Curvature estimates for spacelike hypersurfaces in de Sitter space

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Abstract

Local estimates of the maximal curvatures of admissible spacelike hypersurfaces in de Sitter space for k-symmetric curvature functions are obtained. They depend on interior and boundary data. The curvature function is also assumed to depend on the tilt/slope of the hypersurface and an additional growth condition holds.

Declaration

The work in this thesis is based on research carried out in the Geometry Research Group at the Department of Mathematical Sciences of Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text. Sections 2.1 and 2.4 as well as Chapters 4 and 5 of this thesis have been previously submitted to research journals for publication.

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Contents

Chapter 1

Introduction

The study of the curvature of geometric objects has been one of the major subjects in differential geometry. In the special case of hypersurfaces (submanifolds of codimension one), the extrinsic notion of curvature and how it is influenced by the nature of the ambient space is also a classical topic.

The problem of classification of hypersurfaces in Euclidean space by their curvature values has been widely investigated. With the introduction of new tools from partial differential equations, many conjectures and results have been proven.

One of the equations that appears naturally in differential geometry when one tries to prescribe the Gauss curvature of a hypersurface in Euclidean space is the Monge-Ampère equation. For convex hypersurfaces this is the so called Minkowski problem, and the existence of smooth solutions for the two-dimensional case was established independently by L. Nirenberg and A.V. Pogorelov [19, 21]. Later on, a complete proof for the n-dimensional Minkowski problem was given in [8] by S-Y. Cheng and S-T. Yau.

In a series of papers dedicated to fully nonlinear elliptic equations, L. Caffarelli, L. Nirenberg and J. Spruck [4–6] provide the theory needed to study the prescribed curvature problem for a larger class of curvature functions, namely those that can be represented by a symmetric homogeneous function of the principal curvatures. Since these equations are concave for admissible solutions, in order to carry out the classical method of continuity, it is necessary to obtain $C^{2,\alpha}$-regularity. A fundamental result in the theory of elliptic fully nonlinear equations of second order of concave type (see [10, 13, 17]) is the Evans-Krylov theorem, which guarantees that the needed Hölder estimate of a solution will follow from the C^2 a priori bound.

In [6] Caffarelli et al describe how to obtain the existence of star-shaped hypersurfaces with prescribed k-symmetric curvature in Euclidean space. Firstly it is shown how this problem fits in the frame of concave elliptic fully nonlinear equations and how to get the estimates needed: using barriers for the C^0 bounds, estimating the strict star-shapedness to get the C^1 a priori estimate, and getting bounds for the maximal curvature of the hypersurface to conclude the C^2 interior bound using the maximal principle on a clever test function.

For smooth strictly convex hypersurfaces B. Guan and P. Guan in [14] solved the problem of existence and uniqueness when the prescribed function is defined on $\mathbb{S}^n$ in terms of the inverse of the Gauss map. They parameterise the hypersurfaces by means of the support function, a tool widely used in convex geometry, and from the bounds of the eigenvalues of the inverse of the second fundamental form, they show how to derive the C^0 estimates using the so called Cheng-Yau's lemma, and from C^2 and C^0 estimates, they show a C^2 a priori bound. They also observed that it is not possible to apply the continuity method for the resulting equation, but it is possible to use it for an auxiliary equation and apply degree theory arguments using a *group invariance* assumption, and then they proved the existence of a solution.

The question of existence of hypersurfaces in Riemannian manifolds of constant sectional curvature has also been investigated. Moreover, star-shaped hypersurfaces with given k-symmetric curvature in the sphere is obtain in [18] by Y. Li and V. Oliker. They used C^0, C^1 and curvature estimates proven by M. Barbosa, L. Herbert and V. Oliker in [2]. Also in [2], one can also find the C^0 and C^1 a priori bounds for hypersurfaces of prescribed curvature in hyperbolic space. The remaining curvature bound and existence result were proved by Q. Jin and Y. Li in [16] using similar arguments of W. Sheng, J. Urbas and X. Wang [22].

For spacelike hypersurfaces in Minkowski space and Lorentz manifolds various results have been proved by R. Bartnik and L. Simons, C. Gerhardt, Y. Huang [3, 11, 12, 15], and the references provided in them. The curvature estimates in these cases rely on the Gauss formula, and the Lorentzian nature of de Sitter space requires additional assumptions in the prescription in order to apply the maximum principle.

In this thesis we obtain similar curvature estimates as in [15] in de Sitter space. As in [15] we impose a growth assumption on the right hand side of the equation in terms of the tilt function of the hypersurface, to be defined below.

We introduce in Chapter 2 the fundamental equations of hypersurfaces in Riemannian and Lorentz manifolds. We also provide explicit expressions for hypersurfaces in de Sitter space after providing several examples. In Chapter 3 we give the formulation of the problem in terms of partial differential equations. In Chapters 4 and 5 we present the main results and their corresponding proofs.

Chapter 2

Geometry of hypersurfaces in Riemannian and Lorentz manifolds

We will recall the fundamental formulae for hypersurfaces in Riemannian and Lorentz manifolds. We refer the interested reader to [9, 20] for more details of the topics in this chapter. These formulae relate the concept of curvature of the hypersurface with the curvature of the ambient space. We also give explicit expressions of the second fundamental form of different hypersurfaces. In the case of strictly convex hypersurfaces in Euclidean space the formulae are given when we parameterised the hypersurfaces via the support function.

2.1 Geometric formulae for hypersurfaces in Lorentz manifolds

We will recall some geometric formulae for hypersurfaces in Lorentzian manifolds and at the end we will apply them to the case of spacelike hypersurfaces in de Sitter space.

Let $\{\partial_1, ..., \partial_n, N\}$ be a coordinate frame of a Lorentzian manifold $(\bar{M}, \bar{g})$ and M a Lorentzian (not necessarily spacelike) hypersurface with induced metric g such that $\{\partial_i\}$ span TM, let N be the unit normal field to M and put $\epsilon = \bar{g}(N, N) = \pm 1$. When the induced metric is positive definite, then we say that M is a spacelike hypersurface, then g can be represented by the matrix $g_{ij} = g(\partial_i, \partial_j)$ with inverse denoted by g^{ij}.

The *Gauss formula* for $X, Y \in T\Sigma$ reads

$$D_X Y = \nabla_X Y + \epsilon\, h(X,Y) N, \tag{2.1.1}$$

here D is the connection on $\bar{M}$, ∇ is the induced connection on M and the *second fundamental form* h is the normal projection of D. In a coordinate basis we write

$$h_{ij} = h(\partial_i, \partial_j). \tag{2.1.2}$$

The *shape operator* is obtained by raising an index with the inverse of the metric

$$h^i_j = g^{ik} h_{kj}. \tag{2.1.3}$$

The *principal curvatures* of the hypersurface Σ are the eigenvalues of the symmetric matrix (h^i_j). The tangential projection of the covariant derivative of the normal vector field N on Σ, $\nabla_j N = (D_{\partial_j} N)^\top$ is related to the second fundamental form by the *Weingarten equation*

$$\nabla_j N = -h^i_j \partial_i = -g^{ik} h_{kj} \partial_i. \tag{2.1.4}$$

The *curvature tensor* is defined for $X, Y, Z \in T\Sigma$ as

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z. \tag{2.1.5}$$

The *Christoffel symbols* are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right), \tag{2.1.6}$$

and the curvature tensor in terms of Christoffel symbols is

$$R_{ijk} = R^m_{ijk} \partial_m = \left(\partial_j \Gamma^m_{ik} - \partial_i \Gamma^m_{jk} + \Gamma^m_{js} \Gamma^s_{ik} - \Gamma^m_{is} \Gamma^s_{jk} \right) \partial_m. \tag{2.1.7}$$

Contracting with the metric

$$R_{ijkl} = g\left(R(\partial_i, \partial_j)\partial_k, \partial_l \right) = g_{lm} R^m_{ijk}. \tag{2.1.8}$$

We can also write the curvature tensor of the ambient manifold in terms of the curvature of the surface and the second fundamental form

$$\begin{aligned}\bar{R}_{ijk} &= R^m_{ijk}\partial_m \\ &= D_j(D_i\partial_k) - D_i(D_j\partial_k) \\ &= (\nabla_j + D^\perp_j)(\nabla_i\partial_k + \epsilon h_{ik}N) - (\nabla_i + D^\perp_i)(\nabla_j\partial_k + \epsilon h_{jk}N) \\ &= R_{ijk} + \epsilon h_{ik}\nabla_j N - \epsilon h_{jk}\nabla_i N + \epsilon D^\perp_j(hN)_{ik} - \epsilon D^\perp_i(hN)_{jk},\end{aligned} \tag{2.1.9}$$

where $D^\perp_i(hN)_{jk} = D^\perp_i(h_{jk}N) - \Gamma^r_{ik}h_{rj}N - \Gamma^r_{ij}h_{rk}N$.

