0\}$. For $\lambda > 0$, let $H(t,x) = Ax + t\lambda e$ for $x \in \overline{D}_K$ and $t \in [0,1]$. By hypothesis, $x \neq H(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$. As $\overline{I}_K(x)$ is a wedge containing K we see that $H(t,.) : \overline{D}_K \to X$ is weakly inward for each $t \in [0,1]$. It follows from (P_4) that we have $i_K(A, D_K) = i_K(A + \lambda\hat{e}, D_K) \neq \{0\}$. Hence for each $n \in \mathbb{N}$, there exists $x_n \in \overline{D}_K$ such that $x_n = Ax_n + ne$. As $\overline{D}_K$ and $A(\overline{D}_K)$ are bounded this implies $e = 0$, a contradiction. □

Theorem 5.4.7. *Let K be a wedge in X and let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Let $A : \overline{D}_K \to X$ be weakly inward, P_1-compact on K, with $A(\overline{D}_K)$ bounded. Suppose the following conditions are satisfied.*
(LS) $\quad x \neq tAx$ for $x \in \partial D^1_K$ and $0 \leq t < 1$.
(E) $\quad$ There exists $e \in K \setminus \{0\}$ such that $x \neq Ax + \lambda e$ for $x \in \partial D_K$ and $\lambda > 0$.
Then A has a fixed point in $\overline{D}_K \setminus D^1_K$. The same conclusion remains valid if (LS) holds on ∂D_K and (E) holds on ∂D^1_K.

Proof. Suppose that A has no fixed point in $\partial D_K \cup \partial D^1_K$. It follows from Theorem 5.4.4 and Lemma 5.4.6 that $i_K(A, D^1_K) = \{1\}$ and $i_K(A, D_K) = \{0\}$. By the additivity property (P_3) of index we have

$$i_K(A, D_K \setminus \overline{D^1_K}) = i_K(A, D_K) - i_K(A, D^1_K) = \{-1\}$$

and thus A has a fixed point in $\overline{D}_K \setminus D^1_K$. The proof is exactly similar if the hypotheses are interchanged. □

Our next Lemma gives a new result in that it was previously assumed [51] that K was a cone and that A map into K, a stronger condition than our condition (A_1) below and weakly inward. We note that, for a wedge K, if $K \neq -K$ then $K \setminus -K \neq \emptyset$.

Lemma 5.4.8. *Let K be a wedge in X with $K \neq -K$. Let D be a bounded open set such that $D_K \neq \emptyset$. Let $A, C : \overline{D}_K \to X$ be weakly inward maps such that $A(\overline{D}_K)$ and $C(\overline{D}_K)$ are bounded. Suppose that $I - A - \lambda C$ is A-proper on K for each $\lambda \geq 0$. Suppose the following conditions hold.*

(A_1) $\overline{C(\partial D_K)} \cap (-K \setminus K) = \emptyset$.

(A_2) $\alpha = \inf\{\|Cx\| : x \in \partial D_K\} > 0$.

(A_3) $x \neq Ax + \lambda Cx$ *for* $x \in \partial D_K$ *and* $\lambda \geq 0$.

Then $i_K(A, D_K) = \{0\}$.

Proof. We first prove that for any fixed $e \in K \setminus (-K)$ with $\|e\| = 1$, there exists $\lambda_0 > 1$ such that $x \neq Ax + \lambda_0 Cx + \beta e$ for every $x \in \partial D_K$ and every $\beta \geq 0$. In fact, if not, there exist sequences $\{\lambda_n > 1\}$, $\lambda_n \to \infty$, $\{\beta_n \geq 0\}$ and $\{x_n\} \subset \partial D_K$ such that $x_n = Ax_n + \lambda_n Cx_n + \beta_n e$. Hence, $Cx_n + \lambda_n^{-1}\beta_n e \to 0$, and as $\{Cx_n\}$ is bounded we may assume that $\beta_n/\lambda_n \to b \in [0, +\infty)$. By (A_2), $b \neq 0$. It follows that $Cx_n \to -be$. Since $e \in K \setminus (-K)$ implies $-be \in -K \setminus K$, we have $-be \in \overline{C(\partial D_K)} \cap (-K \setminus K)$, which contradicts (A_1). For this $\lambda_0 > 1$, by Lemma 5.4.6 we have $i_K(A + \lambda_0 C, D_K) = \{0\}$. Let $H(t, x) = Ax + t\lambda_0 Cx$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. By hypothesis, we have $x \neq H(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$. It follows from the homotopy property of index that $i_K(A, D_K) = i_K(A + \lambda_0 C, D_K) = \{0\}$. □

Remark 5.4.9. If $I - A$ is A-proper (on X) and $C : K \to X$ is compact then $I - A - \lambda C$ is automatically A-proper. However this fails when we only have that $I - A$ is A-proper on K. However, if $C : K \to K$ is compact and $I - A$ is A-proper on K then Lemma 5.4.8 holds without assuming (A_1) explicitly.

