PRIME IDEALS OF FIXED RINGS

J. Taylor

A Thesis submitted for the degree of Doctor of Philosophy at the University of Glasgow

Department of Mathematics

University of Glasgow

October, 1993

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Professor K.A. Brown, for his enthusiastic instruction and moral support throughout my period of study.

I would also like to acknowledge the Science and Engineering Research Council for its financial support over the last three years. In addition, I thank the Department of Mathematics for its assistance in financing conference visits.

CONTENTS

SUMMARY

Throughout this thesis, S is a ring, G is a finite group of automorphisms of S and R is the fixed ring S^G. We are concerned here with the correlation between properties of R and properties of S.

In Chapter 2, we discuss certain finiteness conditions for the ring R. D.S. Passman has asked, "Is the fixed ring of kH, where k is a field and H is a polycyclic-by-finite group, Noetherian for any finite group G ?" We produce infinitely many examples for which the answer to this question is "yes". The most substantial result in relation to this borrows from the methods in [L-P1] and is:

2.2.8 COROLLARY *Let H_n be the nth Heisenberg group for some $n \in \mathbb{N}$. Let $g \in Aut(H_n)$ be an automorphism of order 2 such that $x_i{}^g = x_i{}^{-1}z^{u(i)}$, $y_i{}^g = y_i{}^{-1}z^{v(i)}$ and $z^g = z$ for some $u(i), v(i) \in \mathbb{Z}$ $(i=1,\dots,n)$. Let k be a field and S the group algebra kH_n. Now, $G = \langle g \rangle$ acts as k-automorphisms on the ring S and S^G is Noetherian.*

The most important results of this thesis are contained in Chapter 3. We develop the Morita prime correspondence of Chapter 1, §2, to produce results relating $Spec_t R := \{p \in Spec R\colon tr(S) \not\subseteq p\}$ to $Spec_f S := \{P \in Spec S\colon \Sigma_{g\in G}\, g \notin \mathcal{J}(P^o{*}G)\}$ where $P^o{*}G$ is an ideal in the skew group ring $S{*}G$. S. Montgomery has proved many of our results in [Mo2] for the special case where $|G|^{-1} \in S$. First we establish the extent to which members of $Spec_f S$ are determined by their intersections with R.

3.1.13 THEOREM *Let S be a ring and G a finite group of automorphisms of S. Suppose $P \in Spec_f S$ and $Q \in Spec S$ with $\mathcal{J}(P \cap R) = \mathcal{J}(Q \cap R)$. Then P and Q are G-conjugate, so that $Q \in Spec_f S$, and $P \cap R = Q \cap R$.*

We proceed to prove the next result which summarises the close connection between $Spec_f S$ and $Spec_t R$.

3.1.21 THEOREM *Let S be a ring and G a finite group of automorphisms of S.*

(i) Given $P \in Spec_f S$, there are a finite number of primes in $Spec_t R$ minimal over $P \cap R$, $\{p_1, p_2, \ldots, p_m\}$ say, with $m \leq |G|$. Also, $(\cap_i p_i)tr(S)$ is nilpotent modulo $P \cap R$.

(ii) Given $p \in Spec_t R$, there exists $P \in Spec_f S$ such that p is minimal over $P \cap R$. Moreover, P is unique up to its G-orbit in $SpecS$.

In Chapter 4, we restrict our attention to the case where S is a group algebra. The following key lemma establishes precisely what the factor ring $R/tr(S)$ is in certain circumstances.

4.1.1 LEMMA *Let U be a ring, M a semigroup and G a subgroup of $AutM$ of prime order, q. Let G act as U-automorphisms on the semigroup ring $S = UM$. Then*

$$R \,/\, tr_G(S) \cong (U/qU).C_M(G).$$

As an application of this result, we establish bounds for $rk(R)$, the prime rank of R, in certain circumstances.

4.2.10 COROLLARY *Let H be finitely generated torsionfree nilpotent group and k a field of characteristic q. Suppose G is a finite group of automorphisms of H such that the Sylow q-subgroup of G, Q, is normal of order q. Then*

$$\hat{h}(H) \leq rk(R) \leq \hat{h}(H) + \hat{h}(C_H(G)).$$

We conjecture that $rk(R) = \hat{h}(H)$. As evidence to support this, Example 4.2.12 gives infinitely many such examples. Example 4.2.12 is also notable for showing that R need not satisfy the saturated chain condition even when S does.

We conclude this thesis in Chapter 5 with some results on localisation in the ring R. Many of these are inspired by the methods of Warfield in [W1]. We find that, with the necessary hypotheses, $Spec_t R$ has the strong second layer condition.

5.2.5 THEOREM *Let S be a Noetherian ring satisfying the strong second layer condition and G be a finite subgroup of $AutS$ such that R is Noetherian and ${}_RS$ and S_R are finitely generated modules. Suppose $p \in Spec_t R$. Then p has SSLC.*

Finally, we give a result which relates the link graph of R to that of S.

5.3.6 THEOREM *Let S be a ring with the SSLC and let G be a finite group of automorphisms of S. Suppose that $R = S^G$ is Noetherian and ${}_RS$ and S_R are finitely generated. Let d be a symmetric dimension function on $\{R, S\}$. If $p_1, p_2 \in Spec_t R$ with p_1 second layer linked to p_2, then there exist primes $Q_1, \ldots, Q_n$ of S with $n \geq 2$, such that Q_1 lies over p_1, Q_n lies over p_2 and such that Q_i is second layer linked to Q_{i+1} for $1 \leq i \leq n-1$.*

INTRODUCTION

This thesis is devoted to the study of fixed rings and, in particular, the prime ideals in fixed rings.

We deal with the following situation: S is an associative ring with an identity element and G is a finite group of ring automorphisms of S. The fixed ring is defined to be

$$R = \{s \in S: s^g = s \text{ for all } g \in G\}.$$

It is this ring R, sometimes denoted by S^G, we study. Of particular interest to us is the correllation between the properties of S and the properties of R. We often make use of the following ideal of R:

$$tr(S) = \{ \Sigma_{g \in G} s^g: s \in S\}.$$

Generally speaking, as we point out throughout the thesis, the relationship between S and R is well understood when $|G|^{-1} \in S$, principally because $tr(S) = R$ in this case. Our main aim is to generalise results which hypothesise that $|G|^{-1} \in S$ to allow for the possibility that $|G|^{-1} \notin S$.

We begin in Chapter 1 by giving some of the well established results on fixed rings. We discuss the Morita context involving R and the skew group ring $S*G$, which results in a prime correspondence between certain subsets of $SpecR$ and $Spec(S*G)$. This yields some basic results which provide the foundation for much that follows in Chapter 3. We also feature the Bergman-Isaacs Theorem as part of a survey on results establishing the existence of fixed points. §4 is devoted to an examination of finiteness conditions such as the finite generation of S as an R-module. We also quote the well known result of Farkas and Snider that, if S is Noetherian and $|G|^{-1} \in S$, then R is Noetherian. Chapter 1 concludes with a brief summary of results on localisation and GK-dimension, both of which we use in Chapter 5.

In Chapter 2, we elaborate on the finiteness results of Chapter 1, §4. In doing so, we attempt to answer the following question asked by D.S. Passman:

2B QUESTION *Suppose H is a polycyclic-by-finite group and G is a finite group of automorphisms of H. Let k be a field and S be the group algebra kH. Is the fixed ring S^G Noetherian ?*

It is rather easy to deduce from a classical result of E. Noether, stated as Lemma 1.4.4, that if H is Abelian-by-finite, the answer to this question is "yes", so interest centres on the more general polycyclic-by-finite groups. In particular, the question is open for H nilpotent.

With Question 2B in mind, we prove the following corollary on the finite generation of S as an R-module.

2.1.5 COROLLARY *Let S be semiprime with no non-zero nilpotent elements and G be a finite group of automorphisms of S. If $R := S^G$ is left Noetherian, then S is left Noetherian and is a finitely generated R-module.*

As an application of Corollary 2.1.5, we show in Theorem 2.1.7 that, in order to answer Question 2B, it is sufficient to consider only the cases where H is poly-C_∞. Even with this reduction, we are unable to answer Question 2B. What we are able to do is provide infinitely many groups H, nilpotent-by-finite but not abelian-by-finite, each with infinitely many distinct finite automorphism groups G, such that $(kH)^G$ is Noetherian for all fields k. The most substantial of these results is:

2.2.8 COROLLARY *Let H_n be the nth Heisenberg group for some $n \in \mathbb{N}$. Let $g \in Aut(H_n)$ be an automorphism of order 2 such that $x_i{}^g = x_i{}^{-1}z^{u(i)}$,*

$y_i^g = y_i^{-1}z^{v(i)}$ *and* $z^g = z$ *for some* $u(i), v(i) \in \mathbb{Z}$ $(i=1,\ldots,n)$. *Let* k *be a field and* S *the group algebra* kH_n. *Now,* $G = \langle g \rangle$ *acts as* k*-automorphisms on the ring* S *and* S^G *is Noetherian.*

Corollary 2.2.8 is inspired by results of M. Lorenz and D.S. Passman in [L-P1] and T. Hodges and J. Osterburg in [H-O]. [L-P1] contains Theorem 2.2.3, a result that is very similar in substance to Corollary 2.2.8.

Chapter 3 embodies the main results of this thesis. Many of the results here are generalisations of Montgomery's work in [Mo2] which required as a hypothesis that $|G|^{-1} \in S$. Developing the Morita prime correspondence of Chapter 1, §2, we prove, the following result. Here and below, $Spec_f S := \{P \in SpecS: \Sigma_{g\in G} g \notin \surd(P^o * G)\}$ and $Spec_t R := \{p \in SpecR: tr(S) \not\subseteq p\}$ are the open dense subsets of the prime spectra resulting from the Morita correspondence.

3.1.13 THEOREM *Let* S *be a ring and* G *a finite group of automorphisms of* S. *Suppose* $P \in Spec_f S$ *and* $Q \in SpecS$ *with* $\surd(P \cap R) = \surd(Q \cap R)$. *Then* P *and* Q *are* G*-conjugate, so that* $Q \in Spec_f S$, *and* $P \cap R = Q \cap R$.

The main result in Chapter 3, §1 shows that there is a nice relationship between primes in S and primes in R.

3.1.21 THEOREM *Let* S *be a ring and* G *a finite group of automorphisms of* S.

(i) Given $P \in Spec_f S$, *there are a finite number of primes in* $Spec_t R$ *minimal over* $P \cap R$, $\{p_1, p_2, \ldots, p_m\}$ *say, with* $m \leq |G|$. *Also,* $(\cap_i p_i)tr(S)$ *is nilpotent modulo* $P \cap R$.

(ii) Given $p \in Spec_t R$, *there exists* $P \in Spec_f S$ *such that* p *is minimal over* $P \cap R$. *Moreover,* P *is unique up to its* G*-orbit in* $SpecS$.

As we have already seen noted, R is well understood when $|G|^{-1} \in S$. In Chapter 3, §2, we look at the other extreme, namely when S has prime characteristic q and $|G| = q^a$ for some $a \in \mathbb{N}$. Proposition 1.2.12 is essential in providing us with special cases of results in §1. We obtain:

3.2.13 THEOREM *Let S be a ring of characteristic q and G a subgroup of Aut S of order q^a. Then*

(i) Given $P \in Spec_I S$, there exists $p \in Spec_t R$ such that p is the unique prime minimal over $P \cap R$ not containing the trace.

(ii) Given $p \in Spec_t R$, then there exists $P \in Spec_I S$ such that p is minimal over $P \cap R$. Moreover P is unique up to its G-orbit.

In the last section of Chapter 3, we exploit the relationships we have established in the first two sections. We derive a number of applications. For example, we have:

3.3.8 LEMMA *Let S be a ring and G a finite group of automorphisms of S. Suppose $p,q \in Spec_t R$ both lie under $P \in SpecS$, then $ht(p) = ht(q) = ht(P)$.*

Recall that a ring is said to be *Jacobson* if all its prime factors are semiprimitive. We prove:

3.3.23 THEOREM *Let S be a ring and G a finite group of ring automorphisms of S. If S and $R/tr_G(S)$ are both Jacobson rings, R is also Jacobson.*

In Chapter 4 we study the prime ideals of R where S is a group ring. As indicated by Theorem 3.3.23, the factor $R/tr(S)$ plays an important rôle in the study of R. Our key lemma shows that, in certain circumstances, we know exactly what the ring $R/tr(S)$ is. We prove:

4.1.1 LEMMA. *Let U be a ring, M a semigroup and G a subgroup of $AutM$ of prime order, q. Let G act as U-automorphisms on the semigroup ring $S = UM$. Then*

$$R \,/\, tr_G(S) \cong (U/qU).C_M(G).$$

We point out in Corollary 4.1.2 that, under the hypotheses of Lemma 4.1.1, when M is a polcyclic-by-finite group, the factor $R/tr(S)$ is also the group ring of a polycyclic-by-finite group. Our main consequence of Corollary 4.1.2 is:

4.1.15 THEOREM *Let K be a commutative Jacobson ring all of whose field factors are absolute fields. Let H be a polycyclic-by-finite group and S the group ring KH. Suppose G is a group of automorphisms of H of prime order q so that G acts as K-automorphisms on S. Set $R = S^G$. Then*

(i) every maximal ideal M of R intersects K in a maximal ideal of K;

(ii) every primitive ideal of R is maximal;

(iii) for M above, R/M has finite dimension over the absolute field $K/(M \cap K)$.

In particular, every irreducible R-module is finite dimensional over a field factor of K.

In Theorem 4.1.4, we combine Corollary 4.1.2 with Theorem 3.3.23 to show that R is a Jacobson ring when the necessary hypotheses are satisfied.

In Chapter 4, §2, we attempt to answer two questions, the first of these concerns $rk(R)$, the prime rank of R.

QUESTION 4B Suppose H is a nilpotent group, k is a field and S is the group algebra kH. Let G act as k-automorphisms on S and set $R = S^G$. Does $rk(R) = rk(S)$?

Our best result on bounds for $rk(R)$ for is given below:

4.2.10 COROLLARY *Let H be finitely generated torsionfree nilpotent group and k a field of characteristic q. Suppose G is a finite group of automorphisms of H such that the Sylow q-subgroup of G, Q, is normal of order q. Then*

$$\hat{h}(H) \leq rk(R) \leq \hat{h}(H) + \hat{h}(C_H(G)).$$

We do not find any examples for which the answer to Question 4B is "no". On the contrary, Example 4.2.12 gives infinitely many examples for which the answer to Question 4B is "yes". Example 4.2.12 is also notable for the bearing it has on the next question.

QUESTION 4C Suppose H is a nilpotent group and k is a field. Let S denote the group algebra kH. Suppose G acts as k-automorphisms on S. Does S^G satisfy the saturated chain condition?

In Example 4.2.12 we give an infinite number of examples which answer Question 4C negatively.

Chapter 5 is joint work with K.A. Brown. In this final chapter, we investigate some localisations of the ring R. §1 just gives some elementary results concerning the inversion of central regular elements in R. The remaining two sections are devoted to determining which semiprime ideals of R we may localise at. In §2 we find that, with the necessary hypotheses, certain members of $SpecR$ satisfy the strong second layer condition of Chapter 1, §5. We develop ideas of Warfield to prove:

5.2.5 THEOREM *Let S be a Noetherian ring satisfying the strong second layer condition and G be a finite subgroup of AutS such that R is Noetherian and ${}_RS$ and S_R are finitely generated modules. Suppose $p \in Spec_tR$. Then p has SSLC.*

Despite this, in Example 5.2.7, we find a ring, R, the fixed ring of a group algebra of the second Heisenberg group, for which $tr(S)$ is prime but does not have even the second layer condition. Beyond this, we are unable to say anything further about SSLC in R.

In Definition 1.5.2, we explain what we mean by the link graph of a ring. It is the link graph of R in relation to that of S that we study in §3. Again, building on Warfield's ideas, we have:

5.3.6 THEOREM *Let S be a ring with the SSLC and let G be a finite group of automorphisms of S. Suppose that $R = S^G$ is Noetherian and ${}_RS$ and S_R are finitely generated. Let d be a symmetric dimension function on (R, S). If $p_1, p_2 \in Spec_tR$ with p_1 second layer linked to p_2, then there exist primes $Q_1, \ldots, Q_n$ of S with $n \geq 2$, such that Q_1 lies over p_1, Q_n lies over p_2 and such that Q_i is second layer linked to Q_{i+1} for $1 \leq i \leq n-1$.*

The above result is significant in that it allows us to understand links inside $Spec_tR$. Any links from $Spec_tR$ to $SpecR \backslash Spec_tR$ remain unknown. However, our final result is a very nice one which obviously has strong implications for the link graph. It shows that a certain clique is a finite subset of $Spec_tR$.

5.3.11 THEOREM *Let S be a Noetherian ring with finite GK-dimension, G a subgroup of AutS and $R = S^G$. Suppose $P \in SpecS$ such that $C_S(P^o)$ is an Ore set in S, $R/(P \cap R)$ is Noetherian and S/P is finitely generated on both* *sides as $R/(P \cap R)$-modules. Suppose $p_1, \ldots, p_n$ are the primes of R minimal over $P \cap R$. Suppose $p_i \in \mathrm{Spec}_t R$ $(i=1,\ldots,n)$ or, equivalently that $tr(S) \cap C_S(P^o) \neq \emptyset$. Then $C_S(P^o) \cap R = C_R(P \cap R) =: X$, say and X is an Ore set in S. In particular, $\cap_i p_i$ is a localisable semiprime ideal of R.*

Throughout the thesis, we give original references wherever this is possible. Otherwise, we use the books [Mo1], [P1], [P2], [G-W] and [McC-R] for background reference.

Unless otherwise stated, the results contained herein are original results obtained under the supervision of Professor K.A. Brown.

CHAPTER 1

BASIC PROPERTIES OF FIXED RINGS AND SOME TECHNICALITIES

We deal with some basic questions concerning the following scenario. Suppose S is an associative ring with an identity element, denoted by 1, and G is a finite group acting as ring automorphisms on S. For $g \in G$ and $s \in S$, we denote the action of g on s by s^g. Define R to be the set

$$\{r \in S: r^g = r \text{ for all } g \in G\}.$$

Trivially, R is a subring of S. It is called either the *fixed ring* or the *ring of invariants* and is sometimes represented by S^G.

As one would expect, there is a close relationship between S and S^G. We will show in later chapters that certain properties of S are inherited by S^G. There are however, other properties for which the relationship between S and S^G is not so clear.

We begin in §1 by introducing some of the standard terminology used to describe aspects of the theory of group actions on rings. In §2, we review the main properties of the Morita context which relates the skew group ring $S*G$ to the fixed ring S^G. §3 features the Bergman-Isaacs Theorem and discusses other known results on the existence of fixed points. The next section is devoted to established results on finiteness conditions such as the inheritance of the Noetherian condition by the fixed ring.

The remaining two sections in this chapter review many of the properties required in Chapter 5. §5 recalls some of the results on the strong second layer condition and links needed for localising in the ring S^G. Finally, we summarise some of the properties of GK-dimension in §1.6.

§1.1 Preliminaries

To begin with we establish some basic terminology.

1.1.1 DEFINITIONS For $X \subseteq S$ and $g \in G$, define $X^g := \{x^g : x \in X\}$. A subset Y of S is said to be *G-invariant* or, alternatively, *G-stable* if $Y^g = Y$ for all $g \in G$. For $X \subseteq S$, we let X^o denote $\cap_{g \in G} X^g$, the largest G-invariant subset of S containing X. With the above definition, we see that $R := S^G$ is just the set of G-invariant elements of S.

It is possible to manufacture members of R using *the trace map*. This is defined to be $tr: S \to R$ such that $tr(s) = \Sigma_{g \in G}\, s^g$. For $s \in S$ and $h \in G$,

$$(tr(s))^h = (\Sigma_{g \in G}\, s^g)^h = \Sigma_{g \in G}(s^g)^h = \Sigma_{g \in G}\, s^{gh} = \Sigma_{g \in G}\, s^g = tr(s)$$

since $Gh = G$. Thus $tr(S) \subseteq R$. Moreover, tr is easily seen to be an $(R\text{-}R)$-bimodule homomorphism. Consider, for example, the left action: for $r \in R$ and $s \in S$,

$$tr(rs) = \Sigma_{g \in G}(rs)^g = \Sigma_{g \in G}\, r^g s^g = \Sigma_{g \in G}\, rs^g = r\Sigma_{g \in G}\, s^g = rtr(s).$$

Thus, $tr(S)$ is a (two-sided) ideal of R.

While, in general $tr(S)$ may be a proper ideal of R, when $|G|^{-1} \in S$, $tr(S) = R$. We have $tr(|G|^{-1}) = \Sigma_{g \in G}\, (|G|^{-1})^g = \Sigma_{g \in G}\, |G|^{-1} = |G|.|G|^{-1} = 1$. Since $tr(S)$ is an ideal of R, we have that $tr(S) = R$.

When there is a risk of confusion regarding the group acting, the trace map is denoted by tr_G.

Suppose that I is a G-invariant ideal of S. We may define an induced action of G on the ring S/I in the obvious way: for $s \in S$ and $g \in G$, let $(s + I)^g = s^g + I$. The fact that I is G-invariant ensures this action is well-defined.

We denote the group of all ring automorphisms on S by $AutS$. An automorphism $g \in AutS$ is said to be *inner* if there exists a unit $u \in S$ for which $s^g = u^{-1}su$ for all $s \in S$. Otherwise g is said to be *outer*. If all the members of G are inner then G itself is described as *inner*. Similarly, if all the non-identity elements of G are outer, then G is said to be *outer*. For G, a subgroup of $AutS$, it's easy to see that the set of inner automorphisms in G form a normal subgroup of G.

Suppose N is a normal subgroup of G. Certainly, N acts on S. Moreover, the factor group G/N acts on the ring S^N with the property that $S^G = (S^N)^{G/N}$. In particular, we may arrange that the action of G on S is faithful. Thus, if we take $N = \{g \in G: s^g = s \text{ for all } s \in S\}$, N is easily seen to be a normal subgroup of G, and we may just consider the action of G/N on S.

Inextricably related to R is the *skew group ring* $S*G$ which we will denote by T. This is defined to be a free right S-module with basis $\{g: g \in G\}$. Multiplication in T is defined as follows:

$$(gs_1).(hs_2) = gh(s_1)^h s_2 \text{ for } s_1,\ s_2 \in S,\ g,\ h \in G.$$

For a G-invariant ideal A of S, we define an ideal $A*G$ of $S*G$ to be the set of elements in $S*G$ for which the coefficients lie in A. In this case, $T/(A*G) \cong (S/A)*G$ is also a skew group ring. Now, $S*G$ contains all the ingredients required in the formulation of the fixed ring S^G, so it is not surprising that there is a close connection between $S*G$ and S^G. This link is manifested in the Morita context. There is one element of T that, as we shall see, plays an important rôle in the Morita context. This element is the sum of the basis elements of T, $\Sigma_{g \in G}\, g$, and is represented by f. Before we give the details of the Morita context, we first give a generalisation of the skew group ring, namely the crossed product.

1.1.2 DEFINITION Let S be a ring and G a finite group of automorphisms of S. Let $\alpha: G \times G \to S$ be a map with "nice" properties. The crossed product $D = (S,G,\alpha)$ is defined to be the free right S-module with basis $\{\bar{g}: g \in G\}$ with multiplication given by the two relationships:

$$r\bar{g} = \bar{g}r^g \quad \text{and} \quad \bar{g}.\bar{h} = \alpha(g,h)\overline{gh}.$$

The so-called "nice" properties of α are those required to make the multiplication in D associative; see, for example, [P2, pages 2-3].

Related to the concepts of a prime and semiprime ideal, we have the following definition.

1.1.3 DEFINITION Suppose we have a ring S with a finite group G acting on it. An ideal I of S is said to be *G-prime* if for two G-invariant ideals A and B of S, $AB \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$. The ring S is said to be G-prime if 0 is a G-prime ideal.

The next lemma, taken from [P2, Lemma 14.2], illustrates the relationship between this new definition and those of prime and semiprime ideals.

1.1.4 LEMMA *Let S be a ring and G a finite group of automorphisms of S. Let I be a G-invariant ideal of S. Then there is the following hierarchy for I:*

$$\text{prime} \Rightarrow G\text{-prime} \Rightarrow \text{semiprime}.$$

PROOF The first implication is obvious. The second is more substantial. By passing to the factor ring S/I, we may assume that $I = 0$ so that S is a G-prime ring. By Zorn's Lemma, we can find an ideal Q of S maximal with respect to $\cap_{x \in G} Q^x = 0$. We claim that Q is prime. Suppose this is not the case so that there exist ideals A and B of S with $A \supset Q$ and $B \supset Q$ but $AB \subseteq Q$. Then $(\cap_{x \in G} A^x)(\cap_{x \in G} B^x) \subseteq \cap_{x \in G} Q^x = 0$. Since S is G-prime, either $\cap_{x \in G} A^x = 0$ or $\cap_{x \in G} B^x = 0$. If $\cap_{x \in G} A^x = 0$, the maximality of Q yields $A = Q$. This contradiction shows that Q is a prime ideal. We have therefore shown that S is semiprime.

§1.2 The Morita Context

Morita Theory provides us with a powerful tool for investigating the relationship between certain pairs of rings. Most of the results in this section are well known and are due to S.A. Amitsur, W.K. Nicholson and J.F. Watters. However, the detailed calculation of the prime correspondence for the skew group ring / fixed ring pair is original. Much of the background in this section comes from [McC-R, Chapter3].

The definition of Morita context we give here is more restrictive than the general definition as in [McC-R, 1.1.6] but it will be sufficient for our purposes.

1.2.1 DEFINITION Let U be any ring and M a right U-module. We define a *Morita context* to be the *2x2* array

$$C = \begin{bmatrix} U & M^* \\ M_U & V \end{bmatrix}$$

where $M^* = Hom(M_U, U)$ and $V = End(M_U)$. With the following maps, we have that the above array is a matrix ring:

(i) $M^* \times M \to U$ by $(\alpha, m) \mapsto \alpha(m)$;

(ii) $V \times M \to M$ by $(\varphi, m) \mapsto \varphi(m)$;

(iii) $U \times M^* \to M^*$ by $(u, \alpha) \mapsto \lambda_u \circ \alpha$

where $\lambda_u \in End(U_U)$ denotes left multiplication by u;

(iv) $M \times M^* \to V$ by $(m, \alpha) \mapsto \lambda_m \circ \alpha$

where $\lambda_m \in Hom(U_U, M_U)$ denotes left multiplication by m.

(v) $M^* \times V \to M^*$ by $(\alpha, \varphi) \mapsto \alpha \circ \varphi$.

It should also be noted that the *dual* of C,

$$C^* = \begin{bmatrix} V & M_U \\ M^* & U \end{bmatrix}$$

is also a Morita context because M^* is a right V-module,

$(M^*{}_V)^* = Hom(M^*{}_V, V) \cong M_U$ and $End(M^*{}_V) \cong U$.

In these circumstances, we get a bijection between subsets of *SpecU* and *SpecV*. We reproduce [McC-R, Theorem 3.6.2].

1.2.2 THEOREM *Let*

$$C = \begin{bmatrix} U & M^* \\ M_U & V \end{bmatrix}$$

be a Morita context where U is a ring, M is a right U-module, $M^* = Hom(M_U, U)$ and $V = End(M_U)$. Then there is a bijection between the sets of prime ideals $\{P \in SpecU: P \not\supseteq M^*M\}$ and $\{P' \in SpecV: P' \not\supseteq MM^*\}$ given by

$$P \mapsto \{v \in V: M^*vM \subseteq P\}.$$

PROOF [McC-R, Theorem 3.6.2].

Suppose $P \in SpecU$ with $P \not\supseteq M^*M$ and that $P' \in SpecV$ is the corresponding prime. Then we say that U/P and V/P' are *context equivalent*. Since the dual C^* is also a Morita context, we see that context equivalence is symmetric.

Context equivalence preserves a number of properties. For example, as the next proposition shows, it preserves primitivity.

First, we give a definition.

1.2.3 DEFINITION Let C be the Morita context described in 1.2.1. We say C is a *prime context* if

(i) U is a prime ring;

(ii) $M^*m \neq 0$ for all $0 \neq m \in M$;

(iii) $M^*vM \neq 0$ for all $0 \neq v \in V$.

Proposition 1.2.4 appears as [McC-R, Proposition 3.6.5].

1.2.4 PROPOSITION

(i) Prime rings U and V are context equivalent if and only if they belong to a prime context.

(ii) Context equivalence preserves primitivity.

PROOF (i) Suppose that R and S are context equivalent. Then, by definition, there exist rings U and V with $P \in SpecU$ and $P' \in SpecV$ such that $U/P \cong R$ and $V/P' \cong S$ with

$$C = \begin{bmatrix} U & M^* \\ M_U & V \end{bmatrix}$$

a Morita context such that P and P' are corresponding prime ideals. Then it is easily verified that

$$C' = \begin{bmatrix} U/P & M^*/N' \\ M_U/N & V/P' \end{bmatrix}$$

where $N' = \{\varphi \in M^*: \varphi(M) \subseteq P\}$ and $N = \{m \in M: M^*m \subseteq P\}$ is a prime context.

Conversely, suppose that the prime rings U and V belong to a prime context. Then, taking $P = 0$ and $P' = 0$, we see that U and V are context equivalent.

(ii) Suppose now that U is a primitive ring in the prime context

$$C = \begin{bmatrix} U & M^* \\ M_U & V \end{bmatrix}.$$

Let $X = xU$ be a faithful simple U-module and let $N' = \{\varphi \in M^*: x\varphi(M) = 0\}$. We claim that M^*/N' is a faithful simple V-module. First we establish that this module is faithful. Suppose we have $v \in V$ with $M^*v \subseteq N'$. Then $x.M^*vM = 0$. So we have $0 = X.M^*vM$. Since X is a faithful U-module, we must have $M^*vM = 0$. As C is a prime context, we must have $v = 0$, proving M^*/N' is faithful.

Finally, choose any $\varphi \in M^*\backslash N'$. By definition of N', $x\varphi(M) \neq 0$. The fact that X is simple then yields $x\varphi.M = X$. Hence $x = x\varphi.m$ for some $m \in M$. Let $\psi \in M^*$. Then $x.(\psi - \varphi.m.\psi).M = 0$ and so $\psi - \varphi.m.\psi \in N'$. Thus, $\varphi V + N' = M^*$, proving that M^*/N' is simple.

We have established that V is also a primitive ring.

While, in a Morita context, we have this relationship between the two rings, we get stronger results still if two further conditions are fulfilled.

1.2.5 DEFINITION Consider the Morita context

$$C = \begin{bmatrix} U & M^* \\ M_U & V \end{bmatrix}$$

where $M^* = Hom(M_U, U)$ and $V = End(M_U)$. If $MM^* = V$ and $M^*M = U$, we say that U and V are *Morita equivalent*.

In these circumstances, we get a much closer correlation between properties of U and V.

1.2.6 PROPOSITION *The following properties are preserved by Morita equivalence:*

(i) being Artinian;

(ii) being Noetherian;

(iii) being prime;

(iv) being semiprime;

(v) being semiprime right Goldie.

PROOF [McC-R, Propositon 3.5.10].

At this stage we apply the Morita Theory to our particular setting of fixed rings. Now, T is the first ring in our context and we use S as our T-module. In order to view S as a right T-module, we define the following action:

$$s_1.(gs_2) = s_1{}^g s_2$$

where $s_1, s_2 \in S$ and $g \in G$. In fact, we may also regard S as a left T-module using the T-module action defined below

$$(gs_2).s_1 = (s_2s_1)^{g^{-1}}$$

where $s_1, s_2 \in S$ and $g \in G$.

Thus, from 1.2.1, we have the following Morita Context:

$$\begin{bmatrix} T & Hom(S_T, T) \\ S_T & End(S_T) \end{bmatrix}.$$

The next proposition from [McC-R, Proposition 7.8.5] gives us a more concrete view of the above Morita context. Recall from 1.1.1 that $f = \Sigma_{g \in G}\, g \in T$.

1.2.7 PROPOSITION *Let S be any ring, let G be a finite group of automorphisms of S and let T be the skew group ring $S*G$. Then*

(i) $S_T \cong fS$, as right T-modules.

(ii) $Hom\ (\ S_T,\ T\ ) \cong Sf$, as left T-modules, where we identify the element sf ($s \in S$) with left multiplication by sf.

(iii) $End(S_T) \cong S^G := R$, as rings, where S^G acts as left multiplication on S_T.

PROOF For (i), note first that fS is a right ideal of T. For, if $fs_1 \in fS$ and $gs_2 \in T$ where $s_1, s_2 \in S$, $g \in G$, we have $fs_1.gs_2 = fg(s_1)^g s_2 = f(s_1)^g s_2 \in fS$. Now observe that the map $\psi\colon S_T \longrightarrow fS$ such that $\psi(s) = fs$ is an isomorphism of T-modules.

Now we prove (ii). We show that the map $\lambda\colon {}_TS \longrightarrow Hom(S_T, T)$ such that $\lambda(s) = \lambda_s$ where $\lambda_s(x) = sfx$ for all $x \in S$ is an isomorphism. It is clear that λ is an injective T-module map and so we need only show that it is surjective. Let $\alpha \in Hom(\ S_T,\ T\ )$. Then $\alpha(1) = \Sigma s_g g$ for some $s_g \in S$ ($g \in G$). Let $h \in G$. Since $1.h^{-1} = 1$, $\Sigma s_g g = \alpha(1) = \alpha(1.h^{-1}) = \alpha(1)h^{-1} = (\Sigma s_g g)h^{-1} = \Sigma s_g(gh^{-1})$. So we must have that $s_h = s_1$. Since h was arbitrary, $s_g = s_1$ for all $g \in G$. Thus, $\alpha(1) = s_1 f = \lambda_{s_1}(1)$. This proves λ is an isomorphism. The left-handed version of (i) completes the proof of (ii).

Now, for (iii), let $\varphi: S^G \to End(S_T)$ such that $\varphi(r) = \varphi_r$ where $\varphi_r(s) = rs$ for all $s \in S$. As before, the only complication is in showing that φ is onto. Let $\pi \in End(S_T)$. So we have that $\pi(1) = s$ for some $s \in S$. Now, for all $t \in S$, $\pi(t) = \pi(1.t) = \pi(1).t = st$ and so the map π is just left multiplication by s. Let $h \in G$. Then $\pi(1.h) = \pi(1).h = s.h = s^h$ but $s = \pi(1) = \pi(1^h) = \pi(1.h)$. Comparing these expressions we find that s is fixed under the action of G and so $s \in S^G$. This proves that φ is onto.

Using the isomorphisms of Proposition 1.2.7, the Morita context of 1.2.1 relating to T becomes:

$$\begin{bmatrix} T & Sf \\ fS & R \end{bmatrix}$$

with multiplications within the matrix ring (i) to (v) becoming:

(i) $Sf \times fS \to T$ via $(s_1f, fs_2) \to s_1fs_2$ because the pair (uf, fv) is identified with $(\lambda_{uf}, v) \in Hom(S_T, T) \times S_T$ which is mapped to $\lambda_{uf}(v) = ufv \in T$.

(ii) $R \times fS \to fS$ via $(r, fv) \to frv$ because the pair (r, fv) is identified with the pair $(\lambda_r, fv) \in End(S_T) \times S_T$ which is mapped to $\lambda_r(fv) = rfv = frv$.

(iii) $T \times Sf \to Sf$ via $(t, uf) \to (t.u)f$ because the pair (t, uf) is associated with the pair $(t, \lambda_{uf}) \in T \times Hom(S_T, T)$ which is mapped to $t.\lambda_{uf}$ and $t.\lambda_{uf}(1) = t(uf) = (t.u)f$ and so $t.\lambda_{uf} = \lambda_{(t.u)f}$.

(iv) $fS \times Sf \to R$ via $(fs_1, s_2f) \to tr(s_1s_2)$ because the pair (fu, vf) is identified with the pair $(u, \lambda_{vf}) \in S_T \times Hom(S_T, T)$ which is mapped to $u.\lambda_{vf}(-) \in Hom(S_T)$ and $u.\lambda_{vf}(1) = u.vf = (uv).f = \Sigma_{g \in G}(uv)^g = tr(uv)$. Thus, $u.\lambda_{vf} = \lambda_{tr(uv)}$.

(v) $Sf \times R \to Sf$ via $(uf, r) \to urf$ because the pair (uf, r) is identified with $(\lambda_{uf}, \lambda_r) \in Hom(S_T, T) \times End(S_T)$ which is mapped to $\lambda_{uf} \circ \lambda_r = \lambda_{ufr} = \lambda_{urf}$ which is in turn associated with $urf \in Sf$.

In addition, we have:

(vi) $fS \times T \to fS$ via $(fu, t) \to f(u.t)$ since (fu, t) is identified with $(u, t) \in S_T \times T$ which is mapped to $u.t$.

Continuing with this translation of the Morita context, we find an explicit statement of the prime correspondence of Theorem 1.2.2. We define the relevant subsets of $SpecR$ and $SpecT$ first.

1.2.8 DEFINITION Let S be a ring and G a finite group of automorphisms of S. Then we define $Spec_tR$ to be $\{p \in SpecR: tr(S) \not\subseteq p\}$ and $Spec_fT$ to be $\{\hat{P} \in SpecT: f \notin \hat{P}\}$.

1.2.9 THEOREM *Let S be a ring and G a finite subgroup of $AutS$. Let R denote the ring S^G and T denote the skew group ring $S*G$. Then there exists a bijection $\beta: Spec_fT \to Spec_tR$ given by*

$$\beta(\hat{P}) = \{r \in R: rf \in \hat{P}\}.$$

The inverse of this bijection is

$$\beta^{-1}(p) = \{t \in T: tr(S.t.S) \subseteq p\}$$

where the dots denotes the T-module action on S.

PROOF We show that β described above is the Morita correspondence of Theorem 1.2.2. We have that

$$C = \begin{bmatrix} T & Sf \\ fS & R \end{bmatrix}$$

is a Morita context with the maps exhibited prior to the statement of this theorem. Applying Theorem 1.2.2, we get a bijection between $Spec_fT$ and $Spec_tR$ given by $\hat{P} \to \{ r \in R : Sf.r.fS \subseteq \hat{P} \}$. Again considering the above map (v), we see that sf acts as left multiplication by sf and so $(Sf.r).fS = (Srf).fS$. We are then left with the map (i) above which gives that $Srf.fS = SrfS$ and hence this bijection is indeed the map β.

We now show that the inverse of β is $\beta^{-1}: Spec_tR \to Spec_fT$ where $\beta^{-1}(p) = \{ t \in T: tr((S.t)S) \subseteq p \}$. It is routine to check that this concurs with the definition of β^{-1} given in the statement of the theorem.

Recall from 1.2.1 that

$$C^* = \begin{bmatrix} R & fS \\ Sf & T \end{bmatrix}$$

is also a Morita context. According to Theorem 1.2.2, the Morita formulation of the bijection $\varphi\colon Spec_t R \to Spec_f T$ is $\varphi(p) = \{ t \in T\colon fS.t.Sf \subseteq p \}$. Map (vi) shows that the first dot signifies the T-module action on fS, giving $fS.t = f(S.t)$. Then, we have map (iv) to give $f(S.t).Sf = tr((S.t)S)$. It remains to show that φ is actually the inverse of β. Let $\hat{P} \in Spec_f T$ and $p = \beta(\hat{P}) \in Spec_t R$. We claim that $\varphi(p) = \hat{P}$. For $r \in R$,

$$\begin{aligned} fr \in \varphi(p) &\Leftrightarrow tr((S.fr).S) \subseteq p && \text{by definition of } \varphi \\ &\Leftrightarrow tr((tr(S)r.S) \subseteq p && \text{since } S.f = tr(S) \\ &\Leftrightarrow tr(S)rtr(S) \subseteq p && \\ &\Leftrightarrow r \in p && \text{since } tr(S) \not\subseteq p. \end{aligned}$$

We see that $\beta(\varphi(p)) = p = \beta(\hat{P})$ and, since β is a bijection, $\beta^{-1}(p) = \hat{P} = \varphi(p)$.

The map β can be used to exploit information about the ring $S*G$ and relate it to S^G. The skew group ring $S*G$ is generally better understood than S^G. We give some results about the prime ideals in T.

The first lemma, combined from [P2, Lemmas 14.1(i) and 14.2(i)], gives a connection between primes in S and primes in T.

1.2.10 LEMMA *(i) Let $\hat{P} \in SpecT$. Then $\hat{P} \cap S$ is a G-prime ideal of S.*

(ii) An ideal P of S is G-prime if and only if $P = \cap_{g \in G} Q^g$ for some $Q \in SpecS$.

PROOF For (i), let $\hat{P} \in SpecT$. We show that $\hat{P} \cap S$ is a G-prime ideal of S. Suppose A and B are G-stable ideals of S with $AB \subseteq \hat{P} \cap S$. Then $(A*G)(B*G) \subseteq (AB)*G \subseteq (\hat{P}\cap S)T \subseteq \hat{P}$. Since $\hat{P}$ is prime, we have either $A*G \subseteq \hat{P}$ or $B*G \subseteq \hat{P}$. Intersecting to S, these inclusions become $A \subseteq \hat{P} \cap S$ or

$B \subseteq \hat{P} \cap S$, proving the first part.