From the last identity, when the ambient manifold is flat, we obtain the *Codazzi equation* given by the identity

$$\nabla_i h_{jk} = \nabla_j h_{ik}. \tag{2.1.10}$$

Note that the first and second covariant derivatives of the second fundamental form are given by

$$\nabla_l h_{ij} = \partial_l h_{ij} - \Gamma^r_{li}h_{rj} - \Gamma^r_{lj}h_{ir}, \tag{2.1.11}$$

$$\nabla_k\nabla_l h_{ij} = \partial_k(\nabla_l h_{ij}) - \Gamma^r_{kl}\nabla_r h_{ij} - \Gamma^r_{ki}\nabla_l h_{rj} - \Gamma^r_{kj}\nabla_l h_{ir}. \tag{2.1.12}$$

The Gauss Equation expressed in orthonormal coordinates is given by

$$\bar{R}_{ijkl} = R_{ijkl} - \epsilon\left(h_{ik}h_{jl} - h_{il}h_{jk}\right). \tag{2.1.13}$$

When M is a hypersurface of a flat manifold $\bar{R}_{lkij} = 0$, the last equation simplifies to the identity

$$R_{ijkl} = \epsilon\left(h_{ik}\,h_{jl} - h_{jk}\,h_{il}\right). \tag{2.1.14}$$

Note that A is a bilinear symmetric tensor, and the following *Ricci identity* holds

$$\nabla_k\nabla_l A_{ij} - \nabla_l\nabla_k A_{ij} = R_{kljr}A_{ir} + R_{klir}A_{rj}. \tag{2.1.15}$$

In the following we will write $\nabla^r u(\partial_{i_1}, \ldots, \partial_{i_r})$ simply as $\nabla_{i_1 i_2 \cdots i_r} u$.

2.2 Hypersurfaces as graphs in Euclidean space

Consider $\Sigma \subset \mathbb{R}^{n+1}$ parametrised as the graph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$. Then at any point $p \in \Sigma$ there is a neighbourhood $U \subset \mathbb{R}^n$ such that every $y \in \Sigma \cap f(U)$ can be written as $y = (x, f(x))$ for some $x = (x^1, ..., x^n) \in U$.

In these coordinates, the basis for the tangent space $T_p\Sigma$ is given by

$$y_i = \frac{\partial y}{\partial x^i} = (e_i, f_i),$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$. Let $Df = (f_1, ..., f_n)$ be the usual gradient vector of f. Then a unit normal vector field in Σ is given by

$$\hat{n} = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}}.$$

To compute the second fundamental form $h_{ij} = g(D_j y_i, \hat{n})$ we observe that

$$D_j y_i = \frac{\partial^2 y}{\partial x^j \partial x^i} = (0, f_{ij}),$$

then its normal projection is

$$h_{ij} = \langle D_j y_i, \hat{n} \rangle = \frac{f_{ij}}{\sqrt{1 + |Df|^2}}.$$

Since we are using the standard metric of the Euclidean space, then the induced metric on Σ is given by

$$g_{ij} = \langle y_i, y_j \rangle = \delta_{ij} + f_i f_j$$

The inverse of the metric is then

$$g^{ij} = \delta_{ij} - \frac{f_i f_j}{1 + |Df|^2},$$

and the shape operator is given by

$$A^i_j = g^{ik} h_{kj} = \sum_k \left(\delta_{ik} - \frac{f_i f_k}{1 + |Df|^2} \right) \left(\frac{f_{kj}}{\sqrt{1 + |Df|^2}} \right) = \frac{\partial}{\partial x^j} \left(\frac{f_i}{\sqrt{1 + |Df|^2}} \right).$$

2.3 Strictly convex hypersurfaces and support function in Euclidean space

Let $\mathbf{n} : \Sigma \to \mathbb{S}^n$ be the Gauss map of a strictly convex hypersurface $\Sigma \subset \mathbb{R}^{n+1}$. Then $\mathbf{n}$ is an isomorphism with inverse $Y : \mathbb{S}^n \to \Sigma$. The support function of Σ is the function $u : \mathbb{S} \to \mathbb{R}$ given by

$$u(x) = x \cdot Y(x) \qquad x \in \mathbb{S}^n, \tag{2.3.16}$$

where the $\cdot$ is the usual inner product in Euclidean space which will also be denoted by $\langle \cdot, \cdot \rangle$ and we write σ for the induced metric on the unit sphere $\mathbb{S}^n$.

Now we want to recover the hypersurface Σ via the support function. Geometrically, the support function measures the distance from the origin to the plane with unit normal direction x that passes through the point $Y(x)$, equivalently, the distance from the origin to the tangent plane of Σ at $Y(x)$. Then any point $Y(x)$ can be written as the sum of the vector $u(x)x$ and a vector $a(x)$ which lies entirely in the tangent plane of Σ at $Y(x)$, that is

$$Y(x) = u(x)x + a(x) \tag{2.3.17}$$

Let D be the Levi-Civita-connection of the ambient space $\mathbb{R}^{n+1}$, and denote by ∇ the tangential component to the sphere and $\nabla^{\perp}$ the orthogonal component. That is $D = \nabla + \nabla^{\perp}$. Now, if we take the derivative of (2.3.17) in the direction $v \in T_x\mathbb{S}^n$ then we have

$$D_vY = (D_vu)x + u(x)h + D_va. \tag{2.3.18}$$

Now, since Y is a diffeomorphism then DY is an isomorphism between tangent spaces, and then DY has no normal component, that is $\langle D_vY, x\rangle = 0$. On the other hand, since $a(x)$ is orthogonal to x then $\langle a(x), x\rangle = 0$ implies that $0 = \langle D_va, x\rangle + \langle a(x), v\rangle$. Then taking the component in the x- direction of (2.3.18) we obtain that

$$\begin{aligned}\langle D_v Y, x, \rangle &= \langle (D_v u)x, x\rangle + \langle u(x)v, x\rangle + \langle D_v a, x\rangle \\ 0 &= D_v u - \langle a(x), v\rangle \\ D_v u &= \langle a(x), v\rangle.\end{aligned}$$

Since the last equation is valid for every tangent vector h, then we must have that

$$a(x) = \nabla u.$$

Finally, recalling that all derivatives were computed at the point $x \in \mathbb{S}^n$ we can write

$$Y(x) = u(x)x + \nabla|_x u.$$

We use now indices $a, b, i, j, k, l, m = 1, 2, \ldots, n$. Direct computations of Y_a and Y_{ab}, give us the identities

$$g_{ab} = \left(u\sigma_{ma} + \nabla^2_{ma} u\right)\sigma^{mj}\sigma_{jk}\left(u\sigma_{lb} + \nabla^2_{lb} u\right)\sigma^{lk}$$

and

$$h_{ab} = \langle Y_{ab}, x\rangle = u\sigma_{ab} + \nabla^2_{ab} u.$$

Combining these equations we get

$$g_{ab} = h_{am}\sigma^{ml}h_{lb},$$

from which it follows that the inverse of the shape operator B is

$$A^a_b = (B^a_b)^{-1} = h^{ak}g_{kb} = \sigma^{ak}h_{kb}.$$

Then the eigenvalues of the matrix

$$A = \sigma^{-1}\nabla^2 u + u\, I \tag{2.3.19}$$

are of the form $1/\kappa_i$ where κ_i are the principal curvatures of Σ.

Note that the Minkowski problem of finding a strictly convex hypersurface with prescribed Gauss curvature $K > 0$, can be formulated as the problem of finding a solution to the equation

$$\det(A) = \varphi,$$

on the unit sphere $\mathbb{S}^n$ under the condition that $\varphi = 1/K > 0$.

Minkowski found also that a necessary and sufficient condition to solve the problem is that

$$\int_{\mathbb{S}^n} \varphi(x) \langle E_i, x\rangle = 0$$

holds. Moreover, note that from the change of variables formula the following holds:

$$\begin{aligned}
\int_{\mathbb{S}^n} \varphi(x) \langle E_i, x\rangle &= \int_{\mathbb{S}^n} K^{-1}(x) \langle E_i, x\rangle dx \\
&= \int_{\mathbf{n}(\Sigma)} K^{-1}(x) \langle E_i, x\rangle dx \\
&= \int_{\Sigma} K^{-1}(n(y)) \langle E_i, n(y)\rangle |\det d\mathbf{n}| dy \\
&= \int_{\Sigma} \langle E_i, n(y)\rangle dy \\
&= \int_{\text{int}(\Sigma)} \operatorname{div}(E_i) dy = 0.
\end{aligned}$$

The following result is sometimes referred as the Cheng-Yau lemma [8]. This provides the key estimates of second order needed to establish the existence of a solution of the Minkowski problem for dimension $n \geq 3$.

Lemma 1. *Let $M \subset \mathbb{R}^{n+1}$ be a compact convex C^4 hypersurface. Let K be the Gauss curvature function defined on $\mathbb{S}^n$. Then the extrinsic diameter of M can be estimated from above by*

$$c_n \left(\int_{\mathbb{S}^n} \frac{1}{K} \right)^{\frac{n}{n-1}} \left[\inf_{u \in \mathbb{S}^n} \int_{\mathbb{S}^n} \max\left(0, \langle u, w\rangle\right) K(w)^{-1} \right]^{-1},$$

where the positive constant c_n depends only on n.

There exist also a positive constant r depending only on an upper estimate of $\int_{\mathbb{S}^n} K^{-1}$ and a lower estimate of $\int_{\mathbb{S}^n} \max(0, \langle u, w\rangle) K(w)^{-1}$, such that we can always put a ball of radius r inside the hypersurface M.