Our results enable us to prove the existence of eigenvalues of P_γ-compact maps. We illustrate with the following corollary which extends a result from [51].

Corollary 5.4.10. *Let K and D be as in Lemma 5.4.8 with $0 \in D$. Let $C : \overline{D}_K \to X$ be weakly inward, P_0-compact on K, and with $C(\overline{D}_K)$ bounded. Suppose that (A_1) and (A_2)*

of Lemma 5.4.8 hold on ∂D_K. Then there exists $\mu > 0$ and $x \in \partial D_K$ such that $Cx = \mu x$.

Proof. We take $A = \hat{0}$. Since $i_K(A, D_K) = \{1\}$, the hypothesis (A_3) (with $A = \hat{0}$) of Lemma 5.4.8 must fail, that is there is $\lambda \geq 0$ and $x \in \partial D_K$ such that $x = \lambda Cx$. Hence $\lambda > 0$ and $Cx = \mu x$ for $\mu = 1/\lambda$. □

By using Theorem 5.4.4 and Lemma 5.4.8 and an argument similar to that of Theorem 5.4.7, we obtain

Theorem 5.4.11. *Let K be a wedge in X with $K \neq -K$. Let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Suppose $A : \overline{D}_K \to X$ is weakly inward, P_1-compact on K with $A(\overline{D}_K)$ bounded. Suppose $C : \overline{D}_K \to X$ is weakly inward, $C(\overline{D}_K)$ is bounded, and that $I - A - \lambda C$ is A-proper on K for all $\lambda \geq 0$. Assume the following conditions hold.*

(LS) $x \neq tAx$ for $x \in \partial D^1_K$ and $0 \leq t < 1$.

(A_1), (A_2), (A_3) of Lemma 5.4.8 hold on ∂D_K. Then A has a fixed point in $\overline{D}_K \setminus D^1_K$. The same conclusion holds if (LS) holds on ∂D_K and $(A_1) - (A_3)$ hold on ∂D^1_K.

When the map C is compact some other hypotheses can be applied too. The proof uses arguments similar to some of those used in [21].

Lemma 5.4.12. *Let K be a wedge in X such that $\partial K_1 = \{x \in K : \|x\| = 1\}$ is not compact. Let D be a bounded open set in X with $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is weakly inward, and that $A(\overline{D}_K)$ is bounded. Let $C : \overline{D}_K \to X$ be a weakly inward compact map and that $I - A - \lambda C$ is A-proper on K for all $\lambda \geq 0$. Suppose that (A_2) and (A_3) of Lemma 5.4.8 hold. Then $i_K(A, D_K) = \{0\}$.*

Proof. As previously, we can find $e \in \partial K_1$ and $\lambda_0 > 1$ such that

$$x \neq Ax + \lambda_0 Cx + \beta e \quad \text{for } x \in \partial D_K \quad \text{and } \beta \geq 0.$$

By Lemma 5.4.6 we have $i_K(A + \lambda_0 C, D_K) = \{0\}$. Let $H(t, x) = Ax + t\lambda_0 Cx$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. By hypothesis, we have $x \neq H(t, x)$ for $x \in \partial(D_K)$ and $t \in [0, 1]$. It follows from (P_4) that $i_K(A, D_K) = i_K(A + \lambda_0 C, D_K) = \{0\}$. □

By the same method as before we immediately obtain the following result.

Theorem 5.4.13. *Let K be a wedge in X such that ∂K_1 is not compact. Let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Suppose $A : \overline{D}_K \to X$ is weakly inward and such that $A(\overline{D}_K)$ is bounded. Let $C : \overline{D}_K \to X$ be a weakly inward compact map where $I - A - \lambda C$ is A-proper on K for all $\lambda \geq 0$. Suppose that the following conditions hold.*

(LS) holds on ∂D^1_K.

(A_2) and (A_3) of Lemma 5.4.8 hold on ∂D_K.

Then A has a fixed point in $\overline{D}_K \setminus D^1_K$. The same conclusion remains valid if (LS) holds on ∂D_K and $(A_2), (A_3)$ hold on ∂D^1_K.