We prove (ii) now. Suppose P is a G-prime ideal of S. Choose Q maximal in S such that $\cap_{x\in G} Q^x = P$. We claim that Q is prime. To this end, suppose C,D are ideals of S with $C,D \supseteq Q$ and $CD \subseteq Q$. Then $(\cap C^x)(\cap D^x) \subseteq \cap Q^x = \hat{P} \cap S$. Since $\hat{P} \cap S$ is G-prime, we have $\cap C^x \subseteq \hat{P} \cap S$ or $\cap D^x \subseteq \hat{P} \cap S$. The maximality of Q yields $C = Q$ or $D = Q$, proving the lemma. The converse to this direction is easily seen to be true.

M.Lorenz and D.S.Passman have proved the following theorem in [L-P2]. Their paper is fundamental in describing the relationship between primes in S and primes in $S*G$.

1.2.11 THEOREM *Let S be a ring and G be a finite group of automorphisms acting on S. Denote the skew group ring $S*G$ by T. Suppose S is a G-prime ring. Then*

(i) $\hat{P} \in \mathrm{Spec}T$ is minimal if and only if $\hat{P} \cap S = 0$.

(ii) There are finitely many minimal primes of T, say $\hat{P}_1,\dots,\hat{P}_n$, and in fact $n \leq |G|$.

(iii) $N = \hat{P}_1 \cap \dots \cap \hat{P}_n$ is the unique maximal nilpotent ideal of T and $N^{|G|} = 0$.

(iv) If Q is a minimal prime of S, then $\{Q^x : x \in G\}$ is the set of all minimal primes of S and $\cap Q^x = 0$.

PROOF [P2, Theorem 16.2].

As a non-trivial consequence of Theorem 1.2.11, Passman and Lorenz give the next proposition.

1.2.12 PROPOSITION *Let S be a ring and G a finite group of automorphisms of S such that S is a G-prime ring. Suppose S has prime characteristic q and* *that G is a q-group. Then $S*G$ has a unique minimal prime ideal $\hat{P}$ which is necessarily nilpotent.*

PROOF [P2, Proposition 16.4].

In Chapter 3, we look at the properties of the map β defined in 1.2.9 in greater detail.

§1.3 The Existence of Fixed Points

Here we consider in what circumstances a G-invariant subring, X, (not necessarily with an identity element) of S contains a non-zero member of S^G. There are a number of partial results which answer this question to some extent. The most celebrated of these is the Bergman-Isaacs Theorem which we state as Theorem 1.3.2.

First we show that it is not always the case that such a non-zero subring X contains a non-zero fixed element. The following example is due to G.Bergman and may be found in [M1, Example 1.1].

EXAMPLE 1.3.1 *There exists a ring S and a finite group of inner automorphisms G such that S has a non-zero ideal which has no non-zero fixed element.*

Let F be a field of characteristic $p \neq 0$ with an element $\omega \neq 0,1$ of finite multiplicative order, n, say.

Let S be $M_2(F\{x, y\})$, the ring of $2x2$ matrices over the free algebra in two non-commuting indeterminates.

Define G to be the subgroup of $AutS$ generated by the inner automorphisms induced by:

$$A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix}.$$

Let K be the Abelian group generated by A and B. As $|K| = p^2$ and K is normal in G, it is clear that $|G| = np^2$. An easy calculation shows that

$$S^K = \left\{ \begin{bmatrix} a & f(x,y) \\ 0 & a \end{bmatrix} : a \in F,\ f(x,y) \in F(x,y) \right\}.$$

Since K is normal in G, we have that $S^G = (S^K)^{G/K}$ as in 1.1.1. Thus, we find that:

$$S^G = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in F \right\} \cong F.$$

Take X to be the two-sided ideal of S consisting of those members of S whose entries have zero constant term. Then $X^G = S^G \cap X = 0$.

We now give the Bergman-Isaacs Theorem which establishes the existence of non-zero fixed points in a non-zero G-invariant right ideal. It was originally proved by G.M. Bergman and I.M. Isaacs in [B-I].

1.3.2 THEOREM *Let X be a semiprime ring (possibly without an identity element) with G a finite group of automorphisms of X such that X has no additive $|G|$-torsion. Then*

(i) X^G is semiprime;

(ii) if I is a non-zero G-invariant left (right) ideal of X, then $tr(I) \neq 0$.

PROOF [Mo1, Corollary 1.5].

The next definitions are given in [Mo1].

1.3.3 DEFINITION Suppose X is a ring (possibly without an identity element) and G is a finite group of automorphisms acting on X. Suppose the group action has the following properties:

(i) X^G is semiprime;

(ii) If I is a non-zero G-invariant left (right) ideal of X, then $tr(I) \neq 0$.

Then we say that G has a *non-degenerate trace*.

An associated concept is that of a *partial trace function*. A partial trace function is an $(X^G\text{-}X^G)$-bimodule homomorphism of the form: $t_\Lambda : X \to X^G$ such that $t_\Lambda(x) = \Sigma_{g \epsilon \Lambda}\, x^g$ where Λ is a subset of G. Such a function is said to be *non-trivial on X* if $tr(X) \neq 0$.

While the Bergman-Isaacs Theorem is the best known of the results concerning the existence of fixed points, there are others which will also be of use to us. We state three important results here. It's worth observing that the hypotheses of each of these theorems are violated by Example 1.3.1.

The first of these was proved by V.K. Kharchenko in [K].

1.3.4 THEOREM *Let S be a ring with an identity element and no non-zero nilpotent elements. Let G be a finite group of automorphisms acting on S. If L is a non-zero, G-stable (right or left) ideal of S, then $L^G \neq 0$.*

PROOF [P2, Theorem 27.4].

In [C-M], S.Montgomery and M. Cohen proved the following result.

1.3.5 THEOREM *Let X be a ring (possibly without an identity element) and let G be a finite group of automorphisms acting on X. If X has no non-zero nilpotent elements then a non-trivial partial trace function exists.*

PROOF [P2, Corollary 24.11].

S. Montgomery established the final result here. It was first shown in [Mo3].

1.3.6 THEOREM *Let G be a finite group acting on a domain, S, with an identity element. The following are equivalent:*

(i) $tr(S) \neq 0$;

(ii) $tr(I) \neq 0$ for all non-zero right ideals I of S;

(iii) the skew group ring $S*G$ is semiprime.

PROOF [P2, Corollary 27.8]

§1.4 Finiteness Conditions

Of concern to us in this section are the circumstances in which the Noetherian property passes down from S to S^G. We also examine whether or not S is a finitely generated S^G-module.

We first show that S being Noetherian does not always guarantee that S^G is Noetherian. In the same example, due to C.L. Chuang and P.H. Lee, we also show that S need not be a finitely generated S^G-module.

1.4.1 EXAMPLE *There is a commutative Noetherian domain of characteristic zero with an automorphism of order 2, such that S^G is not Noetherian and S is not a finitely generated S^G-module.*

Let $A = \mathbb{Z}[a_1,b_1,a_2,b_2,\dots]$ be a polynomial ring in indeterminates a_i,b_i over $\mathbb{Z}$. Let K be the localisation of A at $2A$. We take S to be the ring $K[[x, y]]$, the ring of formal power series in indeterminates x and y over K. Since K is a principal ideal domain, S is a Noetherian domain. There is an automorphism g on S, given by:

$$x^g = -x,\ y^g = -y,\ a_i{}^g = -a_i + p_{i+1}y,\ b_i{}^g = b_i + p_{i+1}x$$

where $p_i = a_i x + b_i y$.

[Mo1, Example 5.5] shows that S^G is not Noetherian and S is not a finitely generated S^G-module.

Despite this example, we can still answer the Noetherian question positively in a number of cases. As we will see later, when $|G|^{-1} \in S$, S^G turns out to be a well behaved ring and, as the next lemma shows, it is Noetherian if S is Noetherian. This result is well known; its earliest occurrence in the literature appears to be [F-S].

1.4.2 LEMMA *Let S be a right Noetherian ring and G a finite group of automorphisms of S with the property that $tr(S) = R$. (This happens, for example, when $|G|^{-1} \in S$ as observed in 1.1.1.) Then S^G is right Noetherian.*

PROOF Consider the ascending chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_t \subseteq \ldots \qquad -(1)$$

of right ideals of S^G. This yields a chain

$$I_1 S \subseteq I_2 S \subseteq I_3 S \subseteq \ldots \subseteq I_t S \subseteq \ldots \qquad -(2)$$

of right ideals of S. Since S is right Noetherian, the chain *(2)* must terminate so that there exists $j \in \mathbb{N}$ such that $I_j S = I_{j+u} S$ for all $u \in \mathbb{N}$. Applying the trace map to this equation yields $tr(I_j S) = tr(I_{j+u} S)$ for all $u \in \mathbb{N}$. Since tr is a left S^G-module homomorphism, we have that $I_j tr(S) = I_{j+u} tr(S)$ for all $u \in \mathbb{N}$. By hypothesis $tr(S) = S^G$, so $I_j = I_{j+u}$ for all $u \in \mathbb{N}$. We have shown that the chain *(1)* does terminate. Thus S^G is right Noetherian.

Finite generation of S over the fixed ring is also well behaved when the order of the group is invertible in the ring. D.R. Farkas and R.L. Snider have proved the following result, again in [F-S].

1.4.3 THEOREM *Let S be a right Noetherian ring and G be a group of automorphisms of S such that $|G|^{-1} \in S$. Then S is finitely generated as a right S^G-module.*

PROOF [Mo1, Corollary 5.9].

Despite Example 1.4.1, there are some positive results known on the preservation of the ascending chain condition in the absence of a surjective trace map. The oldest and most well known of these was proved by E. Noether in 1926 in response to Hilbert's Fourteenth Problem. It can be found as [H, Theorem 5.1].

1.4.4 THEOREM *Let K be a commutative Noetherian ring and S a (commutative) affine K-algebra. Then*

(i) S^G is an affine K-algebra and therefore Noetherian;

(ii) S is a finitely generated S^G-module.

In a similar vein, we have a theorem of Azumaya and Nakayama in [A-N].

1.4.5 THEOREM *Let S be a simple Artinian ring and G a finite group of outer automorphisms of S. Then*

(i) both S^G and $S*G$ are simple Artinian;

(ii) S is a free S^G-module of rank $|G|$.

PROOF [Mo1, Theorem 2.7].

The final result here gives us more information in the case where S is simple.

1.4.6 THEOREM *Let S be a simple ring and G be a finite group of outer automorphisms of S. Then*

(i) $tr_G(S)$ is the unique minimal non-zero ideal of R;

(ii) R is primitive;

(iii) $(C(S))^G = C(R)$;

(iv) $S*G$ is simple.

PROOF [Mo1, Theorem 2.9] proves the first three parts. [Mo1, Theorem 2.3] proves (iv).

§1.5 Prime Links and The Second Layer Condition

In Chapter 5, we hope to localise S^G at certain semiprime ideals. In view of the correspondence exhibited in Chapter 1, §2, the localisations of S itself are obviously relevant to this matter.

In order to examine the issue in any detail we need to look at the notions of prime links and that of the (strong) second layer condition. This theory is extensive and we only provide a brief overview in this section. For background see, for example, [G-W, Chapters 11 & 12] and [McC-R, Chapter 4].

To begin with we consider inversion of a subset, X, of a ring S. To do this we form a quotient ring where members of the set X are units. It's well known that in order to do this, X must be an Ore set. We define such a set here.

1.5.1. DEFINITION Let S be a ring and X be a non-empty multiplicatively closed subset of S. Then X is said to be a *right Ore set* if, for all $x \in X$, $r \in R$, $xR \cap rX \neq \emptyset$. Similarly, we define a *left Ore set*. We say X is an *Ore set* if it is both a left and right Ore set.

We now describe what we mean by localisation at a semiprime ideal of a ring. Let S be a ring and N a semiprime ideal of S. By localisation at N, we mean inversion of $C_S(N)$, the set of elements of S that are regular modulo N. As we see in Proposition 1.5.3, the prime ideals containing N play an important part in the theory of localisation. With this in mind we give the definition of a second layer link. This terminology is due to Jategaonkar and Muller.

1.5.2 DEFINITION For a Noetherian ring S and P, $Q \in SpecS$, we say that a *second layer link* exists from P to Q or that *P is second layer linked to Q* if there is an ideal A of S containing PQ such that $(P \cap Q)/A$ is non-zero and is torsionfree as a right S/Q-module and as a left S/P-module. The bimodule $(P \cap Q)/A$ is called a *linking bimodule*.

For example, let

$$S = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}, \quad P = \begin{bmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}, \quad Q = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}.$$

Then, $P, Q \in SpecS$, $P \cap Q = I$ while $PQ = 0$. Consider the bimodule $(P \cap Q)/PQ$. As a left S-module, it has annihilator P and, as a right S-module, it has annihilator Q. Thus, we see that P is second layer linked to Q and that $P \cap Q$ is the linking bimodule.

There are generalisations of a linking bimodule which we use in Chapter 5. A non-zero Noetherian bimodule ${}_SB_R$ is called a *bond* if R and S are prime rings and both of the modules ${}_SB$ and B_R are torsionfree.

Suppose $P \in SpecR$ and $Q \in SpecS$. If there exists a non-zero subfactor B' of B such that $l.ann_R(B') = P$ and $r.ann_S(B') = Q$ and B' is a torsionfree $((R/P)-(S/Q))$-bimodule, then B' is said to be a *B-bond* from R/P to S/Q.

A special case of a B-bond is described next. Let $P, Q \in SpecS$. If there exists a R-bond from R/P to S/Q, we say there is an *ideal link* from S/P to S/Q. Such an ideal link from P to itself is said to be *non-trivial* if the R-bond is a subfactor of ${}_RP_R$.

With the definition of a link, we may view $SpecS$ as the vertex set in a (directed) graph, the edges being determined by the links. Such a graph is called the *link graph*. For $P \in SpecS$, the set of vertices in the connected component of the link graph containing P is called the *clique of P* and is denoted by $cl(P)$.

To give an appreciation of the relationship the definition of a link has with localisation at semiprime ideals, we provide the following proposition.

1.5.3 PROPOSITION *Let S be a Noetherian ring which has a left and right denominator set, C. Suppose $P, Q \in SpecS$ and that P is linked to Q. Then $C_S(P) \subseteq C$ if and only if $C_S(Q) \subseteq C$.*

PROOF [McC-Rob, Proposition 4.3.6]

This result shows that, for any $P \in SpecS$, the largest subset of $C_S(P)$ at which it is feasible to localise is $\cap C_S(Q)$ as Q ranges through the clique containing P. One further technical condition guarantees that we may localise at such a subset. Before we give this condition, it is necessary to give two definitions and state Jategaonkar's Main Lemma which was originally proved in [J2, Lemma 2.2].

1.5.4 DEFINITION Let M be a right S-module and let N be a submodule of M. If N has a non-zero intersection with every non-zero submodule of M, then we say that *N is essential in M* and write $N <_e M$.

The second set of definitions was originally given in [St]. Let M be a non-zero S-module. An *affiliated submodule of* M is any submodule of the form $ann_M(P)$ where P is an ideal of S maximal among the annihilators of non-zero submodules of M. An affiliated series for M is a series of submodules of the form

$$0 < M_0 < M_1 < \dots < M_n = M$$

where, for each $i=1,\dots,n$, the module M_i/M_{i-1} is an affiliated submodule of M/M_{i-1}. If $P_i = ann_S(M_i/M_{i-1})$ then the series $P_1,\dots,P_n$ is *the series of affiliated primes of* M corresponding to the given affiliated series.

Recall the example used in 1.5.2. The right S-module S_S has an affiliated series $0 < P < S$ with correseponding primes Q and P.

1.5.5 THEOREM *Let S be a Noetherian ring and let M be a right S-module with affiliated series $0 < U < M$ and corresponding affiliated prime ideals P and Q, such that $U \leq_e M$. Let M' be a submodule of M, properly containing U, such that the ideal $A = ann_S(M')$ is maximal among annihilators of submodules of M properly containing U. Then exactly one of the following alternatives occur:*

(i) $Q \subset P$ and $M'Q = 0$. In this case, M' and M'/U are faithful torsion S/Q-modules.

(ii) Q is linked to P and $(Q \cap P)/A$ is a linking bimodule between Q and P. In this case, if U is torsionfree as a right (S/P)-module, then M'/U is torsionfree as a right (S/Q)-module.

Jategaonkar has introduced the following definitions.

1.5.6 DEFINITION Suppose P is a prime ideal in a Noetherian ring S. Then P is said *to satisfy the right strong second layer condition (SSLC)* if, given the hypotheses of Theorem 1.5.5, the conclusion (i) never occurs.

Similarly, P is said *to satisfy the right second layer condition (SLC)* if, given the hypotheses of Theorem 1.5.5 and the additional hypothesis that U is torsionfree as an (S/P)-module, the conclusion (i) never occurs. Analogously defined are the left SSLC and the left SLC. The ring S is said to satisfy right (left) SLC if all its primes have right (left) SLC. If S has both right and left SLC, it is said to have SLC. Also, the ring S is said to satisfy right (left) SSLC if all its primes have right (left) SSLC. If S has both right and left SSLC, it is said to have SSLC.

This definition is used by Jategaonkar in [J1] for the next theorem.

1.5.7 THEOREM *Let S be a Noetherian ring. Suppose N is a semiprime ideal of S and that X is the set of primes of S minimal over N. Suppose X is closed under the taking of links and satisfies SLC. Then S can be localised at $C_S(N)$.*

PROOF [McC-R, Theorem 4.3.16].

The next proposition provides a useful test for determining whether or not a prime has SLC or SSLC.

1.5.8 PROPOSITION *Let P be a prime ideal in a Noetherian ring S.*

(i) P satisfies the right SSLC if and only if there does not exist a finitely generated uniform right S-module, M, with an affiliated series $0 < U < M$ and corresponding affiliated primes P and Q, such that M/U is uniform, $Q \subset P$ and $MQ = 0$.

(ii) P satisfies the right SLC if and only if there does not exist a finitely generated uniform right S-module, M, with an affiliated series $0 < U < M$ and corresponding affiliated prime ideals P and Q such that U is a torsionfree (S/P)-module, M/U is uniform, $Q \subset P$, and $MQ = 0$.

PROOF [G-W, Proposition 11.3].

There is an easy corollary which shows that we may reduce to the case where M is cyclic rather than just finitely generated.

1.5.9 COROLLARY *Let P be a prime ideal in a Noetherian ring S. Then:*

(i) P satisfies the right SSLC if and only if there does not exist a cyclic right uniform S-module, M, with an affiliated series $0 < U < M$ and corresponding affiliated primes P and Q, such that M/U is uniform, $Q \subset P$ and $MQ = 0$.

(ii) P satisfies the right SLC if and only if there does not exist a cyclic uniform right S-module, M, with an affiliated series $0 < U < M$ and corresponding affiliated prime ideals P and Q such that U is a torsionfree (S/P)-module, M/U is uniform, $Q \subset P$ and $MQ = 0$.

PROOF By Proposition 1.5.8, we only have to prove the "only if" direction.

For (i), suppose that P does not satisfy the SSLC. By Proposition 1.5.8, there exists a finitely generated right S-module, M, with an affiliated series $0 < U < M$ and corresponding affiliated primes P and Q, such that M/U is uniform, $Q \subset P$ and $MQ = 0$. Let $0 \neq m \in M \backslash U$. We claim that $0 < mS \cap U < mS$ is an affiliated series for mS with corresponding affiliated primes P and Q such that $mS/(mS \cap U)$ is uniform, $Q \subset P$ and $mSQ = 0$. By definition, P is maximal among the annihilators of non-zero submodules of M and so is certainly maximal among annihilators of non-zero submodules of mS. Moreover, $ann_{mS}(P) = ann_M(P) \cap mS = U \cap mS$. Since $mS/(mS \cap U)$ embeds in M/U, a similar argument will show that $mS/(mS \cap U)$ is the affiliated submodule of $mS/(mS \cap U)$ with affiliated prime Q. Now, since $mS/(mS \cap U)$ is a submodule of the uniform module M/U, it too is uniform. The last two statements of the claim are obviously true. This proves (i)

For (ii), we need only show that if U is in addition (S/P)-torsionfree, then $U \cap mS$ is (S/P)-torsionfree. This is trivial since $U \cap mS$ is a submodule of U.

Another adaptation of Proposition 1.5.8 will be useful in Chapter 5.

1.5.10 COROLLARY *Let P be a prime ideal in a Noetherian ring S. Suppose X is an Ore set with $X \subseteq C_S(P)$. Then*

(i) P has SLC in S only if PX^{-1} has SLC in SX^{-1};

(ii) P has SSLC in S only if PX^{-1} has SSLC in SX^{-1}.

PROOF We only give the proof for (i) here as the proof for (ii) is contained therein.

Let A denote SX^{-1}. Suppose PX^{-1} does not have SLC in A. By Corollary 1.5.9, there exists a cyclic right A-module, M, such that M is uniform, and an A-submodule U of M such that $0 < U < M$ is an affiliated series with corresponding primes PX^{-1} and Q' such that U is a torsionfree (A/PX^{-1})-module, aA/U is uniform, $Q' \subset PX^{-1}$ and $MQ' = 0$.

By [G-W, Theorem 9.22], $Q' = QX^{-1}$ for some $Q \in SpecS$ with $Q \subset P$.

Let $m \in M$ be such that $M = mA$. We claim that $0 < U \cap mS < mS$ is an affiliated series with corresponding primes P and Q. First, suppose that P is not maximal among annihilators of non-zero submodules of mS. Then P is strictly contained in such a prime ideal $P_1 \in SpecS$ with $P_1 = ann_{mS}(Y)$ for some $0 \neq Y \leqslant mS$. Since Y is contained in a right A-module, for $x \in X$, $Y = Y(xx^{-1}) = (Yx)x^{-1}$ and so $P_1 \cap X = \emptyset$. Then we find that YA is annihilated by $P_1X^{-1} \supset PX^{-1}$. This contradiction shows that P is maximal among annihilators of non-zero submodules of mS. We now show that the affiliated submodule is $U \cap mS$. Certainly, $U \cap mS \subseteq ann_{mS}(P)$. If $y \in ann_{mS}(P)$, $y(PX^{-1}) = 0$ and so $y \in U \cap mS$. Similar consideration shows that $mS/(mS \cap U)$ is the affiliated submodule of $mS/(mS \cap U)$ with affiliated prime Q.

We claim that $U \cap mS$ is a torsionfree (S/P)-module. Suppose this is not the case. Then there exists $0 \neq u \in U \cap mS$ and $x \in C_S(P)$ such that $u(x + P) = 0$. However, $u \in U$ and $x + PX^{-1} \in A/(PX^{-1})$. This contradicts the fact that U is $A/(PX^{-1})$-torsionfree.

In addition, we claim that $mS/(U \cap mS)$ is uniform. Suppose to the contrary that there exist S-submodules C and D of mS, such that $C \supset U \cap mS$, $D \supset U \cap mS$ but that $C \cap D \subseteq U \cap mS$. Now, $(U \cap mS)A \cap mS = ann_{mS}(P) = U \cap mS$, so that $CA \supset (U \cap mS)A$; similarly, $DA \supset (U \cap mS)A = U$. Let $y \in CA \cap DA$. There exists $c \in C$, $d \in D$ and $x \in X$ such that $y = cx^{-1} = dx^{-1}$. Postmultiplying the last equality by x yields that $c = d \in C \cap D \subseteq U \cap aS$. This gives that $y \in (U \cap mS)A = U$ and so $CA \cap DA \subseteq U$. This contradicts mA/U being uniform. Thus, $mS/(mS \cap U)$ is uniform. Similarly, we prove that mS is a uniform S-module.

Since $Q \subset P$ and $(mS)Q = 0$, the "only if" direction of Corollary 1.5.9(ii) shows that P does not have SLC.

We shall make use of one further result about localisation. This is known as Small's Theorem. It was originally stated by L.W. Small in [Sma1] and [Sma2].

1.5.11 THEOREM *Let S be a right Noetherian ring and let N denote the prime radical of S. Then S has a right Artinian right quotient ring if and only if $C_R(N) = C_R(0)$.*

PROOF [McC-R, Corollary 4.1.4].

§1.6 Gelfand-Kirillov Dimension

A dimension function on finitely generated algebras is defined in this section. Throught this section k is a field and S is a k-algebra. The function measures the growth of certain k-vector spaces and is known as Gelfand-Kirillov dimension. It is named after I.M.Gelfand and A.A. Kirillov who together published two influential papers in 1966. See [GK1] and [GK2]. For a detailed discussion of this dimension function, see [K-L].

As space is limited, our treatment will be fairly brief. Basically, it is a well behaved dimension function and we list some of its nice properties here. We use these properties when studying nilpotent group algebras in Chapter 5.

1.6.1 DEFINITION Let k be a field and S a finitely generated k-algebra. Let V be a finite dimensional generating subspace for S so that $\Sigma_{i=1}^{\infty} V^i = S$. Let $d_V(n)$ denote $dim_k(\Sigma_{i=1}^{n} V^n)$. The *Gelfand-Kirilov dimension of S* is defined as follows:

$$GKdim(S) = \lim_{n\to\infty} \frac{\log d_V(n)}{\log n} \quad .$$

It transpires in [K-L, Lemma 1.1] that this definition is independent of the choice of V. While the above definition seems very abstract, it is in fact equivalent to the following definition. There exist constants A,B and c such that the inequalities $An^c \leqslant d_V(n) \leqslant Bn^c$ hold for all but finitely many n if and only if S has finite GK-dimension equal to c. The equivalence of these definitions follows from [K-L, Lemma 2.1].

The following results give a more intuitive feel for the notion of GK-dimension.

1.6.2 LEMMA *Let k be a field and S a finitely generated k-algebra.*

(i) If B is a subalgebra or a homomorphic image of S,

$GKdim(B) \leq GKdim(S)$.

(ii) If B is a subalgebra of S so that S_B is finitely generated, $GKdim(S) = GKdim(B)$,

(iii) If S_1 and S_2 are finitely generated k-algebras, then $GKdim(S_1 \oplus S_2) = max\ \{GKdim(S_i)\}$.

(iv) $GKdim(A/(I_1 \cap I_2 \ldots \cap I_n)) \leq max\ \{\ GKdim(A/I_j)\}$.

PROOF The result (i) is clear from the definition. [K-L, Propositon 5.5] proves (ii). [K-L, Proposition 3.2] gives (iii) and (iv) is just [K-L, Corollary 3.3].

1.6.3 LEMMA *Let k be a field and S a finitely generated k-algebra. If $P \in SpecS$, then $GKdim(S) \geq GKdim(S/P) + ht(P)$.*

PROOF [K-L, Corollary 3.16].

So far we have only discussed the GK-dimension of a finitely generated k-algebra S. We may also define the GK-dimension of a module over the ring S.

1.6.4 DEFINITION Let k be a field and S a finitely generated k-algebra with a finite dimensional generating subspace V with $1_S \in V$. Let M be a finitely generated right S-module so that there exists a finite dimensional subspace F which generates M as an S-module. Thus, we have $M = \cup_{n=0}^{\infty} FV^n$. Let $d_{V,F}(n) = dim_k(FV^n)$. Define

$$GKdim(M_S) = \lim_{n\to\infty} \frac{log\ d_{V,F}(n)}{log\ n} .$$

As before this definition is independent of the choice of V and F.

When $M = S$, we see that $GKdim(S_S) = GKdim(S)$ as defined in 1.6.1.

This definition for modules also has some nice properties which we can take advantage of.

1.6.5 LEMMA *Let k be a field and S a finitely generated k-algebra. Let M be a finitely generated S-module with $\alpha \in End_S(M)$ an injective map. Then*

$$GKdim(M/\alpha(M)) \leq GKdim(M) - 1.$$

PROOF [K-L, Proposition 5.1(e)].

We see how the GK-dimensions of a bimodule on either side compare. Lemma 1.6.6 was proved independently by W. Bohro in [Bo] and T.H. Lenagan in [L].

1.6.6 LEMMA *Let S and T be finitely generated k-algebras and ${}_SM_T$ an $(S-T)$-bimodule which is finitely generated on both sides. Then*

$$GKdim({}_SM) = GKdim(M_T).$$

PROOF [K-L, Corollary 5.4].

1.6.7 DEFINITION Let S be a finitely generated k-algebra with M a finitely generated right S-module. The module M is said to be *GK-homogeneous* if, for all non-zero submodules N of M, $GKdim(N) = GKdim(M)$.

The next result, [K-L, Lemma 5.13], can in some ways be considered to be a converse of Lemma 1.6.5. To prove it, observe that if $N \cap A = 0$ for some non-zero submodule A of M, then A embeds isomorphically in M/N.

1.6.8 LEMMA *Let S be a finitely generated k-algebra and M a finitely generated right S-module which is GK-homogeneous. If $GKdim(M/N) < GKdim(M)$ for some submodule N of M, then N is essential in M.*

In Chapter 5, we are particularly interested in the GK-dimension of group algebras. As the next theorem shows, such algebras have finite GK-dimension only if the group in question is nilpotent-by-finite. It was proved by Gromov in [G].

1.6.9 THEOREM *Let H be a finitely generated group and k a field. Then $GKdim(kH) < \infty$ if and only if H has a nilpotent normal subgroup N such that H/N is finite.*

H. Bass goes further and calculates the GK-dimension of a nilpotent group algebra.

1.6.10 THEOREM *Let H be a finitely generated nilpotent group and let $H_0 = \{1\} < H_1 < \ldots < H_t = H$ be the lower central series. Let k be a field. Then*

$$GKdim(kH) = \Sigma_{i=1}^{t}\ i\hat{h}(H_i/H_{i-1})$$

where $\hat{h}(H_i/H_{i-1})$ is the torsionfree rank of H_i/H_{i-1}.

PROOF [K-L, Theorem 11.14].

§1.7 Additional Remarks

7.1 K.Morita introduced the concepts discussed in §2 in connection with category equivalences. S.A. Amitsur studied the more general Morita context in [A]. W.K. Nicholson and J.F. Watters defined and studied prime contexts in [N-W].

7.2 K. Nagarajan in [Na] gave an example similar to Example 1.4.1 in non-zero characteristic. C.L. Chuang and P.H. Lee raised Nagarajan's example to characteristic zero. It is their version we give as Example 1.4.1

7.3 Theorem 1.4.6 is a compilation of results, each using the same hypotheses. (i) to (iii) were proved by Osterburg in [O] while Miyashita gave an earlier proof of (iii). Azumaya proved (iv) in [Az].

7.4 While one direction of Proposition 1.5.8 is just Theorem 1.5.5, the other direction appears in the literature for the first time as [G-W, Proposition 11.3]. Corollary 1.5.9 is well known but does not seem to be stated explicitly in the literature. The same is true of Corollary 1.5.10.

7.5 I.N. Bernstein in [Be] first made Definition 1.6.4. In [J-S], A. Joseph and L.W. Small studied the properties of GK-dimension when applied to a module.

CHAPTER 2

FINITENESS CONDITIONS

This chapter is devoted to a detailed discussion of the finiteness conditions of Chapter 1, §4.

In Chapter 2, §1, we investigate under what circumstances S is finitely generated as an S^G-module. We find a sufficient condition for this to happen.

We have already seen in Example 1.4.1, that it is not always the case that S^G is Noetherian when S is. There are cases, however, where S^G is known to be Noetherian. For example, Lemma 1.4.2 shows that S^G is Noetherian when $|G|^{-1} \in S$. In Chapter 2, §2, there are two specific questions we will examine. The first of these is a conjecture of S. Montgomery. In [Mo4, Problem 6], Montgomery conjectured that if S is simple and Noetherian, then S^G is Noetherian. We thus ask the following question.

2A QUESTION *Suppose S is a simple Noetherian ring and G is a finite group of automorphisms of S. Is the fixed ring S^G Noetherian ?*

Notice from Lemma 1.4.4 that if H is an Abelian-by-finite group with a finite group of automorphisms G and k is any field, then $(kH)^G$ is Noetherian. This fact led to the next question being asked by D.S. Passman in [P2]. He was concerned with the following scenario. Let H be a polycyclic-by-finite group, G a finite group of automorphisms of H, k a field and kH the group algebra. By a variant of Hilbert's Basis Theorem, kH is Noetherian and Passman asked if the fixed ring $(kH)^G$ is always Noetherian. We call this Question 2B.

2B QUESTION *Suppose H is a polycyclic-by-finite group and G is a finite group of automorphisms of H. Let k be a field and S be the group algebra kH. Is the fixed ring S^G Noetherian ?*

As already noted, if $|G|^{-1} \in k$ then Question 2B has a positive answer and so we consider the case where $|G| = 0$ in k.

§2.1 Finite Generation of S as an S^G-module

It is of interest to know the circumstances in which S is a finitely generated S^G-module. The first case we investigate is that of S being a division ring. We require to state the following definition and lemma. The definition is given in [Mo1].

2.1.1 DEFINITION Let S be a simple ring and let $g \in AutS$ be an inner automorphism. Define the following subset of S:

$$\varphi_g = \{x \in S: s^g = x^{-1}sx \text{ for all } s \in S\}.$$

Then if C denotes the centre of S, we have $\varphi_g = Cx_g$, for any $0 \neq x_g \in \varphi_g$. Now, let G be a group of inner automorphisms of S. The *algebra of the group* is $B := \Sigma_{g \in G} \varphi_g = \Sigma_{g \in G} Cx_g$ where $0 \neq x_g \in \varphi_g$ for all $g \in G$.

2.1.2 LEMMA *Let S be a division ring and let G be a finite group of inner automorphisms of S. Denote S^G by R. Let C be the centre of S and B be the algebra of the group. Then* $S \cong B \otimes_C C_S(B) = B \otimes_C R$.

PROOF [Mo1, proof of Lemma 2.12].

The next lemma appears as [Mo1, Lemma 2.18].

2.1.3 LEMMA *Let D be a division ring, let G be any finite group of automorphisms of D and let $R = D^G$. Then the rank of D as a right R-module, which we denote $[D:R]$, is less than or equal to $|G|$.*

PROOF Let D be a division ring. If G has a proper normal subgroup N, then G/N acts on D^N. By induction on $|G|$, we would have $[D:D^G] = [D:D^N][D^N:D^G] \leq |N||G/N| = |G|$, and we would be finished. Henceforth, we assume that G is simple. Since inner automorphisms of G form a normal subgroup of G by 1.1.1, G is either inner or outer. If G is outer, we may apply Theorem 1.4.5 to give the result. Now we consider the case where G is inner on D. Let B be the algebra of the group. Now, B is finite dimensional over C, the centre of D, and so is a division ring. By the previous lemma, $D \cong B \otimes_C C_D(B) = B \otimes_C D^G$. Thus, D is finite dimensional over D^G with $[D:D^G] = dim_C B \leq |G|$.

We now aim to prove a similar result for S semiprime with no non-zero nilpotent elements. The proof of the theorem is modelled on a result by D.R. Farkas and R.L. Snider in [F-S]. Their result appears in the literature as [P2, Theorem 26.16].

2.1.4 THEOREM *Let S be a semiprime ring with no non-zero nilpotent elements and let G be a finite group of automorphisms of S such that $R:=S^G$ is a left Goldie ring. Then S can be embedded in a free R-module of finite rank.*

PROOF By [Mo1, Theorem 5.7], S is a Goldie ring with semi-simple Artinian quotient ring $Q(S)$. Let e be a primitive central idempotent of $Q(S)$ so that, in view of the hypotheses on S, $eQ(S)$ is a division ring. We first show that $S \cap eQ(S)$ can be embedded in a free R-module of finite rank. We claim that $S \cap eQ(S)$ is an Ore domain with $Q(S \cap eQ(S)) = eQ(S)$, the whole division ring. For, choose $z, s \in S$ with z regular so that $e = z^{-1}s$. Then $s = ze \in S \cap eQ(S)$. For $x \in eQ(S)$, choose $q, w \in S$, q regular, so that $qx = w$. Then $(sq)x = sw$. Since sq is regular in eS, we have $x = (sq)^{-1}(sw) \in Q(S \cap eQ(S))$.

Now let $H = \{g \in G: e^g = e\}$ so that H acts on $S \cap eQ(S)$. Let Γ be a right transversal for H in G. Let $a \in S^H \cap eQ(S)$ and $g \in G$, then $(tr_\Gamma(a))^g = \Sigma_{\gamma\in\Gamma}(a^\gamma)^g = \Sigma_{\gamma\in\Gamma}a^{\gamma g} = tr_\Gamma(a)$ since g permutes the elements of Γ up to elements of H. Thus, $tr_\Gamma(a) \in R$ for all $a \in S^H \cap eQ(S)$. Now G permutes the primitive central idempotents of $Q(S)$ and so, for $g \neq h$ elements of Γ, $e^h \neq e^g$ and $e^h e^g = 0$. Thus, if $x = er \in S^H \cap eQ(S)$, then $tr_\Gamma(x) = \Sigma_{\gamma\in\Gamma}e^\gamma r^\gamma$, and so $etr_\Gamma(x) = \Sigma_{\gamma\in\Gamma}ee^\gamma r^\gamma = er = x$. Thus, $tr_\Gamma: (S^H \cap eQ(S)) \to R$ is an injective left R-homomorphism.

Since $eQ(S)$ is a division ring, $eQ(S)$ is finite dimensional over $(eQ(S))^H$ by Lemma 2.1.3. Let $\{x_1, \ldots, x_n\}$ be a basis for $eQ(S)$ over $(eQ(S))^H$ so that $eQ(S) = \Sigma_{i=1,..,n}x_i(eQ(S))^H$. As $eQ(S) = Q(S \cap eQ(S))$ each $x_i = t^{-1}s_i$ for some $s_i, t \in S \cap eQ(S)$ $(i=1,\ldots,n)$. Thus, $t(eQ(S)) = \Sigma_{i=1,..,n}s_i(eQ(S))^H$. Since $t^{-1} \in eQ(S)$, $teQ(S) = eQ(S)$, and so $eQ(S) = \Sigma_{i=1,..,n}s_i(eQ(S))^H$. Thus, we may assume that $x_i = s_i \in S \cap eQ(S)$.

By Theorem 1.3.5, there exists $\Lambda \subseteq H$ such that $tr_\Lambda: eQ(S) \to (eQ(S))^H$ is non-trivial. We may define $\varphi: (S \cap eQ(S)) \to R^n$ such that $\varphi(a) = \Sigma_{i=1,..,n}{\oplus}(tr_\Gamma(tr_\Lambda(ax_i)))$. It is clear that φ is a left R-module homomorphism and we claim that it is in fact a monomorphism. Suppose that, for $a \in S \cap eQ(S)$, $\varphi(a) = 0$. Then $(tr_\Gamma(tr_\Lambda(ax_i)) = 0$ for $i=1,\ldots,n$. By the preceding paragraph, tr_Γ is injective so we must have $tr_\Lambda(ax_i) = 0$ for $i=1,\ldots,n$. Thus, $tr_\Lambda(ax_i(eQ(S)^H)) = 0$ for $i=1,\ldots,n$ and because $eQ(S) = \Sigma_{i=1,..,n}x_i(eQ(S))^H$, we have that $tr_\Lambda(aeQ(S)) = 0$ which contradicts the fact that tr_Λ is non-trivial on $eQ(S)$ unless $a = 0$. Thus, $S \cap eQ(S)$ can be embedded in a free R-module of finite rank.

Now, let $\{e_1,\ldots,e_m\}$ be all the primitive central idempotents in $Q(S)$. By the above $\Sigma_{i=1,..,n}^{\oplus}(S \cap e_iQ(S))$ is contained in a free R-module of finite rank. Since $Q(S \cap e_iQ(S)) = e_iQ(S)$, each $S \cap e_iQ(S)$ contains an element d_i which is invertible in $e_iQ(S)$. Thus, $d = d_1+\ldots+d_n$ is invertible in $Q(S)$. Now define $f:S \to \Sigma_{i=1,..,n}(S \cap e_iQ(S))$ by $f(r) = rd$. Since $d^{-1} \in Q(S)$, f is injective. Thus, we've shown that S can be embedded in a free R-module of finite rank. This completes the proof of the theorem.

Notice that Example 1.3.1 shows that Theorem 2.1.4 must be close to the best possible. The ring S in 1.3.1 is prime, has no $|G|$-torsion and S^G is a field, yet S is certainly not a finitely generated S^G-module.

Theorem 2.1.4 provides us with a partial converse to the general question discussed in §2.

2.1.5 COROLLARY *Let S be semiprime with no non-zero nilpotent elements and G be a finite group of automorphisms of S. If $R := S^G$ is left Noetherian, then S is left Noetherian and is a finitely generated R-module.*

PROOF Theorem 2.1.4 shows that S embeds in a free R-module of finite rank and so S must be a Noetherian R-module. Hence S is a left Noetherian ring.

Now, we show that Corollary 2.1.5 enables us to make a reduction when dealing with Question 2B. First, we state a well known Lemma.

2.1.6 LEMMA *Let H be a polycyclic-by-finite group and let G be a finite group of automorphisms of H. Then there exists a G-invariant poly-C_∞ subgroup L of H such that $|H:L| < \infty$.*

PROOF [P2, Lemma 21.4(i)] shows that there exists a normal subgroup, N of H, such that N is poly-C_∞ and has finite index in H. Taking $L = \bigcap_{g\in G} N^g$

gives the result.

2.1.7 THEOREM *Let H be polycyclic-by-finite, G a finite subgroup of $AutH$ and k a field. Suppose L is a G-invariant poly-C_∞ subgroup of finite index in H. The group G acts on kL and, if $(kL)^G$ is Noetherian, then so too is $(kH)^G$.*

PROOF If $|G|^{-1} \in k$, then the Theorem is true since, by Lemma 1.4.2, both rings are Noetherian. Henceforth, we assume $char k > 0$ and $|G| = 0 \in k$. Since L is torsion-free, [P1, Theorem 3.4.12] shows that kL is a domain. (We assume $char k > 0$). So, by Corollary 2.1.5, kL is a finitely generated $(kL)^G$-module. As $|H:L| < \infty$, kH is a finitely generated kL-module and so, kH is a finitely generated $(kL)^G$-module. As $(kL)^G$ is Noetherian, we must have that $(kH)^G$, a $(kL)^G$-submodule of kH, is a Noetherian $(kL)^G$-module. Hence $(kH)^G$ is a Noetherian ring.