Proof. We only outline the first part of the proof which uses change of variables formula, Stokes' theorem and the isoperimetric inequality. Since the Gauss map $\mathbf{n} : \Sigma \to \mathbb{S}^n$ is diffeomorphism, the change of variables formula reads

$$\int_{\mathbf{n}(\Sigma)} f = \int_{\Sigma} (f \circ \mathbf{n}) |\det(d\mathbf{n})|.$$

Note that the Gauss curvature is $K = \det(d\mathbf{n})$. Let M be the domain enclosed by Σ. The integration by parts formula is given by

$$\int_M \operatorname{div}(v\mathbf{F}) = \int_M \nabla v \cdot \mathbf{F} + \int_M v \operatorname{div}(\mathbf{F}),$$

and Stokes theorem:

$$\int_M \operatorname{div}(v\mathbf{F}) = \int_\Sigma v\,\mathbf{F} \cdot \mathbf{n}.$$

Apply Stokes' theorem with $\mathbf{F} = \nabla|x|^2 = 2x \in M \subset \mathbb{R}^{n+1}$, $v = 1$ so $\mathbf{F}\cdot\mathbf{n} = 2u(\mathbf{n}(x))$ is the support function when restricted to Σ, i.e.

$$\begin{aligned}\int_M \Delta x &= 2\int_\Sigma \langle x, \mathbf{n}(x)\rangle \\ &= 2\int_\Sigma (u \circ \mathbf{n})(x) \\ &= 2\int_\Sigma (u \circ \mathbf{n})(x)\frac{1}{K}\,|\det(d\mathbf{n}(x))|\,dx \\ &= 2\int_{S^n} \frac{u}{K}.\end{aligned}$$

Recall the isoperimetric inequality

$$|\Sigma|^n \geq n^n |M|^{n-1}\omega_n,$$

where $|\cdot|$ is the corresponding volume and ω_n is the volume of the n-dimensional unit ball. On the other hand note that the volume of the set enclosed by Σ is

$$|\Sigma| = vol(\Sigma) = \int_{\mathbb{S}^n} \frac{1}{K}.$$

Since in R^{n+1} we have

$$\Delta|x|^2 = 2(n+1),$$

and using the fact that for any unit vector $w \in \mathbb{S}^n$, the support function at w satisfies

$$u(w) \geq \frac{L}{2}\max_{v\in S^n}\{0, \langle v, w\rangle\},$$

where L is the extrinsic diameter of M, and we have moved the origin to be the midpoint of the segment of length L joining two suitable points $\{p, q\}$ in M. Then the inequality follows.

□

In 2002, P. Guan and B. Guan (see [14]) initiated the investigation of the existence of strictly convex hypersurfaces Σ in Euclidean space with normalised k-symmetric curvature prescribed by a positive function $\psi : \mathbb{S}^n \to \mathbb{R}^+$. They also made use of the support function and they used a group invariant assumption on the prescription function ψ, namely, those functions which are invariant under an automorphic group G of $\mathbb{S}^n$ without fixed points: $\psi(gx) = \psi(x)$ for all $g \in G$ and $x \in \mathbb{S}^n$.

2.4 Star-shaped hypersurfaces in de Sitter space

Let $\mathbb{R}^{n+2}_1 = (\mathbb{R}^{n+2}, \bar{g})$ be the Minkowski space with metric $\bar{g} = -dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2$ and covariant derivative $\bar{D}$. Then *de Sitter* space is defined as $S_1^{n+1} = \left\{x \in \mathbb{R}^{n+2}_1 : \bar{g}(x,x) = 1\right\}$ with the induced Lorentzian metric which we will denote by g and covariant derivative D. Moreover, any point in S_1^{n+1} can be written as $(r, \xi) \in \mathbb{R}^+ \times \mathbb{S}^n$, with the induced metric

$$g = -dr^2 + \cosh^2(r)\sigma, \tag{2.4.20}$$

where σ is the round metric on $\mathbb{S}^n$, and later we will use $\tilde{\nabla}$ to denote the covariant derivative for the metric σ. The vector field ∂_r will be written separately from any other index notation $\partial_\alpha, \partial_j, ...,$ etc., the latter indices taking values from 1 to n.

Let $u : \mathbb{S}^n \to [0, \infty)$ be a smooth function and consider a spacelike hypersurface in S_1^{n+1} given by the graph $\Sigma = \{(u(\xi), \xi)\}$. The tangent space of the hypersurface at a point $Y \in \Sigma$ is spanned by the tangent vectors $Y_j = u_j\partial_r + \partial_j$, and the covariant derivative ∇ corresponding to the induced metric on Σ is given by

$$G_{ij} = -u_i u_j + \cosh^2(u)\sigma_{ij}. \tag{2.4.21}$$

Since the metric is positive definite, the inverse can be computed

$$G^{ij} = \cosh^{-2}(u)\sigma^{ij} + \frac{\sigma^{i\gamma}u_\gamma \sigma^{j\eta}u_\eta}{\cosh^4(u) - \cosh^2(u)|\tilde{\nabla}u|^2}, \tag{2.4.22}$$

where $\tilde{\nabla}u = \sigma^{ij}u_j\partial_i$ and $|\tilde{\nabla}u| := \sigma^{ij}u_i u_j$. Note that for this to be well defined we need to have $|\tilde{\nabla}u|^2 \neq \cosh^2(u)$, and this is the case when the surface is spacelike. A unit normal vector to Σ at the point Y can be obtained by solving the equation $g(Y_\alpha, \hat{n}) = 0$, and then we get

$$\hat{n} = -\frac{\cosh^2(u)\partial_r + \tilde{\nabla}u}{\sqrt{\epsilon\left(-\cosh^4(u) + \cosh^2(u)|\tilde{\nabla}u|^2\right)}}, \tag{2.4.23}$$

and moreover, since Σ is spacelike, then the following inequality must hold

$$|\tilde{\nabla}u| \leq \cosh(u), \tag{2.4.24}$$

because the unit vector $\hat{n}$ normal to Σ is time-like, that is $g(\hat{n}, \hat{n}) = -1$.

The second fundamental form is the projection of the second derivatives of the parameterisation $D_{Y_\alpha}Y_\beta$ on the normal direction. Notice that

$$D_{\partial_r}\partial_r = 0; \quad D_{\partial_r}\partial_j = \tanh(r)\partial_j; \quad D_{\partial_i}\partial_j = \cosh(r)\sinh(r)\sigma_{ij}\partial_r + \tilde{\Gamma}_{ij}^k\partial_k, \quad (2.4.25)$$

and use them in

$$\begin{aligned} D_{Y_i}Y_j &= D_{u_i\partial_\tau+\partial_i}(u_j\partial_\tau + \partial_j) \\ &= u_ju_jD_{\partial_\tau}\partial_\tau + u_iD_{\partial_\tau}\partial_j + u_{ij}\partial_\tau + u_jD_{\partial_i}\partial_\tau + D_{\partial_i}\partial_j. \end{aligned} \quad (2.4.26)$$

Then $A_{\alpha\beta} = g(D_{Y_\alpha}Y_\beta, \hat{n})$ is given explicitly by

$$A_{ij} = \frac{\cosh^2(u)}{\sqrt{\cosh^4(u) - \cosh^2(u)|\tilde{\nabla}u|^2}}\left(\tilde{\nabla}_{ij}^2u - 2\frac{\sinh(u)}{\cosh(u)}u_iu_j + \sinh(u)\cosh(u)\sigma_{ij}\right). \quad (2.4.27)$$

Then applying the Gauss equation (2.1.13) to the surface as a submanifold of codimension two $\Sigma \subset S_1^{n+1} \subset \mathbb{R}^{n+1,1}$ we have

$$\begin{aligned} 0 = \bar{\bar{R}}_{ijkl} &= \bar{R}_{ijkl} - \epsilon_1(h_{ik}h_{jl} - h_{il}h_{jk}) \\ &= R_{ijkl} - \epsilon_2(A_{ik}A_{jl} - A_{il}A_{jk}) - \epsilon_1(h_{ik}h_{jl} - h_{il}h_{jk}), \end{aligned} \quad (2.4.28)$$

where $\epsilon_1 = \bar{g}(Y, Y) = 1$ and $\epsilon_2 = \bar{g}(\hat{n}, \hat{n}) = -1$.

Chapter 3

Geometric Fully Nonlinear Equations.

In this chapter we will outline the general theory of existence of solutions for fully nonlinear elliptic partial differential equations of concave type. The main reference for the topics in this chapter is [13]. The continuity method is also discussed, along with the a priori estimates needed, which we apply to the question of existence of a hypersurface with prescribed curvature.

Let Ω be a bounded open subset of $\mathbb{R}^n$, $k \geq 0$ an integer and $0 < \alpha < 1$. We recall that the **Hölder space** $C^{k,\alpha}(\bar{\Omega})$, is a Banach space of functions f with norm

$$|f|_{C^{k,\alpha}(\Omega)} = |f|_{C^k(\Omega)} + \max_{|r|=k} |D^r f|_{C^\alpha(\Omega)}, \tag{3.0.1}$$

where

$$|f|_{C^k(\Omega)} = \max_{|r|\leq k} \sup_{x\in\Omega} |D^r f| \quad ; \quad |f|_{C^\alpha(\Omega)} = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}. \tag{3.0.2}$$

3.1 The general equations

The equations that we will be considering are of the form

$$F(x, u, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \tag{3.1.3}$$

where F is a real function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{M}$, and $\mathcal{M}$ is the space of real symmetric $n \times n$ matrices, where $\dim(\mathcal{M}) = n(n+1)/2$.