We conclude this section by giving conditions that assure the existence of at least two nonzero fixed points in K.

Theorem 5.4.14. *Let K be a wedge and let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Let $A : K \to X$ be a weakly inward and P_1-compact on K with $A(K)$ bounded. Suppose that (LS) holds on ∂D^1_K and that (E) holds on ∂D_K. Then A has at least two nonzero fixed points in K.*

Proof. Since $A(K)$ is bounded, by Theorem 5.4.3 there exists a bounded open set D^2 such that $\overline{D}_K \subset D^2_K$ and $i_K(A, D^2_K) = \{1\}$. By (E), Lemma 5.4.6, and property (P_3) we have

$$i_K(A, D^2_K \setminus \overline{D}_K) = i_K(A, D^2_K) - i_K(A, D_K) = \{1\} - \{0\}.$$

Hence A has a fixed point x_1 in $D^2_K \setminus \overline{D}_K$. If A has a fixed point in ∂D^1_K, then the conclusion holds. If $x \neq Ax$ for $x \in \partial D^1_K$, by the proof of Theorem 5.4.4 and (P_3) we have $i_K(A, D_K \setminus \overline{D^1_K}) = \{-1\}$ and therefore A has a fixed point x_2 in $D_K \setminus \overline{D^1_K}$. □

Comments

All the results of this section were published in [43].

5.5 Fixed point theorems for P_γ-compact maps

This section and the following section are new, not part of [43].

In this section we give norm-type expansion and compression theorems. Our method is simpler than that used in [36] and can also be applied to establish norm-type expansion and compression theorems for weakly inward k-γ-contractive maps.

Lemma 5.5.1. *Let K be a wedge in X and D a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is weakly inward and P_γ-compact for some $\gamma \in [0,1)$ such that $A(\overline{D}_K)$ is bounded. Suppose that $x \neq Ax$ for $x \in \partial D_K$ and the following conditions hold.*
(H_1) There exist $a \in (1, \gamma^{-1})$ (if $\gamma = 0$, let $\gamma^{-1} = +\infty$) and $e \in K \setminus \{0\}$ such that $x \neq aAx + \lambda e$ for $x \in \partial D_K$ and $\lambda \geq 0$.
(H_2) $x \neq \beta Ax$ for $x \in \partial D_K$ and $\beta \in (1, a)$.
Then $i_K(A, D_K) = \{0\}$.

Proof. By (H_1) and Lemma 5.4.6 we have $i_K(aA, D_K) = \{0\}$. Let

$$H(t,x) = (1 + (a-1)t)Ax \quad \text{for } x \in \overline{D}_K \quad \text{and } t \in [0,1].$$

It follows from the property (P_4) in Theorem 5.3.8 that

$$i_K(A, D_K) = i_K(aA, D_K) = \{0\}.$$

□

Theorem 5.5.2. *Let K be a wedge in X and let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Let $A : \overline{D}_K \to X$ be weakly inward, P_γ-compact for some $\gamma \in [0,1)$, with $A(\overline{D}_K)$ bounded. Suppose (H_1) and (H_2) hold on ∂D_K and the following condition is satisfied.*
(LS) $x \neq tAx$ for $x \in \partial D^1_K$ and $0 \leq t < 1$.
Then A has a fixed point in $\overline{D}_K \setminus D^1_K$. The same conclusion remains valid if (LS) holds on ∂D_K and (H_1) and (H_2) hold on ∂D^1_K.

Proof. Suppose that A has no fixed point in $\partial D_K \cup \partial D^1_K$. It follows from Theorem 5.4.4 and Lemma 5.5.1 that $i_K(A, D^1_K) = \{1\}$ and $i_K(A, D_K) = \{0\}$. By the additivity property (P_3) of index we have

$$i_K(A, D_K \setminus \overline{D^1_K}) = i_K(A, D_K) - i_K(A, D^1_K) = \{-1\}$$

and thus A has a fixed point in $\overline{D}_K \setminus D_K^1$. The proof is exactly similar if the hypotheses are interchanged. □

If the wedge K in Theorem 5.5.2 satisfies $K \neq -K$, then the condition (H_1) in Theorem 5.5.2 can be replaced by a norm condition and an additional boundary condition. To do this, we first recall the definition of the constant of quasinormality of K.