2.1.8 NOTE Suppose H is a polycyclic-by-finite group. By Lemma 2.1.6, there exists a G-invariant subgroup L of H such that L is poly-C_∞ and H/L is finite. Theorem 2.1.7 shows that if $(kL)^G$ is Noetherian, then so too is $(kH)^G$. Thus, the preceding theorem means we only have to consider the case where H is poly-C_∞ in Question 2B.

§2.2 Does S Noetherian Imply R Noetherian ?

We return to Questions 2A and 2B stated in the introduction to this chapter. Already, we have seen in Note 2.1.8 that Question 2B can be reduced to the case where H is poly-C_∞.

We look first at an example that will have bearing on both questions.

This example is interesting for a number of reasons: the first being that it shows that if S is simple and G is a group of outer automorphisms (see 1.1.1), then R is not necessarily simple. It was conceived by A.E. Zalesskii and O.M. Neroslavskii in [Z-N] as an example of a simple Noetherian ring with zero divisors but no non-trivial idempotents.

2.2.1 EXAMPLE Let k be a field of characteristic 2. Let $S_1 = k(z)[x, x^{-1}]$ where x and z are commuting indeterminates, and let y be the $k(z)$-automorphism of S_1 defined by $x^y = zx$. Let $J = \langle y \rangle$ and let $S = S_1 * J$. Now define g to be the $k(z)$-automorphism of S such that $x^g = x^{-1}$ and $y^g = y^{-1}$. Let $G = \langle g \rangle$, let $T = S*G$ and let $R = S^G$.

We claim that S is simple and G is outer but that R is not simple. We first show that S is simple. Let $H = \langle x, y, z: [x,y] = z, z \text{ central} \rangle$, the first Heisenberg group (see Definition 2.2.6). Then $X := k\langle z \rangle \setminus \{0\}$ is a regular Ore set with $kHX^{-1} \cong S$. Let $P \in SpecS$. By [G-W, Theorem 9.22], $P \cap kH$ is a prime ideal of kH with $(P \cap kH) \cap X = \emptyset$. Since H is nilpotent, the Zalesskii subgroup of H is just $Z(H) = \langle z \rangle$. [P1, Theorem 9.1.17] shows that if $P \cap kH \neq 0$, then $(P \cap kH) \cap X \neq \emptyset$. We conclude that $P \cap kH = 0$ and so, $P = 0$. Thus, S is simple.

Secondly, we claim that G is outer on S. Suppose this is not the case and that there exists a unit $u \in S$, $u = \Sigma_{i=-n,..,n} s_i y^i$ where $s_i \in S_1$ $(i=-n,...,n)$, such that $uw = w^g u$ for all $w \in S$. In particular, $uy = y^{-1}u$. By considering the degree in y of each side in the equation, we see $u = 0$, a contradiction which proves the claim.

That T is simple is immediate from Theorem 1.4.6 (iv). It is this ring, T, that is the subject of [Z-N]. Zalesskii and Neroslavskii show that T has no non-trivial idempotents.

Finally, we show that R is not simple. Since $tr_G(y) \neq 0$, $tr(S)$ is a non-zero ideal of R. Suppose R is simple, so that $tr_G(S) = R$. Thus, there exists $s \in S$, say $s = \Sigma_{i=-t,...,t} s_i y^i$, with $s + s^g = 1$. Hence

$\Sigma_{i=-t,\ldots,t} s_i y^i + \Sigma_{i=-t,\ldots,t} s_i y^{-i} = 1$ but the coefficient of y^0 on the left hand side is $s_0 + s_0 = 0$ because $char k = 2$. This contradiction shows that R is not simple.

Although, as Example 2.2.1 shows, R is not always simple when S is simple and G is outer, we can still give some detailed information about the structure of R. For details see Theorem 1.4.6.

The ring S in Example 2.2.1 satisfies both hypotheses in Question 2A and so it would be helpful to know if S^G is Noetherian. In 1986, T. Hodges and J. Osterburg proved that S^G is Noetherian in [H-O], giving evidence for an affirmative answer to Question 2A. We state their theorem below.

2.2.2 THEOREM *Let $S = k(z)[x, x^{-1}]*\langle y\rangle$ be the ring of Example 2.2.1 where x and z are commuting indeterminates and y is a $k(z)$-automorphism such that $x^y = zx$. Let $G = \langle g\rangle$ be the same subgroup of $AutS$ where g is a $k(z)$-automorphism such that $x^g = x^{-1}$ and $y^g = y^{-1}$. Then the fixed ring, R, is Noetherian.*

In 1989, M. Lorenz and D.S. Passman generalised Theorem 2.2.2 in [L-P1] using similar methods. We give their theorem here. (See Definition 1.1.2 for the definition of a crossed product). This result is the best we have relating to Question 2A.

2.2.3 THEOREM *Let $S = D*\Gamma$ be a crossed product between $\Gamma \cong Z^r$ for some $r \in \mathbb{N}$ and D a division ring. Let $G = \langle\sigma\rangle$ act on S so that D is centralised and the action of σ on Γ is inversion modulo D^*; that is, for $x \in \Gamma$, $x^\sigma = dx^{-1}$ for some $d \in D$. Put $R = S^G$. Suppose $|G| = 2$. Then S is right and left Noetherian as an R-module and consequently R is Noetherian as a ring.*

PROOF We prove that ${}_RS$ is a Noetherian module.

Step1 We define an ordering on Γ. (We consider $\Gamma \cong Z^r$ as an additive group). Let $x = (x_1,\dots,x_r) \in \Gamma$, define $|x|$ to be $\Sigma_i |x_i|$.

For each $s \geq 0$, let $\Gamma_s = \{x = (x_1,\dots,x_r): |x| = s\}$ denote the r-dimensional cube of diameter $2s$ centred at the origin with corners on the axes. Then

$$\Gamma = \dot{\cup}_{s \geq 0} \Gamma_s$$

and we can linearly order the set Γ. To do this we utilise the lexicographical ordering, $\leq_l$, on Γ. Suppose $x = (x_1,\dots,x_r)$, $y = (y_1,\dots,y_r) \in \Gamma$ and that the first difference between x and y occurs at the jth coordinate. Then, if $x_j < y_j$, we say $x \leq_l y$ or, if $y_j < x_j$, we say that $y \leq_l x$.

We now define our linear ordering on Γ. For $x, y \in \Gamma$, define

$$x \leq y \Leftrightarrow \begin{cases} |x| < |y| \\ \text{or } |x| = |y| \text{ and } x \leq_l y. \end{cases}$$

Clearly, Γ contains no infinite decreasing sequence with respect to the ordering $\leq$.

We define certain subsets, Q_ε, of Γ as follows. For each multi-sign $\varepsilon = (\varepsilon_1,\dots,\varepsilon_r) \in \{\pm\}^r$, put

$$Q_\varepsilon = \{(x_1,\dots,x_r) \in \Gamma: x_i \geq 0 \text{ if } \varepsilon_i = +,\ x_i < 0 \text{ if } \varepsilon_i = -\}.$$

Define $e(i)$ to be the element of Γ with 1 in the ith coordinate and zeros everywhere else. Then we let $e(1),\dots,e(r)$ be the canonical $\mathbb{Z}$-basis of Γ.

Claim Let $\emptyset \neq M \subseteq \Gamma$ be a finite subset of Γ. Let $m = (m_1,\dots,m_r) := \max\{M\}$ under the linear ordering definition given above. Suppose $m \in Q_\varepsilon$ where $\varepsilon = (\varepsilon_1,\dots,\varepsilon_r)$. Then

$$m + \varepsilon_i e(i) = \max\{ M + \{e(i), -e(i)\} \}.$$

Proof Clearly, $m + \varepsilon_i e(i) \leq \max\{ M + \{e(i), -e(i)\} \}$. Moreover, for all $x = (x_1,\dots,x_r) \in \Gamma$, $|x \pm e(i)| \in \{|x|+1, |x|-1\}$. In particular, if $x \in M$, then $|x \pm e(i)| \leq |m| + 1$. Now, since $m \in Q_\varepsilon$, $|m| + 1 = |m + \varepsilon_i e(i)|$. Also, for $x \in M$, $|x \pm e(i)| = |m| + 1$ implies $|x| = |m|$. But then we must have $x \leq_1 m$. Thus, if $\varepsilon_i = +$, then $x \pm e(i) \leq_1 m + \varepsilon_i e(i)$. So we may assume that $\varepsilon_i = -$. If $x_j < m_j$ for some $j < i$, then clearly, $x \pm e(i) < m + \varepsilon_i e(i)$. On the other hand, if $x_j \geq m_j$ for all $j < i$, then $x_j = m_j$ for all $j < i$ and $x_i \leq m_i < 0$. Hence, $|x + e(i)| = |x|-1 < |m|+1$, and $x-e(i) \leq_1 m + \varepsilon_i e(i)$. Thus, for all $x \in M$, we have $x \pm e(i) \leq m + \varepsilon_i e(i)$, as we have claimed.

<u>Step 2</u> Leading terms.

For each $0 \neq s \in S$, put

$$\lambda(s) := \max\{ \mathit{Supp}(s) \} \in \Gamma$$

using the ordering of Γ introduced in Step 1. For each non-zero R-submodule I of S, put

$$\lambda I := \lambda(I\setminus\{0\}) \subseteq \Gamma.$$

<u>claim</u> If $I \subseteq J$ are non-zero R-submodules of ${}_RS$, then $\lambda I \subseteq \lambda J$. If $I \subset J$, then $\lambda I \subset \lambda J$.

<u>proof</u> The first assertion is clear. Suppose $I \subset J$ but that $\lambda I = \lambda J$. Pick $s \in J\setminus I$ with $\lambda(s)$ as small as possible. By assumption, $\lambda(s) = \lambda(t) =: x \in \Gamma$ for some $t \in I$. But then for some $d \in D^*$, $x \notin \mathit{Supp}(s-dt)$ and so $\lambda(s-dt) < \lambda(s)$. Since $s-dt \in J\setminus I$, this contradicts the minimality of s.

For each basis vector $e(i) \in \Gamma$, put

$$b_i = \overline{e(i)} + \overline{e(i)}^{\sigma} = tr(\overline{e(i)}) \in R.$$

Note that $\overline{e(i)}^{\sigma} = d_i\overline{e(i)^{\pm 1}}$ for some $d_i \in D^*$ $(i=1,\ldots,r)$. (Here we temporarily revert to the multiplicative notation of C). Thus,

$$\mathit{Supp}(b_i s) \subseteq \mathit{Supp}(s) \pm \{e(i)\}$$

holds for $i=1,\ldots,r$ and for any non-zero $s \in S$. If $\lambda(s) \in Q_\varepsilon$ where $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_r)$, the claim in Step 1 easily implies that $\lambda(b_i s) = \lambda(s) + \varepsilon_i e(i) \in Q_\varepsilon$. Therefore, if $\lambda(s) \in Q_\varepsilon$, as above, then $\lambda(Rs) \supseteq \lambda(s) + \hat{Q}_\varepsilon$ where $\hat{Q}_\varepsilon = (\varepsilon_1\mathbb{N}_0,\ldots,\varepsilon_r\mathbb{N}_0)$. ($\mathbb{N}_0 = \{0,1,2,\ldots\}$).

Step3 Conclusion

Suppose $0 \subset I_1 \subset I_2 \subset \ldots$ is an infinite strictly ascending chain of left R-submodules of S. Then the preceding claim shows that $\lambda I_1 \subset \lambda I_2 \subset \lambda I_3 \subset \ldots$ and so we can select elements $a(i) \in \lambda I_i \backslash \lambda I_{i-1}$. By considering a suitable subchain if necessary, we may assume that all the $a(i)$ belong to the same Q_ε for some $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$. By step 2, we have that $a(i) + \hat{Q}_\varepsilon \subseteq \lambda I_i$ whence $a(t) \notin a(i) + \hat{Q}_\varepsilon$ for $t > i$. Write $a(i) = (\varepsilon_1 a_{1,i}, \ldots, \varepsilon_r a_{r,i})$ with $a_{j,i} \in \mathbb{N}_0$ and put $a(i)^* = (a_{1,i}, \ldots, a_{r,i}) \in \mathbb{N}_0{}^r$.

Consider the (partial) product ordering of $\mathbb{N}_0{}^r$ given by the natural ordering of $\mathbb{N}_0$:

$$(x_1, \ldots, x_r) \text{ precedes } (y_1, \ldots, y_r) \Leftrightarrow x_i \leq y_i \text{ for all } i.$$

Then the set $\{a(i)^*: \text{ for all } i\} \subseteq \mathbb{N}_0{}^r$ has finitely many minimal elements. Again, by considering a suitable subchain, we may assume that these elements are $a(1)^*, \ldots, a(p)^*$ for some $p \in \mathbb{N}$. By [D, Lemma 2.6.2], each $a(i)^*$ majorises at least one of $a(1)^*, \ldots, a(p)^*$.

Finally, let $t > p$. Since $a(t) \notin a(i) + \hat{Q}_\varepsilon$ for $i = 1, \ldots, p$, we must have $a_{j(i),t} < a_{j(i),i}$ for some $j(i) \in \{1, \ldots, r\}$. Consequently, $a(t)^*$ majorises none of $a(1)^*, \ldots, a(p)^*$, contradicting the previous paragraph.

We have therefore proved the theorem.

Although these methods were conceived with a view to answering Question 2A, a minor adaptation gives a result applicable to the question on group rings in Question 2B. This result will deal with one class of examples where S is the group algebra of the nth Heisenberg group. For the remainder of the chapter, we study fixed rings of group algebras of the nth Heisenberg group. We define the Heisenberg groups below.

2.2.4 DEFINITION For $n \in \mathbb{N}$, let H_n denote the nth *Heisenberg group*. Then $H_n = \langle x_i, y_i, z \ (1 \leq i,j \leq n)$: for all i,j

$$[x_i, x_j] = [y_i, y_j] = 1, \ [x_i, y_j] = z^{\delta_{ij}}, \ z \text{ central} \rangle.$$

Note here that H_n is nilpotent of class 2 since $Z(H_n) = \langle z\rangle$ and $H_n/\langle z\rangle \cong \mathbb{Z}^{2n}$. We will give a positive answer to Question 2B for infinitely many groups G, of order 2, acting on kH_n. We first study some automorphisms admitted by H_n.

2.2.5 NOTE Fix $n \in \mathbb{N}$. Consider the $2n$ maps, for $i=1,\dots,n$:

$$\theta_i: x_j \mapsto x_j z^{\delta ij}, \quad y_j \mapsto y_j \ , \quad (j=1,..,n) \quad z \mapsto z;$$
$$\mu_i: x_j \mapsto x_j \ , \quad y_j \mapsto y_j z^{\delta ij}, \quad (j=1,..,n) \quad z \mapsto z.$$

All these maps are automorphisms which commute and have infinite order so that $A := \langle \theta_i, \mu_i: i=1,\dots,n\rangle \cong \mathbb{Z}^{2n}$. Any automorphism of H_n not in A has a non-trivial action on $H_n/\langle z\rangle$.

Consider $\tau \in Aut(H_n)$ such that $\tau(x_i) = x_i^{-1}$, $\tau(y_i) = y_i^{-1}$ and $\tau(z) = z$ for $i=1,\dots,n$. Now, τ has order 2 and $\tau\theta_i\tau = \theta_i^{-1}$ and $\tau\mu_i\tau = \mu_i^{-1}$ for $i=1,\dots,n$. Thus, we may form the semidirect product $A_1 := \langle A, \tau\rangle = A \ \ \langle\tau\rangle$ which is Abelian-by-finite where τ acts by inversion on A. If $\alpha \in A_1 \backslash A$, then $\alpha^2 = 1$ and

$$x_i^{\alpha} = x_i^{-1}z^{u(i)}, \ y_i^{\alpha} = y_i^{-1}z^{v(i)} \text{ and } z^{\alpha} = z$$
$$\text{where } u(i), v(i) \in \mathbb{Z} \text{ for } i=1,\dots,n \qquad -(1).$$

Conversely, all automorphisms of the form (1) lie in $A_1 \backslash A$. We deal with these automorphisms in Corollary 2.2.8.

Now fix $J \subseteq \{1,\dots,n\}$ and define ω_J as follows:

$$\omega_J: x_i \mapsto x_i \ , \ y_i \mapsto y_i^{-1} \text{ for } i \in J,$$
$$x_i \mapsto x_i^{-1}, \ y_i \mapsto y_i \quad \text{for } i \notin J \text{ and } z \mapsto z^{-1}.$$

Then ω_J is an automorphism of H_n of order 2. Let $A_{2,J} = \langle A, \omega_J\rangle$. If $\alpha \in A_2$ has order 2, then

$$\omega_J: x_i \mapsto x_i \ , \quad y_i \mapsto y_i^{-1}z^{u(i)} \quad \text{where } u(i) \in \mathbb{Z} \text{ for } i \in J,$$
$$x_i \mapsto x_i^{-1}z^{u(i)}, \quad y_i \mapsto y_i \quad \text{where } u(i) \in \mathbb{Z} \text{ for } i \notin J$$
$$\text{and } z \mapsto z^{-1} \qquad -(2).$$

We give the result for these automorphisms in Corollary 2.2.10.

With $K \subseteq \{1,\ldots,n\}$, we define another set of automorphisms of H_n as follows. Define λ_K such that:

$$\lambda_K\colon x_i \mapsto y_i \quad , \; y_i \mapsto x_i \qquad \text{for } i \in K,$$

$$x_i \mapsto x_i^{\pm 1},\; y_i \mapsto y_i^{\mp 1} \text{ for } i \notin K \text{ and } z \mapsto z^{-1}.$$

Certainly, each λ_K is an automorphism of order *2* and, with certain provisos, we may combine them with automorphisms in *(2)* and those in *A* to form more automorphisms of order *2*. Elementary considerations show that any such automorphism, *g*, must be of the following form. Let *X,Y* and *Z* partition the set $\{1,\ldots,n\}$. Then

$$
\begin{aligned}
x_i g &= y_i z^{a(i)}, & y_i g &= x_i z^{a(i)} && \text{for } i \in X\\
x_i g &= x_i z^{a(i)}, & y_i g &= y_i^{-1} && \text{for } i \in Y\\
x_i g &= x_i^{-1} \quad, & y_i g &= y_i z^{a(i)} && \text{for } i \in Z
\end{aligned}
$$

$$\text{and } z^g = z^{-1} \text{ where } a(i) \in \mathbb{Z} \text{ for } i=1,\ldots,n \qquad -(3).$$

The fixed ring for an automorphism of the type *(3)* is studied in Corollary 2.2.10. Observe that any automorphism of type *(2)* is certainly of type *(3)* also.

When considering the automorphisms *(1)* in 2.2.5, we make the following definition in order to allow us to adapt Theorem 2.2.3.

2.2.6 DEFINITION Let $k[z,z^{-1}]$ be a Laurent polynomial ring in a commuting indeterminate *z* over a field *k*. For

$$f(z) = a_t z^t + a_{t+1} z^{t+1} + \ldots + a_{t+s} z^{t+s} \in k[z,z^{-1}],$$

with a_t and a_{s+t} non-zero, we define the *length of f, l(f)* to be *s*.

We now base our proof of the next result on the proof of Theorem 2.2.3.

2.2.7 PROPOSITION *Let S be the crossed product $k[z, z^{-1}] * \Gamma$ where k is a field, $\Gamma \cong \mathbb{Z}^r$ and, for $x, y \in \Gamma$, $\bar{x}.\bar{y} = \alpha(x,y)\overline{xy}$ where $\alpha(x,y) \in \langle z\rangle$. Suppose $G = \langle g\rangle$ where $g \in AutS$ has order 2, the action of g on $\Gamma \cong U(S)/\langle z\rangle$ is inversion and $z^g = z$. Then S is a left and right Noetherian S^G-module.*

PROOF As usual, we let R denote S^G. We prove ${}_RS$ is Noetherian.

Step 1 An ordering on a spanning set for S.

Now, S is a free $k\langle z\rangle$-module with basis Γ, and so every element is a sum of terms in

$$B = \{f(z)\bar{x}: f(z) \text{ is a polynomial in } k\langle z\rangle,\ x \in \Gamma\}.$$

We place an ordering on the set B. We have a map $\tau: B \to \mathbb{N}_0 \times \Gamma$ where $\tau(f(z)\bar{x}) = (n,x) \in \mathbb{N}_0 \times \Gamma$ where $n = l(f)$ as defined above.

We impose an ordering on $\mathbb{N}_0 \times \Gamma$ as follows:

$$(n,x) < (m,y) \iff \begin{cases} x < y \text{ where } < \text{ is the ordering of Theorem 2.2.3} \\ \text{or } x = y \text{ and } n < m. \end{cases}$$

Step 2 Leading Terms

We use the fact that S is a free $k\langle z\rangle$-module with basis Γ. For each $0 \neq s \in S$, let $Supp(s) = \{f(z)\bar{x} \in B: f(z) \text{ is the coefficient of } \bar{x} \text{ in } s\}$. Put $\varphi(s) = max(\tau\ (Supp(s))) \in \mathbb{N}_0 \times \Gamma$. For each non-zero R-submodule I of ${}_RS$, put $\varphi I = \varphi(I\setminus\{0\}) \subseteq \mathbb{N}_0 \times \Gamma$.

claim If $I \subseteq J$ are non-zero R-submodules of ${}_RS$, then $\varphi I \subseteq \varphi J$. If $I \subset J$, then $\varphi I \subset \varphi J$.

Proof The first assertion is clear. Suppose that $I \subset J$ and that $\varphi I = \varphi J$. Choose $s \in J\setminus I$ with $\varphi(s)$ minimal. Then there exists $t \in I$ with $\varphi(s) = \varphi(t) =: (n,x) \in \mathbb{N}_0 \times \Gamma$. So s has a term $f(z)\bar{x}$ and t has a term $g(z)\bar{x}$ for some $f(z), g(z) \in k[z,z^{-1}]$ where $l(f) = l(g) = n$. Suppose now that $deg(f) - deg(g) = u \in \mathbb{N}_0$. Then $z^u g(z) \in k\langle z\rangle$ has $l(z^u g(z)) = l(f(z))$ and $deg(f) = deg(z^u g)$. Hence, there exists $c \in k$ such that $\varphi(s - cz^u t) < (n,x)$. Since $cz^u \in U(R)$, we have that $s - cz^u t \in J\setminus I$. This contradicts the minimality of $\varphi(s)$ thereby establishing the claim.

Adopting the notation in the proof of Theorem 2.2.3 and reproducing the argument there, gives that, if $s \in S$ with $\varphi(s) = (n,x) \in \mathbb{N}_0 \times \Gamma$, then

$$\varphi(Rs) \supseteq \varphi(s) + (0, \hat{Q}\varepsilon)$$

where $x \in Q_\varepsilon$. Since $(1+z)^v \in R$ has length v, we also have that $\varphi((1+z)^v s) = (n+m, x)$. Together with the above containment, this gives:

$$\varphi(Rs) \supseteq \varphi(s) + (\mathbb{N}_0, \hat{Q}\varepsilon).$$

Step 3 Conclusion

Suppose $0 \subset I_1 \subset I_2$ is a strictly ascending chain of R-submodules of $_RS$. Then $\varphi I_1 \subset \varphi I_2 \subset \varphi I_3 \subset \ldots$ and so we can select elements $a(i) = (n(i), x(i)) \in \varphi I_i \backslash \varphi I_{i-1}$. We write each $a(i)$ as $(a_{1,i},\ldots,a_{r+1,i})$. Define $a(i)^*$ to be $(n(i),x(i)^*)$ where $x(i)^*$ is as defined in Step 3 of Theorem 2.2.3. By choosing a suitable subchain if necessary, we may assume that all the $x(i)$ belong to the same Q_ε. The set $\{a(i)^*: i=1,2,\ldots\}$ has finitely many minimal members. By the choice of a suitable subchain if necessary, we may assume that these minimal members are $a(1)^*,\ldots,a(p)^*$ for some $p \in \mathbb{N}$. Thus each $a(i)^*$ majorises at least one of $a(1)^*,\ldots,a(p)^*$.

Finally, let $t > p$. Since $a(t) \notin a(i) + (\mathbb{N}_0,\hat{Q}_\varepsilon)$, we must have $a_{j(i),t} < a_{j(i),i}$ for some $j(i) \in \{1,\ldots,r+1\}$. Consequently, $a(t)^*$ majorises none of $a(1)^*,\ldots,a(p)^*$. This contradiction proves the theorem.

This Proposition answers Question 2B for cases *(1)* in 2.2.5 where S is a group algebra of the nth Heisenberg group.

2.2.8 COROLLARY *Let H_n be the nth Heisenberg group for some $n \in \mathbb{N}$. Let $g \in Aut(H_n)$ be an automorphism of order 2 such that $x_i g = x_i^{-1} z^{u(i)}$, $y_i g = y_i^{-1} z^{v(i)}$ and $z^g = z$ for some $u(i),v(i) \in \mathbb{Z}$ $(i=1,\ldots,n)$. Let k be a field and S the group algebra kH_n. Now, G acts as k-automorphisms on the ring S and S^G is Noetherian.*

PROOF Writing S as $k[z, z^{-1}] * (H_n/\langle z\rangle)$, we see that the hypotheses of Proposition 2.2.7 apply.

A more direct approach will answer Question 2B for the automorphisms *(2)* in 2.2.5.

2.2.9 PROPOSITION *Let H_n be the nth Heisenberg group as defined in 2.2.4. Let $J \subseteq \{1,\ldots,n\}$ and define $g \in \mathrm{Aut}(H_n)$ as follows:*

$$x_i^g = x_i z^{u(i)},\quad y_i^g = y_i^{-1} \qquad \text{for } i \in J$$

$$\text{and } x_i^g = x_i^{-1},\qquad y_i^g = y_i z^{u(i)} \quad \text{for } i \notin J$$

$$\text{and } z^g = z^{-1} \quad \text{where } u(1),\ldots,u(n) \in \mathbb{Z}.$$

Let k be any field and $G = \langle g\rangle$. Then $(kH_n)^G$ is Noetherian.

PROOF We adopt the notation of Definition 2.2.4. For $i \in J$, put $w_i = y_i$ and, for $i \notin J$, put $w_i = x_i$. Let $L = \langle z, w_1,\ldots,w_n\rangle$. Then L is an Abelian, G-invariant subgroup of H_n. Theorem 1.4.4 shows that $(kL)^G$ is Noetherian. Let $v_i = x_i^2 z^{u(i)}$ for $i \in J$ and $v_i = y_i^2 z^{u(i)}$ for $i \notin J$. Then,

$$\text{for } i \in J,\quad v_i^g = x_i^2 z^{2u(i)} z^{-u(i)} = x_i^2 z^{u(i)} = v_i$$

$$\text{and, for } i \notin J,\quad v_i^g = y_i^2 z^{2u(i)} z^{-u(i)} = y_i^2 z^{u(i)} = v_i.$$

Thus, $v_i \in (kH_n)^G$ for $i=1,\ldots,n$. Let $H_n' = \langle v_1,\ldots,v_n, L\rangle = \langle x_i^2, y_i, x_j, y_j^2, z : i \in J, j \notin J\rangle$. We claim that $(kH_n')^G$ is just the Laurent polynomial ring

$$(kL)^G [v_1, v_1^{-1}; \tau_1] \ldots [v_n, v_n^{-1}; \tau_n] \qquad -(*)$$

where τ_i denotes conjugation by v_i for $i = 1,\ldots,n$. Certainly, $(kH_n')^G \supseteq (kL)^G [v_1, v_1^{-1}; \tau_1] \ldots [v_n, v_n^{-1}; \tau_n]$.

Suppose now that $r \in (kH_n')^G$, so that

$$r = \Sigma_j\, f_j(z, v_1,\ldots,v_{n-1}, w_1,\ldots,w_n) v_n^j \qquad -(1)$$

where $f_j(z,v_1,\ldots,v_{n-1},w_1,\ldots,w_n) \in k\langle z,v_1,\ldots,v_{n-1},w_1,\ldots,w_n\rangle$ because kH_n' is a free $k\langle z,v_1,\ldots,v_{n-1},w_1,\ldots,w_n\rangle$-module with basis $\{v_n^i : i \in \mathbb{Z}\}$. Now, since $r \in (kH_n')^G$,

$$r = r^g = \Sigma_j \ (f_j(z,v_1,\dots,v_{n-1},w_1,\dots,w_n))^g v_n{}^j \qquad -(2).$$

A comparison of the expressions *(1)* and *(2)* shows that

$$(f_j(z,v_1,\dots,v_{n-1},w_1,\dots,w_n))^g = f_j(z,v_1,\dots,v_{n-1},w_1,\dots,w_n)$$

for all j. We've shown that

$$(kH_n{}')^G = (k\langle z,v_1,\dots,v_{n-1},w_1,\dots,w_n\rangle)^G[v_n,v_n{}^{-1};\tau_n].$$

We may repeat this argument by expressing $k\langle z,v_1,\dots,v_{n-1},w_1,\dots,w_n\rangle$ as a free $k\langle z,v_1,\dots,v_{n-2},w_1,\dots,w_n\rangle$-module. Continuing this way, we establish the claim *(*)*.

By [G-W, Theorem 1.17], $(kH_n{}')^G$ is then Noetherian. Since $|H_n:H_n{}'| < \infty$, $(kH_n)^G$ is itself Noetherian by Theorem 2.1.6.

A corollary to this proposition will answer Question 2B for the nth Heisenberg group when an automorphism of type *(3)* in 2.2.5 is acting.

2.2.10 COROLLARY *Let $n \in \mathbb{N}$ and let H_n be the nth Heisenberg group as defined in 2.2.4. Let X, Y and Z partition the set $\{1,\dots,n\}$. Define $g \in Aut(H_n)$ as follows:*

$$\begin{aligned} x_i{}^g &= y_i z^{a(i)}, & y_i{}^g &= x_i z^{a(i)} && \text{for } i \in X \\ x_i{}^g &= x_i z^{a(i)}, & y_i{}^g &= y_i{}^{-1} && \text{for } i \in Y \\ x_i{}^g &= x_i{}^{-1}, & y_i{}^g &= y_i z^{a(i)} && \text{for } i \in Z \end{aligned}$$

$$\text{and } z^g = z^{-1} \text{ where } a(i) \in \mathbb{Z} \text{ for } i=1,\dots,n.$$

Let k be a field and let $G = \langle g\rangle$. Then $(kH_n)^G$ is Noetherian.

PROOF Let $u_i = x_i y_i{}^{-1}$, $v_i = x_i{}^2 y_i{}^2$ for $i \in X$, $u_i = x_i{}^4$, $v_i = y_i$ for $i \in Y$, $u_i = x_i$, $v_i = y_i{}^4$ for $i \in Z$ and let $w = z^4$. Define $H_n{}'$ to be the subgroup of H_n generated by $u_1,\dots,u_n,v_1,\dots,v_n,w$. It's easy to see that $\tau: H_n{}' \to H_n$ where $\tau(u_i) = x_i$, $\tau(v_i) = y_i$ and $\tau(w) = z$ for $i=1,\dots,n$ is an isomorphism. Now, it's routine to check that, for $i \in X$, $u_i{}^g = u_i{}^{-1}$, $v_i{}^g = v_i w^{a(i)-1}$, for $i \in Y$, $u_i{}^g = u_i w^{a(i)}$, $v_i{}^g = v_i{}^{-1}$ and, for $i \in Z$, $u_i{}^g = u_i{}^{-1}$ and $v_i{}^g = v_i w^{a(i)}$. By the previous lemma with $J = Y$, we have that $(kH_n{}')^G$ is

Noetherian. As $|H_n:H_n'| < \infty$, Theorem 2.1.7 shows that $(kH_n)^G$ is Noetherian.

These results are the best we have for the nth Heisenberg group. They do in fact cover all automorphisms of order *2* of the first Heisenberg group as the next Corollary shows.

2.2.11 COROLLARY *Let $H = \langle x, y, z: [x, y] = z, z \text{ central}\rangle$ be the first Heisenberg group, k any field and φ any automorphism of H of order 2. Let $G = \langle\varphi\rangle$. Then $(kH)^G$ is Noetherian:*

PROOF Note first that if k has characteristic other than *2*, then $(kH)^G$ is Noetherian by Lemma 1.4.2. Thus, we may assume that $char\ k = 2$.

The method of proof here is to describe all the automorphisms of H of order *2* and show that Corollary 2.2.8, Corollary 2.2.10 or Proposition 2.2.9 apply.

Let φ be such an automorphism. Then φ is completely specified by its action on x and y because these elements generate H. Suppose $\varphi(x) = x^r y^t z^u$ and $\varphi(y) = x^l y^m z^n$ for some $l,m,n,r,s,t \in \mathbb{Z}$. Since $\langle z\rangle = Z(H)$ is a characteristic subgroup, φ acts on $H/\langle z\rangle \cong \mathbb{Z}^2$ and so we may associate with φ a member of the set $U := \{X \in GL_2(\mathbb{Z}): detX = \pm 1\}$, dependant on its action on $\mathbb{Z}^2$. Using additive notation for $H/\langle z\rangle$ to identify $x^i y^j + Z$ with $(i, j) \in \mathbb{Z}^2$, we have $\varphi(x^i y^j + Z) = x^a y^b + Z$ where $(i\ j)X = (a\ b)$. Suppose $\varphi(x + Z) = x^r y^t + Z$ and $\varphi(y + Z) = x^l y^m + Z$. Then we find in this case that:

$$X = \begin{bmatrix} r & t \\ l & m \end{bmatrix}.$$

Hence, for any such automorphism φ we may use a triple (X, u, n) where $X \in U$, $u, n \in \mathbb{Z}$ to specify φ.

We now see what the possibilities for such a φ actually are. Since $\varphi^2 = id$, it is the case that $X^2 = 1$. According to [Ne, Pages 179-181], we have that the possibilities for X, up to conjugation, are:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The automorphisms for which the first matrix is the associated matrix are covered by Corollary 2.2.8. If φ is associated with the second matrix, we see that $u = n$ because $\varphi^2(x) = x$, and so, Corollary 2.2.10 gives the result. Finally, suppose that φ is associated with the third matrix. Now $y = \varphi^2(y) = \varphi(y^{-1}z^n) = yz^{-n}z^{-n} = yz^{-2n}$ and so $n = 0$. An application of Proposition 2.2.9 completes the Corollary.

CHAPTER 3

PRIME IDEALS IN THE RING OF INVARIANTS

In this chapter, we are concerned with developing the Morita correspondence of Theorem 1.2.9 and showing how these results may be applied. We adopt the notation of Theorem 1.2.9 so that S is any ring, G is a finite group of automorphisms of S, T denotes the skew group ring $S*G$ and R denotes the fixed ring S^G.

The Morita correspondence between the appropriate subsets of the prime spectra of R and T and its consequences are well understood when $|G|^{-1} \in S$. S. Montgomery has collated the known results in this case in [Mo2]. In §3.1, we make no hypothesis on the order of the group and provide generalisations for many of the results in [Mo2]. For example, we have, in the terminology of Definition 3.1.1:

3.1.9 THEOREM *Let S be a ring and G a finite group of automorphisms of S. Suppose $\hat{P} \in Spec_f T$ and $P \in Spec S$. Then $\beta(\hat{P})$ is minimal over $P \cap R$ if and only if $\hat{P}$ is minimal over P^o*G. In particular, $\beta(\hat{P})$ is minimal over $\hat{P} \cap R$.*

Theorem 3.1.21 generalises what is perhaps the fundamental result in Montgomery's paper, namely [Mo2 Theorem 2.1].

3.1.21 THEOREM *Let S be a ring and G a finite group of automorphisms of S.*

(i) Given $P \in Spec_f S$, there are a finite number of primes in $Spec_t R$ minimal over $P \cap R$, $\{p_1, p_2, \ldots, p_m\}$ say, with $m \leq |G|$. Also, $(\cap_i p_i)tr(S)$ is nilpotent modulo $P \cap R$.

(ii) Given $p \in Spec_t R$, there exists $P \in Spec_f S$ such that p is minimal over $P \cap R$. Moreover, P is unique up to its G-orbit in $Spec S$.

Due to the similarity in content, we use [Mo2] as a model for our results in §3.1.

While Montgomery deals with the special case where $|G|^{-1} \in S$, we devote the second section here to the other extreme case, namely where $|H| = 0$ in S for all non-trivial subgroups H of G. This usually involves the characteristic of S being prime, q say, and G being a q-group. We can then utilise, for example, Proposition 1.2.12. With $Spec_I S = \{P \in SpecS: SfS \cap S \not\subseteq P\}$, we provide a special case of Theorem 3.1.21:

3.2.13 THEOREM *Let S be a ring of characteristic q and G a subgroup of Aut S of order q^a. Then*

(i) Given $P \in Spec_I S$, there exists $p \in Spec_t R$ such that p is the unique prime minimal over $P \cap R$ not containing the trace.

(ii) Given $p \in Spec_t R$, there exists $P \in Spec_I S$ such that p is minimal over $P \cap R$. Moreover P is unique up to its G-orbit.

We close this chapter, in §3.3, with applications of the earlier results. Some relate to the general case of §3.1 while others are in the prime characteristic setting of §3.2. We state two of the more useful applications.

3.3.8 LEMMA *Let S be a ring and G a finite group of automorphisms of S. Suppose $p,q \in Spec_t R$ both lie under $P \in SpecS$. Then $ht(p) = ht(q) = ht(P)$.*

3.3.24 COROLLARY. *Let K be a commutative ring and let S be a K-algebra acted on by G, a group of K-automorphisms. Suppose S satisfies the Nullstellensatz over K. Suppose further that $R/tr_G(S)$ also satisfies the Nullstellensatz over K. Then R must also have this property.*

§3.1 General Situation

We begin this section with a series of definitions which will prove helpful in the discussion.

3.1.1 DEFINITION Recall from 1.1.1 that $f = \Sigma_{g\epsilon G}\, g \epsilon T$, $I = TfT \cap S$ and for $X \subseteq S$, $X^o := \cap_{g\epsilon G} X^g$. There are subsets of the various spectra which we look at. These are:

$$Spec_t R = \{p \in SpecR : tr(S) \nsubseteq p\};$$

$$Spec_I S = \{P \in SpecS : I \nsubseteq P\};$$

$$Spec_f S = \{P \in SpecS : f \notin \surd(P^o*G)\};$$

$$Spec_f T = \{\hat{P} \in SpecT : f \notin \hat{P}\}.$$

We give an easy lemma to show that there is no distinction between $Spec_f S$ and $SpecS$ when $|G|^{-1} \in S$.

3.1.2 LEMMA *Let S be a ring and G a finite group of automorphisms of S with the property that $|G|^{-1} \in S$. Then $Spec_f S = SpecS$.*

PROOF Certainly, $Spec_f S \subseteq SpecS$. Now, let $P \in SpecS$ and suppose that $f \in \surd(P^o*G)$. By Theorem 1.2.11(iii), $\surd(P^o*G)$ is nilpotent modulo P^o*G and so there exists $n \in \mathbb{N}$ such that $f^n \in P^o*G$. Now,

$$f^2 = (\Sigma_{g\epsilon G} g)f = \Sigma_{g\epsilon G}\, gf = \Sigma_{g\epsilon G}\, f = |G|f.$$

So it's easy to see that $f^n = |G|^{n-1}f$. Since $|G|$ is a unit in S, we have that $f \in P^o*G$. Comparing coefficients shows that $1 \in P^o$. This contradiction proves the lemma.

The definitions in 3.1.1 are used in the next two definitions.

3.1.3 DEFINITION There are three equivalence relations which we define on certain spectra.

(i) We already have the notion of *G-conjugate* primes in *SpecS*.

(ii) For $\hat{P}$, $\hat{Q} \in SpecT$ we say that $\hat{P}\ \rho\ \hat{Q}$ if and only if $\hat{P} \cap S = \hat{Q} \cap S$.

(iii) For p, $q \in Spec_t R$, we say that $p \sim q$ if and only if p and q are both minimal over $P \cap R$ for some $P \in SpecS$.

We elaborate on each of these definitions in turn.

The first definition is already understood and is easily seen to be an equivalence relation. For $P \in SpecS$, we let $[P]$ denote the class of all G-conjugates of P. We may also define a partial ordering on $SpecS/G$ as follows. We say $[P] \subseteq [Q]$ if there exists $h \in G$ such that $P^h \subseteq Q$. Of course, we may refer to G-conjugacy on the subset $Spec_f S$.