We say that F is *elliptic* in a subset $\mathcal{U} \subset \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{M}$ if for any $(x, z, p, r) \in \mathcal{U}$ F is differentiable with respect to the variable r and the matrix given by $F^{ij} = \frac{\partial F}{\partial r_{ij}}$ is positive definite. F is said to be elliptic with respect to $u \in C^2(\Omega)$ if in the definition we take the subset $\mathcal{U}$ to be the range of $x \to (x, u, \nabla u, \nabla^2 u)$.

Note as well that the ellipticity of F implies the following *comparison principle* (see [13], Theorem 17.1): Let $u, v \in C^2(\bar{\Omega})$, where Ω is a bounded domain and suppose we have that $u \geq v$ on $\partial\Omega$ and $F(x, u, \nabla u, \nabla^2 u) \leq F(x, v, \nabla v, \nabla^2 v)$ in Ω. If the following hold

(a) F is continuously differentiable with respect to the corresponding variables z, p, r.

(b) F is elliptic with respect to $tu + (1 - t)v$ for all $t \in [0, 1]$.

(c) F is non-increasing in the variable z.

Then $u \geq v$ in Ω.

3.2 Prescribed curvature equations

In this work we will consider solutions to fully nonlinear equations of the form

$$F(A) = f(\lambda_1, \ldots, \lambda_n) = \psi \text{ in } \Omega \subset \mathbb{S}^n, \tag{3.2.4}$$

where A is the second fundamental form of a spacelike hypersurface in de Sitter space dS^n, f is a symmetric function of the eigenvalues of A, and ψ is a function of the position vector and the tilt of the surface (see (4.1.6)).

We will also assume that the hypersurface is the graph over an open set of the sphere of a function. More precisely, let $\Omega \subset \mathbb{S}^n$ be a smooth domain and $u : \Omega \to \mathbb{R}$ a positive smooth function such that the graph

$$\Sigma = \text{graph}(u) = \{(u(\xi), \xi) \mid \xi \in \Omega \subset \mathbb{S}^n\} \subset dS^n \tag{3.2.5}$$

is a spacelike hypersurface in de Sitter space dS^n. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, let $S_k(\lambda) = \Sigma_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$, and define the normalised symmetric polynomial

$H_k(\lambda) = \binom{n}{k} S_k$. In this paper we will be considering the case where f is homogeneous of degree one given by

$$f(\lambda) = H_k^{1/k}(\lambda), \tag{3.2.6}$$

defined in an open convex cone Γ which is symmetric and with vertex at the origin and contains the positive cone $\Gamma^+ = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0, \forall i = 1, 2, \ldots, n\}$.

Then F can be expressed as the k-th root of the sum of the principal minors of the shape operator $A^i_j = G^{ik} A_{kj}$, and from equations (2.4.22) and (2.4.27) we will have $F(A) \equiv F(\xi, u, \nabla u, \nabla^2 u)$. A solution u of (3.2.4) will be called *admissible* if Σ given by (3.2.5) is a spacelike hypersurface and the principal curvatures $\lambda = (\lambda_1, \ldots, \lambda_n)$ belong to Γ.

Under these conditions, the ellipticity and concavity of the nonlinear operator F are established by properties of f. For instance, if $f_{\lambda_i} > 0$ for all $i = 1, 2, \ldots, n$, and $f(\lambda)$ is concave in Γ, then it follows that F is elliptic and concave.

Moreover, let

$$F^{ij} := \frac{\partial F}{\partial a_{ij}}, \tag{3.2.7}$$

then it will follow that F^{ij} is diagonal when A is diagonal, and we have $F^{ij} = \operatorname{diag}(f_1, ..., f_n)$.

A direct computation shows that

$$\sum_{i=i}^{n} \frac{\partial S_k}{\partial \lambda_i} = (n - k + 1) S_{k-1}, \tag{3.2.8}$$

or equivalently

$$\sum_{i=i}^{n} F^{ii} = \sum_{i=i}^{n} f_i = \sum_{i=i}^{n} \frac{\partial H_k}{\partial \lambda_i} = \frac{H_{k-1}}{H_k^{\frac{k-1}{k}}}. \tag{3.2.9}$$

The theory of existence of solutions of such equations has been studied extensively and in more generality in [5] by L. Caffarelli, L. Nirenberg and J. Spruck. In [6], they proved the existence of star-shaped hypersurfaces in Euclidean space with prescribed k-symmetric curvature using a priori estimates needed to carry out the continuity method. The idea behind it is the following: Suppose that one wants to show that a solution of the equation $F(u) = 0$ exists. Consider a one parameter family of problems $F_t(u) = 0$ depending continuously on t such that $F_1(u) = F(u)$ is the problem we wish to solve and $F_0(u) = 0$ is a problem that we know how to solve (in our case when u is a suitable constant). Then define $A = \{t \in [0,1] \,|\, \text{one can solve } F_t(u) = 0\}$. The existence of a solution follows by showing that A is non-empty, open and closed. In suitable functional spaces the openness follows from a version of the inverse function theorem for infinite dimensional vector spaces, and the closedness by establishing suitable a priori estimates.

The existence of a solution for the Dirichlet problem is reduced to obtaining the a priori estimate

$$|u|_{C^{2,\alpha}(\bar{\Omega})} \leq C \tag{3.2.10}$$

for some $0 < \alpha < 1$. Then, as in the quasilinear case we have to establish estimates for $\sup_\Omega |u|$, $\sup_{\partial\Omega} |Du|$, $\sup_\Omega |Du|$, and additionally $\sup_{\partial\Omega} |D^2u|$, $\sup_\Omega |D^2u|$.

In the case when F is a uniformly elliptic fully nonlinear concave equation, the Evans-Krylov theorem [10] [17] gives the following a priori estimate

$$|u|_{C^{2,\alpha}(\bar{\Omega})} \leq C|u|_{C^{1,1}(\bar{\Omega})} \tag{3.2.11}$$

when $\Omega = B_1$ is the unit ball and C depends only on the concavity property of F. See also [7], where the authors present the regularity theory for fully nonlinear elliptic equations in more detail.

Chapter 4

First Curvature Estimate

We obtain similar curvature estimates as in [15] in de Sitter space. As in [15] we impose a growth assumption on the right hand side in terms of the tilt τ (see (4.1.6)).

Theorem 1. *Let $\Omega \subset \mathbb{S}^n$ be a domain in the round sphere, and let $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ be an admissible solution of the boundary value problem*

$$\begin{cases} F(A) = H_k^{\frac{1}{k}}(\lambda(A)) & = & \psi(Y,\tau) & in & \Omega \\ u & = & \varphi & on & \partial\Omega \end{cases},$$

where A is the second fundamental form of a spacelike surface Σ in de Sitter space given by (2.4.27), $\psi \in C^\infty(\bar{\Omega})$, $\psi > 0$ and convex in τ. Assume additionally that

$$\psi_\tau(X,\tau)\tau - \psi(X,\tau) \geq 0$$

for all $X \in S_1^{n+1}$ and $\tau \in [1,\infty)$. Then

$$\sup_\Omega |A| \leq C,$$

where C is a constant depending on n, $\|\varphi\|_{C^1(\bar{\Omega})}$, $\|\psi\|_{C^2(I,\Omega,[1,\infty))}$ and $\sup_{\partial\Omega} |A|$.

4.1 Commutator formula, tilt and height functions

We are now going to prove that if the curvature of the hypersurface is bounded, then the C^2 estimate of the solution will be a consequence of the equation of the second fundamental form (2.4.27). We will need the commutator formula for the second order derivatives of the second fundamental form, given by Ricci's identity (2.1.15), together with the Gauss equation of the surface as a codimension 2 spacelike submanifold of Minkowski space. This is on account of equation (2.4.28), which gives the following

$$R_{ijkl} = -(A_{ik}A_{jl} - A_{il}A_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{4.1.1}$$

where we are using A_{ij} as the second fundamental form of the spacelike hypersurface in de Sitter space, and h_{ij} denotes the second fundamental form of de Sitter space in flat Minkowski space. Substituting in equation (2.1.15) we get

$$\begin{aligned}\nabla_k\nabla_l A_{ij} &= \nabla_l\nabla_k A_{ij} + \sum_r R_{kljr}A_{ir} + \sum_r R_{klir}A_{rj}\\ &= \nabla_l\nabla_k A_{ij} + \sum_r \{-(A_{kj}A_{lr} - A_{kr}A_{lj}) + (h_{kj}h_{lr} - h_{kr}h_{lj})\}\, A_{ir}\\ &\quad + \sum_r \{-(A_{ki}A_{lr} - A_{kr}A_{li}) + (h_{ki}h_{lr} - h_{kr}h_{li})\}\, A_{rj}.\end{aligned}$$