Let K be a wedge in X with $K \neq -K$. Then the quasinormality constant of K, $\delta(K)$, is defined as follows:

$$\delta(K) = \sup\{\delta(u) : u \in K \setminus \{0\}\},$$

where $\delta(u) = \inf\{\|x+u\|/\|x\| : x \in K \setminus \{0\}\}$. Moreover, we have $\delta(K) \in [1/2, 1]$. For more details see Section 1 in Chapter 1.

Corollary 5.5.3. *Let K be a wedge in X with $K \neq -K$ and let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D_K^1} \subset D_K$. Let $A : \overline{D}_K \to X$ be weakly inward, P_γ-compact for some $\gamma \in [0,1]$, with $A(D_K)$ bounded. Suppose the following conditions are satisfied.*

(LS) $x \neq tAx$ for $x \in \partial D_K^1$ and $0 \leq t < 1$.

(h_1) There exists $\sigma > 0$ such that $\|Ax\| \geq \dfrac{(\gamma+\sigma)}{\delta(K)}\|x\|$ for $x \in \partial D_K$, where $\delta(K)$ is the quasinormality constant of K.

(h_2) If $\gamma \in [0,1)$, we assume that $x \neq cAx$ for $x \in \partial D_K$ and $c \in (1, \gamma^{-1})$.

(h_3) $A(\partial D_K) \subset K$.

Then A has a fixed point in $\overline{D}_K \setminus D_K^1$. The same conclusion remains valid if (LS) holds on ∂D_K and (h_1), (h_2) and (h_3) hold on ∂D_K^1.

Proof. For a fixed $\varepsilon \in (0, \sigma(\gamma+\sigma)^{-1})$, there exists $e \in K \setminus \{0\}$ such that

$$\delta(e) \geq (1-\varepsilon)\delta(K).$$

Therefore, we have

$$\|x+e\| \geq \delta(e)\|x\| \geq (1-\varepsilon)\delta(K)\|x\| \quad \text{for all } x \in K.$$

This implies $\|x+\lambda e\| = \lambda\|x/\lambda + e\| \geq \lambda(1-\varepsilon)\delta(K)\|x/\lambda\|$ for every $\lambda > 0$ and $x \in K$. Hence, we have

$$\|x+\lambda e\| \geq (1-\varepsilon)\delta(K)\|x\| \quad \text{for all } x \in K \quad \text{and all } \lambda \geq 0.$$

Let $e \in K \setminus \{0\}$ satisfy the above inequality. We consider two cases. (i) If $\gamma = 1$, we prove

$$x \neq Ax + \lambda e \quad \text{for } x \in \partial D_K \quad \text{and } \lambda \geq 0.$$

In fact, if not, there exist $x_0 \in \partial D_K$ and $\lambda_0 \geq 0$ such that $x_0 = Ax_0 + \lambda_0 e$. Since $Ax_0 \in K$, we have

$$\|x_0\| \geq (1-\varepsilon)\delta(K)\|Ax_0\| \geq (1-\varepsilon)\delta(K)\frac{(1+\sigma)}{\delta(K)}\|x_0\| > \|x_0\|,$$

a contradiction. Hence, when $\gamma = 1$, the result follows from Theorem 5.4.7.

(ii) Now, we assume that $\gamma \in [0,1)$. Let $a \in (\max\{1, (\gamma+\sigma)^{-1}(1-\varepsilon)^{-1}\}, \gamma^{-1})$. Then $a \in (1, \gamma^{-1})$. We prove that

$$x \neq aAx + be \quad \text{for } x \in \partial D_K \quad \text{and } b \geq 0,$$

In fact, if not, there exist $x_0 \in \partial D_K$ and $b_0 \geq 0$ such that $x_0 = aAx_0 + b_0 e$. Since $Ax_0 \in K$, we have

$$\|x_0\| \geq a\delta(e)\|Ax_0\| \geq a\delta(e)\frac{(\gamma+\sigma)}{\delta(K)}\|x_0\| \geq a(1-\varepsilon)(\gamma+\sigma)\|x_0\| > \|x_0\|,$$

a contradiction. Hence, (H_1) in Theorem 5.5.2 holds. It is easy to see that all of other conditions in Theorem 5.5.2 hold. The results follow from Theorem 5.5.2. □

Remark 5.5.4. The following condition used in Theorem 2.1 in [36] implies (h_1) in Corollary 5.5.3.

(h_1') $\inf\{\|Ax\| : x \in \partial D_K\} > \gamma d/\delta(K)$, where $d = \sup\{\|x\| : x \in \partial D_K\}$.