By inspection, the relation ρ on $SpecT$ is an equivalence relation. Lemma 1.2.10 and Theorem 1.2.11(i) show in fact that $\hat{P}\ \rho\ \hat{Q}$ if and only if there exists $P \in SpecS$ such that $\hat{P}$ and $\hat{Q}$ are both minimal over $P^o{*}G$. We denote the ρ-class containing $\hat{P}$ by $[\hat{P}]$. We may also define a partial ordering on $SpecT/\rho$ as follows: $[\hat{P}] \subseteq [\hat{Q}]$ if there exists $\hat{P}_1 \in [\hat{P}]$ and $\hat{Q}_1 \in [\hat{Q}]$ such that $\hat{P}_1 \subseteq \hat{Q}_1$. To see that this actually defines a partial ordering, suppose $[\hat{P}] \subseteq [\hat{Q}]$ and that $[\hat{P}] \supseteq [\hat{Q}]$. By definition, there exist $\hat{P}_1, \hat{P}_2 \in [\hat{P}]$, $\hat{Q}_1, \hat{Q}_2 \in [\hat{Q}]$ with $\hat{P}_1 \subseteq \hat{Q}_1$ and $\hat{P}_2 \supseteq \hat{Q}_2$. Intersecting these inequalities down to S, we find, by Lemma 1.2.10, that $P^o = Q^o$ where $P, Q \in SpecS$ such that $\hat{P}_1 \cap S = \hat{P}_2 \cap S = P^o$ and $\hat{Q}_1 \cap S = \hat{Q}_2 \cap S = Q^o$. The definition of ρ shows that $[\hat{P}] = [\hat{Q}]$. Now, ρ is also an equivalence relation on $Spec_f T$. It's worth noting that Theorem 1.2.11(i) and Proposition 1.2.12 show that when $char\ S = q$ and $|G| = q^a$ (q prime, $a \in \mathbb{N}$) ρ collapses to the trivial relation.

At the moment it is only clear that $\sim$ is a symmetric relation on $Spec_t R$. It is non-trivial to see that the reflexive and transitive properties also hold. We establish these properties in Theorem 3.1.9.

We examine certain topological properties of the spaces *SpecR*, *SpecS* and *SpecT* with respect to the Zariski topologies. These topologies are defined as follows.

3.1.4 DEFINITION We define the Zariski topology on *SpecS*. The closed sets of the Zariski topology on *SpecS* are defined as follows: let Y be a subset of S, then the closed sets are of the form $v(Y) := \{ P \in SpecS : P \supseteq Y \}$ where we may assume that Y is an intersection of primes.

We have a Zariski topology defined on *SpecR* with the closed sets defined to be $u(X) = \{ p \in SpecR : p \supseteq X \}$ where X is a subset of R which we may assume to be an intersection of primes in *SpecR*.

Similarly, for T we have a Zariski topology on *SpecT* with closed sets $w(Z) = \{ \hat{P} \in SpecT: Z \subseteq \hat{P} \}$ where Z is a subset of T which we may assume to be an intersection of primes of T.

Of interest are certain associated topologies. First, we have the topologies on the open subsets $Spec_tR$, $Spec_fS$ and $Spec_fT$ which are induced by the Zariski topologies on *SpecR*, *SpecS* and *SpecT*.

We also define some quotient Zariski topologies. A general explanation of their construction is given here. Suppose σ is an equivalence relation on $Spec\ W$ for some ring W and that $\pi : SpecW \to (Spec\ W)/\sigma$ is the projection map. Then $U \subsetneq (SpecW)/\sigma$ is said to be closed if and only if $\pi^{-1}(U)$ is closed in $Spec\ W$. Thus, we have quotient Zariski topologies on $SpecS/G$, $Spec_fS/G$, $SpecT/\rho$ and $Spec_fT/\rho$. Later, once we have established that $\sim$ is an equivalence relation on $Spec_tR$, we will also have the quotient Zariski topolgy on $Spec_tR/\sim$.

The first result in this section is fundamental as it provides a basis for all else that follows. It is essentially the prime correspondence of the Morita context already stated in Theorem 1.2.9. We expand on that basic result here.

3.1.5 THEOREM *Let S be a ring and G a finite subgroup of $AutS$. Let*

$$\Delta = \{ N < T : N = \cap\hat{P}_i,\ \hat{P}_i \in Spec_f T \}$$

$$\text{and } \Omega = \{ J < R : J = \cap p_i,\ p_i \in Spec_t R \}$$

and define a map $\hat{\beta} : \Delta \to \Omega$ where $\hat{\beta}(\cap\hat{P}_i) = \{ r \in R : rf \in \cap\hat{P}_i \}$. Let β be the following restriction of $\hat{\beta}$, β: $Spec_f T \to Spec_t R$ where $\beta(\hat{P}) = \hat{\beta}(\hat{P})$. Then:

(i) $\hat{\beta}$ is an order preserving, intersection preserving map;

(ii) β is the prime correspondence of the Morita context;

(iii) the inverse of β is $\beta^{-1}(p) = \{t \in T: tr(S.t.S) \subseteq p\}$ where the dot denotes the T-module action on S;

(iv) β is a homeomorphism with respect to Zariski topologies on $Spec_f T$ and $Spec_t R$.

Moreover, the restriction of β to primitive ideals, β_{pv}:$PrimSpec_f T \to PrimSpec_t R$, given by $\beta_{pv}(\hat{P}) = \beta(\hat{P})$ is also a bijection.

PROOF Since $\beta(\hat{P}) = \{r \in R: rf \in \hat{P}\}$, properties (ii) and (iii) are immediate from Theorem 1.2.9. The properties of $\hat{\beta}$ follow from those of β stated in (ii) and (iii).

We now show that β is a homeomorphism. First, we recall the definitions of closed sets in $Spec_t R$ and $Spec_f T$. A closed set in $Spec_t R$ is of the form $u_t(X) := \{p \in Spec_t R: p \supseteq X\}$ where X is a subset of R. A closed set in $Spec_f T$ is of the form $w_f(Z) := \{ \hat{P} \in Spec_f T: \hat{P} \supseteq Z\}$ where Z is a subset of T. Suppose $w_f(Z)$ is such a subset. Clearly, we may assume that $Z = \cap_{\lambda\in\Lambda}(\hat{P}_\lambda)$ where the $\hat{P}_\lambda$ are all the primes in $w_f(Z)$. Now, $\hat{\beta}$ preserves intersections as noted above and so $\hat{\beta}(Z) = \cap_\Lambda\beta(\hat{P}_\lambda)$. With $X = \hat{\beta}(Z)$, it's clear that $\beta(w_f(Z)) = u_t(X)$, a closed set in $Spec_t R$. Thus, β is a continuous map.

We now show that β^{-1} is a continuous map. Let $u_t(X)$ be a closed set in $Spec_t R$. As before, we may assume that X is an intersection of primes in $Spec_t R$ in that $X = \cap_{\gamma\in\Gamma} p_\gamma$ for some $p_\gamma \in Spec_t R$. Let Z denote $\cap_{\gamma\in\Gamma}\beta^{-1}(p_\gamma)$, an intersection of primes in $Spec_f T$. It's easy to see that $\beta^{-1}(u_t(X)) = w_f(Z)$, a closed set in $Spec_f T$. Thus, β^{-1} is a continuous map. Hence, we've shown that β is a homeomorphism.

Finally, Proposition 1.2.4 shows that β is a bijection between the subsets of primitives.

It's worth making an observation regarding β here.

3.1.6 NOTE Let $\hat{P} \in Spec_f T$. Then there are two cases:

either *(i)* $\hat{P} = (\hat{P} \cap S)T$ – then, by definition of β,

$$\beta(\hat{P}) = \hat{P} \cap R$$

or *(ii)* $\hat{P} \supset (\hat{P} \cap S)T$ – $\beta(\hat{P}) \supsetneq \hat{P} \cap R$.

The following lemma is critical in providing us with another characterisation of the map β.

3.1.7 LEMMA *Let S be a ring and G a finite group of automorphisms of S. Suppose $P \in Spec_f S$ and let $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n$ be all the primes of T minimal over $P^o{*}G$, not containing f. Let $p_i \in Spec_t R$ such that $\beta(\hat{P}_i) = p_i$ $(i = 1, \ldots, n)$ and let $N = \cap_i p_i$. Then $(Ntr_G(S))^{|G|} \subseteq P \cap R$.*

NOTE *It follows from the definition of $Spec_f S$ that $n \geq 1$. In addition, $n \leq |G|$ by Theorem 1.2.11(i).*

PROOF Let $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n, \hat{P}_{n+1}, \ldots, \hat{P}_m$ be all the primes of T minimal over $P^o{*}G$ with $p_1, \ldots, p_n$ and N as stated. By Theorem 1.2.11(i), $m \leq |G|$. By construction, we have $f \in \hat{P}_{n+1} \cap \ldots \cap \hat{P}_m$ while, by definition of β, $Nf \subseteq \hat{P}_1 \cap \hat{P}_2 \cap \ldots \cap \hat{P}_n$. Combining these two facts we have that $Nf \subseteq \hat{P}_1 \cap \hat{P}_2 \cap \ldots \cap \hat{P}_m$ so that $NfS \subseteq \hat{P}_1 \cap \hat{P}_2 \cap \ldots \cap \hat{P}_m$. Applying Theorem 1.2.11(iii) gives us that $(NfS)^{|G|} \subseteq P^o{*}G$. Now, for $s \in S$,

$$fsf = fs\Sigma_{g \in G} g = f\Sigma_{g \in G} sg = f\Sigma_{g \in G} g s^g = f\Sigma_{g \in G} s^g = ftr(s)$$

and so $fSf = ftr(S)$. Using this and the fact that $R \subseteq C_T(f)$, we have that

$$(NfS)^2 = NfS.NfS = NfSfNS = Nftr(S)NS = Ntr(S)NfS.$$

Repeating this process, we find that $(Ntr(S))^{|G|-1}NfS = (NfS)^{|G|} \subseteq P^o{*}G$. Thus, $(Ntr(S))^{|G|}f = (Ntr(S))^{|G|-1}NfSf \subseteq P^o{*}G$. Comparing coefficients, $(Ntr(S))^{|G|} \subseteq P^o$. Moreover $N \subseteq R$ and $tr(S) \subseteq R$ giving that $(Ntr(S))^{|G|} \subseteq P \cap R$ as claimed.

The above lemma enables us to distinguish between those primes in $Spec_fS$ and those not in $Spec_fS$.

3.1.8 LEMMA *Let S be a ring, G a finite group of automorphisms of S, $P \in SpecS$. Then the following are equivalent:*

(i) $P \notin Spec_fS$;

(ii) $(tr(S))^n \subseteq P \cap R$ for some $n \in \mathbb{N}$;

(iii) $(tr(S))^{|G|} \subseteq P \cap R$.

PROOF First we prove (i) $\Rightarrow$ (iii). Suppose $P \notin Spec_fS$. Thus we have that $f \in \sqrt{(P^o{*}G)}$ and so, by Theorem 1.2.11(iii), $(fS)^{|G|} \subseteq P^o{*}G$. Now, $(tr(S))^{|G|}f = (fS)^{|G|}f$ as noted in the proof of Lemma 3.1.7. Comparing coefficients we have $(tr(S))^{|G|} \subseteq P^o$ and so $(tr(S))^{|G|} \subseteq P \cap R$.

That (iii) $\Rightarrow$ (ii) is vacuous.

For (ii) $\Rightarrow$ (i), suppose now that $P \in Spec_fS$. Then there exists $\hat{P} \in Spec_fT$ minimal over $P^o{*}G$. Theorem 3.1.5 shows that $\beta(\hat{P})$ contains $\hat{P} \cap R = P \cap R$. Since $tr(S) \not\subseteq \beta(\hat{P})$, $tr(S) \not\subseteq \sqrt{(P \cap R)}$ and so, in particular, $(tr(S))^n \not\subseteq P \cap R$ for all $n \in \mathbb{N}$. This proves the lemma.

We now employ Lemma 3.1.7 to give an interpretation of the map β that is more intuitive than Theorem 3.1.5.

3.1.9 THEOREM *Let S be a ring and G a finite group of automorphisms of S. Suppose $\hat{P} \in Spec_f T$ and $P \in SpecS$. Then $\beta(\hat{P})$ is minimal over $P \cap R$ if and only if $\hat{P}$ is minimal over $P^o{*}G$. In particular, $\beta(\hat{P})$ is minimal over $\hat{P} \cap R$.*

PROOF By definition of β, $\hat{P} \cap R \subseteq p =: \beta(\hat{P})$. Let $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n$ be all the members of $Spec_f T$ minimal over $(\hat{P} \cap S)T$. Let $p_1, \ldots, p_n \in Spec_t R$ be the corresponding primes under the Morita context so that $\beta(\hat{P}_i) = p_i$ $(i=1,\ldots,n)$ and let $N = \cap_i p_i$.

Suppose $\hat{P}$ is minimal over $P^o{*}G$. Then, by Theorem 1.2.11(i), $\hat{P} \cap S = P^o$ and we may assume $\hat{P} = \hat{P}_1$. Suppose that p is not minimal over $\hat{P} \cap R$, so that there exists $q \in SpecR$ with $\hat{P} \cap R \subseteq q \subset p$. Since $tr(S) \not\subseteq p$, $tr(S) \not\subseteq q$. By Lemma 3.1.7, $Ntr(S) \subseteq \sqrt{(\hat{P} \cap R)} = \sqrt{(P \cap R)} \subseteq q$. Since $tr(S) \not\subseteq q$ and q is prime, we have that $N \subseteq q$. Thus, there exists $j \in \{1,\ldots,n\}$ such that $p_j \subseteq q$. Thus, $\beta^{-1}(p_j) \subseteq \beta^{-1}(q) \subset \beta^{-1}(p_1)$ so that $\hat{P}_j \subset \hat{P}_1$. This contradiction proves the reverse direction.

Suppose conversely that $\beta(\hat{P})$ is minimal over $P \cap R$. By the above $Ntr(S) \subseteq \sqrt{(P \cap R)} \subseteq \beta(\hat{P})$ and so, since $tr(S) \not\subseteq \beta(\hat{P})$ and $\beta(\hat{P})$ is prime, there exists $k \in \{1,\ldots,m\}$ such that $p_k \subseteq \beta(\hat{P})$. Since $p_k \supseteq P \cap R$ and $\beta(\hat{P})$ is minimal over $P \cap R$, we have $\beta(\hat{P}_k) = p_k = \beta(\hat{P})$. Thus, $\hat{P} = \hat{P}_k$ is minimal over $P^o{*}G$.

For the last part, $\hat{P} \cap S = Q^o$ for some $Q \in SpecS$ by Lemma 1.2.10. Thus, $\hat{P}$ is minimal over $Q^o{*}G$ by Theorem 1.2.11(ii) and so, by the above, $\beta(\hat{P})$ is minimal over $Q^o \cap R = \hat{P} \cap R$.

This theorem enables us to show that the relation $\sim$ on $Spec_t R$, defined in 3.1.3(iii) is in fact an equivalence relation. We first establish that $\sim$ is reflexive. Let $p \in Spec_t R$ and use Theorem 3.1.5 to find $\hat{P} = \beta^{-1}(p) \in Spec_f T$. By Lemma 1.2.10, there exists $P \in SpecS$ such that $\hat{P}$ is minimal over $P^o{*}G$. Applying Theorem 3.1.9 yields that p is minimal over $P \cap R$ and we have established that $p \sim p$. Now we show that $\sim$ is transitive.

For, if $p,q,a \in Spec_tR$ with $p \sim q$ and $q \sim a$ then by definition of $\sim$ there exist $P,Q \in SpecS$ with p and q both minimal over $P \cap R$ and q and a both minimal over $Q \cap R$. Theorem 3.1.9 shows that $\beta^{-1}(p)$ and $\beta^{-1}(q)$ are both minimal over P^o*G while $\beta^{-1}(q)$ and $\beta^{-1}(a)$ are minimal over Q^o*G. Theorem 1.2.11(i) shows that $P^o = \beta^{-1}(p) \cap S = \beta^{-1}(q) \cap S = \beta^{-1}(a) \cap S = Q^o$. Thus p and a are both minimal over $P \cap R = Q \cap R$ and so $p \sim a$. This shows that $\sim$ is transitive and therefore an equivalence relation. It should be noted that, when $charS = q$ and G is a q-group, then $\sim$ is the trivial relation. In this case, suppose $p \sim q$. Then Theorem 3.1.9 shows that $\beta^{-1}(p)\ \rho\ \beta^{-1}(q)$. But, as already noted in 3.1.3, ρ is trivial in the q-case and so $\beta^{-1}(p) = \beta^{-1}(q)$, proving that $p = q$. We may also define a partial ordering on $(Spec_tR)/\sim$ as follows: $[p] \subseteq [q]$ if and only if there exists $p_1 \in [p]$, $q_1 \in [q]$ such that $p_1 \subseteq q_1$.

We can now prove the following theorem which generalises [Mo2, Theorem 5.1].

3.1.10 THEOREM *Let S be a ring and G a finite subgroup of Aut S. The induced map $\overline{\beta}$: $(Spec_fT)/\rho \to (Spec_tR)/\sim$ such that $\overline{\beta}([\hat{P}]) = [\beta(\hat{P})]$ is a well defined order preserving homeomorphism. Let $\overline{\beta}_{pv}$: $(PrimSpec_fT)/\rho \to (PrimSpec_tR)/\sim$ be the restriction of $\overline{\beta}$. Then $\overline{\beta}_{pv}$ is also an order preserving homeomorphism.*

PROOF Suppose $\hat{P},\hat{Q} \in Spec_fT$ such that $\hat{P}\ \rho\ \hat{Q}$. Thus, by definition of ρ, $\hat{P} \cap S = \hat{Q} \cap S = P^o$ for $P \in Spec_fS$, say. By Theorem 3.1.9, $\beta(\hat{P})$ and $\beta(\hat{Q})$ are both minimal over $P \cap R$ and so $\beta(\hat{P}) \sim \beta(\hat{Q})$. We've thus shown that $\overline{\beta}$ is well defined. Now suppose $[\hat{P}] \subseteq [\hat{Q}]$ so that there exist $\hat{P}_1 \in [\hat{P}]$ and $\hat{Q}_1 \in [\hat{Q}]$ with $\hat{P}_1 \subseteq \hat{Q}_1$. Since β preserves inclusions, $\beta(\hat{P}_1) \subseteq \beta(\hat{Q}_1)$ and so $\overline{\beta}([\hat{P}_1]) = [\beta(\hat{P}_1)] \subseteq [\beta(\hat{Q}_1)] = \overline{\beta}([\hat{Q}_1])$.

We now show that $\overline{\beta}$ is a homeomorphism. First, we show what the respective closed sets are. A closed set in $Spec_t R$ is of the form $u_t(X) := \{p \in Spec_t R: p \supseteq X\}$ where X is a subset of R. We may assume X is an intersection of primes in $Spec_t R$. Let $\varphi: Spec_t R \to (Spec_t R)/\sim$ be the projection map so that the closed sets of $(Spec_t R)/\sim$ are precisely the sets x such that $\varphi^{-1}(x) = u_t(X)$ for some $X \subseteq R$. A closed set in $Spec_f T$ is of the form $w_f(Z) := \{ \hat{P} \in Spec_f T: \hat{P} \supseteq Z\}$ where Z is a subset of T. As above, we may assume that Z is an intersection of primes in $Spec_f T$. Let $\gamma: Spec_f T \to (Spec_f T)/\rho$ be the projection map. Then the closed subsets of $(Spec_f T)/\rho$ are sets z such that $\gamma^{-1}(z) = w_f(Z)$ for some $Z \subseteq R$.

We show that $\overline{\beta}$ is continuous. Let z be a closed set in $(Spec_f T)/\rho$ in that $\gamma^{-1}(z) = w_f(Z)$ for some subset $Z \subseteq T$. We may assume Z is an intersection of ρ-classes of primes in $Spec_f T$ so that $Z = \cap_I \hat{P}_i$ where $\{\hat{P}_i: i \in I\} \subseteq Spec_f T$ is a union of ρ-classes. Observe that, by Theorem 3.1.9, $\hat{P} \rho \hat{Q}$ if and only if $\beta(\hat{P}) \sim \beta(\hat{Q})$ and consequently, $\{ \beta(\hat{P}_i) : i \in I \}$ is a union of $\sim$-classes. Let $X = \hat{\beta}(Z)$, which equals $\cap_I \hat{\beta}(\hat{P}_i)$ since $\hat{\beta}$ preserves intersections. Let $x = \varphi(u_t(X))$, a closed set in $(Spec_t R)/\sim$. We claim that $x = \overline{\beta}(z)$. Let $[p] \in \overline{\beta}(z)$ so that $[p] = \overline{\beta}([\hat{P}])$ for some $\hat{P} \in Spec_f T$ with $\gamma^{-1}([\hat{P}]) \subseteq w(Z)$ where we may assume $p = \beta(\hat{P})$. In particular, $\hat{P} \supseteq Z$ and so $p = \beta(\hat{P}) \supseteq \hat{\beta}(Z) = x$. Thus, $[p] \in x$ and so $\overline{\beta}(z) \subseteq x$. Conversely, suppose $[p] \in x$ so that $p \supseteq X$ and $\hat{Q} := \beta^{-1}(p) \supseteq Z$. Thus, $[p] = \overline{\beta}([\hat{Q}]) \in \overline{\beta}(z)$, proving the equality.

We now show that $\overline{\beta}^{-1}$ is also continuous. Let x be a closed set in $(Spec_t R)/\sim$. By definition, $\varphi^{-1}(x) = u_t(X)$ for some semiprime ideal X of R such that X is an intersection of primes in $Spec_t R$. So we have that $X = \cap_\Lambda p_\lambda$ where we may assume that $\{p_\lambda: \lambda \in \Lambda\}$ is a union of $\sim$-classes. As noted in the preceding paragraph, for $\hat{P}, \hat{Q} \in Spec_f T$, $\hat{P} \rho \hat{Q}$ if and only if $\beta(\hat{P}) = \beta(\hat{Q})$. Thus, the set $\{\beta^{-1}(p_\lambda): \lambda \in \Lambda\}$ is a union of ρ-classes. Let $Z = \cap_{\lambda\in\Lambda} \beta^{-1}(p_\lambda)$ and let $z = \gamma(\omega(Z))$. We claim that $\overline{\beta}^{-1}(x) = z$. Let, $[\hat{P}] \in \overline{\beta}^{-1}(x)$ so that $[\hat{P}] = \overline{\beta}^{-1}([p])$ for some $p \in Spec_t R$ with $\varphi^{-1}([p]) \subseteq u(X)$ where may assume $p = \beta(\hat{P})$. In particular, $p \supseteq X$ and so

$\hat{P} = \beta^{-1}(p) \supseteq \cap_\Lambda \beta^{-1}(p_\lambda) = Z$. Thus, $[\hat{P}] \in z$. Conversely, suppose that $[\hat{P}] \in z$ so that $\hat{P} \supseteq Z$ and $q := \beta(\hat{P}) \supseteq \hat{\beta}(Z) = X$. Thus, $[\hat{P}] \in \overline{\beta}^{-1}(x)$, proving the equality.

The restriction map $\overline{\beta}_{pv}$ is well defined by Theorem 3.1.5 and its properties follow immediately from those of $\overline{\beta}$.

We now show to what extent members of $Spec_f T$ are distinguishable when comparing their intersections with R.

3.1.11 COROLLARY *Let S be a ring, G a finite group of automorphisms of S. Suppose that $\hat{P}, \hat{Q} \in Spec_f T$ and $\surd(\hat{P} \cap R) = \surd(\hat{Q} \cap R)$. Then $[\hat{P}] = [\hat{Q}]$, (and hence $\hat{P} \cap R = \hat{Q} \cap R$).*

PROOF By Lemma 1.2.10, $\hat{P} \cap S = P^o$, $\hat{Q} \cap S = Q^o$ for some $P, Q \in SpecS$. Let $\hat{P} = \hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n$ be all the primes minimal over $P^o * G$ not containing f. (n is finite by Theorem 1.2.11(ii)). Let $p_i = \beta(\hat{P}_i)$ $(i=1,\ldots,n)$ and let $N = \cap p_i$. Then Theorem 3.1.9 says that

$$Ntr(S) \subseteq \surd(\hat{P} \cap R) = \surd(\hat{Q} \cap R) \subseteq \beta(\hat{Q}).$$

Since $\beta(\hat{Q})$ is prime and doesn't contain $tr(S)$, there exists j such that $p_j \subseteq \beta(\hat{Q})$. By Theorem 3.1.9, $\beta(\hat{Q})$ is minimal over $\hat{Q} \cap R$. Since $\surd(\hat{Q} \cap R) = \surd(\hat{P} \cap R) \subseteq p_j$, we have $p_j = \beta(\hat{Q})$. Applying β^{-1} this gives $\hat{P}_j = \hat{Q}$. Intersecting these primes down into S gives that $P^o = Q^o$. Finally, $\hat{P} \cap R = P^o \cap R = Q^o \cap R = \hat{Q} \cap R$.

The above result provides a similar corollary for determining primes in S from their intersection in R. First we state a theorem, due to S. Montgomery, which shows to what extent we can do this in particular situations.

3.1.12 THEOREM *Let S be a ring acted on by a finite group G. Suppose that $P,Q \in SpecS$ and that $P \cap R = Q \cap R$. Then P and Q are in the same G-orbit in any of the following situations:*

(i) S is commutative;

(ii) Every prime ideal of S is generated by its intersection with the centre Z of S;

(iii) $|G|^{-1} \in S$;

(iv) S is a semiprime PI-algebra and either P or Q has the property that the polynomial identity of lowest degree satisfied by the factor ring of S by that ideal is that satisfied by S.

PROOF [Mo2, Proposition 1.1].

It should be noted that the proof of (iii) above is heavily dependent on the Bergman-Isaacs Theorem which we stated as Theorem 1.3.2. We now provide a consequence of Corollary 3.1.11 and go on to show that it provides a generalisation of (iii) without recourse to the Bergman-Isaacs result.

3.1.13 THEOREM *Let S be a ring and G a finite group of automorphisms of S. Suppose $P \in Spec_f S$ and $Q \in SpecS$ with $\surd(P \cap R) = \surd(Q \cap R)$. Then P and Q are G-conjugate, so that $Q \in Spec_f S$, and $P \cap R = Q \cap R$.*

PROOF If $Q \notin Spec_f S$ then Lemma 3.1.8 shows that $(tr(S))^n \subseteq Q \cap R = P \cap R$ and then the reverse direction of Lemma 3.1.8 shows that $P \notin Spec_f S$. Thus, we have $Q \in Spec_f S$ also. By definition of $Spec_f S$ there exists $\hat{P},\hat{Q} \in Spec_f T$ with $\hat{P} \cap S = P^o$ and $\hat{Q} \cap S = Q^o$. Since $\surd(P \cap R) = \surd(Q \cap R)$, $\surd(\hat{P} \cap R) = \surd(\hat{Q} \cap R)$ and so we apply Corollary 3.1.11, to find that $\hat{P}\ \rho\ \hat{Q}$. Thus, $P^o = \hat{P} \cap S = \hat{Q} \cap S = Q^o$ and so P and Q are G-conjugate.

We now give our proof of Theorem 3.1.12(iii).

3.1.14 COROLLARY *Let S be a ring acted on by a finite group of automorphisms G with $|G|^{-1} \in S$. If $P, Q \in SpecS$ with $P \cap R = Q \cap R$ then P and Q are in the same G-orbit.*

PROOF Lemma 3.1.2 shows that $Spec_f S = SpecS$ and so we may apply Theorem 3.1.13 to give the result.

The following example, due to Passman, shows that the hypothesis $\hat{P}, \hat{Q} \in Spec_f T$ is in fact necessary in Corollary 3.1.11 and that it is necessary to insist that $P \in Spec_f S$ in Theorem 3.1.13. This example appears as [Mo2, Exercise 1.2].

3.1.15 EXAMPLE *There exists a prime PI-algebra S of characteristic $q \neq 0$ with an outer automorphism group G of order q, such that T contains two primes $\hat{P}$, $\hat{Q} \in SpecT$ with $f \in \hat{P} \cap \hat{Q}$ satisfying $\hat{P} \cap R = \hat{Q} \cap R$ but that $\hat{P}$ and $\hat{Q}$ are not ρ-equivalent. Also we can find $P, Q \in SpecS$, not G-conjugate with $P \cap R = Q \cap R$.*

Let $A = k[x_1, \ldots, x_q]$ be the commutative polynomial ring in q variables over a field k of characteristic $q \neq 0$, and let $M = (x_1, \ldots, x_q)$, the maximal ideal generated by all the x_i. Let σ be the k-automorphism on A such that $\sigma(x_i) = x_{i+1}$ for $i < q$ and $\sigma(x_q) = x_1$. We use the following notation:

$$S = \begin{bmatrix} A & A \\ M & A \end{bmatrix}, \quad P = \begin{bmatrix} M & A \\ M & A \end{bmatrix}, \quad Q = \begin{bmatrix} A & A \\ M & M \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The automorphism of A, σ, becomes an automorphism of the ring S by letting it act on each entry. Let $\tau \in AutS$ be conjugation by U. Then $\sigma\tau = \tau\sigma$ is an automorphism of S of order q. It is outer since it moves the centre of S. Let $G = \langle \sigma\tau \rangle$. Since $S/P \cong S/Q \cong k$, P and Q are primes of S. Moreover, they are G-stable since τ is inner and σ acts on entries. By Proposition

1.2.12 there exists a unique prime ideal of T, minimal over $P*G$, $\hat{P}$ say. It is clear that, since $T/(P*G) \cong kG$, $\hat{P}/(P*G) \cong aug(kG)$ and so $f \in \hat{P}$. Similarly, $f \in \hat{Q}$, the unique minimal ideal over $Q*G$. We claim that $P \cap R = Q \cap R$ and so $\hat{P} \cap R = \hat{Q} \cap R$. If

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } X^{\tau^{-1}} = \begin{bmatrix} a+c & -a-c+b+d \\ c & -c+d \end{bmatrix}, \quad X^{\sigma} = \begin{bmatrix} a^{\sigma} & b^{\sigma} \\ c^{\sigma} & d^{\sigma} \end{bmatrix}.$$

Thus, $X^{\tau^{-1}} = X^{\sigma}$ forces $b^{\sigma} = -a-c+b+d$. Since σ fixes A/M, $b \equiv b^{\sigma} \ (mod\ M)$ and because $c \in M$, it follows that $a \equiv d \ (mod\ M)$. Thus, $P \cap R \subseteq Q$ and $Q \cap R \subseteq P$. It follows that $P \cap R = Q \cap R$.

That this example works is not reliant on the fact that $|G| = 0$ in S, only that $|G|$ is not a unit in S. In fact R.Guralnick and C.L.Hung have shown that the above example can be lifted to characteristic 0. See [Mo2, Example 1.2] for details.

Theorem 3.1.9 provides us with an intuitive way of viewing the prime correspondence of Theorem 3.1.5. The next lemma gives us a more concrete way of viewing the map β and, in fact, when the trace map is onto, it shows that $p = (\hat{P} + \Sigma_{g \in G} T(g - 1)) \cap R$.

3.1.16 LEMMA *Let S be a ring, G a group of automorphisms of S. If $\hat{P} \in Spec_f T$ and $\beta(\hat{P}) = p$ then $ptr_G(S) \subseteq (\hat{P} + \Sigma_{g \in G} T(g-1)) \cap R \subseteq p$.*

PROOF Let $x \in (\hat{P} + \Sigma_{g \in G} T(g-1)) \cap R$. Then $x = y + t$ for some $y \in \hat{P}$, $t \in \Sigma_{g \in G} T(g-1)$. So $xf = yf + tf = yf \in \hat{P}$ because $(\Sigma_{g \in G} T(g-1)).f = 0$. By Theorem 3.1.5, $x \in p$. Suppose now $r \in R$. Then

$$\begin{aligned} r \in p &\Leftrightarrow rf \in \hat{P} \\ &\Leftrightarrow rfs \in \hat{P} \text{ for all } s \in S \\ &\Leftrightarrow \Sigma_{g \in G} rs^g g \in \hat{P} \text{ for all } s \in S \\ &\Rightarrow \Sigma_{g \in G} rs^g \in (\hat{P} + \Sigma_{g \in G} T(g-1)) \cap R \text{ for all } s \in S \end{aligned}$$

$$\Rightarrow r.tr_G(S) \subseteq (\hat{P} + \Sigma_{g\in G}T(g-1)) \cap R$$

$$\Rightarrow r.tr_G(S) \subseteq p \quad \text{(by first part)}$$

$$\Rightarrow r \in p \quad \text{(since } tr_G(S) \not\subseteq p, \text{ prime)}.$$

Thus, the above must be a chain of equivalences and so

$r.tr_G(S) \subseteq p \Leftrightarrow r \in p \Leftrightarrow r.tr_G(S) \subseteq (\hat{P} + \Sigma_{g\in G}T(g-1)) \cap R$, proving the lemma.

We now consider the connection between primes in S and those in T. The following theorem encapsulates the connection between $SpecS$ and $SpecT$.

3.1.17 THEOREM *Let S be a ring, G a finite subgroup of $Aut\ S$. Define $\alpha:SpecS \to$ (Power set of $SpecT$) such that $\alpha(P) = \{ \hat{P} : \hat{P}$ is minimal over $P^o*G \}$. Then*

(i) α is a closed map with respect to the Zariski topologies.

(ii) The induced map $\bar{\alpha}:SpecS/G \to SpecT/\rho$ is an order preserving map and is a homeomorphism with respect to the quotient Zariski topologies.

(iii) The restricted maps $\alpha_r:Spec_fS \to$ (Power set of $Spec_fT$), where $\alpha_r(P) = \alpha(P) \cap Spec_fT$, and $\alpha_{pv}:PrimSpecS \to$ (Power set of $PrimSpecT$), where $\alpha_{pv}(P) = \alpha(P) \cap PrimSpecT$, are also closed.

(iv) Finally, their quotient maps $\bar{\alpha}_r:Spec_fS/G \to Spec_fT/\rho$ and $\bar{\alpha}_{pv}:PrimSpecS/G \to PrimSpecT/\rho$ are homeomorphisms.

PROOF By [P2, Proposition 16.7] the results for α_{pv} and $\bar{\alpha}_{pv}$ follow immediately from those for α and $\bar{\alpha}$. Since α_r concerns the induced topologies of $SpecS$ and $SpecT$ with respect to the open sets $Spec_fS$ and $Spec_fT$, it again suffices to prove the result for α.

We prove (i) first. Let $v(X)$ be a closed set in $SpecS$ where $X = \cap_{i\in I}P_i$ for some primes $P_i \in SpecS$ and let $P \in v(X)$ so that $P \supseteq X$. Let $\hat{X} = \cap_{i\in I}(P_i{}^o*G)$. If $\hat{P} \in \alpha(P)$ then $\hat{P}$ is minimal over P^o*G. Since $P_i{}^o \subseteq P^o$ for all $i \in I$, $\hat{P} \supseteq X$ and so $\alpha(P) \subseteq v(\hat{X})$. Conversely, let $\hat{Q} \in v(\hat{X})$ so that

$\hat{Q} \supseteq \cap_{i\in I}(P_i{}^{O}*G)$ and $\cap_{i\in I}P_i{}^{O} \subseteq Q^{O} \subseteq Q$ where $\hat{Q} \cap S = Q^{O}$ *for* $Q \in SpecS$. This shows that $\cap_{g\in G}X^{g} \subseteq Q$ and since G is finite, we have $h \in G$ such that $X^{h} \subseteq Q$. Let h denote y^{-1}. Then $X \subseteq Q^{y}$. We've shown $Q^{y} \in v(X)$ and since $\hat{Q} \in \alpha(Q^{y})$ we have that $v(\hat{X}) \subseteq \alpha(v(X))$. Thus, $v(\hat{X}) = \alpha(v(X))$, proving (i).

Now we consider statement (ii) for the map $\overline{\alpha}$. First we show that $\overline{\alpha}$ is well defined and preserves order. Let $P,Q \in SpecS$ with P and Q G-conjugate. Since $P^{O} = Q^{O}$, $\overline{\alpha}([P]) = [\{\hat{P} \in SpecT: \hat{P} \cap S = P^{O}\}] = \overline{\alpha}([Q])$. Now we show $\overline{\alpha}$ preserves inclusions. Suppose $Q \subseteq P$ are primes in $SpecS$ and let $[\hat{P}] \in \overline{\alpha}([P])$. Since $Q^{O}*G \subseteq P^{O}*G \subseteq \hat{P}$, there exists $\hat{Q} \in SpecT$ minimal over $Q^{O}*G$ with $\hat{Q} \subseteq \hat{P}$, by Theorem 1.2.11(i). Since $\hat{Q} \in \overline{\alpha}(Q)$, $[\overline{\alpha}(Q)] \subseteq [\overline{\alpha}(P)]$.

Secondly, we show $\overline{\alpha}$ is a homeomorphism. To do this we examine the closed sets of $SpecS/G$ and $SpecT/\rho$. Let y be a closed set of $SpecS/G$ so that $\pi^{-1}(y) = v(Y)$ for some $Y = \cap_{i\in I}P_i$. Since $\pi^{-1}(y)$ is a collection of G-orbits, we may assume $Y = \cap_{i\in I}P_i{}^{O}$. Conversely any such intersection of G-prime ideals gives rise to a closed set in $SpecS/G$. Now, let z be a closed set in $SpecT/\rho$ so that $\gamma^{-1}(z) = w(Z)$ with $Z = \cap\hat{P}_\lambda$ for some $\hat{P}_\lambda \in SpecT$ where $\gamma: SpecT \to SpecT/\rho$ is the projection map. As before, we may assume $\{\hat{P}_\lambda\}$ is a collection of ρ-orbits. Thus, letting $N_\lambda = \{\ \hat{P} : \hat{P}$ is minimal over $P_\lambda{}^{O}*G\ \}$ where $\hat{P}_\lambda \cap S = P_\lambda{}^{O}$ for $P_\lambda \in SpecS$, we have $Z = \cap\hat{P}_\lambda$. Thus:

$$\begin{aligned} & \hat{P} \supseteq Z \\ \Leftrightarrow\ & \hat{P} \supseteq \cap_\lambda\{\cap\{\hat{P}: \hat{P} \in N_\lambda\}\} \\ \Leftrightarrow\ & \hat{P} \supseteq (\cap_\lambda\{\cap\{\hat{P}: \hat{P} \in N_\lambda\}\})^{|G|} \quad \text{using } \hat{P} \text{ prime} \\ \Leftrightarrow\ & \hat{P} \supseteq \cap_\lambda(P_\lambda{}^{O}*G) \qquad \text{by Theorem 3.1.14.} \\ \Leftrightarrow\ & \hat{P} \supseteq \cap_\lambda P_\lambda{}^{O} \end{aligned}$$

So we may assume that $Z = \cap_\lambda P_\lambda{}^{O}$.

Here, we show that $\overline{\alpha}$ is a continuous map. Let y be a closed set in $Spec_f S/G$, so that $\pi^{-1}(y) = v(Y)$ for some set $Y \subseteq S$ where we may assume that $Y = \cap_i P_i{}^{O}$ for some $P_i \in SpecS$. Let $Z = Y$. With $z = \gamma(\omega(Z))$, we show that $\overline{\alpha}(y) = z$. To this end, let $[P] \in y$ so that $P \in v(Y)$ and so, $P \supseteq Y$. Clearly, for $\hat{P} \in \alpha(P)$, $Z \subseteq \hat{P}$ and hence $\bar{\alpha}([P]) \in z$, proving $\bar{\alpha}(y) \subseteq z$. Now, let $[\hat{Q}] \in z$ so that, without loss of generality, $\hat{Q} \supseteq Z$. Now, $\hat{Q} \cap S = Q^o$ for some $Q \in SpecS$. Since $Z \subseteq \hat{P}$, $P \supseteq Y$. Thus $[\hat{Q}] = \bar{\alpha}([Q]) \in \bar{\alpha}(y)$, proving the opposite inclusion.

Finally, we show that $\bar{\alpha}^{-1}$ is a continuous map. Let z be an arbitrary closed set in $SpecT/\rho$ so that $\gamma^{-1}(z) = \omega(Z)$ for some subset Z of T where, as above, we may assume that $Z = \cap_i P_i{}^o$ for some $P_i \in SpecS$. Put $Y = Z$ and let $y = \pi(v(Y))$. We claim that $\bar{\alpha}^{-1}(z) = y$. For, let $[\hat{P}] \in z$ where we may assume that $\hat{P} \supseteq Z$. With $\hat{P} \cap S = P^o$ for some $P \in SpecS$, we have that $P \supseteq Y$. Thus, $\bar{\alpha}^{-1}([\hat{P}]) = [P] \in y$. Now, let $[Q] \in y$ for some $Q \in SpecS$ with $Y \subseteq Q$. Then, for $\hat{Q} \in SpecT$, minimal over Q^o*G, we have that $Z \subseteq \hat{Q}$, $[\hat{Q}] \in z$ and $\bar{\alpha}([Q]) = [\hat{Q}]$. Consequently, $[Q] \in \bar{\alpha}^{-1}(z)$ and this completes the proof that $\bar{\alpha}^{-1}$ is a homeomorphism.

We may now compose the maps $\bar{\alpha}$ and $\bar{\beta}$ in order to get a map from $Spec_fS/G$ to $Spec_tR/\sim$ as described below.

3.1.18 THEOREM *Let S be a ring and G a finite group of automorphisms of S. Then the map $\varphi: (Spec_fS)/G \to (Spec_tR)/\sim$ given by composing $\bar{\alpha}$ and $\bar{\beta}$ as shown:*

$$\begin{array}{ccccc} (Spec_fS)/G & \xrightarrow{\quad\bar{\alpha}\quad} & (Spec_fT)/\rho & \xrightarrow{\quad\bar{\beta}\quad} & (Spec_tR)/\sim \\ [P] & \longmapsto & [\hat{P}] & \longmapsto & [p] \end{array}$$

so that $\varphi([P]) = \{p : p \text{ is minimal over } P \cap R,\ tr(S) \not\subseteq p\}/\sim$ is an order preserving homeomorphism. Moreover if we restrict φ to the subsets consisting of primitive ideals only, we get a bijection $\varphi_{pv}: (PrimSpec_fS)/G \to (PrimSpec_tR)/\sim$ such that $\varphi_{pv}([P]) = \varphi([P])$.

PROOF This result is immediate from Theorem 3.1.10 and Theorem 3.1.17.