Moreover, notice that by the Codazzi equation and Ricci identity (2.1.15) we get

$$\begin{aligned}\nabla_i\nabla_j A_{kk} =& \nabla_i\nabla_k A_{kj}\\ =& \nabla_k\nabla_i A_{kj} + R_{ikkr}A_{rj} + R_{ikjr}A_{kr}\\ =& \nabla_k\nabla_k A_{ij} + R_{ikkr}A_{rj} + R_{ikjr}A_{kr}.R_{ikkr}A_{rj} + R_{ikjr}A_{kr}.\end{aligned}$$

Using coordinates such that A is diagonal, from equation (4.1.1) we obtain

$$\begin{aligned}\nabla_j\nabla_j A_{kk} = \nabla_k\nabla_k A_{jj} + A_{kk}A_{jj}^2 + h_{jk}h_{jk}A_{jj} - h_{kk}h_{jj}A_{jj}\\ - A_{jj}A_{kk}^2 + h_{jj}h_{kk}A_{kk} - h_{jk}h_{jk}A_{kk}.\end{aligned} \tag{4.1.2}$$

The first covariant derivative of (3.2.4) is given by

$$F^{ij}\nabla_k A_{ij} = \nabla_k\psi,$$

and the second covariant derivative

$$F^{ij}\nabla_k\nabla_k A_{ij} + F^{ij,ml}\nabla_k A_{ij}\nabla_k A_{ml} = \nabla_k\nabla_k\psi. \tag{4.1.3}$$

By multiplication of F^{jj} with (4.1.2) and adding repeated indices

$$F^{jj}\nabla_j\nabla_j A_{kk} = F^{jj}\nabla_k\nabla_k A_{jj} + A_{kk}F^{jj}A^2_{jj} - F^{jj}A_{jj} - F^{jj}A_{jj}A^2_{kk} + A_{kk}\sum_j F^{jj}. \tag{4.1.4}$$

Let $H = \sum_k A_{kk}$, we will use the identities above to compute $F^{jj}\nabla_j\nabla_j H$ that will be used later. From (4.1.4) we have

$$F^{jj}\nabla_j\nabla_j H = F^{jj}\sum_k \nabla_k\nabla_k A_{jj} + HF^{jj}A^2_{jj} - nF^{jj}A_{jj} - F^{jj}A_{jj}\sum_k A^2_{kk} + H\sum_j F^{jj}.$$

Since $H_k^{1/k}$ is homogeneous of degree 1, it holds that $F^{jj}A_{jj} = \psi$, and then

$$F^{jj}\nabla_j\nabla_j H = \sum_k F^{jj}\nabla_k\nabla_k A_{jj} + H\left(F^{jj}A^2_{jj} + \sum_j F^{jj}\right) - \psi\left(n + \sum_j A^2_{jj}\right).$$

Using equation (4.1.3) we can rewrite the first term of the left hand side above and we get

$$F^{jj}\nabla_j\nabla_j H = -\sum_k F^{ij,lm}\nabla_k A_{ij}\nabla_k A_{lm} + \sum_k \nabla_k\nabla_k\psi + H\left(F^{jj}A^2_{jj} + \sum_j F^{jj}\right) - \psi\left(n + \sum_j A^2_{jj}\right). \tag{4.1.5}$$

Now we consider the following parameterisation of the hypersurface

$$Y = \sinh(u(\xi))E_1 + \cosh(u(\xi))\xi, \quad \xi \in \mathbb{S}^n,$$

where $E_1 = (1, 0, ..., 0) \in \mathbb{R}^{n+1,1}$. The tangent space to Σ is spanned by the vectors $Y_i = u_i\left(\cosh(u)E_1 + \sinh(u)\xi\right) + \cosh(u)\xi_i = u_i\partial_r + \partial_i$. We will write $Y_i = \nabla_i$ and $u_i = \partial_i u = \cosh(u)\xi_i u$.

Note that

$$\cosh(u)\partial_r = (\cosh(u)E_1 + \sinh(u)\xi) = \cosh(u)^2 E_1 + \cosh(u)\sinh(u)\xi = E_1 + \sinh(u)Y.$$

The tilt and the height functions are given respectively by

$$\tau = \langle \hat{n}, E_1 \rangle = \frac{\cosh^2(u)}{\sqrt{\cosh^2(u) - |\tilde{\nabla} u|^2}}; \qquad \eta = \langle Y, E_1 \rangle = -\sinh(u), \tag{4.1.6}$$

and

$$\exp[\Phi(u,\xi)] = \frac{A_{11}}{g_{11}} \exp[\alpha(\tau) - \beta\eta].$$

The following proposition provides very useful formulae to be used in the next section and next chapter.

Proposition 4.1.1. *For τ and η defined as above, the following holds:*

1. $\nabla_{ij}\eta = -\tau A_{ij} - \eta g_{ij}$.

2. $\nabla_j \tau = -g^{ik} A_{kj} \nabla_i \eta$.

3. $\nabla_j \nabla_i \tau = -g^{mn} \nabla_n A_{ij} \nabla_m \eta + \tau A_{mj} g^{mn} A_{ni} + A_{ij}\eta$.

Proof. Using the Weingarten equation (2.1.4) we obtain

$$\begin{aligned} \nabla_j \tau = \langle \nabla_j \hat{n}, E_1 \rangle = -\langle A_j^i Y_i, E_1 \rangle & \\ = -g^{ik} A_{kj} \langle Y_i, E_1 \rangle = -g^{ik} A_{kj} \nabla_i \langle Y, E_1 \rangle = -g^{ik} A_{kj} \nabla_i \eta. \end{aligned}$$

From the Gauss formula note that at any point $p \in S_1^{n+1}$ we have $h_{ij} N_p = -g_{ij} p$, and this implies $h_{ij} N\eta = -g_{ij}\eta$. Then it follows using the Gauss formula twice

$$\begin{aligned} \nabla_{ij}\eta &= Y_j(Y_i\eta) - (\nabla_{Y_i} Y_j)\eta \\ &= Y_j(Y_i\eta) - (D_{Y_i} Y_j + A_{ij}\hat{n})\eta \\ &= Y_j(Y_i\eta) - (\bar{D}_{Y_i} Y_j - h_{ij} N + A_{ij}\hat{n})\eta \\ &= Y_j(Y_i\eta) - (Y_j Y_i - h_{ij} N + A_{ij}\hat{n})\eta \\ &= -\tau A_{ij} - \eta g_{ij}, \end{aligned}$$

and from this we have

$$
\begin{aligned}
\nabla_{ij}\tau &= \nabla_j(-g^{mn}A_{ni}\nabla_m\eta) \\
&= -\nabla_j g^{mn}A_{ni}\nabla_m\eta - g^{mn}\nabla_j A_{ni}\nabla_m\eta - g^{mn}A_{ni}\nabla_{mj}\eta \\
&= -g^{mn}\nabla_j A_{ni}\nabla_m\eta - g^{mn}A_{ni}\nabla_{mj}\eta \\
&= -g^{mn}\nabla_n A_{ij}\nabla_m\eta - g^{mn}A_{ni}(-\tau A_{mj} - \eta g_{mj}) \\
&= -g^{mn}\nabla_n A_{ij}\nabla_m\eta + \tau A_{mj}g^{mn}A_{ni} + g^{mn}A_{ni}\eta g_{mj} \\
&= -g^{mn}\nabla_n A_{ij}\nabla_m\eta + \tau A_{mj}g^{mn}A_{ni} + A_{ij}\eta.
\end{aligned}
$$

□

4.2 Proof of Theorem 1

Proof of Theorem 1. Since $\psi = \psi(Y,\tau)$ and from the assumption that ψ is convex in τ and Proposition 4.1.1(3), in an orthonormal frame such that A is symmetric it holds (see [15]) that

$$
\begin{aligned}
\sum_k \nabla_k\nabla_k\psi &\geq \psi_\tau \sum_k \nabla_k\nabla_k\tau + \psi_{\tau\tau}\sum_k (\nabla_k\tau)^2 - C_1H - C_2 \\
&\geq \psi_\tau\left(-\nabla_k H\nabla_k\eta + \tau A_{ki}A_{ki} + H\eta\right) - C_1H - C_2.
\end{aligned}
\tag{4.2.7}
$$

We continue from equation (4.1.5), and we will make use of the last inequality (4.2.7), the concavity of F, the fact that $H \geq 0$ and $\sum_j F^{jj} \geq 0$. Note that at the maximum of H we have $\nabla H \doteq 0$ and $\nabla_j\nabla_i H \dot{\leq} 0$, it also follows $0 \dot{\geq} F^{jj}\nabla_j\nabla_j H$, then

$$
\begin{aligned}
0 &\geq \sum_k \nabla_k\nabla_k\psi + H\left(F^{jj}A_{jj}^2 + \sum_j F^{jj}\right) - \psi\left(n + \sum_j A_{jj}^2\right) \\
&\geq \psi_\tau\left(\sum_k \tau A_{kk}^2 + H\eta\right) - C_1H - C_2 + HF^{jj}A_{jj}^2 - \psi\left(n + \sum_j A_{jj}^2\right) \\
&\geq -C_2 - n\psi + (\psi_\tau\eta - C_1)H + F^{jj}A_{jj}^2H + (\psi_\tau\tau - \psi)\sum_k A_{kk}^2.
\end{aligned}
$$

Since $(\psi_\tau\tau - \psi) \geq 0$ and by the Newton-Maclaurin inequalities $H_{k+1}H_{k-1} \leq H_k^{\frac{1}{2}}$ one can show (see [23]) the following

$$
F^{ij}A_{il}A_{lj} \geq \frac{1}{n}S_k^{1/k}S_1,
$$

and from this it follows that

$$0 \geq -C_2 - n\psi + (\psi_\tau \eta - C_1)\, H + C_3 \psi H^2$$

which implies H is bounded, hence A is bounded. □

Chapter 5

Second Curvature Estimate

In this chapter we give an interior estimate when the growth condition is strict, and the boundary data is spacelike and affine.