Therefore, Corollary 5.5.3 generalizes Theorem 2.1 in [36] in the following ways: (i) $A(\overline{D}_K)$ need not be contained in K and (ii) the condition (h_1) is weaker than (h_1'). Moreover, our method is simpler than that used in Theorem 2.1 in [36].

Corollary 5.5.5. *Let K be a wedge in X with $K \neq -K$ and let D^1, D be bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Let $A : \overline{D}_K \to X$ be weakly inward, P_γ-compact for some $\gamma \in [0, \delta(K))$ ($\delta(K)$ is the quasinormality constant of K), with $A(D_K)$ bounded. Suppose the following conditions are satisfied.*

(S_1) $\|Ax\| \leq \|x\|$ *for* $x \in \partial D^1_K$.

(S_2) $\|Ax\| \geq \|x\|$ *for* $x \in \partial D_K$.

(H_3) $\quad A(\partial D_K) \subset K.$

Then A has a fixed point in $\overline{D}_K \setminus D_K^1$. The same conclusion remains valid if (S_1) holds on ∂D_K and (S_2) and (H_3) hold on ∂D_K^1.

Proof. It is easy to see that (S_1) implies (LS) in Corollary 5.5.3. We prove that (S_3) implies (h_1) and (H_2). Let $\sigma \in (0, \delta(K) - \gamma)$, then we have $\dfrac{(\gamma + \sigma)}{\delta(K)} < 1$. Hence, (S_3) implies (h_1) immediately. Now we prove (H_2), that is, $x \neq cAx$ for $x \in \partial D_K$ and $c > 1$. In fact, if not, there exists $x_0 \in \partial D_K$ and $c_0 > 0$ such that $x_0 = c_0 A x_0$. Then we obtain

$$\|x_0\| = c_0\|Ax_0\| \geq c_0\|x_0\| > \|x_0\|,$$

a contradiction. Hence (H_2) holds. The result follows from Corollary 5.5.3. $\square$

Corollary 5.5.5 generalizes Theorems 2.3 and 2.4 in [36] to weakly inward maps. Our method is simpler than that used in Theorem 2.4 in [36].

5.6 Applications

In this section we consider the existence of positive solutions of the boundary value problem of the form

$$x''(t) + f(t, x, x', x'') = 0, x(0) = x(1) = 0. \tag{5.1}$$

We make the following hypotheses.

(C_1) There exists $r > 0$ such that $f : [0,1] \times [0, \frac{r}{8}] \times [-\frac{r}{2}, \frac{r}{2}] \times [-r, 0] \to \mathbb{R}$ is a continuous function.

(C_2) There exists $k \in (0,1)$ such that

$|f(t,u,v,-s_1) - f(t,u,v,-s_2)| \leq k|s_1 - s_2|$ for $t \in [0,1]$, $u \in [0, \frac{r}{8}]$, $v \in [-\frac{r}{2}, \frac{r}{2}]$, and $s_1, s_2 \in [0, r]$.

(C_3) $\quad f(t,u,v,0) \geq 0$ for $t \in [0,1]$, $u \in [0, \frac{r}{8}]$ and $v \in [-\frac{r}{2}, \frac{r}{2}]$.

(C_4) $\quad f(t,u,v,-r) \geq r$ for $t \in [0,1]$, $u \in [0, \frac{r}{8}]$ and $v \in [-\frac{r}{2}, \frac{r}{2}]$.

(C_5) There exists $\rho \in (0, r)$ such that $f(t,u,v,-\rho) \leq \rho$ for $t \in [0,1]$, $u \in [0, \frac{r}{8}]$ and $v \in [-\frac{r}{2}, \frac{r}{2}]$.

Theorem 5.6.1. *Assume that the conditions (C_1)-(C_5) hold. Then the equation* (5.1) *has a solution x satisfying $0 \leq x(t) \leq \dfrac{r}{8}$ for $t \in [0,1]$ and $x(t) \not\equiv 0$.*

Proof. Let $Y = C[0,1]$ be the space of continuous functions on $[0,1]$ with the maximum norm. It is known that there exists a projection scheme $\Gamma = \{Y_n, P_n\}$ associated with the usual Schauder basis (see Example 5.1.1). Let $K = \{y \in Y : y(t) \geq 0\}$ for $t \in [0,1]$. Then K is a cone in Y and $P_n K \subset K_n$ (see Example 5.1.1).