So far we have neglected to discuss the relationship between $Spec_IS$ and

$Spec_f S$. It's easy to see that $Spec_I S \subseteq Spec_f S$ in general. For, if P is a prime in S with $P \notin Spec_f S$ then there exists $\hat{P} \in SpecT$ minimal over $P^o{*}G$ containing f. Thus, $I = SfS \cap S \subseteq \hat{P} \cap S = P^o \subseteq P$ and so $P \notin Spec_I S$. We've shown that $Spec_I S \subseteq Spec_f S$. In general, however, the containment is strict as the following example shows.

3.1.19 EXAMPLE *There is a ring S and a group acting on S for which $Spec_I S$ is strictly contained in $Spec_f S$.*

Let $S = \mathbb{Q}$, the field of rational numbers and G be a non-trivial finite group acting on S with the trivial action. Now SfS is a proper ideal of T and so $I = SfS \cap S = 0$. Thus, $Spec_I S = \emptyset$. Lemma 3.1.2 shows that $Spec_f S = SpecS$ and so $Spec_f S \neq \emptyset$.

We now reach the climax of this section where we relate certain primes of R to certain primes of S. When the order of the group is invertible in the ring S, Montgomery has proved the following theorem.

3.1.20 THEOREM *Let S be a ring acted upon by a finite group of automorphisms, G. Suppose that $|G|^{-1} \in S$.*

(i) Given $P \in SpecS$, $P \cap R = p_1 \cap p_2 \cap \ldots \cap p_m$, where $m \leq |G|$ and the $\{p_i\}$ are the set of primes in R minimal over $P \cap R$.

(ii) Given $p \in SpecR$, there exists $P \in SpecS$ such that p is minimal over $P \cap R$. Moreover, P is unique up to its G-orbit in $SpecS$.

PROOF [Mo2, Theorem 2.1].

When $|G|^{-1} \in S$, $tr(S) = R$ and so $Spec_t R = SpecR$. Lemma 3.1.2 shows that $Spec_f S = SpecS$. Thus, the following theorem is indeed a generalisation of Montgomery's Theorem. It has no hypothesis on the order of G other than being finite.

3.1.21 THEOREM *Let S be a ring and G a finite group of automorphisms of S.*

(i) Given $P \in Spec_f S$, there are a finite number of primes in $Spec_t R$ minimal over $P \cap R$, $\{p_1, p_2, \ldots, p_m\}$ say, with $m \leq |G|$. Also, $(\cap_i p_i)tr(S)$ is nilpotent modulo $P \cap R$.

(ii) Given $p \in Spec_t R$, there exists $P \in Spec_f S$ such that p is minimal over $P \cap R$. Moreover, P is unique up to its G-orbit in $SpecS$.

PROOF We prove (i) first. Let $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n$ be all the primes in $Spec_f T$ minimal over $P^o{*}G$. The definition of $Spec_f S$ shows that there must be at least one of these and Theorem 1.2.11(ii) shows that $n \leq |G|$. Let $q_i = \beta(\hat{P}_i)$ $(i = 1, \ldots n)$ and set $N = \cap_i q_i$. Lemma 3.1.7 shows that $Ntr(S) \subseteq \surd(P \cap R)$. By Theorem 3.1.9, q_i is minimal over $\hat{P}_i \cap R = P \cap R$ $(i = 1, \ldots, n)$. It remains to show that these are all the members of $Spec_t R$ minimal over $P \cap R$. Let $q \in Spec_t R$ with $P \cap R \subseteq q$. As noted in the previous part, $Ntr(S) \subseteq \surd(P \cap R) \subseteq q$. Since $tr(S) \not\subseteq q$, $N \subseteq q$ and so there exists $j \in \{1, \ldots, n\}$ such that $q_j \subseteq q$. This completes the proof of (i).

For (ii), let $p = \beta(\hat{P})$ for some $\hat{P} \in Spec_f T$. By Lemma 1.2.10, $\hat{P} \cap S = P^o$ for $P \in Spec_f S$. Then Theorem 1.2.11(i) shows that $\hat{P}$ is minimal over $P^o{*}G$. By Theorem 3.1.9, p is minimal over $P \cap R$. Suppose now that p is minimal over $Q \cap R$ for some $Q \in SpecS$. By the proof of (i), $p = \beta(\hat{Q})$ for some $\hat{Q} \in Spec_f T$ minimal over $Q^o{*}G$. Thus, since $\hat{P} = \beta^{-1}(q) = \hat{Q}$, $\hat{P} \cap S = \hat{Q} \cap S$ so that $P^o = Q^o$ and P and Q are G-conjugate.

Letzter introduces the following definitions to explain the relationship between primes in R and those in S.

3.1.22 DEFINITION Let U and V be rings with $U \subseteq V$. Suppose $p \in SpecU$ is minimal over $P \cap U$ for $P \in SpecV$. Then we say that p lies under P and that P lies over p.

We use the above generalisation of [Mo2, Theorem 2.1] to improve on

[Mo2, Lemma 3.1]. This corollary will yield a number of applications in Chapter3, §3. For example it is used extensively in Lemma 3.3.8, a satisfying result on the heights of prime ideals.

3.1.23 COROLLARY *Let S be a ring and G a finite subgroup of Aut S.*

(i) Given $P_1 \subset P_2$ in $Spec_f S$ and $p_2 \in Spec_t R$ lying under P_2, there exists $p_1 \in Spec_t R$ lying under P_1 with $p_1 \subset p_2$.

(ii) Given $p_1 \subset p_2$ in $Spec_t R$ and $P_1 \in SpecS$ lying over p_1, there exists $P_2 \in Spec_f S$ lying over p_2 with $P_1 \subset P_2$.

(iii) Given $p_1 \subset p_2$ in $Spec_t R$ and $P_2 \in SpecS$ lying over p_2, there exists $P_1 \in Spec_f S$ lying over p_1 with $P_1 \subset P_2$.

(iv) Given $p_1 \subset p_2$ in $Spec_t R$ and $q_2 \in Spec_t R$ with $p_2 \sim q_2$, there exists $q_1 \in Spec_t R$ with $p_1 \sim q_1$ and $p_1 \subset q_1$.

PROOF In (i), Corollary 3.1.11 shows that $P_1 \cap R \subset P_2 \cap R$. Let $q_1,\ldots,q_n \in Spec_t R$ be all the minimal primes over $P_1 \cap R$ not containing $tr(S)$. Writing $N = \cap_i q_i$, Theorem 3.1.21(i), says $Ntr(S) \subseteq \surd(P_1 \cap R) \subseteq p_2$. Since $tr(S) \not\subseteq p_2$, $N \subseteq p_2$ and so it follows that there exists $j \in \{1,..,n\}$ such that $q_j \subseteq p_2$. Since p_2 cannot lie under both P_1 and P_2, we have $q_j \subset p_2$. Taking $p_1 = q_j$ gives the required result.

For (ii) and (iii), we have $p_i = \beta(\hat{P}_i)$ for some $\hat{P}_i \in Spec_f T$ and let $\hat{P}_i \cap S = Q_i{}^o$, say, for some $Q_i \in SpecS$ (i = 1,2). For (ii), it's clear from Theorem 3.1.21, that $P_1 = Q_1{}^h$ for some $h \in G$. Since $Q_1{}^o \subseteq Q_2{}^o$, there exists $x \in G$ such that $P_1 \subseteq Q_2{}^x$. Taking $P_2 = Q_2{}^x$ gives (ii).
Similarly for (iii), we have that $P_2 = Q_2{}^k$ for some $k \in G$. Since, $Q_1{}^o \subseteq Q_2{}^o \subseteq PQ_2$, there exists $x \in G$ such that $Q_1{}^x \subseteq P_2$. Taking $P_1 = Q_1{}^x$ gives the result for (ii).

For (iv), we let $\hat{P}_i \in Spec_f T$ be such that $\beta(\hat{P}_i) = p_i$ and $P_i \in SpecS$ such that $P_i{}^o = \hat{P}_i \cap S$ (i = 1,2). Let $a_1,a_2,\ldots,a_n$ in $Spec_t R$ be all the primes lying under P_1 not containing $tr(S)$. Since $P_1 \cap R \subseteq P_2 \cap R \subseteq q_2$, q_2

contains a prime of R minimal over $P_1 \cap R$ and since $tr(S) \not\subseteq q_2$, there exists $j \in \{1,\ldots,n\}$ such that $a_j \subseteq q_2$. Clearly, $a_j \subset q_2$ and since a_j and p_1 are both minimal over $P_1 \cap R$, $a_j \sim p_1$. Thus, we take $q_1 = a_j$.

§3.2 The q-case

Here we are concerned with the special case where S has prime characteristic q and G is a q-group. Occasionally, it is necessary to look at the case where G has order q. In this section we find an explicit formulation for the map β of Theorem 3.1.5 and we state corollaries to the major theorems of Chapter 3, §1.

To provide another characterisation of β, we first require a technical lemma.

3.2.1 LEMMA. *Let C be a G-invariant right Ore set in S. Then C is a right Ore set in T with $TC^{-1} \cong SC^{-1}*G$.*

PROOF. By definition there exists a right ring of fractions of S with respect to C, namely SC^{-1}. To be precise, there is a ring homomorphism $\varphi : S \to SC^{-1}$ satisfying:

(i) $\varphi(x)$ is a unit for all $x \in C$

(ii) each elt. of SC^{-1} has form $\varphi(s)\varphi(x^{-1})$ for some $s \in S$, $x \in C$.

(iii) $\ker\varphi = \{s \in S\colon sc = 0 \text{ for some } c \in C\}$.

Now G acts on SC^{-1} by: $(\varphi(r)\varphi(x)^{-1})^g = \varphi(r^g)\varphi(x^g)^{-1}$ for all $g \in G$. So we may consider the skew group ring $SC^{-1}*G$.

We claim that $SC^{-1}*G$ is a right ring of fractions for $S*G$ with respect to C. For, define $\psi\colon S*G \to SC^{-1}*G\colon \Sigma s_g g \mapsto \Sigma\varphi(s_g)g$. Then it is clear that ψ is a ring homomorphism. We show that it satisfies the required properties for $SC^{-1}*G$ to be a ring of fractions. Property (i) is trivial.

In order to show that (ii) holds we show that a given element is of the required form. Let $t\colon = \Sigma_{g\in G}\varphi(s_g)\varphi(x_g)^{-1}g \in SC^{-1}*G$ where $s_g \in S$, $x_g \in C$ $(g \in G)$. Then

$$t = \Sigma_{g\in G}\varphi(s_g)g\varphi(x_g{}^g)^{-1}$$
$$= \Sigma_{g\in G}\varphi(s_g)g\varphi(r_g)\varphi(y)^{-1}$$

where $\varphi(x_g{}^g)^{-1} = \varphi(r_g)\varphi(y)^{-1}$ for some $r_g \in S$ $(g \in G)$, $y \in C$. Thus,

$$t = [\Sigma_{g\in G}\varphi(s_g)\varphi(r_g{}^{g^{-1}})g]\varphi(y)^{-1}$$
$$= \psi(\Sigma_{g\in G}s_g r_g{}^{g^{-1}}g)\psi(y)^{-1} \text{ as required.}$$

So $S*G$ has a right ring of fractions with respect to C and [G-W, Lemma 9.1] gives us that C is a right Ore set in $S*G$.

We consider now the case where $|G| = q$ (q prime), S is a ring with $char\ S = q$ and

$$\hat{P} \supset (\hat{P} \cap S).T \qquad - (*).$$

Lemma 1.2.10 gives us that $\hat{P} \cap S = \cap_{x\in G}Q^x$ for some $Q \in SpecS$. [P2, Proposition 14.10] shows that, if $Stab_G(Q) = 1$, then $\hat{P} = Q^o*G = (\hat{P} \cap S)T$. Thus, by $(*)$, $Q = Q^g$ where $G = \langle g \rangle$. Let $\overline{S} = S/Q$ and suppose it is right Goldie. Let $Q(\overline{S})$ denote the classical quotient ring of $\overline{S}$. By Proposition 1.2.12, $Q(\overline{S})*G$ has a unique prime ideal with zero intersection with the coefficient ring. We now show that this prime ideal is derived from $\hat{P}$.

3.2.2 LEMMA. *Let S be a ring acted on by a finite group G. Suppose $\hat{P} \in Spec_f T$ has $\hat{P} \cap S = Q$ for some G-invariant prime ideal Q with S/Q right Goldie. Let $\overline{S} = S/Q$, $\overline{T} = T/(Q*G)$ so that $\overline{T} \cong \overline{S}*G$. Let $X = C_S(0).\overline{1} \subseteq \overline{T}$. Write $\hat{P}/(Q*G)$ as $\overline{P}$. Then $\overline{P}X^{-1}$ is a prime ideal of $\overline{T}X^{-1}$ and $\overline{P}X^{-1} \cap Q(\overline{S}) = 0$.*

PROOF. By Lemma 3.2.1, $Q(\overline{S})*G$ is a right ring of fractions for $\overline{S}*G$ with respect to X. We show that, as a right $\overline{S}$-module, $(\overline{S}*G)/\overline{P}$ is X-torsion free. Let $I/\overline{P}$ be the right X-torsion submodule of $(\overline{S}*G)/\overline{P}$.

Then I is a left ideal of $\overline{S}*G$ and, by the right Ore condition, I is a right ideal of $\overline{S}*G$. So $I/\overline{P}$ is a two sided ideal of the prime right Goldie ring $\overline{S}*G/\overline{P}$ and hence contains a regular element. This gives a contradiction which proves the objective. So by [G-W, Theorem 9.20(b)] $\overline{P}X^{-1}$ is a prime ideal in $Q(\overline{S})*G$. Moreover, since $\hat{P} \cap S = Q$, $\overline{P}X^{-1} \cap Q(\overline{S}) = 0$.

3.2.3 LEMMA *Let S be a Noetherian ring of characteristic q and G a group of automorphisms of S of order q. Let $\hat{P} \in Spec_f T$ with $\hat{P} \supset (\hat{P} \cap S).T$. Then there exists a unit $U \in Q(\overline{S})$ such that $\beta(\hat{P}) = \{ r \in R : \overline{r}.\Sigma_{i=1,..q} U^i = 0 \}$ where β is the map defined in Theorem 3.1.5.*

PROOF When $\hat{P} \cap S = \cap_{g \in G} Q^g$ where $Q \neq Q^g$, by [P2, Theorem 14.7], $(\hat{P} \cap S)T$ is a prime ideal. Moreover, this prime ideal has the same intersection with S as $\hat{P}$ and so, by Proposition 1.2.12, they are equal. So here we must have $\hat{P} \cap S = Q$ where $Q = Q^g$. So $\overline{S} = S/Q$; G acts on this ring. In the case of prime Noetherian rings, we have that G is inner on $Q(\overline{S})$, the classical ring of quotients, if and only if G is X-inner by [Mo1, Example 3.7]. Since $Q*G \subset \hat{P}$ and Proposition 1.2.12 shows that $\overline{P}$ is the unique prime ideal of $\overline{S}*G$ with zero intersection with the coefficient ring, $\overline{S}*G$ itself cannot be prime. So, [Mo1, Theorem 3.17(2)] tells us that G must be inner on $Q(\overline{S})$. Suppose there does not exist $U \in Q(\overline{S})$ of order q which induces the action of g. Then by [Yi, Proposition 2.5] $Q(\overline{S})*G$ is prime and so by [G&W 5.11] $Q*G$ is prime and so $\hat{P} = (\hat{P} \cap S)T$, giving a contradiction. So there must exist $U \in Q(\overline{S})$ of order q which induces the action of g.

By observing that $Q(\overline{S})*G = Q(\overline{S})\langle U^{-1}g\rangle$, the ordinary group ring, whose unique prime ideal is its augmentation ideal, we have that

$$\overline{P}X^{-1} = aug(Q(\overline{S})\langle U^{-1}g\rangle) = \Sigma_{g \in G} Q(\overline{S})((U^{-1}g)^i - 1).$$

The proof of [G-W, Theorem 9.22] gives us that $\overline{P}X^{-1} \cap \overline{T} = \overline{P}$. Now let $r \in p$ so that, by Theorem 3.1.5, $\overline{rf} \in \overline{P}$. We may write

$$\overline{rf} = \Sigma_{i=1,\ldots,q} \overline{r} U^i U^{-i} \overline{g}^i \in \overline{P}X^{-1}.$$

Since $U^{-i}g^i \equiv 1 \pmod{\overline{P}}$ by the above, we conclude that $\Sigma_{i=1,..p} \overline{r}\, U^i = 0$.

Conversely, let $r \in R$ with $\overline{r}\Sigma_{i=1,..q} U^{-i} = 0$. Then

$\overline{rf} = \Sigma_{i=1,..q}q\bar{r}U^iU^{-i}g^i \in \bar{P}X^{-1}\cap\bar{T} = \bar{P}$. Thus, $rf \in \hat{P}$ and so $r \in p$. This proves the lemma.

Combining this with Theorem 3.1.5, we've thus shown the following result.

3.2.4 THEOREM *Let S be a Noetherian ring of characteristic q and $|G| = q$, prime. Suppose $\hat{P} \in Spec_f T$.*

If $(\hat{P} \cap S).T = \hat{P}$ then $\beta(\hat{P}) = \hat{P} \cap R$.

If $(\hat{P} \cap S).T \subset \hat{P}$ then $\beta(\hat{P}) = \{r \in R : \bar{r}\,\Sigma_{i=1,..q}\, U^i = 0\}$ for some $U \in Q(\bar{S})$ as described above.

We return to the relationship between $Spec_I S$ and $Spec_f S$ originally discussed at 3.1.17. There we saw that, in general, it is possible to have $Spec_I S$ strictly contained in $Spec_f S$. The following shows this cannot happen when $char\ S = q$ and G is a q-group (q prime).

3.2.5 THEOREM *Let G have order q^a ($a \in \mathbb{N}$) and let S be a Noetherian ring of characteristic q. Let $P \in SpecS$* and *$\hat{P}$ be the unique prime of T minimal over P^o*G. Then the following are equivalent:*

(i) $I \subseteq P$;

(ii) $f \in \hat{P}$;

(iii) $(tr(S))^n \subseteq P \cap R$ for some $n \in \mathbb{N}$.

When these occur, $Stab_G(P) \neq \{1\}$.

PROOF First note that $\hat{P}$ is unique by Proposition 1.2.12 and that $\hat{P} \cap S = P^o$ by Theorem 1.2.11(i). First we establish the equivalence of (i) and (ii). Suppose (i) holds so that $I \subseteq P$. Let Q be a prime of S minimal over I, contained in P and let $\hat{Q}$ be the prime of $S*G$ minimal over Q^o*G. Since $S/\sqrt{I}$ is a semiprime Noetherian ring, [G-W, Exercise 9U] says that we may localize at $Q/\surd I$. First we have that $C(Q^o/\surd I)$ is a G-invariant Ore set in $\overline{S} := S/\surd I$. So $C := C(Q^o/\surd I).\overline{1}$ is an Ore set in $\overline{T} := \overline{S} * G$ by Lemma 3.2.1.

Now $\overline{T}C^{-1} = \overline{S}C^{-1} * G$ is a local ring because $G = q^a$ and $char\ S = q$. This is the case because $J(\ \overline{S}C^{-1}\ )*G \subseteq J(\overline{T}C^{-1})$ by [McC-R, Corollary 10.2.10(v)] and because $\overline{S}C^{-1}*G/\ J(\ \overline{S}C^{-1}\ )*G \cong Q(S/Q^o)*G$ is local by Proposition 1.2.12.

So either (i) $\overline{T}C^{-1}.f.\overline{T}C^{-1} = \overline{T}C^{-1}$

or (ii) $\overline{T}C^{-1}.f.\overline{T}C^{-1} \subseteq J(\overline{T}C^{-1})$.

In case (i),there exist

$$c \in C_{\overline{S}}(Q^o/\surd I),\ s_i, s_i' \in \overline{S}\ (\ i=1,..,n\ )$$

such that $\quad 1 = \Sigma_{i=1,..,n}\ c^{-1}s_i f s_i' c^{-1}$.

Therefore $\quad c^2 = \Sigma_{i=1,..,n}\ s_i f s_i' \in \overline{S}f\overline{S} \cap C_{\overline{S}}(Q^o/\surd I) \subseteq (Q^o/\surd I) \cap C_{\overline{S}}(Q^o/\surd I)$.

This contradiction shows that case (ii) is the only one that can arise. By [G-W, Theorem 9.22], $\hat{Q}C^{-1} \in Spec(TC^{-1})$. Since $(SC^{-1}\ /\ Q^oC^{-1}) \cong (S/Q^o)C^{-1}$ is semisimple Artinian, $\hat{Q}C^{-1}$ is a maximal ideal of TC^{-1}. Since TC^{-1} is local $\hat{Q}C^{-1} = J(TC^{-1})$.

So $f \in (TC^{-1})f(TC^{-1}) \subseteq J(TC^{-1}) = \hat{Q}C^{-1}$. Hence, by [G-W, Theorem 9.22], $f \in \hat{Q}$. Since $\hat{P}$ a prime T containing Q^o*G and $\hat{Q}$ is the unique prime minimal over Q^o*G, we have that $\hat{Q} \subseteq \hat{P}$. Thus, $f \in \hat{P}$ and we have shown that (ii) holds. Rather easier is the implication (ii) $\Rightarrow$ (i). For, if $f \in \hat{P}$ then $I = SfS \cap S \subseteq \hat{P} \cap S = P^o \subseteq P$.

Next we show that (i) and (iii) are equivalent. Since the previous part of the proof establishes that $Spec_I S = Spec_f S$, we have to show that $P \notin Spec_f S$ if and only if $(tr(S))^n \subseteq P \cap R$ for some $n \in \mathbb{N}$. This is just Lemma 3.1.8.

Finally, suppose that (i) to (iii) above hold. If $Stab_G(P) = \{1\}$ then [P2, Corollary 14.10] gives that $\hat{P} = P^o*G$ which does not contain f. Thus $Stab_G(P) \neq \{1\}$.

We use this lemma to study the relationship between $tr(S)$ and I still further .

3.2.6 THEOREM *Let S be a Noetherian ring of characteristic q and G a finite group of automorphisms of S such that G is a q-group. Then there exists $t \in \mathbb{N}$ such that $(tr_G(S))^t \subseteq I$.*

PROOF Let $P_1, P_2, \ldots, P_m$ (for some $m \in \mathbb{N}$) be the primes of S minimal over I. By Corollary 3.2.5, there exist u_i such that $(tr_G(S))^{u_i} \subseteq P_i$ ($1 \leq i \leq m$). Let u be the maximum of the u_is. Then $(tr_G(S))^u \subseteq \cap_{i=1,..,m} P_i = \sqrt{I}$. Also, Since S is Noetherian there exists $w \in \mathbb{N}$ such that $(\sqrt{I})^w \subseteq I$ and so $(tr_G(S))^{uw} \subseteq I$.

We can say more when $a = 1$ in Theorem 3.2.5.

3.2.7 COROLLARY. *Suppose S is a Noetherian ring with char $S = q$ and G a subgroup of Aut$S$ of order q. Let $P \in$ Spec$S$ and let $\hat{P}$ be the unique prime of T minimal over $P^o * G$. Then the following are equivalent:*

(i) $I \subseteq P$;

(ii) $f \in \hat{P}$

(iii) $P = P^g$, the action of g on $Q(S/P)$ is induced by a unit, U, of $Q(S/P)$ with $(U-1)^{q-1} = 0$ and $Q(S/P)*G \cong Q(S/P)\langle U^{-1}g\rangle$, the ordinary group ring.

PROOF Note $\hat{P}$ is unique by Proposition 1.2.12. The equivalence of (i) and (ii) is just Theorem 3.2.5. Let $G = \langle g\rangle$. Suppose that (i) and (ii) hold. Then $P = P^g$ from the theorem. Since $\hat{P} \supset (\hat{P} \cap S).T$, we are in the same situation as case (ii) of Theorem 3.2.4. Adopting the notation there, we have that g is induced by a unit, U, of $Q(\bar{S})$ with $(U-1)^q = 0$. Moreover $Q(S/P)*G \cong Q(S/P)\langle U^{-1}g\rangle$. Now,

$$\bar{f} \in \bar{P}X^{-1} = aug(Q(\bar{S})\langle U^{-1}g\rangle) = \Sigma_{i=1,..,q-1} Q(\bar{S})(U^{-i}g^{i}-1).$$

Now, $\bar{P}X^{-1}$ is a free $Q(\bar{S})$-module with basis $\{U^{-i}g^{-i} - 1 : i=1,\ldots,q-1\}$ and

$\overline{f} = \Sigma_{g\epsilon G}\, g \;\epsilon\; \overline{P}X^{-1}$. Thus, the coefficient of $U^{-i}g^{-i}{}_{-1}$ in the expression for $\overline{f}$ must be U^i for $i = 1,\ldots,q-1$. Hence, we have:

$$\overline{f} = \Sigma_{i=1,\ldots,q}U^i(U^{-i}g^{i-1}{}_{-1}) = \overline{f} + \hat{U}$$

where $\hat{U} = \Sigma_{i=1,\ldots q}U^i$. So $\hat{U} = (U-1)^{q-1} = 0$. Conversely, suppose (iii) holds. Then $\overline{f} = \overline{f} + \hat{U} = \Sigma_{i=1,\ldots,q}U^i(U^{-i}g^{i-1}{}_{-1}) \;\epsilon\; aug(Q(S/P)\langle U^{-1}g\rangle)$ and so $f \;\epsilon\; \hat{P}$.

The next note shows that (iii) does not imply (i) in Corollary 3.2.7 if we omit the hypothesis that $(U - 1)^{q-1} = 0$.

3.2.8 NOTE *There is a ring S, a group of automorphisms G and $P \;\epsilon\; SpecS$ such that $P = P^g$, G is inner on S/P and $S*G$ is a group ring but that $I \not\subseteq P$.*

Take $S = M_2(\mathbb{Z}/2\mathbb{Z})$, $P = 0$, $G = \langle g\rangle$ where g is induced by

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Certainly, $P = P^g$, g is inner on S/P and $S*G$ is an ordinary group ring. Also, $1_S - U \neq 0$ so that all the hypotheses of (iii) hold except $(U - 1_S)^{q-1} = 0$. We now exhibit a non-zero element of I to show that this example does not contradict Corollary 3.2.7. Let

$$X = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Now $0 \neq U = tr(X) = g^{-1}(fX - Xf) \;\epsilon\; I$. This completes the example.

With the additional hypotheses that $char\ S = q$ (q prime) and $|G| = q^a$ ($a \;\epsilon\; \mathbb{N}$), we get a stronger version of Theorem 3.1.17. Since ρ is then the trivial equivalence, we may now construct a bijection between $(SpecS)/G$ and $SpecT$.

3.2.9 COROLLARY *Let S be a Noetherian ring of characteristic q, G a finite group of automorphisms of S of order q^a. Let*

$$\Delta = \{\, N < S : N = \cap P_i,\ P_i \;\epsilon\; SpecS \,\}$$

$$\text{and } \Omega = \{\, J < T : J = \cap \hat{P}_j,\ \hat{P}_j \;\epsilon\; SpecT \,\}.$$

Let $\hat{\alpha} : \Delta \to \Omega$ where $\hat{\alpha}(\cap P_i)/((\cap P_i^O) *G) = N(T / ((\cap P_i^O) *G)$. We may also define $\bar{\alpha}: (SpecS)/G \to SpecT$ by $\bar{\alpha}([P]) = \hat{\alpha}(P)$. Then

(i) $\hat{\alpha}$ is an order preserving, intersection preserving map;

(ii) $\bar{\alpha}$ is a homeomorphism with respect to the Zariski topologies.

(iii) The map α_{pv}: PrimSpecS/G $\to$ PrimSpecT given by $\bar{\alpha}_{pv}([P]) = \bar{\alpha}([P])$ is a homeomorphism.

(iv) If we restrict the domain and codomain of $\bar{\alpha}$, in order to define a new function $\bar{\alpha}_r$: $(Spec_I S)/G \to Spec_f T$ then all of the above properties are preserved.

PROOF For (i), (ii) and (iii), since $\bar{\alpha}$ coincides with the $\bar{\alpha}$ of Theorem 3.1.17, we only have to show that $\hat{\alpha}$ preserves intersections. Let $P_i \in SpecS$ ($i \in J$, J finite). Define $\hat{P}_i = \alpha(P_i)$. Certainly $\hat{\alpha}(\cap P_i) \subseteq \cap \hat{P}_i$. Let $\hat{Q}_l$ $(l \in L)$ be the minimal primes over $(\cap P_i^O)*G$. So $\hat{\alpha}(\cap P_i) = \cap \hat{Q}_l$. Now $\hat{Q}_l \cap S = Q_l^O$ for some $Q_l \in SpecS$ by Lemma 1.2.10. So $\cap P_i^O \subseteq Q_l$. Thus, there exists $i \in J$ such that $P_i \subseteq Q_l$. Moreover, $P_i^O*G \subseteq Q_l^O*G \subseteq \hat{Q}_l$ and so, since $\hat{P}_i$ is the unique minimal prime over P_i^O*G, we have that $\hat{P}_i \subseteq \hat{Q}_l$. Thus, we've shown $\cap_{i \in I} \hat{P}_i = \cap_{l \in L} \hat{Q}_l$ so that $\cap_{i \in I} \alpha(P_i) = \hat{\alpha}(\cap_{i \in I} P_i)$.

For (iv) it remains to show that $\bar{\alpha}_r([P])$ is a member of $Spec_f T$ when $P \in Spec_I S$. Let $P \in Spec_I S$. By Corollary 3.2.7, $Spec_f S = Spec_I S$. By definition of $Spec_f S$, there exists $\hat{P} \in Spec_f T$ minimal over P^O*G. Proposition 1.2.12 yields that $\hat{P}$ is the unique prime of T minimal over P^O*G. Thus, $\bar{\alpha}_r([P]) = \hat{P}$.

Now, we may compose $\bar{\alpha}$ and β to obtain a bijection between $(Spec_I S)/G$ and $Spec_t R$.

3.2.10 THEOREM *Let S be a Noetherian ring of characteristic q, $|G| = q^a$. Then the map φ: $Spec_I S/G \to Spec_t R$ given by $\varphi = \beta \circ \bar{\alpha}$ is an order preserving map which may be extended to intersections. Also, φ is an inclusion* *preserving homeomorphism. Moreover, if we restrict φ to primitive ideals, we get a bijection between these subsets.*

PROOF This comes from Theorem 3.1.17 and Theorem 3.1.5 since $\beta = \overline{\beta}$ and $Spec_I S = Spec_f S$ by Corollary 3.2.9.

We now look at the special case of Theorem 3.1.9. As observed after Theorem 3.1.9, $\sim$ is trivial when $charS = q$ and G is a q-group. We use this fact together with Theorem 3.1.9 to provide a unique identification of $\beta(\hat{P})$ for any $\hat{P} \in Spec_f T$.

3.2.11 COROLLARY *Let S be a ring of characteristic q and G a group of automorphisms of S of order q^a (q prime, $a \in \mathbb{N}$). Suppose $\hat{P} \in Spec_f T$. Then $\beta(\hat{P})$ is the unique prime of R minimal over $\hat{P} \cap R$ not containing $tr(S)$.*

PROOF Theorem 3.1.9 shows that $\beta(\hat{P})$ is minimal over $\hat{P} \cap R$. If $q \in Spec_t R$ is minimal over $\hat{P} \cap R$, then $p \sim q$ but as noted above $\sim$ is trivial on $Spec_t R$ in this case, so $q = \beta(\hat{P})$.

We now take advantage of the fact that ρ is trivial on $SpecT$ in the q-case to see to how a prime in $Spec_f T$ is uniquely determined by its intersection with R.

3.2.12 COROLLARY *Let S be a ring with $char\ S = q$ and G a group of automorphisms of S with $|G| = q^a$ (q prime, $a \in \mathbb{N}$). Suppose that $\hat{P}, \hat{Q} \in SpecT$ with $f \notin \hat{P}$ satisfy $\sqrt{(\hat{P} \cap R)} = \sqrt{(\hat{Q} \cap R)}$. Then $\hat{P} = \hat{Q}$. That is, $\hat{P}$ is entirely determined by its intersection with R.*

PROOF If $f \in \hat{Q}$, then Lemma 3.1.8 shows that $(tr(S))^n \subseteq \hat{Q} \cap R = \hat{P} \cap R$ and the reverse direction of the lemma then shows that $f \in \hat{P}$. This contradiction shows that $\hat{Q} \in Spec_f T$ and so we may apply Corollary 3.1.11. This shows that $\hat{P}$ and $\hat{Q}$ are in the same ρ-class. The observation on ρ in 3.1.2 gives the final part of the result.

We conclude this section by giving the special version of Theorem 3.1.21.

3.2.13 THEOREM *Let S be a Noetherian ring of characteristic q and G a subgroup of Aut S of order q^a. Then*

(i) Given $P \in Spec_I S$, there exists $p \in Spec_t R$ such that p is the unique prime minimal over $P \cap R$ not containing the trace.

(ii) Given $p \in Spec_t R$, there exists $P \in Spec_I S$ such that p is minimal over $P \cap R$. Moreover P is unique up to its G-orbit.

PROOF Given that $Spec_I S = Spec_f S$ by Theorem 3.2.5, this result is immediate from Theorem 3.1.21 and Corollary 3.2.11.

§3.3 Applications

Here we exploit the results of the previous two sections in order to relate the properties of corresponding primes. This section culminates with certain ring theoretic properties which are retained when passing from one ring to another.

The first properties we investigate are those of height and coheight of prime ideals. First we define these concepts.

3.3.1 DEFINITION Let U be a ring. Consider a chain of prime ideals in U:

$$P_0 \subset P_1 \subset \ldots \subset P_n, \quad P_i \in SpecU, \ (0 \leq i \leq n).$$

We define the length of such a chain to be n. If P is a fixed prime, we define the height of P, $ht(P)$, to be the maximum length of any such chain with $P = P_n$. There may not be a chain of maximal length, in which case the height of P is said to be infinite. We may also define the coheight of a prime P, $coht(P)$. This is just the maximal length of a chain above with $P = P_0$. If there does not exist such a chain we say that P has infinite coheight.

3.3.2 LEMMA *Let S be a ring, G a finite subgroup of $Aut(S)$, $\hat{P} \in Spec_f T$ and $p = \beta(\hat{P})$. Then $ht(\hat{P}) = ht(p)$.*

PROOF Let $\hat{P}_0 \subset \hat{P}_1 \subset \hat{P}_2 \subset \ldots \subset \hat{P}_n = \hat{P}$ $(n \in \mathbb{N})$ be a chain of primes in T. Since $\hat{P} \in Spec_f T$, $\hat{P}_i \in Spec_f T$ for all i. Then $p_0 \subset p_1 \subset p_2 \subset \ldots \subset p_n = p$ where $p_i = \beta(P_i)$ $(i=1,\ldots,n)$ is a chain of primes in R by Theorem 3.1.5. So $ht(\hat{P}) \leq ht(p)$. Similarly using β^{-1}, we can prove the opposite inequality.

3.3.3 LEMMA *Let S be a ring, G a finite subgroup of $Aut(S)$ with the trace map surjective. Suppose $\hat{P} \in Spec_f T$ and $p = \beta(\hat{P})$, then $coht(p) \leq coht(\hat{P})$.*

PROOF Let $\hat{P} \in Spec_f T$ and $p = \beta(\hat{P})$. Suppose p has coheight at least n, with $p = p_0 \subset p_1 \subset p_2 \subset \ldots \subset p_n$ $(p_i \in SpecR)$ a chain of primes in $SpecR$. Since $tr_G(S) = R$, $p_i \in Spec_t R$ for $i = 1, \ldots, n$. Thus, we may apply β^{-1} to the p_i to get a chain of primes in T. This chain shows us that $coht(\hat{P}) \geq n$.

It turns out that the hypothesis, $tr : S \to R$ surjective, in Lemma 3.3.3 is necessary as the following example shows.

3.3.4 EXAMPLE *We now give an example where $coht(\hat{P}) < coht(p)$.*
Let $H = \langle x, y, z : [x,y] = z,\ z \text{ central} \rangle$, the first Heisenberg group, k a field of characteristic 2 with an element $\lambda \in k$ such that λ is not a root of unity. Let g be the automorphism of order 2 such that $x^g = x^{-1}$, $y^g = y^{-1}$, $z^g = z$. Let $M := (z-\lambda)S$. Then T has a maximal ideal $M*G$ in $Spec_f T$. However, $\beta(M*G)$ is not maximal.

PROOF We show first that M is maximal. Now, $S/M \cong k[x, x^{-1}][y, y^{-1}; \sigma]$ where $\sigma(x) = \lambda^{-1}x$. Note that S/M is a free $k[x, x^{-1}]$ - module with basis the powers of y. Consider J, a non-zero ideal of S/M. Let h be a non-zero element of J such that all powers of y are positive and h is of minimal degree in y. Then there exists $n \in \mathbb{N}$ such that $h = g_0(x) + g_1(x)y + \ldots + g_n(x)y^n$ where $g_i(x) \in k[x, x^{-1}]$ $(i = 1, \ldots, n)$. Now,

$$xhx^{-1} = g_0(x) + g_1(x)\lambda y + \ldots + g_n(x)\lambda^n y^n.$$

So $\lambda^n h - xhx^{-1} = (\lambda^n - 1)g_0(x) + (\lambda^n - \lambda)g_1(x)y + \ldots + (\lambda^n - \lambda^{n-1})g_{n-1}(x) \in J$ has degree less than h and is non-zero unless $n = 0$. So $n = 0$. By symmetry we may do the same for x. This shows that a minimal element of J is in fact a member of the field. Thus $J = S/M$. This proves the claim.

We now show that g acts as an outer automorphism on S/M. By considering a degree argument, it's clear that there does not exist $u \in S/M$ such that $xu = ux^{-1}$ and we can conclude that g acts as an outer automorphism on S/M.

Thus, by Theorem 1.4.6(iv), $M*G$ is a maximal ideal of T. So, by Theorem 3.1.5, $\beta(M*G) = M \cap R$. Now, since $x + x^{-1} \in tr(S)\setminus M$, $tr_G(S) \not\subseteq M \cap R$ and so $(tr_G(S) + (M \cap R))/(M \cap R)$ is a non-zero ideal of $R/(M \cap R)$. We show that this is in fact a proper ideal. Suppose it is not a proper ideal. Now, G acts on S/M and we may conclude that the map $tr : S/M \longrightarrow (S/M)^G$ is onto. Let $\Sigma_{a,b}\, k_{ab}x^a y^b \in S/M$. Then $tr(\Sigma_{a,b}\, k_{ab}x^a y^b) = \Sigma_{a,b}\, k_{ab}(x^a y^b + x^{-a}y^{-b})$ which clearly cannot equal 1. So $M \cap R$ is strictly contained in a proper ideal is therefore not maximal. Thus $coht(M \cap R) \geqslant 1$ while $coht(M*G) = 0$.

We have a corollary to Theorem 1.2.11 relating height and coheight of primes in S to the height and coheight of the corresponding prime of T.

3.3.5 COROLLARY *Let S be a ring and G a finite group of automorphisms of S. Suppose $P \in SpecS$ and that $\hat{P}$ is minimal over P^o*G. Then $ht(P) = ht(\hat{P})$ and $coht(\hat{P}) \leqslant coht(P)$. Furthermore, there exists $\hat{Q} \in [\hat{P}]$ such that $coht(P) = coht(\hat{Q})$.*

PROOF Suppose $P_0 \subset P_1 \subset \ldots \subset P_n = P$ is a chain of primes in S. Now $P_{n-1}{}^o*G \subset P_n{}^o*G \subseteq \hat{P}$ and since $\hat{P} \cap S = P_n{}^o$, Theorem 1.2.11(i) shows that $\hat{P}$ is not minimal over $P_{n-1}{}^o*G$ and that there exists $\hat{P}_{n-1}$ minimal over $P_{n-1}{}^o*G$ with $\hat{P}_{n-1} \subset \hat{P}_n$. Continuing in this manner we construct a chain of primes in T of length n, proving $ht(P) \leqslant ht(\hat{P})$. Conversely, let $\hat{P}_1 \subset \hat{P}_2 \subset \ldots \subset \hat{P}_m = \hat{P}$ be a chain of primes in T. Since, by Theorem 1.2.11(i), $\hat{P}_j \cap S \subset \hat{P}_{j+1} \cap S$ for $j = 0,\ldots,m-1$, we have a chain $P_0{}^o \subset P_1{}^o \subset \ldots \subset P_m{}^o = P^o$ where $P_j{}^o = \hat{P}_j \cap S$ for $j = 0,\ldots,m$. Since $P_{m-1}{}^o \subset P_m{}^o \subseteq P$, there exists $h \in G$ such that $P_{m-1}{}^h \subseteq P$. Continuing this way, we construct a chain of primes in S of length m, proving $ht(\hat{P}) \leqslant ht(P)$. Thus, $ht(P) = ht(\hat{P})$.

Now we consider the coheight. Let $\hat{P} = \hat{P}_0 \subset \hat{P}_1 \subset \ldots \subset \hat{P}_n$ be a chain in T. As we did above we construct a chain of length n in S, starting with P_n

and working down to P^h for some $h \in G$ where $P^o = \hat{P} \cap S$. Since $S/P \cong S/P^h$, we have $coht(\hat{P}) \leq coht(P^h) = coht(P)$ as claimed. Suppose now that $P = P_0 \subset P_1 \subset \ldots \subset P_n$ is a chain of primes in S. As we did when considering $ht(P)$, we construct a chain $\hat{P}_0 \subset \hat{P}_1 \subset \ldots \subset \hat{P}_n$ in T with $\hat{P}_i$ minimal over $P_i{}^o{*}G$. So $coht(\hat{P}_0) \geq coht(P_0)$. The above inequality gives us that $coht(\hat{P}_0) = coht(P_0)$. Since $\hat{P}$ and $\hat{P}_0$ are minimal over $P_0{}^o{*}G$ they are ρ-equivalent.