Theorem 2. *Let $\Omega \subset \mathbb{S}^n$ be a domain in the round sphere, and let $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ be an admissible solution of the boundary value problem*

$$\begin{cases} F(A) = H_k^{\frac{1}{k}}(\lambda(A)) & = \psi(Y, \tau) \quad in \quad \Omega \\ \qquad\qquad\qquad u & = \varphi \qquad\quad on \quad \partial\Omega \end{cases},$$

where A is the second fundamental form of a spacelike surface Σ in de Sitter space given by (2.4.27), $\psi \in C^\infty(\bar{\Omega})$, $\psi > 0$ and convex in τ. Assume also that

$$\psi_\tau(X, \tau)\tau - \psi(X, \tau) > 0$$

for all $X \in S_1^{n+1}$ and $\tau \in [1, \infty)$, and that the domain $\Omega \subset \mathbb{R}^n$ is C^2, uniformly convex. If the boundary value φ is spacelike and affine, namely φ is the restriction of an affine function on ambient Minkowski space of dimension $n+2$, then for any $\Omega' \subset\subset \Omega$ there is a constant C depending only on n, Ω, $\mathrm{dist}(\Omega', \partial\Omega)$, $\|\varphi\|_{C^1(\bar{\Omega})}$ and $\|\psi\|_{C^2(I,\Omega,[1,\infty))}$, such that

$$\sup_{\Omega'} |A| \leq C.$$

5.1 Proof of Theorem 2

Proof. Consider the function $\gamma = \varphi - u$, $\gamma > 0$ in Ω, and let

$$\Phi(\xi) = \ln(A_{11}) + \alpha(\tau) + \beta \ln(\gamma),$$

with first covariant derivative

$$\nabla_j \Phi = \frac{\nabla_j A_{11}}{A_{11}} + \alpha' \nabla_j \tau + \beta \frac{\nabla_j \gamma}{\gamma}. \tag{5.1.1}$$

The second covariant derivative is:

$$\nabla_j \nabla_j \Phi = \frac{\nabla_j \nabla_j A_{11}}{A_{11}} - \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 + \alpha'' \left(\nabla_j \tau\right)^2 + \alpha' \nabla_j \nabla_j \tau + \beta \frac{\nabla_j \nabla_j \gamma}{\gamma} - \beta \left(\frac{\nabla_j \gamma}{\gamma}\right)^2.$$

Use the commutator formula (4.1.2) and computing $F^{jj} \nabla_j \nabla_j \Phi$, we have

$$\begin{aligned} F^{jj} \nabla_j \nabla_j \Phi = \frac{1}{A_{11}} \big\{ & F^{jj} \nabla_k \nabla_k A_{jj} + F^{jj} A_{kk} A_{jj}^2 + F^{jj} h_{jk} h_{jk} A_{jj} \\ & - F^{jj} h_{kk} h_{jj} A_{jj} - F^{jj} A_{jj} A_{kk}^2 + F^{jj} h_{jj} h_{kk} A_{kk} \\ & - F^{jj} h_{jk} h_{jk} A_{kk} \big\} - F^{jj} \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 + \alpha'' F^{jj} \left(\nabla_j \tau\right)^2 \\ & + \alpha' F^{jj} \nabla_j \nabla_j \tau + \beta F^{jj} \frac{\nabla_j \nabla_j \gamma}{\gamma} - \beta F^{jj} \left(\frac{\nabla_j \gamma}{\gamma}\right)^2. \end{aligned}$$

Note that in coordinates such that $h_{ij} = \delta_{ij}$, some terms in the brackets cancel. Now, using the identity $F^{jj} A_{jj} = \psi$ from the homogeneity of (3.2.6), we can write

$$\begin{aligned} F^{jj} \nabla_j \nabla_j \Phi = & \frac{1}{A_{11}} F^{jj} \nabla_1 \nabla_1 A_{jj} + F^{jj} A_{jj}^2 - \left(A_{11} + \frac{1}{A_{11}}\right) \psi \\ & + \sum_j F^{jj} - F^{jj} \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 + \alpha'' F^{jj} \left(\nabla_j \tau\right)^2 \\ & + \alpha' F^{jj} \nabla_j \nabla_j \tau + \beta F^{jj} \frac{\nabla_j \nabla_j \gamma}{\gamma} - \beta F^{jj} \left(\frac{\nabla_j \gamma}{\gamma}\right)^2. \end{aligned}$$

Using equation (4.1.3) in the last equation we get

$$
\begin{aligned}
F^{jj}\nabla_j\nabla_j\Phi = -\frac{1}{A_{11}}F^{ij,kl}\nabla_1 A_{ij}\nabla_1 A_{kl} + \frac{\nabla_1\nabla_1\psi}{A_{11}} \\
-\left(A_{11}+\frac{1}{A_{11}}\right)\psi + F^{jj}A_{jj}^2 + \sum_j F^{jj} \\
- F^{jj}\left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 + \alpha'' F^{jj}\left(\nabla_j\tau\right)^2 + \alpha' F^{jj}\nabla_j\nabla_j\tau \\
+ \beta F^{jj}\frac{\nabla_j\nabla_j\gamma}{\gamma} - \beta F^{jj}\left(\frac{\nabla_j\gamma}{\gamma}\right)^2. \quad (5.1.2)
\end{aligned}
$$

Then as in [15], by Proposition 4.1.1(3) and using $\psi(Y,\tau)$ we have

$$
\begin{aligned}
\nabla_1\nabla_1\psi &\geq \psi_\tau\nabla_1\nabla_1\tau - C_1A_{11} - C_2 \\
&= \psi_\tau\left(-\sum_r \nabla_r A_{11}\nabla_r\eta + A_{11}^2\tau + A_{11}\delta_{11}\right) - C_1A_{11} - C_2.
\end{aligned}
$$

Then we have the following inequality:

$$
\frac{\nabla_1\nabla_1\psi}{A_{11}} \geq -\frac{\psi_\tau}{A_{11}}\sum_r \nabla_r A_{11}\nabla_r\eta + \psi_\tau A_{11}\tau + \psi_\tau\delta_{11} - C_1 - \frac{C_2}{A_{11}}. \quad (5.1.3)
$$

On the other hand, using the assumption that φ is affine we have

$$
F^{jj}\nabla_j\nabla_j\gamma \geq -C. \quad (5.1.4)
$$

Also we are assuming control over $|\nabla_j\gamma| \leq C$, and then

$$
F^{jj}\nabla_j\gamma\nabla_j\gamma \leq C\sum_j F^{jj}, \quad (5.1.5)
$$

which will be used at the end. If we carry on using inequalities (5.1.4) and (5.1.3) in (5.1.2) we obtain

$$
\begin{aligned}
F^{jj}\nabla_j\nabla_j\Phi \geq -\frac{1}{A_{11}}F^{ij,kl}\nabla_1 A_{ij}\nabla_1 A_{kl} - \frac{\psi_\tau}{A_{11}}\sum_r \nabla_r A_{11}\nabla_r\eta \\
+ \psi_\tau A_{11}\tau + \psi_\tau\delta_{11} - C_1 - \frac{C_2}{A_{11}} + F^{jj}A_{jj}^2 \\
-\left(A_{11}+\frac{1}{A_{11}}\right)\psi + \sum_j F^{jj} - F^{jj}\left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 \\
+ \alpha'' F^{jj}\left(\nabla_j\tau\right)^2 + \alpha' F^{jj}\nabla_j\nabla_j\tau - \beta\frac{C}{\gamma} - \beta F^{jj}\left(\frac{\nabla_j\gamma}{\gamma}\right)^2.
\end{aligned}
$$

Using again Proposition 4.1.1(3), we replace the term $\alpha' F^{jj}\nabla_j\nabla_j\tau$ to get

$$\begin{aligned}F^{jj}\nabla_j\nabla_j\Phi \geq & -\frac{1}{A_{11}}F^{ij,kl}\nabla_1 A_{ij}\nabla_1 A_{kl} - \frac{\psi_\tau}{A_{11}}\sum_r \nabla_r A_{11}\nabla_r\eta \\ & + \psi_\tau A_{11}\tau + (\psi_\tau + \alpha'\psi)\,\delta_{11} - C_1 - \frac{C_2}{A_{11}} + \sum_j F^{jj} \\ & + (1+\alpha'\tau)F^{jj}A_{jj}^2 - \left(A_{11} + \frac{1}{A_{11}}\right)\psi - F^{jj}\left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 \\ & + \alpha'' F^{jj}\left(\nabla_j\tau\right)^2 - \alpha'\sum_r \nabla_r\psi\nabla_r\eta - \beta\frac{C}{\gamma} - \beta F^{jj}\left(\frac{\nabla_j\gamma}{\gamma}\right)^2.\end{aligned}$$