Let $X = \{x \in C^2[0,1] : x(0) = x(1) = 0\}$. Define a map $L : X \to Y$ by $Lx = -x''$. Then L is a linear bounded isometric isomorphism and $L^{-1}y(t) = \int_0^1 k(t,s)y(s)\,ds$ for $t \in [0,1]$, where

$$k(t,s) = \begin{cases} t(1-s) & \text{for } 0 \leq t \leq s \leq 1 \\ s(1-t) & \text{for } 0 \leq s \leq t \leq 1 \end{cases}$$

We define a continuous map $A : \overline{K}_r \to Y$ by

$$Ay(t) = f(t, L^{-1}y, \frac{d}{dt}L^{-1}y, -y),$$

where $K_r = \{y \in K : \|y\| < r\}$. By a simple calculation we obtain that if $y \in \overline{K}_r$, then $0 \leq L^{-1}y(t) \leq \frac{r}{8}$ and $-\frac{r}{2} \leq \frac{d}{dt}L^{-1}y(t) \leq \frac{r}{2}$. Therefore, A is well-defined and the condition (C_1) implies A is continuous.

Define a map $V : \overline{K}_r \times \overline{K}_r \to Y$ by

$$V(x,y) = f(t, L^{-1}y, \frac{d}{dt}L^{-1}y, -x)$$

Note that L^{-1} and $\frac{d}{dt}L^{-1}$ are compact maps. These facts together with the condition (C_2) implies that V satisfies all the conditions of Lemma 1.3.31. Therefore, we have

$$\beta_Y(A(Q)) \leq k\beta_{\overline{K}_r}(Q) \quad \text{for every bounded set } Q \subset \overline{K}_r.$$

It follows from Example 1.3.12 that $\beta_Y(Q) = \beta_{\overline{K}_r}(Q)$ for each bounded set $Q \subset \overline{K}_r$. By Corollary 5.1.13 we see that A is P_γ-compact for every $\gamma \in (k,1)$.

Now, we use Example 2.4.10 to prove A is weakly inward relative to K. Let $y \in \partial K$, that is, $E(y) = \{t \in [0,1] : y(t) = 0\} \neq \emptyset$. Then $(Ay)(t) = f(t, L^{-1}y, \frac{d}{dt}L^{-1}y, 0)$ for every $t \in E(y)$. It follows from (C_3) that

$$(Ay)(t) \geq 0 \quad \text{for every } t \in E(y).$$

It follows from Example 2.4.10 that $Ay \in \overline{I}_K(y)$ and A is weakly inward.

We prove that A satisfies the condition (LS) in Theorem 5.4.7, that is, $y \neq \lambda Ay$ for $y \in \partial K_\rho$ and $\lambda \in (0,1)$. In fact, if not, there exist $y_0 \in \partial K_\rho$ and $\lambda_0 \in (0,1)$ such that $y_0 = \lambda A y_0$. Let $t_0 \in [0,1]$ be such that $y_0(t_0) = \rho$. Then by (C_4) we have

$$\rho = y_0(t_0) = \lambda_0 f(t_0, L^{-1}y(t_0), \frac{d}{dt}L^{-1}y(t_0, -\rho) \leq \lambda_0 \rho < \rho,$$

a contradiction. Finally, we prove that A satisfies the condition (E) in Theorem 5.4.7 with $e(t) \equiv 1$ for $t \in [0,1]$, that is, $y \neq Ay + \beta e$ for $y \in \partial K_r$ and $\beta > 0$. In fact, if not, there exist $y_0 \in \partial K_r$ and $\beta_0 > 0$ such that $y_0 = Ay_0 + \beta_0 e$. Let $t_0 \in [0,1]$ be such that $y_0(s) = \|y_0\| = r$. Then we have

$$r = f(t_0, L^{-1}y(t_0), \frac{d}{dt}L^{-1}y(t_0), r) + \beta_0 e \geq r + \beta_0 e > r,$$

a contradiction. It follows from Theorem 5.4.7 that A has a fixed point $y \in K$ satisfying $\rho \leq \|y\| \leq r$. Let $x = L^{-1}y$. Then it is easy to see that x is a desired solution. □

Remark 5.6.2. Theorem 5.6.1 generalizes Example 1 in [36] in the following ways: (i) the domain of the function f in Theorem 5.6.1 is smaller than that used in Example 1 in [36] so the conditions (C_2), (C_4) and (C_5) are weaker than (C_1)-(C_3) of Example 1 in [36]; and (ii) f in Theorem 5.6.1 may take negative values although it satisfies condition (C_3) while f used in Example 1 in [36] is required to be positive.

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