3.3.6 NOTE The triviality of the ρ-classes when $charS = q$, $|G| = q^a$ ensures that both height and coheight are preserved in the q-case. We exploit this later in 3.3.10.

We use $\overline{\alpha}$ to look at the corresponding result when we restrict to the case where $charS = q$ and G is a q-group.

3.3.7 COROLLARY *Let S be a Noetherian ring of characteristic q, G a finite group of automorphisms of S of order q^a and $\overline{\alpha}$ as defined in 3.2.9. Let $P \in SpecS$ and $\hat{P} := \overline{\alpha}([P])$. Then $ht(P) = ht(\hat{P})$ and $coht(P) = coht(\hat{P})$.*

PROOF This is immediate from Corollary 3.3.5.

Finally, we look at the relationship between primes in S and those in R. The next lemma shows that *height* is constant on $\sim$-classes in $Spec_tR$.

3.3.8 LEMMA *Let S be a ring and G a finite group of automorphisms of S. Suppose $p,q \in Spec_tR$ both lie under $P \in SpecS$. Then $ht(p) = ht(q) = ht(P)$.*

PROOF Let $p_0 \subset p_1 \subset \ldots \subset p_n = p$ ($p_i \in SpecR$) be a chain of primes in R. Corollary 3.1.23(iv) shows that we can find $q_{n-1} \in SpecR$ with $q_{n-1} \sim p_{n-1}$ and $q_{n-1} \subset q$. Repeating this process we find a chain of length n inside q.

So we've shown that $ht(p) \leq ht(q)$. By symmetry $ht(p) = ht(q)$. For the same chain in R, we use Corollary 3.1.23(iii) to find $P_{n-1} \in SpecS$ lying over p_{n-1} with $P_{n-1} \subset P$. Repeating the process gives a chain of length n in $SpecS$ inside P proving $ht(p) \leq ht(P)$. Similarly, Corollary 3.1.23(i) proves the opposite inequality, giving $ht(p) = ht(P)$.

3.3.9 EXAMPLE *We give an example where S is a ring, G a finite group of automorphisms of S and there exists $P \in Spec_t S$ lying over $p \in Spec_t R$ with $coht(P) \neq coht(p)$.* In Example 3.3.4, we take P to be the maximal ideal M and p to be the non-maximal ideal of that example. So $0 = coht(P) < coht(p)$.

We take advantage of the above relationships to look at a ring theoretic property derived from height.

3.3.10 DEFINITION In a ring U, two primes, $P \subset Q$ are said to be neighbouring if there does not exist $W \in SpecU$ with $P \subset W \subset Q$. The ring U is said to satisfy the *saturated chain condition* (SCC) or be *catenary* if neighbouring primes differ in height by 1. This is equivalent to the property that all descending chains of neighbouring primes down from a given prime have the same length.

The following lemma shows that SCC is inherited by R from T when the trace map is onto.

3.3.11 LEMMA *Let S be a ring and G a finite subgroup of $AutS$ such that T has the saturated chain condition. Suppose that the trace map is surjective. Then R also has SCC.*

PROOF Let $p_1 \subset p_2$ be neighbouring primes in $SpecR$. Since β^{-1} preserves order, $\beta^{-1}(p_1) \subset \beta^{-1}(p_2)$ are neighbouring primes in T. By hypothesis,

$ht(\beta^{-1}(p_1)) = ht(\beta^{-1}(p_2)) -1$ and Lemma 3.3.2 completes the proof.

Again we shall show that the hypothesis in Lemma 3.3.11 that the trace map is onto is necessary. We show in Example 4.2.12 a ring S which has SCC, the trace map is not surjective and R does not have SCC.

It is a well known open question whether T has SCC when S does and G is finite. See [L, Remarks(ii)]. We prove an easy positive result here. When $charS = q$ and G is a q-group, we use $\bar{\alpha}$ to show that T has SCC if S has SCC.

3.3.12 COROLLARY *Let S be a Noetherian ring of characteristic q, G a finite group of automorphisms of S of order q^a. If S satisfies the saturated chain condition, then so too does T.*

PROOF Suppose $\hat{P} \subset \hat{Q}$ are neighbouring primes in T. There exist $P,Q \in SpecS$ such that $P^o = \hat{P} \cap S$ and $Q^o = \hat{Q} \cap S$ and since $\hat{P} \subset \hat{Q}$ we have $P^o \subset Q^o$. Since Q is prime, we have that $P^h \subseteq Q$ for some $h \in H$. Without loss of generality we may assume that $h = 1$. Moreover, $\bar{\alpha}([P]) = \hat{P}$ and $\bar{\alpha}([Q]) = \hat{Q}$. Since $\hat{P}$ and $\hat{Q}$ are neighbouring primes and α preserves inclusions, P and Q are neighbouring in $SpecS$. By hypothesis, $ht(P) = ht(Q) - 1$. Lemma 3.3.7 completes the proof.

We now relate the Goldie dimension of corresponding primes. Initially, we investigate the results yielded by the Morita correspondence.

3.3.13 PROPOSITION *Let S be a ring and G a finite group of automorphisms of S. Suppose $\hat{P} \in Spec_f T$ with $\beta(\hat{P}) = \{r \in R: rf \in \hat{P}\} =: p$, say, and suppose $\hat{P} \cap S = P^o$ for $P \in SpecS$.*

(i) If $T/\hat{P}$ is right Goldie, then so too is R/p.

(ii) If S/P is right Goldie, then $T/\hat{P}$ is right Goldie and consequently,

R/p is right Goldie.

In both of these cases, we have $u.dim_R(R/p) \leq u.dim_T(T/\hat{P})$.

PROOF Suppose $T/\hat{P}$ is right Goldie. That R/p is right Goldie is a straight application of [McC-R, Corollary 3.6.7].

Suppose now that S/P is right Goldie. Consider the ring $\overline{T} = T/(P^o*G)$. This ring has a right Artinian right quotient ring, namely $Q(S/P^o)*G$. By Theorem 1.2.11(i), $\hat{P}$ is minimal over P^o*G and so, $\overline{P} := \hat{P}/(P^o*G)$ is a minimal prime of $\overline{T}$. [McC-R, Theorem 4.1.4] shows that the factor ring of $\overline{T}$ by its prime radical is right Goldie. Applying [G-W, Proposition 6.1] yields that $\overline{T}/\overline{P} \cong T/\hat{P}$ is right Goldie.

Finally, we prove the inequality regarding uniform dimensions. The proof of Proposition 1.2.4, together with Proposition 1.2.7 shows that

$$\begin{bmatrix} \dfrac{T}{\hat{P}} & \dfrac{Sf}{Sf \cap \hat{P}} \\ \dfrac{fS}{fS \cap \hat{P}} & \dfrac{R}{p} \end{bmatrix}$$

is a prime context. The proof of [McC-R, Theorem 3.6.6] shows that $u.dim_R(R/p) = u.dim_T(fS/(fS \cap \hat{P}))$. Now, $fS/(fS \cap \hat{P})$ is isomorphic to the cyclic right ideal of $T/\hat{P}$, $(f+\hat{P})T/\hat{P}$. Thus, $u.dim_T(fS/(fS \cap \hat{P}) \leq u.dim_T(T/\hat{P})$. This proves the lemma.

3.3.14 NOTE In some special cases we can improve on Proposition 3.3.13.

A sharper calculation in Proposition 3.3.13 shows the following result. Let S be any ring and G a finite group of automorphisms of S. Let $P \in SpecS$ such that $P^o*G \in SpecT$. Then $p = P \cap R$ is the unique prime lying under P and $u.dim(R/p) \leq |G| u.dim(S/P^o)$.

Using Theorem 3.2.4, we can produce a different inequality, involving $u.dim(S/P^o)$ in the q-case. Let S be a Noetherian ring of characteristic q, G a finite group of automorphisms of S of order q. If $P \in Spec_f S$ and $\hat{P} \in Spec_f T$ is the unique prime of T minimal over P^o*G, then

$u.dim(T/\hat{P}) \leq q(u.dim(S/P^o))$. (Note that by Proposition 1.2.12, $\hat{P}$ is unique and, by definition of $Spec_f S$, $\hat{P} \in Spec_f T$).

We now state our final result on uniform dimension.

3.3.15 PROPOSITION *Let S be a ring of characteristic q, $|G| = q$, $P \in Spec_I S$ and let $p = \varphi([P])$ as defined in 3.1.18. If S/P is right Goldie, R/p is right Goldie and $u.dim(R/p) \leq u.dim(S/P^o)$.*

PROOF Let A be the right T-module $\{s \in S: fs \in \hat{P}\}$. The proof of Proposition 3.3.13 gives us that $u.dim(S_T/A) = u.dim(R/p)$. Thus, $u.dim(R/p) \leq u.dim(S_S/A_S)$.

Let $\hat{P}$ be the prime of T minimal over P^o*G. If $\hat{P} = (\hat{P} \cap S).T$ then $A = P^o$ and we must have $u.dim(R/p) \leq u.dim(S_S/P^o)$. Henceforth, we assume $\hat{P} \supset (\hat{P} \cap S).T$. We adopt the following notation: $\overline{S} := S/P$, $\overline{P} := \hat{P}/(P^o*G)$, $C := C_S(0)$, $Q := \overline{S}C^{-1}$ and $\overline{T} := \overline{S}*G$. By Lemma 3.2.1, $\overline{T}C^{-1} = Q*G$. As in Theorem 3.2.4, g is induced by a unit, U, of Q of order q and $\overline{P}C^{-1}$ is the augmentation ideal of $Q\langle U^{-1}g\rangle$. Consequently, $g \equiv U$ (modulo $\overline{P}C^{-1}$). With $\hat{U} = 1 + U + \ldots + U^{q-1}$, we also have $f \equiv \hat{U}$ (modulo $\overline{P}C^{-1}$).

Let $s \in S$. Then

$$\begin{aligned} s \in A &\Leftrightarrow fs \in \hat{P} \\ &\Leftrightarrow \overline{fs} \in \overline{P} \\ &\Leftrightarrow \overline{fs} \in \overline{P}C^{-1} \qquad (\Leftarrow \text{ is [G-W, Theorem 9.22]}) \\ &\Leftrightarrow (\overline{f} + \overline{P}C^{-1}).(\overline{s} + \overline{P}C^{-1}) = 0_{T/P} \\ &\Leftrightarrow (\hat{U} + \overline{P}C^{-1}).(\overline{s} + \overline{P}C^{-1}) = 0 \\ &\Leftrightarrow \hat{U}\overline{s} = 0_Q. \end{aligned}$$

We consider the map φ of S/P-modules given by $\varphi : S/P \to Q$ such that $\varphi(\overline{s}) = \hat{U}\overline{s}$. This gives rise to the isomorphism: $S/A \cong \hat{U}(S/P)$, a submodule of $Q_{S/P}$. Thus, $u.dim(R/p) \leq u.dim(S_S/A_S) \leq u.dim(Q_{S/P}) = u.dim(S/P)$. Thus, the inequality is satisfied in both cases.

An easy observation will reap many results regarding the ring R/p.

3.3.16 NOTE Let S be a ring, let G a finite group of automorphisms of S, let $R = S^G$ and let $T = S*G$. Suppose $p \in Spec_t R$ and that $\beta^{-1}(p) =: \hat{P} \in Spec_f T$. By Lemma 1.2.10, $\hat{P} \cap S = P^o$ for some $P \in Spec S$. Certainly, $R/(P \cap R)$ embeds in both S/P and $T/\hat{P}$. By Theorem 3.1.9, p is minimal over $P \cap R$. Using these facts, it's easy to see that if P or $\hat{P}$ is completely prime, then p is also completely prime. Similarly, if S/P or $T/\hat{P}$ satisfy a polynomial identity, then so does R/p.

Now, we reach the main results of this section. We exploit the maps α and β to emphasise the close relationship between ring theoretic properties of R and S. The general strategy here is to use α and β to understand primes not containing the trace and to look at the factor $R/tr_G(S)$ separately.

3.3.17 DEFINITION Suppose K is a commutative ring. A ring U is said to be a K-algebra if there exists a ring homomorphism, φ, from K to the centre of U. For a subset X of U, we define $K \cap X$ to be $\varphi^{-1}(X \cap \varphi(K))$.

Usually we may assume that K embeds in the ring W by factoring out $ker\varphi$. When this is not possible, for example in Theorem 3.3.20, we have to consider the map φ.

3.3.18 DEFINITION Let k be a field and W a k-algebra. We say that W *has the endomorphism property over* k if $End_k(V)$ is algebraic over k for all irreducible W-modules V.

3.3.19 DEFINITION Let K be a commutative ring. A K-algebra U is said *to have the primitive property over* K if, whenever P is a primitive ideal of

U, $P \cap K$ is a maximal ideal of K and U/P has the endomorphism property over the field $K/(P \cap K)$.

3.3.20 LEMMA *Let K be a commutative ring and let S be a K-algebra. Suppose that G is a finite group of K-automorphisms of S. If S has the primitive property over K, then so too does $T := S*G$.*

PROOF Let V be an irreducible T-module. By [P2, Proposition 4.10], $V_S = V_1 \oplus \ldots \oplus V_n$ for some $n \in \mathbb{N}$ and irreducible S-modules, $V_1, \ldots, V_n$. By rearranging if necessary, we take $V_1, \ldots, V_t$ to be representatives of the t homogeneous components in V_S. Since S has the primitive property over K, we must have that for $i=1,\ldots,n$, with $P_i = ann_S(V_i)$, $P_i \cap K = M_i$ for some maximal ideals M_i of K. If $M_i \neq M_j$ for some i,j, then $Ann_V(M_i)$ would be a nonzero proper T-submodule of V, which is impossible. Thus, each M_i equals, say, M. Let D_i be the division ring $End_S(V_i)$ for $i=1,\ldots,n$. By hypothesis, $D_1,\ldots,D_n$ are algebraic over K/M. Now,

$$End(V_S) \cong M_{n_1}(D_1) \oplus \ldots \oplus M_{n_t}(D_n)$$

for some $n_1,\ldots,n_t \in \mathbb{N}$. Thus, $End(V_S)$ is algebraic over K/M. Since, $End(V_T)$ embeds into $End(V_S)$, we have that $End(V_T)$ is algebraic over K/M.

3.3.21 THEOREM *Let S be a K-algebra such that K embeds in S and G a finite group of K-automorphisms of S. If S and $R/tr_G(S)$ have the primitive property over K then so too does R.*

PROOF Note first that Lemma 3.3.20 shows that T has the primitive property over K. Let $\varphi: K \to C(R)$, where $C(R)$ denotes the centre of R, and p be a primitive ideal of R. Consider first the case where $tr(S) \subseteq p$. We have to show that $p \cap K$ is maximal. Now we have that $p/tr(S)$ is a primitive ideal of $R/tr(S)$ and, since $R/tr(S)$ has the primitive property,

$$\varphi^{-1}(p/tr(S) \cap (K + tr(S))/tr(S)) = \varphi^{-1}(((p \cap K) + tr(S))/tr(S))$$

is maximal in K. We claim that this is just $p \cap K$. For, if $\varphi(x) \in ((p \cap K) + tr(S))/tr(S)$, we have that $x + tr(S) \in (p \cap K) + tr(S)$ so that there exists $s \in S$, $y \in p \cap K$ such that $x = y + tr(s)$. However, $tr(s) = x - y \in K \cap tr(S) \subseteq K \cap p$. Since $tr(s) \in K \cap p$ and $y \in K \cap p$, we get $x = y + tr(s) \in K \cap p$, proving the claim. Thus, $K/(p \cap K)$ is a field which we denote by k. Also, we have that $(R/tr(S))/(p/tr(S)) \cong R/p$ has the endomorphism property over k.

Henceforth, suppose $tr_G(S) \not\subseteq p$. Then $\beta^{-1}(p) = \hat{P}$ is primitive in $T = S*G$ by Theorem 3.1.5. Since T has the primitive property, $\hat{P} \cap K$ is maximal in K and $T/\hat{P}$ has the endomorphism property over $K/(\hat{P} \cap K)$. By Theorem 3.1.5, we see that $\hat{P} \cap K \subseteq p \cap K \neq R$ and so $p \cap K = \hat{P} \cap K$ is a maximal ideal. Again we adopt the notation that $k = K/(p \cap K)$. It remains to show that R/p has the endomorphism property over k. Let M be an irreducible R/p-module. Now, $M \cong R/X$ for some maximal right ideal X of R. Fix $0 \neq m \in M$ and set $Y = \{ u \in S : m.tr_G(uS) = 0 \}$. As in proof of Proposition 1.2.4(ii), S_T/Y is an irreducible $T/\hat{P}$-module. Taking $m = 1 + X$, $Y = \{ u \in S : tr(uS) \subseteq X \}$.

We show that $Y \cap R = X$. Let $u \in X$. Then $tr_G(uS) = utr_G(S) \subseteq X$. Hence, $X \subseteq Y \cap R$. Conversely, suppose $y \in Y \cap R$. Now, $ytr_G(S) = tr_G(yS) \subseteq X$. If $My \neq 0$, $(My)R = M$ and so, $Mtr_G(S) = (MyR)tr_G(S) = Mytr_G(S) \subseteq MX = 0$. But $tr(S) \not\subseteq p$ and this contradiction shows that $My = 0$ so that $y \in X$.

The above shows that R/X embeds as an R/p-module into S_T/Y. Let $\psi \in End_R(R/X)$ and suppose $\psi(1 + X) = r + X$ for $r \in R$. We have that $rX \subseteq X$ and so, for $u \in Y$, $tr_G(ruS) = rtr_G(uS) \subseteq rX \subseteq X$. This shows that $ru \in Y$ for arbitrary $y \in Y$, so $rY \subseteq Y$. Thus, we may define a map $\psi' \in End(S_T/Y)$ such that $\psi'(s+Y) = rs + Y$. Since ψ' restricted to M is just ψ and ψ' is algebraic over k, ψ is algebraic over k.

3.3.22 DEFINITION A ring is said to be *Jacobson* if all its prime ideals are semiprimitive.

We now prove an analogue of Theorem 3.3.21 for Jacobson rings. Warfield proved this result for the case $|G|^{-1} \in S$ in [W1, Corollary 1.4].

3.3.23 THEOREM *Let S be a ring and G a finite group of ring automorphisms of S. If S and $R/tr_G(S)$ are both Jacobson rings, R is also Jacobson.*

PROOF First note that, by [P2, Theorem 22.3], T is a Jacobson ring. Let $p \in SpecR$. Then either (i) $tr_G(S) \subseteq p$ or (ii) $tr_G(S) \not\subseteq p$. Suppose (i) is the case. Then $p/tr_G(S)$ is a prime ideal of $R/tr_G(S)$. By hypothesis, $R/tr_G(S)$ is a Jacobson ring and so $p/tr_G(S)$ is semiprimitive and so p is semiprimitive. Henceforth assume we are in case (ii). We apply Theorem 3.1.5 and let $\hat{P} = \beta^{-1}(p)$. Then $f \notin \hat{P}$. Since T is a Jacobson ring, $\hat{P}$ is semiprimitive so that $\hat{P} = \cap\{\hat{Q} \in SpecT\text{: } \hat{Q} \text{ primitive in } T,\ \hat{P} \subseteq \hat{Q}\}$.

Now let $A = \cap\{\hat{Q} \in SpecT\text{: } \hat{Q} \text{ primitive in } T,\ \hat{P} \subseteq \hat{Q}, f \notin \hat{Q}\}$ and let $B = \cap\{\hat{Q} \in SpecT\text{: } \hat{Q} \text{ primitive in } T,\ \hat{P} \subseteq \hat{Q},\ f \in \hat{Q}\}$. Then $\hat{P} = A \cap B$. Noting that $AB \subseteq \hat{P}$ and $f \in B$, we must have $A = \hat{P}$. So $p = \beta(\hat{P}) = \beta(A) = \cap\beta\{\hat{Q} \in SpecT\text{: } \hat{Q} \text{ primitive in } T,\ \hat{P} \subseteq \hat{Q}, f \notin \hat{P}\}$, an intersection of primitives by Theorem 3.1.5. Thus, in either case, p is semiprimitive and, since p is arbitrary, R is a Jacobson ring.

In Chapter 4, Example 4.1.7, we give an example where T is Jacobson but R is not Jacobson in order to show that the hypothesis that $R/tr(S)$ is Jacobson is in fact necessary.

We now combine the last two definitions in order to give a non-commutative version of Hilbert's Nullstellensatz.

3.3.24 DEFINITION. U, an algebra over a commutative ring K, is said *to satisfy the Nullstellensatz over K* if U is a Jacobson ring and it has the primitive property over K.

3.3.24 COROLLARY. *Let K be a commutative ring and let S be a K-algebra acted on by G, a group of K-automorphisms. Suppose S satisfies the Nullstellensatz over K. Suppose further that $R/tr_G(S)$ also satisfies the Nullstellensatz over K. Then R must also have this property.*

PROOF This is immediate from Theorems 3.3.21 and 3.3.23.

CHAPTER 4

PRIME IDEALS IN GROUP RINGS

Here we consider the case where, for a commutative ring K and a group H, S is the group ring KH. We take a finite group G of automorphisms of H and we extend the action of G to K-automorphisms of S. Under these hypotheses, we have that $T = KH * G \cong K(H \rtimes G)$ is itself a group ring, this time of the semi-direct product of H by G. In particular, we are interested in the case where K is a field and H is polycyclic-by-finite.

Recall that, in Chapter 3, with hypotheses on S and $R/tr(S)$, we discovered that R inherits some of the properties of S. In §1, we prove the following fundamental result that, in certain circumstances, shows exactly what the factor ring $R/tr(S)$ is.

4.1.2 COROLLARY. *Let H be a polycyclic-by-finite group, K a commutative ring, S the group ring KH and G an automorphism group of H of order q. Then $R/tr_G(S)$ is itself the group ring of a polycyclic-by-finite group over (K/qK), namely $(K/qK)C_H(G)$.*

We go on to establish whether, when S is a group algebra of a polycyclic-by-finite group, R inherits some of the well known properties of the ring S. For example, J.E. Roseblade has shown that, when k is absolute, the primitive ideals of kH are all maximal and have finite codimension. In Theorem 4.1.15, we show that these properties pass down to R under the hypotheses of Corollary 4.1.2. We go on to discuss the following question.

QUESTION 4A Let S be the ring KH where K is a commutative Jacobson ring and H is polycyclic-by-finite. Let G be a finite group of automorphisms of H so that G acts as K-automorphisms on S. Is it the case that S^G satisfies the Nullstellensatz over K ?

Recall that, in Chapter 2, §2, we have already discussed whether or not R is Noetherian when S is the group algebra of a polycyclic-by-finite group and been unable to answer that question fully.

Section 2 is primarily devoted to the study of the prime rank of the ring R but we do also address the following question and answer it negatively.

QUESTION 4C Suppose H is a nilpotent group and k is a field. Let S denote the group algebra kH. Suppose G acts as k-automorphisms on S. Does S^G have SCC ?

§4.1 Key Lemma and Applications

We use Chapter 3 to get information regarding primes of R not containing the trace ideal while in certain circumstances, the following key lemma enables us to understand primes outside the Morita correspondence.

4.1.1 LEMMA. *Let U be a ring, M a semigroup and G a subgroup of $AutM$ of prime order, q. Let G act as U-automorphisms on the semigroup ring $S = UM$. Then*

$$R \,/\, tr_G(S) \cong (U/qU).C_M(G).$$

PROOF Let $r = c_1h_1 + c_2h_2 + \ldots + c_th_t \in R$ $(c_i \in U,\ h_i \in M)$. Since $r = r^g$, $Supp(r) = \{h_1, h_2, \ldots, h_t\}$ is G-invariant and so we may divide it into G-orbits. Let $h_1, h_2, \ldots, h_m$ be representatives of each G-orbit (re-ordering if necessary). Each G-orbit has size 1 or q and we may suppose that the first a orbits are singletons. In other words, $h_1, \ldots, h_a \in C_H(G)$. Now, because $r = r^g$, $h_{a+1}{}^g \in Supp(r)$ and it has coefficient c_{a+1} in r. Thus, $c_{a+1}tr(h_{a+1})$ appears in r. In fact it follows that $r = c_1h_1 + \ldots + c_ah_a + c_{a+1}tr(h_{a+1}) + \ldots + c_mtr(h_m)$. Let $\bar{\ }$ denote images modulo qU in the ring U. We consider the map $\psi\colon R \to \bar{U}C_M(G)$ such that $\psi(r) = \bar{c}_1h_1 + \ldots + \bar{c}_ah_a$. It is clear that ψ is a well defined, surjective map. We now show that ψ is a ring homomorphism. Let $r,s \in R$. By the above,

$$r = c_1h_1 + \ldots + c_ah_a + c_{a+1}tr(h_{a+1}) + \ldots + c_mtr(h_m)$$

and

$$s = d_1l_1 + \ldots + d_bl_b + d_{b+1}tr(l_{b+1}) + \ldots + d_ntr(l_n)$$

for some $h_i, l_j \in M$, c_i, $d_j \in U$. Then $rs = \Sigma_{i=1}^{a}\Sigma_{j=1}^{b}\ c_id_jh_il_j + z$ where

$$z = (c_{a+1}tr(h_{a+1}) + \ldots + c_mtr(h_m))(d_{b+1}tr(l_{b+1}) + \ldots + d_ntr(l_n)).$$

Thus, z is a U-linear combination of terms of the form:

$$\begin{aligned}
&(x + \ldots + x^{g^{q-1}})\ (y + \ldots + y^{g^{q-1}}) \\
= \quad & xy + x^gy^g + \ldots + x^{g^{q-1}}y^{g^{q-1}} \\
+ \quad & x^gy + x^{g^2}y^g + \ldots + x\,y^{g^{q-1}} \\
+ \quad & \ldots \\
+ \quad & x^{g^{q-1}}y + xy^g + \ldots + x^{g^{q-2}}y^{g^{q-1}} \\
= \quad & tr(xy) + tr(x^gy) + \ldots + tr(x^{g^{q-1}}y).
\end{aligned}$$

Thus, $\psi(rs) = \Sigma_{i=1}^{a}\Sigma_{j=1}^{b}\ \bar{c}_i\bar{d}_jh_il_j = \psi(r)\psi(s)$, as required. Finally, we show that $ker\psi = tr(S)$. That $tr(S) \subseteq ker\psi$ is clear. Suppose now that $r \in ker\psi$. Then

$$r = c_1h_1 + \ldots + c_ah_a + c_{a+1}tr(h_{a+1}) + \ldots + c_mtr(h_m)$$

where $q \mid c_j$ $j = 1,\ldots,a$. Thus

$$r = tr((c_1/q)h_1 + \ldots + (c_a/q)h_a + c_{a+1}h_{a+1} + \ldots + c_mh_m) \in tr(S).$$

This proves the lemma.

We consider the case where K is a commutative Noetherian Jacobson ring and H is a polycyclic-by-finite group. When $|G|^{-1} \in K$, the fixed ring is very well understood as explained in chapter 1. For example, it is Noetherian by Lemma 1.4.2 and, by Corollary 3.3.27, satisfies the Nullstellensatz. Thus, we study Question 4A in the case where $|G|$ is not a unit in K. An extreme example of this is when $|G| = 0 \in K$. The following corollary helps us to understand the simplest of these cases, namely when G is cyclic of order q.

4.1.2 COROLLARY. *Let H be a polycyclic-by-finite group, K a commutative ring, S the group ring KH and G be an automorphism group of H of order q. Then $R/tr_G(S)$ itself is the group ring of a polycyclic-by-finite group over (K/qK), namely $(K/qK)C_H(G)$.*

PROOF The proof is immediate from Lemma 4.1.1.

4.1.3 COROLLARY. *Let H be a polycyclic-by-finite group and G a group of automorphisms of H with $|G| = q$, prime. Let K be a commutative Noetherian ring and S the group ring KH. Then all prime factors of R are right Goldie.*

PROOF Let $p \in SpecR$. If $tr_G(S) \subseteq p$, then $p/tr_G(S) \in Spec(R/tr_G(S))$. By Corollary 4.1.2, $R/(tr_G(S)) \cong (K/qK)H$, a Noetherian ring and so R/p is right Goldie. Otherwise $p \nsubseteq tr_G(S)$ and so $\beta^{-1}(p) = \hat{P} \in Spec_f T$. Now, $T = (KH)*G$ is a Noetherian ring and so $T/\hat{P}$ is right Goldie. An application of Lemma 3.3.11 shows that R/p is right Goldie.

We now exploit Corollary 4.1.2 in order to enhance our knowledge of the fixed ring of a group ring. In particular, we show how important the primitive ideals are in the study of these fixed rings when the group, H, is polycyclic-by-finite and the coefficient ring is a Jacobson ring. [McC-R, Corollary 9.4.22] give us that polycyclic-by-finite group rings over K satisfy the Nullstellensatz. Thus $S = KH$ and $T = K(H \rtimes G)$ both satisfy the Nullstellensatz. As stated prior to Corollary 4.1.2, when $|G|^{-1} \in K$, R also satisfies the Nullstellensatz. We now show that this property carries over to R when G is a cycle of order q regardless of whether $|G|$ is a unit in K.

4.1.4 THEOREM. *Let H be a polycyclic-by-finite group and K a commutative Jacobson ring. Let S be the group ring KH. Suppose G is a group of automorphisms of H of prime order q. Then R satisfies the Nullstellensatz over K and, in particular, is a Jacobson ring.*

PROOF As explained above, the group ring $(K/qK)C_H(G)$ satisfies the Nullstellensatz. By Corollary 4.1.2, $R/tr(S) \cong (K/qK)C_H(G)$. Since S and $R/tr(S)$ satisfy the Nullstellensatz over K, Corollary 3.3.26 shows R satisfies the Nullstellensatz over K.

Up until now, we have been considering the extreme cases where the order of the group is a unit in the ring S or where the order of the group is prime. With the following two exceptions, Question 4A in intermediate cases remains open.

4.1.5 COROLLARY *Let S be the group ring KH where H is a polycyclic-by-finite group and K is a commutative Jacobson ring with char $K = q$. Suppose that G is a finite subgroup of Aut H with a Sylow q-subgroup, Q, of order q, normal in G such that $|G/Q|^{-1} \in K$. Then R satisfies the Nullstellensatz. In particular, R is a Jacobson ring.*

PROOF We deal with the proof in two parts. First, the Sylow q-subgroup Q acts on S. By Theorem 4.1.4, S^Q satisfies the Nullstellensatz. Now $R = S^G = (S^Q)^{G/Q}$, and so we consider the action of the q'-group G/Q on the ring S^Q. As, $|G/Q|^{-1} \in S^Q$, the trace map, $tr_{G/Q}: S^Q \to S^G$ is surjective and, by Lemma 3.3.26, R satisfies the Nullstellensatz.

4.1.6 LEMMA *Let H be a finitely generated abelian-by-finite group, k a field and S the group algebra kH. Let G be a finite group of k-automorphisms of S. Then $(kH)^G$ satisfies the Nullstellensatz over k.*

PROOF By Lemma 2.5.1, there exists L, a characteristic torsionfree abelian subgroup of finite index in H. Then, by Theorem 1.4.4, $(kL)^G$ is an affine k-algebra and kL is a finitely generated $(kL)^G$-module. Hence, S is a finitely generated $(kL)^G$-module. So, R is an affine k-algebra. Moreover, as R is contained in kH, it satisfies a polynomial identity. [McC-R, Theorem 13.10.3] shows R has the Nullstellensatz over k.

Given the above results, we conjecture that $(kH)^G$ always satisfies the Nullstellensatz over k. This is not true, however, for an arbitrary ring S which satisfies the Nullstellensatz over k. Recall that in Lemma 3.3.24, we have that for any ring S and any finite subgroup G of $AutS$, R is Jacobson when S and $R/tr(S)$ are Jacobson. We now give an example to show that the hypothesis that $R/tr(S)$ is Jacobson is in fact necessary.

4.1.7 EXAMPLE *We give an example of a Jacobson ring, S, which is a localisation of a group algebra and a group, G, acting on S where the fixed ring is not Jacobson.*

Let $S_1 = kH$ where $H = \langle x, y, z: [x, y] = z; z \text{ central}\rangle$ is the first Heisenberg group as in Example 3.3.4 and k is a field of characteristic 2. Let $G = \langle g\rangle$ where $x^g = x^{-1}$, $y^g = y^{-1}$ and $z^g = z$. Let

$C = \{ \hat{z} \in kZ : \hat{z} \notin (z-1)kZ \}$, a central, G-invariant set of regular elements. Thus we may localize at C and G still acts on the ring $S = S_1C^{-1}$.

We show that S is a Jacobson ring. Let $0 \neq P \in SpecS$. By [G-W, Theorem 9.22], $P \cap S_1$ is a non-zero prime of S_1 and so, by the Zaleskii intersection theorem given in [P1, Theorem 9.1.17], $P \cap kZ$ is a non-zero prime of kZ. Since C is invertible, $P \cap kZ$ must be $(z-1)kZ$. Thus, $P/((z-1)kH)$ is a prime of $S_1/((z-1)kH) \cong k[\, x, x^{-1}; y, y^{-1}]$, a Jacobson ring. Thus P is semiprimitive. Finally, we show that 0 is a semiprimitive ideal. Let $U_i = (kZC^{-1})[\, x, x^{-1} \,]y^i \;\; (i \in \mathbb{Z})$. We have that S is a $\mathbb{Z}$-graded ring in that $S = \Sigma^{\oplus}{}_{n\in\mathbb{Z}}\, U_n$ and $U_iU_j \subseteq U_{i+j}$. By [P2, Theorem 22.6], $J(S)$ is a graded ideal with $J(S) \cap U_n$ nilpotent for all $0 \neq n \in \mathbb{Z}$. But S is a domain and so $J(S) \cap U_n = 0$ for all $n \neq 0$. Let $t \in J(S) \cap U_0$. Then $ty \in J(S) \cap U_1 = 0$. Hence, $J(S) = 0$ and so 0 is semiprimitive. We've thus shown that S is a Jacobson ring.

However, we now show that R factored by the trace ideal is not Jacobson so that R itself is not Jacobson. Simulating the argument in Lemma 4.1.1, we find that $R/tr_G(S) \cong kZC^{-1}$, a local, commutative ring which is not a field and, therefore, not Jacobson. In particular, R is not Jacobson.

We have thus established that primitive ideals play an important role in the structure of R, S and T. We can say more about the primitive ideals in S and T. We first recall the following well-known results for group rings. The first of these was proved by A.E. Zalesskii.

4.1.8 THEOREM *Let H be a finitely generated nilpotent group and k any field. Then every primitive ideal of kH is maximal.*

PROOF This theorem is just [P1, 12.2.11].

When the order of the group G is invertible in S, we can provide a direct analogue of Theorem 4.1.8 for the fixed ring.

4.1.9 LEMMA *Let S be the ring kH where k is a field and H is a finitely generated nilpotent group. Let G be a finite group of automorphisms of H such that $|G|^{-1} \in k$. Then every primitive ideal of $(kH)^G$ is maximal.*

PROOF Let p be a primitive ideal of $R := S^G$. Since $tr(S) = R$, $p \in Spec_t R$. By Theorem 3.1.21(ii), p is minimal over $P \cap R$ for some $P \in Spec_f S$. By Lemma 3.1.18, P is a primitive ideal and so, by Theorem 4.1.8, P is a maximal ideal of S. Let $\hat{P} = \beta^{-1}(p)$. Theorem 3.1.9 shows that $\hat{P}$ is minimal over $P^o{*}G$. Now, $coht(p) \leq coht(\hat{P})$ by Lemma 3.3.3 and $coht(\hat{P}) \leq coht(P)$ by Corollary 3.3.5. Since $coht(P) = 0$, we have $coht(p) = 0$, so that p is maximal.

However, as the following example shows, if we remove the hypothesis on the order of the group, we find that there is no analogue of Theorem 4.1.8 for R .

4.1.10 EXAMPLE *There exists a field k, a finitely generated torsion-free nilpotent group H and a finite subgroup of $AutH$ such that R, the fixed ring of kH, has a primitive ideal M which is not maximal.*

See Example 3.3.4. The ideal M in S, being maximal, is certainly primitive. By Theorem 3.2.10, $\varphi(M) = M \cap R$ is primitive. However, as is shown in Example 3.3.4, $M \cap R$ is not maximal.

P. Hall has proved that, in certain circumstances, the irreducible modules over a group algebra of a polycyclic-by-finite group are finite dimensional. See [P1, Corollary 12.2.10]. We indicate in Theorem 4.1.12 how to deduce the following well-known consequence of this result.

4.1.11 DEFINITION A field k is said to be absolute if it is algebraic over some finite field.

4.1.12 THEOREM (J.E. Roseblade) *Let H be a polycyclic-by-finite group, k an absolute field. Then all primitive ideals of kH are maximal and have finite co-dimension over k.*

PROOF Let S denote the group ring kH. Let P be a primitive ideal of S and suppose that $P = ann_S(M)$ for some irreducible right S-module M. Let L be the kernel of the action of H on L so that H/L embeds in $End_k(M)$. By [P1, Corollary 12.2.10], $dim_k(M) = n$ for some $n \in \mathbb{N}$. Thus, H/L embeds in $GL_n(k)$. But H and so H/L is a finitely generated group. Hence H/L embeds in $GL_n(k_0)$ where k_0 is a subfield of k finitely generated over the prime subfield of k. Since k is absolute, k_0 is finite. Thus, H/L is a finite group. Now M is a $k(H/L)$-module. Since $k(H/L) \cong (kH)/(aug(kL)kH)$ and $P \supseteq aug(kL)kH$, P has finite codimension over k. Thus, the factor S/P is prime Artinian and therefore simple. This shows that P is maximal in S.

Note that the hypothesis that the field k is absolute is necessary. In Example 3.3.4, we have a maximal ideal M of the nilpotent group algebra S such that S/M is infinite dimensional over k.

We now provide a generalisation of Theorem 4.1.12, again proved by Roseblade, and extend it to the fixed ring setting. We make a definition generalising the concept of an absolute field first.

4.1.13 DEFINITION Let K be a commutative Jacobson ring. By a *capital of K*, we mean a factor K/M for some maximal ideal M of K. Now K is said to be *absolutely capital* if all its capitals are absolute fields.

Note that the ring of integers, and any absolute field, are absolutely capital rings.

4.1.14 THEOREM *Let K be a commutative Jacobson ring which is absolutely capital. Let H be a polycyclic-by-finite group and S be the group ring KH. Then*

(i) every primitive ideal M of S intersects K in a maximal ideal of K;

(ii) every primitive ideal of S is maximal;

(iii) if M is a primitive ideal of S then M has finite codimension over $K/(M \cap K)$.

In particular, every irreducible S-module is finite dimensional over a capital of K.

PROOF [R2, Corollary C3] is (i). Suppose now that P is a primitive ideal of S. Now, $P/(K \cap P)$ is a primitive ideal of $(K/(P \cap K))H$ and, by (i), $K \cap P$ is a maximal ideal of K. By hypothesis, $K/(P \cap K)$ is an absolute field and so we may apply Theorem 4.1.12. This proves (iii), and the final statement follows immediately from (iii).

We now provide an analogue of Theorem 4.1.14 for the fixed ring.

4.1.15 THEOREM *Let K be a commutative Jacobson ring which is absolutely capital. Let H be a polycyclic-by-finite group and S the group ring KH. Suppose G is a group of automorphisms of H of prime order q so that G acts as K-automorphisms on S. Set $R = S^G$. Then*

(i) every maximal ideal M of R intersects K in a maximal ideal of K;

(ii) every primitive ideal of R is maximal;

(iii) for M above, R/M has finite dimension over the absolute field $K/(M \cap K)$.

In particular, every irreducible R-module is finite dimensional over a *capital of U.*

PROOF Let m be a primitive ideal of R. Suppose $tr(S) \subsetneq m$. Then $m/tr(S)$ is a primitive ideal of $R/tr(S)$ which is isomorphic to $(K/qK)C_H(g)$ by Lemma 4.1.2. The hypotheses of Theorem 4.1.12 apply to the ring $(K/qK)C_H(g)$ and so the Theorem holds in this case.

Suppose now that $tr(S) \not\subseteq m$. So, by Theorem 3.1.5, $m = \beta(\hat{M})$ for some primitive ideal $\hat{M}$ of T. Since $T = S*G \cong K(H \rtimes G)$, we may apply Theorem 4.1.14 to the ring T. Thus, $\hat{M} \cap K$ is a maximal ideal of K. By definition of β, $m \cap K \supseteq \hat{M} \cap K$ and, therefore, $m \cap K = \hat{M} \cap K$ is a maximal ideal of K. Thus, we have established (i). By Theorem 4.1.14(iii), $T/\hat{M}$ is finite dimensional over the absolute field $K/(K \cap \hat{M})$. Since R/m is a factor of $R/(\hat{M} \cap R)$ and $R/(\hat{M} \cap R)$ embeds in $T/\hat{M}$, R/m is finite dimensional over $K/(K \cap \hat{M})$, proving (iii). Hence R/m is a simple ring and m is a maximal ideal of R. This proves (ii). The final statement is an immediate consequence of (i),(ii) and (iii).

§4.2 The Prime Rank of a Nilpotent Group Algebra

We now investigate the prime rank of R in relation to that of S. First we define the term prime rank.