Now, at the maximum of Φ we have $\nabla\Phi\dot{=}0$ and $\nabla_j\nabla_j\Phi\dot{\leq}0$ and by ellipticity $0\dot{\geq}F^{jj}\nabla_j\nabla_j\Phi$. Then we get

$$-\psi_\tau\sum_r \frac{\nabla_r A_{11}}{A_{11}}\nabla_r\eta = \psi_\tau\sum_r\left(\alpha'\nabla_r\tau + \beta\frac{\nabla_r\gamma}{\gamma}\right)\nabla_r\eta,$$

and since $\nabla_r\psi = \psi_r + \psi_\tau\nabla_r\tau$, we have that

$$-\psi_\tau\sum_r \frac{\nabla_r A_{11}}{A_{11}}\nabla_r\eta - \alpha'\sum_r\nabla_r\psi\nabla_r\eta = \sum_r\left(\beta\frac{\nabla_r\gamma}{\gamma} - \alpha'\psi_r\right)\nabla_r\eta \geq -\frac{C\beta}{\gamma} - C,$$

then,

$$\begin{aligned}F^{jj}\nabla_j\nabla_j\Phi \geq & -\frac{1}{A_{11}}F^{ij,kl}\nabla_1 A_{ij}\nabla_1 A_{kl} - C_1 - \frac{C_2}{A_{11}} - 2\beta\frac{C}{\gamma} - C \\ & + (\psi_\tau\tau - \psi)\,A_{11} + (\psi_\tau + \alpha'\psi)\,\delta_{11} \\ & + (1+\alpha'\tau)F^{jj}A_{jj}^2 - \frac{\psi}{A_{11}} + \sum_j F^{jj} \\ & - F^{jj}\left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 + \alpha'' F^{jj}\left(\nabla_j\tau\right)^2 - \beta F^{jj}\left(\frac{\nabla_j\gamma}{\gamma}\right)^2. \quad (5.1.6)\end{aligned}$$

Case 1: In this case we will use the concavity of F and drop the term with the second derivatives $F^{ij,kl}$ in the inequality (5.1.6), and we will suppose that there is $\mu > 0$ such that

$$A_{nn} \leq -\mu A_{11},$$

this implies that

$$F^{jj}A_{jj}^2 \geq \frac{\mu^2}{n}A_{11}^2\sum_j F^{jj}, \quad (5.1.7)$$

and also

$$F^{nn} \geq \frac{1}{n}\sum_j F^{jj}.$$

Note that

$$F^{jj}\left(\nabla_j\tau\right)^2 = F^{jj}A_{jj}^2(\nabla_j\eta)^2 \le CF^{jj}A_{jj}^2.$$

At the maximum of Φ we have $\nabla_j\Phi = 0$ and from (5.1.1) we have

$$\left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 = \left(\alpha'\nabla_j\tau + \beta\frac{\nabla_j\gamma}{\gamma}\right)^2, \tag{5.1.8}$$

and moreover, for all $\epsilon > 0$ we have

$$\left(\alpha'\nabla_j\tau + \beta\frac{\nabla_j\gamma}{\gamma}\right)^2 < (1+\epsilon)(\alpha')^2(\nabla_j\tau)^2 + (1+\epsilon^{-1})\beta^2\left(\frac{\nabla_j\gamma}{\gamma}\right)^2. \tag{5.1.9}$$

Note now that if $(\alpha'' - (1+\epsilon)(\alpha')^2) < 0$,

$$\left(\alpha'' - (1+\epsilon)(\alpha')^2\right)F^{jj}\left(\nabla_j\tau\right)^2 \ge C_1\left(\alpha'' - (1+\epsilon)(\alpha')^2\right)F^{jj}A_{jj}^2, \tag{5.1.10}$$

then from (5.1.6),

$$\begin{aligned} F^{jj}\nabla_j\nabla_j\Phi \ge & -\beta\frac{C}{\gamma} - C - \beta\frac{C}{\gamma} - C_1 - \frac{C_2}{A_{11}} + (\psi_\tau\tau - \psi)A_{11} \\ & + (\psi_\tau + \alpha'\psi)\delta_{11} + \left\{(1+\alpha'\tau) + C_1\left(\alpha'' - (1+\epsilon)(\alpha')^2\right)\right\}F^{jj}A_{jj}^2 \\ & - \frac{\psi}{A_{11}} + \left\{1 - \left(\beta + (1+\epsilon^{-1})\beta^2\right)\frac{1}{\gamma^2}\right\}\sum_j F^{jj}. \end{aligned} \tag{5.1.11}$$

Now, in order to control the coefficients of $F^{jj}A_{jj}^2$, we solve the following ordinary equation

$$\alpha'' - (\alpha')^2 = 0,$$

and we find solutions of the form

$$\alpha = -\ln(\tau + a),$$

where $a > 0$ to be specified. Moreover, the first and second derivatives are

$$\alpha' = -\frac{1}{\tau + a}, \quad \alpha'' = \frac{1}{(\tau + a)^2},$$

and then it is clear that

$$\alpha'' - (1+\epsilon)(\alpha')^2 = -\frac{\epsilon}{(\tau + a)^2} \le 0,$$

from which we can see that for $\epsilon = a^2/2C_1$ we have

$$
\begin{aligned}
(\alpha'\tau + 1) + C_1(\alpha'' - (1+\epsilon)(\alpha')^2) &= \frac{a}{\tau + a} - \frac{C_1\epsilon}{(\tau+a)^2} \\
&= \frac{a(\tau+a)}{(\tau+a)^2} - \frac{C_1\epsilon}{(\tau+a)^2} > \frac{a^2}{2(\tau+a)^2} \geq C_3 > 0,
\end{aligned}
$$

then from (5.1.11) we get

$$
\begin{aligned}
0 \geq -\beta\frac{C}{\gamma} - C - \beta\frac{C}{\gamma} - C_1 - \frac{C_2}{A_{11}} \\
+ (\psi_\tau\tau - \psi)A_{11} + (\psi_\tau + \alpha'\psi)\delta_{11} \\
+ C_3 F^{jj} A_{jj}^2 \\
- \frac{\psi}{A_{11}} + \left\{1 - \left(\beta + (1+\epsilon^{-1})\beta^2\right)\frac{1}{\gamma^2}\right\}\sum_j F^{jj}.
\end{aligned}
$$

Note $A_{11} \geq \cdots \geq A_{nn}$ and this implies that

$$\sum_j F^{jj} = \frac{1}{\psi^{k-1}} H_{k-1},$$

from this it follows that

$$\sum_j F^{jj} \geq C_4 > 0.$$

Using the growth assumption $\psi_\tau\tau - \psi > 0$, the inequality (5.1.7), and choosing $\beta > 0$ such that $\{1 - (\beta + (1+\epsilon^{-1})\beta^2)\frac{1}{\gamma^2}\} > 0$, we obtain

$$0 \geq -\beta\frac{C}{\gamma} - C - \beta\frac{C}{\gamma} - C_1 - \frac{C_2}{A_{11}} - \frac{\psi}{A_{11}} + \frac{\mu^2}{n} C_3 A_{11}^2.$$

Now we make use of the assumption $\lambda_1 \geq 1$ so that

$$\frac{C(\beta)}{\mu} \geq \gamma A_{11}.$$

Case 2: Looking back at inequality (5.1.6), the assumption for this case is the existence of $\mu > 0$ such that

$$A_{nn} \geq -\mu A_{11},$$

and in this case we will make use of the term with $F^{ij,kl}$. Note also that $A_{jj} \geq -\mu A_{11}$, for all $j = 1, 2, \ldots, n$ since $A_{11} \geq A_{22} \geq \cdot \geq A_{nn}$.

Consider the following partition of the indices $\{1, 2, \ldots, n\}$,

$$I = \{j \,|\, F^{jj} \leq 4\,F^{11}\}, \quad \text{and} \quad J = \{j \,|\, F^{jj} > 4\,F^{11}\}.$$

Now, for $j \in I$, at the maximum, equation (5.1.8) and inequality (5.1.9) hold for any $\epsilon > 0$, namely

$$\left(\alpha' \nabla_j \tau + \beta \frac{\nabla_j \gamma}{\gamma}\right)^2 < (1+\epsilon)(\alpha')^2 (\nabla_j \tau)^2 + (1+\epsilon^{-1})\beta^2 \left(\frac{\nabla_j \gamma}{\gamma}\right)^2, \quad j \in I.$$

For $j \in J$, at the maximum, since $\nabla_j \Phi = 0$ in equation (5.1.1), we have for any $\epsilon > 0$ that

$$\beta^{-1} \left(\alpha' \nabla_j \tau + \frac{\nabla_j A_{11}}{A_{11}}\right)^2 \leq \frac{1+\epsilon}{\beta} (\alpha')^2 (\nabla_j \tau)^2 + \frac{1+\epsilon^{-1}}{\beta} \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2.$$