4.2.1 DEFINITION Let U be a ring. The *prime rank* of U, $rk(U)$, is defined to be the upper bound (if it exists) for the height of a prime in U. If no such bound exists, the prime rank is said to be infinite.

We may now state the question which will pre-occupy us in this section.

QUESTION 4B Suppose H is a nilpotent group, k is a field and S is the group algebra kH. Let G act as k-automorphisms on S and set $R = S^G$. Does $rk(R) = rk(S)$?

While discussing Question 4B, another question naturally arises. Recall the definition of the Saturated Chain Condition in 3.3.10. When S is the group algebra of a nilpotent group, [R1, §2.4 Theorem H3] shows that S satisfies SCC. We ask the following question.

QUESTION 4C Suppose H is a nilpotent group and k is a field. Let S denote the group algebra kH. Suppose G acts as k-automorphisms on S. Does S^G have SCC ?

There is a result of P.F.Smith given in [P1, Theorem 11.4.9] which states that when H is a finitely generated nilpotent group, the prime rank of the group algebra, $rk(kH)$, is the Hirsch length of H, $\hat{h}(H)$. When H is polycyclic-by-finite, $rk(kH) \leqslant \hat{h}(H)$, but in general this inequality can be strict. See [Smi].

We give an easy consequence of Smith's result which shows that, when the order of the group is invertible in k, the fixed ring R also has prime rank $\hat{h}(H)$, answering Question 4B positively.

4.2.2 LEMMA *Let S be any ring and G a finite group of automorphisms of S with $|G|^{-1} \in S$. Letting $R = S^G$, we have that $rk(R) = rk(S)$.*

PROOF *Let $p \in SpecR$.* Since $tr(S) = R$, $p \in Spec_t R$ and by Theorem 3.1.22(ii) there exists $P \in SpecS$ lying over p. By Corollary 3.3.4, $ht(p) = ht(P)$ and so $rk(R) \leqslant rk(S)$.

Now, let $P \in SpecS$. By Lemma 3.1.2, $P \in Spec_f S$. We apply Theorem 3.1.21(i) to find $p \in Spec_t R$ lying under P. Corollary 3.3.8 gives

$ht(p) = ht(P)$ and so $rk(S) \leqslant rk(R)$.

4.2.3 NOTE When H is a polycyclic-by-finite group, kH has prime rank less than or equal to $\hat{h}(H)$, the Hirsch length of H by [Smi]. [P1, Theorem 11.4.9] shows that we get equality when H is nilpotent.

When $|G| = 0$, the situation is not so clear. We concentrate on the simplest case where $chark = q$ and G has prime order q. We can use the correspondence of Theorem 3.2.10 to investigate the prime rank of R.

Consider $p, q \in Spec_t R$ with $q \subset p$. Then $\varphi^{-1}(p) = [P]$, say, and $\varphi^{-1}(q) = [Q]$, say, for some $P, Q \in Spec_I S$. As φ^{-1} preserves order, $Q^o \subset P^o \subset P$ and we may assume $Q \subset P$. Thus we must have $rk(S/Q) > rk(S/P)$. This observation enables us to place an upper bound on the length of a chain of primes in R of the form

$$p_0 \subset p_1 \subset \ldots \subset p_n \quad \text{with } p_i \in Spec_t R \ (0 \leqslant i \leqslant n).$$

For $i = 0,\ldots,n$, let $(\beta\alpha)^{-1}(p_i) = [P_i] \in (Spec_I S)/G$ for some $P_i \in Spec_I S$. From the above,

$$rk(S) \geqslant rk(S/P_0) > rk(S/P_1) > \ldots > rk(S/P_n) \geqslant 0.$$

Thus, $n \leqslant rk(S) \leqslant \hat{h}(H)$ as in Note 4.2.3.

Then, applying Smith's result in [Smi] to $R/tr(S)$, yields an upper bound on the length of a chain of primes in R which contain $tr_G(S)$. Using the fact that $R/tr_G(S) \cong kC_H(G)$, we find that any such chain has length at most $\hat{h}(C_H(G))$.

We find an upper bound for $rk(R)$. Suppose $t = rk(R)$. Let

$$p_0 \subset p_1 \subset p_2 \subset \ldots \subset p_t \quad p_i \in SpecR \ (\ 1 \leqslant i \leqslant t \ )$$

be a chain of maximal length in R. If $p_i \in Spec_t R$ for $i=1,\ldots,t$, then the above shows that $t \leqslant \hat{h}(H)$. Otherwise, there exists u ($1 \leqslant u \leqslant t$) with $tr_G(S) \subseteq p_u$ and $tr_G(S) \not\subseteq p_{u-1}$. As noted earlier $u-1 \leqslant \hat{h}(H)$. Also, since $tr_G(S) \subseteq p_u$, $t-u \leqslant \hat{h}(C_H(G))$. Thus, $t = (t-u) + (u-1) + 1 \leqslant \hat{h}(C_H(G)) + \hat{h}(H) + 1$. So we have found an upper bound for $rk(R)$.

We have proved the following result:

4.2.4 PROPOSITION *Let H be a finitely generated polycyclic-by-finite group, G a subgroup of $AutH$ of order q. Let k be a field with char $k = q$ and S be the group algebra kH. Then*

$$rk(R) \leq 1 + \hat{h}(H) + \hat{h}(C_H(G))$$

The following lemma which follows will allow us to refine this bound for H nilpotent in Theorem 4.2.9. In order to prove this lemma, we first require a series of definitions.

4.2.5 DEFINITION Let H be a finitely generated torsion-free nilpotent group, k a field and I an ideal of kH.

(i) We define I^+ to be $\{ h \in H: h-1 \in I\}$, a normal subgroup of H. It is normal because, if $h \in H$ and $x \in I$, then $h^x - 1 = x^{-1}(h - 1)x \in I$.

If in addition, I is prime, we have:

(ii) The map $\varphi: kH \to k\overline{H}$ where $\overline{H} = H/I^+$ is the canonical epimorphism.

(iii) The function $\lambda(I^\varphi)$ is defined as follows. Let A be the centre of $\Delta(\overline{H}) := \{h \in \overline{H}: h$ has finitely many $\overline{H}$-conjugates$\}$, a characteristic subgroup of $\overline{H}$, so that A is normal in $\overline{H}$. By [R1, §4.1 Lemma 5], $I^\varphi \cap kA = Q^o$ where Q^o denotes the intersection of the (finite) $\overline{H}$-conjugates of $Q \in Spec(kA)$. We define $\lambda(I^\varphi)$ to be $ht_{kA}(Q)$.

Theorem 4.1.12 shows that all maximal ideals of a nilpotent group algebra over an absolute field have finite codimension. As was pointed out after that theorem, the same is not always true of a nilpotent group algebra over a non-absolute field. However, we can still show that maximal ideals with maximal height in S do have finite codimension.

4.2.6 LEMMA *Let H be a finitely generated nilpotent group, k a field and S the group algebra kH. Suppose $P \in SpecS$ such that $ht(P) = rk(S) = \hat{h}(H)$. Then $dim_k(S/P) < \infty$.*

PROOF Let $P \in SpecS$ with $ht(P) = rk(S) = \hat{h}(H)$. Recall the Definition 4.2.5. Clearly $P^{\wp}$ is a faithful prime of $k\overline{H}$ in the sense that $(P^{\wp})^{+} \cap \overline{H} = \langle 1 \rangle$. [R1, §2.4] gives us

$$ht(P) = \lambda(P^{\wp}) + \hat{h}(P^{+}).$$

So $$ht_{kA}(Q) = ht(P) - \hat{h}(P^{+}) = \hat{h}(H) - \hat{h}(P^{+}) = \hat{h}(H/P^{+}).$$ But, $\hat{h}(A) \geq ht(Q) = \hat{h}(H/P^{+}) \geq \hat{h}(A)$ and so we must have equality. In particular, $|\overline{H}:A| < \infty$. Moreover, $ht(Q) = \hat{h}(A)$ and so Q is maximal in kA and, since A is abelian, $dim_k(kA/Q) < \infty$. Now, kH/P is isomorphic to a factor of $k\overline{H}/((P^{\wp} \cap kA)k\overline{H}) \cong kA/(P^{\wp} \cap kA) * \overline{H}/A$ which has finite dimension over k. Thus, kH/P is finite dimensional over k.

We give a consequence of Lemma 4.2.6 which relates to the fixed ring and will help us refine the upper bound for $rk(R)$.

4.2.7 COROLLARY *Let H be a finitely generated nilpotent group, G a subgroup of $AutH$ of order q, k a field of characteristic q and S the group algebra kH. Suppose $p \in Spec_t R$ with $ht(p) = \hat{h}(H)$. Then p is maximal in R.*

PROOF Let $P \in SpecS$ lie over p. By Corollary 3.3.8, $ht(P) = ht(p) = \hat{h}(H)$ so that P is maximal in S and $ht(P) = rk(S)$. From Lemma 4.2.6, $dim_k(S/P) < \infty$. Thus, $R/(P \cap R)$ is finite dimensional over k because it embeds in S/P. Now, R/p is a factor of $R/(P \cap R)$ and is therefore finite dimensional over k. Since R/p is a prime ring, it must be simple Artinian. Hence, p is maximal in R.

We now give a lower bound for $rk(R)$.

4.2.8 LEMMA *Let H be a finitely generated nilpotent group, suppose that k is a field and that S is the group algebra kH. Suppose that G is a finite subgroup of $AutH$ such that $tr(S) \neq 0$. Then $rk(R) \geq \hat{h}(H)$.*

PROOF If $|G|^{-1} \in k$, then the result is true by Lemma 4.2.1 and Note 4.2.2. Henceforth, assume $char\ k = q$, for some prime q. Let Y be the set $\{A : A$ is a normal, G-invariant subgroup of H with $|H/A| < \infty$ such that $q \nmid |H/A|\}$. Since every normal subgroup of finite index contains a G-invariant subgroup of finite index, [Rob, Theorem 9.38] gives:

$$\cap\{A: A \in Y\} = 1 \qquad -(*).$$

Let $A \in Y$. Then, clearly, A is a normal subgroup of H G and so, since $T = S*G \cong k(H \rtimes G)$, we may consider the factor ring

$$T_A := T/(aug(kA)T) \cong k((H/A) \rtimes G) = k(H/A)*G.$$

Since $q \nmid |H/A|$, Maschke's Theorem asserts that $k(H/A)$ is semiprime. Thus, [P2, Theorem 4.2] shows that $J(T_A)^{|G|} = 0$. Let M be the set

$\{N \in SpecT$: N is the inverse image in T of a maximal ideal of T_A, for some A as above$\}$.

Let $W = \cap\{N: N \in M\}$. We claim that $W = 0$. By the above $W^{|G|} \subseteq \cap_{A \in Y} aug(kA)T$ and so, by $(*)$, $W^{|G|} = 0$. By Theorem 1.3.6, T is semiprime. Thus, $W = 0$, proving the claim.

As a consequence of the fact that $W = 0$, we can choose $N \in M$ such that $N \in Spec_f T$. Since $N \in M$, M/N^+ is finite. Thus $ht(aug(kN^+)) = \hat{h}(N^+) = \hat{h}(H)$. [R1, §8.4, Paragraph 5] shows that $ht(N) = \hat{h}(H)$. Now, by Lemma 3.3.2, $rk(R) \geq ht(\beta(N)) = \hat{h}(H)$, proving the Theorem.

We have thus established bounds for $rk(R)$. We state these bounds in our next theorem together with a refinement of the upper bound.

4.2.9 THEOREM *Let S be the group algebra kH where H is a finitely generated torsionfree nilpotent group and k a field of characteristic q. Suppose that G is a group of automorphisms of H of order q. Then*

$$\hat{h}(H) \leq rk(R) \leq \hat{h}(H) + \hat{h}(C_H(G)).$$

PROOF Clearly, we may suppose that the action of G is non-trivial on S. In this case, because $|G| = q$, we have that $tr(S) \neq 0$ and so Lemma 4.2.8 gives a lower bound for $rk(S)$. Now suppose $rk(R) = \hat{h}(H) + \hat{h}(C_H(G)) + 1$. Then, there exists a chain:

$$p_0 \subset p_1 \subset \dots \subset p_n \subset p_{n+1} \subset p_{n+2} \subset \dots \subset p_{n+m+1}$$

where $n = \hat{h}(H), m = \hat{h}(C_H(G))$ and $p_i \in SpecR$ $(1 \leq i \leq n+m+1)$. As noted in the proof of Proposition 4.2.4, $p_0, \dots, p_n \in Spec_t R$. By Corollary 4.2.5, p_n is a maximal ideal of S. This contradiction proves the theorem.

We may improve on the theorem by combining it with Lemma 4.2.2.

4.2.10 COROLLARY *Let H be finitely generated torsionfree nilpotent group and k a field of characteristic q. Suppose G is a finite group of automorphisms of H having a normal Sylow q-subgroup Q of order q. Then*

$$\hat{h}(H) \leq rk(R) \leq \hat{h}(H) + \hat{h}(C_H(G)).$$

PROOF By Theorem 4.2.9, $\hat{h}(H) \leq rk(S^Q) \leq \hat{h}(H) + \hat{h}(C_H(G))$. Lemma 4.2.2 shows that $rk(S^Q) = rk(S^G)$ because $S^G = (S^Q)^{G/Q}$, proving the corollary.

Corollary 4.2.10 is our best result as far as Question 4B is concerned. Example 4.2.12 will provide some examples for which this question is answered positively. As a result we conjecture that the answer to Question 4B is in fact "yes".

Example 4.2.12 will settle Question 4C. We return to the saturated chain condition of 3.3.10. It's easy to see from Lemma 3.3.11 that if T has

SCC then R has SCC when $|G|^{-1} \in k$. However, we give an example to show that R need not have SCC even when S does. In Example 4.2.12, T may or may not have SCC and $|G| = 0 \in k$.

First we require a lemma.

LEMMA 4.2.11 *Let H be a finitely generated, torsion-free nilpotent group and G a finite group of automorphisms of H of prime order q. Let k be a field of characteristic q and S the group algebra kH. If $Z(H)$ is fixed by G then the trace ideal is prime of height 1.*

PROOF First, by Lemma 4.1.2, $R/tr(S) \cong kC_H(G)$, a domain, and so $tr(S)$ is a prime ideal. Suppose there exists $p \in SpecR$ with $0 \subset p \subset tr_G(S)$. Theorem 3.1.21(ii) shows that there exists $P \in Spec_f S$ such that P lies over p. Since H is nilpotent, the Zalesskii subgroup $\mathfrak{Z}(H)$ is just the centre, $Z(H)$. (See [P1, Chapter9, §1] for details of $\mathfrak{Z}(H)$). Now, [P1, Theorem 9.1.17] guarantees that any non-zero prime ideal of S has non-zero intersection with $Z(H)$. Hence, $tr_G(S) \cap kZ \supset p \cap kZ \supseteq P \cap kZ \neq 0$ but as in the proof of Lemma 4.1.1, $tr_G(S) \cap kZ \subseteq tr_G(S) \cap kC_H(G) = 0$. This contradiction proves the lemma.

EXAMPLE 4.2.12 *There exists a countably infinite family of group algebras, each with a finite group of k-automorphisms such that their fixed rings do not have SCC.*

Fix $n \in \mathbb{N}$ and let H_n be the nth Heisenberg group of 2.2.4. Let k be a field of characteristic 2 and let g be the automorphism of H_n of order 2 such that: $x_i^g = x_i^{-1}$; $y_i^g = y_i^{-1}$; $z^g = z$.

We show that $tr_G(S)$ is a prime ideal of R of height 1 and coheight 1. For, by Corollary 4.1.2, $R/tr_G(S) \cong k\langle z\rangle$ and so $tr_G(S)$ is a prime ideal of R. Clearly $tr_G(S)$ has coheight 1. That the height of the trace is 1 is immediate from Lemma 4.2.11.

We now show $rk(R) = 2n + 1$. For Theorem 4.2.9 gives us that $rk(R) = 2n + 1$ or $2n + 2$. Suppose it is the latter so that there is a chain

$$0 =: p_0 \subset p_1 \subset p_2 \subset \ldots \subset p_{2n+2} \quad p_i \in SpecR \ (0 \leq i \leq 2n + 2).$$

By the considerations of Lemma 4.2.4, there exists $j \in \{1,2,\ldots,2n+2\}$ such that $tr_G(S) \subseteq p_j$ but that $tr(S) \not\subseteq p_{j-1}$. Since the coheight of $tr(S)$ is equal to 1, $j \neq 0,\ldots,2n$. Suppose $j = 2n + 1$. If $tr(S) \subset p_{2n+1}$, then $coht(tr(S)) \geq 2$. Thus, we must have $p_{2n+1} = tr_G(S)$, contradicting $ht(tr_G(S)) = 1$. So $j = 2n + 2$. Thus, we may apply Theorem 3.2.10 to p_{2n+1}. By Corollary 4.2.7, p_{2n+1} is maximal in R. This contradiction shows $rk(R) = 2n + 1$.

Let $M := aug(kH) \cap R$. Then $R/M \cong k$ and so M is a maximal ideal of R. We now show that $ht(M) = 2n + 1$. For, let $q_0 = (z-1)S \cap R$,

$q_i = ((z-1)S + (x_1-1)S + (y_1-1)S + \ldots + (x_{i-1}-1)S + (y_{i-1}-1)S + (x_i-1)S) \cap R$

and $q_i' = ((z-1)S + (x_1-1)S + (y_1-1)S + \ldots + (x_i-1)S + (y_i-1)S) \cap R$ $(1\leq i\leq n)$. Now, $x_1+x_1^{-1} = (x_1 - 1) + (x_1 - 1)x_1^{-1} \in q_1\backslash q_0$, $y_i + y_i^{-1} = (y_i - 1) + (y_i - 1)y_i^{-1} \in q_i'\backslash q_i$ for $i=1,\ldots,n$ and $x_i + x_i^{-1} = (x_i - 1) + (x_i - 1)x_i^{-1} \in q_i\backslash q_{i-1}'$ for $i=2,\ldots,n$. Consequently, we have a chain of primes

$$0 \subset q_0 \subset q_1 \subset q_1' \subset q_2 \subset \ldots \subset q_n \subset q_n' = M$$

of length $2n +1$. Thus, $ht(M) = 2n + 1$.

Since G acts trivially on the factor $kH/(aug(kH))$, we have $tr(S) \subseteq M$. Since $coht(tr(S)) = 1$, they are neighbouring primes.

Thus, we've shown that $(kH_n)^G$ has neighbouring prime ideals, one of height $2n + 1$ and the other of height 1. Thus, $(kH_n)^G$ does not have SCC and we have answered Question 4C negatively.

CHAPTER 5

LOCALISATION IN FIXED RINGS

This chapter is joint work with K.A. Brown.

In this chapter, we examine the localisations of the fixed ring S^G, when compared to those of S. We use the preparatory results of Chapter 1, §5 and §6. Some elementary results concerning the inversion of central regular elements are given in §1.

Section 2 is modelled on [W1, §1] where Warfield studies the inheritance of the SSLC in a ring U from a ring V with $U \subseteq V$. Warfield's results apply to the fixed ring situation when $|G|^{-1} \in S$ and we extend these results to cover the possibility that $tr(S) \subset R$. Our best result in §2 is:

5.2.5 THEOREM *Let S be a Noetherian ring satisfying the strong second layer condition and G be a finite subgroup of $AutS$ such that R is Noetherian and ${}_RS$ and S_R are finitely generated modules. Suppose $p \in Spec_tR$. Then p has SSLC.*

Again, when dealing with the rings U and V, Warfield [W1, §6] examines the link graph of $SpecU$ in comparison to that of $SpecV$. As above, Warfield's results apply to the fixed ring case when $|G|^{-1} \in S$ and we extend them to allow for the possibility that tr is not surjective. We obtain:

5.3.6 THEOREM *Let S be a ring with the SSLC and let G be a finite group of automorphisms of S. Suppose that $R = S^G$ is Noetherian and ${}_RS$ and S_R are finitely generated. Let d be a symmetric dimension function on (R, S). If $p_1, p_2 \in Spec_t R$ with p_1 second layer linked to p_2, then there exist primes $Q_1, \ldots, Q_n$ of S with $n \geq 2$, such that Q_1 lies over p_1, Q_n lies over p_2 and such that Q_i is second layer linked to Q_{i+1} for $1 \leq i \leq n-1$.*

As indicated above, Sections 2 and 3 require some strong hypotheses on the ring S^G. For example, we require that S^G is Noetherian and that S is a finitely generated S^G-module. While these hypotheses seem quite strong, they are satisfied when S is the group algebra of the nth Heisenberg group and G is one of the automorphism groups in Corollary 2.2.7, Lemma 2.2.8 and Corollary 2.2.9. In fact, if Question 2B has a positive answer, then any polycyclic-by-finite group and any finite subgroup, G, of $AutS$ has $(kH)^G$ with the required hypotheses, in view of Corollary 2.1.4. Alternatively, if S is a ring which is finitely generated over its affine k-algebra centre, C, then Theorem 1.4.4 gives us that S^G here satisfies the hypotheses.

Throughout §2 and §3, we assess the implications of our results for $S = kH$ where k is a field and H is a finitely generated nilpotent group.

§1 Elementary Results

We begin this chapter with some elementary results on localisation. Initially, we concentrate on inverting central regular elements. We then look at their relationship with the fixed ring of the localized ring. The first lemma applies to any ring, not necessarily a group ring. It concerns localising at regular elements in $C(S)$, the centre of S.

LEMMA 5.1.1 *Let S be a ring and G be a finite group of automorphisms of S. Let $C \subseteq C(S) \cap C_S(0)$ and suppose C is non-empty and is multiplicatively closed. Then C is an Ore set in S and $X := C^G$ is an Ore set in both S and R. Moreover,*

$$SC^{-1} = SX^{-1}.$$

Consequently, with $\hat{S} = SC^{-1}$ and $\hat{R} = RX^{-1}$, G still acts on $\hat{S}$ as C is G-Invariant and so

$$(\hat{S})^G = \hat{R} \text{ and } tr_G(\hat{S}) = tr_G(S)X^{-1}.$$

PROOF Trivially C and X are Ore sets in S and we may localize at them. Also, the elements of X are invertible in R. We show that any element of $\hat{S}$ may be expressed as sx^{-1} for some $s \in S$, $x \in X$. Let $tc^{-1} \in \hat{S}$ with $t \in S, c \in C$. We may define x to be the "multiplicative trace" of c so that $x = \Pi_{g \in G} c^g$. This is a well defined element of X because C is commutative. Let $u = \Pi_{g \in G \setminus \{1\}} c^g$ so that $uc = x$. Thus $tc^{-1} = (tu)(cu)^{-1} = (tu)x^{-1} \in SX^{-1}$. This establishes the first part. Now let $vy^{-1} \in \hat{S}$ with $v \in S, y \in X$. Thus, $tr_G(vy^{-1}) = \Sigma_{g \in G}(vy^{-1})^g = \Sigma_{g \in G}(v^g)y^{-1} \in tr_G(S)y^{-1}$. Hence, $tr(\hat{S}) \subseteq tr(S)X^{-1}$, and the reverse inclusion is clear.

Suppose now $vy^{-1} \in (\hat{S})^G$. Then for all $g \in G$, $vy^{-1} = (vy^{-1})^g = v^g y^{-1}$ and so $v = v^g$. Thus $v \in R$ and $vy^{-1} \in \hat{R}$.

Suppose now that H is a finitely generated torsion-free nilpotent group and that $S = kH$ is the group algebra over some field k. We now investigate the consequences of localizing at the non-zero elements of the centre of kH, that is $kZ \setminus \{0\}$ where $Z = Z(H)$. This is of interest because, as a result of the fact that any ideal of S has non-zero intersection with the centre, the localized ring is then simple.

COROLLARY 5.1.2. *Let S be the group ring of a finitely generated torsion-free nilpotent group H over a field k and G be a finite group of automorphisms of H. Let $C = kZ \setminus \{0\}$, an Ore set in S, and $X = C^G$, an Ore set in R, then*

$$SC^{-1} = SX^{-1}.$$

Consequently, with $\hat{S} = SC^{-1}$ and $\hat{R} = RX^{-1}$, G still acts on $\hat{S}$ since Z is characteristic in H and

$$(\hat{S})^G = \hat{R} \text{ and } tr_G(\hat{S}) = tr_G(S)X^{-1}.$$

PROOF This is a straight application of Lemma 5.1.1.

We now give a lemma which shows that any finite group of k-automorphisms of kH acts as outer automorphisms.

LEMMA 5.1.3. *Let H be a finitely generated torsion-free nilpotent group, k any field. Then every unit of finite order of $kH(kZ\setminus 0)^{-1}$ is central.*

PROOF Let u be a unit of $kH(kZ\setminus 0)^{-1}$ such that $u^n = 1$ for some $n \in \mathbb{N}$. Since H is nilpotent, we construct a chain of subgroups of H:

$$Z =: H_0 < H_1 < H_2 < \dots < H_t \text{ for some } t \in \mathbb{N}$$

where each H_i is normal in H and $H_i/H_{i-1} \cong C_\infty$ for $i=1,\dots,t$. Suppose u is not central. Then there exists $j>0$ and $s\neq 0$ such that $u = \Sigma_{i=-s,\dots,s}\, v_i x^i$ where $v_i \in Q(kZ) * (H_{j-1}/Z)$ $(i= -s,\dots,s)$, either $v_s \neq 0$ or $v_{-s} \neq 0$ and $\langle xH_{j-1}\rangle = H_j/H_{j-1}$. Without loss of generality, we may suppose that $v_s \neq 0$. The expansion of u^n has a term $(v_s x^s)^n$. Now, a simple calculation shows that

$$(v_s x^s)^n = v_s (v_s)^{x^{-s}} (v_s)^{x^{-2s}} \dots (v_s)^{x^{-(n-1)s}} x^{sn}.$$

Since H_{j-1} is normal in H, all the conjugates of v_s belong to the domain $Q(kZ) * (H_{j-1}/Z)$. Hence this term in non-zero. Since it is the only term in the expansion of u^n of degree sn in x, we deduce that u^n is not equal to

1. This contradiction proves the lemma.

We now exploit this lemma below.

5.1.4 LEMMA *Let H be a finitely generated torsion-free nilpotent group, let k be any field and G be a finite group of automorphisms of H. Let $C = kZ\backslash 0$,* $X = C^G$ and let $\hat{S}$ denote $kH.C^{-1}$ and $\hat{R}$ denote $(\hat{S})^G$. Then

(i) *$\hat{S} * G$ is simple;*

(ii) *$tr_G(S)X^{-1}$ is the unique minimal non-zero ideal of $\hat{R}$;*

(iii) *for every non-zero ideal J of R the factor $(tr_G(S) + J)/J$ is X-torsion.*

PROOF From Lemma 5.1.3, G is outer on $\hat{S}$. As noted at the beginning of the section, $\hat{S}$ is simple and we may apply Theorem 1.4.6(iv) to see that $\hat{S} * G$ is simple, proving (i). For (ii), Theorem 1.4.6(i) shows that $tr_G(\hat{S})$ is the unique minimal non-zero ideal of R. Finally, suppose J is a non-zero ideal of R. Then JX^{-1} is an ideal of $\hat{S}$ and so $tr_G(S)X^{-1} \subseteq JX^{-1}$. Let $t \in tr_G(S)$. Then there exists $x \in X$, $j \in J$ with $t = jx^{-1}$ so that $j = tx$. This completes the lemma.

We are now in a position to provide a corollary which gives sufficient conditions for the fixed ring of a localized ring to be Noetherian.

5.1.5 COROLLARY *Let H be a finitely generated torsion-free nilpotent group, let k be any field and let G be a group of automorphisms of H of prime order q. Let $C = kZ\backslash 0$, $X = C^G$ and suppose $C_G(Z) \neq G$. Write $\hat{S}$ for kHC^{-1} and $\hat{R}$ for $(\hat{S})^G$. Then $\hat{R}$ is Noetherian.*

PROOF By Lemma 5.1.4, G is outer on $\hat{S}$ and $\hat{S}*G$ is simple. Let $z \in Z$ be such that $z^g \neq z$. Then since q is prime, $tr(z) \neq 0$ and so $tr(z)$ is a unit in $\hat{R}$.

Lemma 1.4.2 shows that $\hat{R}$ is Noetherian

§5.2 The Strong Second Layer Condition in R

In this section, the objective is to discover which primes of the fixed ring, R, inherit SSLC when the ring S has SSLC.

The results here are motivated by those of R.B. Warfield in [W1]. The results in Warfield's paper relate to the following situation: $U \subseteq V$ is an extension of Noetherian rings where V is finitely generated as a right U-module. The right trace ideal of a right U-module M is defined to be the sum in U of the images $f(M)$ over all $f \in Hom(M_U, U_U)$. Warfield requires as a hypothesis that the right trace ideal of V_U be equal to U. In [W1, Corollary 5.6], we have that when V has SSLC and both the trace ideal of $_UV$ and the trace ideal of V_U are equal to U, then U itself has SSLC.

We are concerned with the case where S is a ring satisfying SSLC and G is a finite group of automorphisms of S. Suppose R is Noetherian and that S_R and $_RS$ are finitely generated. Suppose in addition that the trace map $tr: S \to R$ is surjective. All these occur for example when $|G|^{-1} \in S$ by Lemma 1.4.2 and Theorem 1.4.3. Since $tr \in Hom(_RS_R, {}_RR_R)$, we have that the right trace ideal of S_R and left trace ideal of $_RS$ are equal to R. [W1, Corollary 5.6] shows that in these circumstances, R has SSLC.

Consequently, we concentrate on the case where $tr(S) \subset R$. Our results in this case reduce to Warfield's when the trace map is onto.

First we quote two of Warfield's results.

5.2.1 LEMMA *Let U, V and W be Noetherian rings, such that U and W satisfy the second layer condition. Suppose that $_UA_V$ and $_VB_W$ are Noetherian bimodules which are faithful on each side. Assume that V is prime and that*

A_V and $_VB$ are torsionfree. Then U and W possess Artinian classical quotient rings, and $_UA$ and B_W are torsionfree (that is, $_UA$ is $C_U(0)$-torsionfree and B_W is $C_W(0)$-torsionfree).

PROOF [W1, Lemma 5.2]

Also, we have:

5.2.2 LEMMA *Let U and V be Noetherian rings, and suppose that B is a Noetherian (U-V)-bimodule which is faithful on each side. Suppose also that V has an Artinian classical quotient ring, and that B is torsion-free as a right V-module. Let J be an ideal of V not contained in any minimal prime. Then there exists an ideal K of U, not contained in any minimal prime, such that $KB \subseteq BJ$.*

PROOF [W1, Lemma 5.3].

The next theorem is inspired by the arguments contained in [W1, Lemma 5.4 and Theorem 5.5]. As was pointed out in the introduction, the original form of these results showed that R has SSLC when $tr(S) = R$. Our modified version still gives this result but also handles the case where $tr(S) \subset R$. First we give a definition required in the proof of the theorem.

5.2.3 DEFINITION Let $U \subseteq V$ be rings. Let J be an ideal of U and define the $(U\text{-}V)$-bimodule $\langle J\rangle V$ as follows: $\langle J\rangle V := \cap\{\ker f : f \in Hom(_UV, _U(U/J))\}$. Similarly, we make the definition that $V\langle J\rangle := \cap\{\ker f: f \in Hom(V_U, (U/J)_U)\}$.

We show that $\langle J\rangle V$ is a $(U\text{-}V)$-bimodule. Let $u \in U$, $v \in \langle J\rangle V$ and $w \in V$. Suppose $f \in Hom(_UV, _U(U/J))$. Then $f(uv) = uf(v) = u0 = 0$. Since f was arbitrary, we conclude that $uv \in \langle J\rangle V$. We now consider the element $vw \in V$. Define a map $g: V \to (U/J)$ by $g(x) = f(xw)$ for all $x \in V$. It is easily seen that $g \in Hom({}_UV, {}_U(U/J))$ and so, by definition of $\langle J\rangle V$, $0 = g(v) = f(vw)$. Again, since f is arbitrary, we see that $vw \in \langle J\rangle V$. Thus, $\langle J\rangle V$ is a $(U\text{-}V)$-bimodule. Similarly, $V\langle J\rangle$ is a $(V\text{-}U)$-bimodule.

We now give a lemma concerning Definition 5.2.3.

5.2.4 LEMMA *Let U and V be Noetherian rings with $U \subseteq V$. Let $P \in SpecU$. Then the $(U\text{-}V)$-bimodule $V/\langle P\rangle V$ is torsionfree as a left U/P-module.*

PROOF Let $T/\langle P\rangle V$ be the torsion submodule of $V/\langle P\rangle V$ as a left U/P-module. Since $V/\langle P\rangle V$ is Noetherian as a right V-module, T is finitely generated as a right V-module by $t_1,\dots,t_n$, say. By definition of T and because U/P is prime Goldie, there exists a regular element $y + P \in U/P$ such that $(y + P)t_i = 0$ for $i=1,\dots,n$. Thus, $(y + P).(T/\langle P\rangle V) = 0$ and so, $yT \subseteq \langle P\rangle T$. Let $f \in Hom({}_UV, {}_U(U/P))$. Then $f(yT) = 0$. Thus, $yf(T) = 0$ and so, $(y + P)f(T) = 0$. Since $y + P$ is regular in U/P, $f(T) = 0_{U/P}$. But f was arbitrary, so $T \subseteq \langle P\rangle V$. This proves the lemma.

We now give the main result of this section.

5.2.5 THEOREM *Let S be a Noetherian ring satisfying the strong second layer condition and G be a finite subgroup of AutS such that R is Noetherian and ${}_RS$ and S_R are finitely generated modules. Suppose $p \in Spec_tR$. Then p has SSLC.*

PROOF Suppose that $p \in Spec_tR$ and that p does not have SSLC. Then, by Corollary 1.5.7, there exists a cyclic uniform R-module M such that $q = ann_R(M)$ is prime and $p = ass(M_R) \supset q$. Consider the map $\bar{t}:S \to R/q$ such that $\bar{t}(s) = tr(s) + q$. It is an $(R\text{-}R)$-bimodule homomorphism which is non-zero since $tr(S) \not\subseteq q$. Since $\langle q\rangle S$ and $S\langle q\rangle$ lie inside $ker\bar{t}$, they must be proper subbimodules of S. Equivalently, $S/(\langle q\rangle S)$ and $S/(S\langle q\rangle)$ are non-zero $(R\text{-}S)$- and $(S\text{-}R)$-bimodules respectively.

Let $D = r.ann_S(S/(\langle q\rangle S))$, $E = l.ann_S(S/(S\langle q\rangle))$ and $\bar{R} = R/q$. Henceforth, we let $\bar{\ }$ denote modulo q in R. Lemma 5.2.4 shows that $S/(\langle q\rangle S)$ is a torsion-free left R/q-module. Since $\bar{R}$ is prime Noetherian, $S/(\langle q\rangle S)$ is also faithful as a left $\bar{R}$-module. By definition it is a faithful right (S/D)-module. Similarly, $(S/S\langle q\rangle)$ is a torsion-free, faithful right $\bar{R}$-module and a faithful left S/E-module.

By Lemma 5.2.1, $S/(\langle q\rangle S)$ is a torsion-free S/D-module and S/D has an Artinian quotient ring. We denote S/D by $\bar{S}$ and the $(\bar{R}\text{-}\bar{S})$-bimodule $S/(\langle q\rangle S)$ by B.

Since M is cyclic with annihilator q, we may assume that $M = R/K$ for some right ideal K of R with $q \subseteq K$. Suppose $B = \bar{K}B$. Then

$$S = KS + \langle q\rangle S$$

and so
$$\begin{aligned} tr(S) &= tr(KS) + tr(\langle q\rangle S) \\ &\subseteq Ktr(S) + q \qquad \text{by definition of } \langle q\rangle S \\ &\subseteq Ktr(S) + K \subseteq K. \end{aligned}$$

This shows that $Mtr(S) = 0$, contradicting the fact that $tr(S) \not\subseteq q$. THerefore $\bar{K}B \subset B$ - that is $B/(\bar{K}B)$ is a non-zero left S-module.

Let L be the right ideal of R with $K \subseteq L$ such that $L/K = ann_M(p)$, the first layer of M. Suppose $\bar{L}B = \bar{K}B$. Then $LS + \langle q\rangle S = KS + \langle q\rangle S$ and so $LS \subseteq KS + \langle q\rangle S$. Taking the trace of this, we find $Ltr(S) \subseteq Ktr(S) + q \subseteq K$. But this gives that $tr(S) \subseteq p$, a contradiction. Thus, we must have $\bar{L}B \neq \bar{K}B$. Let $\bar{J} = ann(B/(pB))_{\bar{S}}$. By Lemma 5.2.2, $\bar{J}$ is not contained in a minimal prime of $\bar{S}$. Note also that $\bar{J}$ annihilates the non-zero bimodule $(\bar{L}B)/(\bar{K}B)$ on the right.

We choose a right submodule C of B containing $\bar{K}B$ which is maximal such that $C \cap \bar{L}B = \bar{K}B$. Then (B/C) as a right $\bar{S}$-module is an essential extension of $(\bar{L}B + C)/C \cong (\bar{L}B)/(\bar{K}B)$. Since $\bar{S}$ has an Artinian quotient ring, Theorem

1.5.11 shows the prime radical of $\bar{S}$ is localisable. Hence, Proposition 1.5.2 shows the set of minimal prime ideals *of* $\bar{S}$ is closed with respect to links. Since $\bar{J}$ annihilates $(\bar{L}B)/(\bar{K}B)$, and since $\bar{S}$ satisfies SSLC, it follows from [Jat, Theorem 9.1.12], that there exists an ideal $\bar{N}$ of $\bar{S}$, not contained in any minimal prime, such that $(B/C)\bar{N} = 0$. (See also [G&W, Theorem 11.4]).

Hence, $B\bar{N} \cap \bar{L}B \subseteq \bar{K}B$. Letting $\bar{H} = ann_{\bar{R}}(B/B\bar{N})$, we see from Lemma 5.2.2 that $\bar{H} \neq 0$, or equivalently, that:

$$H \supset q \qquad -(*).$$

Now,

$$(\bar{H} \cap \bar{L})B \subseteq \bar{H}B \cap \bar{L}B \subseteq C \cap \bar{L}B \subseteq \bar{K}B$$

and hence $$(H \cap L)S + \langle q\rangle S \subseteq KS + \langle q\rangle S.$$

Equivalently, $(H \cap L)S \subseteq KS + \langle q\rangle S$, and taking the trace we find

$$(H \cap L)tr(S) \subseteq Ktr(S) + tr(\langle q\rangle S) \subseteq K + q \subseteq K.$$

If $H \cap L \not\subseteq K$, then $(H \cap L) + K/K$ is a non-zero submodule of L/K and so $tr(S) \subseteq ann(L/K) = p$. This contradiction shows that $H \cap L \subseteq K$. So $(H+K)/K \cap L/K = 0$. By uniformity of R/K, $(H+K)/K = 0$, and so $M\bar{H} = (R/K)H = 0$. This contradicts $(*)$ and the faithfulness of $M_{(R/q)}$ and so proves the theorem.

The next lemma will enable us to show that not all prime ideals of such fixed rings satisfy even the second layer condition.

5.2.6 LEMMA *Let R be a prime Noetherian ring with a unique minimal non-zero ideal, J. Suppose that J is a prime ideal. Then J does not have the SLC.*

PROOF Let U be a uniform right submodule of R/J and E the injective hull of U in R. Since E is divisible and R is prime Noetherian, E is a faithful R-module. In particular $EJ \neq 0$. Choose $m \in E$ such that $mJ \neq 0$ and let $M' = mR$ and let $U' = ann_{M'}(J)$. Thus, $U' \neq 0$ and $M'/U' \neq 0$. Choose M such that $M/U' \subseteq M'/U'$ is uniform with prime annihilator B. By Theorem 1.5.3, either $B \subset J$ or B is second layer linked to J. Suppose the latter is the case and that the link is given by $(B \cap J)/I$ where $BJ \subseteq I \subset B \cap J$. Thus, by minimality of J, $B \cap J = J$ and $BJ = J$. This is a contradiction and so we must have $B \subset J$. Thus $B = 0$ and we've shown that J does not have SLC.

We now give an example of a fixed ring of a nilpotent group algebra not satisfying the SLC.

5.2.7 EXAMPLE Let $H = \langle\, x,y,z\colon [x,y] = z,\ z \text{ central} \,\rangle$, the first Heisenberg group, k a field of characteristic 2, and $G = \langle g\rangle$, the subgroup of $AutS$ of order 2 such that $x^g = x^{-1}$, $y^g = y^{-1}$ and $z^g = z$. By Lemma 4.1.2, $R/tr(S) \cong kC_H(G)$, a domain, and so $tr(S)$ is a prime ideal of R. We aim to show that $tr(S)$ does not have SLC. Denote $Z(H)$ by Z and $kZ\setminus\{0\}$ by C. Since $kZ \subseteq kC_H(G)$ and, by the proof of Lemma 4.1.1, $tr(S) \cap kC_H(g) = 0$, $C \subseteq C_R(tr(S))$ and so, by Corollary 1.5.8, it suffices to show that $tr(S)C^{-1}$ does not have SLC in RC^{-1}. We introduce the following notation: $\hat{R} = RC^{-1}$, $J = tr(S)$, $\hat{J} = JC^{-1}$. By Theorem 1.4.6(i), $\hat{J}$ is the unique minimal non-zero ideal of $\hat{R}$. Lemma 5.2.6 shows that $\hat{J}$ and hence J does not have SLC.