From these two inequalities we can get

$$\begin{aligned}
\beta F^{jj} \left(\frac{\nabla_j \gamma}{\gamma}\right)^2 + F^{jj} \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 &\leq + \frac{1+\epsilon}{\beta} (\alpha')^2 \sum_{j \in J} F^{jj} (\nabla_j \tau)^2 + \frac{1+\epsilon^{-1}}{\beta} \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 \\
&\quad + \beta \sum_{j \in I} F^{jj} \left(\frac{\nabla_j \gamma}{\gamma}\right)^2 (1+\epsilon)(\alpha')^2 \sum_{j \in I} F^{jj} (\nabla_j \tau)^2 \\
&\quad + (1+\epsilon^{-1})\beta^2 \left(\frac{\nabla_j \gamma}{\gamma}\right)^2 + \sum_{j \in J} F^{jj} \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 \\
&\leq 4n\{\beta + (1+\epsilon^{-1})\beta^2\} F^{11} \left(\frac{\nabla_j \gamma}{\gamma}\right)^2 \\
&\quad + (1+\epsilon)(1+\beta^{-1})(\alpha')^2 F^{jj} (\nabla_j \tau)^2 \\
&\quad + \{1 + (1+\epsilon^{-1})\beta^{-1}\} \sum_{j \in J} F^{jj} \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2.
\end{aligned}$$

Using the last two estimates in (5.1.6) at the maximum we obtain

$$\begin{aligned}
0 \geq & -\frac{1}{A_{11}} F^{ij,kl} \nabla_1 A_{ij} \nabla_1 A_{kl} - C_1 - \frac{C_2}{A_{11}} - 2\beta \frac{C}{\gamma} - C \\
& + (\psi_\tau \tau - \psi) A_{11} + (\psi_\tau + \alpha' \psi) \delta_{11} \\
& + (1 + \alpha' \tau) F^{jj} A_{jj}^2 - \frac{\psi}{A_{11}} + \sum_j F^{jj} \\
& - F^{jj} \left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 + \alpha'' F^{jj} (\nabla_j \tau)^2 - \beta F^{jj} \left(\frac{\nabla_j \gamma}{\gamma}\right)^2.
\end{aligned}$$

Solving $\alpha'' - (\alpha')^2 = 0$ as in Case 1, we obtain (5.1.10) then

$$\begin{aligned} 0 \geq & -\frac{1}{A_{11}} F^{ij,kl} \nabla_1 A_{ij} \nabla_1 A_{kl} - C_1 - \frac{C_2}{A_{11}} - 2\beta \frac{C}{\gamma} - C \\ & + (\psi_\tau \tau - \psi) A_{11} + (\psi_\tau + \alpha' \psi) \delta_{11} - \frac{\psi}{A_{11}} \\ & - 4n\{\beta + (1+\epsilon^{-1})\beta^2\} F^{11} \left(\frac{\nabla_j \gamma}{\gamma} \right)^2 + \sum_j F^{jj} \\ & + \{(1+\alpha'\tau) + C_1 \left(\alpha'' - (1+\epsilon)(1+\beta^{-1})(\alpha')^2 \right) \} F^{jj} A_{jj}^2 \\ & - \{1 + (1+\epsilon^{-1})\beta^{-1}\} \sum_{j \in J} F^{jj} \left(\frac{\nabla_j A_{11}}{A_{11}} \right)^2, \end{aligned}$$

and moreover for $\epsilon = \epsilon(a)$, there is $C_0 > 0$ such that the last term is improved by

$$\begin{aligned} 0 \geq & -\frac{1}{A_{11}} F^{ij,kl} \nabla_1 A_{ij} \nabla_1 A_{kl} - C_1 - \frac{C_2}{A_{11}} - 2\beta \frac{C}{\gamma} - C \\ & + (\psi_\tau \tau - \psi) A_{11} + (\psi_\tau + \alpha' \psi) \delta_{11} - \frac{\psi}{A_{11}} \\ & - 4n\{\beta + (1+\epsilon^{-1})\beta^2\} F^{11} \left(\frac{\nabla_j \gamma}{\gamma} \right)^2 + \sum_j F^{jj} \\ & + \{(1+\alpha'\tau) + C_1 \left(\alpha'' - (1+\epsilon)(1+\beta^{-1})(\alpha')^2 \right) \} F^{jj} A_{jj}^2 \\ & - \{1 + C_0 \beta^{-1}\} \sum_{j \in J} F^{jj} \left(\frac{\nabla_j A_{11}}{A_{11}} \right)^2. \end{aligned} \tag{5.1.12}$$

It is also known (see for instance Lemma 2.20 and Lemma 2.21 in [1]) that for any symmetric matrix η_{ij} we have

$$F^{ij,kl} \eta_{ij} \eta_{kl} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \eta_{ii} \eta_{jj} + \sum_{i \neq j} \frac{f_i - f_j}{\lambda_i - \lambda_j} \eta_{ij}^2,$$

and whenever F is concave, then the second term of the right hand side of the equation is non-positive and it should be read as a limit when $\lambda_i = \lambda_j$. Then, using this Lemma, the Codazzi equation (2.1.10) and since $1 \notin J$ we have the following inequality

$$\begin{aligned} -\frac{1}{\lambda_1} F^{ij,kl} \nabla_1 A_{ij} \nabla_1 A_{kl} & \geq -\frac{2}{\lambda_1} \sum_{j \in J} \frac{f_1 - f_j}{\lambda_1 - \lambda_j} |\nabla_1 A_{1j}|^2 \\ & = -\frac{2}{\lambda_1} \sum_{j \in J} \frac{f_1 - f_j}{\lambda_1 - \lambda_j} |\nabla_j A_{11}|^2. \end{aligned}$$

Then following from (5.1.12) we get

$$0 \geq -C_1 - \frac{C_2}{A_{11}} - 2\beta\frac{C}{\gamma} - C + (\psi_\tau\tau - \psi)\, A_{11} + (\psi_\tau + \alpha'\psi)\,\delta_{11}$$
$$+ C_3 F^{jj} A_{jj}^2 - \frac{\psi}{A_{11}} + \sum_j F^{jj} - 4n\{\beta + (1+\epsilon^{-1})\beta^2\}F^{11}\left(\frac{\nabla_j\gamma}{\gamma}\right)^2$$
$$- \left(1 + C_0\beta^{-1}\right)\sum_{j\in J} F^{jj}\left(\frac{\nabla_j A_{11}}{A_{11}}\right)^2 - \frac{2}{\lambda_1}\sum_{j\in J}\frac{f_1 - f_j}{\lambda_1 - \lambda_j}|\nabla_j A_{11}|^2. \quad (5.1.13)$$

Put $\delta = C_0\beta^{-1}$, and recall that since $j \in J$ we have $f_j > 4f_1$. If $\lambda_j > 0$ then the equation

$$(1-\delta)f_j\lambda_1 \geq 2f_1\lambda_1 - (1+\delta)f_j\lambda_j \quad \text{for} \quad j \in J, \quad (5.1.14)$$

holds with $\delta = \frac{1}{4}$. If $\lambda_j \leq 0$, then since $\lambda_n \geq -\mu\lambda_1$ and thus $\lambda_j \geq -\mu\lambda_1$ for all $j = 1, 2, \ldots, n$, then we have $|\lambda_j| \leq \mu\lambda_1$. This implies that (5.1.14) is also satisfied if $\delta = 1/4$ and $\mu = 1/5$. Recall that this choice implies a value for β which depends on $\sup_\Omega |\tilde{\nabla} u|$.

Equation (5.1.14) implies the inequality

$$-\frac{2}{\lambda_1}\frac{f_1 - f_j}{\lambda_1 - \lambda_j} \geq (1 + C_0\beta^{-1})\frac{f_j}{\lambda_1^2}, \quad j \in J,$$

for β sufficiently small, and then we can drop the last two terms in (5.1.13)

$$0 \geq -C_1 - \frac{C_2}{A_{11}} - 2\beta\frac{C}{\gamma} - C$$
$$+ (\psi_\tau\tau - \psi)\, A_{11} + (\psi_\tau + \alpha'\psi)\,\delta_{11} + C_3 F^{jj} A_{jj}^2$$
$$- \frac{\psi}{A_{11}} + \sum_j F^{jj} - 4n\{\beta + (1+\epsilon^{-1})\beta^2\}F^{11}\left(\frac{\nabla_j\gamma}{\gamma}\right)^2$$

Now, recall from (5.1.5) we get

$$0 \geq -C_1 - \frac{C_2}{A_{11}} - 2\beta\frac{C}{\gamma} - C + (\psi_\tau\tau - \psi)\, A_{11} + (\psi_\tau + \alpha'\psi)\,\delta_{11}$$
$$+ C_3 F^{jj} A_{jj}^2 - \frac{\psi}{A_{11}} + \sum_j F^{jj} - 4n\{\beta + (1+\epsilon^{-1})\beta^2\}C\frac{F^{11}}{\gamma^2},$$

which gives us at the end an estimate of the type

$$C_4\lambda_1 + C_3F^{11}\lambda_1^2 \leq C\left(1 + \frac{1}{\gamma} + \frac{F^{11}}{\gamma^2}\right),$$

which concludes the proof of the theorem.

□

Chapter 6

Conclusions

The curvature estimates obtained for the curvature equation of spacelike hypersurfaces in de Sitter space work for a class of prescribing functions that also depend on the slope or tilt of the hypersurface and with a given growth rate. This dependency makes it possible to control in particular the term $-\psi A_{11}$ that appears when no dependency in τ is assumed together with the growth assumption.

This result helps us to address the existence of such spacelike hypersurfaces.

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