We now concentrate on the case where H is a finitely generated torsionfree nilpotent group and S is the group algebra kH over some field k. The following theorem shows all we need to know about localisation in such a ring. It is the culmination of the work of many people, among them Roseblade, P. Smith and M. Smith who proved [P1, Theorem 11.3.12], and Nouaze and Gabriel who proved [P1, Theorem 11.2.8].

5.2.8 THEOREM *Let H be a finitely generated nilpotent group and let G be a finite group of automorphisms of H. Let $S = kH$ so that G acts as* *k-automorphisms on S. Then SpecS has SSLC and all the cliques in SpecS are singletons.*

PROOF [P1, Theorem 11.3.12] shows that S is a polycentral ring. We then apply [P1, Theorem 11.2.8] to see that S is in fact an AR ring. Finally, [J, Theorem 8.1.9] shows that the S satisfies the strong second layer condition and that the cliques of S are singletons .

5.2.9 NOTE Suppose $(kH)^G$ is Noetherian where H is a finitely generated torsionfree nilpotent group and G is a finite subgroup of $AutH$. By Corollary 2.1.4, kH is a finitely generated $(kH)^G$-module. The results of this section leave a rather unclear impression as to which primes in the fixed ring have SSLC. Theorem 5.3.4 shows that all primes in $Spec_t R$ have SSLC but Example 5.2.6 shows that it is possible for $tr(S)$ not to have even SLC. One may be tempted to conjecture that certain primes of R, perhaps those containing $tr(S)$, say, do not have the SLC. Recall however, that when H is an Abelian group, then R is a commutative ring and so all primes have SSLC. It is in this rather unsatisfactory state that we are forced to leave this question.

§5.3 The Link Graph in *SpecR*

Again we consider the case where H is a finitely generated torsion-free nilpotent group, k is a field and S is the group algebra kH. The group G acts as k-automorphisms on S. Suppose R is Noetherian. Then Corollary 2.1.4 shows that S is a finitely generated R-module. We have seen in Note 5.2.9 that all primes in $Spec_t R$ satisfy SSLC. We now go on to show a necessary condition for two primes in $Spec_t R$ to be linked. Namely, p_1 is second layer linked to p_2 only if there exists $P \in Spec_f S$ lying over both p_1 and p_2.

This is a corollary to the more general result, Theorem 5.3.6.

In this section we borrow heavily from the methods of Warfield in [W1, §6]. Often we quote directly from this paper but frequently we have to adapt the results there to allow for the possibly that $tr(S) \subset R$.

Here we give one of Warfield's results. Recall Definition 3.1.22 for the definition of lying over.

5.3.1 LEMMA *Let U and V be Noetherian rings such that U is a subring of V and V_U is finitely generated. If Q is a minimal prime of U there exists a prime of V which lies over Q. Moreover, if P is any prime of V which lies over Q, there exists an $({}_VV_U)$-bond from V/P to U/Q.*

PROOF [W1, Lemma 6.1]

Next, we adapt [W1, Lemma 6.2].

5.3.2 LEMMA *Let S* be *a* Noetherian ring *and G a finite group of automorphisms of S such that $R := S^G$ is Noetherian and S_R and ${}_RS$ are finitely generated. Let $q_1, q_2 \in Spec_tR$ with $Q_1, Q_2 \in Spec_fS$ lying over q_1 and q_2 respectively. Suppose q_1 is second layer linked to q_2 where I is an ideal of R such that $q_1q_2 \subseteq I \subset q_1 \cap q_2$ and $(q_1 \cap q_2)/I$ is the link from q_1 to q_2. Then there exists an ideal K of S such that $K \subseteq Q_1^o \cap Q_2^o$, $K \cap R \subseteq I$ and if $P \in SpecS$ is minimal over K, then there is a prime p of R containing I with an $({}_SS_R)$-bond from S/P to R/p.*

PROOF Let $S\langle I\rangle = \{ s \in S: f(s) \in I$ for all $f \in Hom(S_R, R_R) \}$. This coincides with the definition in 5.2.3 and so $S\langle I\rangle$ is an $(S\text{-}R)$-subbimodule of S. Clearly, $SI \subseteq S\langle I\rangle$. We may therefore regard $S/(S\langle I\rangle)$ as an $S\text{-}(R/I)$-bimodule.

Suppose $SJ \subseteq S\langle I\rangle$ for some ideal J of R with $J \not\subseteq I$. Then $tr(S)J \subseteq tr(SJ) \subseteq I \subset q_1 \cap q_2$. So that $tr(S)J \subseteq q_i$ and, since q_i is prime and doesn't contain $tr(S)$, we have that $J \subseteq q_i$ $(i=1,2)$. Hence $J \subseteq q_1 \cap q_2$ and $(I+J)/I$ is a non-zero ideal contained in $(q_1 \cap q_2)/I$ and $tr(S).((I+J)/I) = 0$. But $tr(S) \cap C_R(q_1) \neq \emptyset$ and $(q_1 \cap q_2)/I$ is R/q_1-torsionfree as a left module. This contradiction establishes that $J \subseteq I$ and hence that

$S/(S\langle I\rangle)$ *is a faithful right* (R/I)*-module.* $\quad - (1)$

Now let $K = ann_S(S/(S\langle I\rangle))$. Let $a \in K$; then $SaS \subseteq S\langle I\rangle$ and so, $tr(SaS) \subseteq I \subseteq q_1 \cap q_2$. Theorem 3.1.5 shows that $a \in \beta^{-1}(q_1) \cap \beta^{-1}(q_2)$ and so $a \in Q_1{}^o \cap Q_2{}^o$, establishing $K \subseteq Q_1{}^o \cap Q_2{}^o$. Note that K is the largest two sided ideal of S contained within the left ideal $S\langle I\rangle$. Hence, $S(K \cap R) \subseteq S\langle I\rangle$ and therefore $K \cap R \subseteq I$, by *(1)*. Now observe that $S/(S\langle I\rangle)$ is an (S/K)-(R/I)-bimodule which is faithful and finitely generated on each side. Denote $S/(S\langle I\rangle)$ by A and let P be a prime of S minimal over K. By [G-W, Proposition 7.5], there exists a left affiliated series for A:

$$A_0 = 0 < A_1 < A_2 < \ldots < A_m := A$$

for some subbimodules $A_0, A_1, \ldots, A_m$ where $P_1, \ldots, P_m$ are the corresponding affiliated primes. We also have that each A_i/A_{i-1} is a torsionfree left (S/P_i)-module. [G-W, Proposition 2.14] says that since P is minimal over $K = l.ann_S(A)$, there exists $j \in \{1,..,m\}$ such that $P_j = P$.

Now consider a right affiliated series for the $(S/P$-$R/I)$-bimodule A_j/A_{j-1}:

$$B_0 = 0 < B_1 < B_2 < \ldots < B_k = A_j/A_{j-1}.$$

for some subbimodules B_i of A_j/A_{j-1}. By [G-W, Proposition 7.7], each factor B_j/B_{j-1} is a torsionfree left (S/P)-module. Consider the factor B_k/B_{k-1}: by definition it is a faithful R/p-module for some $p \in SpecR$ with $I \subseteq p$. Thus, the $(S/P$-$R/p)$-bimodule B_k/B_{k-1} is faithful and torsionfree on both sides.

This time we use Warfield's result in its original form.

5.3.3 LEMMA *Let U and V be Noetherian rings such that U is a subring of V. Let Q_1, Q_2 be minimal prime ideals of U such that there is an ideal link from Q_1 to Q_2. Then there exist primes P_1 and P_2 of V such that P_1 lies over Q_1 and P_2 lies over Q_2, and such that there is an ideal link from P_1 to P_2 in S.*

PROOF [W1, Lemma 6.3(i)]

We now explain what is meant by a symmetric dimension function and define such a function for R and S when R is a Noetherian subring of the nilpotent group algebra $S = kH$.

5.3.4 DEFINITION A collection X of Noetherian rings is said to possess a symmetric dimension function if there exists a function d assigning to each prime factor ring of each ring $R \in X$ an element of a fixed totally ordered set such that d satisfies the following conditions:

(i) If P and Q are prime ideals of a ring $R \in X$ such that $Q \supset P$ then $d(R/Q) < d(R/P)$.

(ii) If R and S are prime factors of rings in X, and if there exists a bond from R to S, then $d(R) = d(S)$.

We extend such a function d to arbitrary factor rings R of rings in X by setting $d(R)$ equal to the maximum of $d(R/P)$ for P ranging over the minimal primes of R.

Suppose X is a collection of algebras with finite GK-dimension over a fixed field k. For each prime factor R of an algebra in X let $d(R)$ denote the GK-dimension of R. It follows from Lemma 1.6.3 that the dimension fuction d satisfies property (i) above while Lemma 1.6.6 and [K-L, Lemma 5.3] establish (ii). Hence X possesses a symmetric dimension function. Now extend d to arbitrary factors of algebras in X as in 5.3.4. It is an open question whether this extension of d is equivalent to GK-dimension for Noetherian rings. G.M. Bergman discusses this in [B] and in fact produces an example of a non-Noetherian ring where this fails.

We use the above definition in the next lemma. Recall from 1.5.2 that an ideal link from $P \in SpecU$ to itself is *non-trivial* if the linking module is a subfactor of ${}_UP_U$.

5.3.5 LEMMA *Let U be a Noetherian ring possessing a symmetric dimension function d. Let P_1 and P_2 be minimal primes of U such that there is an ideal link from P_1 to P_2. Suppose further that $d(U/P_1) = d(U/P_2) = d(U)$. If the ideal link from P_1 to P_2 is nontrivial then there exist primes $Q_1,\dots,Q_n$ of U with $n \geqslant 2$, such that $Q_1 = P_1$ and $Q_n = P_2$, and such that Q_i is second layer linked to Q_{i+1} for $1 \leqslant i \leqslant n-1$.*

PROOF [W1, Lemma 6.5]

The above results enable us to prove our main result on the link graph in $Spec_tR$.

5.3.6 THEOREM *Let S be a ring with the SSLC and let G be a finite group of automorphisms of S. Suppose that $R = S^G$ is Noetherian and ${}_RS$ and S_R are finitely generated. Let d be a symmetric dimension function on (R, S). If $p_1,p_2 \in Spec_tR$ with p_1 second layer linked to p_2, then there exist primes $Q_1,\dots,Q_n$ of S with $n \geqslant 2$, such that Q_1 lies over p_1, Q_n lies over p_2 and such that Q_i is second layer linked to Q_{i+1} for $1 \leqslant i \leqslant n-1$.*

PROOF By Theorem 3.1.19(ii), there exist primes $P_1,P_2 \in Spec_fS$ lying over p_1 and p_2 respectively. Let the link between p_1 and p_2 be given by

$(p_1 \cap p_2)/I$ for some ideal I of R with $p_1 p_2 \subseteq I \subset p_1 \cap p_2$. Let $d(R/p_1) = \alpha$, say, then $d(R/p_2) = \alpha$ by 5.3.4(ii). We apply Lemma 5.3.2 to find an ideal K of S, contained in $P_1^o \cap P_2^o$, with $K \cap R \subseteq I \subset p_1 \cap p_2$ such that if $P \in SpecS$ is minimal over K, then there is a bond between S/P and a prime factor of R/I. Since the minimal primes of R/I are p_1/I and p_2/I, it follows from 5.3.4(ii) that $d(S/P) \leqslant \alpha$ for every prime P/K of S/K and so $d(S/K) \leqslant \alpha$. From Lemma 5.3.1 and 5.3.4 we have that if $p_3/(K \cap R)$ is a prime of $R/(K \cap R)$, then $d(R/p_3) \leqslant \alpha$. We therefore conclude, again using 5.3.4 that p_1 and p_2 are primes minimal over $R \cap K$ and that $d(R/(R \cap K)) = \alpha$. Moreover, there is a link form $p_1/(R \cap K)$ to $p_2/(R \cap K)$ so we work in S/K.

We are now in the situation where $d(R/(K \cap R)) = \alpha$ and $d(S/K) \leqslant \alpha$. We show that P_1/K is a prime of S/K lying over $p_1/(K \cap R)$. This is the same as showing that $p_1/(K \cap R)$ is minimal over $(P_1/K) \cap (R/(K \cap R)) = (P_1/K) \cap ((R + K)/K) = ((P_1 \cap R) + K)/K$. But $((P_1 \cap R) + K)/K \cong (P_1 \cap R)/((P_1 \cap R) \cap K) = (P_1 \cap R)/(K \cap R)$ because $K \subseteq P_1$. Since p_1 is minimal over $P_1 \cap R$, we've shown what we set out to. We now apply Lemma 5.3.1 to see there is a bond from $(S/K)/(P_1/K)$ to $(R/K \cap R)/(p_1/K \cap R)$. Hence $d((S/K)/(P_1/K)) = d(R/p_1) = \alpha$ and P_1/K is a minimal prime of S/K. Thus, each prime of S/K lying over $p_1/(K \cap R)$ is a minimal prime of codimension α, and similarly for each prime of S/K lying over $p_2/(K \cap R)$.

By Lemma 5.3.3, there exist primes A_1/K, A_2/K of S/K such that A_i/K lies over $p_i/(K \cap R)$ $(i=1,2)$ and such that there is an ideal link from A_1/K to A_2/K in S/K. By Lemma 5.3.5 there exist $Q_1/K, \ldots, Q_m/K \in Spec(S/K)$ such that Q_i/K is second layer linked to Q_{i+1}/K $(i=1,\ldots,n-1)$ and $A_1/K = Q_1/K$ and $A_2/K = Q_m/K$. This proves the theorem.

At this stage we make note of the results of P. Loustaunau and J. Shapiro in [L-S]. They prove in [L-S, Theorem 3.3] that, when $|G|^{-1} \in S$, R

inherits SLC from the skew group ring T, and some structure on the links is preserved when passing from T to R. It follows that, in the setting of Theorem 5.3.6, if the cliques of S are finite, then the same is true of the intersections of the cliques of R with $Spec_t R$.

When S is the group algebra of a nilpotent group, we know from Lemma 5.2.8 that the cliques in S are singletons. This fact, together with Theorem 5.3.6, yields some strong information regarding the links in $Spec_t R$. For example, it shows that when the link graph of $Spec R$ is intersected down to $Spec_t R$, the cliques are subsets of the ~-classes.

5.3.7 COROLLARY *Let S be the group algebra kH where H is a finitely generated torsion-free nilpotent group and k is a field. Let G be a finite subgroup of $Aut H$ such that $R = S^G$ is Noetherian. Suppose $p_1, p_2 \in Spec_t R$ with p_1 second layer linked to p_2, then $p_1 \sim p_2$.*

PROOF First observe that S_R and ${}_R S$ are finitely generated modules by Corollary 2.1.4. Now, by Theorem 1.6.9, $GKdim(S) < \infty$ and by Lemma 1.6.2(ii)$GKdim(R) = GKdim(S)$. Thus, $\{R, S\}$ has a symmetric dimension function, namely d, as defined in 5.3.4 and so we may apply Theorem 5.3.6 to find primes $Q_1, \ldots, Q_n$ of S with $n \geq 2$, such that Q_1 lies over p_1, Q_n lies over p_2 and such that Q_i is second layer linked to Q_{i+1} for $1 \leq i \leq n-1$. However, by Lemma 5.2.8, the cliques of S are singletons and so $Q_1 = Q_n$. Thus, p_1 and p_2 are both minimal over $Q_1 \cap R$ and so $p_1 \sim p_2$, proving the corollary.

We give the q-case as a special instance of Corollary 5.3.7.

5.3.8 COROLLARY *Let S be the group algebra kH where H is a finitely generated torsion-free nilpotent group and k is a field of characteristic q. Let G be a finite subgroup of $Aut H$ of prime order q such that $R = S^G$ is*

Noetherian. Suppose $p_1, p_2 \in Spec_t R$ with p_1 second layer linked to p_2. Then $p_1 = p_2$.

PROOF From Corollary 5.3.7, $p_1 \sim p_2$. As observed prior to Theorem 3.1.7, $\sim$ is the trivial relation in this case and so $p_1 = p_2$.

Suppose H is a finitely generated torsionfree nilpotent group, k is a field and that S is the group algebra kH. Let G be a finite group of k-automorphisms acting on S. With $R = S^G$ and assuming that R is Noetherian, the above results have given us some insight into the link graph of R. We review this information here.

Suppose that $p \in Spec_t R$. Theorem 5.3.4 shows that p has SSLC. Suppose now that $p' \in SpecR$ is second layer linked to p. Corollary 5.3.7 shows that two possibilities arise. Either:

(i) $p \sim p'$

or (ii) $tr(S) \subsetneq p'$.

If all the primes second layer linked to p fall into the category (i) above and the same is true of all the primes linked to these primes and so on, we find that the clique of p is a subset of its $\sim$-class. By Theorem 3.1.18, $[p]$, and so the clique of p, is finite.

In certain circumstances, we find that this is indeed what happens. First we need a lemma which exploits GK-dimension.

5.3.9 LEMMA *Let S be a Noetherian k-algebra of finite GK-dimension. Let R be a subalgebra of S, $P \in SpecS$ and denote S/P by $\overline{S}$ and $R/(P \cap R)$ by $\overline{R}$. Suppose $\overline{R}$ is Noetherian and that $\overline{S}$ is finitely generated on both sides as an $\overline{R}$-module. Then*

(i) $GK(\overline{R}s) = GK(\overline{R}) = GK(\overline{S})$ for all $0 \neq s \in \overline{S}$. In particular, $\overline{R}$ is GK-homogeneous;

(ii) $\overline{R}$ has an Artinian quotient ring;

(iii) $\mathcal{C}_{\bar{R}}(\bar{0}) = \mathcal{C}_{\bar{S}}(\bar{0}) \cap \bar{R} =: X$, say, and $Q(\bar{S}) = \bar{S}X^{-1}$.

PROOF Let $C = Q(\bar{S})$ which exists and is a simple Artinian ring by Goldie's Theorem. For the bimodule ${}_{\bar{R}}C_C$ form an $(\bar{R}\text{-}C)$-bimodule composition series:

$$0 = C_0 \subset C_1 \subset C_2 \subset \ldots \subset C_m = C \qquad (*)$$

where C_i/C_{i-1} are simple bimodules and $Q_i := l.ann(C_i/C_{i-1}) \in Spec\bar{R}$ $(i=1,\ldots,m)$. By [W2, Lemma 2], there exist $(\bar{R}\text{-}\bar{R})$-bimodules $L_0 = \bar{S} \supset L_1 \supset \ldots \supset L_m \supseteq Q_m\bar{S}$ such that the $R/Q_m\text{-}R/Q_i$-bimodules L_{i-1}/L_i are torsion-free $(i=1,\ldots,m)$. In particular, they are faithful and we immediately have $GK(R/Q_m) = GK(R/Q_i)$ $(i=1,\ldots,m)$ by [K-L, Lemma 5.3] and Lemma 1.6.6. Note also that $Q_1Q_2\ldots Q_mC = 0$ and since C is a faithful $\bar{R}$-module, $Q_1Q_2\ldots Q_m = 0$. Thus, every minimal prime of $\bar{R}$ is one of the Q_is.

We now show that every annihilator prime of $\bar{R}$ is minimal. Let $Q = l.ann_{\bar{R}}(Y)$ be an annihilator prime. Consider the $\bar{R}\text{-}C$-bimodule series for the bimodule C: $0 \subset YC \subset C$ where $l.ann_{\bar{R}}(YC) = Q$. We may refine this to a bimodule composition series of C as in $(*)$. This gives a new series with $C_1 \subseteq YC$ and so $Q_1 \supseteq Q$. However, Q contains a minimal prime Q_j, say. Although $Q_j \subseteq Q \subseteq Q_1$, the argument of the first paragraph gives us that $GK(R/Q_1) = GK(R/Q_j)$ and so $Q_1 = Q = Q_j$. We've shown that Q is a minimal prime of $\bar{R}$.

We now prove (i) above. Let $0 \neq s \in \bar{S}$. Now, $GK_R((Rs + P)/P) = GK_S(RsS + P)/P)$ by Lemma 1.6.6 and $\bar{S}$ is GK-homogeneous by [K-L, Lemma 5.12] and so $GK_S((RsS + P)/P) = GK(\bar{S})$. We have established (i).

[G&K, Theorem 2.7] shows that, as a consequence of the preceding claim, $\bar{R}$ has an Artinian quotient ring, proving (ii).

Finally, we aim to prove (iii). To begin with we show that $\mathcal{C}_{\bar{S}}(\bar{0}) \cap R = \mathcal{C}_{\bar{R}}(\bar{0})$. It is evident that $\mathcal{C}_{\bar{S}}(\bar{0}) \cap R \subseteq \mathcal{C}_{\bar{R}}(\bar{0})$. Now let $d \in \mathcal{C}_{\bar{R}}(\bar{0})$, $0 \neq s \in \bar{S}$ and suppose $ds = 0$. Now $(Rs + P)/P \cong \bar{R}/K$ where $K = l.ann_R(s + P)$. Since $d \in \mathcal{C}_{\bar{R}}(\bar{0}) \cap K$, Lemma 1.6.5 shows that $GK_R((Rs + P)/P) < GK(\bar{R})$. However as argued when proving (i), $GK_R((Rs + P)/P) = GK_S((RsS + P)/P) = GK(\bar{S})$. But

$GK(\bar{R}) = GK(\bar{S})$, a contradiction which shows that $ds \neq 0$ and that d is regular in $\bar{S}$.

Now, we show that X is Ore in $\bar{S}$ and that $Q(\bar{S}) = \bar{S}C^{-1}$. Let $s \in \bar{S}$, $x \in X$ and $\bar{J} = r.ann_{\bar{R}}(s + x\bar{S})$ so that $\bar{R}/\bar{J} \cong (s\bar{R} + x\bar{S})/x\bar{S}$. Now

$$GK_R((s\bar{R} + x\bar{S})/x\bar{S}) \leq GK_R(\bar{S}/x\bar{S}) < GK_R(\bar{S}) = GK_S(\bar{S})$$

by Lemma 1.6.5. Thus by Lemma 1.6.2(ii), $GK(\bar{R}/\bar{J}) < GK(\bar{R})$. By previous claims, $GK(\bar{R}) = GK(\bar{R}/\bar{N})$ where $\bar{N} = N(\bar{R})$. Moreover, by Lemma 1.6.8, $\bar{R}/\bar{N}$ is GK-homogeneous. Also $GK((\bar{R}/\bar{N})/((\bar{J} + \bar{N})/\bar{N})) \leq GK(\bar{R}/\bar{J}) < GK(\bar{R}) = GK(\bar{R}/\bar{N})$. By [K-Le, 5.13], $(\bar{J} + \bar{N})/\bar{N}$ is an essential right ideal of $\bar{R}/\bar{N}$. Thus $\bar{J} \cap C(\bar{N}) \neq \emptyset$. Since $\bar{R}$ has an Artinian quotient ring, Theorem 1.5.11 gives $C_{\bar{R}}(\bar{N}) = C_{\bar{R}}(\bar{0})$ and so there exists $y \in C_{\bar{R}}(\bar{0})$ such that $sy = xt$ for some $t \in \bar{S}$.

This proves the lemma.

One further lemma is required before we give the main theorem.

5.3.10 LEMMA *Let U be a prime Goldie. Suppose I is a right ideal of U with $l.ann_U(I) = 0$. Then $I \cap C_U(0) \neq \emptyset$.*

PROOF We show first that we may assume U is simple Artinian. Let Q denote the simple Artinian ring $Q(U)$. If $\alpha = c^{-1}u \in l.ann_Q(IQ)$ for $c \in C_U(0)$ and $u \in U$, then $0 \neq u \in l.ann_U(I)$. Therefore, $l.ann_Q(IQ) = 0$. But there exists $e = e^2 \in IQ$ such that $IQ = eQ$ and so $(1-e)IQ = 0$. Thus, $1-e = 0$ and so $e = 1$, giving $IQ = Q$ and so $I \cap C_U(0) \neq \emptyset$.

5.3.11 THEOREM *Let S be a Noetherian ring with finite GK-dimension, G a subgroup of $AutS$ and $R = S^G$. Suppose $P \in SpecS$ such that $C_S(P^o)$ is an Ore set in S, $R/(P \cap R)$ is Noetherian and S/P is finitely generated on both sides as $R/(P \cap R)$-modules. Suppose $p_1,\ldots,p_n$ are the primes of R minimal over $P \cap R$. Suppose $p_i \in Spec_tR$ $(i=1,\ldots,n)$ or, equivalently that*

$tr(S) \cap C_S(P^o) \neq \emptyset$. Then $C_S(P^o) \cap R = C_R(P \cap R) =: X$, say and X is an Ore set in S. In particular, $\cap_i p_i$ is a localisable semiprime ideal of R.

PROOF We first prove the equivalence. By Lemma 5.3.9, $R/(P \cap R)$ has an Artinian quotient ring and so, by Theorem 1.5.11,

$$C_R(P \cap R) = C_R(\cap_i p_i) = \cap_i C_R(p_i) \qquad -(1).$$

Suppose that $p_i \in Spec_t R$ $(i=1,\ldots,n)$. Since $tr(S) \not\subseteq p_i$, $tr(S) \cap C_R(p_i) \neq \emptyset$ $(i=1,\ldots,n)$ and so $tr(S) \cap C_R(\cap_i p_i) \neq \emptyset$. Lemma 5.3.9 shows that $C_R(P \cap R) = C_S(P^o) \cap R$ and we have that $tr(S) \cap C_S(P^o) \neq \emptyset$. Conversely, suppose that $tr(S) \cap C_S(P^o) \neq \emptyset$. This yields $tr(S) \cap C_R(P \cap R) \neq \emptyset$ and hence, since $C_R(P \cap R) = \cap_i C_R(p_i)$, $tr(S) \cap C_R(p_i) \neq 0$ for each i so that $tr(S) \not\subseteq p_i$ $(i=1,\ldots,n)$, proving the stated equivalence.

Now let J be a G-invariant right ideal of S with $(J + P^o)/P^o$ essential as a right ideal in S/P^o. We show that

$$tr(J) \cap C_R(P \cap R) \neq \emptyset \qquad -(2).$$

Since, $C_R(P \cap R) = C_R(\cap_i p_i) = \cap_i C(p_i)$ and $tr(J)$ is a right ideal in R, Lemma 5.3.10 shows that it is enough to prove that $l.ann_R((tr(J) + p_i)/p_i) \subseteq p_i$ for $i=1,\ldots,n$. Fix i, let $p = p_i$ and $\hat{P} = \beta^{-1}(p) \in Spec_f T$. Since $(J + P^o)/P^o$ is essential in S/P^o, $J \cap C_S(P^o) \neq 0$. This gives that $J \cap C_T(P^o*G) \neq \emptyset$ because $C_S(P^o) \subseteq C_T(P^o*G)$. But the Noetherian ring $(T/(P^o*G)) \cong (S/P^o)*G$ has an Artinian quotient ring, namely $Q(S/P^o)*G$ as shown in Lemma 3.2.1 and so, by Theorem 1.5.11, $C_T(P^o*G) = \cap_i C_T(\hat{P}_i)$ where $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m$ are all the primes of T minimal over P^o*G. Since $\hat{P}$ equals one of $\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m$, $C_T(P^o*G) \subseteq C_T(\hat{P})$ so that $J \cap C_T(\hat{P}) \neq \emptyset$. Suppose $r \in l.ann_R((tr(J) + p)/p)$. Then $rtr(J) \subseteq p$. By Theorem 3.1.5, $rtr(J)f \subseteq \hat{P}$. Since $frJf = frtr(J)$ as in the proof of Lemma 3.1.7, we have $frJf \subseteq \hat{P}$. This yields $(frJ)(fS) = (frJ)(fT) \subseteq \hat{P}$ and, since $\hat{P} \in Spec_f T$ and frJ is a right ideal of T, we must have $frJ \subseteq \hat{P}$. Now, since, $J \cap C_T(\hat{P}) \neq \emptyset$, we have $fr \in \hat{P}$ and so, by Theorem 3.1.5, $r \in p$. Thus, we have proved (2).

We show that X is an Ore set in S. Let $s \in S$, $x \in X$. Let $K = \{v \in S: sv \in xS\}$. Since $C_S(P^o)$ is an Ore set in S and $X \subseteq C_S(P^o)$, we have that $K \cap C_S(P^o) \neq \emptyset$. Setting $J = \cap_{g \in G} K^g$, we conclude that $J \cap C_S(P^o) \neq \emptyset$. Now, J is a G-invariant right ideal of S with $(J + P^o)/P^o$ essential in S/P^o. We apply (2) above to find that there exists $y \in tr(J) \cap C_R(P \cap R)$. So, $sy = xt$ for some $t \in S$, proving the claim. The rest of the theorem is immediate from this claim.

We provide two corollaries to this theorem. The first requires a technical lemma. For this lemma, recall what is meant by the term GK-homogeneous in Definition 1.6.7.

5.3.12 LEMMA *Let U be a Noetherian, GK-homogeneous ring. Then there exists $m \in \mathbb{N}$ such that $GKdim(U/P) = m$ for all minimal primes P of U.*

PROOF Let $P_1, \ldots, P_t$ be the distinct maximal right annihilator ideals of U. Then $P_i = r.ann(X_i)$ for $i = 1, \ldots, t$ where $X_1, \ldots, X_t$ are non-zero ideals of U. Note that $X := \Sigma_i X_i$ is in fact a direct sum. If X is not essential as a right ideal in U, there exists an annihilator ideal of U not contained in any P_i. Thus, $X \leq_e U_U$. Since U is GK-homogeneous, $GKdim(X_i) = m$ for $i=1, \ldots, t$ where $m = GKdim(U)$. Fix $i \in \{1, \ldots, t\}$. Let T be the torsion submodule of X_i as a right U/P_i-module. Now, since U is left Noetherian, $T = \Sigma_{j=1,\ldots,n} Tt_j$ for some $t_1, \ldots, t_n \in T$. Thus, there exists $c \in U/P_i$, regular, such that $t_j c = 0$ for $j=1, \ldots, n$. Hence, $Tc = 0$. Since X_i is a faithful U/P_i-module, we conclude that $T = 0$ and so, X_i is a torsionfree right U/P_i-module. By [G-W, Corollary 6.26], X_i^n contains an isomorphic copy of U/P. We thus have that $GKdim(U/P_i) \leq GKdim(X_i^n) = GKdim(X_i)$. By virtue of the fact that X_i is a right U/P_i-module, we have that $GKdim(X_i) \leq GKdim(U/P_i)$. We've thus shown that $GKdim(U/P_i) = m$ for $i=1, \ldots, t$.

Let Q be a minimal prime of U and suppose $GKdim(U/Q) < m$. Form an affiliated series for U_U as follows:

$$0 < X_1 < X_1 \oplus X_2 < \ldots < X < Y_1 < \ldots. < Y_s = U_U$$

for some right ideals $Y_1, \ldots, Y_s$ of U with corresponding affiliated primes $P_1, \ldots, P_t, Q_1, \ldots, Q_s$. Since $Q_s Q_{s-1} \ldots Q_1 P_t \ldots P_1 = 0$ and Q is a minimal prime with $GKdim(U/Q) < m$, there exists $j \in \{1, \ldots, s\}$ such that $Q_j = Q$. By [G-W, Theorem 10.13(b)], all the above affiliated primes are minimal. Since X is an essential right ideal of U, Theorem 1.5.3 shows that each Q_i is in the clique of one of the P_js. By 5.3.4(ii), $m = GKdim(U/P_{j(i)}) = GKdim(U/Q_i)$ for $i=1, \ldots, s$. In particular, $GKdim(U/Q) = m$. This contradiction proves the lemma.

Thus, we have the following consequence of Theorem 5.3.11.

5.3.13 COROLLARY *Let S be a Noetherian ring with finite GK-dimension, G a subgroup of AutS and $R = S^G$. Suppose $P \in SpecS$ such that $C_S(P^o)$ is an Ore set in S, $R/(P \cap R)$ is Noetherian and S/P is finitely generated on both sides as $R/(P \cap R)$-modules. Suppose, also, that $GKdim(S/P) > GKdim(R/tr(S))$. Then $C_S(P^o) \cap R = C_R(P \cap R) =: X$, say and X is an Ore set in S. In particular, $\sqrt{(P \cap R)}$ is a localisable semiprime ideal of R.*

PROOF Let $p \in SpecR$ and suppose p is minimal over $P \cap R$. By Lemma 5.3.9(i), $GKdim(R/(P \cap R)) = GKdim(S/P)$ and $R/(P \cap R)$ is GK-homogeneous. Lemma 5.3.12 shows that $GKdim(R/p) = GKdim(R/(P \cap R))$ which is equal to $GKdim(S/P)$. Thus, $GKdim(R/p) > GKdim(R/tr(S))$ and we conclude that $tr(S) \not\subseteq p$. Theorem 5.3.11 finishes the proof.

We conclude this section with one last consequence of Theorem 5.3.11.

5.3.14 COROLLARY *Let S be a Noetherian ring with finite GK-dimension, G a subgroup of $AutS$ and $R = S^G$. Suppose $P \in SpecS$ such that $C_S(P^O)$ is an Ore set in S, $R/(P \cap R)$ is Noetherian and S/P is finitely generated on both sides as $R/(P \cap R)$-modules. Suppose $p_1,\ldots,p_n$ are the primes of R minimal over $P \cap R$. Suppose $p_i \in Spec_t R$ $(i=1,\ldots,n)$ or, equivalently that $tr(S) \cap C_S(P^O) \neq \emptyset$. Then $C_S(P^O) \cap R = C_R(P \cap R) =: X$, say, and X is an Ore set in S. Moreover, $RX^{-1} = (SX^{-1})^G$ and RX^{-1} is Noetherian.*

PROOF All but the final sentence comes from Theorem 5.3.11. Certainly $RX^{-1} \subseteq (SX^{-1})^G$. Suppose now that $sx^{-1} \in (SX^{-1})^G$. Then $sx^{-1} = (sx^{-1})^g = s^g(x^g)^{-1} = s^g x^{-1}$ for all $g \in G$. Thus, $s = s^g$ for all $g \in G$ and so $s \in R$, establishing the reqired inclusion. Finally, $tr\colon SX^{-1} \to RX^{-1}$ is surjective and, therefore, Lemma 1.4.2 completes the proof of the corollary.

Theorem 5.3.11 shows that, when the hypotheses of the theorem apply, if $p \in Spec_t R$ and all primes of R minimal over $P \cap R$ are in $Spec_t R$, then the clique of p is a subset of the $\sim$-class of p. In particular, $cl(p)$ is finite. We conjecture here that, when the hypotheses of Theorem 5.3.11 are fulfilled, $p \in Spec_t R$ has $cl(p) = [p]$ where $[p]$ is the $\sim$-class of p.

REFEREFNCES

[A-N] G. Azumaya and T. Nakayama, *On irreducible rings*, Ann. of Math. (1947), 946-965.

[Am] S.A. Amitsur, *Rings of quotients and Morita context*, J. Algebra 17 (1971) 273-98.

[Az] G.A. Azumaya, *New foundation of the theory of simple rings*, Proc. Japan Acad. Ser. A Math. Sci. 22 (1946), 325-332.

[B-I] G.M. Bergman and I.M. Isaacs, *Rings with fixed point free group actions*, Proc. London Math. Soc. 23 (1971), 70-82. (Corrigendum 24 (1972), 192).

[Ba] H. Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. London Math. Soc. 25 (1972), 603-614.

[Be] I.N. Bernstein, *Modules over a ring of differential operators. Study of the fundamental solutions of equations with constant coefficients*, Funkcional Anal. i Prilozen 5 (1971), 89-101.

[Bo] W. Bohro, *On the Joseph-Small additivity principle for Goldie ranks*, Compositio Math. 47 (1982), 3-29.

[C-L] C.L. Chuang and P.H. Lee, *Noetherian rings with involution* Chinese J. Math. 5 (1977), 15-19.

[C-M] M. Cohen and S. Montgomery, *Trace functions for finite automorphism groups of rings*, Arch. Math. (Basel) 35 (1980), 516-527.

[D] J. Dixmier, *Enveloping algebras*, North-Holland (Amsterdam) 1977.

[F-S] D.R. Farkas and R.L. Snider, *Noetherian fixed rings*, J. Algebra 69 (1977), 347-353.

[G] M. Gromov, *Groups of polynomial growth and and expanding maps*, Inst. Hautes Etudes Sci. Publ. Math. 53 (1981), 53-73.

[GK1] I.M. Gelfand and A.A. Kirillov, *Sur les corps liesaux algebres enveloppantes des algebres de Lie*, Inst. Hautes Etudes Sci. Publ. Math. 31 (1966), 5-19.

[GK2] I.M. Gelfand and A.A. Kirillov, *Fields associated with enveloping algebras of Lie algebras*, Dokl. Acad. Nauk SSSR 167 (1966), 407-409.

[G&K] A. Goldie and G. Krause, *Associated series and regular elements of Noetherian rings*, J. Algebra 105 (1987).

[G-W] K.R. Goodearl and R.B. Warfield, Jr, *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press (Cambridge) 1989.

[H] M. Hochster, *Invariant theory of commutative rings*, Contemp. Math. 43 (1985), 161-179.

[H-O] T.J. Hodges and J. Osterberg, *A rank two decomposable projective module over a Noetherian domain of Krull dimension one*, Bull. London Math. Soc. 19 (1987), 139-144.

[J1] A.V. Jategaonkar, *Localisation in Noetherian Rings*, Cambridge University Press (Cambridge) 1986.

[J2] A.V. Jategaonkar, *Solvable Lie algebras, polycyclic-by-finite groups, and bimodule Krull dimension*, Comm. Algebra 10 (1982) 19-69.

[J-S] A. Joseph and L.W. Small, *An additivity principle for Goldie rank*, Israel J. Math. 31 (1978), 89-101.

[K] V.K. Karchenko, *Algebras of invariants of free algebras*, Algebra i Logika 17 (1978), 478-487. (English translation (1979) 316-321).

[K&L] G.R. Krause and T.H. Lenagan, *Growth of Algebras and Gelfand-Kirillov dimension,* Pitman (London) 1985.

[L1] T.H. Lenagan, *Enveloping algebras of solvable Lie algebras are catenary*, Contemp. Math. 130 (1992) 231-236.

[L2] T.H. Lenagan, *Gelfand-Kirillov dimension an affine PI rings,* Comm. Algebra 10 (1981), 87-92.

[L-P1] M. Lorenz and D.S. Passman, *The structure of G_0 for certain polycyclic group algebras and Related Algebras*, Contemp. Math. 93 (1989), 283-302.

[L-P2] M. Lorenz and D.S. Passman, *Prime ideals in crossed products of finite groups*, Israel J. Math 33 (1979), 89-132.

[L-S] P.Loustaunau and J. Shapiro, *Localisation in a Morita context with applications to fixed rings*, J. Algebra 143 (1991), 373-387.

[McC-R] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience (New York) 1987.

[Mo1] S. Montgomery, *Fixed Rings of Finite Automorphism Groups of Associative Rings*, Lecture Notes in Math., Springer-Verlag (Berlin) 1980.

[Mo2] S. Montgomery, *Prime ideals and group actions in noncommutative algebras*, Contemp. Math. 88 (1989), 103-124.

[Mo3] S. Montgomery, *Automorphism groups of rings with no nilpotent elements*, J. Algebra 60 (1979), 238-248.

[Mo4] S. Montgomery, "Group actions on rings: some classical problems", *Methods in ring theory*, (ed. F. van Oystaeyen, Riedel, Dordrecht) 1984, 327-346.

[Mor] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Diagaku, Sect. A, 6 (1958), 83-142.

[Na] K. Nagarajan, *Groups acting on Noetherian rings*, Nieuw. Arch. Wisk. 16 (1968), 25-29.

[Ne] M. Newman, *Integral Matrices*, Academic Press (London) 1972.

[N-W] W.K. Nicholson and J.F. Watters, *Normal radicals and normal classes of rings*, J. Algebra 59 (1979), 5-15.

[P1] D.S. Passman, *The Algebraic Structure of Group Rings*, Wiley-Interscience (New York) 1977.

[P2] D.S. Passman, *Infinite Crossed Products*, Academic Press (London) 1987.

[R1] J.E. Roseblade, *Prime ideals in group rings of polycyclic groups*, Proc. London Math. Soc. (3) 36 (1978), 385-447. (Corrigenda 38 (1979), 216-218).

[R2] J.E. Roseblade, *Group rings of polycyclic groups*, J. Pure Appl. Algebra 3 (1973), 307-328.

[Rob] D.J.S. Robinson, *Finiteness Conditions and Generalised Soluble Groups*, Springer-Verlag (Berlin) 1972.

[Sma1] L.W. Small, *Orders in Artinian rings*, J. Algebra 4 (1966), 13-41.

[Sma2] L.W. Small, *Orders in Artinian rings II*, J. Algebra 9 (1968), 266-273.

[Smi] P.F. Smith, *On the dimension of group rings*, Proc. London Math. Soc. (3) 25 (1972) 288-302.

[St] J.T. Stafford, *On regular elements of Noetherian rings*. In "Ring Theory, Proceedings of the 1978 Antwerp Conference" (F. van Oystaeyen, Ed.), pp 257-277, Decker (New York) 1979.

[W1] R.B. Warfield, Jr, *Noetherian ring extensions with trace conditions*, Trans. Amer. Math. Soc. 331 (1992), 449-463.

[W2] R.B. Warfield, Jr, *Prime ideals in ring extensions*, J. London Math. Soc. (2), 28 (1983), 453-460.

[Y] Yi Zhong, *Homological dimension of skew group rings and crossed products*, preprint, University of Glasgow (1992) to appear in J. Alg.

[ZN] A.E. Zalesskii and O.M. Neroslavskii, *There exists a simple Noetherian ring with divisors of zero but without idempotents*, Comm. Algebra 5 (1977), 231-245 (Russian).